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PROCEEDINGS

TWO PARTS — AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852 — PART 2

American Society of Civil Engineers

OCTOBER
1936

TRANSACTIONS NUMBER

1936

VOLUME 62 No. 8, PART 2

PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 62 | OCTOBER, 1936 | NO. 8, PART 2

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1936

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Entered as Second-Class Matter, December 28, 1931,
at the Post Office at Albany, N. Y., under the
Act of March 3, 1879. Acceptance for
mailing at special rate of postage
provided for in Section 1103,
Act of October 3, 1917,
authorized on
July 5, 1918

Published at 99–129 North Broadway, Albany, N. Y.,
by the American Society of Civil Engineers

NOTE.—The Society is not responsible for any statement made or opinion expressed in its publications.

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Page 62, Fig. 28(*b*), the equation, "$M = 10\,\frac{(36)}{(14)}\,\frac{1}{5} = 13.2$", should read,

"$M = 10\,\frac{(36)^2}{(14)^2}\,\frac{1}{5} = 13.2$".

Page 424, Equation (5), "$q = dv = KHv = \phi K\,\sqrt{1 - K}\,H^{\frac{3}{2}}$," should read, "$q = dv = KHv = \phi K\,\sqrt{1 - K}\,H^{\frac{3}{2}}\,\sqrt{2g}$."

Page 424, Fig. 17, "$H = d$", should read "$(H - d)$".

Page 425, Line 28, "45%", should read, "45°".

Page 426, Equation (10), "$g^2 = m^2\,2gH^3 = \phi^2 K\,(1 - K)\,2gH^3$", should read, "$q^2 = m^2\,2gH^3 = \phi^2 K^2\,(1 - K)\,2gH^3$."

Page 427, Line 12, for "The factor, 3" etc., read, "The factor, β", etc.

Page 427, Line 14, for "Equation (3)", read, "Equation (4)".

Page 428, Fig. 20, "Values of ζ", should read "Values of β".

Page 428, Third line from bottom, "(Fig. 22)", should read "(Fig. 23)".

Page 428, Second line from bottom "(Fig. 23)", should read "(Fig. 22)".

Page 429, Line 1, for "abc'''", read "abc''".

Page 430, Equation (23), "$\beta = \frac{1}{1 - \frac{d}{R_0}} \times \frac{1}{\log \frac{1 - d}{R_0}}$", should read,

"$$\beta = \frac{\frac{d}{R_0}}{1 - \frac{d}{R_0}} \times \frac{1}{\log\left(1 - \frac{d}{R_0}\right)}$$".

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* Published in *Proceedings* only.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1923

WEIGHTS OF METAL IN STEEL TRUSSES

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With Discussion by Messrs. Arthur M. Shaw, Joseph G. Shryock, Albert F. Reichmann, Robert W. Abbett, George C. Diehl, F. G. Jonah, Clarence D. Foight, A. H. Fuller, William E. Wilbur, W. N. Downey, J. R. Grant, Theron M. Ripley, T. Kennard Thomson, H. H. Allen, John Venable Hanna, and J. A. L. Waddell.

Synopsis

From the ten sets of curves in this paper a designer can read the ratio of the weight of metal in a truss (per linear foot) to the total vertical load (live, impact, and dead loads per linear foot) for which the truss is to be proportioned. They apply to simple truss, cantilever, and arch spans, for railways, highways, and combinations of both, designed either with carbon steel or with silicon steel.

Modifying formulas or data are given to cover uneconomic proportions, the use of high-alloy steels, and variations in the specified intensities of working tensile stress for carbon-steel structures. There are also four sets of curves showing total weights of metal per linear foot for various classes of bridges. These curves are inserted for the purpose of enabling the computer to determine his trial total load.

The paper concludes with a table indicating the results of the "spot-checking" of the percentage ratios, shown on the diagrams, by means of a score of spans for which the truss weights had previously been accurately determined. Only the last two of these cases were utilized in the preparation of the ratio curves; hence the others serve as an absolutely unbiased check on the accuracy of the diagrams.

Introduction

If a bridge computer were able (generally within a minute or so) to find by diagram, for spans of any reasonable length and for the usual types of

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Note.—Published in February, 1935, *Proceedings*.

[1] Cons. Engr., New York, N. Y.

structure, the ratio of weight of metal per linear foot in a truss to the total vertical load per linear foot for which that truss is to be designed, he would be saved much time and trouble. The writer has attempted to supply that need by collating extensive office records accumulated in practice, and plotting them in the form of curves for the use of the designer.

The results obtained are sufficiently accurate for preparing preliminary estimates of cost and for determining dead loads. In fact, they are so accurate that, when properly modified for variations in the intensities of working tensile stress due to using design specifications other than those recommended by the writer,[2] it is probable that no recomputation will ever be necessary because of any serious discrepancy between the assumed and the calculated dead loads. It would be futile to attempt to attain greater accuracy, because each bridge of any importance has some special individuality of its own, tending to cause slight variations from the most accurate of any curves of weight-of-metal ratios that could be made. Furthermore, each designer has personal idiosyncracies that affect the weights of the trusses he computes, and there is quite a perceptible difference in the metal weights between structures which are truly first-class in every particular and those of only mediocre excellence, or those that have been "trimmed" to the limit.

Extent and Limits of This Investigation

It will be noted that Figs. 1 to 5 include simple truss spans, cantilevers of both Type *A* and Type *C* (Fig. 6), and arches; but they embrace neither suspension bridges nor bascule spans. A little thought will convince any one with experience in bridge designing that it would not be feasible to plot the last two types.

The curves cover trusses of both carbon steel and silicon steel; and formulas are given herein for converting values of the weights of metal for silicon steel to the corresponding weights for all feasible alloy steels, up to those having an elastic limit of 100 000 lb per sq in. They embody single-track and double-track railway bridges, modern highway bridges of 20-ft clear roadway without sidewalks (the cheapest legitimate type of structure), and those of 40-ft clear roadway with two 5-ft sidewalks (the most common dimensions for first-class structures). For any other cross-section of floor, the desired "percentage ratio" can be found by interpolation or extrapolation, remembering that, in establishing the equivalent total width of floor, the sum of the widths of the sidewalks must be divided by two.

In the preparation of Figs. 1 to 5, the live loads for railroad bridges were those identified as Class 60 in "Bridge Engineering,"[3] and those for highway bridges were taken from curves presented elsewhere by the writer.[4] It will be noted that only three standard live loads have been utilized.

In Figs. 1(*e*), 1(*f*), 3(*a*), and 5, the truss weights include the weight of the metal in the bents above the arch rings. The reason for inserting

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[2] "Bridge Engineering," by J. A. L. Waddell, M. Am. Soc. C. E., Vol. 1, Chapters XIV and LXVIII, John Wiley & Son, New York, N. Y., 1916.

[3] *Loc. cit.*, p. 103.

[4] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 823 (see Figs. 1, 2, and 3).

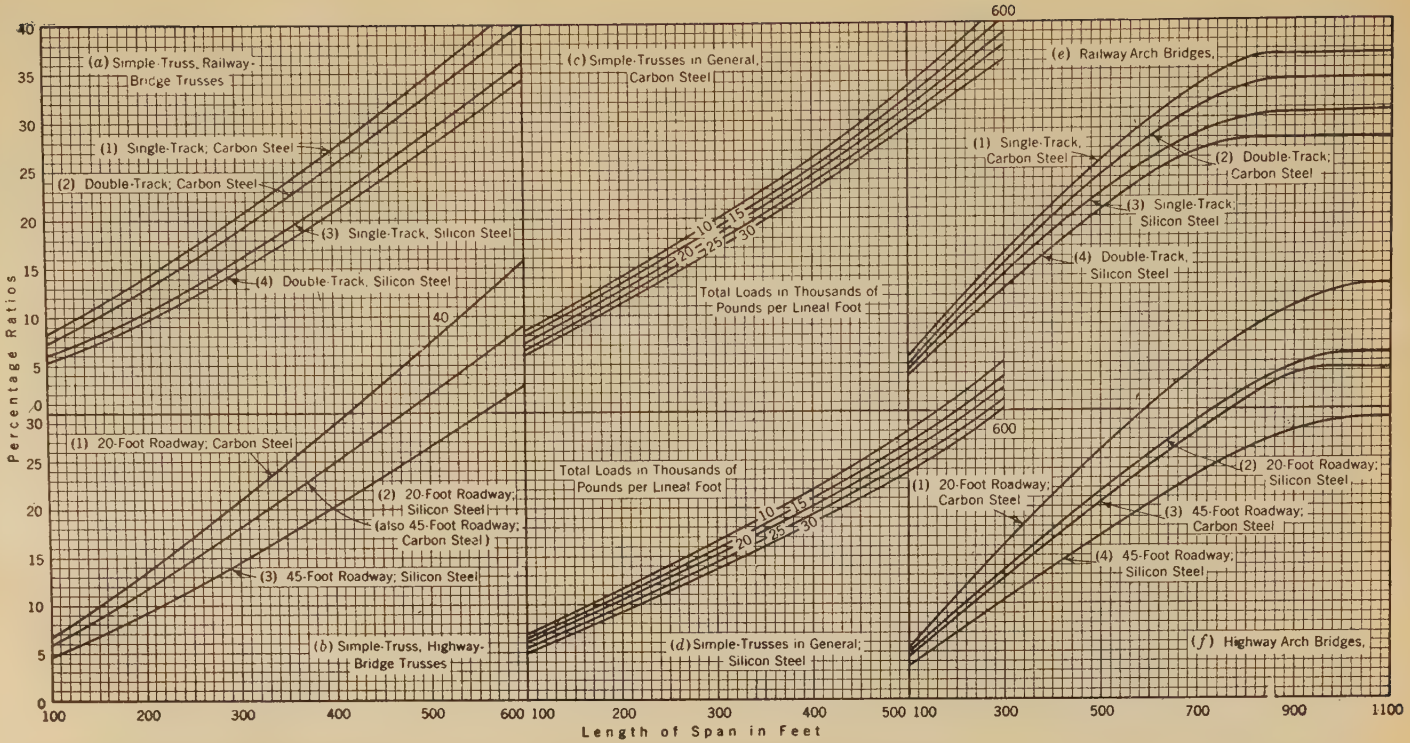

FIG. 1.—PERCENTAGE RATIOS OF WEIGHTS OF METAL PER LINEAR FOOT, FOR TWO BRIDGE TRUSSES, IN RELATION TO THE TOTAL LOADS PER LINEAR FOOT CARRIED BY SUCH TRUSSES.

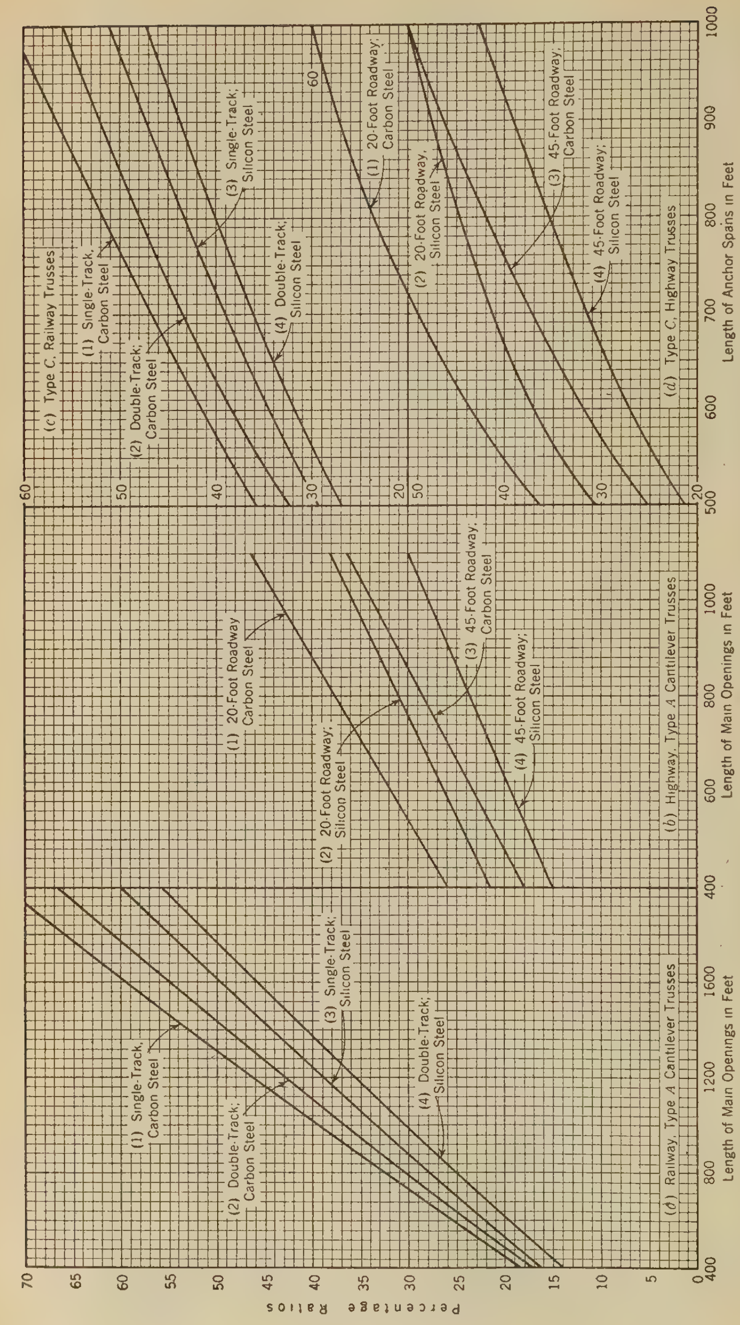

Fig. 2.—Percentage Ratios of Weights of Metal per Linear Foot, for Two Cantilever Bridge Trusses, in Relation to the Total Loads per Linear Foot Carried by **Such Trusses.**

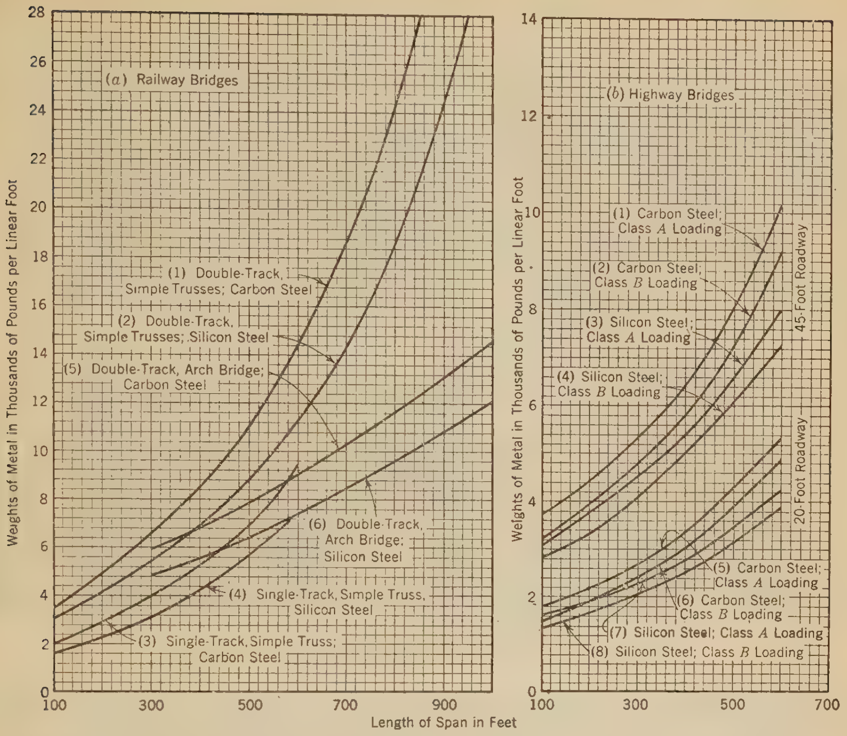

FIG. 3.—TOTAL WEIGHTS OF METAL PER LINEAR FOOT OF SPAN, IN ARCHES AND SIMPLE TRUSS SPANS.

Figs. 3(*a*), 3(*b*), 4, and 5 is to aid computers in finding the trial total load per linear foot for any proposed structure; after doing which the "percentage ratio" can be ascertained from one of the first ten diagrams and another trial made, if necessary.

EXAMPLE

As an example of how a computer would proceed with a new problem, consider the design of a 450-ft, simple-truss, highway span, with Class *A* loading, a 30-ft clear roadway, two 8-ft sidewalks, and trusses of silicon steel. The first step is to determine the character and weight per linear foot of the flooring for both roadway and sidewalks. Assume that this weight is found to be 4 300 lb. The next step is to determine the approximate weight of metal (generally carbon steel) in the floor system and hand-rails, as well as that in the lateral system, which should be of silicon steel. From a prepared table[5] the weight of metal in the floor system and hand-rails is found to be about 1 650 lb per lin ft, and that of the lateral system to be about 540 lb per lin ft. The live load would be found by proportion[6] to be 3 360 lb per lin ft, and the impact,[7] 13.3% of 3 360 or, say, 450 lb per lin ft. The truss

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[5] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 896, Table 2.

[6] *Loc. cit.*, p. 893, Fig. 1.

[7] *Loc. cit.*, p. 895, Fig. 3.

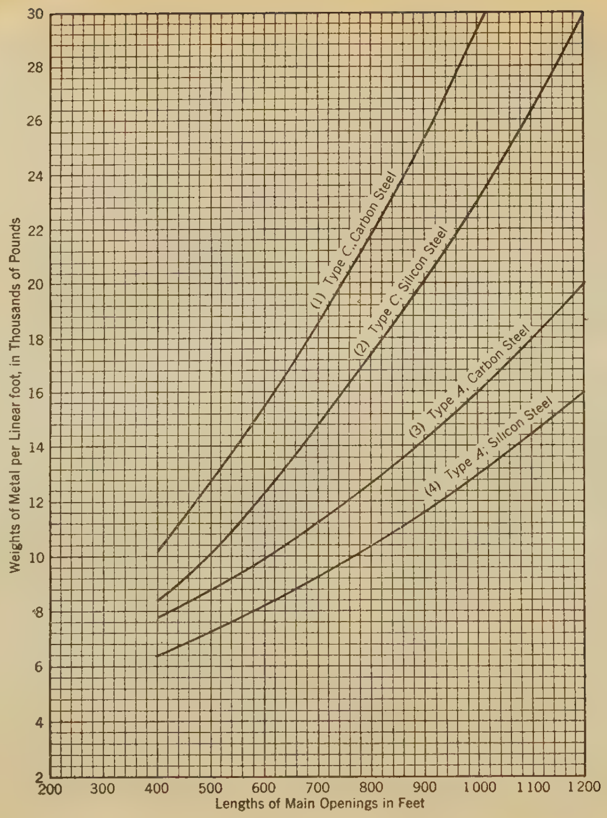


FIG. 4.—TOTAL WEIGHT OF METAL PER LINEAR FOOT OF SPAN, IN DOUBLE-TRACK, RAILWAY, TYPE *A* AND TYPE *C* CANTILEVER BRIDGES.

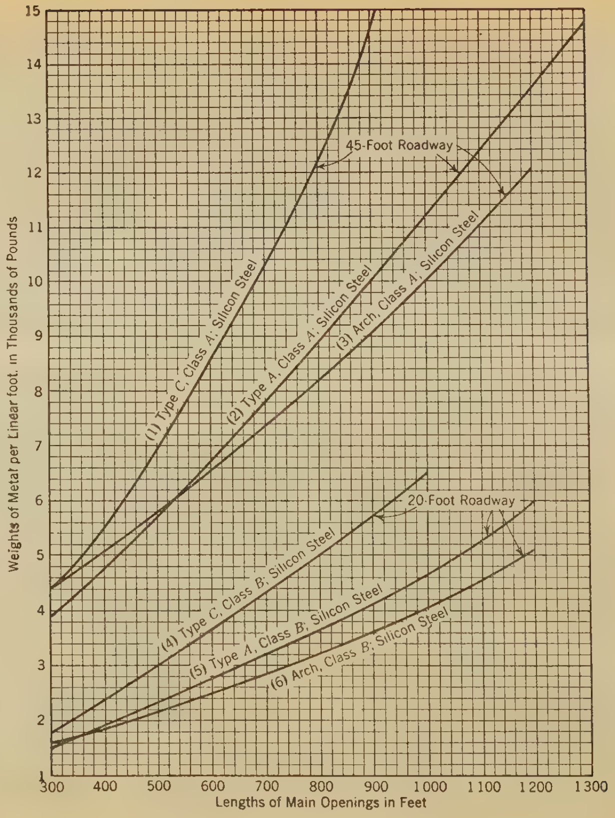
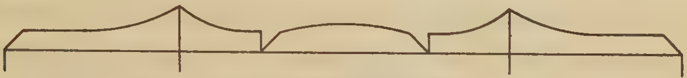


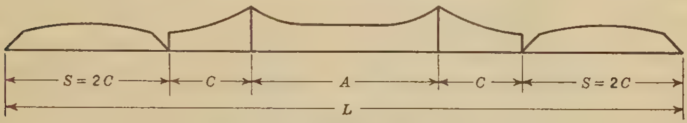
FIG. 5.—TOTAL WEIGHT OF METAL PER LINEAR FOOT OF SPAN, IN HIGHWAY BRIDGES, INCLUDING TYPE *A* AND TYPE *C* CANTILEVERS.

weight might be assumed at 3 000 lb per lin ft. Adding these quantities makes the total load 13 300 lb per lin ft.

From Fig. 1(*b*) the percentage ratio for a 45-ft equivalent roadway is found to be 0.237 and that for a 20-ft roadway, 0.285; the equivalent roadway width is 38 ft. Interpolation between these values yields 25%, which, applied to the total load of 13 300 lb, makes the check truss weight, 13 300 × 0.25 = 3 325 lb. This indicates that the assumed truss weight (3 000 lb) was too small, and the value is raised to 3 400 lb, making the total load 13 700 lb. As before, this would yield 13 700 × 0.25 = 3 425 lb, which checks closely enough to serve as the final truss weight.



(*a*) Type *A*, Cantilever Bridge



(*b*) Type *C*, Cantilever Bridge

FIG. 6.

TOTAL LOADS

The "total loads" per linear foot comprise live load, impact load, and dead load, the latter including the flooring and the total weight of metal. Of course, the truss dead load excludes the metal on the piers or in the anchorages. In determining the equivalent uniform live load per linear foot and the corresponding impact load for any cantilever arm, the length of span to be used is the sum of the length of one cantilever arm and that of the suspended span. In finding the average uniform live load plus impact per linear foot for the entire structure, it is not correct to take the direct average for the three distinct parts of it; each value computed must be multiplied by the total length that it governs, the three results added, and the sum divided by the total length of structure. The same method is to apply to the finding of the average total truss load per linear foot, and of the average weight of truss metal per linear foot, for any cantilever bridge.

MODIFICATIONS OF WEIGHT-CURVE FINDINGS

In determining the weights of metal for the various curves, economic proportions of layout were assumed. If uneconomic designs must be utilized, the weights found from these curves will need modification, as shown in the text following.

Type C Cantilevers.—For greatest economy of metal, the length of the anchor span should be 0.36 times the total length of the structure. If the proportion differs from this, the average weight of truss metal per linear foot will have to be multiplied by one of the quantities given in Table 1.

TABLE 1.—MODIFICATION FACTORS FOR UNECONOMIC LAYOUTS OF TYPE *C* CANTILEVERS

Ratio of length of anchor span to total length of structure	Multiplier for weight given by diagram	Ratio of length of anchor span to total length of structure	Multiplier for weight given by diagram
0.31	1.10	0.35	1.01
0.32	1.08	0.36	1.00
0.33	1.06	0.37	1.05
0.34	1.04	0.38	1.19

Arch Bridges.—For the greatest economy of metal in two-hinged and three-hinged arch bridges the ratios of rise to span are as follows:

Solid-rib structures ....0.175 to 0.225
Braced-rib structures ....0.200 to 0.250
Spandrel-braced structures, with hinge above. .0.225 to 0.275

For hingeless arches each of the foregoing values must be increased by 0.05. The variations in weights of metal in arches caused by the adoption of uneconomic ratios of rise to span length have never yet been determined. For this investigation it is not worth while to make special designs for a number of arch bridges in order to find them—especially in view of the fact that the economic ratio for any case has a fairly wide range (0.05), which is about 11% either above or below the average of the economic ratios.

Uneconomic Truss Depths.—The variations in weight of metal in simple trusses caused by the adoption of uneconomic truss depths almost invariably occur in deck spans, due to a restricted vertical distance between clearance line and grade; and, in these cases, parallel chords are always used.

The correcting ratio, r, is determined thus: Let w = weight of metal, in pounds per linear foot, for a truss of economic depth, d; and w' = weight for any other depth, d', less than d.

The weight, w, is about equally divided between the chords and the web; the chord weight varies almost inversely as the depth; and the web weight varies almost directly with the depth; hence,

$$w' = \frac{w}{2} \times \frac{d}{d'} + \frac{w}{2} \times \frac{d'}{d} \quad \text{.....................(1)}$$

and,

$$r = \frac{w'}{w} = \frac{1}{2}\left(\frac{d}{d'} + \frac{d'}{d}\right) \quad \text{.....................(2)}$$

WEIGHTS FOR ALLOY-STEEL TRUSSES

To find the weight of metal for any alloy-steel truss, with an elastic limit, s_y, all that is necessary is to ascertain from Figs. 1 to 5 the weight of truss metal for silicon steel, and multiply it by:

$0.3 + 0.7r$, for medium spans (less than 500 ft);
$0.25 + 0.75r$, for fairly long spans (500 ft to 1 000 ft); and
$0.2 + 0.8r$, for very long spans (more than 1 000 ft).

In the foregoing values, $r = \frac{45\,000}{s_y}$ (45 000 being the elastic limit for silicon steel). For example: What is the linear weight of truss metal in a simple-truss, double-track railway span of 500 ft, when an alloy steel of 70 000 lb elastic limit is utilized, the total load per linear foot being 25 000 lb for a silicon-steel bridge?

For a silicon-steel bridge, Fig. 1(a) yields a ratio of 0.272; therefore, the weight of truss steel is $0.272 \times 25\,000 = 6\,800$. Furthermore, $r = \frac{45\,000}{70\,000} = 0.643$; and $0.3 + 0.7r = 0.75$. Therefore, the weight required $= 6\,800 \times 0.75 = 5\,100$ lb.

ADAPTATION TO OTHER SPECIFICATIONS

The curves can be utilized for design specifications other than those of "Bridge Engineering," because the main factor in modifying the weight of metal is the comparative averages of the ratios of the principal intensities of working stresses in the two cases. The final average is almost exactly that of the two specified intensities of working tensile stress. The intensity specified in "Bridge Engineering" for carbon steel is 16 000 lb per sq in., and that for any more modern set of bridge specifications is likely to be greater—in most cases as much as 18 000 lb per sq in., and in extreme cases, 20 000 lb per sq in. If the new intensity is denoted by s_t, then $\frac{16\,000}{s_t}$ will be the value of the ratio, r.

If Figs. 1 to 5 indicate a weight, w, for the writer's specifications, the corresponding weight for the other specifications will be:

$w' = w\,(0.3 + 0.7r)$ for short spans (less than 500 ft);
$w' = w\,(0.25 + 0.75r)$ for medium spans (500 ft to 1 000 ft); and,
$w' = w\,(0.2 + 0.8r)$ for long spans (more than 1 000 ft).

For instance, if the intensity of tensile stress for carbon steel in the other specifications is 18 000 lb per sq in., for a span of medium length, r will be $\frac{16\,000}{18\,000} = 0.889$; and $w' = w\,(0.25 + 0.75 \times 0.889) = 0.917\,w$.

In addition to a double checking by the writer of all the calculations made for this paper, the curves have been "spot-checked" by his firm's office force through reference to recorded results of both actual and computed weights of metal, for structures engineered during the last two decades by both himself and his present firm; and the outcome of this spot-checking has been eminently satisfactory, as can be seen by reference to Table 2. As the last two cases were the only ones of this table used in preparing the first ten of the diagrams given herein, the other seventeen cases provide an unbiased check upon their reliability.

In the Portsmouth (N. H.) lift-span, for the sake of æsthetics, the chords were made parallel, whereas the top chords of the two flanking spans, of like length, are polygonal. The lack of economy in the parallel chords (see Column (11), Table 2) accounts for the pronounced variation between the computed ratio and the one found from the diagram. The Jacksonville (Fla.), the Tombigbee River, and the Cape Cod Canal Bridges were very carefully computed; but they have not yet (1935) been built.

TABLE 2.—COMPUTATIONS TO CHECK THE ACCURACY OF FIGS. 1 TO 5

Structure	Type of bridge	Type of span	Width of road-way, in feet*	FOOT-WALKS		Span length, in feet	Total load of bridge, in pounds per linear foot	TRUSS METAL		PER-CENTAGE RATIOS	
				Number	Width, in feet			Weight, in pounds per linear foot	Kind of metal	Com-puted	From the curves
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Springfield, Mass.	Highway.	Fixed....	54	2	8	163.3	5 730	858	Carbon	15.0‡	12.9
Saratoga Lake, N. Y.	Highway.	Fixed....	20	1	5	200.0	6 500†	800†	Carbon	12.4†	12.8
Tombigbee River, Ala.	Railway..	Fixed....	One*	..	..	207.0	14 100	2 160	Carbon	15.3	14.5
Tombigbee River, Ala.	Railway..	Lift......	One*	..	..	161.0	15 940	1 810	Carbon	11.4	11.3
Albany, N. Y.	Highway.	Fixed....	42	2	6	222.0	18 320	2 174	Carbon	11.9	12.0
Albany, N. Y.	Highway.	Lift......	42	2	6	341.0	19 400	3 090	Silicon	16.0	16.0
Jacksonville, Fla.	Highway.	Fixed....	40	2	6	264.0	14 280	2 400	Carbon	16.8	16.3
Jacksonville, Fla.	Highway.	Fixed....	40	2	6	264.0	13 680	1 800	Silicon	13.2	12.5
Portsmouth, N. H.	Highway.	Fixed....	28	1	6	297.1	4 340	870	Carbon	20.0	20.6
Portsmouth, N. H.	Highway.	Lift......	28	1	6	297.1	4 660	996	Carbon	21.3‡	20.6
San Mateo, Calif.	Highway.	Fixed....	27	..	..	297.5	7 620	1 620	Carbon	21.3	20.0
Newark Bay, N. J.	Railway..	Lift......	Two*	..	..	299.0	26 450	3 700	Silicon	14.0	14.2
Newark Bay, N. J.	Railway..	Lift......	Two*	..	..	210.8	27 690	2 630	Silicon	9.5	9.5
Bath, Me.	Railway-highway	Fixed....	One* 20	..	..	330.0	21 460	4 024	Silicon	18.7	18.6
Lexington, Mo.	Highway.	Fixed....	20	..	..	408.0	6,952	2 040	Carbon	29.3	29.3
Lexington, Mo.	Highway.	Fixed. .	20	..	..	246.0	6 290	1 092	Carbon	17.3	17.0
Cape Cod Canal, Mass.	Railway..	Lift......	One*	..	..	544.0	14 920	4 880	Silicon	32.7	32.2
Charleston, S. C.	Highway.	Cantilever	20	..	..	640.0	6 284‡	1 726‡	Silicon	27.5‡	27.2
Charleston, S. C.	Highway.	Cantilever	20	..	..	1 050.0	7 841‡	2 971‡	Silicon	37.9‡	37.0

* Number of tracks indicated for railway bridges.
† Average for two trusses of different live loadings in the same span.
‡ See reference to this item in the text.

The large variation recorded for the Springfield (Mass.) Bridge (Column (11), Table 2) is due to three causes:

(*a*) The bridge is a deck-span structure;
(*b*) The truss depth is only seven-tenths of the economic depth; and,
(*c*) The top chords of the four trusses carry, in bending, a portion of the flooring and its live load, each chord acting as one of the stringers.

Referring to the Charleston (S. C.) Bridges, all the values in Columns (8), (9), and (11), Table 2, were determined by the method previously outlined. The total loads, the weights of metal in trusses, and the computed "percentage ratios" are the averages for the total length of structure; that is, they cover the trusses of the suspended span, the two cantilever arms, and the two anchor arms of a Type *A* cantilever bridge.

A good idea of the accuracy of the "percentage ratios" given on the diagrams may be obtained from the following analysis of Table 2:

(*A*) The variations, regardless of plus or minus signs, between the computed and the diagrammed ratios, omitting from consideration the Portsmouth lift-span, average 0.4 per cent.

(*B*) The corresponding average of the variations, when the plus and minus signs are duly considered, is 0.3%—showing that, in general, the diagrams "under-run" by that amount. The cause of this under-run, undoubtedly, is that in a few cases the truss weights were somewhat greater than usual, owing to some slight abnormality in the governing conditions, whereas all the curves were determined, as closely as practicable, for truly economic structures.

When one considers all the causes that may affect the weight of metal in any truss, especially the unavoidable idiosyncrasies of bridge designers, an average variation of less than 0.5% in truss weights is very small indeed; and an extreme variation (unaccounted for) of less than 1.5% is by no means excessive.

A useful deduction can be drawn from the computations of the 264-ft span of the proposed Jacksonville Bridge. They were made for four cases—carbon steel, silicon steel, ordinary reinforced concrete flooring, and open-grating flooring. The corresponding truss weights were, respectively, 1 200, 900, 930, and 750 lb per lin ft, indicating a saving of metal from the lighter flooring of 22.5% in the trusses of the carbon-steel structure and one of 16.7% in those of the silicon-steel structure. There was also, of course, a further saving of metal in the floor system. From the sum of the money values of these two savings, in any case, must be deducted the extra cost of the flooring itself. In this case the showing favored the open-grating floor. The longer the span the greater is the net saving thereby; and it is evidently much more pronounced in carbon-steel structures than in silicon-steel structures.

DISCUSSION

ARTHUR M. SHAW,[8] M. AM. SOC. C. E. (by letter).—Methods for the quick determination of the weights of metal in bridge trusses are made available in this paper. The curves should effect a tremendous saving in time and effort which heretofore have been necessary in making preliminary studies of proposed structures. By the intelligent use of the graphs presented, it should be possible for the bridge designer to eliminate, promptly, several span-length combinations and types of trusses which, at first, might appear to be worthy of consideration.

Although the paper obviously is of primary importance to the bridge designer, the methods suggested may be used to great advantage by other engineers, especially those engaged in the location of railways and highways. In "easy" country, the locating engineer is not greatly concerned with the type of structure later to be adopted for crossing the various streams encountered. His main problems have to do with securing a reasonable alignment which will permit dropping, by an easy gradient, down to an elevation not far above flood level and at a point resulting in the least possible length of bridge and approaches. Although he gathers data regarding flood flow, navigation demands, and (perhaps) makes something of a study of foundation conditions, in general, his activities and interests are quite fully divorced from those of the designing engineer of the bridge department.

The situation is far different when locating a railway or a highway in a more rugged region. The engineer must have available for convenient use in the field the means of determining, with reasonable accuracy, the cost of various alternate types of structures of varying span lengths, if he is to secure "the best location which the country affords."

As a concrete illustration of the foregoing, the writer recalls a railway location survey which was made in Northern Mexico about 1906. Within a distance of less than thirty miles, the projected line crossed three large arroyos. One of these arroyos was nearly a mile wide, with adjoining topography which permitted easy development to a low crossing but, for the other two, there was no alternative between high viaducts of considerable length and low crossings involving excessively heavy work and objectionable alignment.

The line was constructed as originally located, with a low crossing of the first arroyo (which was obviously proper) and high crossings at the other two, although the writer is not yet satisfied that the most economical location was secured. A serious effort was made to balance the costs of alternate plans, and for this purpose, the engineer was furnished with a certain amount of assistance in the form of assumed cost data, but these data were of doubtful authenticity and were neither sufficiently flexible nor comprehensive. Had the material then been available, which the author has presented, the problems could have been studied intelligently, and it is reasonable to assume that a more economical location would have been secured.

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[8] Cons. Engr., New Orleans, La.

The locating engineer has neither the time nor the facilities for making a careful analysis of alternate structures at each stream crossing, and, in many instances, he would not be competent to make such an analysis even under favorable circumstances; but with all the aids with which he may now be supplied, he should be able to estimate weights and costs with a degree of accuracy which will enable him to follow the true principles of economics. He probably will not select the type of bridge and the arrangement of span lengths which, later, will be found to be the most economical, but his line will be located so that the most economical crossing can be secured. He will have a sound basis for his decision, instead of founding it on a "hunch," or on personal bias, as has been done too often.

JOSEPH G. SHRYOCK,[9] M. AM. SOC. C. E. (by letter).—The bridge engineer should find this paper valuable because it gives him at once the percentage ratio of the weight of metal per linear foot of truss in terms of the known total dead load and live load for all usual spans and types.

As a matter of interest, the writer made a check with the author's tables on an actual highway bridge. The roadway width was 20 ft; it was designed with the K-type of truss, for H-20 loading; its span was 330 ft; and the unit stress for the conventional reinforced concrete floor was 18 000 lb per sq in. Table 1(b) shows a ratio of 23%, which, adjusted for the unit stress intensity, gives 20.4%, whereas a detailed estimate gave a ratio of 20%, which is remarkably close—a variation of only 0.4 per cent.

In the last paragraph of his paper, the author makes a comparison on the Jacksonville Bridge of the saving of 22.5% in the trusses by the use of an open-grating flooring in place of the ordinary reinforced concrete floorslab. The writer made a comparative design and estimate, using a solid steel interlocking channel floor, with an asphalt plank wearing surface instead of the conventional concrete slab, on the aforementioned 330-ft highway bridge, which showed a saving of 22.6% in the carbon steel trusses, in addition to the saving of metal in the floor system. These combined savings more than offset the additional cost of the light-weight metal deck. As the spans increase, the actual savings in cost, of course, become more pronounced.

Some interesting bridges have recently been built by European engineers of chrome copper rustless steel, with physical properties comparable to silicon steel, and with a corrosion resistance four times that of the copper-bearing steel used in the United States. Recent developments in the metallurgy of steel and in the art of welding will doubtless play an important part in the design and construction of the great bridges of the future, and engineers in this country should watch with interest the remarkable progress being made in Europe.

ALBERT F. REICHMANN,[10] M. AM. SOC. C. E. (by letter).—Valuable data are offered in this paper, and the author has presented them in a convenient form which will enable a bridge computer or designer, readily and quickly, to secure the approximate weights of various types of trusses. This assembled information should be of great assistance in preparing preliminary estimates of cost and in determining preliminary dead loads. However, due to the many combinations of loading and the personal equation entering into a design, most engineers are reluctant to use weight curves to obtain dead load weights for final computations.

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[9] Vice-Pres., Director, and Chf. Engr., Belmont Iron Works, Philadelphia, Pa

[10] Vice-Pres., Am. Bridge Co., Chicago, Ill.

In the collection of curves presented with this paper, the author has omitted the continuous type of truss span. This type of span has gained wide recognition in the past decade (1925–35), and has proved economical for long-span bridges. Curves covering the weight of continuous spans might well be added to those of the cantilever and arch spans. With this addition, a designing engineer would be able to determine the approximate weights of the different types of spans, and thus be able to make preliminary estimates and comparisons of cost of various types of spans more readily.

Mr. Waddell is to be commended for contributing to the Engineering Profession a useful set of weight curves. They should prove of distinct value to engineers engaged in making preliminary studies of different bridge projects.

ROBERT W. ABBETT,[11] ASSOC. M. AM. SOC. C. E. (by letter).—The greatest value of the data presented in this paper will be in the field of highway bridge design. Heretofore, several writers have developed formulas for the weights of metal in such structures, but the results from them vary over such a wide range that they are little better than guesswork. In general, the paper will be a distinct aid to younger engineers who have not yet developed the judgment required in preliminary bridge design.

Some pitfalls occur in the diagrams and their use should be tempered with reason. For example, the author states that,

> "* * * they [the results from the graphs] are so accurate that, when properly modified for variations in the intensities of working tensile stress due to using design specifications other than those recommended by the writer,[2] it is probable that no recomputation will ever be necessary because of any serious discrepancy between the assumed and the calculated dead loads."

This statement implies one conclusion that is dangerous, namely, that if there is no serious discrepancy between the assumed and the calculated dead loads no further consideration is necessary. The writer would like to call attention to the fact that both the assumed and the calculated dead loads are considered as being uniformly distributed over the span. In bridges having spans in the longer range included by the author the actual distribution of weight may become an important design consideration. This is particularly true in the case of the cantilever and arch types and to a lesser degree in long-span, curved-chord, simple trusses. Recomputation should always be made on the basis of the actual distribution of weight in the case of large and important structures.

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[11] Asst. Prof. of Bldg. Constr., Thompson-Starrett Foundation, Dept. of Civ. Eng., Union Coll., Schenectady, N. Y.

In cases where concentrated live loads are specified, the practice in determining an equivalent uniform load is not standardized. The author recommends the following method for railroad truss bridges:[12]

> "If one will figure on cars preceding as well as following the locomotives and will compute the maximum moments at the quarter points of the spans, then substitute them for *M* in the proper formula ($M = \frac{3}{32} w l^2$), he will obtain the best averages for the equivalent load curves."

Presumably, the same procedure would be utilized for highway bridges in determining a uniform live load to be used with the diagrams.

Suspension bridges and bascule spans have been excluded from the scope of the investigation, and the author is correct in stating that it is not feasible to plot these types. Recognizing this fact, the diagrams provide a quick method for the preliminary design of all types of steel bridges with the exception of the simple-span, steel beam, the rigid frame, and the continuous truss.

The first two of these types are used for short spans and to estimate their weights is not an outstanding difficulty in design. The continuous truss, on the other hand, is becoming such an important type that it promises to replace the series of simple spans almost entirely, and perhaps the cantilever as well. If this is true there is a definite need for a diagram of weights for the continuous bridge similar to those presented in this paper.

GEORGE C. DIEHL,[13] M. AM. SOC. C. E. (by letter).—The bridge engineer who has not had extensive experience and training is enabled, easily and quickly, to determine the most economical form of steel bridge construction, by the aid of this paper. Table 2 is especially interesting in showing the great accuracy of the charts which the author has prepared. The last paragraph of the paper is especially engaging, as it indicates the possibility of lighter floorings on highway bridges and shows how the open-grating floor will save in the trusses alone, one-fifth the cost if carbon steel is used, and one-sixth in case of silicon steel. Similar charts should be prepared which would show with equal facility and clearness the lessened cost of floor systems if the lighter floors were used and also the relative cost of the different types of floors.

In the United States the perils to life and limb, as well as the property damage, on the public highways are still on the increase. More than 36 000 deaths occurred in 1934, and nearly 1 000 000 people were injured. Many of these hazards occur by skidding on bridges. It is important, of course, that a bridge should be economical in its cost, but it is even more important that it should be safe horizontally as well as vertically. Safety from automobile accidents can be secured at lower cost. This applies particularly to new bridges. The lighter weight of open flooring makes its use practicable on existing structures, when it is found necessary to eliminate slippery roadways and to prevent accidents caused by ice and snow.

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[12] "Bridge Engineering," by J. A. L. Waddell, M. Am. Soc. C. E., Vol I, p. 166.

[13] Cons. Engr., New York, N. Y.

A valuable addition to the paper would be a curve of relative cost of bridges with the conventional types of solid floor as contrasted with open, non-skid flooring.

F. G. JONAH,[14] M. AM. SOC. C. E. (by letter).—The paper offers in a condensed, workable form, a set of curves platted from information accumulated during years of practice and experience. The author is to be commended for his work of assembling and classifying these data and passing it on in a condensed graphical form to the Engineering Profession.

Figs. 1 to 5 will be of great value to bridge engineers (especially to those of less experience) in making preliminary studies and selecting economical types of design for proposed bridges. They will give bridge estimators reliable information that has not been available in such concise and workable form. The percentage-ratio weights of a number of the more recent carbon-steel, simple-truss railway trusses constructed on the "Frisco Lines" have been "spot"-checked and found to agree very closely with the curves of this paper.

CLARENCE D. FOIGHT,[15] M. AM. SOC. C. E. (by letter).—There is no doubt but that the author has made a valuable contribution to bridge designers. In using his curves, however, the designer should know the height ratio, the number of panels, and the type of diagonal bracing that have been used in the design of the bridges from which the curves were derived. Should the most economic height ratio for any span be known, there will be a difference in truss weight for different types of trusses, and for trusses of the same type having different numbers of panels. On a number of occasions, the writer has compared different types and variously proportioned simple-span trusses and has found that a comparison of several trusses having different numbers of panels and different height proportions is generally required before one is able to determine the most economic proportions.

Fig. 7 is a tool for such comparisons, which may help to explain discrepancies in truss weights derived from the author's empirical curves when the truss is not a truly economic structure.

The curves, applying to three common simple-span truss types, are referred to the rational formula,

$$w' = \frac{w_c \times 3.4\,(1 + p)\,L\,\dfrac{K}{n^2}}{f - 3.4\,(1 + p)\,L\,\dfrac{K}{n^2}} \quad \text{.................... (3)}$$

in which w' = average weight of metal per linear foot of truss (not including the members in Fig. 7 which are shown dotted); w_c = average weight per linear foot, carried by truss (includes floor system, all bracing, railing, uniform live load, uniform live load equivalent to a concentration of load, and impact); p = ratio of details to gross section; f = average allowable fiber stress in gross section; and L = span length. Equation (3) can be relied upon to give the truss weight to an accuracy of within 5 per cent.

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[14] Chf. Engr., Frisco Lines, St. Louis, Mo.

[15] Foundation Designer, Jones & Laughlin Steel Corp., South Side Works, Pittsburgh, Pa.

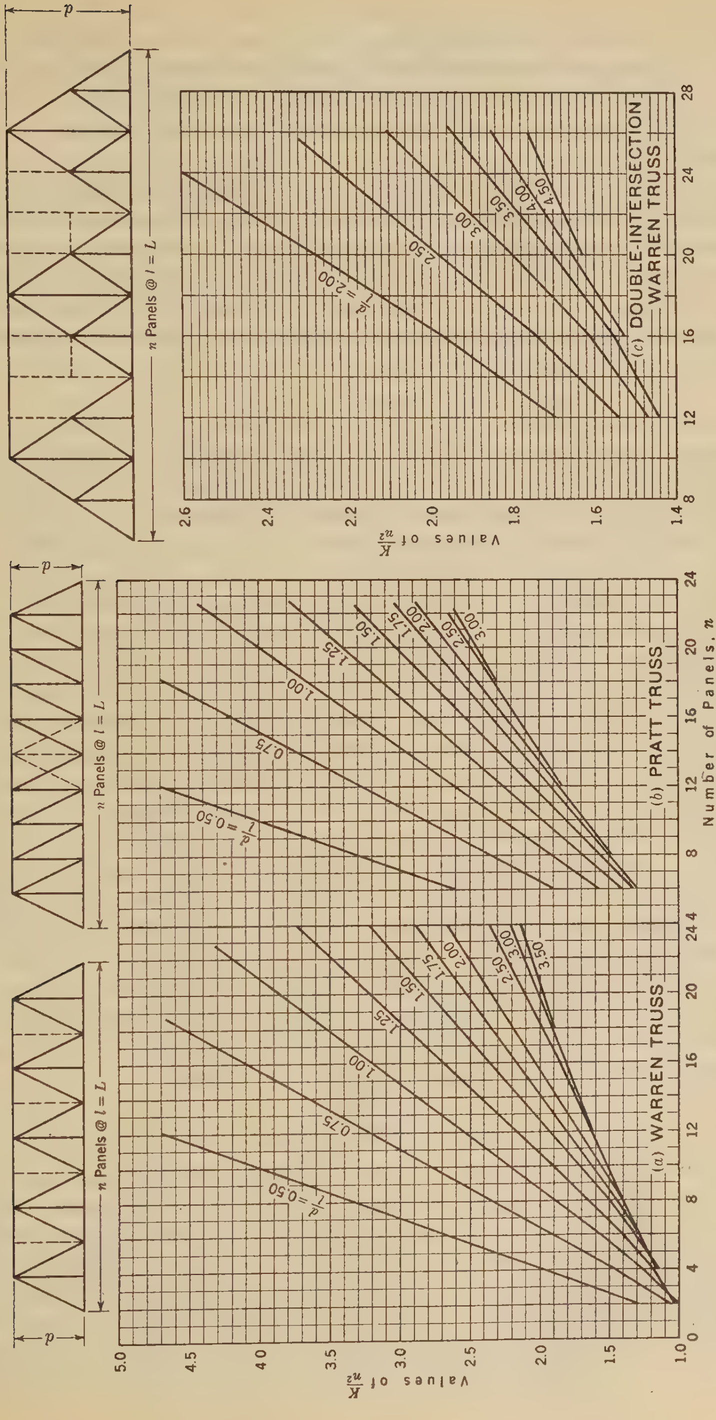

FIG. 7.—COEFFICIENT CURVES.

The proportion of details to the gross section, *p*, will generally vary from 0.30 to 0.35 for pin-connected trusses, and from 0.35 to 0.40 for riveted trusses. The average allowable fiber stress in the gross section, *f*, will be about 75% of the allowable tensile stress in the net section; that is, when a stress of 16 000 lb per sq in. is used, *f* will be approximately 12 000 lb per sq in.

Reference to Fig. 7 will demonstrate that the most economic height proportion varies with the number of panels and the type of bracing. In Table 2, the weight of the Springfield, Mass., Bridge from the curves is 15% lighter than as computed. In the case of the Portsmouth lift-span, the error is 8 per cent. Since the author has not given the number of panels, the depth of truss, or the type of bracing to which his curves apply, large errors may result from variations in these factors.

A. H. FULLER,[16] M. AM. SOC. C. E. (by letter).—The data given in this paper will be of inspiration and of value in office and classroom. Their usefulness will increase as the various results are compared with other available information. The experienced designer will have his own weight data, and will find this paper of value as it touches a field beyond the limit of his experience and as a comparison with his own results.

The young designer and the student who is studying bridge design will find in the paper a marvelous organization of desirable information which has not previously been available. Such men, in considering weight data, should keep in mind the following points: (1) That only a trained designer, working with known specifications, can expect to approach precision in his estimates; (2) that data are now available by which the man with limited experience may make quite a satisfactory preliminary estimate of weight of various types of structures for any loading and specification; (3) that, for those with limited experience in estimating details, the preliminary estimates are likely to be as close to actual weights as computations based upon main sections only; and (4) that weight data, in proper form, are useful in determining the preliminary economic layout of spans for multiple-span bridges.

In 1932 the writer, with Frank Kerekes, M. Am. Soc. C. E., prepared an article[17] on dead loads for simple truss highway and railway bridges. In order to meet the double requirement of making preliminary estimates of weight and to have a basis for the determination of economic span lengths of simple-span bridges he chose the often used expression:

$$w = CL + F \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

in which (in the author's notation), w = weight of steel, in pounds per linear foot of bridge; L = span length, in feet; and C and F are constants. Values of $C = 4$ and $F = 500$ were used for through, riveted, highway bridges with a 20-ft roadway and 8-in. concrete floor-slab on steel stringers, and for the H-15 loading. Values of $C = 10$ and $F = 1\,200$ were adopted for single-track, through, riveted, railway bridges, with E-60 loading and ordinary open floor. The spans in all cases were limited to lengths from 100 to 300 ft. Suggestions were given for modifying the constants to provide for variations in thickness and type of floor, width of roadway, and live load.

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[16] Prof., Civ. Eng., and Head of Dept., Iowa State Coll., Ames, Iowa.

[17] "Analysis and Design of Steel Structures", by A. H. Fuller and Frank Kerekes, D. Van Nostrand Company, Inc., New York, 1933.

The writer has been much interested in making comparisons between results as given by Equation (4) and as taken from Figs. 1 to 5 of the paper. The author's total weights, as taken from Fig. 3 for Class B highway bridges of carbon steel, are considerably heavier than the structures upon which the writer based the constants for his formula. The author's Class A bridges are likewise heavier than Equation (4), with its proposed reduction factor, would suggest for H-20 loading. The railroad bridges cited in the paper are lighter than the writer's examples by about 10 per cent. The writer believes that Equation (4), with proper constants, is more flexible than the curves in Fig. 3 for making a preliminary estimate of weight.

Differences in loads, specifications, and personal equations in designers, are reflected in the differences in weights between those given by the author's curves and those compiled by the writer. Perhaps no appreciable good would result in an attempt to reconcile them. The author states that he gives the actual weights only as a start for using the percentage data in Figs. 1 and 2. These percentage data, with supporting discussion, form the true basis for the paper. The writer finds a much closer comparison with the percentage data than with specific weights. In fact, he would need a definite separation of truss and bracing weights (which are available only in a few instances) to determine whether his percentages were higher or lower than those given by the author. In Equation (4), the term, $C\,L$, includes the trusses and that part of the bracing which varies with the span length; and the factor, F, includes the stringers, floor-beams, and the part of the bracing that is independent of the span.

Using the product, $C\,L$, as representing the weight of trusses for highway bridges, the writer derives percentages of truss load to total load which are uniformly about 10% higher than those of the author. This 10% is a fair value for the part of the bracing that varies with the span length. The check, then, is remarkably close for highway bridges. For railroad bridges, the writer found a greater discrepancy, but in the same direction. The effect of the bracing will reduce the differences to negligible magnitudes.

The writer accepts the percentages of the paper as a definite contribution to the problem of making and adjusting estimates for weights of bridges. The extension of data from simple spans of moderate length to simple spans of great lengths and to cantilever and arch bridges makes this paper a notable addition to the literature on this subject.

WILLIAM E. WILBUR,[18] M. AM. SOC. C. E. (by letter).—For some years the writer has been using curves presented by the author,[2] or others similarly constructed, for preliminary estimates, as well as for obtaining an assumed dead load for designing. This paper which brings the information up to date, constitutes a welcome addition to the available tools of the designing engineer.

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[18] Asst. Engr., Harrington & Cortelyou, Kansas City, Mo.

The device of expressing the weight of the truss as a percentage of the total load results in a considerable simplification of the diagrams. The process could readily have been carried one step further, by expressing the truss weight as a percentage of the superimposed load; that is, the total load exclusive of the weight of truss. When this is done, the weight of the truss is obtained directly, without the necessity of first assuming its weight before entering the diagram, as was done in the example given by the author. In Fig. 8, the author's diagrams, Fig. 1(*c*) and Fig. 1(*d*), giving weights of simple trusses of carbon and silicon steel, have been replotted on this basis.

The weights given in the paper agree quite closely with the experience of the writer. Assuming the same basic unit stresses, the minor differences of specifications will have no great effect on the weights of properly designed trusses. For ordinary highway bridges of carbon steel and moderate span, with a basic unit stress of 16 000 lb per sq in., the writer has found the following formula quite satisfactory:

$$w = \frac{1}{2} L + \frac{w_t L}{1\,600} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which (conforming to the notation of the paper), w = weight of truss, in pounds per linear foot; L = span length, in feet; and w_t = load on one truss, including weight of truss, in pounds per linear foot. In terms of the plotting used by the author (Fig. 1(*c*)), this gives a straight-line variation. The author's lines are slightly curved, which is more accurate for the longer spans and heavier loadings.

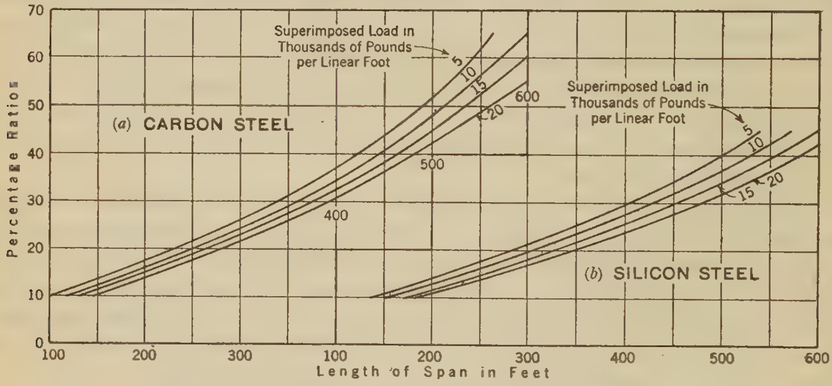

FIG. 8.—PERCENTAGE RATIOS OF WEIGHTS OF METAL FOR SIMPLE BRIDGE TRUSSES IN GENERAL.

Attention should be called to the fact that the loads per foot, given in the author's diagrams, are the total for two trusses, rather than for one truss. The minimum given, 10 000 lb per ft, is rather too heavy for short, light spans. For highway bridges of 20-ft roadway with concrete floor, designed by specifications for *H*-15 loading, advanced by the American Association of State Highway Officials, the total load per foot of span for short spans is only slightly more than 5 000 lb.

In his previous paper on "Economic Proportions and Weights of Modern Highway Cantilever Bridges."[19] the author gave, for structures of combined silicon and carbon steel, an estimate of the percentage of each kind of steel in the structure. This information is of great value in making preliminary estimates; and the writer would suggest that the value of the paper will be considerably increased if the author will include, in his closing discussion, such data for the structures covered in this paper.

W. N. Downey,[20] Assoc. M. Am. Soc. C. E. (by letter).—The Engineering Profession is indebted to the author for considerable information concerning the weights of metal in bridges; his paper is a notable contribution, covering as it does such a variety of types. The following discussion is restricted to comments on the author's weights of simple-span highway bridges. The curves submitted for comparison with Figs. 1 to 5 were designed to carry live loads and impact loads in accordance with the recommendations of the American Association of State Highway Officials (1931). The allowable unit stress in tension was taken at 16 000 lb per sq in. on the net section except where noted. The designs provided for a concrete floor and a future surface item, the combined weight of which was 115 lb per sq ft of floor area.

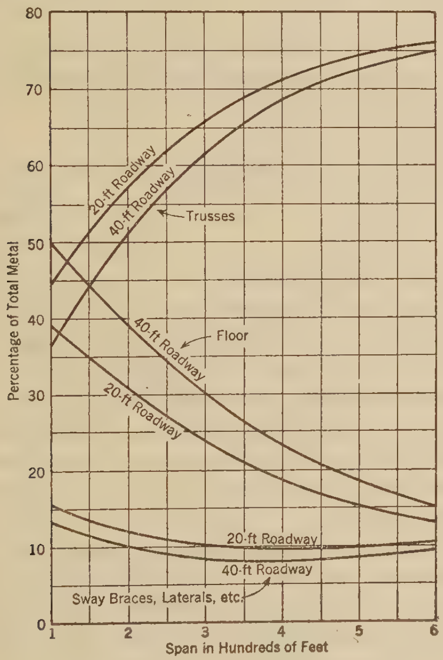

Fig. 9.— Distribution of Metal.

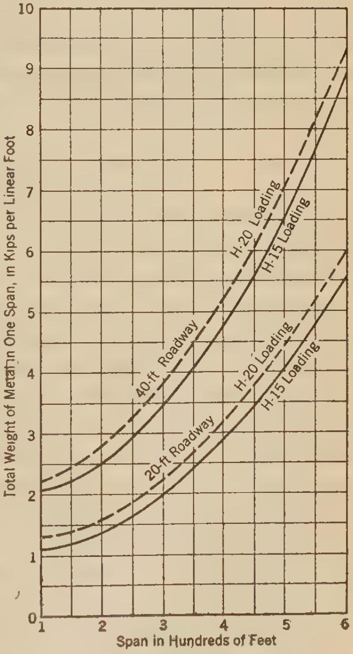

Fig. 10.—Total Weights of Metal. (1 Kip = 1 000 Pounds.)

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[19] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 888.

[20] Louisville, Tenn.

The writer obtained a close check on the percentage ratios of Fig. 1(*b*), his values being about 1% higher than the ratios shown by the author. This is gratifying since it gives a check on the portion of the bridge weight that is most difficult to estimate, namely, the trusses. In this connection, Fig. 9 shows the distribution of the metal between the major elements of the span. The weights of hand-rails, expansion devices, and shoes are not included in the weights given in the paper.

Presumably, the total weights of metal given by Fig. 3 (*b*) include the weights of the hand-rails and expansion dams. The total weights obtained by the writer for bridges with 20-ft roadway, are in good agreement with Fig. 3(*b*) for the longer spans, but he gets somewhat smaller weights for spans of less than 300 ft.

Frequently, one has available the weight of metal in a bridge of a certain span length and roadway width, but desires the weight of metal for another bridge of the same span having a different roadway width. From a study of data at hand the relation expressed by Equation (6) was found to give good results. The weight of metal may be determined for any wider roadway from the weight for a 20-ft roadway by the following equations:

$$w_w = C\,w_{20} \quad\quad (6)$$

in which w_w = weight of structural steel, in pounds per square foot of roadway for the wider roadway; w_{20} = weight of structural steel, in pounds per square foot of roadway for the 20-ft roadway; and,

$$C = 1 - \frac{R'}{100}\left[0.10 + \frac{R'}{100} \times \frac{L}{500}\right] \quad\quad (7)$$

in which R' = the width, in feet, of the wider roadway; and L = the span, in feet, of the bridge.

Attention is called to the fact that, in Equation (6), w_{20} must be the weight of metal, in pounds per square foot of roadway, for a bridge with a 20-ft roadway. Suppose, for instance, that for a certain span, the weight of metal is known for a bridge with a 30-ft roadway and it is required to find the weight for a bridge having a 40-ft roadway. First, using Equations (6) and (7) and the known weight, w_w, for the 30-ft roadway, compute the weight, w_{20}, for a 20-ft roadway bridge. Then, with the value of w_{20} known, apply Equations (6) and (7) again to find the weight of metal, in pounds per square foot, in the bridge with a 40-ft roadway.

Fig. 10 shows the total weights of metal in the spans as determined by the writer and is included for comparison with the values given in Fig. 3(*b*). Fig. 11 permits the determination of desirable values that are not directly determinable by Figs. 1 to 5 of the paper. It yields the weights of the floor system and, incidentally, shows the effect on the floor weight of changing the panel length.

As is demonstrated in the paper, the depth of truss affects its weight materially. Coefficients for the determination of the truss depths for bridges with 20-ft roadways may be read from Fig. 12. For roadways wider than

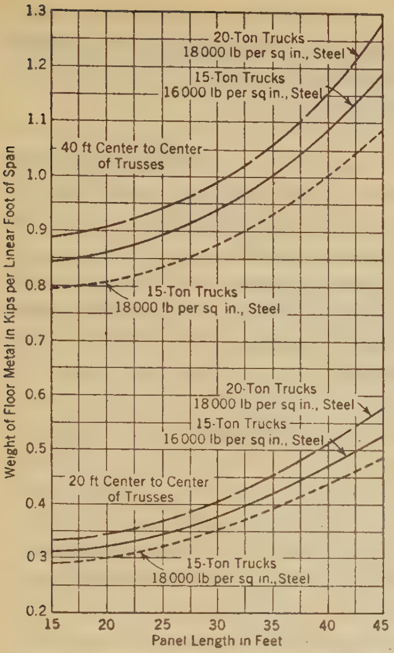

FIG. 11.—WEIGHT OF METAL IN FLOOR FOR ONE SPAN.

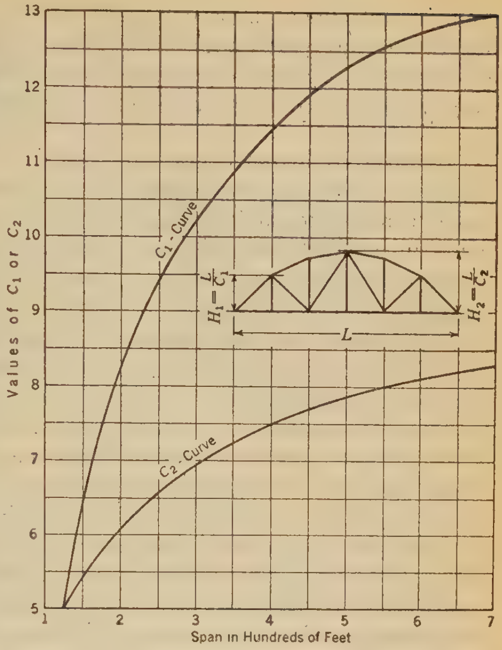

FIG. 12.—DEPTHS OF TRUSSES WITH POLYGONAL TOP CHORDS AND 20-FOOT ROADWAY.

20 ft add 1 ft to the center depth for each 5 ft of additional roadway width. Trusses of these proportions have been found to be pleasing in appearance and economical of metal.

J. R. GRANT,[21] M. AM. SOC. C. E. (by letter).—The Engineering Profession is deeply indebted to the author for making available to bridge engineers the curves of truss weights and other valuable information included in this excellent paper. It is essential, particularly during the early stages of a bridge project, to have reliable approximate weights of the steel part of the structure on which to base estimates of the cost of alternative plans. The floor can be computed quickly and the bracing can be estimated, so that with the truss weights readily available from the curves, an approximate estimate of the weight of steel in the superstructure is soon obtained.

The writer checked some of the percentages given by Figs. 1 to 5 for a number of spans for which he had the weights available, and although there was considerable variation in some cases it was usually not difficult to account for the difference between the computed percentage ratio and that from the curves. They should enable any bridge engineer, who is familiar with the design of the class of structure proposed, to make a fairly close estimate of the truss weights. It is essential that due allowance be made for the various factors, besides unit stresses and variation from the economic depth, which will have an influence on these weights. In considering combined railway and highway structures, the writer found that the percentage ratio from the curves for the railway part of the bridge, when increased or decreased by one-third the difference between this percentage and that for the highway part, checked very well with the computed or actual weight percentage.

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[21] Structural Engr., Vancouver, B. C., Canada.

THERON M. RIPLEY,[22] M. AM. SOC. C. E. (by letter).—In this paper the author has added another rung to the ladder by which the general practitioner can rise to an intelligent answer to many questions without spending hours in detailed computations to secure a general result.

An example of the value of service, as covered by papers such as this, was recently proved by a preliminary report submitted by the writer. The basis of this report was the question: "Can that highway be completed for $400 000"? The report was wanted (as usual) "at once"—"right away"—"as soon as possible."

This section of highway, less than a mile long, will cross four railroads—under one and over three—requiring one single-span, single-track railroad bridge and one 6-span, highway bridge with a 40-ft roadway. Preliminary plans and estimates had been made by various municipal and railroad organizations over a period of several years up to 1931; item prices and unit quantities spread over a wide range in these former estimates. For a preliminary report such as that required for a case of this kind, the handbooks of a fabricating-erecting steel company will supply weights of steel which agree, with sufficient accuracy, with the weights determined by Fig. 3 of the paper.

This question of "loads" for highway bridges has become of greater importance as the class of traffic has changed. Live loads (moving loads) have increased from an average maximum, say, of 4 tons per unit to an average maximum of 20 tons. or more, per unit; and speeds have increased from 3 to 25 miles per hr, with impact "entering the picture" in a geometrical ratio.

At the beginning of the Twentieth Century, the sale of small bridges identified as "town-bought-bridge-company-competitive" designs, was a "racket" in the State of New York. Structures were sold, not necessarily to the lowest bidder, or to the most reliable firm, but to the most generous payer. As the entire cost must come out of the bridge, the keen competition for work on the part of the bridge companies, and for money on the part of the buyers, created a situation in which the steel in the structure became of minor importance.

Another type of this class of bridge, was the so-called "bedstead design," found principally in the mountainous regions. It was composed of white-painted railings of 1-in. and 1½-in. pipe, and resembled nothing so much as the white iron bedstead found in mountain inns.

The flimsy, spider-web trusses of these bridges are no longer considered "good business" as damage claims are difficult to contest and expensive to settle. The bridge engineer has come into his own on highway bridges of practically all lengths and it is with information such as that supplied in this paper that the engineer who has devoted his time to hydraulics, or some other specialty, may be saved much time in telling his client whether a general plan is possible with the money available. In this manner, subsequently, another case is provided for the bridge expert.

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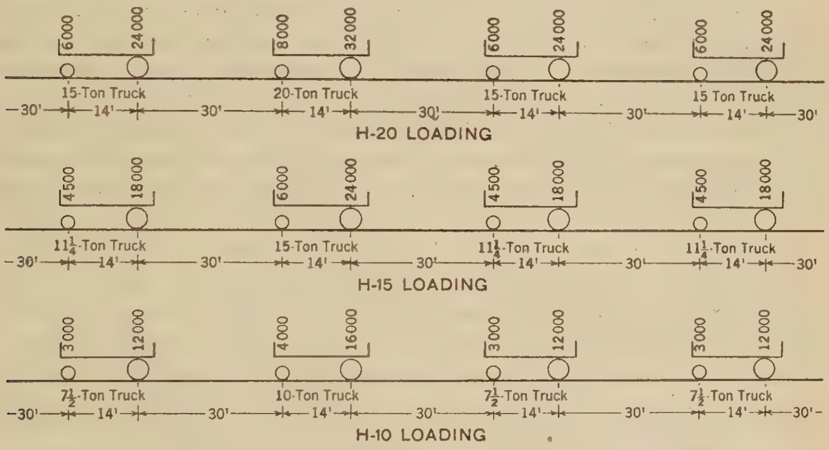
[22] Cons. Engr., Buffalo, N. Y.

An idea of the present condition of thought upon this matter of loads for highway bridges may be gathered from a study of the following specifications[23]:

(*a*) The highway loading shall be of three classes, namely, *H*-20, *H*-15, and *H*-10, and may be either truck train loadings or equivalent loadings. Loadings *H*-15 and *H*-10 are 75% and 50%, respectively, of Loading *H*-20. The truck train loading shall be as shown in Fig. 13 and shall be used for loaded lengths of less than 60 ft. It shall consist of one truck of the gross weight indicated by the loading class followed by, or preceded by (or both followed and preceded by) a line of trucks of indefinite length, each of the following or preceding trucks having a gross weight of three-fourths that indicated by the loading class.


6000
24000
15-Ton Truck
8000
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20-Ton Truck
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—30'—14'—30'—14'—30'—14'—30'—14'—30'
H-20 LOADING
4500
18000
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—30'—14'—30'—14'—30'—14'—30'—14'—30'
H-15 LOADING
3000
12000
7½-Ton Truck
4000
16000
10-Ton Truck
3000
12000
7½-Ton Truck
3000
12000
7½-Ton Truck
—30'—14'—30'—14'—30'—14'—30'—14'—30'—
H-10 LOADING

FIG. 13.—SPECIFIED TRUCK TRAIN LOADING (1 KIP = 1 000 POUNDS).

(*b*) If the loaded width of the roadway exceeds 18 ft, the specified loads shall be reduced 1% for each foot of loaded roadway width in excess of 18 ft, with a maximum reduction of 25 per cent.

The *H*-20 loading referred to in Specification (*a*) is shown in Fig. 13. Loadings *H*-15 and *H*-10 differ only in the applied loads, the spacing of trucks being the same for all classes. Specification (*b*) indicates the permissible reduction in "load intensity." If the author has used the *H*-15 and *H*-10 loadings as his Class *A* and Class *B* loadings (which is what they are termed in the Standard Specifications[23]), it is believed that his paper is not consistent with present practice in the Engineering Departments of the State of New York. As far as the writer can learn this State is designing, and approving the design of, bridges under Class *H*-20 loadings and using 20 ft as the distance between trucks.

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[23] Standard Specifications for Highway Bridges and Incidental Structures, by the Am. Assoc. of State Highway Officials, 1931; Specifications 5.2.7, p. 174, and 5.2.10, Fig. 5, p. 176.

Highway bridges are now being built with 40-ft roadways, consisting of reinforced concrete pavements 12 in. thick and, where they cross a railroad, the steel floor systems are cement-coated. Under such construction the dead weight of reinforced concrete per linear foot of bridge, is about 3.83 tons; to carry this concrete the floor system will weigh about 0.54 ton per lin ft of bridge. Adding impact loads, wind pressure, or the weights of curbs, sidewalks, snow, pipe lines, etc., indicates a wide departure from the spider-web type of truss.

Loadings for highway bridges present an open question at the present time. Various types of steel floors are being studied in order to reduce the tremendous weights of solid concrete or block pavement. Live loads are pure assumption based, perhaps, upon the hope that the bridge will endure even if the truck operator insists on transporting all the freight in the country in unregulated loads and at, practically, unregulated speeds.

As a large majority of the bridges being built upon highways (those carrying the obstruction over the highway as well as those carrying the highway over the obstruction) are of 300 ft, or less, in span length, it is hoped that the author will describe somewhat in more detail loadings and other factors that affect Figs. 1 to 5. The final design must always be made by the bridge engineer, as guessing and "skinning" are ruled out when dealing with present-day, and probable future, traffic weights, volumes, and speeds.

T. Kennard Thomson,[24] M. Am. Soc. C. E. (by letter).—Recalling vividly an experience of more than forty years ago, the writer does not hesitate to state that, had it been available, this paper would have been worth considerable money to him in times past. He had been assigned, suddenly, the task of designing and estimating the cost of a bridge over the East River in New York City, to be composed of six tracks, two roadways, and two sidewalks. The maximum time allotted to the task was one week.

The only truss bridge of comparable length at that time was the Firth of Forth Bridge in Scotland which had been described[25] by the late Sir Benjamin Baker, Hon. M. Am. Soc. C. E. Designing the floor system carefully and the lateral system proportionally, the writer used the dead-load weights of the Forth Bridge in solving the stresses for the bridge graphically. By successive repetitions of this procedure a schedule of dead loads was determined which seemed satisfactory.

Four bridge companies were requested to check the design and the quantities, and the only organization that could be interested required the services of a squad of men for four months to produce the complete design.

The foregoing account demonstrates roughly, the tremendous amount of time that can be saved by the use of curves, such as those in the paper. Even in those days, the designer had at least one standard handbook, produced by steel manufacturers. They had an infinitely easier job than the late James B. Eads, F. Am. Soc. C. E., for example, who had to invent his own structural shapes and fight to have them rolled.

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[24] Cons. Engr., New York, N. Y.

[25] "Bridging the Firth of Forth," by Sir Benjamin Baker, Lond., 1887.

The paper is the result of such tremendous amount of strenuous and careful work, that all bridge designers to whom it is made available should be grateful for its presentation.

H. H. ALLEN,[26] M. AM. SOC. C. E. (by letter).—The results of the studies and investigations reported in this paper are borne out remarkably well by the extensive office records available to the writer. He has used the nine Type *A* cantilever bridges which he (with Wilson T. Ballard, M. Am. Soc. C. E.) presented in the discussion of the author's paper entitled "Economic Proportions of Weights of Modern Highway Cantilever Bridges,"[27] as a basis for the data submitted herewith, and, in addition, one other Type *A* cantilever bridge designed and constructed by the writer's firm, namely, the cantilever bridge across the Kanawha River, at St. Albans, W. Va.

Table 3 is an arrangement of these ten Type *A* cantilever bridges in a manner similar to Table 2 of the paper. In addition, Table 3 contains similarly arranged data for the suspended spans of the Type *A* cantilevers. Data for four simple spans are also included, namely, the 156-ft simple spans that form the approaches at each end of the cantilever bridge across the Ohio River, at Ashland, Ky.; the 144-ft simple approach span to the cantilever bridge across the Ohio River, at Huntington, W. Va.; the 300-ft lift span of the James River Bridge, at Newport News, Va.; and the 208-ft approach spans of the same bridge.

All the Type *A* cantilever bridges, except the one at Ashland, Ky., were designed for the use of medium structural grade steel with a basic working stress of 18 000 lb per sq in. The Ashland Bridge was designed for the use of silicon steel and heat-treated eye-bars in the main members of the trusses, with a basic working stress of 24 000 lb per sq in.

The percentage ratios in Table 3 (Columns (10) and (11)) are based on an adjustment in the weight of the trusses, as described in the author's paper, in order properly to evaluate the truss weights on the basis of a working stress of 16 000 lb per sq in. instead of the 18 000 lb per sq in. used. The computed percentage ratios for the ten cantilever bridges agree very closely with those taken from Figs. 1, 2, 3, and 5. All these computed ratios are higher than those taken from the plotted curves, except the cantilever bridge at St. Albans, W. Va., which is 2.7% lower. This can be explained as being due to the use of rolled sections throughout the trusses for this bridge, thereby reducing the weights of details in the trusses materially.

The larger variations between the percentage ratios computed and taken from the curves of the paper for the nine suspended spans can be explained by the additional erection material included in the weights of the trusses for these spans, as compared with the weights of similar simple spans erected on falsework. The close agreement between the computed percentage ratios and those taken from Figs. 1 to 5 for the four simple spans listed, is remarkable.

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[26] Vice-Pres., The J. E. Greiner Co., Baltimore, Md.

[27] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 931.

TABLE 3.—WEIGHTS OF METAL MODERN HIGHWAY BRIDGES

Structure	Type of span	Width of roadway, in feet	Foot-walks		Main span, length, in feet	Total load, in pounds per linear foot	Truss Metal		Percentage Ratios		Unit stress, in kips† per square inch
			No.	Width, in feet			Weight, in pounds per linear foot	Kind of metal	Computed	From the curves	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Huntington, W. Va. (Ohio River)	Cantilever	22	1	8	700	9 408*	3 165	Carbon	33.7	32.8	16
Bellaire, Ohio (Ohio River)	Cantilever	22	1	6	700	8 993*	3 210	Carbon	35.7	33.2	16
Pt. Pleasant, W. Va. (Kanawha River)	Cantilever	20	1	4.25	600	7 452	2 462	Carbon	33.0	30.6	16
Ashland, Ky. (Ohio River; Long A Arm)	Cantilever	22	1	5	739	7 800	2 426	Silicon	31.1	28.2	24
Portsmouth, Ohio (Ohio River)	Cantilever	22	1	6	700	7 090	2 790	Carbon	39.4	32.0	16
Cabin Creek, W. Va. (Long A Arm)	Cantilever	20	1	4	450	7 130	1 785	Carbon	25.0	26.2	16
Cabin Creek, W. Va. (Short A Arm)	Cantilever	20	1	4	450	7 295	1 860	Carbon	25.5	26.2	16
Ashland, Ky. (Ohio River; Short A Arm)	Cantilever	22	1	5	739	7 565	2 222	Silicon	29.0	28.2	24
Norfolk, Va. (Elizabeth River)	Cantilever	36	2	10	1 198.5	22 620*	9 280	Carbon	41.0	38.6	16
St. Albans, W. Va. (Kanawha River)	Cantilever	20	1	4.25	450	7 145	1 695	Carbon	23.7	26.4	16
Huntington, W. Va. (Ohio River)	Suspended	22	1	8	350	8 312*	2 290	Carbon	27.54	24.0	16
Bellaire, Ohio (Ohio River)	Suspended	22	1	6	350	7 844*	2 247	Carbon	28.70	24.1	16
Pt. Pleasant - Henderson, W. Va.	Suspended	20	1	4.25	300	6 511	1 673	Carbon	25.7	20.6	16
Ashland, Ky. (Final) (Ohio River)	Suspended	22	1	5	350	7 119	1 774	Silicon	24.9	20.7	24
Portsmouth, Ohio (Ohio River)	Suspended	22	1	6	350	6 068	1 970	Carbon	32.4	23.7	16
Cabin Creek (Kanawha River)	Suspended	20	1	4	200	6 389	1 086	Carbon	17.0	12.9	16
Norfolk, Va.	Suspended	36	2	10	493.5	18 878*	4 780	Carbon	25.4	31.3	16
Ashland, Ky. (original design)	Suspended	22	1	5	350	6 516	1 460	Silicon	22.4	20.7	24
St. Albans, W. Va.	Suspended	20	1	4.25	200	6 443	1 100	Carbon	17.1	13.0	16
Ashland, Ky.	Simple	20	1	5	156	6 720	760	Carbon	10.3	10.5	16
Huntington, W. Va.	Simple	22	1	8	144	12 394*	1 370	Carbon	11.0	9.3	16
James River	Lift	22	..	....	300	5 998	1 260	Carbon	21.0	20.8	16
James River	Simple	22	..	....	208	6 860	940	Carbon	13.7	13.8	16
Typical spans	Simple	20	..	....	200	5 556	780	Carbon	14.3	13.5	16
Typical spans	Simple	45	..	....	200	13 048	1 645	Carbon	12.5	11.5	16
Typical spans	Simple	20	..	....	300	6 197	1 397	Carbon	22.5	21.0	16
Typical spans	Simple	45	..	....	300	14 259	2 804	Carbon	19.7	18.0	16
Typical spans	Simple	20	..	....	400	6 985	2 155	Carbon	30.8	29.0	16
Typical spans	Simple	45	..	....	400	15 783	4 287	Carbon	27.2	25.0	16

* Including electric street railway.
† 1 kip = 1 000 lb.

It will be noted, however, that in each case the computed percentage of truss weights is slightly higher than that taken from the curves, except in the case of the 208-ft James River spans for which the computed percentage ratio is 0.1% less than the ratio taken from the curves. This truss is of the Warren type with the intermediate posts omitted, which may explain the reduction in truss weights.

The fact that the computed percentage ratios are uniformly higher than those taken from the curves led to a further investigation by the writer. Fig. 14 shows the ratio of truss weights to total weight carried for simple highway trusses on which are plotted Curves *A*, *B*, and *C*, corresponding to Curves 1, 2, and 3, Fig. 1(*b*). The curves shown in dashed lines on Fig. 14, immediately above Curves *A* and *B*, respectively, are plotted by collating office records which include the weights of steel subdivided into floor, bracing, and trusses for ideal typical simple highway spans for Class *A* loading, varying by 50 ft in length from 200 ft to 500 ft, for roadway widths varying between 16 and 36 ft by 4-ft variations, with additional weights for one or two sidewalks of 6 ft, 8 ft, or 10 ft. The curves of Fig. 14, shown in dashed lines, were obtained by using the percentage of truss weights of these typical, ideal, highway, simple-truss spans compared with the total weight of these spans, and agree more closely with the computed percentage ratios in Table 3 than those of the paper. This may be attributed to the personal equation of the designer, as expressed by the author:

> "* * * each designer has personal idiosyncrasies that affect the weights of trusses he computes, and there is quite a perceptible difference in the metal weights between structures which are truly first-class in every particular and those of only mediocre excellence, or those that have been 'trimmed' to the limit."

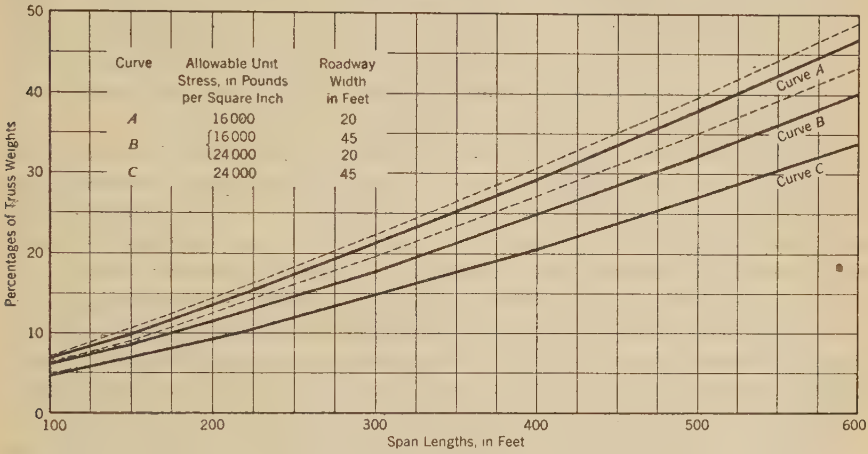

FIG. 14.—RATIO OF TRUSS WEIGHTS TO TOTAL WEIGHT CARRIED, SIMPLE HIGHWAY TRUSSES.

Regardless of the relatively small differences obtained from the results of the author's investigations and the data submitted by the writer, the writer feels that this method of approaching the problem of quickly estimating the weights of various types of trusses is an excellent one. These curves, of course, will need to be somewhat modified where the use of rolled sections reduces the weight of truss details.

JOHN VENABLE HANNA,[28] M. AM. SOC. C. E. (by letter).—In the early days of the Kansas City Terminal Railway Company's project for a new Union Station, and trackage in connection therewith, the writer was under the necessity of making, quickly, preliminary estimates of cost for a considerable number of structures. These structures were to carry urban highway traffic as well as railway traffic. In addition to the Union Station Building and the trackage immediately required for it, additional approach tracks were needed, which it was thought highly desirable should be entirely free from grade crossings. To estimate the cost of the structures necessary for this purpose, it was essential to determine weights of steel with a reasonable approach to the actual weights resulting from detailed designs. A method of estimating these weights, such as that set forth in the author's paper, would have been most welcome.

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[28] Chf. Engr., Kansas City Terminal Ry., Kansas City, Mo. Mr. Hanna died on April 30, 1935.

No doubt many engineers have been in the position of needing reliable methods of estimating weights quickly, and many will be in that position hereafter. In the light of the writer's experience, he believes that this paper will be of great value to many members of the profession.

J. A. L. Waddell,[29] M. Am. Soc. C. E. (by letter).—The two exceedingly close checks of the writer's findings reported by Mr. Shryock are very gratifying; and his allusion to the European chrome-copper rustless steel, having a corrosion resistance four times as great as that of the copper-bearing steel used in the United States, is both important and timely.

In respect to Mr. Reichmann's suggestion of preparing special curves for continuous trusses, the writer would reply that the way to utilize the curves of the paper for any continuous-span layout is to compute the truss weights as if the spans were non-continuous, then correct the results by means of data published elsewhere.[30]

Replying to Professor Abbett, the curves in Figs. 1 and 2 are intended mainly for rapid preliminary estimates, and are sufficiently accurate in ordinary cases for contractors to use in tendering; but, when there is close competition, it would be advisable to make special computations of truss weights, mainly because the idiosyncrasies of the designer affect the totals.

Mr. Diehl states that "a valuable addition to the paper would be a curve of relative cost of bridges with the conventional types of solid floor as contrasted with open, non-skid flooring." The writer has made a lengthy, complicated, and elaborate investigation of this subject which has not yet (1935) been published. It contains the information mentioned by Mr. Diehl.

Mr. Foight offers a method of comparing truss weights which may prove to be a bit misleading to some readers, causing them to think that the curves are greatly in error. The "percentages" treated in the paper are those of truss weights compared with total vertical loads; hence, the variation in the Springfield Bridge is 2.1% — not 15 per cent. That variation is entirely due to an uneconomic truss depth. As modern highway bridges are generally designed with economic panel lengths and economic truss depths, no "large errors" can "result from variations in these factors."

Professor Fuller's four cautions to "the young designer and the student who is studying bridge design" are directly to the point, and should be given thorough consideration by all tyros in such work. The writer has never had much luck in determining weights of metal in bridges by formulas, although for half a century he has been using quite freely various "formulas of reduction" of his own in passing from known weights of metal for certain conditions to the corresponding weights for differing conditions.

Replying to Mr. Wilbur, the writer deemed it more logical to make his percentage ratios apply to the grand total vertical load than to that load, less the truss weight; but Mr. Wilbur's supplementary diagrams, based on his assumption of omitting the truss weight itself, are most acceptable. He is correct in stating that it would have been better in Fig. 1 (*c*) and Fig. 1 (*d*) to add to each a curve for a total load of 5 000 lb per lin ft — in truth, the writer has since discovered this fact and has plotted such curves to aid in his own subsequent work. This is a case in which "one's hindsight is better than his foresight."

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[29] Cons. Engr., New York, N. Y.

[30] "Economics of Bridgework", by J. A. L. Waddell, M. Am. Soc. C. E., Chapter XI.

TABLE 4.—APPROXIMATE AVERAGE PERCENTAGES OF SILICON STEEL MEMBERS IN TRUSSES OF COMBINED SILICON STEEL AND CARBON STEEL.

Roadway width (clear span), in feet	(*a*) PERCENTAGES IN SIMPLE TRUSS BRIDGES, FOR THE FOLLOWING SPAN LENGTHS, IN FEET:					(*b*) PERCENTAGES IN TYPE A CANTILEVER BRIDGES FOR THE FOLLOWING MAIN SPAN LENGTHS, IN FEET:					
	200	300	400	500	600	500	600	700	800	1 000	1 200
20.......	20	70	75	80	82	55	65	70	75	80	85
45.......	35	80	85	90	92	70	75	80	85	90	95

In accordance with Mr. Wilbur's suggestion that the writer include the percentages of silicon steel in structures of combined silicon steel and carbon steel, an attempt is made to furnish the desired information for the trusses in a satisfactory manner. This is especially difficult because of the diverse individual preferences of bridge designers. On that account, tables are preferable to curves; hence, the information is presented in Table 4 as percentages of weight of silicon-steel members compared with that of the truss as a whole. The actual weight of silicon-steel material, however, is from 80 to 90% of the amounts given, since some of the details on silicon-steel members, such as lacing, tie-plates, and rivets, are of carbon steel. Some of the percentages in Table 4 may vary, either up or down, in extreme cases by as much as 10% of their values, due to varying conditions and personal peculiarities of the computer, but generally that variation will be less than 5 per cent.

Mr. Wilbur is correct in stating that the loads per foot given in the diagrams are the totals for two trusses, rather than for one truss.

The writer's method of determining the percentage ratios for combined railway-and-highway bridges would be as follows: Let P = the percentage ratio given by the railway-bridge diagram; p = the percentage ratio for the highway-bridge diagram; W = the equivalent uniform live load, plus impact per linear foot, for the railway portion of the bridge; w = the equivalent uniform live load, plus impact, for the highway portion of the bridge; and, p' = the desired percentage ratio for the combined bridge. Then,

$$p' = \frac{W\,P + w\,p}{W + w} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Mr. Grant's formula would be:

$$p' = P - \frac{1}{3}(P - p) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

If the railway is double-tracked (Class 60); if the clear width of the highway deck is 30 ft with Class A loading; if the structure is of carbon steel; and if the span length is taken as 400 ft,[31] W = 15 912 lb per lin ft; w = 2 475 lb per lin ft, and the curves given in the paper make $P = 25.5$ and $p = 27$. Hence, by Equation (8),

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[31] "Bridge Engineering", by J. A. L. Waddell, M. Am. Soc. C. E., pp. 106, 129, 117, and 131.

$$p' = \frac{15\,912 \times 25.5 + 2\,475 \times 27}{15\,921 + 2\,475} = 25.7\%$$

and by Equation (9),

$$p' = 25.5 - \frac{1}{3}(25.5 - 27) = 26\%$$

The difference between Mr. Grant's percentage ratio and that of the writer is 0.3.

A similar computation for a single-track railway, a 20-ft clear roadway, and the same span length, gives 27.7% by Equation (9) and 27.3% by Equation (8), a difference of 0.4. Either difference is so small that it would scarcely show on the diagram.

Mr. Ripley's discussion reminds the writer of the days in the early Eighties when he used to consort with the "highwaymen" and compete for highway bridges. Fortunately for him, a call to Japan in 1882 took him out of that objectionable business; and upon his return to the United States four years later, he "declared war" upon all its disreputable practices and initiated what later proved to be a revolution in highway-bridge designing, letting, and construction.

The writer cannot describe in greater detail, "the loadings and other factors that affect Figs. 1 to 5", because the live loads and impacts were presented in another paper,[32] which is readily available. The percentage ratios are given in the present paper; and the manner of using them, both directly and with modifications, is fully explained. The Class A loading used is based on an 18-ton truck, and the Class B loading on a 12-ton truck.

The writer is much indebted to Mr. Allen for his trouble in computing the "percentage ratios" of the numerous bridges he has designed and built, as a check on curves of the paper. The close agreement found is a source of deep satisfaction to the writer. Of course, as Mr. Allen points out, one cannot expect perfect coincidence between the findings of any two engineers, because of the difference in their personal equations. His "percentage ratios" show a marked uniformity, indicating on the part of his computers a close adherence to specifications and the application of the principles of true economy in designing.

A further analysis of Table 2 of the paper, omitting from consideration the abnormal cases of the Springfield Bridge and the Portsmouth lift-span, indicates an average plus variation of 0.19%, an average minus variation of 0.42%, and an algebraic average of all variations equal to minus 0.25 — only one-fourth of 1 per cent. As this analysis is based upon seventeen cases, and in view of the fact that all the fifteen engineers who discussed the paper endorse the curves in general, it may be concluded that the findings of the paper are reliable; nevertheless, it would well pay any engineer who designs many bridges to plot, for the use of his office, new percentage-ratio curves based upon the specifications he utilizes and the general characteristics of his computers.

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[32] "Economic Proportions and Weights of Modern Highway Cantilever Bridges", by J. A. L. Waddell, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 888.

The writer's experience recently has led him to construct a diagram that should prove useful, especially to beginners, in utilizing Figs. 1 and 2. The operation in so doing involves an assumption of a truss weight, checking it, assuming another and checking that, etc., until there is an exact agreement between the assumption and the check. The final weight can be found correctly from the first check weight by computing the difference between it and the assumption, and multiplying the result by the proper factor taken from the curve shown in Fig. 15. The product should be either added to or subtracted from the weight assumed, in order to determine the correct truss weight.

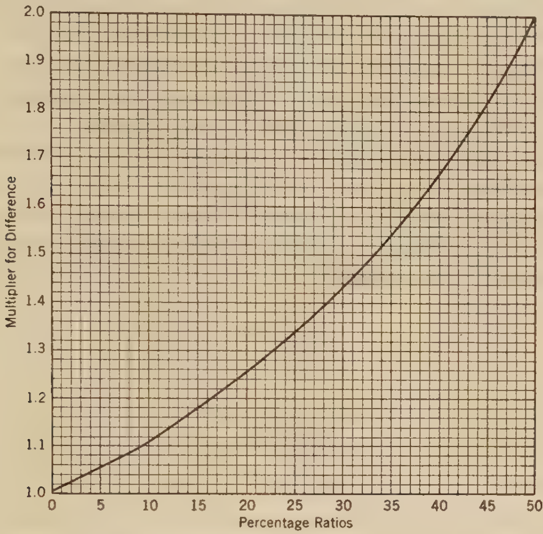

FIG. 15.—CORRECTIONS FOR ASSUMED TRUSS WEIGHTS.

As a check on the validity of this method, consider the case of a 400-ft, simple-truss bridge of 45-ft clear roadway, in silicon steel, for which Fig. 1 (*b*) gives 20 as the percentage ratio. The make-up of total vertical load (in pounds per linear foot) is as follows:

Live load plus impact	3 440	
Flooring	5 470	
Hand-rails	120	
Floor system	1 575	
Lateral system	740	
Trusses (assumed)	2 600	
Total vertical load	13 945	
The percentage ratio of 20 gives		2 789
Assumption		2 600
Difference		189

For a percentage ratio of 20, Fig. 15 gives a multiplier of 1.25, which yields the correction: $189 \times 1.25 = 236$; and the correct truss weight, $2\,600 + 236 = 2\,836$. With this truss weight, the new total vertical load will be 14 181. Applying the percentage ratio of 20, $W = 2\,836$, which checks.

Had the assumption been 3 000 lb per lin ft, the total vertical load would have been 14 345 lb per lin ft. Applying a percentage ratio of 20, $W = 2\,869$; a value of 3 000 was assumed, indicating a difference of 131. Multiplying 131 by 1.25 yields a correction of 164 which, subtracted from 3 000, leaves 2 836 — a check on the computation. This method is generally much quicker than the one involving a series of guesses and substitutions.

In conclusion, the writer desires to express to the gentlemen who have done him the honor to discuss this paper (both individually and collectively) his deep appreciation of their courtesy. This applies particularly to the generous comments by Messrs. Shaw, Jonah, Thomson, and Hanna, who approve the paper wholeheartedly.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1924

AN ASYMMETRIC PROBABILITY FUNCTION

BY J. J. SLADE,[1] JR., ESQ.

WITH DISCUSSION BY MESSRS. GORDON R. WILLIAMS, H. ALDEN FOSTER, R. D. GOODRICH, F. T. MAVIS, L. STANDISH HALL, ARNE FISHER, ARTHUR W. KEMPERT, AND J. J. SLADE, JR.

SYNOPSIS

In engineering problems that require statistical analysis, such as studies of rainfall and run-off, the data are frequently too meager to allow the use of the elaborate methods developed by Karl Pearson, or the Danish statisticians; and yet, the asymmetries in the frequency distributions associated with these problems are usually sufficiently marked to place the investigation in a field definitely outside that of the simple Gaussian theory. Unfortunately, where the basic probability function is other than the "normal curve", the determination of the constants involved, as well as the graduation of the data, generally becomes a difficult task. Furthermore, unless the data are very complete, the final results will seldom have meaning because of the magnitude of the probable errors of the parameters involved.

The purpose of this paper is to introduce a function that differs as little as it seems possible to let it differ from the "normal" in its general characteristics, while allowing it an unlimited degree of "skewness". At the same time, it is an easy curve to apply, as such things go, and one for the application of which existing tables may be used. The paper is subdivided into the following parts:

Section I contains a critical discussion of the various methods in use at present in the analysis of frequency distributions. The purpose of this section is, first of all, to contrast and to compare the function introduced with those already in use so that both its merits and defects will be exposed at the beginning. It will thus be seen to contain a justification for the intro-

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NOTE.—Published in October, 1934, *Proceedings*.

[1] Asst. Prof. of Eng. Mechanics, Rutgers Univ., New Brunswick, N. J.

duction of a new function in a field that literally teems with such studies. It is hoped that this paper will also help the engineer to a better understanding of a subject the approach to which is barricaded by so many mathematical hurdles.

The requirements of a function that is to be one degree more general than the Gaussian, are discussed in Section II. This function is introduced in Section II and is subjected to detailed mathematical analysis, and the constants of the curve are expressed in terms of the moments of the distribution which it is to fit. Finally, it is proved that the curve is a true generalization of the Gaussian, becoming identical with it, in fact, when the skewness parameter vanishes.

In Section III the formulas derived in Section II are collected and a procedure for their use is outlined. A number of examples are given which serve both to illustrate the manipulation of the new curve and to compare the results obtained by its use with those obtained by the use of other functions.

In Section IV, finally, the most general homograde function is discussed. Its parameters are expressed in terms of the bounds and standard deviation of the statistics, and a procedure for its application is outlined.

NOTATION

The following symbols used in the paper are summarized herewith for the guidance of discussers:

$\left.\begin{matrix} a \\ c \\ d \end{matrix}\right\}$ = parameters of the frequency function.
b = a parameter that corresponds to the maximum lower deviation of the statistics from their arithmetic mean.
g = a parameter that corresponds to the maximum upper deviation of the statistics from their arithmetic mean.
$\lambda_1, \lambda_2, \lambda_3$, etc. = parameters of the observed frequency distribution.
X = magnitude of the variate. This may be the absolute magnitude, but usually it stands for serial numbers of the classes into which the variate has been grouped.
M = arithmetic mean of the quantities, X.
x = deviations of the variate from arithmetic mean, $= X - M$.
y = frequency ordinate corresponding to the deviation, x.
Y = probability of the occurrence of deviations less than, or equal to, the deviation, x.
z = a variable abscissa; also transformed argument of probability integral.
u = a "dummy" variable of integration.
$\left.\begin{matrix} f \\ F \\ K \end{matrix}\right\}$ = functional symbols.
ϕ = symbol of frequency function.

ϕ_0 = symbol of generic frequency function: A first approximation to ϕ.

$\phi_1, \phi_2, \ldots$ = successive derivatives of ϕ_0.

$o_1, o_2, \ldots$ = magnitudes of observations.

$x_1, x_2, \ldots$ = distinct magnitudes of observations.

k = the last number of a finite series of parameters; also, k = a substitution factor when so identified in the text.

m = the last number of a finite series of distinct magnitudes.

n = the last number of a finite series of observations.

N = the total number of items in a statistical series $= S_0$.

$\left.\begin{matrix} r \\ s \end{matrix}\right\}$ = indices of summation.

λ = an index which takes on the values, 0, 1, 2, 3.

S_λ = sum of the λ powers of the deviations from the arithmetic mean $= \Sigma x^\lambda$.

μ_λ = λth moment of the distribution $= \dfrac{S_\lambda}{S_0} = \dfrac{\Sigma x^\lambda}{N}$. Thus, $\mu_2 = \sigma^2$, the square of the standard deviation, or, if the mean is taken as the unit, $= (CV)^2$, the square of the Hazen-Foster coefficient of variation.

β = the coefficient of skewness. This is identical with $\sqrt{\beta_1}$ of the Pearson theory, and with (CS), the coefficient of skew, of the Hazen-Foster development. (In engineering literature the divisor of the moments is usually taken as $N - 1$ instead of N. This, no doubt, comes from least square usage, but in this connection the writer finds no justification for it.)

σ = the standard deviation $= \sqrt{\mu_2} = M\,(CV)$.

$\left.\begin{matrix} k \\ l \\ M_\lambda \\ v \\ w \end{matrix}\right\}$ = substitution factors.

t = a parameter connected with β by the equation, $t^3 + 3t - \beta = 0$.

$\left.\begin{matrix} A \\ B \end{matrix}\right\}$ = conjugate components of the real root of $t^3 + 3t - \beta = 0$.

ε = a small quantity that vanishes with t.

$O(\ )$ = "terms of the order of", a symbol denoting a small quantity which vanishes with the infinitesimal enclosed in the brackets.

e = base of Naperian logarithms.

Σ = sign of summation.

I.—THEORIES AND METHODS IN USE

1.—The Normal Curve.—The most satisfactory basis for a great variety of statistical studies is the Gaussian "normal curve"[2]:

$$y = \frac{1}{\sqrt{2\pi\mu_2}}\, e^{-\frac{x^2}{2\mu_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which, y = the frequency ordinate corresponding to the variation, x; $\mu_2 = \sigma^2$ = the second moment; e = Naperian base of logarithms; and,

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[2] "The Mathematical Theory of Probability," by Arne Fisher, p. 198, Macmillan, 1930.

x = a variation from the arithmetical mean of the statistics. This function has been the basis of elaborate theories of errors and correlation, principally, no doubt, because its properties are mathematically simple. This is not all, however. There seems to be considerable theoretical justification for its use, although this is in no way comparable to the justification derived from experimental verification of the results obtained from its use. Under certain conditions it gives reliable information; this is a matter subject to controlled experiment. Its use is indicated whenever the following assumptions can be made regarding the variation of the entity considered[3]:

(*a*) The most probable (or most frequent) value of a large number of observations is their arithmetic mean.

(*b*) Small variations are more likely to occur than large ones.

These or equivalent assumptions relate to the variate; in addition, of course, the foundations of probability theory are assumed.

As long as the variation comes fairly well within this set the normal curve represents the data surprisingly well, and more significant than mere "fitting" of the given data is the fact that the information obtained by its use (as in extrapolating, for instance) is surprisingly accurate. The single parameter of "spread" seems adequate to take care of the possible configurations of such a set.

In engineering problems, however, these assumptions can seldom be made outside the case of direct and precise measurements. Usually the mean, the mode, and the median do not coincide.

Since the relation between negative and positive variations is not known when the symmetry that leads to the Gaussian distribution is lacking, a general frequency function cannot be developed entirely from *a priori* considerations. This leads to investigation of all the mathematical expressions that share with the probability function its characteristics properties, among which, *a fortiori,* must be the curve that is sought; and, since there is no method of generating all such curves, the investigator must finally be content to try as many of them as he can discover and to let experience be the final arbiter of the success of the search.

The extension of the theory to include skew probability functions has been the subject of much ingenious mathematical research.

2.—Pearson's Curves.—One of the first extensions of the theory, and perhaps still the most successful, is due to Karl Pearson,[4] who embodied the asymmetric assumptions regarding the frequency of the variations into the very general differential equation:

$$\frac{dy}{dx} = \frac{y\,(x + a)}{F\,(x)} \quad \ldots\ldots\ldots\ldots (2)$$

which merely states that the curve has a maximum at the point, $x = -a$, and that it approaches the axis of x asymptotically as the frequency vanishes.

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[3] For a complete proof that the normal law follows from the principle of the arithmetic mean and the theorem on compound probabilities, see "Advanced Calculus", by E. B. Wilson, Ginn & Co., 1912, p. 386.

[4] "Frequency Curves and Correlation", by W. P. Elderton, Lond., 1927.

The normal curve is obtained from the family generated by Equation (2) simply by letting $F(x) = $ a constant. It is the simplest one of all; but this mode of generation is not altogether satisfactory because, whereas Equation (2) may contain all frequency curves, it is quite likely that it also contains many curves not even remotely related to frequency functions. Equation (2) does not provide a means of distinguishing between them; for, certainly, the problem of frequency distribution is not merely one of fitting, as well as possible, a curve to a set of data; there must be some kind of theory connected with it, as strong, if possible, as that which supports the normal curve. This is a desideratum not likely to be attained, but one which must be aimed at, nevertheless, in searching for general probability functions.

Another difficulty with Pearson's set-up is a practical one. The form of the function, $F(x)$, that appears in Equation (2) must be limited severely in order to integrate the equation at all. Pearson assumed that $F(x)$ can be developed in a Maclaurin series and that it is permissible to drop all the terms of the series after the square. This is merely a roundabout and unscientific manner of assuming that $F(x)$ is a quadratic.

However, the family of curves generated by Equation (2) under the assumption that,

$$F(x) = b_0 + b_1 x + b_2 x^2 \dots\dots\dots\dots\dots\dots\dots (3)$$

has proved to be very useful. One of the principal objections to the family generated by Equation (2) is the variety of types into which it is divided. The criteria for this or that type are artificial and cumbrous. The objection is not altogether æsthetic; in order definitely to place a distribution in this or that category it is necessary to compute the fourth moment; and when the data are meager (as happens frequently in problems investigated by the engineer), a fourth moment is as meaningless as it is laborious to compute. Any accidental irregularity in the distribution is exaggerated dangerously in the higher moments. Even the third moment is quite unreliable when the observations are few. Nothing beyond the standard deviation is reliable for less than 150 items.

Of course, this family contains curves that may be specified completely by three moments (two when $\mu_1 = 0$; that is, when the origin is at the mean). One such is the curve classified as Type III; but this curve degenerates into an L-shaped figure for certain ranges of the parameters, which suffices to throw it out of the class of curves that represent possible homograde frequency distributions.

3.—The General Theory.—A general theory of frequency functions has been developed, following the researches originated by Laplace, which is extremely elegant mathematically, but quite impracticable in fact. (The best treatise, in English, on the theory of frequency curves as developed by the Danish school of statisticians, is that presented by Arne Fisher[5].) The theory is based on a fundamental property of frequency functions, namely, that if one has a set of observations, $o_1, o_2, \ldots, o_n$, and a set of numbers, $x_1, x_2, \ldots, x_m$, which represent the distinct values of the observations, then, if $\phi(x)$ is the frequency function for the o's and $f(x)$ is any single-valued function at all, one may obviously write:

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[5] "The Mathematical Theory of Probability", by Arne Fisher, Macmillan, 1930.

$$\sum_{r=1}^{r=n} f(o_r) = \sum_{r=1}^{r=m} f(x_r)\ \phi(x_r) \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now, $\phi(x)$ is a function not only of the different values of the variate, but also of certain statistical parameters (the moments, for example, would be one such set); and, to a great extent, it is a matter of convenience what functions of the observations these parameters should be. For instance, they may be defined in this manner: Call the parameters, $\lambda_1, \lambda_2, \ldots, \lambda_k$, and take two convenient functions, $F(z, \lambda_1 \ldots, \lambda_k)$ and $\sum_{r=1}^{r=n} K(z, o_r)$. Now, if these two functions are equated and required to be equal for all values of z, a relation may be established in this manner between the parameters, λ_r, and the observations, o_r, because, by expanding both sides of the equation,

$$F(z, \lambda_1 \ldots, \lambda_k) = \sum_{r=1}^{r=n} K(z, o_r) \ldots\ldots\ldots\ldots\ldots (5)$$

into series of powers of z and equating like powers, a relation is established between the parameters, λ_r, and the observations, o_r, that must hold for Equation (5) to be satisfied identically. By making use of Equation (4), Equation (5) may be written in this manner:

$$F(z, \lambda_1, \ldots, \lambda_k) = \sum_{r=1}^{r=m} K(z, x_r)\ \phi(x_r) \ldots\ldots\ldots\ldots (6)$$

or, passing to the limit in some reasonable manner:

$$F(z, \lambda_1, \ldots, \lambda_k) = \int_{-\infty}^{\infty} K(z, x)\ \phi(x)\ dx \ldots\ldots\ldots\ldots (7)$$

This is an integral equation in which, from Equation (5), both $F(z, \lambda)$ and $K(z, x)$ are known, and from which $\phi(x)$ may be determined by the rules established in the theory of such equations. The success with which efforts to solve Equation (7) will be rewarded, will depend on the ingenuity used in defining Equation (5).

Thiele obtained a solution of Equation (7) in terms of his famous semi-invariants.[6] In Thiele's solution, $\phi(x)$ is given as a definite integral which, unfortunately, cannot be integrated; so that the net result is an elegant mathematical process which ultimately gets nowhere. Even if one could make direct use of Thiele's solution of Equation (7) there are certain circumstances that render it too general for ordinary purposes. In a manner somewhat analogous to the case of Pearson's differential equation, one may say that although Equation (7) certainly includes all frequency functions, it includes no theory of actual frequency functions. A fairly arbitrary function, $\phi(x)$, may satisfy Equation (7) without being remotely near an actual frequency function.

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[6] "The Mathematical Theory of Probability", by Arne Fisher, 1930, p. 191.

4.—The Gram-Charlier Series.—A much more practical development of the frequency function is the series named after Gram and Charlier. If one writes:

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8a)$$

and,

$$\phi_k(x) = \frac{d^k \phi_0(x)}{dx^k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8b)$$

then the frequency function may be represented thus,

$$\phi(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots\ldots\ldots (9)$$

In this series the values of c_n may be obtained by making use of the orthogonal properties of the ϕ's, or they may be obtained by the usual method of least squares.[7] W. F. Osgood has written an elementary discussion of series of orthogonal functions and their connection with the principle of least squares.[8] Here, again, the criticism is that Equation (9) is too general; it represents a statement in mathematical analysis rather than one in the theory of frequency functions. In fact, if one is satisfied with some condition slightly less than an absolute mathematical approach to a limit, Equation (9) may be used to represent, very closely, quite fantastic functions—even discontinuous ones—provided they and their slopes vanish at infinity. Clearly, this includes some functions not at all connected with frequency functions.

Furthermore, if $\phi(x)$ deviates appreciably from $\phi_0(x)$ (that is, if it has a marked skew), it may take thousands of terms of Equation (9) to get even a fair representation—and this is a process which is practically impossible. The function, $\phi(x)$, need not deviate greatly from $\phi_0(x)$ before waves and negative frequencies appear by the use of a few terms of Equation (9), and they certainly do not belong in an actual frequency curve. The reason for these spurious phenomena will be obvious to the reader acquainted with Fourier analysis.

A generating function, $\phi_0(x)$, may be defined by a relation other than Equation (8) such that, to begin with, $\phi(x)$ and $\phi_0(x)$ will be fairly similar. Then, $\phi(x)$ may be closely represented by a few terms of Equation (9). This is so because each term represents a correction to the sum of the terms that precede it. Studies have been made of various forms of the generic function, $\phi_0(x)$[9], but the objection to the use of Equation (9) must always remain: That this series, regardless of the form of the generic, $\phi_0(x)$, may be made to represent almost anything. It offers no theory of frequency functions.

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[7] "The Mathematical Theory of Probability", by Arne Fisher, 1930, pp. 199 *et seq.*

[8] "Advanced Calculus", by William Fogg Osgood, Macmillan, 1925.

5.—Transformations of the Variate.—For greatly skewed distributions a logarithmic transformation of the variate has sometimes proved useful.[10] This transformation really furnishes a new generic function, defined, according to Fisher, by

$$\phi_0(x) = \frac{1}{n\sqrt{2\pi}} e^{-(\log \frac{x-m}{n})^2} \ldots\ldots\ldots\ldots\ldots(10)$$

to be used in Equation (9), and, therefore, it needs no further discussion than that of Article 4. The transformation has been suggested by the fact that when the skew of the frequency is great, that of the logarithms of the variate is more nearly normal. Special mention is made of this transformation because of its relation to the partly bounded function introduced in this paper. As objections to it other than those listed in Article 4 the following may be stated:

(*a*) The use of Equation (10) implies the solution of a cubic, distinct for each problem.

(*b*) After the transformation is accomplished one still has to determine the constants of the series by the method of least squares or by some equivalent process.

6. — Empirical Curves. — So-called empirical curves are not essentially different from any of the foregoing because, as a matter of fact, all the curves are empirical; but they suffer from the fact that they are usually devised for particular problems and the range of their applicability is generally quite limited. Furthermore, they are farther from an approximation of any theory of frequency functions than any of the foregoing. It is the integral, or duration, curve (for many purposes more important than the frequency function) which is often represented by means of purely arbitrary functions. Not much can be said in their favor except, perhaps, that they may be easy to use. The principal objection to their use lies in the fact that it is not possible to estimate the reliability of the parameters entering in them. The estimate of this reliability is an important phase of a statistical investigation often neglected in engineering applications.

7.—Graphical Methods. — The use of "probability" paper and other graphical devices is an undesirable practice. These methods convey to the eye a simplicity which does not really exist. The process of using these methods is not essentially different from that of fitting a curve to the data, except for the fact that one is more likely to fit the wrong curve graphically than analytically. A paper marked off in non-rectangular co-ordinates does to the data exactly what a flat map of a curved earth does to geography. One gets an exaggerated idea of the entire problem. A straight line that seems to fit the data closely may not be a good fit at all when one remembers that small deviations in certain regions of the paper may represent quite large errors. Fitting a straight line to data that plot approximately as a straight line is like using the method of least squares without weighting for functional distortion.

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[9] See, for instance, "Generalizations of the Normal Curve of Error", by Luis R. Salvosa, Dissertation, Univ. of Michigan, Edwards Bros., Inc., Ann Arbor, Mich.

[10] "The Mathematical Theory of Probability", by Arne Fisher, 1930, pp. 236 *et seq.*

One notable exception is presented by the closely allied semi-graphical methods of Dr. J. C. Kapteyn[11] and R. D. Goodrich, M. Am. Soc. C. E.[12] Their procedures are fundamentally sound. The weakness in their methods is due to the impossibility of computing the probable errors of the constants in their functions. There is no graphical process that gives an indication of these errors.

II—The Partly-Bounded Function

1.—Desiderata.—From the discussion of Section I it is seen that only one function furnishes something of a theory of variations, and that is the normal curve, Equation (1). The succeeding developments, although no doubt including the general theory of frequency functions within their scope, are far too general in character; in spite of all that may be said against them, the Pearson functions still remain the most useful. The introduction of a new function will be justifiable only in so far as it is able to call forth a minimum of objections that must surely be raised against it. The various requirements for it will now be listed and discussed. These requirements embody the writer's theory of the most general homograde probability function:

(*a*) It must be a simple extension of the normal curve; that is, the normal curve must be one of its possible forms, and for no range of the parameters that specify it must it lose its characteristic shape. It must be smooth, without waves, and without negative frequencies. It is true that waves will be exhibited legitimately by multi-modal distributions, but these must be considered as the superposition of several uni-modal distributions.

(*b*) It should be capable of assuming indefinite skewness. Since the factors that cause skewness in the frequency of a set of variations are not known, deviations from the normal may be assumed in some way to be connected with the fact that, usually, the variate is definitely bounded, the first deviation from the normal occurring from the fact that the variate has a minimum value, but no definite maximum. In a representative curve one may require that its skewness be a simple function of its end point. The function with one end point is here termed "partly bounded," that with two end points, "totally bounded".

(*c*) The curve must be specifiable completely by moments no higher than the third. Should it have more parameters than may be so specified, then it must be possible to find some extra-statistical means of determining them. Since the probable errors in the parameters increase greatly with the order of the moments necessary to specify them, it becomes necessary to search for some physical substitutes for these moments. Two such substitutes, it will be shown, are the upper and lower bounds of the variation, when these bounds can be determined *a priori.*

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[11] "Skew Frequency Curves in Biology and Statistics", by J. C. Kapteyn and M. J. van Uven, Hoitsema Bros., Groningen, 1916.

[12] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 1 *et seq.*

(*d*) The curve must be simple to apply. Considering all the uncertainty that lies at the base of the theory of frequency functions, the information derived from their use is certainly not worth the expenditure of great labor in their application.

2.—The Partly Bounded Function.—Consider, first, the function:

$$y = a\,e^{-c^2[\log d(x+b)]^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

which is defined from $x = -b$ to $x = \infty$. This function which is a first extension of the normal, admits of complete mathematical analysis. The study of it will be a guide in determining the parameters of the totally bounded function of Section IV.

Notice, in the first place, that Equation (11) is similar to the logarithmically transformed function, Equation (10), differing from it only in the power, c^2, to which it is raised. The difference is fundamental, however. It is just this added parameter, *c,* that makes Equation (11) more than a "normal" for the logarithms of the variate and which brings about a remarkable simplification of the formulas. It is this *c,* indeed, that makes Equation (11) a true generalization of Equation (1), as will be shown subsequently.

The discussion that follows immediately is a mathematical investigation of the properties of this function. By the usual method of moments[4] the parameters, *a, b, c,* and *d,* are determined in terms of the moments, μ_1, μ_2, μ_3, of the data and the total frequency, *N*. The investigation, which is given here for the first time, is developed with sufficient detail to enable the reader not well versed in mathematical manipulations to follow each step of the reasoning. It must be kept in mind that the analysis justifies the use of this curve only in so far as it shows that the function Equation (11) meets all the aforementioned requirements.

The reader who is willing to take for granted the accuracy of the following analysis may omit the remainder of this section and proceed to Section III, where the formulas obtained in Section II are tabulated and their application is discussed.

3.—The Parameters.—Let S_λ = the power sums of the variations; that is, let,

$$S_\lambda = \Sigma\, x^\lambda,\ (\lambda = 0, 1, 2, 3) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

so that, for instance, $S_0 = \Sigma\, x^0 = N$, the total frequency. Now, require these power sums to be equal to the corresponding moments of the function Equation (11). Thus,

$$S_\lambda = a\int_{-b}^{\infty} x^\lambda\, e^{-c^2[\log d(x+b)]^2}\,dx \qquad \ldots\ldots\ldots\ldots(13)$$

To integrate, let $z = \log d\,(x + b)$; then, $x = d^{-1} e^{z} - b$; $dx = d^{-1} e^{z}\, dz$; and,

$$x^{\lambda} = d^{-\lambda}\,[e^{\lambda z} - \lambda\, b\, d\, e^{(\lambda-1) z} + \ldots (b\, d)^{\lambda}] \ldots\ldots\ldots\ldots (14)$$

Substituting in Equation (13):

$$S_{\lambda} = \frac{a}{d^{\lambda+1}} \int_{-\infty}^{\infty} [e^{\lambda z} - \lambda\, b\, d\, e^{(\lambda-1) z} + \ldots]\, e^{z - c^{2} z^{2}}\, dz \ldots (15)$$

Equation (15) is a sum of integrals of the type, $\int_{-\infty}^{\infty} e^{k z - c^{2} z^{2}}\, dz$.

Completing the square of the exponent in the integrand:

$$\int_{-\infty}^{\infty} e^{k z - c^{2} z^{2}}\, dz = \int_{-\infty}^{\infty} e^{-c^{2}\left(z^{2} - \frac{k}{c^{2}} z + \frac{k^{2}}{4 c^{4}} - \frac{k^{2}}{4 c^{4}}\right)} dz$$

$$= e^{\left(\frac{k}{2c}\right)^{2}} \int_{-\infty}^{\infty} e^{-c^{2}\left(z - \frac{k}{2c^{2}}\right)^{2}} dz = \frac{\sqrt{\pi}}{c}\, e^{\left(\frac{k}{2c}\right)^{2}} \ldots\ldots (16)$$

by a well-known result of the integral calculus.[13] Using this result in Equation (15):

$$S_{\lambda} = \frac{a}{d^{\lambda+1}} \frac{\sqrt{\pi}}{c} \Bigg[e^{\left(\frac{\lambda+1}{2c}\right)^{2}} - \lambda\, b\, d\, e^{\left(\frac{\lambda}{2c}\right)^{2}}$$

$$+ \ldots (b\, d)^{\lambda}\, e^{\frac{1}{4 c^{2}}} \Bigg] \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

This gives four equations (one each for λ = 0, 1, 2, and 3) with which to determine the constants, a, b, c, and d, in terms of the power sums, S_0, S_1, S_2, and S_3.

To solve Equation (17) let,

$$l = e^{\frac{1}{4 c^{2}}}; \quad M_{\lambda} = S_{\lambda} \frac{c\, d^{\lambda+1}}{a \sqrt{\pi}} \ldots\ldots\ldots\ldots\ldots (18)$$

then Equation (17) may be written:

$$M_0 = l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

$$M_1 = l^{4} - b\, d\, l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19b)$$

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[13] "Advanced Calculus", by F. S. Woods, p. 153, Ginn & Co., 1926.

$$M_2 = l^9 - 2\,b\,d\,l^4 + (b\,d)^2\,l \qquad \text{(19c)}$$

and,

$$M_3 = l^{16} - 3\,b\,d\,l^9 + 3\,(b\,d)^2\,l^4 - (b\,d)^3\,l \qquad \text{(19d)}$$

Solving for the powers of l:

$$l = M_0 \qquad \text{(20a)}$$

$$l^4 = M_1 + b\,d\,M_0 \qquad \text{(20b)}$$

$$l^9 = M_2 + 2\,b\,d\,M_1 + (b\,d)^2\,M_0 \qquad \text{(20c)}$$

and,

$$l^{16} = M_3 + 3\,b\,d\,M_2 + 3\,(b\,d)^2\,M_1 + (b\,d)^3\,M_0 \qquad \text{(20d)}$$

Now, $\frac{M_\lambda}{M_0} = d^\lambda\,\mu_\lambda$, ($\lambda = 1, 2, 3$), in which, according to the usual convention, μ_λ is the λth moment. Dividing Equations (20b), (20c), and (20d) by Equation (20a):

$$l^3 = d(\mu_1 + b) \qquad \text{(21a)}$$

$$l^8 = d^2(\mu_2 + 2b\mu_1 + b^2) \qquad \text{(21b)}$$

and,

$$l^{15} = d^3(\mu_3 + 3b\mu_2 + 3b^2\mu_1 + b^3) \qquad \text{(21c)}$$

a set from which a has been eliminated. The usual simplification may now be introduced, which in no way affects the generality of the results, obtained by choosing the origin at the mean so that $\mu_1 = 0$. Now, let $k = \frac{l^2}{d}$, so that Equations (21) become:

$$kl = b \qquad \text{(22a)}$$

$$k^2\,l^4 = \mu_2 + b^2 \qquad \text{(22b)}$$

and,

$$k^3\,l^9 = \mu_3 + 3\,b\,\mu_2 + b^3 \qquad \text{(22c)}$$

Eliminating b and the combination, kl, from Equations (22):

$$\mu_3 + 3\,\frac{\mu_2^{1.5}}{\sqrt{l^2 - 1}} = \frac{\mu_2^{1.5}}{(l^2 - 1)^{1.5}}\,(l^6 - 1)$$

that is,

$$\frac{\mu_3}{\mu_2^{1.5}} = \frac{l^6 - 1 - 3\,(l^2 - 1)}{(l^2 - 1)^{1.5}} = \sqrt{l^2 - 1}\,(l^2 + 2) \qquad \text{(23)}$$

an equation in which all the parameters of the curve except c have disappeared. Let $\beta = \frac{\mu_3}{\mu_2^{1.5}}$ and $t = \sqrt{l^2 - 1}$. Then, Equation (23) becomes:

$$t^3 + 3\,t - \beta = 0 \qquad \text{(24)}$$

a cubic with one real root equal to:

$$t = A^{\frac{1}{3}} + B^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

in which, $A = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + 1}$; and $B = \frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} + 1}$.

In terms of t all the parameters of the curve have simple expressions. For $l^2 = t^2 + 1$ and from Equation (18), $\frac{1}{2\,c^2} = \log\,(t^2 + 1)$; or,

$$c = [2 \log\,(t^2 + 1)]^{-0.5} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

and,

$$b = k\,l = \frac{\sqrt{\mu_2}}{\sqrt{l^2 - 1}} = \frac{\sqrt{\mu_2}}{t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

By definition, $k = \frac{l^2}{d}$. Therefore, from Equation (27):

$$d = \frac{t}{\sqrt{\mu_2}}\,l^3 = \frac{t\,(t^2 + 1)^{1.5}}{\sqrt{\mu_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

and, finally, since by Equation (17), $S_0 = N = \frac{a\sqrt{\pi}}{c\,d}\,e^{\frac{1}{4c^2}}$:

$$a = \frac{S_0 c\,d}{\sqrt{\pi}}\,e^{-\frac{1}{4c^2}} = \frac{S_0 c\,d}{\sqrt{\pi}}\,e^{-\log\sqrt{t^2+1}}$$

$$= \frac{N}{\sqrt{\pi}}\;\frac{1}{\sqrt{2\log\,(t^2+1)}}\;\frac{t\,(t^2+1)^{1.5}}{\sqrt{\mu_2}}\;\frac{1}{\sqrt{t^2+1}}$$

$$= \frac{N}{\sqrt{2\,\pi\,\mu_2}}\;\frac{t\,(t^2+1)}{\sqrt{\log\,(t^2+1)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

4.—The Duration Curve.—The probability that the variate lies between $-\,b$ (the zero of the function) and x is expressed by:

$$Y = a \int_{-b}^{x} e^{-c^2[\log d\,(x+b)]^2}\,dx \quad \ldots\ldots\ldots\ldots (30)$$

or, making use of the transformation at the beginning of Article 3, Section II,

$$Y = \frac{a}{d} \int_{-\infty}^{\log d\,(x+b)} e^{z - c^2 z^2}\,dz$$

$$= \frac{a}{d}\,e^{\frac{1}{4c^2}} \int_{-\infty}^{\log d\,(x+b)} e^{-c^2\left(z - \frac{1}{2c^2}\right)^2}\,dz$$

Now, let $u = c\ (z - \frac{1}{2\,c^2})$, $du = c\,dz$, and the duration curve becomes:

$$Y = \frac{a}{c\,d}\, e^{\frac{1}{4c^2}} \int_{-\infty}^{c\left[\log d\,(x+b) - \frac{1}{2c^2}\right]} e^{-u^2}\,du$$

$$= \frac{N}{\sqrt{\pi}} \int_{-\infty}^{c \log \frac{d\,(x+b)}{t^2+1}} e^{-u^2}\,du \quad \ldots\ldots\ldots\ldots\ldots(31)$$

This is the ordinary probability integral with argument, $c \log \dfrac{d\,(x + b)}{t^2 + 1}$.

5.—*The Elements of the Curve.*—A few points on the curve are of special interest:

(*a*) The End Point: When $x = -\,b$, the curve obviously ends because there are no real values of y for $x < -\,b$. Since, for every real value of t there is one and only one real value of β and, conversely, it is a mere matter of convenience whether t or β is designated the coefficient of skewness. With this in mind, it is seen from Equation (27) that the skewness of the curve is a simple function of its end point, as was required of it in Article 1, Section II. In general, however, the end point of the curve does not correspond to the zero of the data. If one is quite sure of the zero, then this value may be taken for b, in terms of which, a, c, and d may be computed by Equations (26) to (29). When this is done the third moments of the curve and data do not quite agree, but it must be remembered that the probable error in the third moment is large when the distribution is not normal, and, therefore, agreement in the third and higher moments is not a good criterion of fit.

(*b*) The Mode: To obtain the mode,

$$\frac{dy}{dx} = -2\,a\,c^2\, \frac{\log d\,(x+b)}{x+b}\, e^{-c^2\,[\log d\,(x+b)]^2} = 0 \quad \ldots.(32)$$

For finite values of $x > -\,b$ this expression vanishes when the logarithmic factor vanishes; that is, when $d\ (x + b) = 1$. So that the mode is at:

$$x = \frac{1}{d} - b = b\left[\frac{1 - (t^2 + 1)^{1.5}}{(t^2 + 1)^{1.5}}\right] \ldots\ldots\ldots\ldots\ldots(33)$$

(*c*) The Median: To obtain the median, notice that,

$$\int_{-\infty}^{0} e^{-u^2}\,du = \int_{0}^{\infty} e^{-u^2}\,du$$

so that,

$$u = c \log \frac{d\,(x + b)}{t^2 + 1} = 0$$

or,

$$d\,(x + b) = t^2 + 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

and,

$$x = \frac{\sqrt{\mu_2}}{t\sqrt{t^2 + 1}} - \frac{\sqrt{\mu_2}}{t} = b\left[\frac{1 - \sqrt{t^2 + 1}}{\sqrt{t^2 + 1}}\right] \ldots\ldots\ldots (35)$$

Dividing Equation (35) by Equation (33):

$$\frac{\text{Median}}{\text{Mode}} = \frac{1 - (t^2 + 1)^{0.5}}{1 - (t^2 + 1)^{1.5}} \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

which, for all values of *t*, is less than 1. From Equation (36) it is seen that the median always lies between the mode and the mean.

6.—*The Normal Curve as a Limiting Form.*—The next step is to inquire what shape this curve will have as *t* approaches 0; that is, as the skewness vanishes:

$$\underset{t \to 0}{\text{limit}}\; a = \underset{t \to 0}{\text{limit}}\; \frac{N}{\sqrt{2\,\pi\,\mu_2}}\;\frac{t\,(t^2 + 1)}{\sqrt{\log\,(t^2 + 1)}}$$

$$= \underset{\to 0}{\text{limit}}\; \frac{N}{\sqrt{2\,\pi\,\mu_2}}\;\frac{t\,(t^2 + 1)}{\sqrt{1 - \dfrac{t^2}{2} + \dfrac{t^4}{3} - \ldots}} = \frac{N}{\sqrt{2\,\pi\,\mu_2}} \ldots\ldots (37)$$

and,

$$\underset{t \to 0}{\text{limit}}\; c\,d = \underset{t \to 0}{\text{limit}}\; \frac{t\,(t^2 + 1)^{1.5}}{\sqrt{\mu_2}\;\sqrt{2\log\,(t^2 + 1)}}$$

$$= \underset{t \to 0}{\text{limit}}\; \frac{t\,(t^2 + 1)^{1.5}}{\sqrt{2\,\mu_2}\;t\;\sqrt{1 - \dfrac{t^2}{2} + \ldots}} = \frac{1}{\sqrt{2\,\mu_2}} \ldots\ldots\ldots (38)$$

so that, for small values of *t*, $cd = \dfrac{1}{\sqrt{2\,\mu_2}} + \varepsilon$; and,

$$d = \frac{1}{c}\left(\frac{1}{\sqrt{2\,\mu_2}} + \epsilon\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

in which, ε is a small quantity that vanishes with *t*.

Now, $b\,d = (t^2 + 1)^{1.5} = 1 + \dfrac{3}{2}\,t^2 + \ldots$ by the binomial theorem; or,

$$b\,d = 1 + 0\,(t^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

in which, $0\,(t^2)$ is a small quantity that vanishes with t^2. Then:

$$\left.\begin{aligned}\lim_{t\to 0} c\log d\,(x+b) &= \lim_{t\to 0}\log\,[d\,(x+b)]^c\\ &= \lim_{c\to\infty}\log\left[\frac{x}{c}\left(\frac{1}{\sqrt{2\,\mu_2}}+\epsilon\right)+1+0\,(t^2)\right]^c\\ &= \log e^{\frac{x}{\sqrt{2\mu_2}}} = \frac{x}{\sqrt{2\,\mu_2}}\end{aligned}\right\}\quad\ldots\ldots(41)$$

The existence of this limit (Equation (41)) may be shown by applying the rule of L'Hospital.[14] Combining these various results:

$$\lim_{\beta\to 0} a\,e^{-c^2\,[\log d\,(x+b)]^2} = \frac{1}{\sqrt{2\,\pi\,\mu_2}}\,e^{-\frac{x^2}{2\mu_2}}\quad\ldots\ldots(42)$$

III.—THE USE OF THE FUNCTION

1.—The Computation of the Parameters.—To fit the curve to a set of data compute the moments, μ_2 and μ_3, in the usual way; then compute $\beta = \dfrac{\mu_3}{\mu_2^{1.5}}$, followed by A, B, and t (Equation (25)). It will be found convenient to compute immediately the quantities, $t^2 + 1$; $t\,(t^2 + 1)$; $\sqrt{t^2 + 1}$; and $\sqrt{2.303\ \ldots\ \log_{10}\,(t^2 + 1)}$. (The logarithms of the last section are to the base, e. Consequently, $\log x = 2.303\ \ldots\ \log_{10} x$.)

In terms of these quantities the parameters are:

$$\sqrt{2\,\pi}\;a = \frac{N}{\sqrt{\mu_2}}\;\frac{t\,(t^2+1)}{\sqrt{2.303\,\log_{10}\,(t^2+1)}}$$

$$b = \frac{\sqrt{\mu_2}}{t}$$

$$\sqrt{2}\;c = \frac{1}{\sqrt{2.303\,\log_{10}\,(t^2+1)}}$$

and,

$$d = \frac{1}{\sqrt{\mu_2}}\,t\,(t^2+1)\,\sqrt{t^2+1}$$

Since Equation ($8a$) is the most usual form in which the normal curve is tabulated, the quantities, $\sqrt{2\,\pi}\;a$ and $\sqrt{2}\;c$, are computed instead of a and c.

Table 1 gives the values of these parameters for selected values of β for $\sigma = 1$. If σ is not 1, this table may be used with $\dfrac{x}{\sigma}$ and $\dfrac{a}{\sigma}$.

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[14] "Advanced Calculus", by F. S. Woods, p. 18.

TABLE 1.—VALUES OF THE PARAMETERS FOR $\sigma = 1$

β	t	$\sqrt{2\pi}\,a$	b	$\sqrt{2}\,c$	d	β	t	$\sqrt{2\pi}\,a$	b	$\sqrt{2}\,c$	d
0.0	0	1.0000	......	....	....	2.2	0.643	1.5454	1.555	1.6996	1.0812
0.2	0.067	1.0029	14.925	14.9030	0.0674	2.4	0.690	1.6328	1.449	1.6021	1.2384
0.4	0.133	1.0180	7.518	7.5180	0.1366	2.6	0.735	1.7217	1.361	1.5223	1.4036
0.6	0.197	1.0438	5.076	5.1020	0.2085	2.8	0.777	1.8130	1.278	1.4550	1.5781
0.8	0.261	1.0861	3.831	3.8960	0.2881	3.0	0.818	1.9065	1.223	1.3976	1.7620
1.0	0.322	1.1310	3.106	3.2331	0.3733	3.2	0.857	2.0023	1.167	1.3479	1.9564
1.2	0.382	1.1858	2.621	2.7122	0.4679	3.4	0.895	2.1008	1.118	1.3041	2.1615
1.4	0.438	1.2466	2.281	2.4161	0.5707	3.6	0.931	2.2001	1.074	1.2656	2.3750
1.6	0.493	1.3145	2.027	2.1427	0.6841	3.8	0.966	2.3007	1.035	1.2315	2.5977
1.8	0.545	1.3865	1.834	1.9600	0.8057	4.0	1.000	2.4021	1.000	1.2012	2.8284
2.0	0.596	1.4652	1.678	1.8136	0.9406	...	.....	......	.....	......	......

Table 2 has been constructed to give deviations from the mean for selected frequencies (given herein as percentages of time), and is similar to tables[15] constructed by H. Alden Foster, M. Am. Soc. C. E., for Pearson's Type III curve. Only extreme values are given, as this curve plots fairly straight on logarithmic probability paper. In using such a table, however, it must be remembered that appearance, particularly of a graph plotted on logarithmic probability paper, is no test of the goodness of fit.[16]

TABLE 2.—SHORT TABLE OF EXTREME UPPER VALUES FOR $(CV) = 1$

Coefficient of skew = β	%-OF-TIME									
	90	10	1	0.1	0.01	0.001	0.0001	0.00001	0.000001	0.0000001
0.0.....	—1.28	+1.28	2.33	3.09	3.72	4.26	4.75	5.20	5.61	6.00
0.2.....	—1.25	+1.31	2.49	3.42	4.21	4.91	5.58	6.20	6.78	7.38
0.4.....	—1.23	+1.32	2.64	3.72	4.70	5.62	6.50	7.36	8.19	9.03
0.6.....	—1.20	+1.33	2.79	4.06	5.25	6.41	7.56	8.73	9.89	11.07
0.8.....	—1.16	+1.32	2.91	4.36	5.80	7.24	8.71	10.26	11.81	13.44
1.0.....	—1.12	+1.31	3.04	4.70	6.40	8.20	10.10	12.00	13.90	16.50
1.2.....	—1.09	+1.30	3.16	5.03	7.03	9.16	11.48	14.03	16.74	19.74
1.4.....	—1.05	+1.29	3.25	5.32	7.60	10.19	13.00	16.15	19.22	23.60
1.6.....	—1.03	+1.28	3.36	5.66	8.29	11.25	14.66	18.56	22.89	27.87
1.8.....	—1.00	+1.26	3.45	5.96	8.91	12.32	16.34	21.02	26.34	32.50
2.0.....	—0.97	+1.24	3.53	6.24	9.54	13.43	18.11	23.66	29.99	37.78
2.2.....	—0.94	+1.22	3.60	6.50	10.12	14.49	19.85	26.34	33.91	43.08
2.4.....	—0.91	+1.20	3.65	6.76	10.71	15.58	21.57	29.30	38.03	49.21
2.6.....	—0.89	+1.18	3.71	6.99	11.28	16.64	23.48	31.84	42.57	55.04
2.8.....	—0.87	+1.16	3.75	7.21	11.82	17.70	25.18	34.80	46.94	61.26
3.0.....	—0.84	+1.14	3.79	7.42	12.34	18.77	27.15	37.85	51.00	67.78
3.2.....	—0.82	+1.12	3.83	7.61	12.83	19.72	28.77	40.89	55.60	74.70
3.4.....	—0.81	+1.11	3.86	7.78	13.33	20.80	30.61	43.91	60.27	81.75
3.6.....	—0.79	+1.09	3.88	7.95	13.78	21.78	32.35	46.83	64.90	88.87
3.8.....	—0.77	+1.07	3.90	8.11	14.21	22.64	34.28	49.58	70.08	95.92
4.0.....	—0.76	+1.05	3.92	8.26	14.69	23.61	35.72	52.69	74.44	103.93

A point that will be found useful in plotting the curve is the mode, $x = \frac{1}{d} - b$. If there is sufficient skewness, the end point, $x = -b$, will also help.

2.—Examples.—

Example *a.*—A problem worked out in full detail will illustrate the application of these formulas. The first problem analyzed is that given by

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[15] "Theoretical Frequency Curves and Their Application to Engineering Problems", by H. Alden Foster, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 142.

[16] "Probability and Its Engineering Uses", by Thornton C. Fry, Chapter IX, N. Y., D. Van Nostrand Co., 1928.

Elderton,[17] which is based upon a set of data to which he fits a transition Type III curve. This particular problem was chosen because it is one in which Pearson's curve breaks up into an L-shape. Furthermore, Pearson's Type III curve may be compared fairly with the writer's in that both are specified by the same parameters. The data are listed in Table 3.

TABLE 3.—DATA UPON WHICH EXAMPLE *a* IS BASED ($N = 251$).

x (1)	Observed frequency (2)	Frequency from Pearson's Type III curve (3)	Mid-ordinates of writer's curve (4)
0.400	...	...	0
1	44	59	62
1.575	...	...	125
2	134	111	109
3	45	45	49
4	12	20	19
5	8	9	8
6	3	4	3
7	1	2	2
8	3	1	1

Although the frequencies of Column (3), Table 3, are quite close to the observed frequencies, the mid-ordinates of Pearson's curve show it to be a poor representative of this distribution, as may be seen by referring to the illustration in Elderton's book.[17] Elderton gives the elements: The mean at $x = 2.335$; $\mu_2 = 1.442$ ($\sqrt{\mu_2} = 1.201$; $\mu_2^{1.5} = 1.730$); and, $\mu_3 = 3.607$. Consequently, $\beta = \dfrac{3.607}{1.730} = 2.085$ and, by Equation (25):

$$A = 1.042 + \sqrt{2.022} = 1.042 + 1.421 = 2.463$$

$$B = 1.042 - 1.421 = -0.379$$

and,

$$t = 1.345 - 0.724 = 0.621$$

Of course, this value of t could have been obtained closely enough from Table 1.

Next, compute the quantities: $t^2 + 1 = 1.345$; $t\,(t^2 + 1) = 0.861$; $\sqrt{t^2 + 1} = 1.178$; and, $\sqrt{2.303\ldots \log\,(t^2 + 1)} = 0.574$. With these values, the parameters of the curve may be computed, as follows:

$$\sqrt{2\pi}\;a = \frac{251 \times 0.861}{1.201 \times 0.574} = 313$$

$$b = \frac{1.201}{0.621} = 1.934$$

$$\sqrt{2}\;c = \frac{1}{0.574} = 1.741$$

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[17] "Frequency Curves and Correlation", by W. P. Elderton, p. 90, 1927.

and,

$$d = \frac{0.861 \times 1.178}{1.201} = 0.844$$

The ordinates of the curve are computed by constructing Table 4, each column being constructed from preceding columns. The values in Column (6) are from tables of Equation (8*a*) compiled by Messrs. F. C. Mills and D. H. Davenport.[18] The mode is at $x = \frac{1}{d} - b = -0.75$; $X = 2.335 - 0.75 = 1.575$; and, $y = 313 \times 0.3989 = 125$. The end point is at $x = -1.934$; and $X = 2.335 - 1.934 = 0.4$.

TABLE 4.—Computation Table for Determining Ordinates of Frequency Curve

$x = X - M$ (1)	$x + b$ (2)	$d(x + b)$ (3)	log (Column (3)) (4)	$2.303 \times \sqrt{2} \times c$(Column (4)) (5)	ϕ_0 (Column (5)) (6)	$\sqrt{2\pi}\, a \times$ (Column (6)) (7)
−1.335	0.6	0.506	−0.296	−1.188	0.199	62
−0.335	1.6	1.350	0.130	0.522	0.348	109
+0.665	2.6	2.195	0.341	1.370	0.156	48.8
1.665	3.6	3.040	0.483	1.940	0.061	19.1
2.665	4.6	3.881	0.589	2.365	0.025	7.8
3.665	5.6	4.725	0.674	2.705	0.010	3.1
4.665	6.6	5.570	0.746	2.995	0.005	1.6
5.665	7.6	6.410	0.807	3.240	0.002	0.6
6.665	8.6	7.252	0.860	3.450	0.001	0.3

Example *b*.—The next problem is one given by Fisher,[19] who makes the statement that the statistical assistants working on the problem were unable to fit the data by means of Pearson's curves.

Fisher uses the logarithmically transformed function, (Equation (10)), as a generic function. After calculating the values of m and n for this particular problem, he then determines the constants, c_0, c_3, and c_4, by the method of least squares. Unfortunately, he does not give his computed ordinates; nor does he give the ordinates for the distribution,[20] so that his curve and that of the writer for this problem can be compared only by sight from slightly different perspectives. However, a glance at Fisher's diagram[20] and at Fig. 1 of this paper is sufficient to show that the latter is at least as good a fit. Furthermore, the work required to fit these data by the present method is quite negligible compared to that which Fisher must do to obtain a result that is "satisfactory for all practical purposes".

The observed data and the ordinates computed by the present method are given in Table 5. The elements of this distribution are the following: The mean is at $x = 5.830$, with $\mu_2 = 7.245$ ($\sqrt{\mu_2} = 2.695$; $\mu_2^{1.5} = 19.502$);

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[18] "Problems and Tables in Statistics", by F. C. Mills and D. H. Davenport, Henry Holt & Co., 1925.

[19] "The Mathematical Theory of Probability", by Arne Fisher, 1930, p. 258.

[20] *Loc. cit.*, Fig. 4.

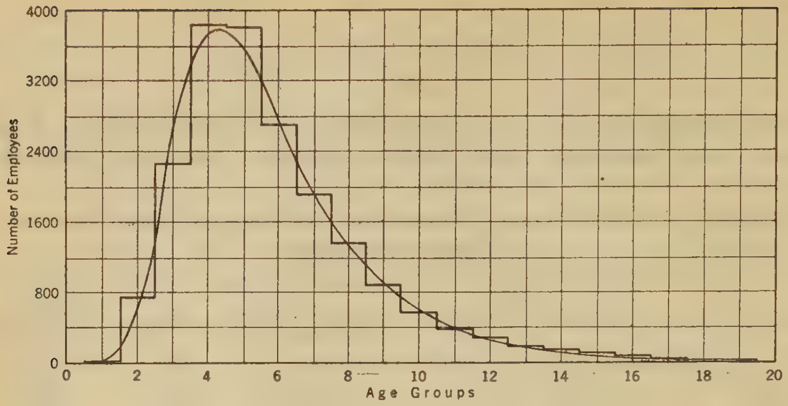

FIG. 1

$\beta = 1.405$; $t = 0.440$; $t^2 + 1 = 1.194$; $t\,(t^2 + 1) = 0.525$; $\sqrt{t^2 + 1} = 1.093$; $\sqrt{2.303 \log (t^2 + 1)} = 0.421$; $b = 6.118$; $c\sqrt{2} = 2.377$; and, $d = 0.2131$.

TABLE 5.—OBSERVED DATA AND ORDINATES

x	Observed frequencies	Computed mid-ordinates	x	Observed frequencies	Computed mid-ordinates
0.42	1	0	12	272	260
1	9	31	13	186	166
2	745	833	14	141	107
3	2 264	2 498	15	110	70
4	3 828	3 492	16	72	45
5	3 801	3 419	17	43	30
6	2 711	2 821	18	17	19
7	1 918	2 066	19	14	12
8	1 339	1 432	20	3	8
9	884	958	21	2	5
10	533	628	22	...	4
11	380	405	23	...	3

Example *c*.—For this problem the writer has taken a 49-yr rainfall record of Chapel Hill, N. C. In order of magnitude the yearly rainfalls are as shown in Table 6. The elements are: $\beta = 0.522$; $\sqrt{\mu_2} = 7.49$; the mean $= 47.72$; $t^2 + 1 = 1.032$; $\sqrt{2\pi}\,a = 6.17$; $b = 42.1$; $\sqrt{2}\,c = 5.14$; and $d = 0.025$.

TABLE 6.—YEARLY RAINFALL, IN INCHES, IN ORDER OF MAGNITUDE

31.88	40.73	46.86	50.39	54.06
32.17	40.82	47.31	50.82	54.54
37.99	40.83	47.37	51.07	54.87
38.47	42.00	47.84	51.38	55.92
38.58	42.41	48.52	52.14	56.85
39.19	43.34	48.56	52.21	58.19
39.42	43.62	48.87	52.81	61.34
39.75	44.81	49.06	53.17	62.25
40.22	45.20	49.42	53.30	67.15
40.45	46.55	50.26	53.57	.....

This record has been plotted cumulatively according to the usual convention[21] (against 0.5, 1.5, 2.5, etc.), the circles in Fig. 2 representing these points. The ordinates for the smooth curve have been computed according to the data in Table 7.

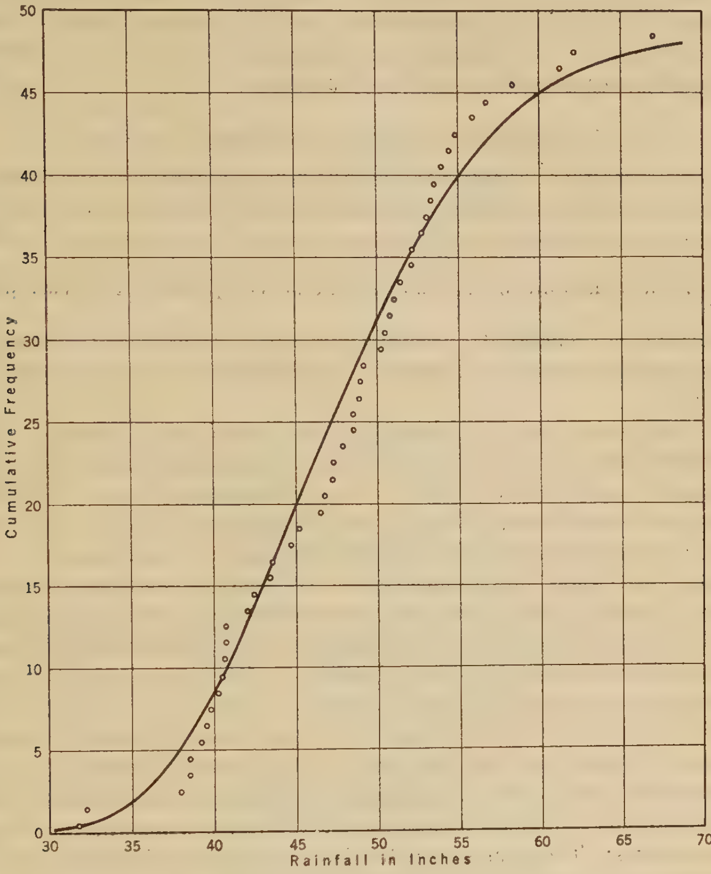

FIG. 2

TABLE 7.—COMPUTATION OF ORDINATES IN FIG. 2

X (1)	$x = X - M$ (2)	$x + b$ (3)	$d\,(x + b)$ (4)	$\frac{d\,(x + b)}{t^2 + 1}$ (5)	log (Column (5)) (6)	11.81 (Column (6)) (7)	Y (8)	$N\,Y$ (9)
30....	—17.72	24.38	0.608	0.589	—0.230	—2.72	0.003	0.15
35....	—12.72	29.38	0.734	0.711	—0.148	—1.75	0.030	1.47
40....	— 7.72	34.38	0.858	0.831	—0.080	—0.95	0.171	8.38
45....	— 2.72	39.38	0.984	0.952	—0.021	—0.25	0.401	19.65
50....	2.28	44.38	1.109	1.073	0.031	0.37	0.644	31.53
55....	7.28	49.38	1.234	1.195	0.077	0.92	0.821	40.25
60....	12.28	54.38	1.359	1.316	0.119	1.41	0.921	45.15
65....	17.28	59.38	1.485	1.439	0.158	1.87	0.969	47.49

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[21] "Duration Curves", by H. Alden Foster, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 99 (1934) p. 1213.

Column (7), Table 7, gives $2.303 \sqrt{2c} \log \dfrac{d(x+b)}{t^2+1}$ for the corresponding values of x. To obtain the values in Column (8) it is necessary to select the values of the probability integral (as given by Mills and Davenport,[14] for instance) corresponding to the values in Column (7). Then, 0.5 + (Column (7)) = (Column (8)), because the tables are calculated so as to give the area from 0 up to the ordinate, x.

IV.—THE MOST GENERAL FREQUENCY FUNCTION

1.—The Range of Variation.—As has been pointed out in Section III, instead of computing the third moment of the distribution, one may determine, somehow, the lower limit of the variation; then, through the relation, $b = \dfrac{\sigma}{t}$, the constants of the curve can be computed. The two processes will not lead, in general, to precisely the same result, but the discrepancy should come well within the margin of probable errors, if the present theory is correct.

It is not always easy to determine this lower point, however. In considering the heights of a group of men, for instance, one would not be justified in taking zero as the lower limit of the variation; 1 ft or 2 ft would be closer to the actual limit. Even if one were to take 0 as this lower limit, however, one would be doing something more rational than taking $-\infty$, as is always done when the normal curve is used in connection with such statistics. In fact, even in the most symmetrical cases ordinarily considered, the normal curve gives the poorest estimate of the range of the variation.

Unlimited variations do occur: There is a finite chance that a run of n heads will occur in tossing a coin no matter how large n may be; but in physical fluctuations these chances do not usually occur because the phenomenon in question does not exist in these extreme aspects.

Consequently, the problem at hand is to determine the range of variation of a statistical series under consideration. For this determination there are two alternatives: The range may be determined statistically (which is, in effect, Pearson's procedure), or, it may be determined from purely physical considerations. The statistical determination is a purely formal process which leads close to the true answer only when the data are abundant, but quite a fair estimate may be made from non-statistical considerations of the range of fluctuation of a great many physical phenomena. For instance, a stream, will certainly not run less than dry, and the greatest possible flood that can occur on it is certainly some function of its drainage area, geographical and geological location, etc.

In what follows it will be assumed that this estimate may be made. Let b = the maximum lower fluctuation from the mean of the statistics, and g = the maximum upper fluctuation.

2.—The Totally Bounded Function.—The construction of a function with both lower and upper limits is a simple generalization of the function already discussed; but the analysis of it is not straightforward. The preceding analysis, however, places one in a position to determine the parameters of the totally bounded function in terms of its end points. Since it is no longer a question of determining these parameters by the method of moments (however desirable such a determination may be), the integral curve will be dealt with directly.

The probability integral will be taken in the form in which it is most usually tabulated, namely,

$$Y = \frac{N}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}}\, du \quad \dots\dots\dots\dots (43)$$

For the partly bounded function, by substituting $\frac{\sigma}{b}$ for t in the preceding formulas:

$$z = c \log d\ (x + b) \quad \dots\dots\dots\dots (44a)$$

in which,

$$d = \frac{\sigma}{b^2} \sqrt{b^2 + \sigma^2} \quad \dots\dots\dots\dots (44b)$$

and,

$$\frac{1}{c} = \sqrt{\log \frac{\sigma^2}{b^2}\ (b^2 + \sigma^2)} \quad \dots\dots\dots\dots (44c)$$

From Equations (43) and (44), by an obvious extension, the following set of formulas is obtained for the totally bounded curve:

$$z = pc \log d \left(\frac{x + b}{g - x}\right) \quad \dots\dots\dots\dots (45a)$$

in which,

$$d = \frac{g^2}{b^2} \sqrt{\frac{\sigma^2 + b^2}{\sigma^2 + g^2}} \quad \dots\dots\dots\dots (45b)$$

$$\frac{1}{c} = \sqrt{\log \frac{g^2}{b^2} \left(\frac{\sigma^2 + b^2}{\sigma^2 + g^2}\right)} \quad \dots\dots\dots\dots (45c)$$

and,

$$p = \sqrt{\frac{g - b}{g + b + 2\,\sigma}} \quad \dots\dots\dots\dots (45d)$$

The factor, p, has been introduced to make the second moment of the curve agree with that of the statistics. The determination has been empirical and, consequently, this factor may be found to require further modification. The considerations that have guided the writer in the present determination are the following: p must approach 1 as g approaches ∞ , and $-$ 1 as b approaches ∞. Furthermore, the product, pc, must remain finite and determinate as g approaches b.

With these elements the curve represents the most general homograde probability function, according to the present theory. It begins at a distance, b, below the mean and ends at a distance, g, above it, these points coinciding with the absolute limits of the statistics, and its mean and standard deviation coincides with those of the observed distribution. By inspection, it is seen that the partly bounded function discussed in Sections II and III is obtained by permitting either g or b recede to ∞ , the other remaining finite (in one case, the curve is right-skewed; in the other, left-skewed).

The form of this function must now be determined as g becomes equal to b; that is, as the curve becomes symmetrical, but bounded. When $g = b$, obviously $d = 1$. Assume that g is nearly equal to b so that $g^2 - b^2$ is small; then,

$$\log \frac{g^2}{b^2}\left(\frac{\sigma^2 + b^2}{\sigma^2 + g^2}\right) = \log\left[1 + \frac{\sigma^2}{b^2}\left(\frac{g^2 - b^2}{g^2 + \sigma^2}\right)\right]$$

$$= \frac{\sigma^2}{b^2}\left(\frac{g^2 - b^2}{g^2 + \sigma^2}\right) + O\left([g^2 - b^2]^2\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(46)$$

Consequently,

$$pc = \sqrt{\frac{g - b}{g + b + 2\sigma}\,\frac{\sigma^2}{b^2}\,\frac{g^2 + \sigma^2}{g^2 - b^2}} + O\left(g^2 - b^2\right) \quad\ldots\ldots\ldots(47)$$

and,

$$\lim_{g \to b} pc = \frac{b}{2\sigma}\sqrt{\frac{b^2 + \sigma^2}{b^2 + \sigma b}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

Therefore, for symmetrical, bounded fluctuations the argument becomes,

$$z = \frac{b}{2\sigma}\sqrt{\frac{b^2 + \sigma^2}{b^2 + \sigma b}}\log\left(\frac{x + b}{b - x}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(49)$$

from which the normal curve is obtained by letting b recede to ∞ , when one obtains, simply,

$$z = \frac{x}{\sigma} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50)$$

It would be highly desirable to determine the parameters of this general function in terms of the moments (for this, a fourth moment would be

TABLE 8.—THE PROBABILITY INTEGRAL

Probability Y, in %-of-time	Parameter, z	Probability Y, in %-of-time	Parameter, z	Probability Y, in %-of-time	Parameter, z
0.0000001	6.00	0.1	3.09	50	0.00
0.000001	5.61	1	2.33	60	−0.25
0.00001	5.20	10	1.28	70	−0.52
0.0001	4.75	20	0.84	80	−0.84
0.001	4.26	30	0.52	85	−1.04
0.01	3.72	40	0.25	90	−1.28
				95	−1.64

required since the condition of termination at the upper end has been added) because, whether or not the present theory is tenable, this function represents the most flexible and stable curve of any in use at present. The writer, however, has found unsurmountable mathematical difficulties on the road to this goal. Table 8 is a short solution of the probability integral.

Example *d.*—To illustrate the use of the most general function given by Equations (45) the distribution of the flow peaks of the Tennessee River, at Chattanooga Tenn., will be considered. In the 57-yr daily flow record of the Chattanooga Station there are 2 440 peaks, ranging from a minimum of 3 360 cu ft per sec, to a maximum of 361 000 cu ft per sec. Their arithmetic average was found to be 49 550 cu ft per sec.

These peaks were grouped into classes of intervals of 5 000 cu ft per sec, so that $X = 1$ is the mid-ordinate (in class units) of the 0—5 000 class, or 2 500; $X = 2$ is the mid-ordinate of the 5 000—10 000 class, or 7 500; etc. The zero flow is at $X = 0.5$, and the mean is $M = 10.41$. The limits, b and g, were taken as 10 and 120 class units, respectively. This makes the range of fluctuation approximately from 0 to 650 000 cu ft per sec. No justification is offered for the selection of these limits. They are used merely for illustration. The standard deviation was computed to be 9.459 class units.

Substitution in Equations (45*b*), (45*c*), and (45*d*), gives $d = 16.47$, $c = 1.257$, and $p = 0.86$. In Table 9 three ordinates are solved to illustrate

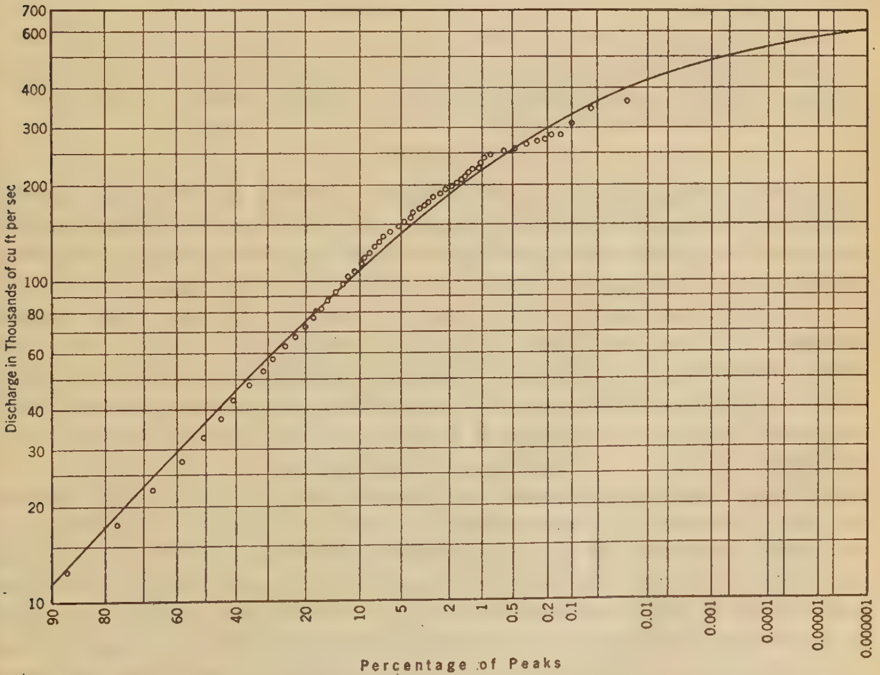

FIG. 3.—FLOW PEAKS, TENNESSEE RIVER AT CHATTANOOGA, TENN., 1875–1931.

the computation. Fig. 3 shows the observed cumulative frequencies[21] compared with the theoretical curve plotted on extreme value logarithmic probability paper.

TABLE 9.—COMPUTATION OF PROBABILITY

X (1)	$x = X - M$ (2)	$x + b$ (3)	$g - x$ (4)	$\frac{x+b}{g-x}$ (5)	$\frac{d(x+b)}{(g-x)}$ (6)	log Column (6) (7)	pc × Column (7) (8)	ϕ_0 (9)	Probability Y, in %-of-time (10)
5....	—5.410	4.59	125.41	0.0366	0.6027	—0.506	—0.548	0.29116	70.88
10....	—0.41	9.59	120.41	0.0796	1.3107	+0.2708	+0.293	0.61409	38.59
90....	79.59	89.59	40.41	2.2170	36.5060	3.597	3.89	0.99995	0.005

Column (9), Table 9, was obtained from the tables of Mills and Davenport.[18] If only a few values on the curve are required, Table 8 may be used as follows: Suppose it is required to find the flood likely to occur 0.01% of the time (assuming the floods equally spaced in time). Opposite 0.01 in Table 8, the value, 3.72, of z is found. The corresponding value of x may now be found by Equation (45a), arranging the computation as follows:

Y....................	0.01	$gw - bd$............	3 585.3
z....................	3.72	$w + d$.............	47.72
$v = \frac{z}{pc}$..............	3.442	$\frac{gw - bd}{w + d} = x$.......	75.1
$w = e^v$.............	31.25		

The last item, $x = 75.1$, is the deviation in class units from the mean. In cubic feet per seconds this flood is, 75.1 × 5 000 + 49 550 = 425 050.

CONCLUSION

Equations (45) were constructed on the assumption that the most general homograde frequency function is characterized by its standard deviation and the maximum possible lower and upper fluctuations from its mean; that the normal is the limiting form of this function when the range is unlimited in both directions; and that the general shape of the normal is always preserved as closely as possible by this most general curve. If the curve becomes unlimited on one side, the other side remaining limited, the partly bounded function of Equation (11) is obtained, the analysis of which has suggested the form of the general function. If the two limits are equal, the symmetrical curve of Equation (49), is obtained.

As in the case of the normal, the errors of sampling of this distribution are those introduced in determining the mean and the standard deviation, the errors introduced in the determination of the end points being non-statistical in nature. Although this does not do away with the difficulties encountered in analyzing skewed frequency distributions, it does shift part of the responsibility from the shoulders of the statistician to those of the engineer or physicist, who may set limits to the problem from extra-statistical considerations.

ACKNOWLEDGMENT

For valuable criticism and suggestions, as well as for having introduced him to a fertile and wide field of analysis, the writer wishes to take this opportunity of expressing his indebtedness to Thorndike Saville, M. Am. Soc. C. E.

While employed as Associate Statistical Engineer by the United States Geological Survey on the work of the Mississippi Valley Committee, the writer has had an opportunity to develop the totally bounded function described in this paper.

DISCUSSION

Gordon R. Williams,[23] Jun. Am. Soc. C. E. (by letter).—The purpose of this discussion is to compare the author's "totally bounded function" with the graphical method in the study of flow peaks. To the mathematician the graphical method is inexcusably crude, but the engineer finds in its use certain practical features that may overshadow what the mathematician considers its undesirable attributes from his standpoint of pure science. It is not proposed to delve into theory herein except to state that it is questionable whether a probability function should be applied to a study of natural phenomena which may result from the combination by summation, multiplication, and cyclical variation of various causes and conditions in a more or less indeterminate and discontinuous sequence.

Any purely mathematical treatment of stream-flow data presumes at the outset that all the data are equally reliable. Unfortunately, this is not the case. Early records are usually less reliable than present-day records, and the accurate determination of extremely high flows is more difficult than that of lower flows. These are facts familiar to the experienced hydrographer. The probable range of variation from the true discharge cannot be computed by any statistical means. If a graphical method is used to interpret these data, the curve can be shifted to give more or less weight to certain points, depending on their accuracy.

To use the author's method it is necessary to list all the flow peaks down to the lowest on record. Of course, it is a minor consideration that this is a laborious task, but it is important to consider what significance these small peaks have. There are few rivers in the United States to-day in which the low flows are not influenced by some works of Man, and, consequently, these flows would be so modified, at least in magnitude, as to have no relation to the frequency and magnitude of the higher flows. There may also be certain peculiarities of the natural drainage area and river valley which would make it doubtful whether one relation would suffice for the full range of flows. Most graphical methods do not attempt to utilize flows below the so-called "one-year flood"; that is, the smallest of the annual maximum floods.

After all the flows have been listed, it is necessary to assign an upper limit, which the author states must be determined by some extra-statistical means. The determination of this upper limit is a matter which is receiving careful consideration from many leading hydraulic engineers to-day. If it is possible to determine this limit accurately, the graphical method as well as the statistical method will benefit. For the graphical method, the upper limit will furnish an asymptote that will greatly increase the accuracy of this method, in the range between the observed data and the limit.

From studies of relatively long records it appears that any method that utilizes a record of less than 30 or 40 years will have to be adjusted for long climatic cycles. For example, any study that utilized the records of the period, 1914 to 1934, would lead in many places to a large under-estimate of the flood potentialities of the region. Phenomena that are influenced by such climatic cycles cannot be studied by mathematical means alone.

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[23] Asst. Hydr. Engr., Water Resources Branch, U. S. Geological Survey, Washington, D. C.

All the methods for analyzing flood and run-off phenomena have weaknesses, but a solution of the problem will be obtained only through a combined consideration of the valuable features of each method.

H. ALDEN FOSTER,[23] M. AM. SOC. C. E. (by letter).—An excellent discussion of probability curves in general, and of the relative merits of the several types of curves which have been proposed, is contained in this paper. The author's criticisms of the Pearson curves are worthy of note, and should be helpful to any engineer who has occasion to use these curves in practical work. His comments on empirical curves and graphical methods are also significant.

Professor Slade presents a new formula for statistical analysis, and develops it mathematically, with numerical examples. This formula is proposed as a substitute for other curves (particularly those of Pearson) which have been in use for many years. He specifies several desiderata which any probability function should satisfy in order to be mathematically consistent and suitable for practical use. The proposed formula is shown to satisfy these requirements; but whether it is superior in these respects to the Pearson curves has not been clearly demonstrated. In the writer's opinion, all these desiderata are satisfied by Pearson's Type I and Type III curves, as well as can be desired for any practical engineering uses.

Requirement (*d*), that "the curve must be simple to apply", does not seem to be well substantiated from a practical standpoint. Doubtless, the proposed curve is easier to handle than some other types of probability functions; but it appears doubtful whether the average engineer would be able to make ready use of it, due to the mathematical operations involved.

As has been shown by the writer,[24] the practical application of the Pearson curves (Types I and III) can be made very simple due to the fact that, with these curves, the distance from the mean to any point on the duration curve is directly proportional to the coefficient of variation. This relation does not appear to hold for the author's formula—at least, he does not mention it. It was by virtue of this relation that the writer was able to prepare tables from which the Pearson curves could be plotted directly, as soon as the proper values of the coefficient of variation (CV) and coefficient of skew (CS) were determined.

The author has given (see Table 2) what appears to be a similar table for his curve, showing variations from the mean at several percentages-of-time, for different values of the coefficient of skew. This table is computed for a coefficient of variation, CV, equal to unity; but there is nothing to show that it can be used for any other value of CV.

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[23] Res. Engr., Parsons, Klapp, Brinckerhoff & Douglas, North Platte, Nebr.

[24] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 158.

Examination of Table 2 indicates that the author's "partly-bounded function" gives results very close to the Pearson Type III curve for percentages-of-time greater than 0.1%; but for more extreme values of the probability scale the proposed formula gives increasingly larger values of the variation than by the Pearson curve.

To apply the new formula to a practical engineering problem, as shown in the examples given by the author, requires the computation of several constants in addition to the coefficients of variation and skew. Computation of ordinates of the frequency curve can then be made directly, by means of Equation (11); but the solution of the duration curve requires the use of a table of the probability integral which, although in common use by statisticians, is not always available to the practicing engineer. The proper method of using the table, even if available, might easily lead the average engineer into difficulties.

The foregoing comments apply particularly to the "partly-bounded function" proposed by the author. The "totally-bounded function" appears to involve still greater difficulties from a practical standpoint.

The writer has attempted to examine this paper from the viewpoint of the practical engineer. In many respects, the paper reflects the attitude of the expert mathematician or statistician rather than that of the engineer. It is quite possible that the author's formula may have great potential value in general statistical analysis. The writer does not consider himself qualified to discuss it from that angle.

R. D. GOODRICH,[25] M. AM. SOC. C. E. (by letter).—A notable contribution to the knowledge and literature of asymmetric probability functions is contained in this paper. Little need be added to the discussion, in Section I, of the general probability functions and of the curves in use at the present time. The author has emphasized clearly the practical difficulties and limitations to the use of Pearson's, Thiele's, and the Gram-Charlier series for many engineering problems. They are all mathematically "elegant"; but mathematical elegance has less appeal to the engineer than practicability.

It is well to call attention, as the author has, to the desirability of estimating the reliability of the parameters used in any method. Perhaps still more important, however, is some knowledge of the reliability of the results to be obtained from the final curve or equation. That this seldom is possible with purely empirical methods is quite true.

The writer disagrees with the author's statement to the effect that "the use of * * * and other graphical devices is an undesirable practice." On the contrary, graphical methods are frequently most desirable for the presentation and analysis of engineering and numerous other data. No doubt the author's point is that the value of graphical methods is largely lost if their limitations are not kept clearly in mind. His criticism of the writer's semi-graphical method[12] for the analysis of skew frequency distributions can be met, at least to a large extent, by adapting the method of least squares to the determination of the parameters of the function. These calculations make it possible, with little computation, to determine the probable errors of the parameters and also that of any value of the variable as read from the curve for a given percentage of time.

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[25] Prof., Civ. Eng., and Dean, Coll. of Eng., Univ. of Wyoming, Laramie, Wyo.

An interesting graphical method for the application of least squares to the fitting of curves to data has been written by Professor E. O. Waters.[26] The writer now makes frequent use of an analytical method for Case I of Professor Water's paper, to fit the curves of his method[12] to flood data, with satisfactory results.

Professor Slade has given an excellent statement of the character desirable for an asymmetric probability function, and has presented a form for this function which meets his specifications to a remarkable degree. The solution of the equation involving this function, when partly bounded, in terms of the power sums, or moments of the data, needs no discussion.

One great advantage of the proposed function is the fact that it includes the normal curve; but a disadvantage is that this curve is a limit as to the skewness in one direction, the skewness always being toward the finite limit of the curve. This is evident from a comparison of Equations (32) and (33) and also from Equation (36), and is stated by the author in the text following Equations (45). An example of flood data for which the mode is greater than the mean, and for which there is no apparent upper limit, is that of the Mackenzie River, in Australia.[27] In this distribution the range of observations above the mean is only 85% of the range below it, although the number of observations above and below the mean is practically the same.

TABLE 10.—COMPARISON OF RESULTS, ANALYTICAL AND GRAPHICAL METHODS

				GRAPHICAL RESULTS (GOODRICH)				SLADE		PEARSON	
Values of the variate, X (1)	Observed frequency (2)	Frequency in percentage of total (3)	Total cumulative frequency (percentages) (4)	Total cumulative frequency (percentages) (5)	Frequency in percentage of total (6)	Frequency number (7)	Difference between observed and graphical frequencies (8)	Frequency number (9)	Difference between observed and computed frequencies (10)	Frequency number (11)	Difference between observed and computed frequencies (12)
1....	44	17.6	17.6	14.0	14.0	35	+9	62	−18	59	−15
2....	134	53.6	71.2	72.0	58.0	145	−11	109	+25	111	+23
3....	45	18.0	89.2	89.0	17.0	43	+2	49	−4	45	0
4....	12	4.8	94.0	94.8	5.8	15	−3	19	−7	20	−8
5....	8	3.2	97.2	97.1	2.3	6	+2	8	0	9	−1
6....	3	1.2	98.4	98.2	1.1	3	0	3	0	4	−1
7....	1	0.4	98.8	98.8	0.6	2	−1	2	−1	2	−1
8....	3	1.2	100.0	99.2	0.4	1	+2	1	+2	1	+2
Total..	250	100.0	.....	......	100.0	250	30	.....	57	.....	51
Means.	...	.....	.....	......	......	.....	3.75	.....	7.13	.....	6.38

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[26] "Graphical Methods for Least Square Problems," *Proceedings*, Applied Mechanics Div., A. S. M. E., 1928.

[27] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 20.

The use of the new equation as an expression of the mathematical frequencies of a skew distribution, as illustrated by the computations for the data of Table 3, involves the computation of ten auxiliary quantities before the four constants of the equation can be determined. Using skew frequency paper, the writer can plot a curve[12] agreeing more closely with the data, comparatively, with little computation. All that is necessary to plot the diagram is to prepare the first four columns of Table 10.

Values of X from Column (1), Table 10, are then plotted against the cumulative values of skew frequency in Column (4) on standard, skew frequency paper, and an average, straight line is drawn by eye through the plotted points. Average values of total frequency are then read from the straight-line curve and entered in Column (5), Table 10. The determination of the data in the remaining columns is obvious. The sum of the differences between the observed and computed frequencies by the graphical method is only three-fifths as great as that resulting from the use of the analytical methods propounded by either Professor Slade or Professor Pearson, while the time and labor required for the graphical solution is only a fraction of that necessary for any analytical solution. Furthermore, the equation of the integral frequency curve requires the determination of only two constants from the diagram, the slope and intercept, which, with the total number of observations, give an equation of the form:

$$t = n\, e^{i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

in which, $i = -\, k\, x^{-c}$. If the writer's equation involving the lower limit of the frequency curve, $-b$ in the author's notation, is used, and the values of the index and modulus are determined by least squares, very much closer agreement can be obtained between the observed and computed frequencies. Furthermore, the probable errors of these parameters can be determined also. Contrary to the author's statement, the computations for the necessary factors can be made graphically by Professor Water's method, as previously mentioned, although the writer prefers analytical methods for this work. The computation of two means and three other quantities similar to the author's moments, are all the factors necessary for the determination of the two parameters of Equation (51). The computations, therefore, are only about one-half those required by the author's method. The maximum deviation below the mean is not determined statistically by the writer's method. Theoretically, at least, Professor Slade's method has an advantage in this one respect, in that, for the partly bounded function, this quantity can be determined statistically.

It can be demonstrated that the writer's equation can be fitted to a set of observations much closer than by the use of either the author's or Pearson's functions. Furthermore, an equation with four constants, including the number of observations, n, and the lower limit, b, can be fitted without the use of products or moments of the third degree which are required in the author's solution. The use of third or higher moments is a disadvantage, especially for the shorter series of observations, on account of the increase in the magnitude of the errors in the resulting power sums for these higher moments.

This fact is recognized and mentioned by the author. The suitability of the value selected for the lower limit of the curve may be tested by use of the coefficient of correlation.

While the duration curve in Fig. 2 approximates the data fairly well, the curve crosses the evident trend of the data in the central part of the diagram instead of following it. With the writer's method[12] a curve can be fitted which has sharper curvature at each end so that the data are more closely approximated by the curve. The question, therefore, arises as to whether or not the new function is sufficiently flexible to become generally useful. The question is one that can only be answered after a great many trials with a great variety of distributions. The rainfall data used show unmistakable evidence of a disturbing influence so that the series is not especially well adapted to illustrate the use of the proposed function. Study of physical conditions might disclose two distinct distributions of storms due to separate sets of causes, which are superposed to form the records given.

The most general function proposed is a logical extension of the partly bounded function and should meet the needs of those who find it necessary to investigate frequency distributions requiring treatment with a more general formula. The writer is of the opinion that extra-statistical considerations are not only justified as an aid in these investigations, but that the physical conditions influencing any phenomena should always be studied, and the results of such studies should be included in and made a part of the investigation. No other method except that of probabilities, is available in many investigations dealing with problems of water supply, river regulation, and flood control, and to pass by the aids offered by the use of such tools as the material in this paper, would appear to be the poorest of judgment on the part of the engineer who does not avail himself of them.

While the writer does not see any great practical advantage in the use of this new function over the semi-graphical methods heretofore proposed, the advantages in many cases over the older functions used for analytical methods are quite obvious. It would be of interest and value to those who contemplate the use of these functions to have the formulas given for the standard errors of the moments and constants necessary for the solution of the equations. The author is to be commended for his able contribution to the literature of this subject.

F. T. MAVIS,[28] ASSOC. M. AM. SOC. C. E. (by letter).—The mathematics of statistical analysis is likely to become so intriguing that the objectives and the limitations of the analysis may be obscured by algebraic manipulation and integration. Undue attention to analytical formulation may conceivably lead to tacit assumptions and hypotheses and to conclusions which appear to be contradictory. The author states, for example (in the sentence preceding Table 2), that "* * * it must be remembered that appearance, particularly of a graph plotted on logarithmic probability paper, is no test of goodness of fit," and later, in Example *b*, he states "* * * a glance at Fisher's diagram[20] and at Fig. 1 of this paper is sufficient to show that the latter is at least as good a fit." Back of the first statement undoubtedly lies not only the tacit assumption of something like Pearson's test of goodness of fit,[29] but also the assumption that the type curve chosen must be expressible in equations of a particular form.

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[28] Associate Director, in Chg. of Laboratory, Iowa Inst. of Hydr. Research; Prof. and Acting Head, Dept. of Mechanics and Hydraulics, State Univ. of Iowa, Iowa City, Iowa.

Under "Graphical Methods," the author appears to have blamed the tools rather than the improper or unskillful use of them in his statement that the use of "probability" paper and other graphical devices is an undesirable practice, and that these methods convey to the eye a simplicity which does not really exist. According to Professor Slade, "the process of using these methods is not essentially different from that of fitting a curve to the data, except for the fact that one is more likely to fit the wrong curve graphically than analytically." To one who has had some practice in using both graphical and numerical methods of curve fitting the foregoing dicta are somewhat disturbing, and the writer would be much interested in the arguments supporting the author's statements.

Admittedly, there are cases in which graphical methods are less convenient than numerical methods, and there are border-line cases in which the choice of the one or the other is largely a matter of personal preference, or the whim of the moment. The person equally skilled in the use of both will ordinarily make his choice between them on the basis of: (1) The limits of precision demanded to be consistent with the precision of the given data; (2) the relative amounts of time required to arrive at a solution by the two methods; and (3) the chance taste of the computer at the moment. It is the writer's opinion that the engineer who has frequent occasion to analyze statistical data would do well to acquire skill in the use of both graphical and numerical (or analytical) methods. It is not unlikely that the crudest of methods—graphical or numerical—will satisfy the engineer who at infrequent intervals follows the simplest procedure described in a book which he finds handy.

Any set of observational data to which statistical methods may be applied, must be considered as a sample which may approximate but never precisely define every larger set of the same data. An arrangement showing the frequencies of values in classes arranged in order is called a frequency distribution. For a given sample and given class intervals there is a unique frequency distribution which may be represented graphically by a frequency polygon, or by a histogram which is made up of frequency rectangles. An estimate of the limit that would probably be approached if the class intervals were made smaller and the number of observations larger, without limit, is represented by the frequency curve—a graphical counterpart of the author's frequency functions. It is to this frequency function that the author is directing his analysis, an analysis which is based naturally on the assumption that the variables are continuous instead of discrete.

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[29] "Handbook of Mathematical Statistics," by H. L. Rietz, p. 78.

If due consideration is given to the effects of errors in making observations, and if data are reported with a precision consistent with the probable error of a single observation, all hydrologic variables must be considered discrete rather than continuous variables. If increments of the variables represent a constant numerical difference—as in the case of annual rainfall shown in Fig. 2—linear graduations of the variable appear consistent with the data. If increments represent constant percentage differences, as might be anticipated in the case of run-off measurements, logarithmic graduations of the variable would appear to be preferred. A wide variety of regular functional co-ordinates can be devised by trial to represent cumulative frequencies as abscissas if the values of the variable are plotted as ordinates. If graphical methods (see under heading, "Graphical Methods") "* * * convey to the eye a simplicity which does not really exist" is it not equally true that by assuming a continuous function for a small sample of discrete variables that the analysis may be needlessly complicated? Is it good practice to apply to small samples of observations, which are most common in engineering studies, the more elegant methods devised by the statistician on the basis of assumptions of an infinite number of observations of a continuous variable?

The writer does not clearly understand (see Section 7) why "* * * one is more likely to fit the wrong curve graphically than analytically." The statement appears to imply one or more assumptions which are not necessarily true: (1) That the same curve cannot be fitted by graphical as by numerical methods; (2) that the correct curve is known or even determinable; and (3) that the tests of goodness of fit can not be applied, within consistent limits of the data, to either an analytical expression or to its graphical counterpart.

The writer does not wish to imply that the analytical methods described by the author do not have their proper sphere of application, nor that after many more years of records are available they might conceivably be helpful in analyzing rainfall and run-off data. Perhaps by that time the mathematician and the engineer may have discovered even greater simplifications in both graphical and numerical methods.

L. Standish Hall,[30] Assoc. M. Am. Soc. C. E. (by letter).—This paper follows a series of papers, covering the same subject, which have been presented to the Society from time to time. Considering the mathematical nature of the subject-matter, the author has handled the text in a very readable manner.

The curves presented are of a sufficiently flexible nature to fit a given set of data more satisfactorily than any other yet prepared. This is particularly true with regard to maximum and minimum values. However, from the standpoint of logic, rather than statistics, it is evident that many short records of hydrological functions cannot be expanded by any formula to give a reliable indication of the probable happening in a 100-yr or 1 000-yr period.

The author anticipates this difficulty for he states (see "Synopsis") that, "unless the data are very complete, the final results will seldom have meaning because of the magnitude of the probable errors of the parameters involved." In discussing end points he concludes, furthermore, that it may be necessary to "set limits to the problem from extra-statistical considerations." These two quotations state clearly the nature of the difficulty encountered in applying statistical methods to short records. Strictly speaking, it is the writer's opinion, after careful analysis of many records, that the distribution of data cannot be determined with accuracy for a 100-yr period until at least a 200-yr record is available.

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[30] Chf. Hydrographer, East Bay Municipal Utility Dist., Oakland, Calif.

The point in question can be best illustrated perhaps by an example taken from the record of the seasonal run-off of the Mokelumne River in California. A record (partly estimated from neighboring streams) is available for a period of 39 yr, from 1895–96 to 1933–34. A duration curve drawn from these data is shown in Fig. 4(*a*). This is the curve of natural run-off from 1895–96 to the season, 1933–34, measured and estimated. The mean annual run-off for the 39-yr record is 744 000 acre-ft; but a consideration of rainfall records, which cover about twice the period of time, indicates that the mean seasonal run-off of the Mokelumne River over a longer period of time would be about 800 000 acre-ft. Therefore, undoubtedly, the period of record includes a preponderance of run-off below normal. From Fig. 4(*a*), as drawn, it is probable that the minimum run-off in a 100-yr period can be determined with a greater degree of accuracy than the maximum run-off.

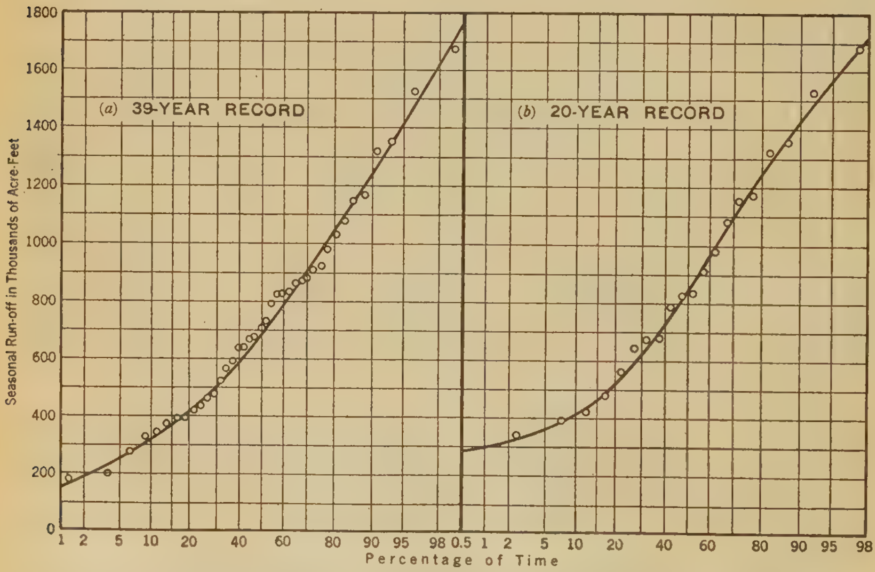

FIG. 4.—MOKELUMNE RIVER AT CLEMENTS, CALIF.; DURATION CURVE OF RUN-OFF FOR THE SEASONAL YEARS ENDING SEPTEMBER 30, 1934.

This is perhaps more evident by comparing the curve in Fig. 4(*a*) with a similar curve prepared from the first 20 yr of record as depicted in Fig. 4(*b*). The latter is the curve of natural run-off from 1895–96 to the season, 1914–15, measured and estimated. From Fig. 4(*a*) it would appear that once in 100 yr a seasonal run-off of less than 150 000 acre-ft could be expected. From Fig. 4(*b*), the minimum in the same period of time is 300 000 acre-ft. There would have been no justification on the basis of the 20-yr record of estimating a minimum of 150 000 acre-ft.

In connection with reservoir studies it is often probable that two, three, or four dry years in succession will create a more serious draft on storage than a single dry year. The average probable minimum yearly flows during such dry periods can be determined readily from the duration curve. The location of the proper percentage can be determined mathematically according to the laws of chance. The probability of several successive occurrences in a period of time is the *n*th root of the period selected. Thus, the two dryest successive years in a 100-yr period would be less than $\sqrt{0.01}$, or 0.10; that is, two years in succession would occur in a 100-yr period of lesser volume than the run-off, which would normally occur once in 10 yr. The average of two successive years can be assumed to be equal to the median point or to the run-off which would occur 5% of the time. Similarly, the average three, four, five, and six successive dry years in a 100-yr period can be found at the ordinates, 11%, 16%, 20%, and 23%, respectively.

The probable dryest periods of annual flow of the Mokelumne River ranging from 1 yr to 6 yr and the actual occurrences based on the 39-yr record in Fig. 4(*a*), and also on the first 20-yr record in Fig. 4(*b*), have been listed in Table 11. It is seen that the year, 1924, was not the dryest year to be expected in 100 yr; neither were the years, 1930 and 1931, the two dryest years in succession to be expected. However, the three years, 1929–31, were about equal to the dryest period of this length to be expected, and the years, 1929 to 1934, were appreciably dryer than the normal 6-yr period to be expected in 100 yr.

TABLE 11.—DRYEST PERIODS OF ANNUAL FLOW OF MOKELUMNE RIVER, RANGING FROM ONE YEAR TO SIX YEARS

Occurrence	39-YEAR RECORD				20-YEAR RECORD			
	Estimated		Measured		Estimated		Measured	
	Percentage from Fig. 4 (*a*)	Mean seasonal run-off, in acre-feet	Seasonal years	Mean seasonal run-off, in acre-feet	Percentage from Fig. 4 (*b*)	Mean seasonal run-off, in acre-feet	Seasonal years	Mean seasonal run-off, in acre-feet
Single year	0.5	120 000	1924	182 000	0.5	280 000	1898	346 000
Two years	5	250 000	1930–31	319 000	5	360 000	1912–13	408 000
Three years	11	330 000	1929–31	323 000	11	425 000	1898–1900	528 000
Four years	16	380 000	1931–34	400 000	16	475 000	1897–1900	640 000
Five years	20	415 000	1930–34	407 000	20	520 000	1896–1900	669 000
Six years	23	440 000	1929–34	395 000	23	555 000	1898–1903	708 000

If the first 20-yr record is considered alone, it will be noted that the estimated occurrences for all periods from 1 yr to 6 yr are considerable lower than the actual record. This could be accepted as an indication that the dryest cycle of years to be expected had not yet been experienced. This diagnosis was amply borne out by the succeeding nineteen years of record.

In the case of the full 39-yr record, the close agreement between the measured and the estimated occurrence for periods from 3 to 6 yr would indicate that, at present, the minimum year can be determined with a reasonable degree of accuracy. In fact, if a wet cycle of years should follow, the record for 60 yr might indicate a revision of the 100-yr minimum upward.

A method of determining the probable accuracy of the maximum and minimum end-points is a necessary adjunct to the use of probability methods to statistics.

ARNE FISHER,[31] ESQ. (by letter).—In several places in this paper Professor Slade uses such terms as "homograde series," "homograde frequency functions," and "partly bounded functions"; and in Part IV he discusses a certain function which he terms "the most general frequency function" which, in a sense, would seem to imply that this type of function includes all other known types, such as the Gaussian or Laplacean exponential error curve, the Poisson exponential, Gram's series, and the error curves introduced by Kapteyn and Bachelier. It should be noted that the author does not state what he understands by the word, "homograde." The technical terms, "homograde" and "heterograde," were first introduced and defined in the theory of statistics by Charlier and have a perfectly definite meaning.[32] The term, "homograde," as used by Professor Slade apparently stands for something quite different and his so-called "homograde functions" appear to be certain frequency functions defined by Equations (45) or Equation (11) of his paper. These equations are nothing but special cases of the frequency function:

$$y = \frac{1}{n \sqrt{2 \pi}} e^{-0.5 t^2} \quad \text{......................(52)}$$

in which, $t = \left[\frac{h(x) - m}{n} \right]$, which in the case of $h(x) = x$ reduces to the well-known law of error by Laplace and Gauss.

In 1856, the Danish Minister of Finance, C. G. Andrae, demonstrated[33] that although the spread of "hits" of rifle bullets, when measured in reference to a vertical or horizontal diameter, follow the well-known exponential error law, it will be found, on the other hand, that the spread of the hits when measured from the center of the target will be characterized by the following skew frequency curve:

$$P(x) = \frac{2x}{a^2} e^{-0.5x^2} \quad \text{......................(53)}$$

where $P(x)$ is the probability that a bullet will strike a target at a distance, x, from the center. Furthermore, if $P_1(x)$ is the normal frequency or probability curve,

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[31] Actuary, Western Union Telegraph Co., New York, N. Y.

[32] "Mathematical Theory of Probabilities", by Arne Fisher, p. 128, Macmillan Co., New York, 1930.

[33] "Bestemmelsen af Skudsikkerheden ved Skydning mod verticale Skiver", by C. G. Andrae, 1856.

$$P_1(x) = \frac{1}{n\sqrt{2\pi}} e^{-0.5x^2 : n^2} \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

the probability curve for, say, x^2, will differ from Equation (54)[34].

In general, therefore, if a certain statistical variate, x, is normally distributed (that is, if it is subject to the Laplacean probability curve, or the error law of Gauss), a function of x, say $H(x)$, will not necessarily be distributed normally. On the other hand, it is also true that if the variate, x, is not normally distributed, a function of x, say $h(x)$, may be so distributed. Therefore, it should be possible, at least in theory, to throw a skew or non-normal curve into normal form through a proper transformation of the variate, x.

As a matter of fact, in 1889[35], Thiele, a Danish astronomer and actuary, discussed the general problem of determining the frequency curves, not only for x but for various functions of x. He mentions that, if the distribution of the variate, x, as actually observed, is found to be skew, or to deviate from the normal probability curve, x may be considered as a "fictitious" observation of the "pseudo-normal" variate, $t = h(x)$, which latter function is supposed to follow the Gaussian error law; that is, if $t = h(x)$ is distributed according to a known normal error law of exponential form, it is possible to find the frequency distribution of the inverse function, $x = g(t)$.

A brief outline of some of the more elementary parts of Thiele's work (based on a communication to the writer from the late Dr. J. P. Gram, in 1903) is, as follows (for numerical details see Table 18):

1.—Arrange the observed frequencies, $F(x)$, in class intervals of the variate, x.

2.—Accumulate the values, $F(x)$ (that is, in the form $\sum_{-\infty}^{x} F(x)$), successively, or step by step, from lower to upper limits of class intervals.

3.—Calculate the accumulated relative frequencies, or $\sum_{-\infty}^{x} \phi(x) = \frac{\sum_{-\infty}^{x} F(x)}{N}$, in which, $\sum_{-\infty}^{\infty} F(x) = N$ equals the size of the observed sample; that is, the total number of observed individuals.

4.—The curve under Step (3) will be a monotonely increasing function, gradually rising from zero to an upper limit which never can be greater than 1.

5.—It is evident, therefore, that a unique, or a one-to-one, correspondence exists between $\sum_{-\infty}^{x} \phi(x)$ and the probability integral, $I(t) = \int_{-\infty}^{t} e^{-0.5t^2}\, dt$, so that to every value of x in $\sum_{-\infty}^{x} \phi(x)$ there will be a corresponding value of t in $I(t)$.

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[34] "The Mathematical Theory of Probabilities," by Arne Fisher, 1930, p. 236.

[35] "Almindelig Iagttagelseslaere" ("A General Doctrine of Observations"), by T. N. Thiele, 1889.

6.—Plot on co-ordinate paper the values of t corresponding to x. The form of the graph or curve drawn through these points will give a general idea of the function of x that will be normally distributed, although x itself may not be normally distributed.

7.—The functional expression of $t = h(x)$ may assume a great variety of forms, some of which are:

$$t = ax + b = (x - M) : \sigma \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

which is the usual form of t in the exponential curve of error;

$$t = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n \ldots\ldots\ldots\ldots\ldots\ldots (56)$$

special forms of which are ax^2, ax^3, etc.; and, $t = \log x$, or,

$$t = [\log (x - a) - m] : n \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

8.—When it is known that there is both a lower finite limit,, x_0, and an upper finite limit, x_n, the following form is very useful:

$$t = h(x) = \left[\log \left(\frac{x - x_0}{x_n - x} \right) - m \right] : n \ \ldots\ldots\ldots\ldots (58)$$

A correlation coefficient, for instance, has a lower limit of -1 and an upper limit of $+1$. A probability can never be less than zero or greater than 1. At any one instant, the busy selectors over a telephone exchange can never be less than zero or greater than the number of subscribers. (In Poisson's formula there can be theoretically an infinite number of calls.)

Among these various transformations the writer first proposes to consider Equation (57) because it is the one that he has discussed somewhat in detail elsewhere and the one which Professor Slade has criticized. Under the heading, "Transformations of the Variate," the author cites Equation (10) as the definition of a new generic function credited to the writer. This statement is in error; no where has the writer made use of the function as claimed in Professor Slade's Equation (10). The generator that the writer used[30] was of the form:

$$\phi_0 (x) = \frac{N}{n \sqrt{2 \pi}} e^{-0.5[h(x)]^2} \ldots\ldots\ldots\ldots\ldots\ldots (59)$$

in which, $h(x) = \left[\dfrac{\log x - m}{n} \right]$.

Moreover, the writer has emphasized the fact that the logarithmic transformation in question, presupposes a lower limit, a, located at a distance of u units from the mean value of the variate (or first-order semi-invariant, $\lambda_1(x)$, of x) and defined by the relation, $\lambda_1(x) - u = a$. The value of u is the real root in the cubic equation,[38]

$$\lambda_3 u^3 - 3 \lambda_2^2 u^2 - \lambda_2^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (60)$$

in which, λ_2 and λ_3 denote the semi-invariants of second and third order.

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[38] "The Mathematical Theory of Probabilities," by Arne Fisher, 1930, p. 239.

The logarithmically transformed generic function, as the writer uses it, becomes, therefore, Equation (59), with $h(x) = \left[\frac{\log (x - a) - m}{n}\right]$, and not the function that Professor Slade attributes to the writer in his Equation (10).

This feature is important in connection with the author's "partly bounded function," which he defines by Equation (11). In introducing this allegedly "new" function Professor Slade writes,

> "Notice, in the first place, that Equation (11) is similar to the logarithmically transformed function, Equation (10), differing from it only in the power, c^2, to which it is raised. The difference is fundamental, however. It is just this added parameter, c, that makes Equation (11) more than a 'normal' for the logarithms of the variate and which brings about a remarkable simplification of the formulas. It is this c, indeed, that makes Equation (11) a true generalization of Equation (1)."

In this case the author has magnified the mathematical symbol, c, out of all proportion; his Equation (11) is identical to the logarithmically transformed generator in Equation (59) which he has criticized in his discussion of some of the writer's work.

As proposed by Professor Slade in Equation (11) his function is of the form:

$$y = A\ e^{-[g(x)]^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

in which, $g(x) = c\ [\log\ (x + b) + \log\ d]$; whereas the generator, as the writer has used it, is of the form of Equation (59), with $h(x) = \left[\frac{\log (x - a) - m}{n}\right]$, which simply states that the quantity, $\log\ (x - a)$, is normally distributed around a mean value, m, with a dispersion (or standard deviation) equal to n.

This generator is algebraically identical to the "partly bounded function" in the paper by Professor Slade; in fact, $A = N\ (n \sqrt{2\pi})^{-1}$; $c^2 = 1:2n^2$; $\log d = - m$; and, $b = - a$ (or u). The author's criticism of the type of generator in Equation (59), therefore, also applies to his own "partly bounded function" (Equation (11)).

By way of illustration it may be of interest to apply the methods advocated by the writer to Professor Slade's "Example *a*." The author uses moments, whereas the writer has preferred to use semi-invariants, which, as a rule, are easier to handle in numerical calculations. However, when the moments are taken around the mean as origin, it is found that the first, second, and third moments, μ_1, μ_2, and μ_3, are equal to the semi-invariants, λ_1, λ_2, and λ_3; therefore, $\mu_1 = \lambda_1 = 2.335$; $\mu_2 = \lambda_2 = 1.442$; and $\mu_3 = \lambda_3 = 3.607$.

The equation for determining the value of u is, therefore: $3.607u^3 - 3\ (1.422)^2\ u^2 - 1.442^3 = 0$; or,

$$3.607\ u^3 - 6.238\ u^2 - 2.998 = 0 \ldots\ldots\ldots\ldots\ldots (62)$$

for which the real root is found to be $u = 1.948$ (with $u^2 = 3.795$). Consequently, the lower limit, a, equals $2.335 - 1.948$, or 0.387.

Moreover,[36] $n^2 = \log (1 + 1.442:3.795) = \log 1.381 = 0.323$; $n = 0.5683$; $m = \log (1.948) - 1.5 (0.323) = 0.6668 - 0.4845 = 0.1823$; and, $N = 251:e^{0.1823+0.1615} = 251:1.416 = 177.93$; so that finally:

$$\phi_0 (x) = \frac{177.93}{0.5683 \sqrt{2 \pi}} e^{-0.5[h(x)]^2} = \frac{313.1}{\sqrt{2 \pi}} e^{-0.5[h(x)]^2} \quad \ldots\ldots(63)$$

in which, $h(x) = \left[\frac{\log (x - 0.387) - 0.1823}{0.5683}\right]$.

The computation of the ordinates is given in Table 12. It will be noticed that the final results are exactly the same as those in Table 4 and are reached with less effort than that involved in the method proposed in the author's paper. Professor Slade's Objection (*a*), under the heading, "Transformations of the Variate," can be met by the statement that his (Slade's) method requires the identical solution of the same type of cubic, as expressed in his own Equation (24). Moreover, it seems that his (Slade's) theoretical development, involving Equations (12) to (42), is unnecessarily unwieldy and considerably more complicated than the writer's (Fisher's) theory.

TABLE 12.—COMPUTATIONS FOR DETERMINING ORDINATES OF FREQUENCY CURVE (SEE TABLE 4)

x (1)	log (x–0.387) (2)	(2)–0.182 (3)	(3):0.568=t (4)	ϕ_0 (t) (5)	313.1(5) (6)
1	–0.489	–0.671	–1.181	0.1986	62.2
2	0.478	0.296	0.521	0.3483	109.1
3	0.960	0.778	1.370	0.1561	48.9
4	1.285	1.103	1.942	0.0605	18.9
5	1.529	1.347	2.371	0.0240	7.5
6	1.725	1.543	2.717	0.0099	3.1
7	1.889	1.707	3.005	0.0044	1.4
8	2.030	1.848	3.254	0.0020	0.6
9	2.153	1.971	3.470	0.0010	0.3
10	2.263	2.081	3.664	0.0005	0.2

In "Example *b*" the author has quoted certain data from the writer's investigations and states "that the work required to fit these data by the present [Slade's] method is quite negligible compared to that which Fisher must do to obtain a result that is 'satisfactory for all practical purposes'."

In the light of what has been said before in reference to the writer's own Equation (59), which is algebraically identical to, and easier to handle than, the author's "partly bounded function," the foregoing and somewhat dogmatic assertion by Professor Slade becomes invalid in so far as the generating function, $\phi_0(x)$, is concerned.

In Table 13, the values of the symbols, b, c, and d, as used by Professor Slade, are compared with the corresponding values of the symbols, a (or u), m, and n, that enter into the formulas that the writer has used.[37] Referring to Column (3), Table 13, $\lambda_1 (x) - u = a$. If the origin is taken at the mean, or at $\lambda_1(x)$, it is found that $\lambda_1(x) = 0$, or $a = - u$.

TABLE 13.—COMPARISON OF SYMBOLS—SLADE AND FISHER METHODS

SLADE		FISHER		Remarks
Symbol (1)	Value (2)	Symbol (3)	Value (4)	(5)
b	6.118	a (or $-u$)	6.118	Identical
c	1.681	n	0.4205	....
c^2	2.826	$1:2n^2$	2.826	Identical
d	0.2131	m	1.5462	....
$\log d$	—1.5462	$-m$	—1.5462	Identical

The logarithmically transformed variate (on the basis of age intervals of 1 yr) becomes,[37]

$$t = h(x) = [\log (x - 13.1) - 2.645] : 0.420 \ldots\ldots\ldots (64)$$

with a generating function of the form:

$$\phi_0 (x) = \frac{1}{0.420 \sqrt{2 \pi}} e^{-0.5[h(x)]^2} \ldots\ldots\ldots\ldots\ldots\ldots (65)$$

in which, $h(x) = \left[\frac{\log (x - 13.1) - 2.645}{0.420}\right]$.

TABLE 14.—COMPUTED MID-ORDINATES (SEE TABLE 5)

Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)	Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)	Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)	Ages (inclusive) (1)	Observed frequencies (2)	Computed mid-ordinates (3)
13 to 15	1	1	31 to 33	2 711	2 798	49 to 51	272	260	67 to 69	17	19
16 to 18	9	31	34 to 36	1 918	2 064	52 to 54	186	166	70 to 72	14	13
19 to 21	745	831	37 to 39	1 339	1 428	55 to 57	141	107	73 to 75	3	8
22 to 24	2 264	2 492	40 to 42	884	958	58 to 60	110	69	76 to 78	2	5
25 to 27	3 828	3 486	43 to 45	533	624	61 to 63	72	45	79 to 82	....	4
28 to 30	3 801	3 426	46 to 48	380	405	64 to 66	43	29	> 83	......	6

The computed mid-ordinates, using the correctly computed generating function (Equation (65)), are listed in Table 14.

It is easily recognized that the generating function itself (or Professor Slade's "partly bounded function") is not powerful enough to represent the observations (which are not only skewed, but peaked or "top heavy"). Indeed this becomes evident if one computes the fourth-order semi-invariant, which serves to characterize the peaked character, or the flatness, of the curve. For this very reason it becomes necessary to resort to a more refined graduation by means of a Gram series of the form:

$$c_0 \phi_0 (x) + c_3 \phi_3 (x) + c_4 \phi_4 (x) \ldots\ldots\ldots\ldots\ldots (66)$$

in which, $\phi_0 (x)$ is the generating function as computed in Equation (65) and

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[37] "The Mathematical Theory of Probabilities", by Arne Fisher, p. 258 ($m + \log 3 = 2.645$).

$\phi_3(x)$ and $\phi_4(x)$, its third and fourth-order derivatives. This further graduation may be carried out by least squares,[37] and results in the distribution shown in Table 15.

TABLE 15.—FREQUENCY DISTRIBUTION BY GRAM SERIES

Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)	Ages (inclusive) (1)	Computed frequencies (2)
13 to 15	0	28 to 30	3 707	43 to 45	519	58 to 60	88	73 to 75	21
16 to 18	72	31 to 33	2 899	46 to 48	342	61 to 63	66	76 to 78	16
19 to 21	696	34 to 36	1 996	49 to 51	233	64 to 66	50	79 to 82	13
22 to 24	2 369	37 to 39	1 284	52 to 54	163	67 to 69	38	>83	24
25 to 27	3 714	40 to 42	816	55 to 57	118	70 to 72	29	....	....

Fig. 5, which demonstrates the superiority of the Gram series, affords a comparison between: (1) The original observations; (2) the computed frequencies based on a single generator, Equation (65); and, (3) the frequencies as computed from a Gram series. An application of the so-called test of "goodness of fit," as devised by Pearson and R. A. Fisher also shows that a single generating function does not suffice to describe the observations. The test for "goodness of fit" (which might better be termed a test for "poorness of fit") gives altogether too much weight, however, to the infrequent and rare observations toward the ends of the curve. A superior, although rather lengthy, test is offered through the so-called "error critique" by Thiele. This "error critique," which utilizes an ingenious system of weighting, also shows the superiority of the Gram series.

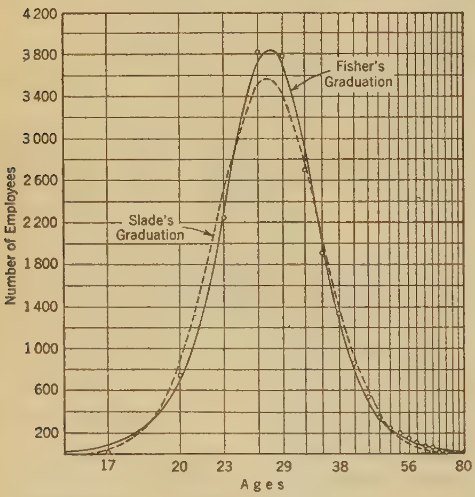

FIG. 5

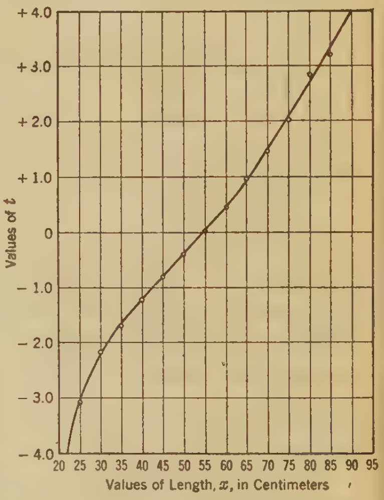

FIG. 6

In reference to "Example *c*," giving forty-nine individual observations on rainfall, the writer recommends the following simple procedure in preference to the tedious method proposed by Professor Slade. If the Naperian logarithms of the forty-nine individual measures of rainfall are arranged in intervals of 0.1, the resulting frequency values are as shown in Table 16.

TABLE 16.—FREQUENCIES OF LOGARITHMS OF INCHES OF YEARLY RAINFALL

Logarithm of rainfall, z (1)	$F(z)$ (2)	Logarithm of rainfall, z (1)	$F(z)$ (2)	Logarithm of rainfall, z (1)	$F(z)$ (2)
3.4 to 3.5	2	3.8 to 3.9	11	4.1 to 4.2	2
3.5 to 3.6	0	3.9 to 4.0	14	4.2 to 4.3	1
3.6 to 3.7	7	4.0 to 4.1	4	> 4.3	0
3.7 to 3.8	8	....	..	....	...

An elementary and rapid calculation yields the value, $\lambda_1(z) = M = 3.852$, for the mean, and $\sqrt{\lambda_2} = \sigma = 0.164$ as the value for the dispersion (or standard deviation). The simple normal exponential frequency curve, therefore, takes on the form of Equation (59) thus,

$$\phi_0(x) = \frac{49}{0.164\sqrt{2\pi}}\,e^{-0.5t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (67)$$

in which, $t = \left[\dfrac{\log x - 3.852}{0.164}\right]$. The resulting values, comparable to those in Table 7, are listed in Table 17.

TABLE 17.—COMPUTATION OF FREQUENCIES TO COMPARE WITH TABLE 7

x (1)	$\log x$ (2)	(2) − 3.852 (3)	(3) : 0.164 = t (4)	$I(t)$* (5)	49(5) (6)
30	3.401	—0.451	—2.750	0.00298	0.14
35	3.555	—0.297	—1.811	0.03507	1.72
40	3.689	—0.163	—0.988	0.16158	7.91
45	3.807	—0.045	—0.274	0.39204	19.20
50	3.912	0.060	0.366	0.64281	31.50
55	4.007	0.155	0.945	0.82766	40.55
60	4.093	0.241	1.470	0.92922	45.53
65	4.174	0.321	1.957	0.97482	47.76
80	4.382	0.530	3.232	0.99938	48.97

* Taken from "Table of Probability Integral," in "Statistical Methods," by C. B. Davenport, Table IV, p. 119, John Wiley & Sons, New York, 1904.

The fact that the logarithms of the measures of rainfall have been regarded as the observed quantities leads immediately to the Laplacean probability function, or normal error curve, without going through the more complicated calculations of the logarithmic transformation. This fact, however, leads one to question the undue stress that Professor Slade has put on the existence of upper and lower limits, situated within a finite range, instead of extending from minus infinity to plus infinity in the normal exponential error law. If one were to insist on definite upper and lower limits of the direct measures of rainfalls, why should one not also insist on definite limits, or a finite range of variation, for the logarithms of the measures?

The writer has referred briefly to Thiele's "pseudo-normal" variates, and has mentioned, as a special case, the transformation,

$$t = h\,(x) = \left[\,\log\left(\frac{x - x_0}{x_n - x}\right) - m\,\right] : n \;\ldots\ldots\ldots\ldots\;(68)$$

in which, x_0 and x_n represent the upper and lower limits of the observations. This transformation can be shown to be identical to that proposed by Professor Slade in his "Most General Function," Equation (45*a*). To be sure the latter contains apparently five constants as against four in Equation (68), but the fifth constant, corresponding to p in Equation (45*a*), is contained in the value of m. Moreover, in Professor Slade's Equation (45*d*) it appears that the constant, p, is expressed in terms of the other constants, so that after all there are only four constants.

TABLE 18.—OBSERVATIONS ON THE LENGTHS OF OAT STALKS, IN CENTIMETERS

Values of x, in centimeters (1)	$F(x)$ (2)	(x) (3)	$\Sigma F(x)$ (4)	$\Sigma \phi(x)$ (5)	Anti-probability of $I(t)$ (6)	log (x) (7)	log $(95-x)$ (8)
< 25..........	1	25	1	0.001	—3.090	1.398	1.845
25 to 30..........	12	30	13	0.013	—2.227	1.477	1.813
30 to 35..........	32	35	45	0.045	—1.696	1.544	1.778
35 to 40..........	62	40	107	0.106	—1.248	1.602	1.740
40 to 45..........	106	45	213	0.211	—0.803	1.653	1.699
45 to 50..........	138	50	351	0.348	—0.391	1.699	1.653
50 to 55..........	159	55	510	0.506	0.015	1.740	1.602
55 to 60..........	182	60	692	0.687	0.487	1.778	1.544
60 to 65..........	155	65	847	0.840	0.994	1.813	1.477
65 to 70..........	92	70	939	0.932	1.491	1.845	1.398
70 to 75..........	49	75	988	0.980	2.054	1.875	1.301
75 to 80..........	18	80	1 006	0.998	2.879	1.903	1.176
80 to 85..........	1	85	1 007	0.999	3.090	1.929	1.000
> 85..........	1	∞	1 008	1.000	∞	1.954	0.699

Value of x, in centimeters (1)	Column (7) minus Column (8) (9)	Column (9) minus 0.129 (10)	$\frac{\text{Column (10)}}{0.214} = t$ (11)	$I(t)$ (12)	$\Delta I(t)$ (13)	1 008 times Column (13) (14)	Observed $F(x)$ (15)
< 25.....	—0.477	—0.576	—2.692	0.0036	0.0036	3.6	1
25 to 30.....	—0.336	—0.465	—2.173	0.0149	0.0113	11.4	12
30 to 35.....	—0.234	—0.363	—1.696	0.0449	0.0300	30.2	32
35 to 40.....	—0.138	—0.267	—1.248	0.1060	0.0611	61.6	62
40 to 45.....	—0.046	—0.175	—0.818	0.2067	0.1007	101.5	106
45 to 50.....	0.046	—0.083	—0.388	0.3490	0.1423	143.4	138
50 to 55.....	0.138	0.009	0.042	0.5168	0.1678	169.1	159
55 to 60.....	0.234	0.105	0.491	0.6883	0.1715	172.9	182
60 to 65.....	0.336	0.207	0.967	0.8332	0.1449	146.1	155
65 to 70.....	0.447	0.318	1.486	0.9314	0.0982	99.0	92
70 to 75.....	0.574	0.445	2.079	0.9812	0.0498	50.2	49
75 to 80.....	0.727	0.598	2.794	0.9974	0.0162	16.3	18
80 to 85.....	0.929	0.800	3.738	0.9999	0.0025	2.5	2
> 85.....	1.255	....	....	......	......	.....	...

"Example *d*," which Professor Slade has chosen to illustrate his "most general frequency function" is too fragmentary to be of much use. No observed statistical data are given, except in a rough graphical representation in Fig. 3 of the paper.

As an illustration of this function and also of the identically same type of transformation (Equation (68)) mentioned by Gram, the writer presents, in

Table 18, Columns (1) and (2), some observations[38] by Dr. Giltay, of Wageningen, Holland, on lengths, in centimeters, of 1 008 oat stalks.

Column (1) shows the length, in centimeters, of the 1 008 stalks; Column (2) gives the frequencies, $F(x)$, in the intervals shown in Column (1); Column (4) contains the accumulated absolute frequencies; Column (5) lists the accumulated relative frequencies; and in Column (6) are the values of t in the probability integral, $I(t)$, corresponding to x in $\sum_{-\infty}^{x}\phi(x)$. This value of t is found by entering the table of the probability integral inversely which explains the heading for Column (6) as the anti-probability of $I(t)$.

The points, t, corresponding to x are plotted as small circles in Fig. 6. If a linear relation, $t = h(x) = (x - M) : \sigma$, is assumed, $t = (x - 54) : 11.8$, approximately. However, this gives a poor fit, because a glance at the curve, $t = h(x)$, shows that it has a pronounced point of inflection in its central region and tapers downward more rapidly than the straight line; and, likewise, it rises more rapidly upward than the straight line. Consequently, the normal error curve that implies a linear relation does not apply to the observations as they stand. The geometrical form of $t = h(x)$, moreover, leads to the assumption of finite upper and lower limits, or to postulate that,

$$t = h(x) = \left[\log\left(\frac{x - x_0}{x_n - x}\right) - m\right] : n \;\ldots\ldots\ldots\ldots(69)$$

The next step, therefore, is to estimate the numerical values of these upper and lower limits, or to find an approximation of x_0 and x_n. It is possible to do so by several algebraic approximations; but this is a tedious process, and it is preferable to use graphical methods of which a great variety is available. A very rapid process has been suggested by the Netherlands mathematician, M. J. Van Uven.[38] An application of this process leads to the choice of x_0 as 0 and x_n as 95, so that,

$$t = h(x) = \left[\log\left(\frac{x}{95 - x}\right) - m\right] : n = k\log\left(\frac{x}{95 - x}\right) + c \ldots(70)$$

The calculation of the logarithms of the fractions, $x : (95 - x)$, is shown in Columns (7), (8), and (9), of Table 18.

The next step is now to find the values of the constants, k and c (or of m and n), in Equation (70). This is most easily done by the method of least squares. If the interval from $x = 35$ to $x = 70$ (in which the majority of the observations fall) is chosen, the anti-probability of $I(t)$ as shown in Column (6), Table 18, may be considered as an observed value of t that is to be expressed by the linear relation,

$$t = k\log\left(\frac{x}{95 - x}\right) + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots(71)$$

The coefficients of k, that is,

$$\log\,[x : (95 - x)] = a_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(72)$$

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[38] Koninklijke Akademie Van Wetenschappen te Amsterdam, Vol. 19, No. 4, pp. 533-545, from which the present example relating to oat stalks is taken.

are given in Column (9), Table 18, and 1.000 may be taken as a provisional or preliminary value of c. Hence, the following schemata of observation equations are obtained:

a_x	c	$o(x)$	a_x	c	$o(x)$
−0.234	1.000	−1.696	0.138	1.000	0.015
−0.138	1.000	−1.248	0.234	1.000	0.487
−0.046	1.000	−0.803	0.336	1.000	0.994
0.046	1.000	−0.391	0.447	1.000	1.491

From these observation equations one finds (in the notation of Gauss) that: $[aa] = \Sigma a^2 = 0.465$; $[ac] = \Sigma ac = 0.783$; $[ao] = \Sigma ao = 1.704$; $[cc] = \Sigma c^2 = 8.000$; and, $[co] = \Sigma co = -1.151$. Consequently, $0.465\,k - 0.783c = 1.704$; and, $0.783\,k + 8.000c = -1.151$; or $k = 4.6786$ and $c = -0.6018$. Therefore, $n = (4.6786)^{-1} = 0.214$, and $m = 0.6018 : 4.6786 = 0.129$; or t reduces to:

$$t = \left[\log\left(\frac{x}{95 - x}\right) - 0.129\right] : 0.214 \quad \ldots\ldots\ldots\ldots (73)$$

The calculation of the graduated curve proceeds now (see Columns (10) to (15)) as a simple continuation of Columns (1) to (9) of Table 18. The detailed calculations are self-explanatory, and the final column (Column (14)) which gives the graduated curve, shows a reasonable agreement with the observed and ungraduated data in Column (2), or Column (15). (See Fig. 7.) The same curve type has been discussed by Van Uven, and it is of interest to compare his form with Equation (71). Van Uven writes t as,

$$t = \lambda \log\left[\left(\frac{x - x_0}{x_n - x} : \frac{x_m - x_0}{x_n - x_m}\right)\right] \ldots\ldots\ldots\ldots\ldots (74)$$

in which, x_m, a fifth constant, analogous to the constant, p, in Professor Slade's Equation (45a), equals the value of the median of the observations. It is evident that the quantity, $\log\left(\frac{x_m - x_0}{x_n - x_m}\right)$, must fall very close to the value of m in Equation (69), because of the fact that t is assumed to be normally distributed, which implies that the mean value, m, coincides with the value of t that corresponds to the median, x_m, of the observations. The median of the observed values of $F(x)$ falls close to 54.5 cm, so that $\log\left(\frac{x_m - x_0}{x_n - x_m}\right) = \log\left(\frac{54.5}{40.5}\right) = 0.1289$, which checks with the previously calculated value of $m = 0.129$.

The restriction of Van Uven to let the mean in the transformed curve equal the particular value of t that corresponds to the median of the observations is analogous, although not equivalent, to the introduction of the author's constant, p, to make the second moment of the final curve agree with that of the statistics, which at its best is nothing more than a kind of artificial *tour de force.*

In the opening paragraphs of his "Synopsis" Professor Slade refers to "a justification for the introduction of a new function in a field that literally teems with such studies." Enough has probably been said in the preceding paragraphs to demonstrate that there is little, if any, newness in the formulas presented; they are, to all practical purposes, identical with those known from the old methods of logarithmic transformations of the variate.

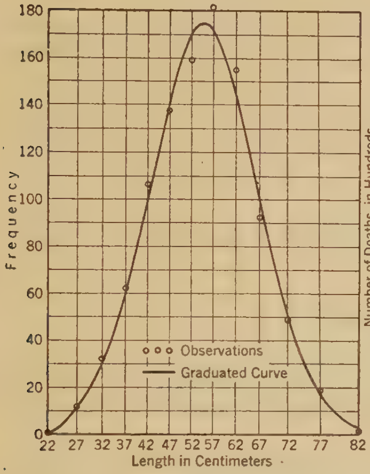

Fig. 7

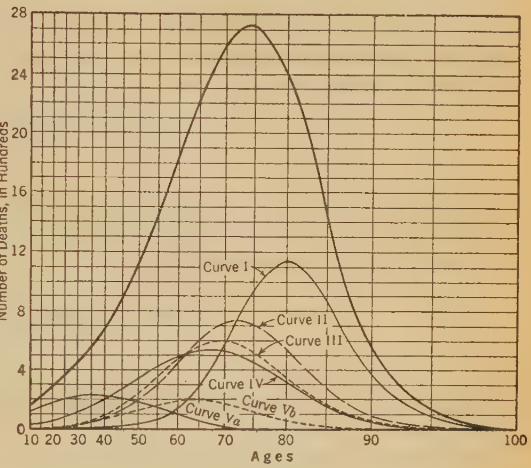

Fig. 8

All such methods go back to Thiele, who always took a critical attitude to the Gaussian law of error with its implied postulate of the arithmetic mean as the most probable value. As early as 1866, Thiele introduced the alternative postulate of the geometric mean as the most probable value, thus antedating McAllister, Fechner, Pearson, and other writers by many years.[39] Thiele's investigations led him to the obvious fact that if the geometric mean of the observations yields the most probable value, the arithmetic mean of the logarithms of the observations must yield the most probable value of the logarithms of the observations, so that if the logarithms of the observations are assumed to be normally distributed, a law of error can be found for the observations themseleves by a logarithmic transformation of the variate. Such a law must necessarily lead to the geometric mean of the observations as the most probable value.

In the text following Equation (7) Professor Slade states "in Thiele's solution, $\phi(x)$ is given as a definite integral which, unfortunately, cannot be integrated; so that the net result is an elegant mathematical process which ultimately gets nowhere."

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[39] "The Mathematical Theory of Probabilities", by Arne Fisher, p. 288.

The solution to which the author refers is the one in which the frequency function is expressed as a definite integral equation of the form:

$$\phi\,(x) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{h(\omega i)}\, e^{-x\omega i}\, d\,\omega \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(75)$$

in which,

$$h\,(\omega i) = \frac{\lambda_1}{1\,!}\, i\,\omega + \frac{\lambda_2}{2\,!}(i\,\omega)^2 + \ldots + \frac{\lambda_n}{n\,!}\,(i\,\omega)^n$$

If we let the complex quantity, $\omega\, i = \omega\sqrt{-1}$, take the special form, ω, and also let all the semi-invariants be equal, or $\lambda_1 = \lambda_2 = \lambda_3 = \ldots \lambda_n = m$; then the solution for $\phi(x)$ is:

$$P\,(x) = \frac{e^{-m}\, m^x}{x\,!} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(76)$$

which is the well-known error law of Poisson.[40]

Also, if all the semi-invariants of higher order than the second are found to vanish, the solution becomes,

$$\phi_0(x) = (2\pi\,\lambda_2)^{-0.5}\, e^{-h(x)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(77)$$

with $h(x) = (x - \lambda_1)^2 : 2\lambda_2$, or the normal error curve of Gauss.[41] In fact, one of the properties of the normal curve is that all its semi-invariants above the second order equal zero.

Thus, one finds that, not only the normal error law, but also Poisson's celebrated exponential, are contained as special cases in Thiele's general functions.

By resorting to an expansion in Gram's series and letting,

$$\phi(x) = c_0\,\phi_0(x) + c_1\,\phi_1(x) + c_2\,\phi_2(x) + \ldots c_n\,\phi_n(x) \ldots\ldots(78)$$[42]

one can also solve Thiele's integral equation and use the solution for the numerical determination of the constants, c, in the Gram expansion, where $\phi_0(x)$ is the normal or Laplacean error curve in Equation (77).

Of course, as Professor Slade points out, Thiele's general formula may contain other functions than frequency functions. To the writer this long and well-known fact actually enhances rather than detracts from the value of Thiele's work, for the more general a formula is, the greater is its applicability in mathematical analysis. Professor Slade, on the other hand, uses this circumstance as a criticism by stating that Thiele's solution is "too general" for what he somewhat ambiguously terms "ordinary purposes." Such an argument can scarcely be considered as mathematically sound. If accepted, it would imply, among other things, that a Fourier series should be considered as inferior to a Taylor power series, although the Fourier series is able to express all functions that can be expanded by Taylor's formula and many other functions as well.

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[40] "The Mathematical Theory of Probabilities," by Arne Fisher, p. 265.

[41] *Loc. cit.*, p. 198.

[42] *Loc. cit.*, pp. 209–210.

The same argument of being "too general" is again used by Professor Slade in his criticism of Gram's series. He states also that "if $\phi(x)$ deviates appreciably from the normal curve, $\phi_0(x)$ (or is markedly skew), it may take thousands of terms to get even a fair representation." The writer doubts this statement. Of course, if one were foolish enough to select a generating function that did not even remotely resemble the rough observations, one might drop such an attitude of skepticism; but, as emphasized both by Thiele and Gram, the choice of the generating function is to be guided by the observations themselves. For instance, one should not always let the mean and the dispersion in the generator be equal to the mean and the dispersion of the observations. In greatly skewed or one-sided distributions, such as the classical example of *Ranunculus Bulbosus* by deVries,[43] the variate in the normal generator should not be taken as:

$$t = (x - 0.631) : 0.958 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

but as,

$$z = [\log (x - 4) - 0.044]\ 0.544 \ldots\ldots\ldots\ldots\ldots\ldots (80)$$

and Gram's series should be written as:

$$F(x) = c_0\, \phi_0(z) + c_1\, \phi_1(z) + c_2\, \phi_2(z) + \ldots\ldots\ldots\ldots (81)$$

Furthermore, the objection toward the possible occurrence of negative frequencies mentioned by Professor Slade can easily be overcome if, like Thiele and Charlier, one were to write the frequency function as,

$$F(x) = e^{\Sigma a_i (x)^i} = e^{\Sigma c_i H_i(x)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (82)$$

(for $i = 0, 1, 2, 3, \ldots n$), in which, $H_i(x)$ represents the Hermite functions of the order, i. Taking the logarithms on both sides of Equation (82):

$$c_0 + c_1\, H_1(x) + c_2\, H_2(x) + \ldots c_n\, H_n(x) = \log F(x) \ldots\ldots\ldots (83)$$

Multiplying both sides of Equation (83) with the ith derivative of the generator, $\phi_0(x)$, and integrating over the entire range:

$$c_i \int_{-\infty}^{\infty} H_i\,(x)\ \phi_i\,(x)\ dx = c_i\,(-1)^i\, i! = \int_{-\infty}^{\infty} \log F\,(x)\ \phi_i\,(x)\ dx$$

or,

$$c_i = \frac{(-1)^i}{i\,!} \int_{-\infty}^{\infty} \log F\,(x)\ \phi_i\,(x)\ dx \ldots\ldots\ldots\ldots\ldots\ldots (84)$$

for $i = 0, 1, 2, 3, \ldots n$.[44]

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[43] "The Mathematical Theory of Probabilities", by Arne Fisher, p. 226.

[44] *Loc cit.*, pp. 201–202.

The determination of the constants in the series through this method will never result in negative frequencies. In fact, the method outlined herein is one that can be highly recommended for curves of rainfall, especially since excellent tables of the Hermite functions are now available, progressing by intervals of 0.01.

Article 6, Section II, of Professor Slade's paper is devoted to a discussion to prove the self-evident fact that as t approaches 0, or as the skewness vanishes, Equation (11) will converge toward the normal Laplacean curve as a limiting form. In order to show this, the author goes through a lengthy mathematical analysis of limits involving certain results originally presented by the French mathematician, L'Hospital. Of course, such complicated networks of formulas for limits become wholly superfluous under the methods advocated by Thiele because, if the skewness, or (what is the same thing) the third (and higher) order of semi-invariants vanish, an obvious corollary is derived from Thiele's theory that the resulting curve always is of Laplacean normal form.

A most interesting point that Professor Slade has very briefly touched upon is that of compound frequency curves (see "II—The Partly Bounded Function," Requirement (*a*)), a somewhat difficult problem from the standpoint of mathematics, but at the same time the one that is probably the most important in actual practice. Every frequency distribution observed in Nature is really a compound frequency curve, composed of several subsidiary error curves, either of normal or non-normal types.

In this connection, it is of inteest to point to the remark which the French mathematician, Poincare, made to his German colleague, Lindemann; "The mathematicians," Poincare said, "imagine that the law of errors is a fact of observation, and the observers fancy that it is a theorem demonstrated by mathematics." Neither one of these beliefs is true. As a matter of fact, the Russian mathematician, Tschuprov, in discussing whether or not it is possible to demonstrate by means of empirical methods the existence or non-existence of the exponential law of error, reached the conclusion that none of the numerous (and at times vast) collections, derived from precision and astronomical measurements, offers proof as to the actual existence of the postulated Gaussian error law. Tschuprov himself leaned toward the opinion that the observed statistical samples, especially the larger samples, tended to discredit the actual existence in Nature of the celebrated law of error such as had been derived from purely mathematical speculations.

The Danish biologist, W. Johannsen, furthermore demonstrated by means of experimental research work in plant physiology that many of the biological variation curve types, discussed by Galton and Karl Pearson as normal error curves or skew curves, in reality were compound frequency curves, composed of two or more mathematical frequency curves. This important fact led Johannsen to the introduction in biological research of his celebrated "pure lines."

In other places the writer has made use of this idea of compound frequency distributions and "pure lines" in the construction of mortality tables from statistics, giving the deaths by sex, attained ages at death, and cause of death, but without knowing the therewith correlated number of living persons exposed to the risk of death.[45] In a sense this purely synthetic method constitutes a solution to the problem of determining the probable values of a series of proper fractions, where only the numerators are known, but where every one of the denominators are missing. Of course, it is impossible to solve this problem by mere mathematical or quantitative methods. The only approach toward a solution seems to be by means of partly quantitative and partly qualitative methods in the sense that the numerators, which show the observed statistics on death, do not only possess the quantitative attributes of attained ages at death, but also the qualitative attributes of the causes of death. These two elements in combination with certain biological hypotheses make at least an approximate solution possible by means of mathematical synthesis.

In Fig. 8 the result of such a construction is shown in relation to the mortality among Australian males for the period 1907–1915. The compound curve represents the well-known d_x column, or "curve of death," as found in all first-class mortality tables; the subsidiary causal or biological components give the death curves from specified groups of causes of death.

As pointed out by Dr. Raymond Pearl, the synthesis as proposed by the writer moves along entirely different lines than the earlier frequency curve study by Karl Pearson in his interesting volume on "The Chances of Death," for which Dr. Pearl states that "actually there is no slightest reason to suppose that it [Pearson's method] represents any *biological* reality.[46]

Moreover, the synthesis is of so general a nature that it may equally well be applied to engineering problems. Mr. Bassett Jones,[47] for example, has made use of it in the design and construction of high-speed elevators, and it seems to be equally well adapted for the study of rainfall by taking into consideration the therewith related geographical, climatic, and geological elements.

A question which is of the utmost importance to engineers and observers, who usually deal with limited samples, is that of the mean (or probable) errors of the constants or parameters. Earlier formulas for probable errors of the constants in certain types of frequency curves derived by graphical methods[48] are defective because their derivation rests on the implied assumption that the constants are mutually free or uncorrelated. This implication is faulty. The constants in such curves are correlated in the sense that they all depend on the lower (or upper) limits of the variate; in fact, the constants are mathematical functions of the limits, which in the graphical process are chosen subjectively or by the mere aid of the eye, as it were. The least bit of change in such limits will influence the other constants.

The obstacle of correlation between the constants exists also in the logarithmic transformation with a lower limit (or mathematical zero) in the generator as defined by Equation (57). The writer as well as Mr. Sture

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[45] "Frequency Curves," by Arne Fisher, Macmillan Co., N. Y., 1922, and also article in 1925 *Skandinavisk Aktuarietidskrift.*

[46] "The Biology of Death", by Raymond Pearl, Philadelphia, Pa., 1922, p. 101.

[47] "Power Calculations for Elevators by the Method of Probabilities", by Bassett Jones, *General Electric Review,* 1930, pp. 545–555 and 650–653.

[48] See, for example, *Transactions,* Am. Soc. C. E., Vol. 91 (1927), p. 1.

Nydell have succeeded, however, in overcoming this obstacle and have been able to give the following approximate formulas for the mean (or standard) errors of u (or a), m, and n:

$$E_m\ (u) = (1 + t)\ t^{-1}\ [(8 + 106t + 332t^2 + 479t^3) : 12s]^{0.5}$$

$$E_m\ (m) = [(4t^{-1} + 54 + 306t + 895t^2 + 1\,598t^3 + 1\,753t^4) : 6s]^{0.5}$$

$$E_m\ (n) = n^{-1}\ [t\ (8 + 106t + 332t^2 + 479t^3) : 12s]^{0.5}$$

in which E = the mean error; s, the number of observations (or size of sample); and $t = \lambda_2 : u^2$.

In Professor Slade's "Example *a*," one finds $\lambda_2 = 1.442$; $u = 1.948$; $t = 0.3800$; $m = 0.1823$; $n = 0.5680$; and $s = 251$, so that $E_m\ (u) = 0.7324$; $E_m\ (m) = 0.5367$; and $E_m\ (n) = 0.2192$, which shows that the value of the logarithmic mean, m, is decidedly uncertain because its mean error is approximately three times the value of m itself.

ARTHUR W. KEMPERT,[49] ESQ. (by letter).—This interesting paper raises several issues which it would be desirable to discuss somewhat in detail. In the first place, the paper bears, in parts, a close resemblance to the earlier works by Thiele, Gram, Jorgensen, and Fisher. Secondly, it raises the moot questions of small samples and chance errors in relation to various pseudographical methods. Last, but not least, there arises the important question as to whether or not the entire problem of the statistical analysis of rainfall, instead of being viewed as a homogeneous (or homograde) mass phenomenon, might not better be thought of as a heterogeneous object and analyzed by the aid of compound frequency curves.

A point very properly emphasized by Professor Slade is that "the use of probability paper and other graphical devices is an undesirable practice," especially when drawn to logarithmic or semi-logarithmic scales. The practice of a graphical process, no matter how carefully and judiciously it may be performed, is not reliable as a rule when applied to a limited number of observations or to a small sample. Investigators who have occasion to use statistical methods should be extremely cautious, therefore, lest they be led astray by the broad claims for some of the graphical schemes as a general procedure for the analysis of small samples. A case in point is the paper by R. D. Goodrich, M. Am. Soc. C. E. (cited by the author) in which the assertion is made[50] that the graphical method proposed "is developed especially for the examination of records consisting of from about 10 to 50 observations."

As an illustration of the danger of applying graphical methods to small samples of observations, the following data relating to some hitherto unpublished investigations on yeast wine cultures may be of interest.

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[49] Statistical Asst. and Draftsman, Western Union Telegraph Co., New York, N. Y.

[50] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 42.

A series of 149 samples, each containing 50 individual observations, was taken from certain pure yeast cultures present in a blend of high-grade vintages. The statistical object in question was found to vary from an observed low value of 8 to a high value of about 27. Table 19 gives the frequency distribution in: (1) The 32d sample (chosen at random); (2) a similar distribution for the 87th sample (also chosen at random); and (3) the frequency distribution of all 149 samples combined. If the 32d sample is fitted graphically, choosing the lower limit[51] at 10, the following equation is obtained:

TABLE 19.—FREQUENCY DISTRIBUTION, VARIATIONS IN PURE YEAST CULTURES IN A BLEND OF HIGH-GRADE WINES

Intervals of the variate (1)	Sample No. 32 (2)	Sample No. 87 (3)	149 samples combined (4)	Intervals of the variate (1)	Sample No. 32 (2)	Sample No. 87 (3)	149 samples combined (4)
8 to 10..........	0	0	5	18 to 20.........	8	12	1 644
10 to 12..........	2	0	78	20 to 22.........	3	5	846
12 to 14..........	8	0	615	22 to 24.........	1	2	270
14 to 16..........	15	17	1 742	24 to 26.........	0	1	52
16 to 18..........	13	13	2 195	26 to 28.........	0	0	3
Total.........	..	..	.....	....	50	50	7 450

$$t:100 = 1 - (10)^{-0.0000224\,(R-10)^{2.355}} \qquad (85)$$

When fitted to a Gram series:

$$G(x) = 50\,[\phi_0\,(t) - 0.048\,\phi_3\,(t)] \qquad (86)$$

with $t = (x - 16.2) : 2.624$. The mean error of c_3 (or of -0.048) equals 0.057 and shows that no reliance can be placed on the empirically determined skewness in the small sample of 50. The mean error in the arithmetic mean exceeds 0.7 and the mean error in the dispersion (standard deviation) is greater than 0.3.

The situation is quite different in the case of the combined sample of 7 450 individual observations. This sample can be fitted to a Gram series of the following order (expressed in definite integral form):

$$G(x) = 7\,450 \left[\int_{-\infty}^{t} \phi_0\,(t)\,dt - 0.045\,\phi_2\,(t) \right] \qquad (87)$$

in which $\phi_0\,(t)$ is the generating function and $\phi_2\,(t)$ its second derivative, with $t = (x - 17.31) : 2.68$, which results in the frequency distribution given in Table 20, Columns (1) to (3).

The empirically determined skewness of -0.045 has a mean error of only 0.0047, about one-tenth of the magnitude of c_3, which is an indication of its reliability in contradistinction to the small sample of 50 in which the mean error of c_3 was greater than c_3 itself.

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[51] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 4, Equation (*A*).

A comparison between the relative frequencies of $\Delta G(x)$ and the corresponding relative frequencies for the small sample (No. 32) as fitted to the Goodrich formula,[51] is given in Columns (6) and (7), Table 20. The two curves are unlike and demonstrate the danger inherent in any attempt to evaluate a presumptive frequency curve from a small sample.

TABLE 20.—FREQUENCY DISTRIBUTIONS FOR A COMBINED SAMPLE OF 7 450 OBSERVATIONS

Values of x (1)	$G(x)$ (2)	$\Delta G(x)$ (3)	Observed data (4)	Intervals (5)	Gram (6)	Goodrich (7)	Type I (8)	Type II (9)	Type III (10)	Compound (11)	Observed (12)
<10	12	12	5	<10 to 10	0.001	0.000	3	0	0	3	5
12	107	95	78	10 to 12	0.013	0 058	76	2	0	78	78
14	764	657	615	12 to 14	0.088	0.205	569	45	2	616	615
16	2 451	1 687	1 742	14 to 16	0.226	0.285	1 388	334	27	1 749	1 742
18	4 648	2 197	2 195	16 to 18	0.295	0.243	1 110	911	171	2 192	2 195
20	6 278	1 630	1 644	18 to 20	0.219	0.138	291	911	447	1 649	1 644
22	7 083	805	846	20 to 22	0.108	0.054	24	334	489	847	846
24	7 363	280	270	22 to 24	0.038	0.014	1	45	223	269	270
26	7 437	74	52	24 to 26	0.010	0.003	0	2	42	44	52
28	7 450	13	3	26 to 28	0.002	0.000	0	0	3	3	3
Totals.	.....	7 450	7 450	....	1.000	1.000	3 462	2 584	1 404	7 450	7 450

In view of the close correspondence between the 7 450 observations and the Gram frequency curve, many investigators are likely to conclude from purely empirical considerations that the material in question is homogeneous; such an inference, however, would be unwarranted. With the aid of experimental laboratory analysis, based on the concept of Johannsen's "pure lines" and his own theory of probability synthesis, Mr. Arne Fisher[52] has demonstrated that the observed mass phenomenon, instead of being of a single type, actually is composed of three distinctive types, namely, a Bordeaux vintage (Capestang), a Burgundy vintage (Beaune type), and a vintage from the Crimean Peninsula, characterized by the statistical parameters given in Table 21. This leads to a compound frequency curve, such as shown in Table 20, Columns (8) to (12).

TABLE 21.—COMPOUND DISTRIBUTION OF THREE TYPES

Type	Name	Area	Mean	Dispersion	Skewness
I....................	Capestang........	3 462	15.6	1.800	None
II....................	Beaune...........	2 584	18.0	1.910	None
III..................	Crimea...........	1 404	20.2	2.060	None
Total compound...........	....	7 450	17.306	2.664	—0.045

A few remarks seem necessary in regard to the difficult problem of the calculation of mean (or probable) errors of the frequency constants. In the case of the regular Gram series this problem offers no serious difficulty because of the orthogonal properties of the individual terms of the series which, in the terminology of Thiele, makes the constants, *c*, mutually free (independent or uncorrelated) of each other.

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[52] "Frequency Curves," by Arne Fisher, New York, Macmillan Co., 1922.

In the case of the logarithmically transformed frequency functions as described by Professor Slade, and also in the case of a graphical process, there arises, however, the formidable obstacle that the constants are not mutually free, but are bound or correlated, in the sense that their values will depend upon the lower (or upper) limits of the variate. Any change whatever in the positions of these limits will influence the other constants—the *c* and the *d*, in Professor Slade's formula, or in other formulas that depend on lower (or upper) limits.

Because of the fundamental fact that the constants are bound or correlated, it would seem, therefore, that prior attempts[53] to evaluate the mean (or probable) errors largely must be looked upon as failures. The fundamental implications in such proofs are that the constants are not correlated. Proponents of graphical methods admit that "it does not seem possible to derive any formula for the probable error of the index, *c*,"[54] and, in another place, in discussing the errors in the upper and lower limits, it is also stated that "it does not seem possible to derive any formulas applicable to them."[55] Admissions such as those just quoted naturally lead one to regard many graphical methods in a light of skepticism, as it were.

As one of his requirements, Professor Slade states that "the curve must be specifiable completely by moments no higher than the third," and for higher moments "it becomes necessary to search for physical substitutes." This sound advice seems to be in complete accord with the following rules by Thiele[56]: (1) For first and second semi-invariants (or moments) rely exclusively on the observations; (2) for semi-invariants of higher order than the sixth rely exclusively on theory; and (3) for intermediate semi-invariants (or moments) rely partly upon theory and partly upon the observations.

In conclusion, it may be said that some slight inconsistencies have occurred in the author's treatment of his subject. In one place he states that graphical methods "convey to the eye a simplicity which does not really exist," yet under Example *b*, in comparing his own results with those of Fisher, the author employs a rather crude graphical method of comparison. As matters turn out, Professor Slade's graduated curve in Fig. 1 does not provide nearly as good a fit as Fisher's graduation when subjected to Pearson's test of "goodness of fit."

J. J. SLADE, JR.,[57] ESQ. (by letter).—In developing the function presented in his paper the writer has necessarily had to give considerable weight to the partly bounded function because it was by means of his analysis of it that he was able to construct the elements of the totally bounded curve. In his work, however, he has not had occasion to make use of it. This former function is merely a limiting case of the latter, in which one of the bounds becomes infinite, and in the problems that he has considered the bounds, obviously, have been finite. Perhaps one of the reasons that much of the work of statisticians has been unavailable to the engineer for hydrological investigations is that statisticians are interested usually in graduating the middle range of a frequency distribution, the extreme outer 1%, say, of the distribution being of relatively slight importance to them. The hydrologist, on the other hand, is frequently interested mostly in this outer 1% of the range. It is for this reason that one must insist on the determination of the limits of the range. It is absurd for a probability function to give a finite probability to a Noah's flood. In modern times, there simply is not enough energy resident in the atmosphere to produce anything like it.

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[53] As, for example, *Transactions*, Am. Soc. C. E., Vol. 91 (1927), pp. 97–102.

[54] *Loc. cit.*, p. 99.

[55] *Loc. cit.*, p. 100.

[56] "The Mathematical Theory of Probabilities," by Arne Fisher, p. 217.

[57] Asst. Prof. of Eng. Mechanics, Rutgers Univ., New Brunswick, N. J.

Equations (45), as given in the writer's paper, are based on an easy generalization of Equations (44). It is apparent, at once, that the factor, p, belongs therein and that it is not merely "a kind of artificial *tour de force,*" as Mr. Fisher expresses it. In the paragraph following these equations, the writer states that their form will probably need modification. A more extended analysis has shown that this is indeed the case. Their correct form, when $b < g$, is:

$$z = p\,c \log d \left(\frac{b + x}{g - x}\right) \tag{88a}$$

$$d = \frac{g}{b} \sqrt{\frac{g^2 (\sigma^2 + b^2) - 4\,\sigma^2\,b\,g}{b^2 (\sigma^2 + g^2) - 4\,\sigma^2\,b\,g}} \tag{88b}$$

$$\frac{1}{c} = \sqrt{\log \frac{g^2 (\sigma^2 + b^2) - 4\,\sigma^2\,b\,g}{b^2 (\sigma^2 + g^2) - 4\,\sigma^2\,b\,g}} \tag{88c}$$

and,

$$p = \sqrt{\frac{g - b}{g + b}} \tag{88d}$$

When $b = g$, then,

$$z = \frac{\sqrt{b^2 - 3\,\sigma^2}}{2\,\sigma} \log \left(\frac{b + x}{b - x}\right) \tag{88e}$$

In his discussion Mr. Williams states that "any purely mathematical treatment of stream-flow data presumes at the outset that all the data are equally reliable," and, from this, he infers that there are limitations in the analytical method which are partly absent in the graphical treatment. The analysis of a physical problem begins when all the data relevant to the problem are known, and these data necessarily include a knowledge of the reliability of the observations. The purely mathematical treatment begins, first of all, with the proper weighting of these observations, for which various methods are well known and widely used. The writer can not agree, therefore, with the idea that the analytical method is inferior to the graphical because the free-hand curve "can be shifted to give more or less weight to certain points."

Mr. Williams' objection to listing the complete set of peaks in a stream-flow record is well taken. In the first place, it is difficult to define a peak, particularly in the range of small flows, as he observes; second, it is difficult to estimate above what datum the peak should be taken (that is, statistically, a peak is an accidental irregularity above a smoothed hydrograph, and it is difficult to smooth the hydrograph properly); and, third, peaks are not uncorrelated quantities and, consequently, a simple analysis of their distribution is not possible. In presenting this paper the writer limited himself to the analysis of the generalized probability function, and he did not attempt to offer a method for the analysis of stream-flow characteristics. The illustration of the peaks of the Tennessee River record was given merely to show that an extremely skewed and platykurtic distribution could be closely fitted by means of this curve. Using the corrected Equations (88) and an estimate of the extreme flood (415 000 cu ft per sec) based on the following method, a much closer fit is obtained.

The use of the partial series of peaks above the "one-year flood" is open, at least statistically, to a fourth objection, namely, that an error, which may be serious, is introduced in assigning a probability to the "one-year flood." Because of the indeterminacy of this error the writer has not been able to make an estimate of the reliability of the probabilities obtained from a curve fitted to such series.

The writer concurs with Mr. Foster's opinion that Pearson's curves satisfy all the desiderata presented in his paper for a generalized frequency curve, provided all the types in Pearson's development are included. Except in the border cases one type cannot be substituted for another, and in order to select the proper type for a given distribution it is necessary to use Pearson's criterion or something equivalent (implying the computation of moments through the fourth). The statistical characteristics beyond the standard deviation computed from small samples (say, samples with less than 100 items) are so unreliable that they are useless. This statement needs no mathematical proof (although such could be readily given); the experiment may be tried of drawing small samples from a fairly large population, and, from these samples, the mean, standard deviation, and the skewness may be computed. It will then be found that, although the first two characteristics will vary, they will show a remarkable uniformity, nevertheless, compared with the wide fluctuations in the latter.

Table 22 meets the objections to the labor involved in graduating distributions by means of this function. If greater detail is needed than is given in this table, then, by perfectly straightforward computation and the use of tables of the probability integral, Equations (88) may be utilized to fill in this detail. Tables of the probability integral have been published frequently and distributed widely; Mr. Foster's statement that these tables are "not always available to the practicing engineer" and that "even if available, might easily lead the average engineer into difficulties" is, therefore, quite disturbing. The answer to this objection is probably that the "average engineer" should not attempt, with his small equipment, to analyze a highly specialized and difficult problem. Undoubtedly, it is true that some kind of an answer may always be found easily to any statistical problem, but it is in the interest of "practical purposes" that this answer should be questioned and its reliability tested. It is unfortunate that statistical analysis does not offer many short cuts, particularly since a wide variety of important problems yield to no other kind of analysis.

Both Professors Goodrich and Mavis disagree with the writer's statements regarding the undesirability of graphical methods of analysis. These statements the writer intends, of course, to apply to the graphical analysis of statistical problems specifically, because he is well aware of the value of graphical analysis in general. Without entering into elaborate arguments, his objections to the graphical method may be stated briefly, as follows: If an n-parameter curve is to be fitted to a set of data, then exactly n independent statements must be used to determine these parameters. If a statistical series is capable of furnishing four independent, significant characteristics (for instance, the mean, standard deviation, skewness, and kurtosis, or, as the writer prefers for small samples, the mean, standard deviation, and the two end points of the range of variation), then these statements are sufficient to determine reliably a four-parameter curve; more parameters may be determined by making use of moments higher than the fourth, or by the method of least squares; but if the significant characteristics of the distribution are only four, the added work is meaningless; that is, there is a definite point beyond which the flexibility of a curve is undesirable. Now, a free-hand curve is a many-parameter curve (theoretically the number of parameters is infinite), and there seems to be no way of making the order of its flexibility agree with the flexibility that the statistics can significantly stand.

Professor Goodrich's method undoubtedly gives closer fits in a great many cases than are obtained by the function the writer proposes (the partly bounded function is obviously not flexible enough to represent the variations in the samples given as illustrations), but this added flexibility is scarcely a desirable feature when dealing with inadequate samples.

In functional analysis (as distinct from statistical analysis) the probable errors of the constants of a curve fitted to a set of data may be computed by the methods developed from the theory of least squares because of the reasonable assumption (in most cases, at any rate) that the variations in the observations from the true functional relation follow a symmetrical law closely represented by the Gaussian normal. In fitting a curve to a statistical series, however, the errors arise in an entirely different manner; they are errors in the characteristics of the distribution function itself. These errors are seldom small quantities of the first order, and so one is scarcely justified in computing the probable errors of the constants of the curve from them, even though by means of them the difficulties of correlation emphasized by Messrs. Fisher and Kempert are obviated (since statistical characteristics are mutually independent). In 1902, Pearson gave[58] the probable errors of statistical characteristics. The writer uses these values, reducing the order

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[58] "On the Mathematical Theory of the Errors of Judgment," *Philosophical Transactions*, Royal Soc. (London), A, 198, 1902, pp. 235–299.

TABLE 22.—VALUES OF t FOR VARIOUS VALUES OF θ AND λ.

%-of-time	PROBABILITY FOR THE INDEX VALUE, λ =:													
	1	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
(a) $\theta = 1$														
0.0000001	......	......	2.00	2.50	2.96	3.37	3.73	4.01	4.28	4.68	4.96	5.18	5.32	5.45
0.000001	......	......	2.00	2.49	2.94	3.33	3.66	3.94	4.16	4.51	4.76	4.94	5.06	5.155
0.00001	......	......	2.00	2.485	2.92	3.28	3.58	3.82	4.02	4.31	4.525	4.66	4.76	4.845
0.0001	......	......	2.00	2.48	2.88	3.205	3.47	3.67	3.84	4.07	4.24	4.35	4.42	4.49
0.001	......	......	2.00	2.46	2.82	3.10	3.32	3.48	3.61	3.78	3.90	3.98	4.04	4.08
0.01	......	......	2.00	2.42	2.72	2.94	3.10	3.22	3.30	3.42	3.50	3.55	3.58	3.615
0.1	......	......	1.99	2.34	2.55	2.69	2.78	2.84	2.89	2.95	2.98	3.01	3.03	3.05
1.0	......	......	1.96	2.15	2.22	2.26	2.28	2.29	2.30	2.31	2.31	2.31	2.32	2.32
10	......	......	1.71	1.53	1.44	1.39	1.36	1.35	1.33	1.32	1.31	1.30	1.29	1.29
20	......	......	1.37	1.09	0.99	0.94	0.92	0.90	0.885	0.87	0.86	0.86	0.85	0.85
30	......	......	0.955	0.70	0.63	0.59	0.57	0.56	0.55	0.54	0.54	0.53	0.53	0.53
40	......	......	0.49	0.345	0.30	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.25	0.25
50	......	......	0	0	0	0	0	0	0	0	0	0	0	0
60	......	......	−0.49	−0.345	−0.30	−0.29	−0.28	−0.27	−0.27	−0.26	−0.26	−0.26	−0.25	−0.25
70	......	......	−0.955	−0.70	−0.63	−0.59	−0.57	−0.56	−0.55	−0.54	−0.54	−0.53	−0.53	−0.53
80	......	......	−1.37	−1.09	−0.99	−0.94	−0.92	−0.90	−0.885	−0.87	−0.86	−0.86	−0.85	−0.85
85	......	......	−1.56	−1.30	−1.20	−1.15	−1.12	−1.11	−1.09	−1.08	−1.07	−1.06	−1.06	−1.05
90	......	......	−1.71	−1.53	−1.44	−1.39	−1.36	−1.35	−1.33	−1.32	−1.31	−1.30	−1.29	−1.29
95	......	......	−1.86	−1.80	−1.75	−1.73	−1.71	−1.69	−1.68	−1.67	−1.66	−1.66	−1.65	−1.65
(b) $\theta = 2$														
0.0000001	......	3.00	3.95	4.73	5.32	5.76	6.05	6.25	6.38	6.52	6.59	6.60	6.595	6.57
0.000001	......	3.00	3.93	4.67	5.18	5.55	5.78	5.93	6.05	6.14	6.17	6.15	6.15	6.12
0.00001	......	3.00	3.89	4.555	5.005	5.30	5.48	5.595	5.65	5.72	5.72	5.72	5.68	5.675
0.0001	......	2.995	3.84	4.41	4.77	4.98	5.10	5.18	5.21	5.24	5.245	5.20	5.18	5.15
0.001	......	2.99	3.75	4.20	4.455	4.595	4.66	4.70	4.71	4.70	4.695	4.66	4.62	4.61
0.01	......	2.97	3.59	3.895	4.05	4.11	4.13	4.125	4.125	4.10	4.075	4.04	4.00	3.99
0.1	......	2.92	3.31	3.43	3.47	3.475	3.45	3.435	3.41	3.38	3.34	3.32	3.30	3.28
1.0	......	2.72	2.75	2.70	2.66	2.62	2.58	2.56	2.54	2.51	2.48	2.46	2.46	2.45
10	......	1.765	1.53	1.44	1.39	1.37	1.36	1.34	1.33	1.32	1.32	1.31	1.30	1.31
20	......	1.02	0.92	0.88	0.86	0.85	0.85	0.84	0.84	0.84	0.85	0.85	0.83	0.83
30	......	0.43	0.48	0.48	0.48	0.48	0.49	0.49	0.49	0.50	0.50	0.50	0.50	0.51
40	......	0.04	0.12	0.16	0.17	0.19	0.20	0.20	0.20	0.21	0.22	0.22	0.22	0.22
50	......	−0.42	−0.185	−0.13	−0.10	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.03	−0.03	−0.03
60	......	−0.72	−0.465	−0.36	−0.36	−0.34	−0.32	−0.32	−0.31	−0.30	−0.29	−0.285	−0.29	−0.28
70	......	−0.97	−0.73	−0.66	−0.62	−0.60	−0.585	−0.58	−0.57	−0.56	−0.55	−0.545	−0.55	−0.55
80	......	−1.17	−1.01	−0.945	−0.91	−0.90	−0.89	−0.88	−0.87	−0.87	−0.86	−0.86	−0.86	−0.855
85	......	−1.26	−1.15	−1.11	−1.08	−1.07	−1.07	−1.06	−1.06	−1.055	−1.04	−1.04	−1.05	−1.05
90	......	−1.34	−1.30	−1.28	−1.27	−1.27	−1.27	−1.27	−1.27	−1.27	−1.26	−1.26	−1.27	−1.28
95	......	−1.41	−1.485	−1.51	−1.53	−1.55	−1.56	−1.57	−1.58	−1.58	−1.59	−1.59	−1.61	−1.60

TABLE 22.—(*Continued*).

% - of - time	PROBABILITY FOR THE INDEX VALUE, λ =:													
	1	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
							(*c*) $\theta = 3$							
0.0000001	......	4.47	5.70	6.525	7.055	7.325	7.45	7.50	7.50	7.425	7.31	7.20	7.13	7.025
0.000001	......	4.45	5.60	6.325	6.75	6.925	7.00	7.025	7.00	6.90	6.78	6.70	6.605	6.55
0.00001	......	4.43	5.46	6.07	6.35	6.475	6.525	6.50	6.45	6.35	6.25	6.15	6.075	6.00
0.0001	......	4.38	5.27	5.725	5.90	5.95	5.95	5.925	5.86	5.75	5.65	5.545	5.48	5.405
0.001	......	4.30	4.995	5.275	5.355	5.35	5.30	5.25	5.20	5.08	5.00	4.92	4.86	4.80
0.01	......	4.14	4.585	4.70	4.68	4.625	4.58	4.52	4.47	4.36	4.30	4.225	4.19	4.125
0.1	......	3.81	3.95	3.92	3.86	3.78	3.71	3.67	3.63	3.54	3.48	3.45	3.42	3.38
1.0	......	3.10	2.98	2.88	2.79	2.735	2.69	2.66	2.62	2.58	2.54	2.51	2.51	2.48
10	......	1.53	1.435	1.40	1.37	1.35	1.34	1.34	1.335	1.32	1.31	1.32	1.31	1.30
20	......	0.81	0.815	0.815	0.82	0.825	0.825	0.825	0.825	0.825	0.83	0.83	0.84	0.825
30	......	0.32	0.40	0.42	0.44	0.45	0.47	0.47	0.48	0.48	0.48	0.51	0.51	0.51
40	......	—0.04	0.07	0.12	0.14	0.16	0.17	0.18	0.19	0.19	0.20	0.22	0.22	0.22
50	......	—0.32	—0.20	—0.15	—0.12	—0.10	—0.08	—0.08	—0.06	—0.06	—0.05	—0.04	—0.03	—0.03
60	......	—0.57	—0.45	—0.395	—0.365	—0.35	—0.33	—0.33	—0.31	—0.30	—0.30	—0.29	—0.29	—0.27
70	......	—0.78	—0.685	—0.64	—0.615	—0.60	—0.585	—0.58	—0.57	—0.57	—0.56	—0.54	—0.54	—0.55
80	......	—0.98	—0.93	—0.90	—0.89	—0.88	—0.87	—0.87	—0.86	—0.86	—0.855	—0.855	—0.85	—0.85
85	......	—1.075	—1.06	—1.05	—1.04	—1.04	—1.04	—1.05	—1.04	—1.04	—1.04	—1.04	—1.03	—1.04
90	......	—1.17	—1.19	—1.21	—1.22	—1.23	—1.23	—1.24	—1.24	—1.25	—1.25	—1.255	—1.25	—1.26
95	......	—1.28	—1.37	—1.42	—1.46	—1.49	—1.50	—1.53	—1.53	—1.55	—1.565	—1.57	—1.57	—1.58
							(*d*) $\theta = 4$							
0.0000001	4.00	5.875	7.225	7.995	8.35	8.405	8.375	8.28	8.20	7.945	7.75	7.54	7.395	7.29
0.000001	4.00	5.80	7.02	7.63	7.85	7.88	7.80	7.68	7.60	7.35	7.135	6.975	6.85	6.75
0.00001	4.00	5.73	6.75	7.20	7.30	7.275	7.165	7.045	6.92	6.68	6.525	6.36	6.26	6.155
0.0001	4.00	5.585	6.39	6.65	6.675	6.57	6.45	6.32	6.225	6.005	5.87	5.73	5.625	5.55
0.001	3.99	5.37	5.90	5.995	5.935	5.80	5.685	5.55	5.46	5.295	5.15	5.03	4.97	4.91
0.01	3.97	4.995	5.23	5.19	5.07	4.93	4.815	4.71	4.64	4.50	4.39	4.30	4.25	4.20
0.1	3.90	4.35	4.32	4.18	4.055	3.94	3.86	3.78	3.72	3.61	3.56	3.48	3.44	3.42
1.0	3.51	3.26	3.08	2.95	2.865	2.79	2.74	2.69	2.66	2.60	2.56	2.535	2.52	2.48
10	1.47	1.43	1.39	1.375	1.365	1.35	1.34	1.33	1.34	1.32	1.32	1.31	1.31	1.31
20	0.38	0.73	0.77	0.79	0.80	0.81	0.805	0.81	0.83	0.815	0.83	0.82	0.82	0.84
30	—0.20	0.29	0.36	0.40	0.42	0.44	0.44	0.45	0.47	0.47	0.49	0.48	0.48	0.48
40	—0.52	—0.04	0.06	0.10	0.13	0.15	0.16	0.16	0.19	0.19	0.20	0.20	0.21	0.22
50	—0.705	—0.30	—0.21	—0.16	—0.13	—0.11	—0.10	—0.09	—0.07	—0.06	—0.04	—0.04	—0.04	—0.03
60	—0.82	—0.52	—0.44	—0.40	—0.37	—0.35	—0.34	—0.33	—0.32	—0.31	—0.29	—0.29	—0.28	—0.28
70	—0.90	—0.72	—0.66	—0.63	—0.61	—0.60	—0.59	—0.59	—0.57	—0.57	—0.55	—0.55	—0.55	—0.55
80	—0.95	—0.90	—0.89	—0.88	—0.87	—0.87	—0.87	—0.87	—0.86	—0.865	—0.85	—0.86	—0.86	—0.85
85	—0.97	—1.00	—1.02	—1.02	—1.02	—1.03	—1.035	—1.03	—1.03	—1.04	—1.04	—1.035	—1.03	—1.03
90	—0.98	—1.095	—1.15	—1.18	—1.19	—1.21	—1.22	—1.23	—1.23	—1.25	—1.25	—1.25	—1.26	—1.25
95	—0.99	—1.21	—1.32	—1.38	—1.42	—1.45	—1.48	—1.50	—1.51	—1.54	—1.55	—1.56	—1.565	—1.58

TABLE 22.—(*Continued*).

% - of - time	Probability for the Index Value, λ =:													
	1	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
							(*e*) $\theta = 5$							
0.0000001	5.00	7.195	8.575	9.22	9.33	9.21	9.09	8.86	8.66	8.28	8.00	7.78	7.575	7.44
0.000001	4.995	7.06	8.25	8.70	8.70	8.55	8.36	8.15	7.98	7.63	7.37	7.16	6.99	6.88
0.00001	4.99	6.89	7.82	8.07	7.995	7.79	7.62	7.41	7.25	6.92	6.69	6.54	6.38	6.26
0.0001	4.95	6.64	7.275	7.375	7.20	6.975	6.80	6.60	6.45	6.19	6.01	5.86	5.71	5.625
0.001	4.91	6.25	6.60	6.52	6.31	6.09	5.92	5.76	5.645	5.41	5.25	5.145	5.03	4.95
0.01	4.80	5.67	5.70	5.52	5.31	5.12	4.99	4.85	4.75	4.60	4.46	4.40	4.30	4.25
0.1	4.495	4.74	4.57	4.37	4.19	4.03	3.94	3.86	3.77	3.65	3.58	4.53	3.48	3.46
1.0	3.62	3.35	3.15	3.02	2.90	2.825	2.77	2.725	2.68	2.62	2.58	2.56	2.52	2.51
10	1.36	1.38	1.375	1.36	1.36	1.34	1.34	1.33	1.33	1.32	1.32	1.32	1.29	1.30
20	0.49	0.69	0.75	0.77	0.79	0.79	0.80	0.81	0.81	0.82	0.815	0.83	0.83	0.82
30	0	0.27	0.35	0.39	0.415	0.42	0.44	0.45	0.46	0.47	0.48	0.49	0.495	0.495
40	—0.30	—0.04	0.05	0.10	0.12	0.13	0.15	0.17	0.17	0.19	0.19	0.20	0.20	0.22
50	—0.51	—0.29	—0.20	—0.16	—0.13	—0.12	—0.10	—0.08	—0.075	—0.07	—0.05	—0.05	—0.04	—0.04
60	—0.66	—0.50	—0.43	—0.40	—0.37	—0.36	—0.34	—0.33	—0.32	—0.31	—0.30	—0.29	—0.29	—0.29
70	—0.77	—0.69	—0.65	—0.63	—0.61	—0.60	—0.59	—0.58	—0.57	—0.57	—0.56	—0.56	—0.55	—0.55
80	—0.86	—0.87	—0.87	—0.87	—0.86	—0.87	—0.86	—0.86	—0.86	—0.86	—0.85	—0.85	—0.85	—0.85
85	—0.90	—0.96	—0.99	—1.01	—1.01	—1.025	—1.03	—1.03	—1.025	—1.035	—1.03	—1.03	—1.03	—1.04
90	—0.93	—1.06	—1.12	—1.16	—1.18	—1.20	—1.21	—1.22	—1.22	—1.24	—1.25	—1.25	—1.25	—1.25
95	—0.96	—1.17	—1.29	—1.36	—1.41	—1.445	—1.47	—1.49	—1.50	—1.53	—1.55	—1.55	—1.56	—1.58
							(*f*) $\theta = 6$							
0.0000001	5.97	8.435	9.75	10.20	10.10	9.85	9.57	9.245	8.95	8.545	8.185	7.90	7.70	7.525
0.000001	5.96	8.235	9.28	9.54	9.35	9.05	8.745	8.46	8.21	7.81	7.52	7.275	7.08	6.955
0.00001	5.935	7.95	8.72	8.75	8.50	8.20	7.92	7.655	7.425	7.07	6.82	6.60	6.46	6.34
0.0001	5.875	7.55	7.98	7.88	7.585	7.30	7.02	6.80	6.60	6.31	6.08	5.91	5.795	5.695
0.001	5.75	6.98	7.10	6.90	6.595	6.32	6.08	5.90	5.73	5.50	5.31	5.18	5.075	5.00
0.01	5.505	6.18	6.025	5.78	5.48	5.265	5.075	4.935	4.815	4.66	4.50	4.405	4.345	4.27
0.1	4.95	4.995	4.72	4.50	4.27	4.105	3.99	3.89	3.825	3.69	3.615	3.54	3.50	3.445
1.0	3.68	3.395	3.18	3.05	2.93	2.85	2.78	2.74	2.70	2.65	2.59	2.56	2.54	2.52
10	1.30	1.35	1.35	1.36	1.34	1.34	1.33	1.325	1.33	1.32	1.31	1.31	1.31	1.31
20	0.49	0.67	0.73	0.77	0.78	0.785	0.79	0.80	0.81	0.81	0.815	0.81	0.82	0.84
30	0.04	0.26	0.33	0.385	0.405	0.42	0.44	0.45	0.455	0.47	0.475	0.475	0.485	0.50
40	—0.24	—0.04	0.04	0.09	0.12	0.13	0.15	0.16	0.17	0.18	0.19	0.20	0.205	0.21
50	—0.44	—0.28	—0.21	—0.16	—0.135	—0.12	—0.105	—0.09	—0.08	—0.07	—0.06	—0.05	—0.05	—0.04
60	—0.59	—0.48	—0.43	—0.39	—0.37	—0.36	—0.35	—0.34	—0.325	—0.31	—0.31	—0.30	—0.30	—0.29
70	—0.71	—0.665	—0.64	—0.615	—0.60	—0.60	—0.59	—0.585	—0.57	—0.57	—0.56	—0.555	—0.56	—0.54
80	—0.81	—0.84	—0.86	—0.855	—0.86	—0.86	—0.86	—0.86	—0.855	—0.86	—0.86	—0.855	—0.86	—0.85
85	—0.85	—0.94	—0.98	—0.99	—1.00	—1.01	—1.02	—1.02	—1.03	—1.03	—1.03	—1.03	—1.04	—1.02
90	—0.89	—1.03	—1.10	—1.14	—1.16	—1.19	—1.20	—1.215	—1.22	—1.24	—1.235	—1.24	—1.25	—1.25
95	—0.94	—1.15	—1.265	—1.34	—1.39	—1.43	—1.46	—1.48	—1.49	—1.52	—1.53	—1.545	—1.56	—1.57

TABLE 22.—(*Continued*).

% - of - time	PROBABILITY FOR THE INDEX VALUE, λ =:													
	1	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
							(*g*) $\theta = 7$							
0.0000001	6.94	9.60	10.8	11.03	10.75	10.34	9.94	9.56	9.23	8.70	8.325	8.05	7.80	7.625
0.000001	6.915	9.30	10.20	10.22	9.875	9.45	9.08	8.75	8.42	7.975	7.645	7.38	7.19	7.02
0.00001	6.85	8.91	9.48	9.35	8.93	8.51	8.175	7.86	7.60	7.20	6.915	6.72	6.52	6.42
0.0001	6.75	8.35	8.60	8.31	7.895	7.525	7.21	6.97	6.72	6.40	6.155	6.00	5.83	5.75
0.001	6.525	7.60	7.54	7.18	6.80	6.47	6.235	6.00	5.82	5.555	5.36	5.23	5.13	5.03
0.01	6.125	6.59	6.30	5.95	5.62	5.36	5.17	5.02	4.89	4.675	4.54	4.43	4.35	4.30
0.1	5.32	5.20	4.85	4.57	4.345	4.17	4.05	3.94	3.85	3.72	3.62	3.57	3.52	3.47
1.0	3.73	3.44	3.22	3.06	2.94	2.87	2.80	2.75	2.71	2.66	2.61	2.58	2.53	2.53
10	1.26	1.33	1.34	1.345	1.35	1.33	1.33	1.34	1.32	1.315	1.32	1.32	1.29	1.30
20	0.48	0.66	0.72	0.755	0.775	0.78	0.80	0.805	0.81	0.815	0.82	0.82	0.825	0.83
30	0.06	0.25	0.33	0.38	0.405	0.415	0.435	0.45	0.45	0.47	0.47	0.48	0.48	0.495
40	—0.21	—0.04	0.035	0.09	0.11	0.13	0.14	0.16	0.16	0.17	0.19	0.21	0.21	0.21
50	—0.41	—0.27	—0.21	—0.16	—0.14	—0.12	—0.11	—0.09	—0.09	—0.08	—0.065	—0.04	—0.04	—0.035
60	—0.56	—0.47	—0.42	—0.39	—0.37	—0.355	—0.35	—0.33	—0.33	—0.31	—0.31	—0.30	—0.30	—0.28
70	—0.68	—0.65	—0.63	—0.615	—0.605	—0.595	—0.59	—0.575	—0.58	—0.57	—0.57	—0.55	—0.55	—0.54
80	—0.78	—0.83	—0.85	—0.855	—0.86	—0.86	—0.86	—0.85	—0.85	—0.85	—0.85	—0.84	—0.85	—0.84
85	—0.83	—0.92	—0.97	—0.99	—1.00	—1.01	—1.02	—1.02	—1.03	—1.03	—1.03	—1.03	—1.03	—1.025
90	—0.87	—1.015	—1.09	—1.13	—1.16	—1.18	—1.19	—1.20	—1.21	—1.225	—1.24	—1.235	—1.25	—1.25
95	—0.915	—1.13	—1.255	—1.33	—1.38	—1.42	—1.45	—1.47	—1.49	—1.51	—1.535	—1.54	—1.56	—1.57
							(*h*) $\theta = 8$							
0.0000001	7.895	10.70	11.78	11.73	11.30	10.76	10.26	9.82	9.44	8.87	8.45	8.15	7.85	7.70
0.000001	7.83	10.30	11.00	10.775	10.31	9.78	9.30	8.93	8.59	8.10	7.74	7.455	7.25	7.05
0.00001	7.74	9.775	10.15	9.775	9.29	8.78	8.355	8.03	7.73	7.31	6.99	6.75	6.57	6.45
0.0001	7.55	9.06	9.10	8.64	8.15	7.71	7.35	7.075	6.83	6.47	6.21	6.03	5.87	5.765
0.001	7.245	8.15	7.90	7.42	6.975	6.60	6.32	6.10	5.90	5.64	5.40	5.27	5.14	5.055
0.01	6.675	6.95	6.545	6.07	5.75	5.45	5.225	5.06	4.92	4.725	4.56	4.47	4.40	4.32
0.1	5.61	5.38	4.97	4.63	4.41	4.22	4.08	3.97	3.88	3.75	3.64	3.58	3.52	3.50
1.0	3.78	3.47	3.24	3.08	2.975	2.88	2.81	2.78	3.72	2.66	2.60	2.575	2.56	2.54
10	1.225	1.32	1.335	1.34	1.34	1.33	1.33	1.33	1.33	1.33	1.30	1.31	1.31	1.31
20	0.48	0.65	0.71	0.75	0.77	0.78	0.79	0.80	0.80	0.825	0.81	0.825	0.83	0.82
30	0.07	0.25	0.32	0.37	0.40	0.41	0.43	0.44	0.45	0.47	0.46	0.485	0.485	0.495
40	—0.195	—0.04	0.03	0.08	0.11	0.13	0.14	0.155	0.17	0.18	0.19	0.20	0.21	0.195
50	—0.385	—0.27	—0.21	—0.17	—0.14	—0.125	—0.11	—0.09	—0.09	—0.07	—0.07	—0.045	—0.045	—0.04
60	—0.53	—0.46	—0.42	—0.40	—0.37	—0.36	—0.34	—0.34	—0.32	—0.32	—0.31	—0.30	—0.30	—0.28
70	—0.66	—0.64	—0.63	—0.61	—0.60	—0.60	—0.59	—0.58	—0.58	—0.57	—0.575	—0.56	—0.55	—0.54
80	—0.76	—0.82	—0.84	—0.85	—0.85	—0.86	—0.86	—0.86	—0.86	—0.86	—0.86	—0.85	—0.85	—0.84
85	—0.81	—0.91	—0.955	—0.98	—0.99	—1.01	—1.01	—1.02	—1.02	—1.02	—1.04	—1.03	—1.03	—1.03
90	—0.855	—1.00	—1.08	—1.13	—1.15	—1.18	—1.19	—1.210	—1.22	—1.23	—1.25	—1.24	—1.25	—1.24
95	—0.91	—1.12	—1.24	—1.32	—1.37	—1.42	—1.44	—1.47	—1.48	—1.51	—1.53	—1.55	—1.56	—1.56

TABLE 22.—(*Continued*).

% - of - time	PROBABILITY FOR THE INDEX VALUE, λ =:													
	1	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
							(*i*) $\theta = 9$							
0.0000001......	8.92	11.70	12.60	12.34	11.72	11.02	10.49	10.01	9.60	8.98	8.525	8.19	7.91	7.72
0.000001.......	8.74	11.20	11.71	11.30	10.67	10.02	9.50	9.10	8.745	8.22	7.775	7.50	7.27	7.14
0.00001........	8.59	10.55	10.69	10.11	9.54	8.95	8.495	8.15	7.835	7.38	7.04	6.82	6.60	6.50
0.0001.........	8.34	9.695	9.52	8.90	8.35	7.83	7.45	7.17	6.90	6.55	6.25	6.07	5.92	5.80
0.001..........	7.90	8.57	8.17	7.595	7.11	6.695	6.375	6.17	5.95	5.65	5.44	5.30	5.185	5.09
0.01...........	7.13	7.20	6.70	6.18	5.78	5.49	5.28	5.115	4.97	4.76	4.595	4.47	4.405	4.345
0.1............	5.845	5.46	5.02	4.70	4.43	4.23	4.09	4.00	3.895	3.77	3.66	3.59	3.53	3.49
1.0............	3.79	3.47	3.25	3.10	2.98	2.89	2.82	2.78	2.72	2.68	2.61	2.59	2.55	2.54
10.............	1.20	1.30	1.32	1.32	1.33	1.32	1.32	1.34	1.31	1.32	1.31	1.32	1.32	1.32
20.............	0.47	0.635	0.71	0.75	0.76	0.77	0.79	0.80	0.80	0.82	0.81	0.83	0.82	0.82
30.............	0.07	0.24	0.32	0.36	0.39	0.40	0.43	0.445	0.45	0.47	0.47	0.49	0.49	0.49
40.............	—0.18	—0.04	—0.03	0.07	0.10	0.12	0.13	0.16	0.16	0.18	0.18	0.20	0.22	0.22
50.............	—0.37	—0.27	—0.21	—0.17	—0.145	—0.13	—0.11	—0.095	—0.09	—0.07	—0.07	—0.045	—0.045	—0.04
60.............	—0.52	—0.46	—0.42	—0.40	—0.37	—0.36	—0.36	—0.33	—0.33	—0.32	—0.31	—0.29	—0.30	—0.28
70.............	—0.64	—0.64	—0.62	—0.61	—0.60	—0.60	—0.585	—0.58	—0.57	—0.56	—0.56	—0.54	—0.54	—0.55
80.............	—0.745	—0.81	—0.83	—0.85	—0.85	—0.86	—0.85	—0.85	—0.85	—0.845	—0.86	—0.86	—0.85	—0.84
85.............	—0.795	—0.90	—0.95	—0.98	—0.99	—1.01	—1.01	—1.01	—1.02	—1.02	—1.03	—1.03	—1.03	—1.02
90.............	—0.84	—0.995	—1.07	—1.12	—1.15	—1.175	—1.19	—1.20	—1.21	—1.22	—1.24	—1.235	—1.24	—1.235
95.............	—0.89	—1.11	—1.24	—1.32	—1.37	—1.415	—1.44	—1.46	—1.48	—1.50	—1.53	—1.54	—1.56	—1.555
							(*j*) $\theta = 10$							
0.0000001......	9.72	12.70	13.36	12.9	12.07	11.36	10.70	10.17	9.71	9.06	8.60	8.26	7.96	7.74
0.000001.......	9.60	12.09	12.36	11.78	10.94	10.29	9.68	9.21	8.81	8.25	7.85	7.58	7.31	7.125
0.00001........	9.40	11.30	11.20	10.51	9.76	9.155	8.65	8.25	7.90	7.43	7.10	6.86	6.64	6.47
0.0001.........	9.05	10.30	9.89	9.18	8.50	7.99	7.55	7.24	6.97	6.57	6.305	6.075	5.93	5.795
0.001..........	8.48	9.01	8.425	7.75	7.175	6.80	6.45	6.20	5.99	5.69	5.47	5.325	5.20	5.09
0.01...........	7.54	7.45	6.825	6.29	5.86	5.575	5.325	5.12	4.985	4.755	4.61	4.50	4.40	4.335
0.1............	6.01	5.575	5.09	4.74	4.46	4.275	4.10	4.00	3.91	3.76	3.67	3.60	3.52	3.48
1.0............	3.78	3.49	3.26	3.11	2.98	2.91	2.83	2.77	2.72	2.67	2.62	2.59	2.55	2.54
10.............	1.18	1.29	1.32	1.33	1.325	1.33	1.32	1.32	1.32	1.325	1.33	1.33	1.31	1.30
20.............	0.46	0.63	0.70	0.74	0.76	0.78	0.78	0.80	0.80	0.81	0.815	0.83	0.81	0.82
30.............	0.08	0.24	0.31	0.36	0.39	0.41	0.43	0.43	0.45	0.46	0.47	0.49	0.49	0.48
40.............	—0.175	—0.04	0.03	0.07	0.10	0.13	0.14	0.15	0.16	0.18	0.19	0.20	0.20	0.20
50.............	—0.36	—0.27	—0.21	—0.17	—0.145	—0.125	—0.11	—0.10	—0.09	—0.07	—0.06	—0.05	—0.05	—0.045
60.............	—0.51	—0.46	—0.42	—0.39	—0.37	—0.36	—0.35	—0.34	—0.32	—0.32	—0.31	—0.30	—0.30	—0.29
70.............	—0.63	—0.63	—0.625	—0.61	—0.60	—0.59	—0.59	—0.58	—0.575	—0.57	—0.56	—0.55	—0.55	—0.55
80.............	—0.73	—0.80	—0.83	—0.84	—0.85	—0.855	—0.86	—0.86	—0.85	—0.85	—0.85	—0.85	—0.85	—0.85
85.............	—0.785	—0.895	—0.95	—0.98	—0.99	—1.00	—1.01	—1.02	—1.02	—1.02	—1.02	—1.02	—1.02	—1.035
90.............	—0.83	—0.985	—1.07	—1.12	—1.14	—1.17	—1.19	—1.20	—1.21	—1.21	—1.23	—1.24	—1.24	—1.25
95.............	—0.89	—1.10	—1.23	—1.31	—1.26	—1.41	—1.435	—1.46	—1.48	—1.50	—1.53	—1.53	—1.55	—1.56

of the moments to any desired degree by a suitable recursion formula (Equations (22), for instance), and computes two curves, one with the errors added to the characteristics as computed from the sample and the other with the errors subtracted. This gives a range within which the curve characteristic of the population has something better than an even chance of lying. The important and difficult subject of errors of sampling has not yet received adequate attention from engineers.

Professor Mavis raises an important question with regard to the representation of finite and discrete variates by means of a smooth frequency curve. Few phenomena in applied mathematics have been so misunderstood as the nature of a frequency function. A smooth curve does not imply that the variate is continuous or that the variations are infinite in number, although in so-called derivations, still abundant in treatises, such notions appear implicitly and explicitly. The fact is that the ordinates of a frequency curve do not give frequencies. The frequencies are always given by the area included between the curve, the class interval, and the two class limits. These frequencies, therefore, are always discrete. It is true that the mid-ordinate of a class (when the class interval is taken as 1) is usually numerically close to the frequency of the class, so that this ordinate is often taken as the frequency; but the frequency is always given by an area. When the variate is continuous and the number of variations infinite, one may only estimate the frequency of the values of the variate within the class limits; in the continuous case, the frequency of a single value is always zero.

In calling the totally bounded function the most general homograde function, the writer is stating a definition and, consequently, cannot be falling into a logical error. He is, in effect, formulating a physical law. This formulation is empirical, and it will take experience and not mathematics to prove it right or wrong. It is well known that the small errors of precise measurements are distributed, to a high order of approximation, according to the normal law. The mean and the standard deviation are the two characteristics necessary to determine such distributions and, for them, there are no other independent characteristics. When the causes producing variations are not infinite in number and are not mutually independent, then these two characteristics do not suffice and two others must be introduced. The writer then considers the four-parameter generalization of the Gaussian normal the most general homograde distribution, more complicated distributions being merely superpositions of this type.

Thiele's observations, as mentioned by Mr. Fisher, regarding the "pseudo-normal" variate, $t = h(x)$, are not discoveries in the theory of statistics, but merely special cases of theorems from the general theory of curves. Any two curves, provided they satisfy certain conditions of uniformity, may always be transformed into one another by suitable transformations. A theory of "pseudo-normal" variates, therefore, is quite as general as the integral equation or the Gram-Charlier formulations. The great advantage of such an observation (and the writer has availed himself of it) is that tables of the probability integral may be used with the more general function.

The writer was in error in attributing Equation (10) to Mr. Fisher, and he did not notice, until it was too late to acknowledge it, that the partly bounded function is exactly equivalent to his generic function; he was confused, on reading Mr. Fisher's treatise, by the discussion of the "mathematical zero." The writer objects to the use of it as a generic function, at least when dealing with small samples, for reasons already stated.

The work of the Danish statisticians is known to the writer only through Mr. Fisher's work and he was not aware of Thiele's generalization, Equation (68). Thiele. however, does not show how to compute the constants of this curve in terms of the statistical characteristics; in this respect, at any rate, the writer has shown some originality. The awkwardness in the writer's presentation, referred to by Mr. Fisher, was to a large extent, unavoidable. In the first place, the writer has definite limitations of expression and knowledge; in the second place, the "obvious" detail in the analysis and the choice of mathematical procedure was deemed desirable in presenting a paper of this sort to engineers instead of mathematicians. In defense of the analysis of Article 6, Section II, which Mr. Fisher considers superfluous, the writer should like to state that the approach of the logarithmic function to the limit, $\frac{x}{\sigma}$, as the skewness vanishes, is by no means self-evident (although it may be well known to mathematicians). If the transformation chosen had been the simple "pseudo-normal" variate, $t = h(x) = ax^3$, then for no value of the parameter, a, would the resulting curve have reduced to the normal, Thiele's theory notwithstanding. Thiele's theory leads to the Gaussian for the vanishing of the higher parameters simply because of the form he chose for his semi-invariants. Another form would have led to a different fundamental function. The theory of integral equations teaches one that the normal functions of a problem depend on the kernel of the equation and not on whether the equation arises in the theory of statistics, or in mechanics, or in some other manner. Thus, it happens that Equations (75) and (78) are mathematically equivalent. Both represent any function, whether frequency or not, which is continuous and vanishes, together with its first derivative, at infinity in both directions.

Mr. Fisher is inaccurate in inferring that the writer's statements regarding the generality of these two equations implies that a Fourier series is inferior to a Taylor series. What they do imply is that both the Fourier and the Taylor series are mathematical devices which sometimes offer a comparatively easy means of solving a difficult problem — more often than not the solution thus obtained will be purely formal and of little practical value.

Elsewhere,[59] the writer has proposed a method for determining the maximum and minimum floods from a record of yearly floods. In this method the constants have been determined only for use with yearly floods and as it stands, it is not applicable to the example furnished by Mr. Hall. The form of the equations has been suggested by theory—physical and statistical—

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[59] Report to the Mississippi Valley Committee, July, 1935.

but their use in their present state can be justified only by the consistency of the results obtained by means of them in the problems attempted by the writer. The following quantities are used: g = maximum upper deviation from the mean, where the mean is taken = 1; b = maximum lower deviation from mean; g_o = maximum observed upper deviation; b_o = maximum observed lower deviation; A = drainage area, in square miles; M = mean flood, in feet per second; and, N = number of years in record. From these the following two quantities are computed: $\frac{0.4\, g_o}{b_o}$ = storm index; and, $\sqrt[3]{\frac{A}{M}}$ = flood index.

Using logarithms to the base 10, compute:

$$x = 6.7 - \left(0.5 \log N + 1.5 \log A + \frac{0.4\, g_o}{b_o}\right) + \sqrt[3]{\frac{A}{M}} \quad \ldots\ldots(89)$$

from which the maximum upper deviation is computed by the formula:

$$g = g_o\,(1 + 10^x)\left(0.7\sqrt[3]{\frac{A}{M}} + 0.8\right) \ldots\ldots\ldots\ldots\ldots(90)$$

In cubic feet per second, the maximum flood will be given by $M(1 + g)$.

For the maximum lower deviation the same two equations are used, except that where g_o appears, $\frac{2\, b_o}{(1 - b_o)}$ is substituted, and where b_o appears, $\frac{g_o}{(2 + g_o)}$ is substituted. Equation (90), then, gives a quantity, b', say, from which,

$$b = \frac{b'}{(2 + b')} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(91)$$

and the minimum flood is given in cubic feet per second by $M(1 - b)$.

The late Allen Hazen, M. Am. Soc. C. E., presented[60] an example that shows the effect of the large flood of 1913 on the yearly flood record of the Hudson River, at Mechanicsville, N. Y. If the 23-yr record[61] ending in 1912 is taken, then $N = 23$; $A = 4\,500$; $M = 41\,700$; $g = 0.43$; and, $b = 0.37$. Substituting these values in Equations (89) and (90), it is found that $g = 2.225$, from which $3.225 \times 41\,700 = 134\,600$ cu ft per sec for the maximum flood. If the 34-yr record[62] ending in 1923 is taken, then the quantities are: $N = 34$; $A = 4\,500$; $M = 44\,000$; $g = 1.56$; and $b = 0.41$; and thus $g = 2.285$ and the maximum flood is 144 600 cu ft per sec. Although the two graphs show quite different characteristics, the values computed from them for the maximum flood are seen to be fairly close. Likewise, from two records of the Arkansas River,[63] the maximum flood is computed to be close to 50 000 cu ft per sec. These two rivers have equal drainage areas, but quite different flood and storm indices. (The storm index as herein given is a very variable quantity and characteristic of the particular record only.)

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[60] "Flood Flows," by Allen Hazen, Wiley & Sons, New York, 1930.

[61] *Loc. cit.*, Fig. 38, p. 89.

[62] *Loc. cit.*, Fig. 15, p. 78.

[63] *Loc. cit.*, Fig. 29, p. 89, and Fig. 34, p. 84.

To use Table 22: (*a*) Find the mean, M, and the standard deviation, σ; (*b*) estimate b and g; and, (*c*) compute $\lambda = \frac{b}{\sigma}$ and $\theta = \frac{g}{b}$. Table 22 gives values of t corresponding to percentages of time for various values of λ and θ. First-difference interpolations will usually be necessary. Then, compute (*d*) $X = (\sigma t + M)$, which is plotted against the corresponding %-of-time.

Table 2 of the paper simply gives the particular case in which g is infinite. It may be stated explicitly that the tables are computed for $\sigma = 1$, but that, because of the homogeneity of the function, any standard deviation is taken into account in Step (*d*) of the foregoing instructions. When b is greater than g, the function is left skewed. Because of the symmetry in b and g of the logarithmic linear fractional transformation, the same tables may be used in that case by merely interchanging b and g.

For instance, taking the 34-yr record for the Hudson River, $g = 2.285$; and, $b = 0.525$. Hazen gave the value, 0.37, for the standard deviation, CV, in this case, and if an attempt is made to fit the curve with this value, it is found to be too far off the main body of the statistics—the 1913 flood pulls it up considerably. Even this poor fit gives to this flood a probability of being equalled or exceeded but once in about 1 000 yr, so that since this flood contributes vastly more than its share to the standard deviation, one may conclude that a better estimate of this value may be made by taking only the forty-two values that exclude it. In this way the standard deviation is found to be 0.26. (For all these computations the flood magnitudes have been scaled from the diagrams previously cited,[60] and a 10-in. slide-rule has been used.)

To compute the standard error in the standard deviation, first calculate,

$$\beta = \left(\frac{\sigma^3}{b^3} + \frac{3\,\sigma}{b}\right)\left(\frac{g - b}{g + b}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (92)$$

from which the required error is given by,

$$\epsilon(\sigma) = \tfrac{1}{2}\,\sigma\sqrt{\frac{3\,\beta^2 + 4}{2\,N}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (93)$$

Equation (93) is obtained from Pearson's relations for the error in the moments in connection with the recursion formulas, Equations (22), and modified so as to be applicable to the general function. These expressions are not exact, but they give values for the error comparable to those obtained by the cumbersome exact formulas. Using the foregoing values for the Hudson River, the error is found to be 0.04. The curve for $\sigma = 0.26 + 0.04 = 0.3$ will be computed. With this value, $\lambda = 1.75$ and $\theta = 4.36$. From Table 22(*d*) by interpolating between the columns for $\lambda = 1.5$ and $\lambda = 2.0$, Column (1) of Table 23 is obtained. Column (2) is obtained from

Table 22(*e*) by the same interpolation. Then, adding to Column (1) Table 23, 0.36 times the differences between the entries in Columns (1) and (2), Column (3) is obtained. Column (4) = 0.3 × Column (3) + 1, and is plotted against the %-of-time values given in Column (5), Table 23. The curve is shown in Fig. 9. For $\sigma = 0.26$ the curve is slightly flatter and crosses the plotted curve at about the mean.

TABLE 23.—COMPUTATION OF PROBABILITY CURVE.

(1)	(2)	(3)	(4)	(5)
6.41	7.66	6.86	3.06	0.000001
6.22	7.36	6.63	2.95	0.00001
5.99	6.96	6.34	2.90	0.0001
5.63	6.42	5.55	2.77	0.001
5.11	5.66	5.31	2.60	0.01
4.34	4.66	4.46	2.34	0.1
3.17	3.25	3.20	1.96	1
1.41	1.38	1.40	1.42	10
0.75	0.72	0.74	1.22	20
0.32	0.31	0.32	1.10	30
0.01	0.00	0.01	1.00	40
—0.25	—0.24	—0.25	0.92	50
—0.48	—0.47	—0.48	0.86	60
—0.69	—0.67	—0.69	0.79	70
—0.90	—0.87	—0.89	0.73	80
—1.10	—0.97	—1.05	0.69	85
—1.12	—1.09	—1.11	0.67	90
—1.26	—1.23	—1.25	0.63	95

Equations (89) and (90) should not be used for areas of less than 1 000 sq mile nor for records of less than about 20 yr.

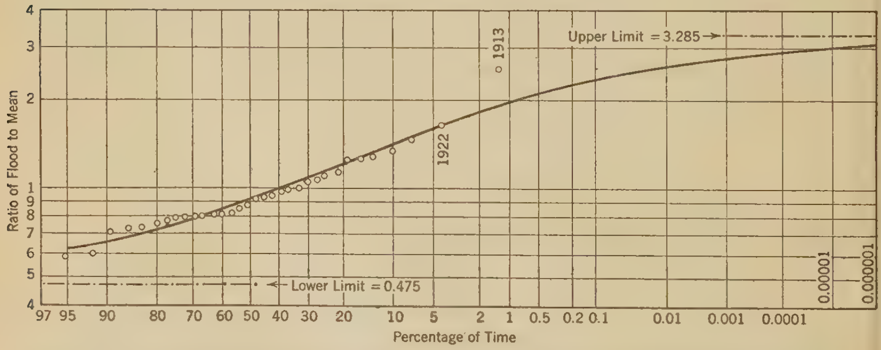

FIG. 9.—YEARLY FLOOD RECORD, 1881–1923, HUDSON RIVER AT MECHANICSVILLE, N. Y.

Acknowledgment.—Equations (89) and (90) were developed while the writer was employed by the United States Geological Survey on the work of the Mississippi Valley Committee. The writer wishes to express his great indebtedness to Messrs. D. Abramowitz and A. Mandel for constructing Table 22 and for much other assistance, and to Thorndike Saville, M. Am. Soc. C. E., for making their labor available and for valuable criticism and encouragement.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1925

ANALYSIS OF CONTINUOUS STRUCTURES BY TRAVERSING THE ELASTIC CURVES

BY RALPH W. STEWART,[1] M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. GARRETT B. DRUMMOND, AUSTIN H. REEVES, E. G. PAULET, ADOLPHUS MITCHELL, DAVID M. WILSON, W. H. KIRKBRIDE, R. B. KETCHUM, A. FLORIS, IVAN M. NELIDOV, FANG-YIN TSAI, AND RALPH W. STEWART.

SYNOPSIS

A method of analyzing the moments in the members of continuous frames by a geometrical solution of the alignment of their elastic curves, is presented in this paper. Memorized or copied slope-deflection equations are not used, and a series of rules for the signs of moments, rotations, and deflections are unnecessary.

PRINCIPLES INVOLVED

Three basic principles are involved[2]:

(1) The angular change of the tangents at any two points on the elastic curve of a flexed beam is equal to the area between the two corresponding sections on the $\frac{M}{EI}$- diagram.

(2) The curvature mentioned in Principle (1) may be represented as an angle which, in radians, is numerically equal to the corresponding $\frac{M}{EI}$- area. This angle is platted opposite the center of gravity of the part of the $\frac{M}{EI}$- diagram under consideration.

(3) For any unit of the $\frac{M}{EI}$- diagram a triangular traverse of the cor-

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NOTE.—Published in October, 1934, *Proceedings.*

[1] Engr. of Structural Design, City of Los Angeles, Los Angeles, Calif.

[2] *Civil Engineering,* February, 1934, p. 88.

responding unit of the elastic curve can be constructed. This triangle is composed of the tangents and the chord of the elastic curve. The angle between the tangents is given by Principle (2). Each angle between a tangent and the chord is directly proportional to the opposite side. This is because in beam flexure the angles are so small that they may be taken as equal to their sines.

The traversing triangle thus constructed can be solved if one angle and one side are known.

APPLICATIONS TO STRUCTURES

In the cantilever beam (Fig. 1), the traversing triangle is ABC. The angle, Δ, is equal to the area of the moment diagram $= \dfrac{Ml}{2EI}$. Angle $CAB = \dfrac{2}{3}\Delta$ and Angle $CBA = \dfrac{1}{3}\Delta$. The deflection equals:

$$d = \Delta \times \frac{2}{3} l = \frac{Ml^2}{3EI} = \frac{Pl^3}{3EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

As an alternate solution the full length of the beam may be multiplied by $\dfrac{2}{3}\Delta$ as shown by Fig. 1(b).

Fig. 2 represents a span of a continuous beam in flexure which involves end slopes, end moments, and end translation as indicated. In dividing the moment diagram into units it will simplify the solution to add the dotted line, AB, and treat the moment diagram as being composed of two triangles, ABD and BAC, each running the full length of the beam. The triangle, ABE, is a positive addition to the real M_1-triangle and a negative addition to the real M_2-triangle and, therefore, cancels its effect on the final deflection and end slope of the span.

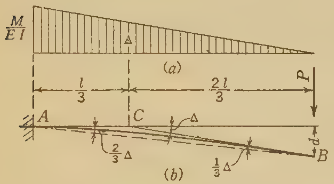
$\frac{M}{EI}$
Δ
(a)
$\frac{l}{3}$
$\frac{2l}{3}$
P
A
C
Δ
$\frac{2}{3}\Delta$
$\frac{1}{3}\Delta$
d
B
(b)

FIG. 1.

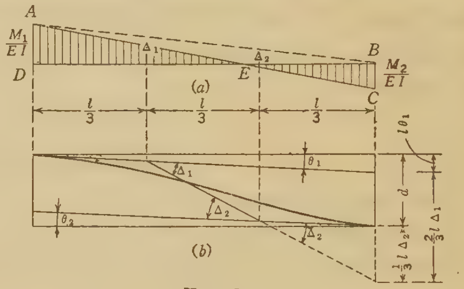
A
$\frac{M_1}{EI}$
Δ_1
Δ_2
B
$\frac{M_2}{EI}$
D
E
(a)
C
$\frac{l}{3}$
$\frac{l}{3}$
$\frac{l}{3}$
$l\theta_1$
θ_1
Δ_1
Δ_2
θ_2
Δ_2
d
$\frac{2}{3}l\Delta_1$
$\frac{1}{3}l\Delta_2$
(b)

FIG. 2.

From the center of gravity of Triangle ABD drop a vertical to the tangent through the left end of the elastic curve and lay off Δ_1 equal to the area of Triangle ABD. From the center of gravity of Triangle BAC drop a vertical to the lower leg of Δ_1 and lay off Δ_2 equal to Area BAC. Produce the upper leg of Δ_2 to the right end of the beam and draw the horizontal line indicating the end slope, θ_2.

From Fig. 2(*b*), it is now possible to read the following equations:

$$\theta_1 + \Delta_1 = \theta_2 + \Delta_2 \quad \ldots\ldots\ldots\ldots (2)$$

and,

$$d = l\,\theta_1 + \frac{2}{3}\,\Delta_1 l - \frac{1}{3}\,\Delta_2 l \quad \ldots\ldots\ldots\ldots (3)$$

Solving:

$$-\,\Delta_1 = 2\,\theta_1 + \theta_2 - \frac{3\,d}{l} \quad \ldots\ldots\ldots\ldots (4)$$

Equating Δ_1 to its value, $\frac{M_1 l}{2\,EI}$, and solving for M_1:

$$-\,M_1 = 2\,E\,\frac{I}{l}\left(2\,\theta_1 + \theta_2 - 3\,\frac{d}{l}\right) \quad \ldots\ldots\ldots\ldots (5)$$

Equation (5) will be recognized as a standard slope-deflection equation. It will not be used in subsequent solutions, but is introduced to illustrate the utility of the traverse method in deriving it. The line, $\Delta_1\,\Delta_2$ (Fig. 2(*b*)), is not tangent to the elastic curve, but crosses it at an angle, due to the addition of Triangle *ABE* to the moment diagram. If the true, shaded, moment triangles were used for locating and evaluating the Δ's, the line, $\Delta_1\,\Delta_2$, would be tangent to the elastic curve at its point of contraflexure, but the distances from the ends of the beam to the Δ's would be inconvenient and would make the solution complicated.

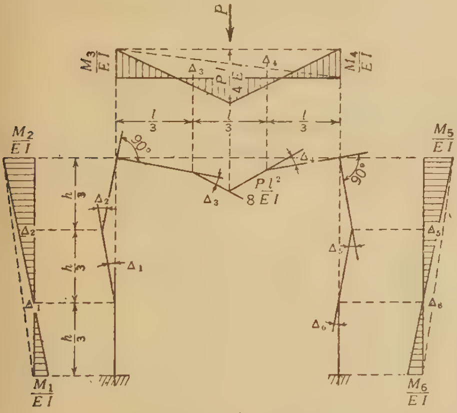

FIG. 3.

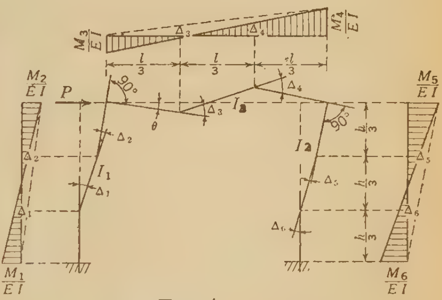

FIG. 4.

In order to illustrate the solution of the moments in a continuous frame by the elastic curve traverse without encumbering the problem with avoidable complications, reference is made to Fig. 3. This diagram shows a symmetrical frame with fixed base columns carrying a concentrated load at the center. The columns and the beams have equal lengths and cross-sections.

The true moment triangles in the beam are shown by the shaded areas. In order to simplify the traverse, the unshaded trapezoidal area between the closing line and the top of the diagram is added to the shaded center

$\frac{M}{EI}$ - triangle, thereby forming a "simple-beam" $\frac{M}{EI}$ - triangle, the middle ordinate of which is $\frac{P\,l}{4\,EI}$. The combined area of the shaded end $\frac{M}{EI}$ - triangles is also increased by the area of this same trapezoid. The resultant curvature in the beam will then be the same as if the true shaded moments only were used, and the diagram may be treated as composed of three $\frac{M}{EI}$ - triangles, comprising the end $\frac{M}{EI}$ - triangles the base of which runs the full length of the beam, and the simple-beam $\frac{M}{EI}$ - triangle the height and area of which are $\frac{P\,l}{4\,EI}$ and $\frac{P\,l^2}{8\,EI}$, respectively. This device is used and discussed in treatises on "slope deflection", to which reference may be made, if further study is desired.[3]

The deflection curves may now be traversed. Each angle in the traverse is equal to the area of its corresponding $\frac{M}{EI}$ - triangle. Beginning at the bottom of the left column the traverse is as follows (the signs for the angles needing no discussion as they are the same as would be used for azimuth in a land survey):

$$-\Delta_1 + \Delta_2 + 90° + \Delta_3 - \frac{P\,l^2}{8\,EI} + \Delta_4 + 90° + \Delta_5 - \Delta_6 = 180° \quad (6)$$

in which, Δ_2, Δ_3, Δ_4, and Δ_5 are all equal since their values of $\frac{M}{EI}$ and the lengths of members are equal. There being no side-sway the triangles along the upper two-thirds of the columns are isosceles; consequently, $\Delta_2 = 2\,\Delta_1$ and $\Delta_5 = 2\,\Delta_6$. Therefore, Equation (6) reduces to $3\,\Delta_2 = \frac{P\,l^2}{8\,EI}$, or,

$$\Delta_2 = \frac{P\,l^2}{24\,EI} \quad \text{.........................} (7)$$

Substituting $\frac{M_2\,l}{2\,EI}$ for Δ_2, the corner moments are $\frac{P\,l}{12}$. The moments at the bases of the columns are $\frac{P\,l}{24}$, and the center moment is $\frac{P\,l}{4} - \frac{P\,l}{12} = \frac{P\,l}{6}$. As shown by Equation (6) the signs of the corner moments are opposite the signs of the center moment and column base moments.

The joint rotations do not enter into the solution, but the equations are formed from the curvature units in the members. This distinguishes the traverse method from the slope-deflection method in which standard equations involving joint rotations are applied.

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[3] *Bulletin 108*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.; Concrete Engineers Handbook, by Hool and Johnson; etc.

Fig. 4 illustrates the application of the traverse method to an unsymmetrical frame having fixed base columns and subjected to a horizontal force, *P*. The material in the frame is assumed to be uniform and *E* may be eliminated from the equations.

From the $\frac{M}{I}$-diagram, $\Delta_2 = \frac{M_2\, h}{2\, I_1}$ and $\Delta_3 = \frac{M_3\, l}{2\, I_3}$; but, since $M_2 = M_3$, $\Delta_2 = \frac{I_3\, h}{I_1\, l}\, \Delta_3 = k_1\, \Delta_3$; similarly, $\Delta_5 = k_2\, \Delta_4$. The angles and courses in the deflected frame constitute a traverse for which the following equations may be written: $\Delta_1 - \Delta_2 + 90° - \Delta_3 + \Delta_4 + 90° + \Delta_5 - \Delta_6 = 180°$; or,

$$\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4 + \Delta_5 - \Delta_6 = 0 \ldots\ldots\ldots\ldots\ldots (8)$$

The vertical deflection of the right end of the beam with reference to the left end is zero; or,

$$l\, \theta - \frac{2}{3}\, l\, \Delta_3 + \frac{1}{3}\, l\, \Delta_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Since, $\theta = \Delta_1 - \Delta_2$, Equation (9) becomes:

$$\Delta_1 - \Delta_2 - \frac{2}{3}\, \Delta_3 + \frac{1}{3}\, \Delta_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Similarly,

$$\Delta_6 - \Delta_5 - \frac{2}{3}\, \Delta_4 + \frac{1}{3}\, \Delta_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Since the lateral deflections of the column tops are equal,

$$\frac{2}{3}\, \Delta_1\, h - \frac{1}{3}\, \Delta_2\, h = \frac{2}{3}\, \Delta_6\, h - \frac{1}{3}\, \Delta_5\, h$$

from which,

$$2\, \Delta_1 - \Delta_2 = 2\, \Delta_6 - \Delta_5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Since the sum of all column moments $= Ph$,

$$M_1 + M_2 + M_5 + M_6 = Ph$$

from which,

$$\Delta_1\, I_1 + \Delta_2\, I_1 + \Delta_5\, I_2 + \Delta_6\, I_2 = \frac{P\, h^2}{2} \ldots\ldots\ldots\ldots\ldots (13)$$

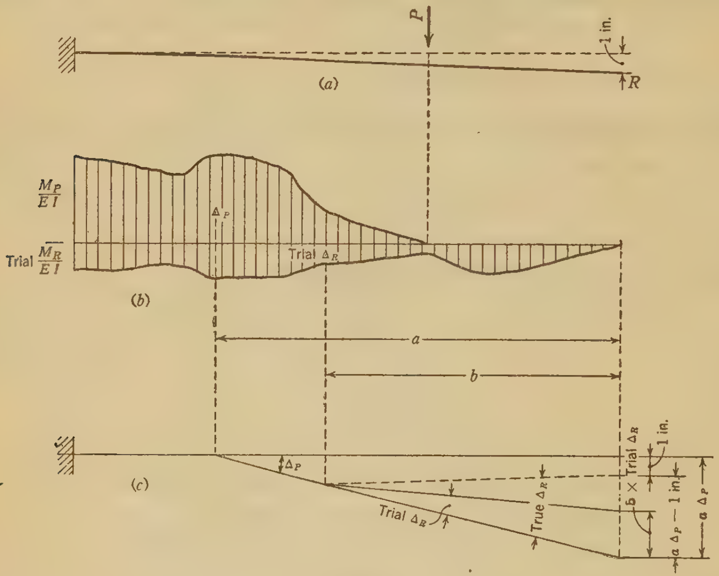
Substituting $\frac{\Delta_2}{k_1}$ for Δ_3 and $\frac{\Delta_5}{k_2}$ for Δ_4, in Equations (8) to (12), solving for Δ_2, and then substituting for Δ_2 its value, $\frac{M_2\, h}{2\, I_1}$:

$$M_2 = \frac{P\, h}{2} \times \frac{1}{1 + \frac{1}{3\, k_1} - \frac{1}{6\, k_2} + q\left(1 + \frac{1}{3\, k_2} - \frac{1}{6\, k_1}\right)} \ldots\ldots (14)$$

in which, $q = \dfrac{2 + k_1}{2 + k_2}$. If $I_1 = I_2$, then $k_1 = k_2$ and $q = 1$, and Equation (14) reduces to a much simpler form.

In practical problems of this type the work is greatly simplified, as the numerical values of constants, such as k_1, k_2, I_1, I_2, and $\dfrac{Ph^2}{2}$, are inserted in the initial equations.

The use of the traverse method for the variable moment of inertia and a settled support is illustrated by Fig. 5, which represents a beam fixed at one end and with a support which has settled 1 in. at the other. The moment of inertia and the modulus of elasticity may both be variable. From Maxwell's theorem, the final distortion of the beam from the combined action of the external forces may be treated as the sum of the distortions, due to each force acting separately.



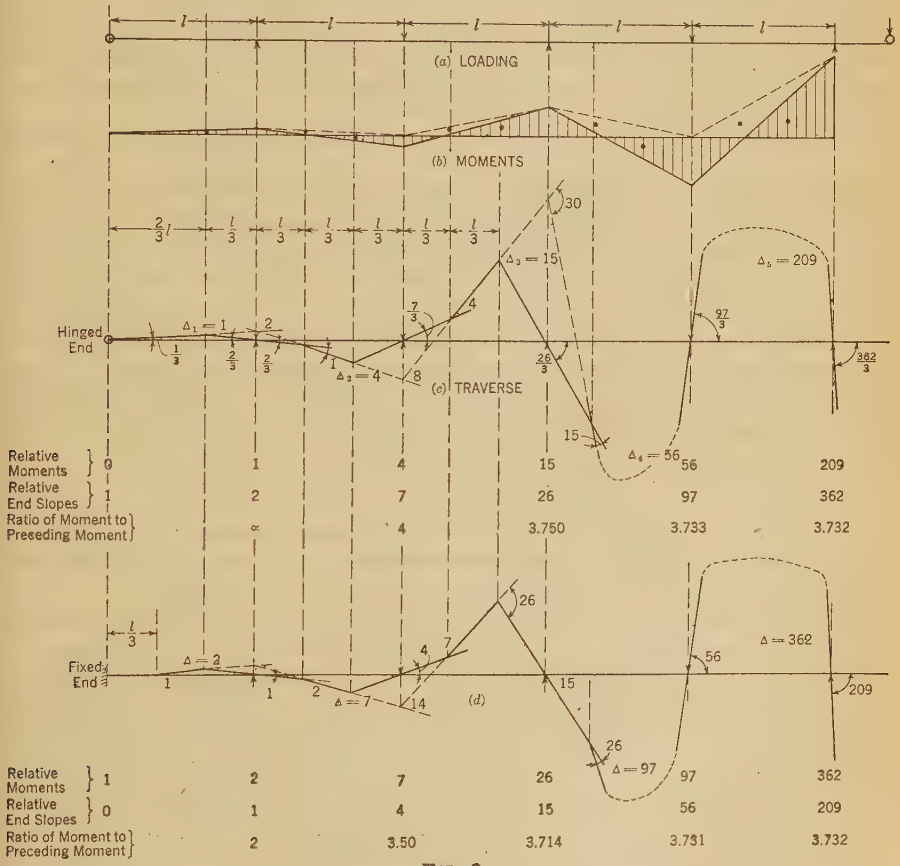
P
1 in.
(a)
R
$\frac{M_P}{EI}$
Δ_P
Trial $\frac{M_R}{EI}$
Trial Δ_R
(b)
a
b
(c)
Δ_P
Trial Δ_R
True Δ_R
$b \times$ Trial Δ_R
1 in.
1 in.
$a\,\Delta_P - 1$ in.
$a\,\Delta_P$

FIG. 5.

With the force, P, acting alone, the deflection of the right end of the beam would be $a\,\Delta_P$ (Fig. 5(c)), in which, Δ_P is the area of the $\dfrac{M}{EI}$ - diagram for the beam considered as a cantilever supporting the load, P, and a is the distance from the right end to the center of gravity of this $\dfrac{M}{EI}$ - area. Assume a trial reaction, R, compute the appurtenant $\dfrac{M}{EI}$ - diagram due to the cantilever uplift of the trial reaction acting alone, and locate its center of gravity, distant from the right end, as shown below the horizontal line in Fig. 5(*b*). The Trial Δ_R and its uplift at the end of the beam ($= b \times$ Trial Δ_R) can now be added to the traverse as shown in Fig. 5(*c*). The true value of R necessary to raise the right end of the beam to its settled position is now given by the proportion, $\dfrac{\text{True } R}{\text{Trial } R} = \dfrac{b\,\Delta_R - 1 \text{ in.}}{b \times \text{Trial } \Delta_R}$. With the value of R determined, the moments may be computed by statics.

TRAVERSE METHOD A SUBSTITUTE FOR THEOREM OF THREE MOMENTS

Fig. 6 represents a series of continuous equal spans with a moment at the right end. The traverse method furnishes a quick and easy solution for all the moments and all the slopes of the elastic curve over the supports by



l
(a) LOADING
(b) MOMENTS
2/3 l
l/3
Hinged End
Δ₁ = 1
2
1/3
2/3
Δ₂ = 4
8
7/3
4
Δ₃ = 15
30
26/3
15
Δ₄ = 56
56
97/3
Δ₅ = 209
362/3
(c) TRAVERSE
Relative Moments: 0, 1, 4, 15, 56, 209
Relative End Slopes: 1, 2, 7, 26, 97, 362
Ratio of Moment to Preceding Moment: ∝, 4, 3.750, 3.733, 3.732
Fixed End
Δ = 2
1
2
Δ = 7
14
4
7
26
15
Δ = 97
56
Δ = 362
209
(d)
Relative Moments: 1, 2, 7, 26, 97, 362
Relative End Slopes: 0, 1, 4, 15, 56, 209
Ratio of Moment to Preceding Moment: 2, 3.50, 3.714, 3.731, 3.732

FIG. 6.

the following construction: Beginning at the left end of Fig. 6(*c*) draw a traverse triangle for the flexure of the elastic curve of the left span. A low altitude for this triangle should be used, but no particular scale for the angles is necessary. Produce the long leg of the diagram to Point 2 over the second support and produce the short leg to the point marked 1 at the one-third span length of the second span. Draw Line 2-1-4-8. Through Point 4 draw Line 4-4 passing through the third support. Draw Line 8-4-15, etc. The result is a series of "pennant" diagrams. Each angle marked Δ is a measure of the moment at the support to its right. The position of Δ above or below the horizontal through the supports shows whether the moment is positive or negative. All angles in Fig. 6(*c*) can be evaluated quickly by simple summation and by determining the products of angles and horizontal distances. For example, to find Δ_2,

$$\frac{2}{3}(l) + 1\left(\frac{2}{3}l\right) - \Delta_2\left(\frac{1}{3}l\right) = 0$$

Therefore, $\Delta_2 = 4$.

Each line through a support is tangent to the elastic curve. Its angle with the horizontal is the measure of the slope of the elastic curve at the support and is obtained by summation. For example, the relative slope over the third support is, $\frac{2}{3} + 1 - 4 = -\frac{7}{3}$. In Fig. 6(*c*) the value of each angle is shown at the angle point. Signs are omitted as unnecessary because the picture of the flexure discloses the direction of moments. Below Fig. 6(*c*) are shown the relative magnitude of the moments at the supports, the relative magnitude of the end slopes, and the ratio of each moment to the preceding moment. Fig. 6(*d*) shows the traverse for the same system with the left end fixed. Note that the numerical series for moments in Fig. 6(*c*) is the same as that for end slopes in Fig. 6(*d*), and *vice versa*. The moment at any support is now known in terms of the moment at the right end. For example, in Fig. 6(*c*), if the right end moment is 3 000, the moment at the second support will be $\frac{1}{209} \times 3\,000$.

If the number of spans is infinite the ratio of each moment to the preceding moment will be 3.73205. The difference between this ratio and the ratios at the fourth support in Fig. 6 is negligible in practical design.

The application of the traverse method to unequal span lengths and loaded spans can be demonstrated by a problem solved previously[4] by L. H. Nishkian and D. B. Steinman, Members, Am. Soc. C. E. For the present purpose, Fig. 7 need not be, and is not, drawn to scale. Free-hand sketching is recommended in practical work. It is known that negative moments will occur over the supports. Sketch in the four moment triangles, Δ_1, Δ_2, Δ_3, and Δ_4, illustrating the existence of such moments. Draw simple moment diagrams for the loading of each span using either the upper lines of the negative moment triangles for bases, as shown, or, if desired, using the horizontal line through the supports as a base. It will make no difference in the computations which is used.

Draw verticals through the centers of gravity of the negative moment triangles (at the third points of the beam) and through the centers of gravity of the simple moment diagrams, which, in this case, are at the centers of the beam. Draw the T-lines (transposition lines[5]) as shown.

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[4] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 10, Fig. 11.

To construct the traverse, Fig. 7(c), draw Line $R_1\ A_1$, making a small angle with the unsprung beam. Draw Line $A_1\ \Delta_1$ which will pass over the second support and extend it to the intersection with the T-line at $\Delta_1 + \Delta_2$.

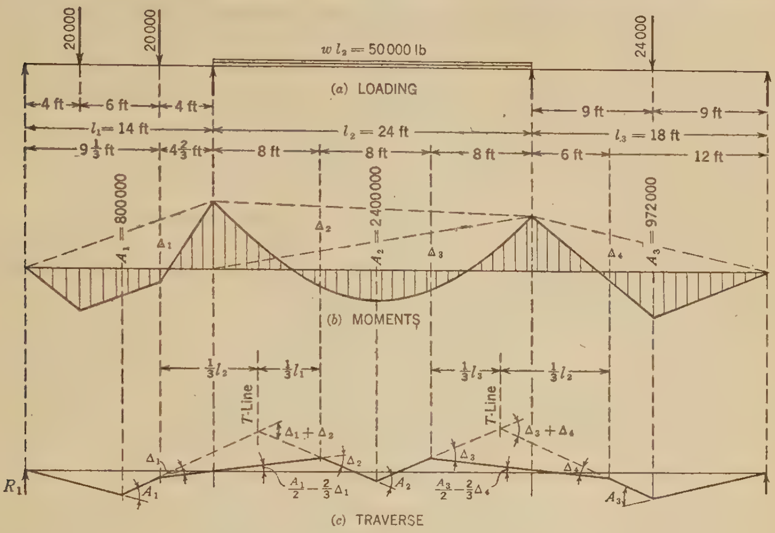
20000
20000
$w\,l_2 = 50000$ lb
24000
(a) LOADING
4 ft
6 ft
4 ft
9 ft
9 ft
$l_1 = 14$ ft
$l_2 = 24$ ft
$l_3 = 18$ ft
$9\frac{1}{3}$ ft
$4\frac{2}{3}$ ft
8 ft
8 ft
8 ft
6 ft
12 ft
$A_1 = 800000$
Δ_1
Δ_2
$A_2 = 2400000$
Δ_3
Δ_4
$A_3 = 972000$
(b) MOMENTS
$\frac{1}{3}l_2$
$\frac{1}{3}l_1$
$\frac{1}{3}l_3$
$\frac{1}{3}l_2$
T-Line
T-Line
$\Delta_1 + \Delta_2$
$\Delta_3 + \Delta_4$
R_1
Δ_1
A_1
Δ_2
$\frac{A_1}{2} - \frac{2}{3}\Delta_1$
A_2
$\frac{A_3}{2} - \frac{2}{3}\Delta_4$
Δ_3
Δ_4
A_3
(c) TRAVERSE

FIG. 7.

Draw Line $\Delta_1\ \Delta_2$ passing through the support and extended to the vertical through the one-third length of the second span. Draw the line from $\Delta_1 + \Delta_2$ through Δ_2 to the vertical through the center of the span at A_2. Draw Line $A_2\ \Delta_3$ to pass over the third support, intersecting the T-line at $\Delta_3 + \Delta_4$. Continue this line sequence to the right end of the beam. Fig. 7(c) can now be solved for the values of the Δ-angles by giving the A-angles values that equal the respective areas of the simple moment diagrams and observing that $\frac{\Delta_1}{\Delta_2} = \frac{l_1}{l_2}$ and $\frac{\Delta_4}{\Delta_3} = \frac{l_3}{l_2}$. Solving for these Δ-angles and reducing them to their corresponding moments, the moment over the second support is found to be 97 166 ft-lb and the moment over the third support, 92 666 ft-lb. The T-lines are on the centers of gravity of the combined abutting negative moment triangles and this is what accounts for their mechanical significance.

The traverse method can be used for spans of variable moment of inertia by the use of trial end moments to locate the Δ-points and to obtain the relative $\frac{M}{EI}$- areas for equal end moments. Tables introduced[6] by Walter Ruppel, Assoc. M. Am. Soc. C. E., will facilitate this work for the cases to which the tables apply. Tables that give $\frac{M}{EI}$ - areas for simple moments are unknown to the writer. Settlement of supports and side-sway can be incorporated in the geometry of a traverse.

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[5] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 8.

[6] *Loc. cit.*, pp. 167–187.

For the design of a member in a building of many stories and many bays, traverses in the form of Fig. 6 can be distributed vertically and horizontally to the columns and girders at the ends of the member. If the columns are flexible as compared with the girders they may be assumed to be fixed at floors above and below the joint in question, while the girders may be assumed to have a ratio of moment at the joint in question to moment at the joint one panel length distant, based on the factors indicated by Fig. 6 modified by an allowance for the joint stiffening effect of the columns. It will be found that a member can be designed quite satisfactorily by using moderately accurate judgment as to these moment ratios. If precision is desired and time is available the traverses may be extended two panel lengths, or more, from the member under consideration.

For the derivation of the basic formula for moments in single-span beams, fixed at one or at both ends, the traverse method offers alternate solutions which require a small fraction of the effort involved in the method of successive integrations found in textbooks on mechanics.

Conclusions

The analysis of continuous structures by traversing their elastic curves offers solutions that are easily understood and for which diagrams that clearly illustrate the problems can be easily drawn. The key constants for the analysis are simple $\frac{M}{EI}$ - and cantilever $\frac{M}{EI}$ - diagrams the construction of which is taught in the first lessons on beam stresses.

The method relieves the analyst from remembering or holding for reference the various forms of the three-moment equations or the slope-deflection equations. It requires no more analytical effort or labor of computation than these or other previously used methods.

DISCUSSION

Garrett B. Drummond,[7] Esq. (by letter).—The paper by Mr. Stewart is timely if for no other reason than that it emphasizes again what is the fundamental theory of indeterminate structures—the fact of continuity.

It is important to consider the limitations of the method as presented. The theory of continuity is based upon certain accepted assumptions: (1) It is assumed that the neutral axes of all members at a joint meet in a point; (2) distortions due to shear and direct thrust are negligible; (3) the relative rotation of the two ends of a member, for short lengths of the axis, is proportional to $\frac{l}{I}$ for the respective sections; and (4) the intensity and duration of the loading do not affect the elastic properties of the materials, thus permitting the assumption that for short lengths of beams the differential rotation of the two ends is directly proportional to the bending moment in the length considered.

These assumptions cause this method of analysis to be applicable in the strictest sense only to those materials which follow Hooke's law, and for steel only within the elastic limit. In structures of reinforced concrete or timber, Assumption (4) is not applicable since the ratio of stress intensity to deformation varies with both the intensity and the duration of stress. Assumption (3) does not apply to reinforced concrete.

However, it is probable that the assumption of a constant value of I in reinforced concrete beams will result in errors not exceeding 5 per cent. Such an error will not seriously affect the analysis of a reinforced concrete beam, although such limitations should be recognized.

For the determination of slopes and deflections, the writer finds it more convenient to utilize the method of stress areas. In this method, which obviates the assumption of the beam formula, the load is the area under the curve of fiber stress in the outer fibers, divided by $E\ c$, assuming $E\ c$ to be constant. The angular change of the tangents at any two points on the elastic curve is equal to the area between the two corresponding sections on the $\frac{f}{E\ c}$ diagram. The deflection at any point is equal to moment about that point, considering the beam to be loaded with the $\frac{f}{E\ c}$ diagram.

The method of stress areas is convenient when the fiber stress is known, and the slope and deflection are desired; or when the slope or deflection is known and the fiber stress is desired.

Fig. 8 represents a simply supported beam of constant cross-section, with a concentrated load at the center. The resulting curve of fiber stress will be as shown. The slope at the supports becomes $\frac{1}{4}\left(\frac{f\ l}{E\ c}\right)$, the same as the end

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[7] Asst. Prof. of Math., Oklahoma Agri. and Mech. Coll., Stillwater, Okla.

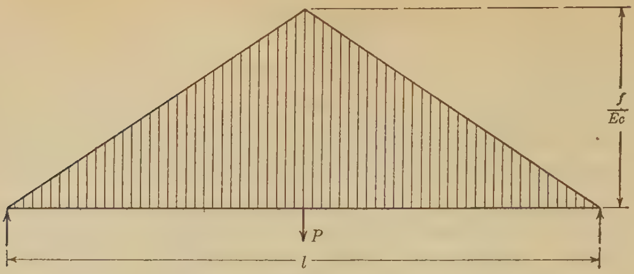

FIG. 8.

shear with $\frac{f}{E\,c}$ as a load. The deflection at the center is equal to the moment of this load at the center, or $\left(\frac{1}{12}\right)\left(\frac{f\,l^2}{E\,c}\right)$.

AUSTIN H. REEVES,[8] ASSOC. M. AM. SOC. C. E. (by letter).—A particularly valuable feature of the method described in this paper is that it provides a complete picture of the action of any loaded structure, no matter how complicated. However, some of the problems can be solved more speedily by other methods. For example, the writer solved the problem shown in Fig. 3 within 2 min, and the one shown in Fig. 4 within 5 min, by the conjugate point method.[9]

A check by the method presented[10] by Hardy Cross, M. Am. Soc. C. E., was made on both problems shown in Fig. 6 and also on the one shown in Fig. 7, and the results given by the author were found to be exact. The writer is in agreement with the "Conclusions."

This paper is a noteworthy contribution to the knowledge of rigid frame design.

E. G. PAULET,[11] JUN. AM. SOC. C. E. (by letter).—A geometrical solution for the analysis of continuous frames is presented in this paper, which is termed the "traverse method." In the "Synopsis", the author states that "memorized or copied slope-deflection equations are not used, and a series of rules for the signs of moments, rotations, and deflections are unnecessary." Notwithstanding that the three basic principles stated by the author would have to be remembered by any one unfamiliar with the moment-area method, the first principle, being one of the moment-area method, is sufficient to derive the fundamental slope-deflection equations at any time. Furthermore, the moment-area method furnishes the end moments of fully restrained beams subject to any type of loading, should one not have these values immediately available for use in conjunction with the slope-deflection equations.

When the slope-deflection method first came to the attention of the writer (whose mind, to that time, had been rigidly imbued with the "work methods" expounded by Europeans), he experienced some confusion at first, due to the signs of moments, rotations, and deflections. The solution of a few simple examples of frames ended the confusion and, to this day, the writer considers the slope-deflection method as the most forceful, next to the "work methods", for the rigorous analysis of rigid frames.

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[8] Newark, N. J.
[9] *Transactions*, Am. Soc. C. E., Vol. 90 (June, 1927), p. 1.
[10] *Loc. cit.*, Vol. 96 (1932), p. 1.
[11] Bridge Design Engr., State Highway Comm., Baton Rouge, La.

The author further states, following Equation (7):

> "As shown by Equation (6) the signs of the corner moments are opposite the signs of the center moment and column base moments.
>
> "The joint rotations do not enter into the solution, but the equations are formed from the curvature units in the members."

Equation (6) furnishes no criterion for the signs of the moments, because the signs of the Δ's depend upon the relative position of the bending moment diagrams, a position which can be chosen arbitrarily and erroneously. To visualize the elastic line of the deflected simple frame and write the Δ's with their proper signs in the equation-types, Equations (6) and (8), may be an easy task for the experienced engineer; however, a complex frame may present an untrue picture to him and leave him with a feeling of false assurance as to the correctness of the results.

In this respect, the traverse method has not as direct an approach to a problem as the slope-deflection method, the equations of which will furnish the correct signs and values of moments, rotations, and deflections, the signs being in accord with those adopted for deriving the equations.

The first and second principles of the traverse method will be helpful in finding the deflection of one end of a member, or series of members, relative to the other end, after the frame has been solved by the moment-distribution method.

Fig. 5 shows the application of the author's method to a beam of variable moment of inertia, and with a settled support. In the writer's opinion, problems of this kind are more directly solved by the moment-area method. Assuming that the correct arrangement of bending moment diagrams for a loaded continuous beam of uniform moment of inertia is established, the traverse method lends itself to a simple and rapid solution of such a problem.

In Fig. 6(*c*), the closing line of the "pennant" diagrams (Lines 2-1-4-8, 8-4-15-30, etc.) of the unloaded spans intersects the base line at a point of zero moment, through which (after the moments over the supports of the loaded span have been computed by the traverse method and platted to any convenient scale), the closing line of the bending moment diagrams for the unloaded spans may be drawn directly, and the moment at any point read to that scale.

From Fig. 6(*c*), it is noted with interest that the series for the relative moments and end slopes may also be found, beginning from the left end of the continuous beam, with the relations:

$$M_n = 2\,M_{n-1} + S_{n-1} \qquad \text{(15)}$$

and,

$$S_n = 3\,M_{n-1} + 2\,S_{n-1} \qquad \text{(16)}$$

in which, M_n = relative moments at the support under consideration; S_n = relative end slopes at that support; M_{n-1} and S_{n-1} = relative moments and end slopes, respectively, at the support preceding the one under consideration; and, at the first support, $M_1 = 0$, and $S_1 = 1$. For the case shown in Fig. 6(*d*), M reads S, and S reads M, in Equations (15) and (16).

Adolphus Mitchell,[12] Jun. Am. Soc. C. E. (by letter).—The method of analyzing continuous structures described in this paper, appears to hold no advantage over the slope-deflection method. On applying the two methods to multiple-story frames, one finds that whereas the slope-deflection method yields a simultaneous equation for each joint, the traverse method yields a simultaneous equation for each joint and member. The result is that an already large number of equations is doubled. Even in the simple problems solved by the author, the writer finds the slope-deflection method the easier.

Most designers are in the habit of assuming that the available foundation either offers no restraint or that it offers full restraint (fixed). As it is usually known that neither of these extremes is the case, the designer might wish to make some intermediate assumption. This can be done by comparing the geometry of the traverse for hinged and fixed conditions at End A of any member, AB. If f_{AB} is the percentage of fixation at End A:

$$\Delta_{BA} = \frac{2}{f}\,\Delta_{AB} \qquad \text{(17)}$$

$$\theta_{BA} = \frac{1}{6}\,(4 - f)\,\Delta_{BA} \qquad \text{(18)}$$

and,

$$\theta_{AB} = \frac{1}{3}\,(1 - f)\,\Delta_{BA} \qquad \text{(19)}$$

For a hinged terminal at A, $f = 0$, and for a fixed terminal at A, $f = 1$.

The author has solved the problem illustrated by Fig. 3, assuming the column terminals to be fixed. Suppose the column terminals are 50% fixed and that it is desired to determine the corner moment. Applying Equations (17), (18), and (19), $\Delta_2 = 4\,\Delta_1$ and $\theta_{21} = \frac{7}{12}\,\Delta_2 = \theta_{34}$. Since M_2 must be equal to M_3, $\Delta_2 = \Delta_3$. Equating ordinates at the right end of Beam 34:

$$\frac{7}{12}\,\Delta_2\,l + \Delta_2\left(\frac{2}{3}\,l\right) - \frac{Pl^2}{8\,EI}\left(\frac{l}{2}\right) + \Delta_2\left(\frac{l}{3}\right) = 0$$

Solving for Δ_2,

$$\Delta_2 = \frac{3\,Pl^2}{76\,EI} = \frac{M_2}{EI}\left(\frac{l}{2}\right)$$

and, hence,

$$M_2 = \frac{3}{38}\,Pl$$

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[12] Structural Designer, State Highway Dept., Santa Fé, N. Mex.

This idea of fixation or restraint is the basis of the method introduced by T. F. Hickerson,[13] M. Am. Soc. C. E., which the writer prefers to that proposed in this paper. In most of the papers on this subject complete tables are given permitting application to members of variable moment of inertia with relative ease. The author presents no such tables and, thereby, his method suffers another handicap.

David M. Wilson,[14] Assoc. M. Am. Soc. C. E. (by letter).—Methods of analyzing continuous structures which are fundamental to an understanding of this important subject may be divided into two classes: (1) The Maxwell-Mohr Method of Work; and, (2) Special Methods, such as (*a*) Moment-Area Method; (*b*) Slope-Deflection Method; and (*c*) Moment-Distribution Method (Cross Method).

The Maxwell-Mohr method of work is based upon the principle of the conservation of energy. It is the most general method available and may be used in all cases.

The special methods in the foregoing classification apply only in the analysis of straight structural members in which bending is the cause of the primary deformations, those due to shear and direct stress being neglected. They may be developed from the method of work, or they may be derived independently. The special methods have extensive application in practice because they are convenient to use. However, one should remember that, in many cases, deformations due to shear and direct stress cannot be neglected safely and, therefore, the results obtained by special methods will not be significant.

The author has presented a method of analyzing the moments in the members of continuous frames which is based directly upon the principles of the moment-area method. It is of special interest because it pictures the approximate deformations of the structure being studied.

In general, every structural engineer has his own favorite schemes of analysis. The method proposed will undoubtedly be of value to the designer who uses the moment-area method in preference to all other methods, whereever it is applicable.

The moment-distribution method is the most workable of the special methods because the solution of simultaneous equations is not required. Furthermore, the unknown moments are determined directly without first solving for rotation angles as required by the slope-deflection method, or for delta angles as required by the author's method.

In many cases, a combination of the special methods is desirable. For example, the fixed-end moments used in moment distribution may be determined conveniently by the application of the principles of area moments. At the same time, a thorough knowledge of slope deflection is of great value in understanding the steps in moment distribution.

In order to compare the proposed method with the moment-distribution method, the writer solved the problem shown in Fig. 7 by each method. Referring to Fig. 7(*c*), the following equations from the author's method may be written since the vertical deflection of the beam at any support with reference to an adjacent support is zero:

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[13] "Structural Frameworks", by T. F. Hickerson, Univ. of North Carolina Press, 1934.

[14] Associate Prof. of Civ. Eng., Univ. of Southern California, Los Angeles, Calif.

$$24\left(\frac{A_1}{2} - \tfrac{2}{3}\,\Delta_1\right) - 16\,\Delta_2 + 12\,A_2 - 8\,\Delta_3 = 0 \ldots\ldots\ldots\ldots (20)$$

and,

$$24\left(\frac{A_3}{2} - \tfrac{2}{3}\,\Delta_4\right) - 16\,\Delta_3 + 12\,A_2 - 8\,\Delta_2 = 0 \ldots\ldots\ldots\ldots (21)$$

Furthermore, from Fig. 7(*b*), $\Delta_1 = 7\,M_1$; $\Delta_2 = 12\,M_1$; $\Delta_3 = 12\,M_2$; and, $\Delta_4 = 9\,M_2$. Therefore, $\frac{\Delta_1}{\Delta_2} = \frac{7}{12}$, and $\frac{\Delta_4}{\Delta_3} = \frac{3}{4}$.

Eliminating Δ_1 from Equation (20) and Δ_4 from Equation (21); substituting for A_1, A_2, and A_3 their respective values; and solving the resulting equations: $\Delta_2 = 1\,164\,000$, and $\Delta_3 = 1\,112\,400$. Therefore, $M_1 = \frac{\Delta_2}{12} = 97\,000$ ft-lb, and $M_2 = \frac{\Delta_3}{12} = 92\,700$ ft-lb.

It is to be noted that the unknowns determined by solution of the simultaneous equations are not the desired moments. In a problem involving a large number of unknowns, the work necessary to solve the simultaneous equations would be prohibitive. On the other hand, the complete solution of an identical problem by the moment-distribution method is a relatively simple process,[15] in which the necessary work is approximately proportional to the number of unknowns.

There is no one simple method that can be applied in analyzing all continuous structures. Furthermore, no analysis is ever absolutely exact. Certain assumptions, based upon the elastic theory of structures, must be made before a problem can be solved by any of the available methods. The validity of these assumptions must be considered carefully in interpreting the results of the analysis. This requires sound engineering judgment at all times. The problem, therefore, is definitely one for the trained engineer with a comprehensive understanding of the behavior of structures under stress.

W. H. KIRKBRIDE,[16] M. AM. SOC. C. E. (by letter).—An interesting new method of stress analysis is introduced in this paper. The illustrations used by the author include portal frames the members of which meet at right angles.

Special methods, such as "slope deflection" and "end moment distribution," are also available for the analysis of such frames. The slope-deflection method, while in quite general use for rectangular frames, is not accepted as suitable for the analysis of frames having sloping tops. The end moment distribution method is also particularly adapted to rectangular frames. It does not permit of the derivation of a general formula in terms of symbols, which can be compiled into a handbook, and becomes less convenient when applied to frames the joints of which are deflected from their original position, due to the action of the loads.

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[15] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

[16] Chf. Engr., S. P. Co., Pacific Lines and Subsidiary Lines, San Francisco, Calif.

It will be interesting to know whether the traverse method is applicable to frames with sloping tops.

R. B. KETCHUM,[17] M. AM. SOC. C. E. (by letter).—The various methods of analyzing continuous structures have involved the work of distortion, or the geometry of distortion, and Mr. Stewart's paper is a distinct contribution to the geometrical method of attack.

The paper seems to have possibilities of expansion in a wide range of problems and would be a valuable check in graphical methods of drawing elastic curves and studying complicated loading conditions that have many unknowns, for which "cut-and-try" methods are used. The writer would be interested to know whether the author has applied this method to frames with sloping tops.

A. FLORIS,[18] ESQ. (by letter).—The methods for the analysis of statically indeterminate structures can be grouped into two main categories: Those in which forces are used as the redundants and those in which moments are utilized for the same purpose. They are termed the "force" and "deformation" methods, respectively.

The author's analysis is pre-eminently a deformation method with the moments at the corners and over the supports as the redundants. By replacing the deformed structure by the traverse (that is, by a set of straight lines), its flexure can be more readily visualized. The intersections of these lines determine the angles corresponding to the bending moment areas of simple beams and cantilevers divided by the stiffness factor, $E\,I$. This interpretation of the elastic curve has the additional advantage that many of the aforementioned angles can be determined by inspection. The seemingly great number of unknown angles is thus reduced considerably, and the solution of complicated problems is rendered more simple. The merit of the present analysis is based on this fact. As a further development of the traverse method, Mohr's theorem pertaining to rotations and deflections of loaded beams is utilized.

The traverse method is theoretically interesting and practically valuable. It is hoped, therefore, that the method will be used by engineers engaged in the design of statically indeterminate structures.

IVAN M. NELIDOV,[19] ASSOC. M. AM. SOC. C. E. (by letter).—This is a method by which moments in a continuous structure can be found with the aid of a traverse, the deflection angles of which correspond to $\frac{1}{E\,I}$ times the moment areas of the moments involved.

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[17] Dean, School of Mines and Eng., Univ. of Utah, Salt Lake City, Utah.

[18] Dipl.-Engr., Los Angeles, Calif.

[19] Senior Engr. of Hydr. Structure Design, State Dept. of Public Works, Sacramento, Calif.

The first consideration in analyzing a structure is the balance between the unknown factors and the available equations. If a structure consists of m members with j interior joints and s supports, the total number of unknowns is: $n = 5\,m$. The number of available equations is:

1.—m-equations of geometry of the type,

$$\Sigma\ (\theta + \Delta) = 0 \text{......................} (22)$$

2.—m-equations of geometry of the type,

$$\Sigma\ d = 0 \text{..............................} (23)$$

3.—$\Sigma\ (x - 1)\ j$-conditions of rigidity in the joints, of the type,

$$\theta_k = \theta_{k+1} \text{..........................} (24)$$

(in which, x is the number of members meeting at the joint).

4.—j-conditions of equilibrium in the joints, of the type,

$$\Sigma\ M_k = 0 \text{............................} (25)$$

5.—s-conditions at the supports, of the type,

$$\theta_s = a \text{..............................} (26)$$

or,

$$M_s = b \text{..............................} (27)$$

6.—A certain number of conditions of rotation at the joints, of the type,

$$\theta_k = c \text{..............................} (28)$$

7.—A certain number of conditions of deflection at the joints, of the type,

$$d_k = d \text{..............................} (29)$$

8.—A certain number of conditions resulting from the symmetry of load and the structure.

Conditions 1 to 8 reduce the original number of the unknowns, but since there are more conditions ordinarily than equations, the deficient number of equations should be supplied from the conditions of the equilibrium of the external and the internal forces.

As a simple illustration of the application of Mr. Stewart's method and of Conditions 1 to 8, the following example is proposed: Find the moments in the frame shown in Fig. 9, which is symmetrical about its center line and symmetrically loaded by a concentrated load at the apex. It will be seen that $m = 4$, $j = 3$, and $s = 2$. The total number of unknowns is $n = 5\,m = 20$, which is reduced, due to symmetry of the frame and loading, to 10. The

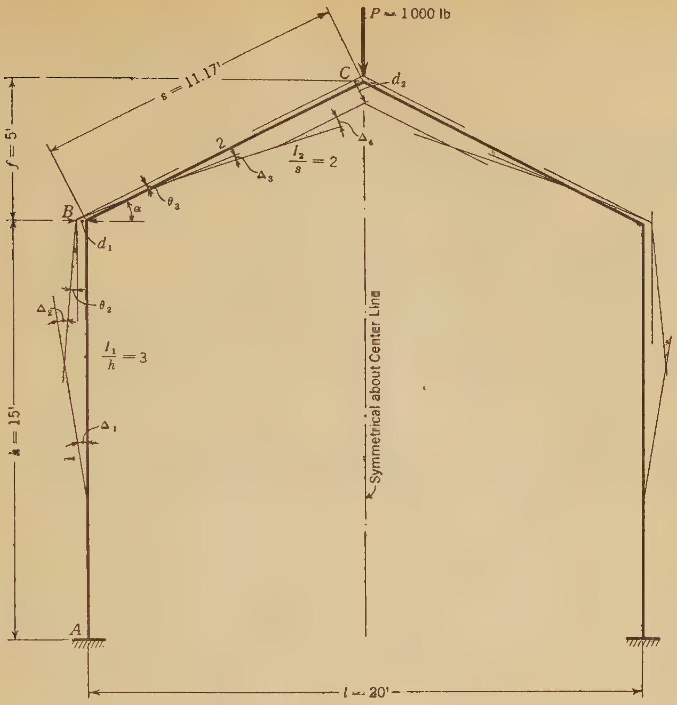

P = 1000 lb
C
d_2
s = 11.17'
f = 5'
2
$\frac{I_2}{s} = 2$
Δ_4
Δ_3
θ_3
α
B
d_1
θ_2
Δ_2
$\frac{I_1}{h} = 3$
Δ_1
1
h = 15'
A
Symmetrical about Center Line
l = 20'

FIG. 9.

available formulas are: Two of the type of Equation (22),

$$\theta_1 + \Delta_1 + \Delta_2 + \theta_2 = 0 \quad \ldots\ldots (30)$$

and,

$$\theta_3 + \Delta_3 + \Delta_4 + \theta_4 = 0 \quad \ldots\ldots (31)$$

two of the type of Equation (23),

$$\theta_1 h + \frac{2}{3} h \Delta_1 + \frac{h}{3} \Delta_2 + d_1 = 0 \quad \ldots\ldots (32)$$

and,

$$\theta_3 s + \frac{2}{3} s \Delta_3 + \frac{s}{3} \Delta_4 + d_2 = 0 \quad \ldots\ldots (33)$$

one of the type of Equation (24),

$$\theta_3 = - \theta_2 \quad \ldots\ldots (34)$$

one of the type of Equation (25),

$$\Delta_3 = C_3 \Delta_2 \quad \ldots\ldots (35)$$

one of the type of Equation (26),

$$\theta_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

one of the type of Equation (28),

$$\theta_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

and, one of the type of Equation (29),

$$d_2 = K_2 \, d_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

in which, $C_3 = \frac{I_1 \, s}{h I_2}$ and $K_2 = - \frac{1}{\sin \alpha}$.

One more equation should be added to Equations (30) to (38) and this results from the condition of equilibrium of the joint, B, as follows:

Since $\Sigma \, X = 0$:

$$- T_2 \cos \alpha + V''_1 + V'_2 \sin \alpha = 0 \ldots\ldots\ldots\ldots (39)$$

and $\Sigma \, Y = 0$:

$$T_2 \sin \alpha + V'_2 \cos \alpha - \frac{P}{2} = 0 \ldots\ldots\ldots\ldots (40)$$

In Equations (39) and (40) T_2 is the normal stress in Member 2; V' is the shear at the left support, and V'' is the shear at the right support, with the subscript corresponding to the number of the member. After T_2 has been eliminated the tenth equation is obtained:

$$V''_1 \sin \alpha + V'_2 - \frac{P}{2} \cos \alpha = 0 \ldots\ldots\ldots\ldots (41)$$

In order to be able to use Equation (41), the shears, V, should be expressed in terms of Δ:

$$V''_1 = \frac{M'_1 + M''_1}{h} = \frac{2 \, E I_1}{h^2} (\Delta_1 - \Delta_2) \ldots\ldots\ldots (42)$$

and,

$$V'_2 = - \frac{M'_2 + M''_2}{s} = - \frac{2 \, E I_2}{s^2} (\Delta_3 - \Delta_4) \ldots\ldots\ldots (43)$$

Reaction shears and reaction moments are used, with a clockwise action as positive. After substituting the expressions for V the last of the necessary equations is obtained, as follows:

$$\frac{2 \, E I_1}{h^2} (\Delta_1 - \Delta_2) - \frac{2 \, E I_2}{s^2} (\Delta_3 - \Delta_4) - \frac{P}{2} \cos \alpha = 0 \ldots\ldots (44)$$

Solving the remaining five equations (Equations (30) to (33)) and Equation (44), the following expression is obtained for d_1:

$$d_1 = \frac{\dfrac{P}{6\,E} \cos\alpha\,(1 + C_3)\,h s^2}{-\dfrac{I_1\,s}{h}\left[\dfrac{3}{\sin\alpha} + \dfrac{s}{h}\,(4 + C_3)\right]\sin\alpha - \dfrac{I_2\,h}{s}\left[3\,C_3\dfrac{s}{h} + \dfrac{1 + C_3}{\sin\alpha}\right]} \quad \ldots (45)$$

Furthermore:

$$\theta_2 = d_1\left(-\frac{1}{\sin\alpha} + s\,C_3\,\frac{1}{h}\right)\frac{1}{\frac{2}{3}s\,(1 + C_3)} \quad \ldots\ldots\ldots\ldots (46)$$

$$\Delta_4 = \theta_2\,(1 + 2\,C_3) - 3\,C_3\frac{d_1}{h} \quad \ldots\ldots\ldots\ldots (47)$$

$$\Delta_2 = 3\,\frac{d_1}{h} - 2\,\theta_2 \quad \ldots\ldots\ldots\ldots (48)$$

and,

$$\Delta_1 = -\,\Delta_2 - \theta_2 \quad \ldots\ldots\ldots\ldots (49)$$

Substituting numerical values in Equations (45) to (49): $d_1 = -\dfrac{1}{E}\,954.199$; $d_2 = \dfrac{1}{E}\,2\,132.289$; $\theta_2 = \dfrac{1}{E}\,57.252$; $\theta_3 = -\dfrac{1}{E}\,57.252$; $\Delta_1 = \dfrac{1}{E}\,248.092$; $\Delta_2 = -\dfrac{1}{E}\,305.343$; $\Delta_3 = -\dfrac{1}{E}\,458.015$; and, $\Delta_4 = \dfrac{1}{E}\,515.267$.

The corresponding reaction moments will be:

$$M'_1 = \frac{2\,E\,I_1\,\Delta_1}{h} = \frac{2\,E\,3 \times 1}{E}\,248.092 = 1\,488 \text{ ft-lb} = M_a$$

$$M''_1 = -\,\frac{2\,E\,I_1\,\Delta_2}{h} = -\,\frac{2\,E\,3 \times 1}{E}\,(-\,305.343) = 1\,832 \text{ ft-lb} = M_b$$

$$M'_2 = \frac{2\,E\,I_2\,\Delta_3}{s} = \frac{2\,E\,2 \times 1}{E}\,(-\,458.015) = -\,1\,832 \text{ ft-lb} = M_b$$

and,

$$M''_2 = -\,\frac{2\,E\,I_2\,\Delta_4}{s} = -\,\frac{2\,E\,2 \times 1}{E}\,515.267 = -\,2\,061 \text{ ft-lb} = M_c$$

The foregoing example demonstrates that Mr. Stewart's method has unquestionable advantages when the supports are prevented from being displaced laterally or when these displacements are symmetrical. On the other hand, when these conditions are not fulfilled, the method becomes as complicated as the slope-deflection method, if not more so.

The writer also wishes to note the importance of revealing physical manifestation of any evolved theory, in order to simplify its understanding. The author did this in a clear and concise manner in another paper[20] when he demonstrated the similarity between the elastic line and a railway transition spiral; and also when he mentioned that the concept of the method is the product of a slope and a distance, rather than the moment of a quantity times its lever arm. These explanations appeared to the writer as being very helpful in his attempt to grasp the method.

Fang-Yin Tsai,[21] Assoc. M. Am. Soc. C. E. (by letter).—The idea behind this method is certainly novel and ingenious. However, the writer does not consider that it furnishes "a quick and easy solution" for continuous structures, as claimed by the author.

In illustrating the application of the method, Mr. Stewart has intentionally selected several unusually simple cases most of which are rarely, if ever, met in actual practice. Undoubtedly, this was done merely to demonstrate the simplicity of the procedure, but it creates the erroneous impression that the same simple procedure can be applied also to any continuous structure in general. For instance, the simple solution of the two-legged bent in Fig. 3 does not apply when the beam and columns do not have the same length, or when the loading is not symmetrical. It may be also noted that the solution for the two-legged bent in Fig. 4 is somewhat lengthy, and does not seem as simple and direct as that by the slope-deflection method. Even for the extremely simple case of the continuous beam (see Fig. 6), the author's claim that the solution for all moments is quick and easy, does not seem to be justified. For a simple case, such as this one, the solution by the graphical method of fixed points, or conjugate points, will be found to be the quickest and easiest of all.

The method illustrated in Fig. 5 evidently applies only for the particular case of yielding supports and its application to continuous structures will be found very difficult if not impossible. Although Mr. Stewart states that "settlement of supports * * * can be incorporated in the geometry of a traverse", he has not explained how this can be done. This problem could be solved readily by Equation (3) and Fig. 2, from which it is seen that the traverse must also pass through the supports after they have settled. For illustration, the writer has solved the problem in Fig. 7, assuming that Supports B and C have settled by an amount of $d_1 = 0.2$ in. and $d_2 = 0.1$ in., respectively, relative to the elevation of Supports A and D. An 18-in 47-lb steel I-beam is assumed, with a moment of inertia, $I = 768.6$ in.^{4}, and a modulus of elasticity, $E = 30\,000\,000$ lb per sq in. The traverse is shown in Fig. 10 (c) for which the equation is,

$$\theta_1 - A_1 + \Delta_1 + \Delta_2 - A_2 + \Delta_3 + \Delta_4 - A_3 + \theta_4 = 0 \ldots\ldots (50)$$

---

[20] "Improved Method of Finding Beam Deflections", by Ralph W. Stewart, M. Am. Soc. C. E., *Civil Engineering*, February, 1934.

[21] Prof. of Structural Eng., Dept. of Civ. Eng., National Tsing Hua Univ., Peiping, China.

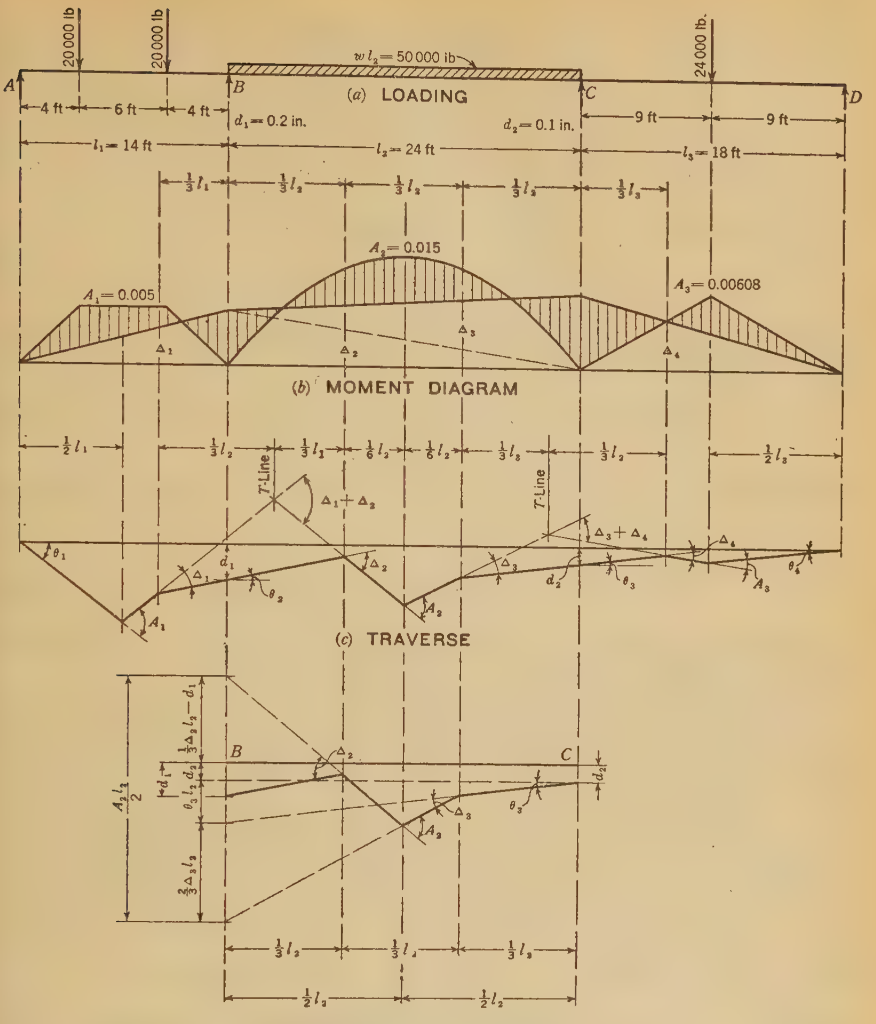

20 000 lb
20 000 lb
$wl_2 = 50\,000$ lb
24 000 lb
A
B
C
D
(a) LOADING
4 ft
6 ft
4 ft
$d_1 = 0.2$ in.
$d_2 = 0.1$ in.
9 ft
9 ft
$l_1 = 14$ ft
$l_2 = 24$ ft
$l_3 = 18$ ft
$\frac{1}{3}l_1$
$\frac{1}{3}l_2$
$\frac{1}{3}l_3$
$A_1 = 0.005$
$A_2 = 0.015$
$A_3 = 0.00608$
Δ_1
Δ_2
Δ_3
Δ_4
(b) MOMENT DIAGRAM
$\frac{1}{2}l_1$
$\frac{1}{6}l_2$
$\frac{1}{2}l_3$
T-Line
$\Delta_1 + \Delta_2$
$\Delta_3 + \Delta_4$
θ_1
θ_2
θ_3
θ_4
d_1
d_2
A_1
A_2
A_3
(c) TRAVERSE
$\frac{A_2 l_2}{2}$
$\frac{1}{3}\Delta_2 l_2 - d_1$
d_1
d_2
$\theta_3 l_2$
$\frac{2}{3}\Delta_3 l_3$
$\frac{1}{2}l_2$
(d) REFERRING TO EQUATION (56)

FIG. 10.

From the geometry of the moment diagram and the traverse, the following relations can be found:

$$\Delta_2 = \frac{l_2}{l_1}\,\Delta_1 \qquad (51)$$

$$\Delta_3 = \frac{l_2}{l_3}\,\Delta_4 \qquad (52)$$

$$\theta_1 = \frac{A_1}{2} - \frac{\Delta_1}{3} + \frac{d_1}{l_1} \qquad (53)$$

and,

$$\theta_4 = \frac{A_3}{2} - \frac{\Delta_4}{3} + \frac{d_2}{l_3} \dots\dots\dots\dots\dots\dots\dots\dots (54)$$

Substituting Equations (51) to (54) and the various numerical values (noting that areas of the $\frac{M}{EI}$-diagrams due to loading are: $A_1 = 0.005$, $A_2 = 0.015$, and $A_3 = 0.00608$) in Equation (50),

$$-0.1889 + 2.381\ \Delta_1 + 2\ \Delta_4 = 0 \dots\dots\dots\dots\dots\dots (55)$$

Another equation may be obtained from the geometry of the traverse in the center span (Fig. 10 (*d*)) as follows:

$$\frac{2}{3}\ \Delta_3\, l_2 + \theta_3\, l_2 + d_2 + \frac{1}{3}\ \Delta_2\, l_2 - d_1 - \frac{1}{2}\, A_2\, l_2 = 0 \dots\dots\dots (56)$$

Noting that $\theta_3 = -\frac{A_3}{2} + \frac{2\ \Delta_4}{3} + \frac{d_2}{l_3}$, the following equation is obtained after proper substitution:

$$-0.1043 + 0.571\ \Delta_1 + 1.556\ \Delta_4 = 0 \dots\dots\dots\dots\dots\dots (57)$$

Solving Equations (55) and (57) simultaneously, $\Delta_1 = 0.00331$; $M_B = -76\,000$ ft-lb; $\Delta_4 = 0.00551$; and $M_C = -98\,000$ ft-lb.

The writer has also solved the same problem by the equation of three moments and the slope-deflection method, both of which are found much more simple than the method proposed in this paper. This seems quite natural since, in applying the method, the equations must be formed from the geometry of the traverse instead of by writing the slope-deflection or three-moment equations directly, to which the author objects. The writer would appreciate a suggestion for simplifying the solution of the foregoing problem.

In the "Conclusions" of the paper, Mr. Stewart states that the key constants of the analysis are simple $\frac{M}{EI}$-diagrams and cantilever $\frac{M}{EI}$-diagrams. It is evident that he has neglected entirely the centers of gravity of the diagrams, which are absolutely necessary, as emphasized by the author himself (see Principle (2)). This oversight is natural since the loading in all his illustrations is symmetrical for every span and, consequently, all the centers of gravity of the simple moment diagrams are known to be at the center of every span.

In discussing the application of the method to the case of variable moment of inertia, Mr. Stewart mentions the tables presented by Walter Ruppel,[*] Assoc. M. Am. Soc. C. E., and states that lists of $\frac{M}{EI}$-areas for simple moments are unknown to him. The simple moment areas for various loading, as well as their centers of gravity, are given in Mr. Ruppel's tables. Elsewhere,[22] the writer has derived the relations between the various $\frac{M}{EI}$ - diagrams and the coefficients in Ruppel's tables, as follows (see Fig. 11):

$$A_L = \frac{pl}{I'} \qquad (58)$$

$$A_R = \frac{ql}{I'} \qquad (59)$$

$$A_0 = \frac{Wl^2}{I'}(sp + tq) \qquad (60)$$

$$u = u \qquad (61)$$

$$v = v \qquad (62)$$

and,

$$g = \frac{tq}{sp + tq} \qquad (63)$$

in which, A_L = area of $\frac{M}{I}$ - diagram due to a moment of unity applied at the left end of a simple beam; A_R = area of $\frac{M}{I}$ - diagram due to a moment of unity applied at the right end of a simple beam; A_0 = area of $\frac{M}{I}$ - diagram due to loading on a simple beam; ul = abscissa of the center of gravity of A_L from the left support, L, of a simple beam; vl = abscissa of the center of gravity of A_R from the right support, R, of a simple beam; gl = abscissa of the center of gravity of A_0 from the left support, L, of a simple beam; I' = minimum moment of inertia of beam; W = total load on span; l = span length; p, q, u, and v = beam coefficients in Ruppel's tables, which depend on the shape of the beam only; and s and t = load coefficients in Ruppel's tables, which depend on both the shape of the beam and the type of loading.

In Equations (58) to (63), the modulus of elasticity, E, has been omitted since it is always assumed to be constant for the entire structure. It may be noted that, for a beam with a moment of inertia varying in any manner, the following relations are always valid:

$$pu = qv \qquad (64)$$

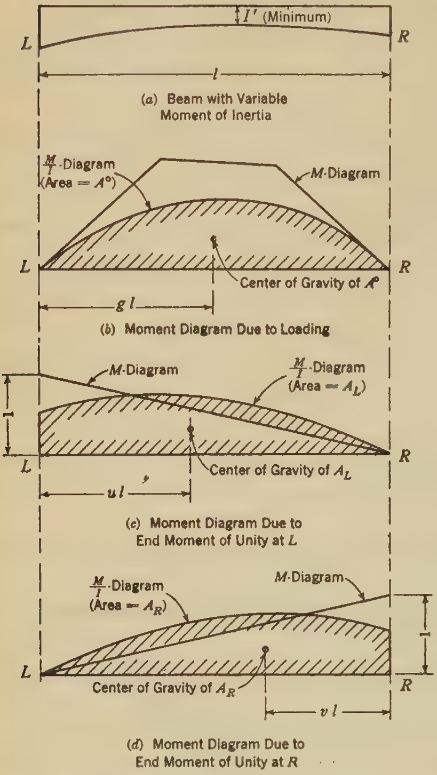
and,

$$A_L u = A_R v \qquad (65)$$

---

[22] "Theorem of Three Moments in General Form," The Science Repts, National Tsing Hua Univ., Peiping, China, Series A, Vol. II, pp. 19–36, April, 1933.

which follow directly from Maxwell's theorem of reciprocal deflections (angular). Therefore, to analyze, by any method, a continuous structure with a moment of inertia varying in any manner, five independent constants or coefficients must be known (three beam coefficients and two load coefficients) for every span of the structure considered as a simply supported beam; and these five coefficients may be expressed in various ways to suit any particular method of analysis. With the aid of Equations (58) to (63), Ruppel's tables can be utilized in applying the method to the case of variable moment of inertia.


I' (Minimum)
L
R
l
$\frac{M}{I}$-Diagram (Area = $A°$)
M-Diagram
L
R
Center of Gravity of $A°$
$g\,l$
M-Diagram
$\frac{M}{I}$-Diagram (Area = A_L)
1
L
R
Center of Gravity of A_L
$u\,l$
M-Diagram
$\frac{M}{I}$-Diagram (Area = A_R)
1
L
R
Center of Gravity of A_R
$v\,l$

(a) Beam with Variable Moment of Inertia

(b) Moment Diagram Due to Loading

(c) Moment Diagram Due to End Moment of Unity at L

(d) Moment Diagram Due to End Moment of Unity at R

FIG. 11.

Mr. Stewart also states that,

"From Maxwell's theorem, the final distortion of the beam from the combined action of the external forces may be treated as the sum of the distortions, due to each force acting separately."

As far as the writer knows, the statement is just one aspect of the general principle of superposition, which has not yet been attributed to Maxwell. The writer has also searched in the two volumes of "Maxwell's Scientific Papers"[23] and has failed to find anything like the theorem explicitly stated therein.

RALPH W. STEWART,[24] M. AM. SOC. C. E. (by letter).—The discussion has brought up criticisms and questions relating to specific items not satisfactorily covered in the paper. It has also raised the broader question as to whether the traverse method is a special method of limited scope, or whether it is a general method which may be used for the analysis of structures for which slope deflection and end-moment distribution are not suitable. To answer the latter query it will be necessary first to show its application to some cases of single-span beams.

Figs. 12(*a*), 12(*b*), and 12(*c*) show the simple moment areas and the positions of their centers of gravity for all conditions of normal loading; Figs. 12(*d*), 12(*e*), 12(*f*), 12(*g*), and 12(*h*) show traverse diagrams for beams of constant cross-section having various conditions of end restraint. The double hatching indicates a fixed end; the single hatching a restrained (but not fixed) end; and, absence of hatching, a hinged end. In the following equations, A is the area of the simple $\frac{M}{EI}$-diagram and Δ, the area of a triangular $\frac{M}{EI}$-diagram (not shown), with a base extending the full length of the beam and an altitude which is the end moment due to restraint, divided by E and I. If the beam has a constant cross-section and is of the same material throughout, and movements of supports are not involved, both E and I may be omitted, and the moment areas used instead of the $\frac{M}{EI}$-areas.

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[23] Pub. by Librairie Scientifique J. Hermann, Paris, by arrangement with the Cambridge Univ. Press, 1890.

[24] Engr. of Structural Design, City of Los Angeles, Los Angeles, Calif.

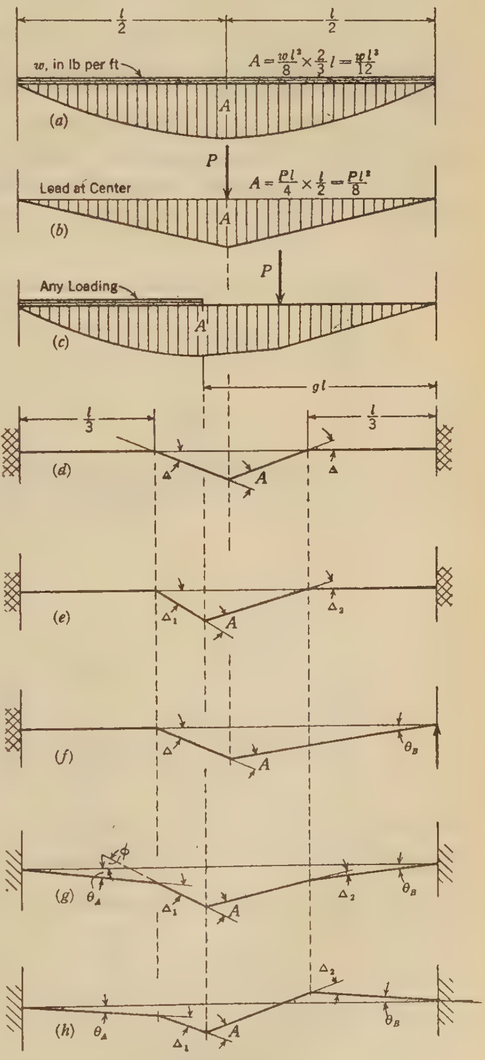
$\frac{l}{2}$
$\frac{l}{2}$
w, in lb per ft
$A=\frac{wl^2}{8}\times\frac{2}{3}l=\frac{wl^3}{12}$
A
(a)
P
Load at Center
$A=\frac{Pl}{4}\times\frac{l}{2}=\frac{Pl^2}{8}$
A
(b)
P
Any Loading
A
(c)
gl
$\frac{l}{3}$
$\frac{l}{3}$
(d)
Δ
A
Δ
(e)
Δ_1
A
Δ_2
(f)
Δ
A
θ_B
φ
(g)
θ_A
Δ_1
A
Δ_2
θ_B
Δ_2
(h)
θ_A
Δ_1
A
θ_B

FIG. 12.

For these beams of constant section all Δ-angles in the traverse are at the one-third length points and $\Delta = \frac{Ml}{2}$, in which M is the appurtenant end moment.

For cases in which a traverse forms a single triangle, it is often more convenient to use the angle relationships of the triangle for obtaining values of the unknowns than to write a traverse equation. In Fig. 12(*d*), the isosceles triangle formed by the traverse gives $\Delta = \frac{1}{2} A$, and from this, $\frac{Ml}{2} = \frac{wl^3}{24}$ for uniform loading, or M (the end moment) $= \frac{1}{12} wl^2$. In Fig. 12(*e*), $\Delta_1 \left(\frac{l}{3}\right) = A\left(g\,l - \frac{l}{3}\right)$; and $\Delta_1 = \frac{M_1 l}{2}$. Therefore, $M_1 = \frac{2A}{l}(3g - 1)$. In Fig. 12(*f*), $\Delta\left(\frac{2}{3}l\right) = A\,g\,l$; and, $\Delta = \frac{M_1 l}{2}$. Therefore, $M_1 = \frac{3Ag}{l}$. If the load is as in Fig. 12(*b*), then $g = \frac{1}{2}$ and $A = \frac{Pl^2}{8}$, giving $M_1 = \frac{3}{16} Pl$. In Fig. 12(*g*) (assuming counter-clockwise rotations to be negative, as in slope deflection), for symmetrical loading from left to right:

$$\theta_A + \tfrac{2}{3}\,\Delta_1 - \tfrac{1}{2}\,A + \tfrac{1}{3}\,\Delta_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

and right to left,

$$-\,\theta_B - \tfrac{2}{3}\,\Delta_2 + \tfrac{1}{2}\,A - \tfrac{1}{3}\,\Delta_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (67)$$

If the loading is unsymmetrical, use $A\,g$ instead of $\frac{1}{2}\,A$ in Equation (66) and $A(1 - g)$ in Equation (67). Solving these equations for Δ_1 and Δ_2, and substituting the M-values of the Δ-angles, standard slope deflection equations are obtained. When the beam and loading are symmetrical, $\phi = \dfrac{A}{2}$ and may often be used to advantage in solutions. In formulas such as Equations (66) and (67) only the coefficients of l are used as l, if written in each term, will be eliminated.

In Fig. 12(h) a dominating moment in an adjoining span has reversed the direction of the end slope, θ_B. This will be discussed subsequently.

Fig. 13 illustrates the traverse of a beam of variable section and complete dissymmetry having fixed ends. All dimensions and properties of this beam have been presented[25] by George E. Large, Assoc. M. Am. Soc. C. E., including the computations for the A and Δ locations and magnitudes.

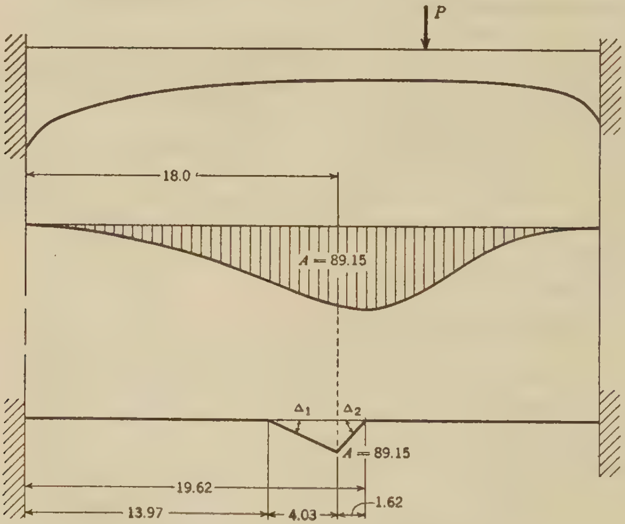

FIG. 13.

From the triangle in the traverse: $\Delta_1 = \dfrac{1.62}{5.65} \times 89.15 = 25.56$; $M_1 = \dfrac{25.56}{9.12} \times 10 = 28.00$; $\Delta_2 = \dfrac{4.03}{5.65} \times 89.15 = 63.59$; and $M_2 = \dfrac{63.59}{12.27} \times 10$

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[25] "The Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 102; also, *Bulletin No. 66*, Ohio State Univ.

= 51.83. This solution gives the fixed-end moments by a simpler procedure, involving less computation, than the previously published solution.[26] Problems of the fixed-end moments in restrained beams cannot be solved by slope deflection or by end moment distribution. Their solution as herein demonstrated shows that the traverse method is applicable to fundamentals.

Frequently, in a continuous beam, the direction of rotation at a support cannot be forecast by inspection, and the question arises as to how to apply the traverse. This condition is illustrated by Fig. 14, in which the slopes over the intermediate supports are purposely drawn in the wrong direction. The procedure for drawing the traverse lines is described in connection with Fig. 7. Before the sketch is completed, it announces the error in assumption by the disproportionate sizes of the A-angles. Disregarding this, now obvious, error in the diagram, write the traverse equation from left to right and from right to left for the full length of the beam; combine the numerical terms and the coefficients of the respective Δ-angles; and obtain: $28\,\Delta_1 + 30\,\Delta_2 = 2\,160$; and, $-12\Delta_1 - 26\,\Delta_2 = -1\,620$. (The signs in these equations follow the rules for slope deflection.) Solving, $M_B = 6.85$ and $M_C = 17.61$. Continuing the solution it is found that $\theta_B = -\dfrac{8.80}{EI}$ and

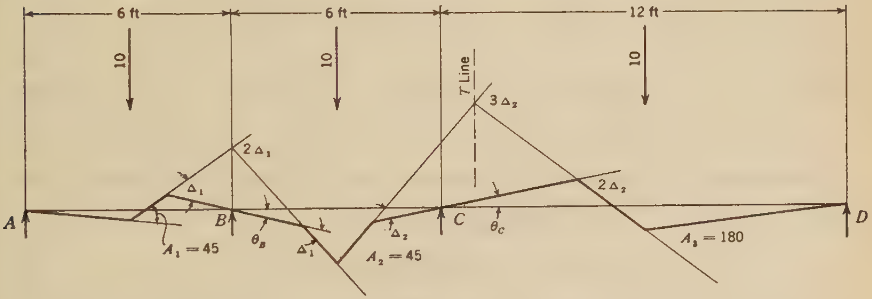

FIG. 14.

$\theta_C = +\dfrac{19.57}{E\,I}$. The solution was not affected by the error in the diagram. Attention is invited to the fact that a mechanical solution of this problem could be obtained by drawing the A-angles to a suitable scale on separate slips of tracing paper and manipulating them with their legs intersecting on the T-lines until proper closure is obtained. The result would be rather rough, but probably more accurate than that obtained by using the arbitrary coefficients of various building ordinances.

The fact that a computed traverse can be checked quickly by plotting, and that errors in assumed direction of joint rotations are disclosed while the traverse is being drawn, is an advantage of no small importance, which is not available with other methods.

The discussion of Mr. Nelidov shows comprehensive understanding of the traverse method and also shows that it can be applied to frames with sloping roof members. In order to do the method justice, however, it will be necessary to show a shorter and simpler solution of the problem that he presented. Fig. 15 shows (in its deflected position) the same frame and loading as Fig. 9, the right half indicating the elastic curvature and the left half, the traverse diagram. The value of Δ_3 results from the relative stiffness factors of the column and the rafter. The value of Δ_4 is obtained by summation of angles from Δ_1 upward.

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[26] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 105, Equations (86) and (87).

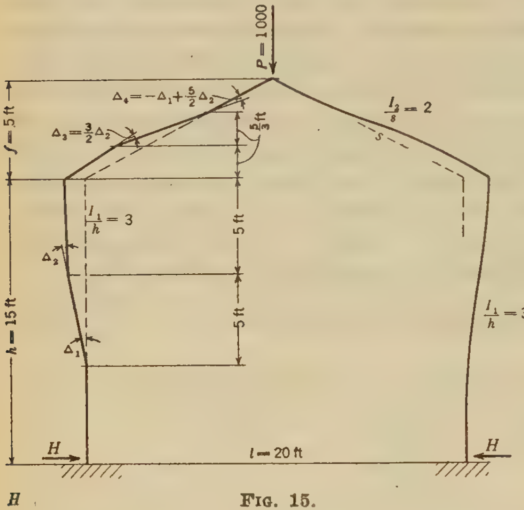
P = 1000
$\Delta_4 = -\Delta_1 + \frac{5}{2}\Delta_2$
$\Delta_3 = \frac{3}{2}\Delta_2$
$\frac{I_2}{s} = 2$
$\frac{5}{3}$ ft
s
f = 5 ft
$\frac{I_1}{h} = 3$
5 ft
5 ft
Δ_2
Δ_1
h = 15 ft
$\frac{I_1}{h} = 3$
H
l = 20 ft
H
H

FIG. 15.

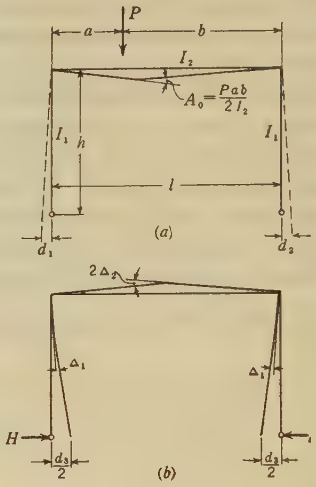
P
a
b
I_2
$A_0 = \frac{Pab}{2I_2}$
I_1
h
I_1
l
(a)
d_1
d_2
$2\Delta_2$
Δ_1
Δ_1
H
$\frac{d_3}{2}$
$\frac{d_3}{2}$
(b)

FIG. 16.

Beginning at Δ_1 the following traverse equation for the horizontal movements of Δ_4 may now be written:

$$-13\tfrac{1}{3}\ \Delta_1 + 8\tfrac{1}{3}\ \Delta_2 + 2\tfrac{1}{2}\ \Delta_2 = 0 \ldots\ldots\ldots\ldots (68)$$

For the left half of the frame two moment-shear equilibrium equations—one for the column and one for the rafter—may be written as follows:

$$M_1 + M_2 = 15\ H \ldots\ldots\ldots\ldots (69)$$

and,

$$M_3 + M_4 = 500 \times 10 - 5\,H \ldots\ldots\ldots\ldots (70)$$

in which H equals the horizontal thrust caused by the load. After transforming M-values to Δ-values, Equations (68), (69), and (70) are easily solved, giving $H = 220.1$, and moments which agree with those found by Mr. Nelidov. (Moment-shear equilibrium formulas of the type of Equations (69) and (70) are used in the analysis of wind stresses in buildings. The principle is that the shear in a column is equal to the sum of its top and bottom moments divided by its height.)

No trigonometric functions are necessary in this solution, the number and size of equations are less impressive, and the labor is reduced.

Several discussers thought that the original paper was confined too much to symmetrical structures and loading. The following illustrations will show that the traverse method readily solves unsymmetrical cases. Fig. 16 is introduced because its solution may be checked conveniently by other methods or by handbooks. It will serve to illustrate the method of approach to the general case which succeeds it.

If the columns in Fig. 16(*a*) were on rollers the beam would deflect as if it were simply supported. The joint rotations would be equal to the end slopes of the simple beam, and the spread, $d_0 = d_1 + d_2$, at the bottoms of the columns would be equal to the sum of the end slopes times the height of the frame, which is equal to the area of the $\frac{M}{EI}$ - diagram for the beam (considered as simply supported) times the height of the frame. For equilibrium, since the column bases do not spread, a force, H, must be exerted with a magnitude sufficient to produce inward deflections of the columns equal to the spread, d_0, as indicated in Fig. 16(*b*). Thus, in Fig. 16(*a*),

$$d_1 + d_2 = d_0 = Ah = \frac{P\,a\,b\,h}{2\,I_2}$$

and, in Fig. 16(*b*),

$$d_3 = 2\,(\Delta_1 \times \tfrac{2}{3}\,h) + 2\,\Delta_2\,h = \frac{2\,H\,h^3}{3\,I_1} + \frac{H\,h^2\,l}{I_2}$$

Equating d_0 to d_3; letting k = the ratio of the stiffness of the beam to the stiffness of the column $= \frac{I_2\,h}{I_1\,l}$; and solving for H,

$$H = \frac{3\,P\,a\,b}{2\,h\,l\,(2\,k + 3)} \qquad \text{(71)}$$

which is a handbook formula.[27] With H known, the moments are statically determinate. This solution takes into account the effect of side-sway.

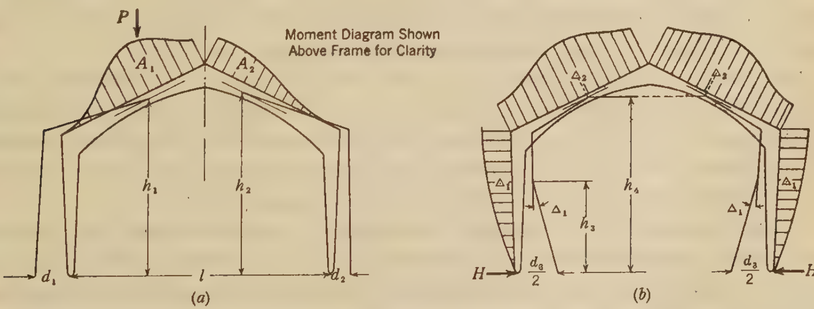

FIG. 17.

Fig. 17 shows unsymmetrical loading for a hinged-base, gable roof frame of variable moment of inertia. Following the method of approach indicated for Fig. 16, the equations by which this frame may be solved are, as follows:

---

[27] "Rahmenformeln," von A. Kleinlogel, Wilhelm Ernst & Son, Berlin.

In Fig. 17(*a*),

$$d_1 + d_2 = d_0 = A_1 h_1 + A_2 h_2 \ldots\ldots\ldots\ldots\ldots (72)$$

For a trial H in Fig. 17(*b*), compute,

$$d_3 = 2(\Delta_1 h_3 + \Delta_2 h_4) \ldots\ldots\ldots\ldots\ldots (73)$$

and,

$$\text{True } H = \frac{d_0}{d_3} \times \text{trial } H \ldots\ldots\ldots\ldots\ldots (74)$$

With H computed, the problem becomes statically determinate. This solution also includes the effect of side-sway.

These applications of the traverse method show that it is not restricted in its scope as are slope deflection and end-moment distribution. The latter methods require the assistance of some other type of analysis to obtain the fixed-end moments needed. They are also unsuitable for frames with sloping roof members. The traverse method is applicable to the entire field of flexure, in which moments only and straight members only are involved. Even where axial shortening of members is involved the traverse can be passed through the altered positions of the joints as shown by Fig. 10, and solved. The method, therefore, appears to be entitled to the adjective, "general," rather than "special."

Mr. Paulet feels that the bending-moment diagram should be available as a guide, so that the traverse can be sketched. Proper observance of laws governing the angles will make it as easy or easier, to draw the traverse independently.

Note that in all diagrams each obtuse angle which is the supplement of each A-angle presents its opening toward the load, and if there is restraint to resist free rotation of an adjoining joint, there will be a Δ-angle of opposite direction between the A-angle and the joint. Using this rule the writer finds it easier to draw the traverse first, the moment diagram being either omitted or drawn as a by-product of the traverse. It was noted with pleasure that Mr. Nelidov omitted the moment diagram from his Fig. 9.

Mr. Paulet gives Equations (15) and (16) to express the series for relative moments and slopes, respectively, in Fig. 6 (*c*), with the letters exchanged for the series in Fig. 6(*d*). The fact that a single simple equation—namely, $M_{n+1} = 4 M_n - M_{n-1}$, in which M = either moment or slope—expresses the law of progression for all four series, is of more interest.

With reference to Mr. Paulet's feeling that for single spans the moment-area method is preferable to the traverse method, it is suggested that Fig. 13, and the supporting text, be examined. The traverse method is simply an amplification of the moment-area method by expressing moment-area relationships as a geometrical traverse, which clarifies the computations, discloses "short cuts" which would not be evident except for the traverse diagram, provides a graphical check, and also an easily understood method of treating settled supports.

The statement by several discussers that the traverse method compares unfavorably in speed with other methods merits a brief investigation. The solution for Fig. 3 was presented in complete detail to illustrate the principle of the traverse. A computer familiar with the method would not write Equation (6), but would analyze the traverse as follows: In Fig 3 where two lines intersects to form a Δ or *A*-angle, note that the intersection displays both an acute angle (the Δ-angle or the *A*-angle) and an obtuse angle which is the supplement of the Δ-angle or the *A*-angle. Designate angles with obtuse openings that face the interior of the frame as negative and the angles with obtuse openings that face outward, as positive. The sum of the negative angles must equal the sum of the positive angles. Noting that each column traverse forms an isosceles triangle, the computer can add the angles mentally and write, $3\,\Delta_2 = \frac{P\,l^2}{8\,E\,I}$. The subsequent steps can also be performed by mental arithmetic.

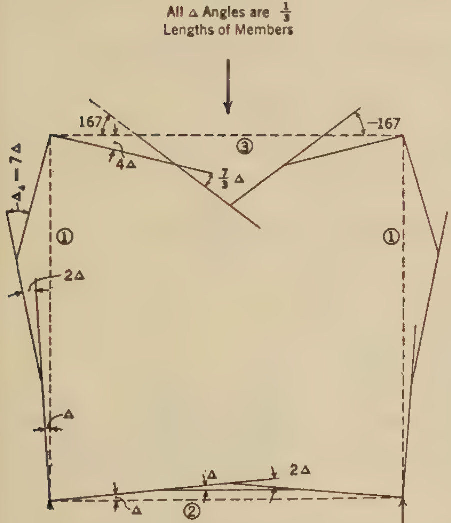
All Δ Angles are $\frac{1}{3}$ Lengths of Members
167
−167
③
4Δ
$\frac{7}{3}$Δ
$\Delta_1 - 7\Delta$
①
①
2Δ
Δ
Δ
2Δ
Δ
②

FIG. 18.

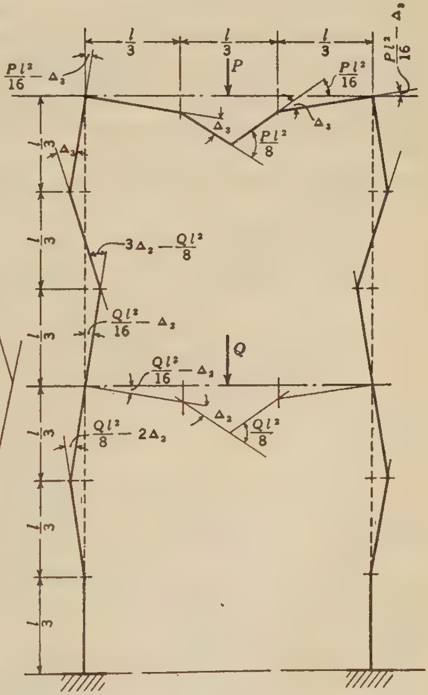
$\frac{l}{3}$
$\frac{l}{3}$
$\frac{l}{3}$
P
$\frac{Pl^2}{16} - \Delta_3$
$\frac{Pl^2}{16}$
$\frac{Pl^2}{16} - \Delta_3$
Δ_3
Δ_3
$\frac{Pl^2}{8}$
Δ_3
Δ_2
$\frac{l}{3}$
$3\Delta_2 - \frac{Ql^2}{8}$
$\frac{Ql^2}{16} - \Delta_2$
Q
$\frac{Ql^2}{16} - \Delta_2$
Δ_2
$\frac{Ql^2}{8}$
$\frac{Ql^2}{8} - 2\Delta_2$

FIG. 19.

A good illustration of the speed of the traverse method is given by Fig. 18 which is the demonstration problem used by T. Y. Lin, Jun. Am. Soc. C. E., in his paper "A Direct Method of Moment Distribution."[28] Let Δ equal the curvature in the bottom member due to one of its end moments. Then Δ will also equal the rotation at each bottom corner. From the stiffness ratios the curvature due to the moment in the bottom of the column will be $2\,\Delta$, as shown. The fixed-end moment of 1 000 at the end of the top member gives a ϕ-angle of 167, as shown. The solution is now given by the following equations: Traversing from the bottom to the top of the column, $-3\,\Delta - 4\,\Delta + \Delta_4 = 0$; and, therefore, $\Delta_4 = 7\,\Delta$. The upper joint rotation, by adding angles from the bottom to the top of the column, is equal to $4\,\Delta$; the curvature due to the end moment in the beam, by stiffness ratios, will be $\frac{7}{3}\,\Delta$; and the closure of the triangle under the end of the upper beam gives, $\frac{19}{3}\Delta = 167$, from which $\Delta = 26.32$, and the bottom and top moments in the column are, respectively, 105.2 and 368.2. This is a direct solution which appears to the writer to be more expeditious than the previous solution cited and will compare well with solutions by slope deflection or by progressive end-moment distribution.

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[28] *Proceedings*, Am. Soc. C. E., December, 1934, p. 1460.

Fig. 19 illustrates a two-story frame with members all of which have equal stiffness factors and joint rotations with unknown directions. Assume the direction of joint rotations as shown, and write, by inspection, all the angle values shown on the diagram, the Δ-value at the lower third point of the upper column being obtained by balancing moments about the adjoining joint.

Upward and downward traverses of the upper column now give:

$$\left(\frac{Q\,l^2}{16} - \Delta_2\right) - \frac{2}{3}\left(3\,\Delta_2 - \frac{Q\,l^2}{8}\right) + \frac{1}{3}\,\Delta_3 = 0$$

and,

$$\left(\frac{P\,l^2}{16} - \Delta_3\right) - \frac{2}{3}\,\Delta_3 + \frac{1}{3}\left(3\,\Delta_2 - \frac{Q\,l^2}{8}\right) = 0$$

Solving,

$$\Delta_2 = \frac{P\,l^2}{224} + \frac{11\,Q\,l^2}{224}$$

The rotation at the left end of the lower beam is $-\frac{P\,l^2}{224\,E\,I} + \frac{3\,Q\,l^2}{224\,E\,I}$. Therefore, if $P < 3Q$, the rotation is positive, as shown, and if $P > 3Q$, the rotation is counter-clockwise.

Continuing the solution it is found that if $Q > 5P$, the left top corner of the frame will rotate contra-clockwise. It is observed that slope-deflection sign rules may be applied to the traverse method of solving problems involving unknown directions of joint rotations.

Mr. Mitchell showed an application of the traverse method to Fig. 3, assuming that the column bases are 50% fixed. The traverse method, however, offers a shorter solution than that given by him, as follows: For the columns fully fixed at the base, $\Delta_1 = \frac{1}{2}\,\Delta_2$. For 50% fixation, Δ_1 will become equal to $\frac{1}{4}\,\Delta_2$ and a rotation will occur at the bottom of the column which will be two-thirds of the change in the value of $\Delta_1 = \frac{1}{6}\Delta_2$.[29] This angle of rotation will present its obtuse opening outward the same as Δ_1. Equating the sum of the negative angles around the frame to the sum of the positive angles, one may write at once $3\frac{1}{6}\,\Delta_2 = \frac{P\,l^2}{8}$, from which $M = \frac{3}{38}\,P\,l$, agreeing with Mr. Mitchell's result, but doing away with his Equations (17), (18), and (19), and his traverse equation.

The writer cannot agree with Mr. Mitchell that the traverse method "holds no advantage over the slope-deflection method", nor with his findings as to the number of equations required.

Professor Tsai indicated in his discussion that he had fully grasped the traverse method. His equations showing how simple moment areas and the positions of their centers of gravity may be derived from (although not "given in") the Ruppel tables[6] constitute a useful contribution to the practical application of the method. His comment that he could use the equations of three moments involving settled supports or the slope-deflection method with greater facility than the traverse method to solve Fig. 10, indicates that for this problem it is easier to apply ready made formulas than to work from basic principles, provided the formulas are memorized.

Professor Wilson has also grasped the traverse method, but leans to the end-moment distribution method as "the most workable of the special methods". In this connection it is worth noting that Hardy Cross, M. Am. Soc. C. E., has stated[30] that for single bents he preferred the column analogy method[31] to the end moment-distribution method. The traverse solutions for single-span frames, including arched frames as given herein, should certainly compete with the column analogy, especially for frames of variable moment of inertia to which the Ruppel tables are applicable.

The writer thanks Professor Drummond, Mr. Reeves, Mr. Kirkbride, Professor R. B. Ketchum, and Mr. Floris for their interest and the last three particularly for their encouraging comments. He feels that considering the versatility and general applicability of the traverse method, it should gain recognition.

It is superior to slope deflection in the following respects: (1) It provides an easy graphical verification of the computed analysis; (2) it uses basic principles rather than memorized formulas ("the anæsthetics of thought"); (3) it does not require the assistance of some other method; and (4) the entire elastic deformation of a structure may be pictured from the solution by the traverse method.

It is superior to end-moment distribution in Items (3) and (4), and also in its utility in deriving a specific formula which may be recorded for future use.

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[29] See properties of 1:2:3 diagrams in "Improved Method of Finding Beam Deflections," by Ralph W. Stewart, M. Am. Soc. C. E., *Civil Engineering*, February, 1934.

[30] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p 145.

[31] *Bulletin No. 215*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1926

RELATION BETWEEN RAINFALL AND RUN-OFF FROM SMALL URBAN AREAS

By W. W. Horner,[1] M. Am. Soc. C. E., and
F. L. Flynt,[2] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. Franklin F. Snyder, Merrill M. Bernard, LeRoy K. Sherman, and W. W. Horner and F. L. Flynt.

Synopsis

The results of research into the relation between rainfall and run-off from small urban areas in St. Louis, Mo., are here presented as specific studies of the run-off from parts of two different city blocks tributary to street inlets and from both roofs and ground surface of another entire city block. The information submitted results from measurements of rainfall and storm flow for practically all heavy rains occurring from 1914 to 1933. The ratio of run-off to rainfall, defined in several ways, is shown to vary over a wide range.

Rainfall rates at each of the locations studied are reduced and developed into frequency diagrams, and these three rainfall studies, with one other, are combined into a master frequency study for the general region. Run-off is also studied as an independent phenomenon; the run-off frequency curves are developed in a form similar to the rainfall diagrams.

The two sets of curves are considered to be comparable as representing equivalent probabilities of occurrence. Ratios are then developed between corresponding values. Within certain limits, it is suggested that these ratios may be applied to proper rainfall frequency curves for other localities and will give approximate run-off values for similar conditions of surface.

Suggestions are offered as to how the values determined for specific blocks in St. Louis might be modified further to be applicable to (*a*) different

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Note.—Published in October, 1934, *Proceedings*.

[1] Cons. Engr., St. Louis, Mo.

[2] Civ. Engr., Sewer Design Dept., St. Louis, Mo.

surface slopes; (*b*) other percentages of impervious area; and (*c*) other typical soils. For Class (*c*), adaptation factors are determined by sprinkling, at definite rates, a number of segregated areas of bare soil and of turf.

INTRODUCTION

In 1908, one of the writers had developed an application of the rational method of sewer design for the City of St. Louis. The form of treatment and the mechanics of the application were considered satisfactory; but there was little information then available as to necessary ratios between rainfall and run-off under specific conditions. A proper development of the hydrology of urban drainage required the provision of dependable factors indicating the relation between rainfall and run-off for a wide range of situations.

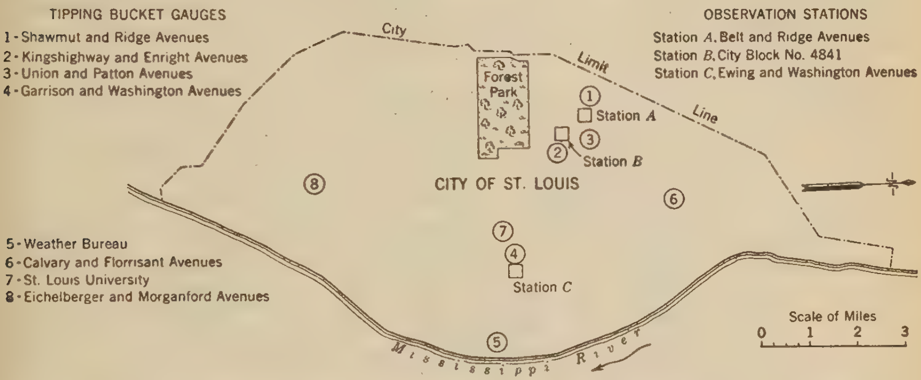
TIPPING BUCKET GAUGES
1 - Shawmut and Ridge Avenues
2 - Kingshighway and Enright Avenues
3 - Union and Patton Avenues
4 - Garrison and Washington Avenues
5 - Weather Bureau
6 - Calvary and Florrisant Avenues
7 - St. Louis University
8 - Eichelberger and Morganford Avenues
OBSERVATION STATIONS
Station *A*. Belt and Ridge Avenues
Station *B*. City Block No. 4841
Station *C*. Ewing and Washington Avenues
City Limit Line
Forest Park
Station *A*
Station *B*
Station *C*
CITY OF ST. LOUIS
Mississippi River
Scale of Miles
0 1 2 3
①
②
③
④
⑤
⑥
⑦
⑧

FIG. 1.—LOCATIONS OF OBSERVATION STATIONS AND TIPPING-BUCKET RAIN GAUGES.

Funds were made available by the City of St. Louis for a research program. In 1910, three tipping-bucket rain gauges were installed in the 500-acre Clarendon drainage area (see Points 1, 2, and 3, Fig. 1), and about ten water-level gauges in its main trunk sewer. The information thus secured was analyzed in part in 1915 and, again, in 1920, but not exhaustively. Tentative values were deduced and the resulting run-off factors have been used in St. Louis, but, heretofore, no thorough study has been completed to a degree justifying publication of the results.

When the characteristics of the flow in the main trunk sewer came to be scrutinized, it was apparent that a closer view of the basic relations might be secured from an examination of the run-off phenomena from smaller areas. Therefore, a study was undertaken of two inlet areas at Belt and Ridge Avenues and at City Block No. 4841 (Stations *A* and *B*, Fig. 1), involving the run-off from entire city blocks, exclusive of that from the roofs of residential buildings. Later, a gauge was installed in a lateral sewer near Ewing and Washington Avenues (Station *C*, Fig. 1), and the entire run-off of a third block, including that from roofs, was made available for analysis. The result of observations in these three city blocks, during heavy rains, is presented herein.

In the course of this more detailed study, a better acquaintance with the effect of soil characteristics and conditions upon run-off became desirable. Impressed with the simplicity of the sprinkling tests, made by the Miami Conservancy District, a similar series of tests were conducted in St. Louis, using somewhat more elaborate and refined methods. The result of a part of this work has been published.[3] All the data are given in the unpublished Appendices filed in Engineering Societies Library, 29 West 39th Street, New York, N. Y. Included with these records is also the result of a similar series of tests, conducted under the supervision of one of the writers, in connection with the studies for storm-sewer design for Dallas, Tex.

PART 1.—STUDY OF THE RUN-OFF FACTOR BY THE UNIT-GRAPH METHOD

Instruments and Installations.—For each of the three city blocks (Stations *A*, *B*, and *C*, Fig. 1). the pertinent information as to surface condition, slopes, etc., is shown in Figs. 2 to 5, inclusive, and in Table 1. During the period of the tests, a few changes took place in the buildings and

TABLE 1.—BASIC INFORMATION PERTAINING TO AREAS *A*, *B*, AND *C*, FIG. 1

Type of area	Total Area Tributary to the Gauge								Area in Entire Block		
	As of 1917				As of 1933						
	Subdivided, in square feet	Total in:			Subdivided, in square feet	Total in:			Subdivided, in square feet	Total in:	
		Square feet	Acres	Percentage		Square feet	Acres	Percentage		Square feet	Percentage
(*a*) Area *A* (Belt and Ridge Avenues; See Fig. 2)											
Impervious:											
Streets and sidewalks	16 117	42 255	0.97	42	......	....	1.07	49	.....	....	.....
Alleys	6 098										
Roofs and porches	4 356										
Sheds	3 485										
Paved yards and walks	12 199										
Pervious	......	57 935	1.33	58	......	....	1.13	51	......	....	.....
Total	......	....	2.30	100	......	....	2.20	.....	......	....	.....
(*b*) Area *B* (City Block No. 4841; See Fig. 3)											
Impervious:											
Streets and sidewalks	4 872	40 916	....	29	......	46 021	....	32	64 030	199 737	51.8
Alleys	14 400								14 400		
Roofs and porches	1 932								81 126		
Sheds	7 673								8 691		
Paved yards and walks	12 039								31 490		
Pervious	......	101 442	....	71	......	96 137	....	68	......	186 341	48.2
Total	......	142 358	3.27	100	......	142 158	3 27	100	......	386 078	100.0
(*c*) Area *C* (Ewing and Washington Avenues; See Fig. 4)											
Impervious:											
Buildings	58 588	136 576	3.14	72.4	59 370	136 904	3.14	72.1	......	....	.....
Sheds	7 072				6 618						
Walks	39 888				39 888						
Alleys	7 725				7 725						
Streets	23 303				23 303						
Pervious	......	52 064	1.20	27.6	......	52 846	1.22	27 9	......	....	.....
Total	......	188 640	4.34	100.0	......	189 750	4.36	100.0	......	....	.....

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[3] *Municipal and County Engineering*, December, 1922.

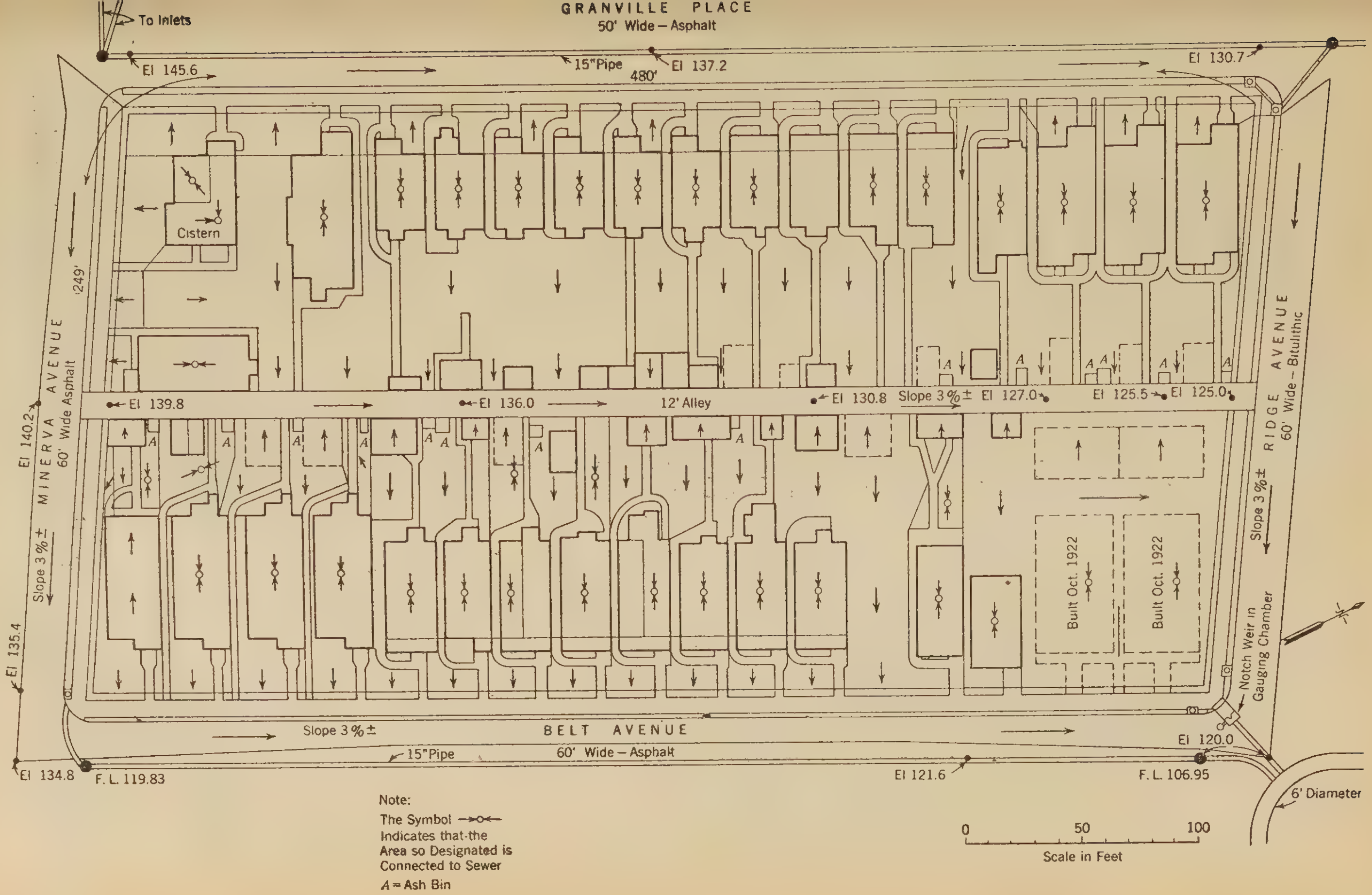

GRANVILLE PLACE
50' Wide – Asphalt
To Inlets
El 145.6
15"Pipe
480'
El 137.2
El 130.7
MINERVA AVENUE
60' Wide Asphalt
'249'
El 140.2
Slope 3%±
El 135.4
El 134.8
F. L. 119.83
Cistern
El 139.8
El 136.0
12' Alley
El 130.8
Slope 3%±
El 127.0
El 125.5
El 125.0
A
Built Oct. 1922
Built Oct. 1922
RIDGE AVENUE
60' Wide – Bitulithic
Slope 3%±
Notch Weir in Gauging Chamber
BELT AVENUE
60' Wide – Asphalt
Slope 3%±
15"Pipe
El 121.6
El 120.0
F. L. 106.95
6' Diameter
0
50
100
Scale in Feet
Note:
The Symbol →o← Indicates that the Area so Designated is Connected to Sewer
A = Ash Bin

FIG. 2.—AREA TRIBUTARY TO STATION *A* AT BELT AND RIDGE AVENUES (SEE TABLE 1 (*a*)).

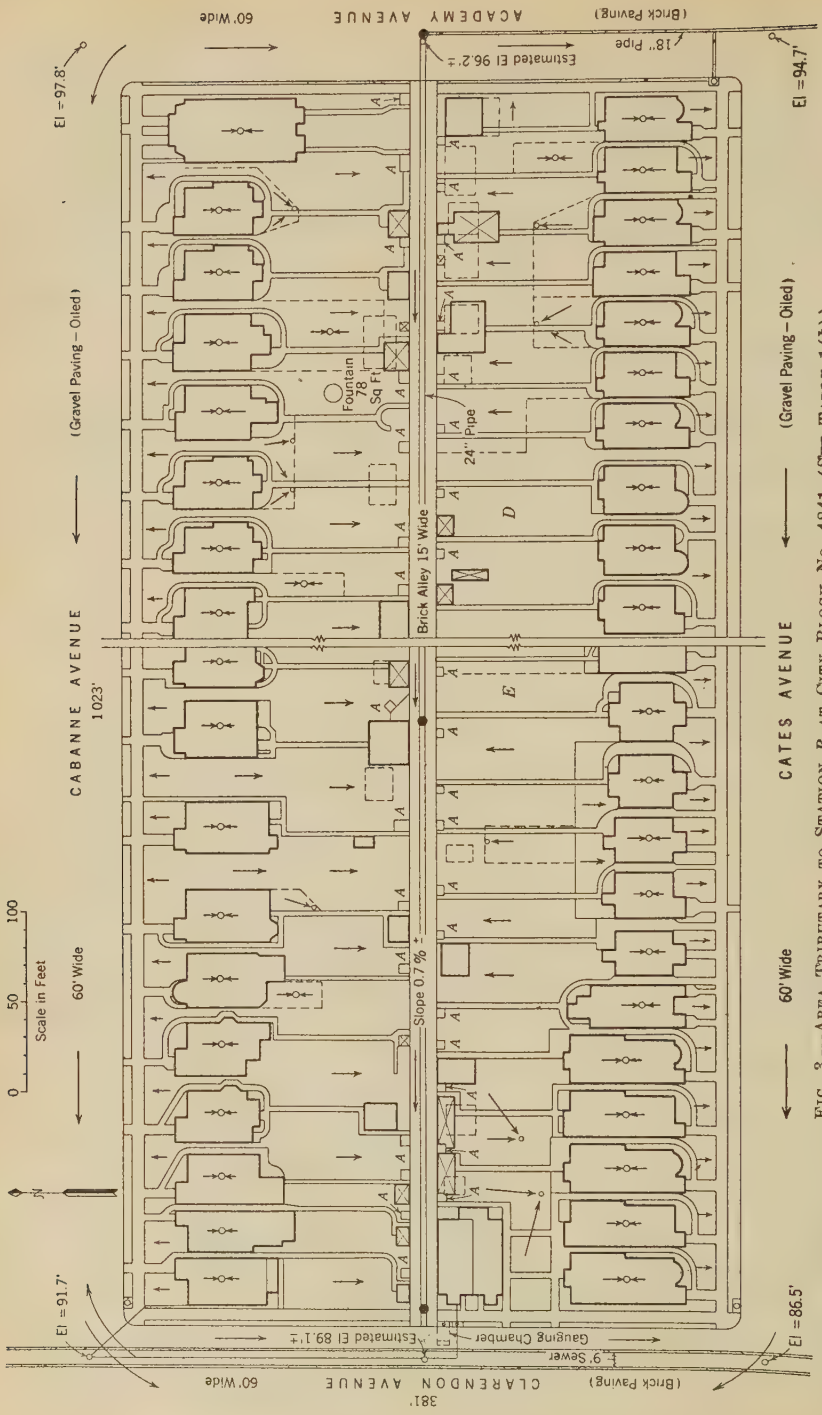

ACADEMY AVENUE
60' Wide
(Brick Paving)
18" Pipe
Estimated El 96.2'±
El = 97.8'
El = 94.7'
(Gravel Paving – Oiled)
CABANNE AVENUE
1 023'
(Gravel Paving – Oiled)
CATES AVENUE
Fountain 78 Sq Ft
24" Pipe
Brick Alley 15' Wide
Slope 0.7% ±
A
D
E
Scale in Feet
0
50
100
60' Wide
60' Wide
N
El = 91.7'
El = 86.5'
Estimated El 89.1'±
Gauging Chamber
9" Sewer
CLARENDON AVENUE
60' Wide
(Brick Paving)
381'

FIG. 3.—AREA TRIBUTARY TO STATION *B* AT CITY BLOCK No. 4841 (SEE TABLE 1(*b*)).

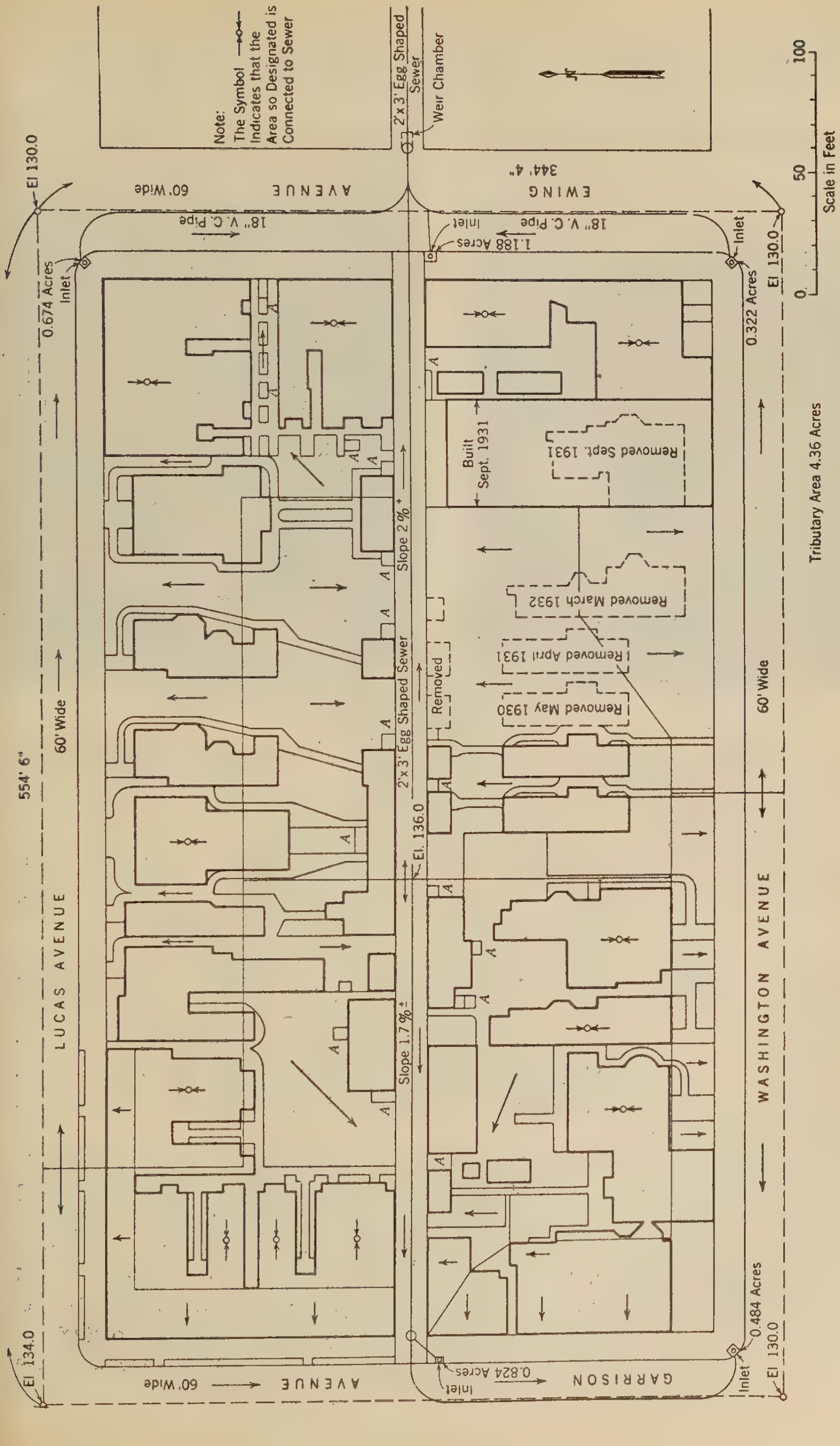

Note:
The Symbol —⊸— Indicates that the Area so Designated is Connected to Sewer
2'x3' Egg Shaped Sewer
Weir Chamber
El 130.0
EWING AVENUE
60' Wide
344' 4"
18" V.C. Pipe
Inlet
1.188 Acres
0.674 Acres
0.322 Acres
Scale in Feet
0
50
100
LUCAS AVENUE
60' Wide
554' 6"
Built Sept. 1931
Removed Sept. 1931
Slope 2%±
Removed March 1932
Removed April 1931
Removed May 1930
Removed
2'x3' Egg Shaped Sewer
El. 136.0
Tributary Area 4.36 Acres
WASHINGTON AVENUE
Slope 1.7%±
A
El 134.0
GARRISON AVENUE
0.824 Acres
0.484 Acres
El 130.0

FIG. 4.—AREA TRIBUTARY TO STATION *C* AT EWING AND WASHINGTON AVENUES (SEE TABLE 1(*c*)).

garages, the only one of importance resulting from the erection of an apartment house at the corner of Belt and Ridge Avenues. The data for the beginning and the ending of the test period are shown in Table 1.

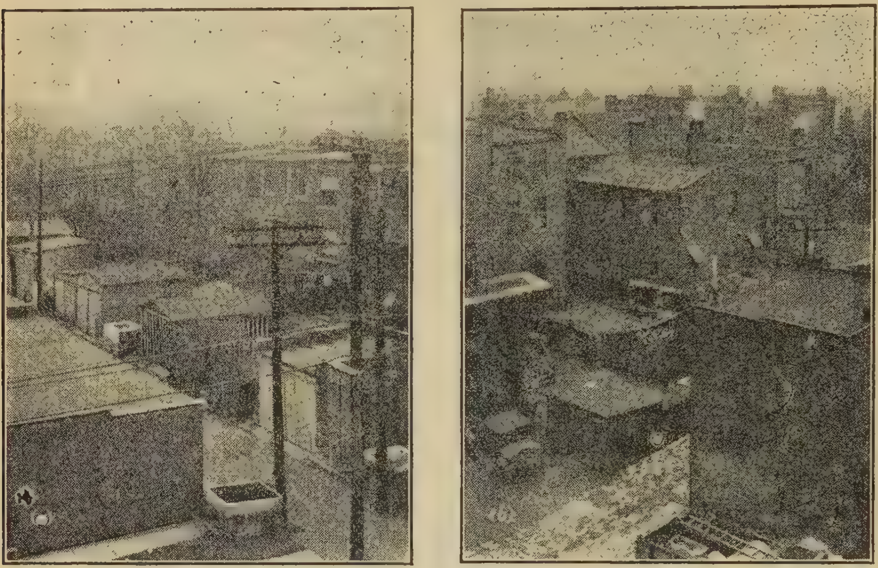


FIG. 5.—TYPICAL APPEARANCE OF PROPERTY INVOLVED: (a) AREA *A*; (b) AREA *C*.

Description of Instruments for Inlet Area Tests.—At each location the run-off was measured by recording the head on a 90°, V-notch weir of $\frac{3}{8}$-in. steel plate with bronze edges. At Station *A* and for Station *B*, the weir was installed in a special chamber built as an extension of the bowl of the catch-basin. At Station *C*, the weir was installed in a chamber built into the brick sewer itself. In each instance, the head on the weir was measured by a pressure bulb connected with a recording clock gauge.

In Fig. 2, 30% of the area above the inlet at the northwest corner of Belt and Minerva Avenues is assumed to be tributary to the gauge at Belt and Ridge Avenues for all rains. Referring to Fig. 3, the area marked, *D*, was judged to yield practically no run-off. The yard marked, *E* (Fig. 3), was found to have several small sumps, so that water from this yard did not reach the gauge. Consequently, it was assumed that water from all storms would probably stand.

At Station *C* (see Fig. 4), the rainfall was measured by a standard tipping-bucket gauge on a building adjacent to the upper edge of the block. At Stations *A* and *B*, home-made rain gauges were installed in such a way that the rain water falling into the gauge was measured as head on an additional pressure bulb, which actuated an extra pen installed on the clock gauge. These rain gauges were calibrated by introducing known quantities of water into the stand-pipe and noting the chart records. To supplement these gauges, three tipping-bucket rain gauges were also available at Shawmut and Ridge Avenues, at Kingshighway and Enright, and at Union and Patton Avenues, all within a mile of the two home-made gauges.

Pervious Area

At Area *A*, Fig. 2, the pervious part consists of small front lawns, some good turf, and some bare soil. At Area *B*, Fig. 3, nearly all the pervious part is well turfed. At Area *C*, Fig. 4, the pervious part consists generally of some small separated plots of rather hard-packed soil.

Reduction of Records.—An examination of a typical chart for Belt and Ridge Avenues (Fig. 6) shows the rain pen moving from the outer circle toward the center, and the weir pen moving outward from the center. Celluloid verniers, indicated by the shaded rectangular areas, make it possible to read the time scales to the nearest minute. Since the rain pen is a home-made addition to the gauge, it does not follow the printed time lines, and the apparent time of the rain record must be corrected to clock time. Fig. 7 shows the curves obtained by plotting the information obtained from charts, such as Fig. 6. The records for City Block No. 4841 were similarly reduced and plotted; the tipping-bucket rain gauge at Station *C* necessitated some differences in the detail of handling its records.

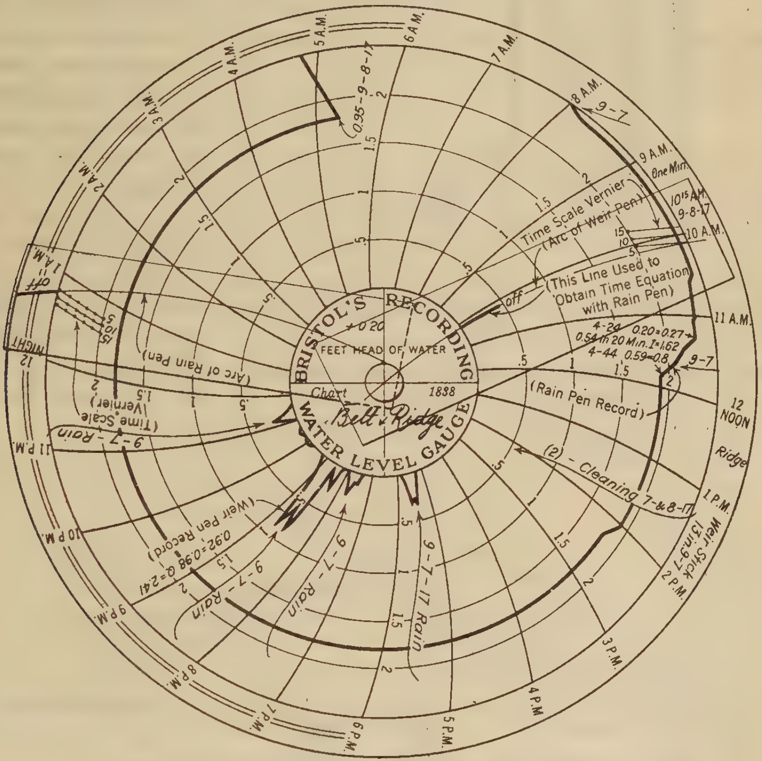


Fig. 6.—Rainfall and Run-Off Record, Station *A*; Rain of September 7, 1917.

Run-Off Characteristics.—To introduce the factors that enter into the rainfall-run-off problem, the following discussion of a hypothetical case is offered.

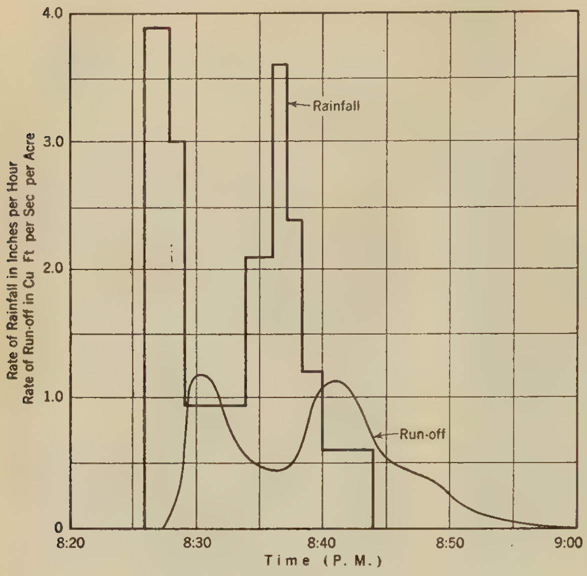


FIG. 7.—RAINFALL-RUN-OFF RATE CURVES; RAIN OF SEPTEMBER 7, 1917, AT STATION *A*.

Fig. 8 represents the case of an inclined plane with a pervious surface, exposed to a rainfall of uniform rate. Let i = intensity, or rate of rainfall, in inches per hour; a = rate of absorption, in inches per hour; e = rate of evaporation in inches per hour; I = residual rainfall available for run-off = $i - (a + e)$; f = depth of water film on the surface due to rainfall; f_c = critical depth of film when run-off begins; V = velocity of water in the

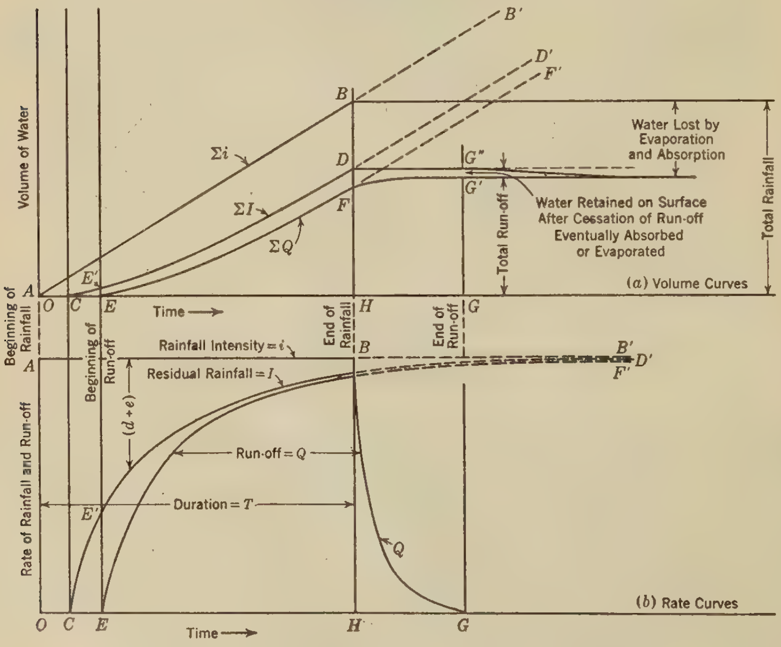

B'
D'
F'
B
D
F
Σi
ΣI
ΣQ
G"
G'
Water Lost by Evaporation and Absorption
Water Retained on Surface After Cessation of Run-off Eventually Absorbed or Evaporated
Total Run-off
Total Rainfall
Volume of Water
(a) Volume Curves
A
E'
O
C
E
Time
End of Rainfall
H
End of Run-off
G
Beginning of Rainfall
Beginning of Run-off
Rainfall Intensity = i
Residual Rainfall = I
(d + e)
Run-off = Q
Duration = T
Q
Rate of Rainfall and Run-off
(b) Rate Curves
O C E H G
Time

FIG. 8.—RELATION BETWEEN RAINFALL AND RUN-OFF.

moving film; Q = rate of run-off; and t = duration of a rain. Point A represents the beginning of the rainfall. For a time, the rate, $(a + e)$, is greater than i and no water can accumulate on the surface; but since i is constant and $(a + e)$ is decreasing, eventually $(a + e) = i$, as at Point C, Fig. 8, when the surface film begins to form and I begins to have a positive value. The water in this film is subject to two opposing forces: (1) A component of the force of gravity which tends to cause motion on the plane; and (2) surface tension which tends to oppose this motion. As the rate, $(a + e)$, decreases, the depth of film, f, increases until the critical depth, f_c, is attained, as at Point E, Fig. 8, when run-off begins. Under the assumed conditions, run-off would begin simultaneously over the entire area.

Frictional resistance, which tends to reduce acceleration, increases with the velocity and decreases with the depth. Therefore, the values of f and V at any given point on the plane depend on a balance between these opposite effects combined with the effects of absorption and evaporation.

If the assumed conditions continued for an indefinite time, the value of $(a + e)$ would approach zero as a limit, and the values of I and Q would approach i as a limit, as indicated in the diagrams by Points B', D', and F', Fig. 8. From this time on, the product, $f \times V$, at all points on the plane would be constant, as would the total volume of water in transit, represented by the intercept on the ordinates between the lines, CD and EF, on Fig. 8(*a*).

If, however, rainfall should cease after a time, t, the value of i would drop immediately to zero as indicated by the line, BH, in Fig. 8(*b*). The water in transit, represented by the line, DF, in Fig. 8(*a*), would still be available for run-off which would continue at a decreasing rate until the moving film was reduced to the critical depth, f_c, as at Point G, when run-off would cease; the water retained would eventually be absorbed or evaporated.

Actual Areas.—Many factors not considered in the hypothetical case are involved in an actual inlet area. It is not a plane surface, but a collection of small contiguous planes; the slopes may range from horizontal for tennis courts, to vertical for building walls; the permeability of the component surfaces may likewise vary over a wide range, from asphalt street paving to grass plots and cultivated ground. The actual area may contain well-defined watercourses, such as gutters and rivulets.

Depressed areas may be present into which water drains and thus becomes trapped; this drainage never reaches the inlet, but eventually is absorbed or evaporated. Such a condition may be designated as retention by pondage. If the depression is shallow, it may soon be filled and then contribute its overflow to the run-off; or, the depression may have an outlet that is too small to drain the water as fast as it falls. This water will appear in the total run-off, but its detention will have an effect upon the rate of run-off.

Although an actual inlet area may be composed of elemental plane surfaces, it is impracticable to analyze run-off from each small area separately. The entire inlet area is here taken as a unit, and the run-off from it is studied as a whole.

Unit Graph.—Fig. 9 is a typical example of a rainfall-run-off graph as derived from the local records. A study of these graphs shows that the reduction in height of the run-off peaks, as compared with the rainfall peaks, is not entirely due to the loss of water by absorption and evaporation. It is due partly to the fact that run-off is spread over a greater time interval than rainfall. If some method could be found to separate these two rate-reducing effects of loss by absorption and of distribution of the run-off in time, it would be possible to measure the absorption and evaporation characteristics of the areas under investigation.

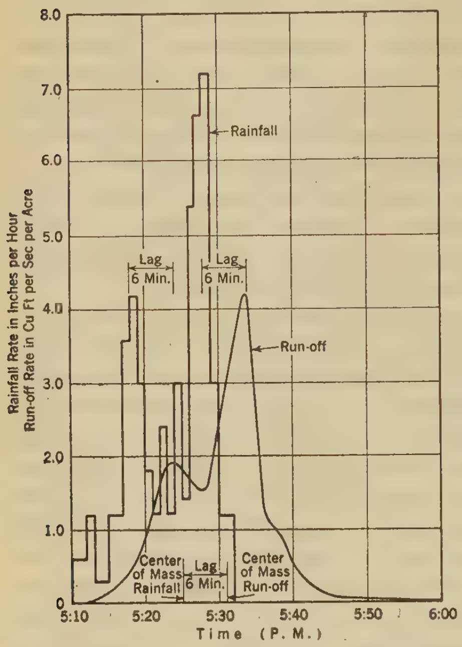


FIG. 9.—RATE CURVES FOR RAINFALL AND RUN-OFF, STATION C; RAIN OF SEPTEMBER 7, 1920.

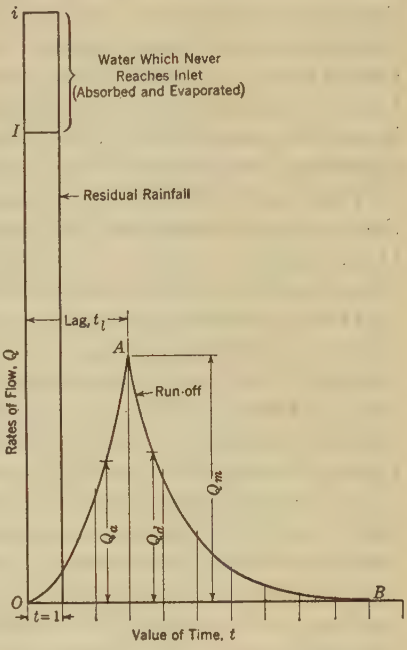


FIG. 10.—UNIT GRAPH REPRESENTING THEORETICAL RUN-OFF RESULTING FROM RAINFALL ON UNIT AREA FOR UNIT TIME.

Following is an outline of a method of analysis based upon a theoretical unit graph showing the probable distribution in time of the run-off resulting from 1 min of uniform rainfall.

In Fig. 10, the rectangle, Oi, represents the volume of 1 min of uniform rainfall. The rectangle, OI, represents the volume of this rainfall which will ultimately reach the sewer.

The shape of the run-off unit graph was derived from a study of the records of a few short rains of fairly uniform intensities, the equations for the two branches being entirely empirical. The quantities, Q_a and Q_d, respectively, are the ascending and descending instantaneous values of Q resulting from a rainfall of unit duration on a unit area; Q_m is the maximum value of

Q_a and Q_d; t is the time measured from the beginning of the rainfall; t_l is the lag, or the value of t at Q_m; and j and k are arbitrary constants. For the increasing values of Q, the equation of the line, OA, is:

$$Q_a = Q_m \left(\frac{t}{t_l}\right)^j \ldots\ldots\ldots\ldots (1)$$

For decreasing values, the equation of the line, AB, is:

$$Q_d = \frac{Q_m}{k^{t-t_l}} \ldots\ldots\ldots\ldots (2)$$

From Equations (1) and (2) it is seen that when $t = t_l$, $Q_a = Q_d = Q_m$.

Integrating and adding Equations (1) and (2), the area under the unit run-off curve is found to be:

$$I = Q_m \int_0^{t_l} \left(\frac{t}{t_l}\right)^j dt + Q_m \int_{t_l}^{\infty} \frac{dt}{k^{(t-t_l)}} = Q_m \left(\frac{t_l}{j+1} + \frac{1}{\log_e k}\right) \ldots (3)$$

Since $I = Pi$, the value of Q_m is found to be:

$$Q_m = \frac{Pi}{\left(\frac{t_l}{j+1} + \frac{1}{\log_e k}\right)} \ldots\ldots\ldots\ldots (4)$$

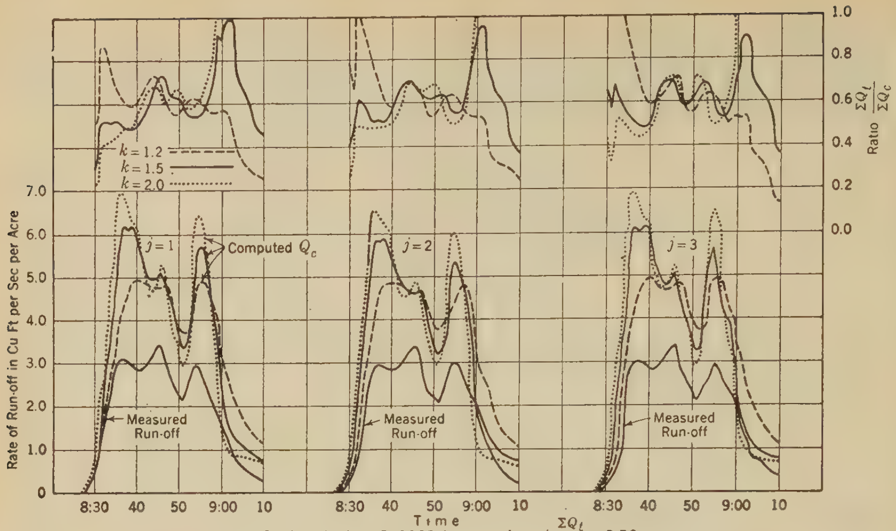
The values, j and k, in Equations (1) to (4) can be determined only by trial. After many rains had been investigated, it was found that, for Stations A and B, values of $j = 1.0$ and $k = 1.2$ and, for Station C, values of $j = 2$ and $k = 2$ gave the best results (see Fig. 11).

TABLE 2.—COMPUTATION OF 100% RUN-OFF CURVE, OR VALUES OF Q_c (SEE FIG. 12)

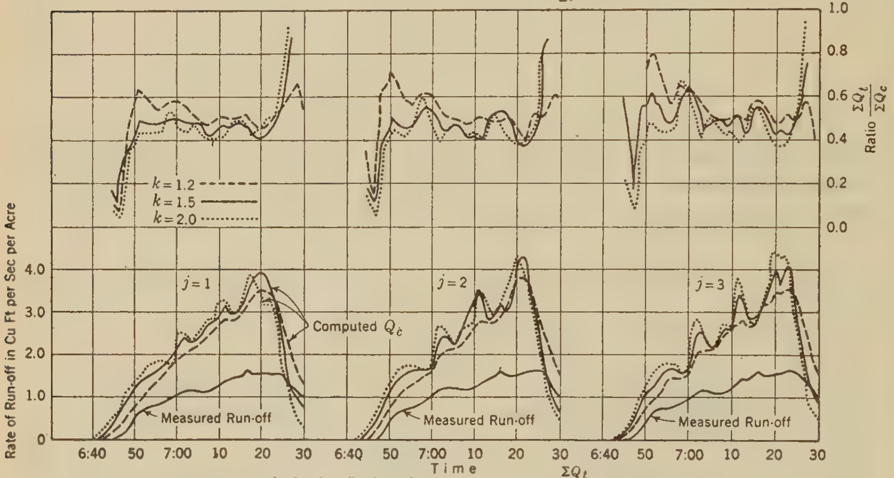
(Assume $t_l = 3$; $j = 2$; and $k = 2$)

Time, t, in minutes of run-off	Quantities of run-off, Q, at successive values of time, t, in minutes					100% run-off, Q_c	Time, t, in minutes of run-off	Quantities of run-off, Q, at successive values of duration period, I, in minutes					100% run-off, Q_c
	1	2	3	4	5			1	2	3	4	5	
	and corresponding values of rainfall intensity, i, in inches per hour							and corresponding values of rainfall intensity, i, in inches per hour					
	2	4	6	3	1			2	4	6	3	1	
1......	0.092	.....	.....	.....	.....	0.092	10.....	0.007	0.026	0.077	0.077	0.052	0.239
2......	0.364	0.184	.....	.....	.....	0.548	11.....	0.004	0.013	0.039	0.039	0.026	0.121
3......	0.820	0.728	0.276	.....	.....	1.824	12.....	0.002	0.007	0.020	0.020	0.013	0.062
4......	0.410	1.640	1.092	0.138	.....	3.280	13.....	0.001	0.004	0.010	0.010	0.007	0.032
5......	0.205	0.820	2.460	0.546	0.046	4.077	14.....	.....	0.002	0.005	0.005	0.004	0.016
6......	0.103	0.410	1.230	1.230	0.182	3.155	15.....	.....	0.001	0.003	0.003	0.002	0.009
7......	0.052	0.205	0.615	0.615	0.410	1.897	16.....	.....	.....	0.002	0.002	0.001	0.005
8......	0.026	0.103	0.308	0.308	0.205	0.950	17.....	.....	.....	0.001	0.001	.....	0.002
9......	0.013	0.052	0.154	0.154	0.103	0.476	18.....	.....	.....	.....	.....	.....	....

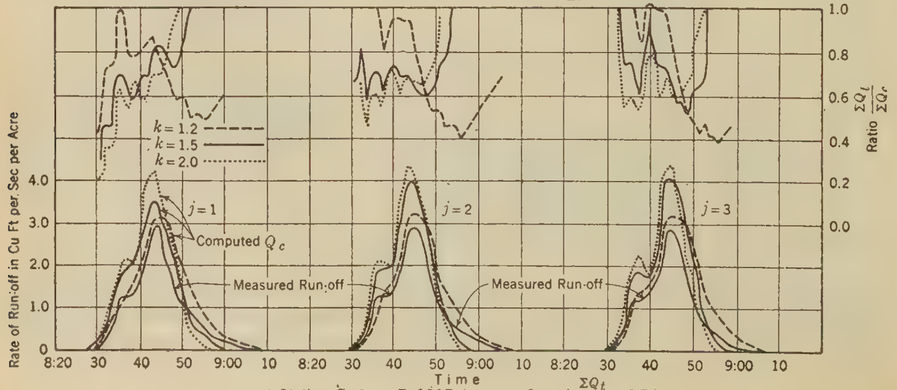
Fig. 12 and Table 2 show the method of computing the ordinates and plotting the 100% run-off or Q_c-curve for a short hypothetical rain. The same method may be applied to any rainfall record to find the computed 100%



(*a*) Station *A*, Aug. 8, 1923; Lag, $t_l = 4$; and $\frac{\Sigma Q_t}{\Sigma i} = 0.56$



(*b*) Station *B*, Aug. 27, 1921; $t_l = 8$; and $\frac{\Sigma Q_t}{\Sigma i} = 0.49$



(*c*) Station *C*, Sept. 7, 1917; Lag, $t_l = 6$; and $\frac{\Sigma Q_t}{\Sigma i} = 0.74$

FIG. 11.—VARIATIONS OF COMPUTED 100% RUN-OFF FOR DIFFERENT VALUES OF *j* AND *k*.

run-off for any area for which the lag is known. The computed Q_c-curve shows the rate-reducing effect of the distribution of run-off, in time, separated from the effect due to absorption and evaporation. A comparison of the measured run-off curve with the 100% run-off curve shows the effect of absorption and evaporation.

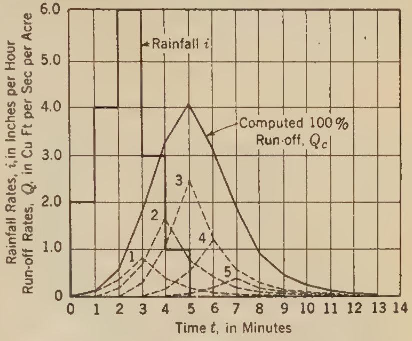


FIG. 12.—HYPOTHETICAL RAIN, SHOWING APPLICATION OF UNIT GRAPH (SEE TABLE 2).

Lag.—The term, "lag", as used herein has reference only to the difference in phase between salient features of the rainfall and run-off rate curves; its numerical values (which are generally somewhat less than the "time of concentration") are difficult to determine with the desired accuracy owing to the limitations of the recording instruments. (A full discussion of this subject is included in the record manuscript filed in Engineering Societies Library.) Observed values, subject to correction, are taken from the time difference between salient features of the rainfall and run-off rate curves; or between the centers of mass of the rainfall and run-off rate curves.

The comparatively wide range in the lag at each location led to the inference that the lag was a variable, its value being determined more by rainfall characteristics than by the characteristics of the drainage area. Diagrams were made of the relation between the lag and various rainfall characteristics (see Fig. 13), but no evidence of correlation was found. On all

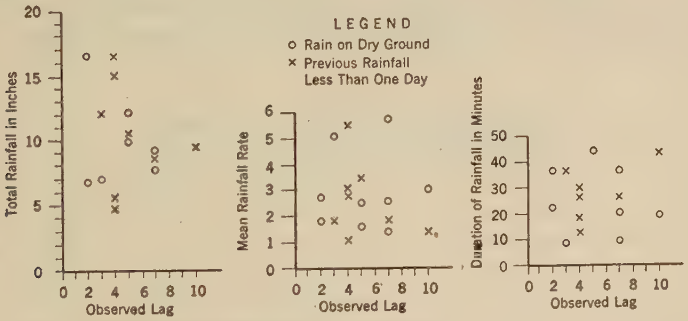


FIG. 13.—TYPICAL CORRELATION DIAGRAMS FOR LAG AT STATION *A*.

these diagrams, the values of the lag for each location tended to be concentrated about certain modal values. A statistical study (see Fig. 14) indicated that the mean, the median, and the modal values of the lag for each location were approximately equal, which suggested that the value was nearly constant for each station, and that the variations from the mean value were probably chance variations which were partly explained later by the discovery of small unsuspected time errors. Each of the plotted points through which the curves in Fig. 14 were drawn, represents the number of storms investigated which had the corresponding observed value of lag, or less. Values of the lag as observed at the three observation stations are given in Table 3. Under certain circumstances, the true lag may vary slightly from the adopted values in Table 3, but the recording instruments available for this work are not sufficiently sensitive to measure the variation with any assurance of accuracy.

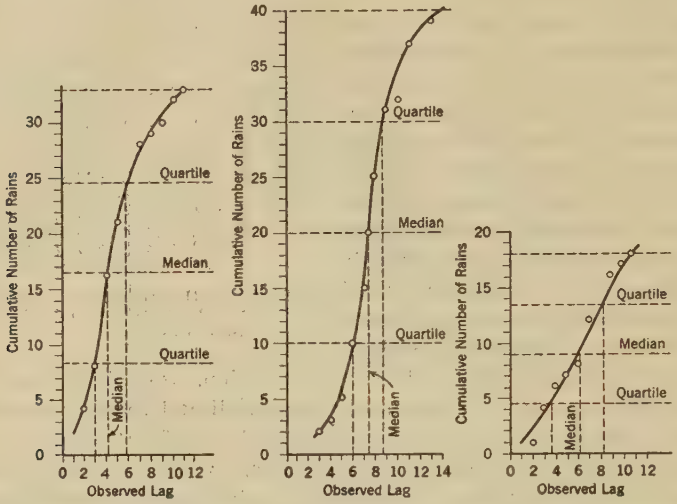


Fig. 14.—Ogive Curves Showing Variation in Observed Value of Lag; (*a*) Station *A*; (*b*) Station *B*; (*c*) Station *C*.

As only three different areas are represented in the local records, it is impossible to establish any definite rules for the determination of the lag for a given inlet area. The two stations for which the extreme values of the lag are indicated (Station *A*, 4 min; and Station *B*, 8 min) have widely different characteristics. The conditions of size and shape which tend to increase the lag are combined with conditions of slope which have the same tendency. This makes it difficult to separate their effects, but it does give an idea of the maximum range of value of the lag for similar areas.

TABLE 3.—Values of Lag, t_l, in Minutes, Observed at the Three Observation Stations (See Fig. 1)

Station	Mean	Median	Mode	Adopted value
A	5.3	4.1	4.0	4
B	8.3	7.5	8.0	8
C	6.4	6.1	7.0	6

Drainage Area *C* (Fig. 4) includes several independent inlet areas and considerable roof drainage, and the storm water from all parts of the block reaches the sewer after a comparatively short run. At the other stations, the flow is entirely over the surface to the point of measurement. For this reason a direct comparison cannot be made with the other locations, but the results indicate that the lag for a completely sewered area is practically the same as that for a simple inlet area of comparable size.

The majority of inlet areas, at least in St. Louis, would have lags somewhere between 4 and 8 min. For any ordinary inlet area, a lag of 5 min could not be very far from the true value.

DETERMINATION OF THE RUN-OFF FACTOR

The 100% Run-Off Graph.—The rates of 100% run-off, computed by the same method as shown in Fig. 12 and Table 2, have been plotted as broken lines in Fig. 15 for a typical storm at each station. The lines represent the theoretical value of the run-off rate for an absolutely impervious area if there is no evaporation. Their close similarity to the measured run-off curves is obvious; the infrequent dissimilarities are due to several causes. For example, the factors, j and k, in the formulas may be slightly in error due to the difficulty of ascertaining these values. There is a possibility that the rate of rainfall for single minutes may be in error, although the average rate for several minutes may be correct. Furthermore, there may be slight time errors which would tend to shift the rainfall and run-off peaks one way or the other. All these causes would affect the shape of the Q_c or 100% run-off curve and, in some cases, their effects might be cumulative and appreciable.

Except as otherwise stated, the run-off factor, P, is defined as the ratio between ΣQ_t, the volume of water that has run off from the given area to a given time, and ΣQ_c, the volume of water that would have run off from the area during the same time had there been no evaporation or absorption. It is indicated most conveniently (see Fig. 16) by the ratio between corresponding ordinates of the mass curves of the measured run-off and the 100% run-off. (In Figs 16(a) and 16(b), the ΣQ_c-curve would eventually meet the Σi-curve if the entire record had been plotted.)

Typical Storms.—The storm of September 8, 1926, on Area A (see Fig. 15(a)) begins with a period of high intensity lasting for about 6 min; then it falls suddenly to medium values which gradually decrease with some minor fluctuations until the end of the storm.

The ordinate to the mass curve, $\frac{\Sigma i}{60}$ (Fig. 16(a)) for any given time is proportional to the area of the rainfall-rate curve from the beginning of the rain to the given time. Similarly, the $\frac{\Sigma Q_c}{60}$ and $\frac{\Sigma Q_t}{60}$ -curves are the mass curves for the 100% run-off and the measured run-off, respectively.

The graph of the run-off factor, $P = \frac{\Sigma Q_t}{\Sigma Q_c}$, is obtained from the ratio of corresponding ordinates of the mass curves. The comparatively large difference between the maximum and final values of the run-off factor in this case is rather unusual and is due to the great and sudden change in rainfall intensity early in the storm.

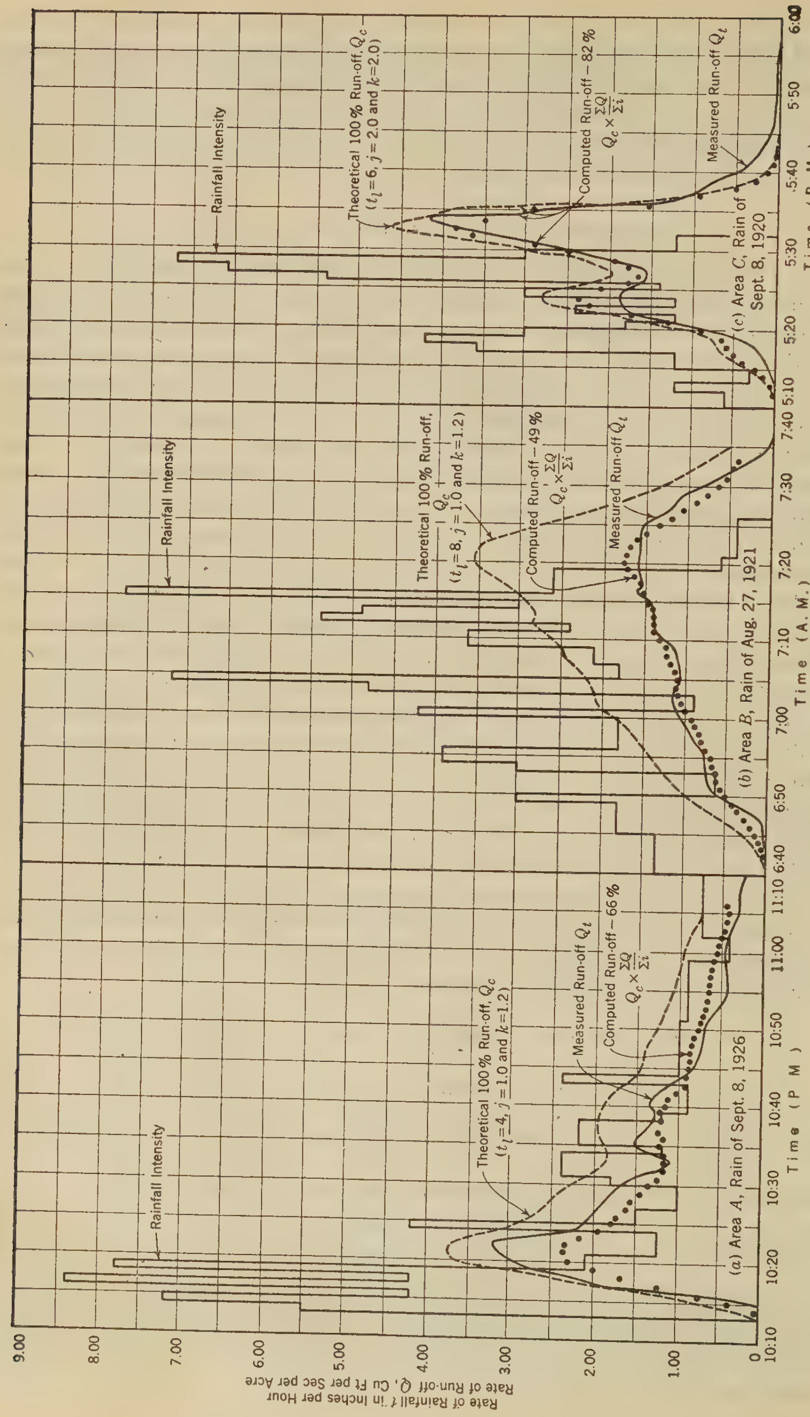


FIG. 15.—RATES OF RAINFALL, i, AND RUN-OFF, Q.

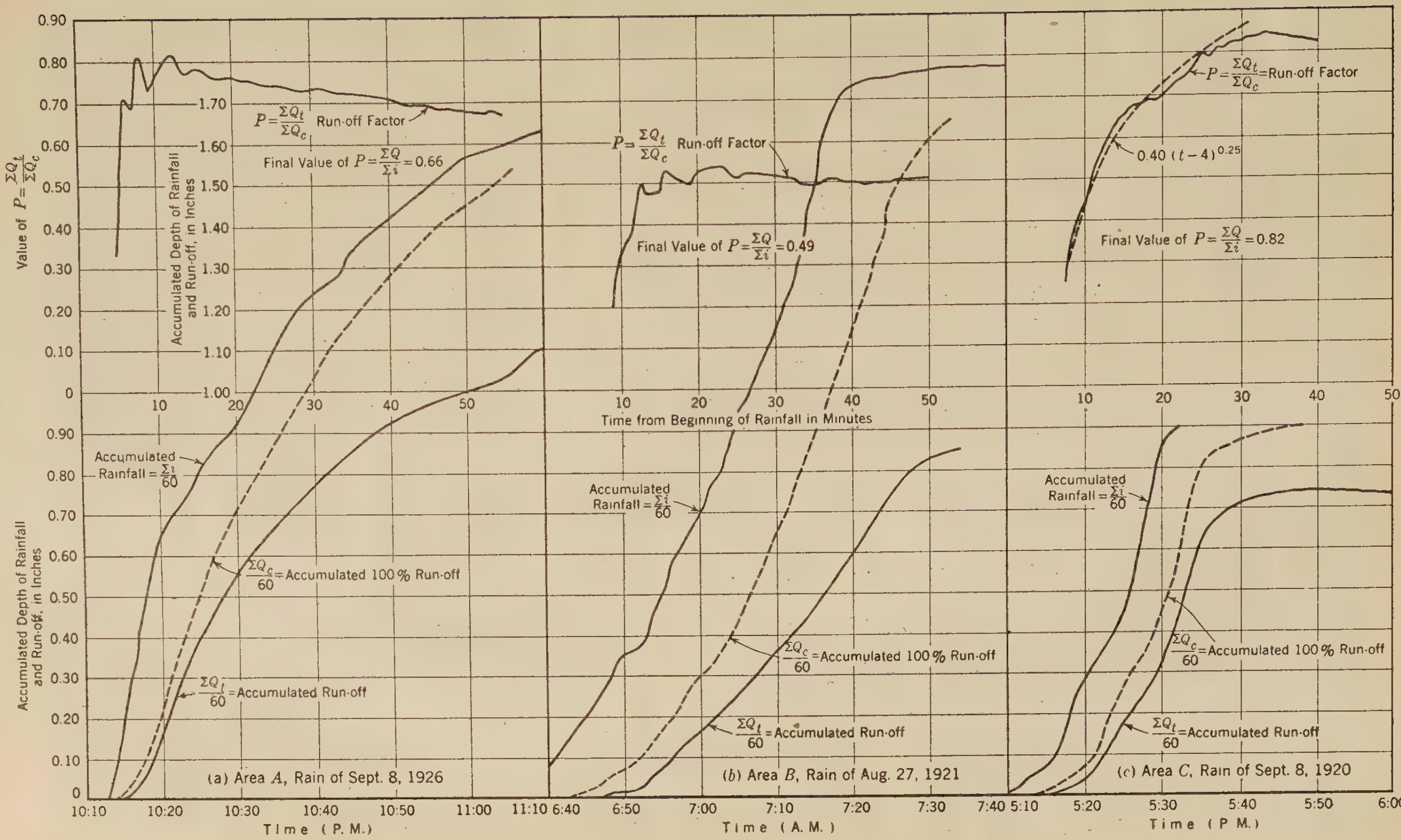


FIG. 16.—ACCUMULATED DEPTHS OF RAINFALL AND RUN-OFF AND VALUES OF P.

An interesting type of rainfall (which seldom occurs more than once a year in St. Louis and then only between August 15 and September 15) is represented by the storm of August 27, 1921, on Area *B* (Figs. 15(*b*) and 16(*b*)). This type of storm, with its steadily increasing intensity, results in higher run-off rates than any other storm of comparable magnitude.

The run-off factor rises to a maximum value within about 20 min and remains practically constant for the remainder of the storm, indicating that once a certain degree of saturation has been reached, further increases in rainfall rates have little effect. The low value of *P* and the smoothness of the curve are characteristic of this area, which is large and flat.

The rain of September 8, 1920, on Area *C* (Fig. 15(*c*)), also begins with a low rainfall intensity and builds up to a climax. The high value of the run-off factor is characteristic of this location which is a completely sewered city block and not an inlet area as are the others.

Complete Data.—Curves similar to Fig. 15(*a*), prepared by the unit-graph method for all representative storms at each location, have been filed with the record manuscript in Engineering Societies Library. On these diagrams is shown, in step curves, the appropriate rainfall rate information; in dotted lines, the 100% run-off curve, Q_c, of the unit-graph method, using the values of j and k appropriate to the particular block under study; and the actual run-off rate curve as determined from the sewer gauges (Q_t). For ready reference, the end values are given on each chart; that is, $\frac{\Sigma Q_t}{\Sigma i}$, or total run-off divided by total rainfall. Notes of other prior precipitation that would have affected ground conditions are also given.

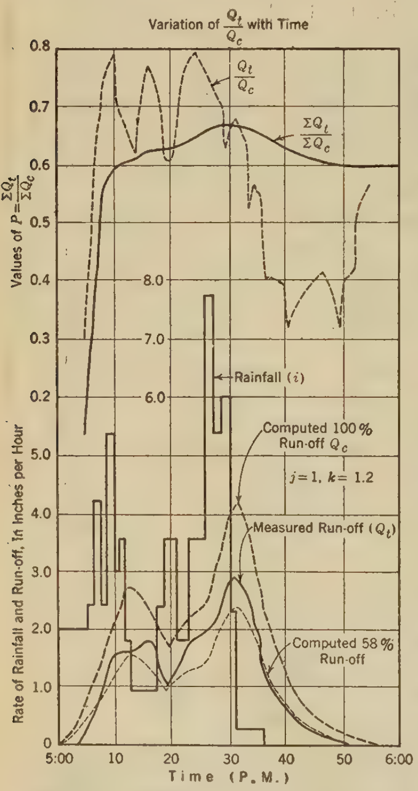


FIG. 17.—VARIATION OF Q_t/Q_c WITH TIME; AREA A, SEPTEMBER 8, 1920 (HEAVY RAIN 5 HOURS PREVIOUS; $= 4$ AND $\frac{\Sigma Q_t}{\Sigma i} = 0.58$).

Unit-Graph Method of Determining Run-Off Factor.—These diagrams have been studied in a number of ways. The form of the rainfall-rate curve having been altered by the unit-graph application, it was expected that the ratio of simultaneous values of the measured run-off, Q_t, to the calculated run-off, Q_c, which could then be determined, would be found to have a fairly simple series of values and a definite relationship to certain remaining variables.

Ratio of Instantaneous Values.—In Fig. 17, the rate ratio, $\frac{Q_t}{Q_c}$, is compared with the volumetric ratio, $\frac{\Sigma Q_t}{\Sigma Q_c}$. The former is much more irregular than the latter, due to the causes enumerated under the heading, "Determination of the Run-Off Factor: The 100% Run-Off Graph". It will be noted in Fig. 17 that when the slope of the run-off curve is zero (at the peaks and valleys) as at 5:13 P.M., 5:19 P.M., and 5:31 P.M., the value of $\frac{Q_t}{Q_c}$ is nearly the same as that of $\frac{\Sigma Q_t}{\Sigma Q_c}$. A practical comparison can be made by ignoring exact time relations and comparing Q_t with Q_c only for generally corresponding peak values.

The results of such a study of all the data are plotted as percentile curves on Fig. 18(*c*). The values cover a wide range and no method has been found to co-ordinate them to other variables. In Fig. 18(*b*), the values are separated in accordance with time of occurrence after the beginning of the rain, but without distinctive results. It seemed possible that more definitely characteristic values for each block might result from the use of average peak values of 5-min duration rather than extreme peak values. The rates thus obtained are shown on Fig. 18(*d*) and may be compared to Fig. 18(*c*).

There is a marked correlation between the ratio of these 5-min peak rates and the final value of the run-off factor, $\frac{\Sigma Q_t}{\Sigma i}$. Correlation diagrams, of which Fig. 19 is an example, indicate that the final value of $P_1 = \frac{\Sigma Q_t}{\Sigma i}$, when applied to the 100% run-off curve, will give a computed peak run-off which should approximate the true value, except for peaks occurring early in the storm. This final value of P_1 can be determined easily and accurately by the ratio of the total run-off to the total rainfall. It is not affected by time errors or by uncertainty as to values of j and k in the unit-graph formulas, as are the other ratios studied.

If each of the ordinates of the 100% run-off curve (for a given storm and location), as computed by means of the unit-graph formula, is multiplied by the final value of the run-off factor for the given storm, the resulting curve usually agrees fairly well with the measured run-off curve, as shown by the circled points in Fig. 15.

The agreement is better in the longer storms, and for the later parts of storms, especially if the maximum rainfall intensity occurs after 15 or 20 min of rainfall when the run-off factor curve has flattened out, as in Fig. 15(*b*). When the maximum rainfall intensity occurs early in the storm, the agreement is not so good (see Fig. 15(*a*)).

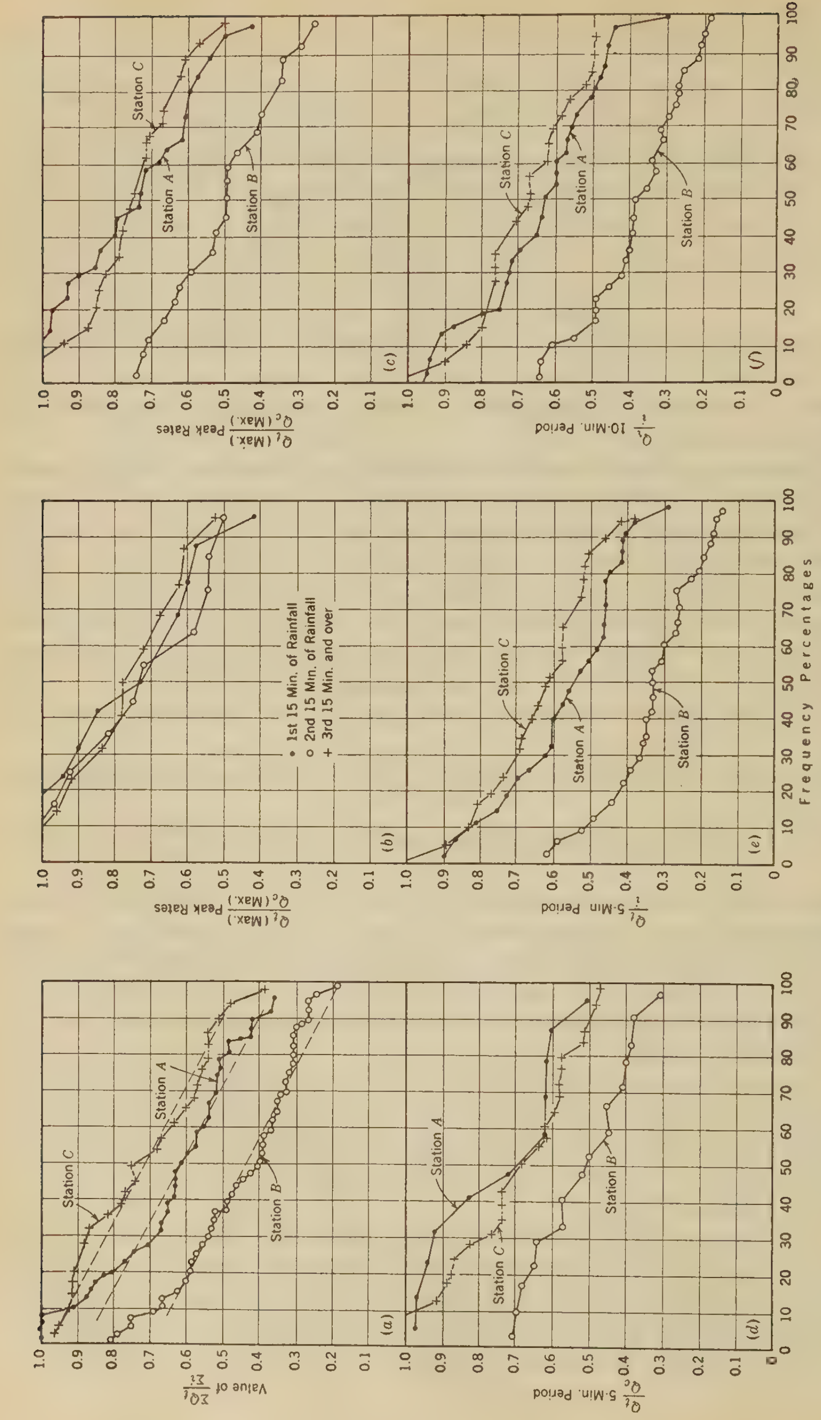


FIG. 18.—PERCENTILE CURVES OF RAINFALL-RUN-OFF RATIOS.

TABLE 4.—DATA USED IN FREQUENCY STUDY OF RAINFALL AND RUN-OFF RATES

Date	RAINFALL							RUN-OFF							RUN-OFF / RAINFALL							ΣQ / Σi
	Duration Period, in Minutes							Duration Period, in Minutes							Duration Period, in Minutes							
	5	10	15	20	30	40	60	5	10	15	20	30	40	60	5	10	15	20	30	40	60	
(a) STATION A, BELT AND RIDGE AVENUES																						
9/15/1914	3.12	2.91	(13) 2.80	(11) 2.47	(7) 2 40	(6) 2 16	(3) 1.77	1.89	1.57	1.41	(17) 1.47	1.4	(9) 1.21	(5) 0.95	0.61	0.54	0.50	0.60	0.59	0.56	0.54	0.58
5/ 2/1915	(18) 4.08	3.06	2.40	2.21	(13) 1.78	(16) 1.23	(19) 0.84	2.14	1.84	(18) 1.61	1.35	(14) 1.12	0.87	(17) 0.64	0.52	0.60	0.67	0.61	0.63	0.71	0.76	0.66
6/20/1915	3.24	2.93	(19) 2.64	(19) 2.27	(14) 1.74	....	....	1.53	1.40	1.24	1.09	0.83	(17) 0.89	(12) 0.73	0.47	0.48	0.47	0.48	0.48	....	....	0.46
6/20/1915	(15) 4.44	(12) 3.44	(14) 2.80	2.23	(15) 1.71	....	....	1.62	1.43	1.17	0.93	....	....	....	0.36	0.42	0.42	0.42	....	....	....	0.46
6/27/1915	2.88	2.76	2.42	2 22	(18) 1.65	....	....	1.36	1.31	1.19	1.07	0.81	....	....	0.47	0.47	0.49	0.48	0.49	....	....	0.51
8/ 2/1915	(12) 5.28	(10) 3.72	(16) 2.72	(12) 2.46	(19) 1.50	(17) 1.17	....	(9) 2.81	(11) 2.46	(9) 2.12	(9) 1.81	(10) 1.28	....	....	0.53	0.66	0.78	0.74	0.85	....	....	0.75
6/ 2/1916	(17) 4.20	(15) 3.40	(9) 3 28	(7) 2.87	....	....	....	2.06	(18) 1.93	(15) 1.96	(11) 1.80	(8) 1.35	....	....	0.49	0.57	0.60	0.63	....	....	....	0.70
8/ 2/1916	(4) 6.12	(8) 4.26	(8) 3.32	(8) 2.76	....	....	....	2.00	1.58	1.24	1.05	0.74	....	....	0.33	0.37	0.37	0.38	....	....	....	0.40
8/11/1916	(5) 6.12	(3) 4.86	(3) 3.68	....	....	....	....	(12) 2.74	(12) 2.36	(13) 2.00	(14) 1.62	....	....	....	0.45	0.49	0.54	....	....	....	....	0.58
8/12/1916	3.00	2.58	2.44	(18) 2.31	(12) 1.80	....	....	1.23	1.20	1.09	0.99	0.81	....	....	0.41	0.47	0.45	0.43	0.45	....	....	0.49
8/14/1916	3.78	2.94	2.28	2.10	(17) 1.68	....	....	1.93	1.74	1.37	1.16	(19) 1.00	0.80	....	0.51	0.59	0.60	0.55	0.60	....	....	0.65
8/14/1916	2.04	1.92	1.64	1.50	1.26	1.08	(18) 0.85	1.34	1.27	1.14	1.01	0.85	0.75	0.53	0.66	0.66	0.70	0.67	0.67	0.69	0.62	0.56
7/27/1917	....	....	....	....	....	(3) 2.36	(5) 1.58	....	....	....	....	....	(8) 1.29	(8) 0.88	....	....	....	....	....	0.55	0.56	....
9/28/1919	....	....	....	....	....	(6) 2.25	(2) 2.04	....	....	....	....	....	(12) 1 00	(7) 0.89	....	....	....	....	....	0.44	0.44	....
4/19/1920	(13) 5.04	(17) 3.27	2.56	2.03	(8) 2.25	(10) 1.61	(11) 1.15	1.71	1.44	1.27	1.13	(17) 1.06	(14) 0.95	(14) 0.69	0.34	0.44	0.50	0.56	0.47	0.59	0.60	0.57
9/ 8/1920	3.73	(16) 3.30	2.58	(10) 2.55	....	....	....	1.66	1.53	....	....	....	....	....	0.45	0.46	....	....	....	....	....	0.58
9/ 8/1920	(9) 5.64	(9) 4.26	(7) 3.53	(5) 3.09	(2) 2.94	(4) 2.30	(6) 1.53	(16) 2.50	(13) 2.22	(11) 2.01	(10) 1.81	(6) 1.65	(7) 1.34	(6) 0.90	0.44	0.52	0.57	0.59	0.56	0.58	0.59	0.58
9/11/1920	....	....	....	....	....	1.11	(15) 1 00	....	....	....	....	....	0.78	(16) 0.66	....	....	....	....	....	0.70	0.66	....
4/25/1921	(14) 5.04	(14) 3.42	(18) 2.67	(16) 2.34	(10) 1.91	(11) 1.56	(10) 1.22	(5) 3.03	(7) 2.59	(8) 2.17	(8) 1.95	(7) 1.63	(6) 1.42	(3) 1.07	0.60	0.76	0.81	0.83	0.85	0.91	0.88	0.93
4/26/1921	1.98	1.74	1.48	1.17	....	....	....	1.53	1.42	1.28	1.14	0.94	0.78	0.53	0.77	0.82	0.86	0.97	....	....	....	....
6/27/1921	(7) 5.88	(18) 3.27	2.20	....	....	....	....	(4) 3.12	(5) 2.69	(10) 2.07	(15) 1.61	(16) 1.10	....	....	0.53	0.82	0.94	....	....	....	....	1.00+
8/27/1921	(8) 5.76	(6) 4.38	(2) 3.86	(2) 3.53	(3) 2.92	(2) 2.43	(4) 1.71	(2) 3.66	(2) 3.19	(2) 3.00	(2) 2.68	(2) 2.26	(2) 1.90	(2) 1.33	0.64	0.73	0.78	0.76	0.77	0.78	0.78	0.86
4/14/1922	(6) 5.91	3.00	....	....	....	....	....	(10) 2.80	(15) 2.20	(19) 1.61	1.25	0.86	....	....	0.47	0.73	....	....	....	....	....	0.86
8/22/1922	(2) 6.48	(7) 4.38	(11) 3.05	(17) 2.34	1.44	(14) 1.32	(17) 0.89	(8) 2.81	(8) 2.57	(12) 2.01	(13) 1.63	(12) 1.17	(18) 0.89	....	0.43	0.59	0.66	0.70	0.81	0.67	....	0.76
8/ 8/1923	(1) 7.20	(1) 5.94	(1) 5.48	(1) 5.10	(1) 4.54	(1) 3.09	(1) 2.24	(3) 3.19	(3) 3.05	(3) 2.94	(1) 2.81	(1) 2.58	(1) 2.08	(1) 1.45	0.44	0.51	0.54	0.55	0.57	0.67	0.65	0.59
6/18/1924	2.16	1.56	1.64	1.77	1.34	1.05	....	1.48	1 33	1.18	1.12	0 90	0.72	....	0.69	0.85	0.72	0.63	0.67	0.69	....	0.59
8/24/1924	(19) 3.87	(11) 3.72	(10) 3.16	(9) 2.61	....	(12) 1.55	(13) 1.06	1.97	(19) 1.91	(16) 1.77	(16) 1.53	(13) 1.13	0.87	....	0.51	0.51	0.56	0.59	....	0.56	....	0.65
6/28/1925	3.64	2.76	2.22	1.65	1.40	1.06	0.77	(15) 2.54	1.88	1.42	1.17	0.83	....	....	0.70	0.68	0.64	0.71	0.71	....	....	0.67
8/11/1925	(16) 4.32	2.47	1.72	1.34	1.21	1.08	0.75	(18) 2.27	1.48	1.14	0.85	....	....	....	0.53	0.60	0.66	0.63	....	....	....	0.60
9/12/1925	3.12	2.76	2 60	2.19	....	....	....	(11) 2.80	(6) 2.65	(6) 2.31	(6) 2.04	....	....	....	0.90	0.96	0.89	0.93	....	....	....	....
9/12/1925	3.72	(13) 3.42	(12) 2.83	(13) 2.42	(4) 2.81	....	....	(6) 2.97	(4) 3.01	(4) 2.67	(4) 2.32	....	....	....	0.80	0.88	0.94	0.96	....	....	....	....
5/19/1926	3.84	2.64	2.08	1.66	1.41	....	....	(19) 2.25	(16) 2.04	(17) 1.76	(18) 1.41	(15) 1.11	(16) 0.89	....	0.59	0.77	0.85	0.85	0.79	....	....	0.84
8/23/1926	(11) 5.28	(4) 4.47	(4) 3.64	(6) 3.05	(9) 2.20	....	....	(1) 3.74	(1) 3.37	(1) 3.05	(3) 2.63	(3) 1.95	(3) 1.52	....	0.71	0.75	0.84	0.86	0.89	....	....	1.00+
8/31/1926	3.76	2.57	1.80	....	....	....	....	(13) 2.69	1.91	1.45	1.13	0.77	....	....	0.72	0.74	0.81	....	....	....	....	0.86
9/ 8/1926	(3) 6.36	(5) 4.38	(6) 3.55	(4) 3.09	(5) 2.56	(8) 1.97	(7) 1.52	(7) 2.97	(10) 2.52	(7) 2.27	(7) 2.02	(5) 1.73	(4) 1.45	(13) 0.69	0.47	0.57	0.64	0.65	0.68	0.74	0.45	0.66
5/ 7/1927	3.16	(19) 3.12	(15) 2.75	(15) 2.35	(16) 1.70	....	....	1.57	1.53	1.42	1.36	0.77	0.82	(15) 0.67	0.50	0.49	0.52	0.58	0.45	....	....	0.61
5/28/1927	3.82	2.88	2.32	....	....	....	....	1.90	(17) 1.93	1.60	(19) 1.38	....	....	....	0.50	0.67	0.69	....	....	....	....	0.73

TABLE 4.—(*Continued*)

Date	Rainfall							Run-Off							Run-Off / Rainfall							$\frac{\Sigma Q}{\Sigma i}$
	Duration Period, in Minutes							Duration Period, in Minutes							Duration Period, in Minutes							
	5	10	15	20	30	40	60	5	10	15	20	30	40	60	5	10	15	20	30	40	60	
(*a*) STATION *A*, BELT AND RIDGE AVENUES (*Continued*)																						
6/19/1928	2.67	2.31	2.04	1.78	1.39	....	....	(17) 2.30	(14) 2.21	(14) 1.98	(12) 1.72	(9) 1.34	(10) 1.10	(10) 0.76	0.86	0.96	0.97	0.97	0.96	....	....	0.85
6/19/1928	3.48	3.12	(17) 2.68	(14) 2.40	(11) 1.90	(13) 1.53	(14) 1.05	(14) 2.61	(9) 2.53	(5) 2.35	(5) 2.18	(4) 1.76	(5) 1.42	(4) 0.99	0.75	0.81	0.88	0.91	0.93	0.93	0.94	......
7/ 5/1928	(10) 5.58	(2) 4.89	(5) 3.60	(3) 3.42	(6) 2.53	(7) 2.00	(8) 1.49	1.62	1.47	1.40	1.33	(11) 1.21	(11) 1.01	(11) 0.74	0.29	0.30	0.39	0.39	0.48	0.51	0.50	0.37
4/20/1929	....	....	....	....	....	(15) 1.32	(9) 1.23	....	....	....	....	....	(15) 0.94	(9) 0.84	....	....	....	....	....	0.71	0.68	......
5/18/1929	....	....	....	....	....	(18) 1.17	(16) 0.91	....	....	....	....	....	0.76	(18) 0.61	....	....	....	....	....	0.65	0.67	......
5/30/1929	....	....	....	....	....	(9) 1.95	....	....	....	....	....	....	(13) 0.95	....	....	....	....	....	....	0.49	....	......
9/14/1930		Imperfect rainfall record						....	....	....	....	....	(19) 0.88	(19) 0.61	....	....	....	....	....	....	....	......
9/ 1/1931	2.90	2.75	2.26	1.96	1.45	....	...	1.35	1.29	1.23	1.13	0.93	....	....	0.47	0.47	0.54	0.58	0.64	....	....	0.67
9/ 1/1931	1.87	1.69	1.56	1.50	1.36	(19) 1.13	(12) 1.08	1.26	1.23	1.19	1.16	(18) 1.03	Too small		0.67	0.73	0.76	0.77	0.76	....	....	......
(*b*) STATION *B*, CITY BLOCK NO. 4841																						
9/ 5/1914	(7) 5.24	(11) 3.80	....	....	....	....	....	0.76	0.67	0.57	0.48	....	....	....	0.15	0.18	....	....	....	....	....	0.25
9/15/1914	(8) 5.08	2.97	2.50	2.07	(16) 1.63	(7) 2.06	(8) 1.56	0.97	0.92	0.86	0.80	0.72	0.64	(14) 0.50	0.17	0.31	0.34	0.39	0.44	0.31	0.32	0.35
6/13/1915	(2) 6.84	(2) 5.76	....	....	....	....	....	1.26	1.17	1.02	0.86	0.61	....	....	0.18	0.20	....	....	....	....	....	0.31
6/20/1915	(12) 4.75	(14) 3.52	(13) 2.77	(12) 2.33	(13) 1.77	(9) 1.94	(7) 1.62	(16) 1.29	1.17	0.93	0.75	0.60	(16) 0.66	(16) 0.49	0.27	0.33	0.34	0.32	0.34	0.34	0.30	0.35
6/27/1915	2.98	2.76	2.52	(16) 2.21	1.61	(15) 1.42	(14) 1.01	0.75	0.72	0.69	0.64	0.53	0.43	....	0.25	0.26	0.27	0.29	0.33	0.30	....	0.31
7/ 7/1915	3.12	2.82	(14) 2.73	(11) 2.56	(11) 1.91	...	...	1.12	0.94	0.95	0.92	0.75	0.59	....	0.36	0.33	0.35	0.36	0.39	....	....	0.36
8/ 2/1915	(11) 4.92	(7) 4.24	(4) 3.86	(8) 2.90	(8) 2.10	(16) 1.42	(16) 0.95	(15) 1.32	(16) 1.19	1.02	0.91	0.73	....	..	0.27	0.28	0.26	0.31	0.35	....	....	0.34
8/20/1915	....	....	....	....	....	0.75	0.72					....	....	0.46	....	....	...	....	....	....	0.64	......
5/28/1915	2.88	2.64	2.12	1.83	1.26	....	....	(14) 1.40	(14) 1.26	(14) 1.10	(14) 0.96	0.74	0.60	0.43	0.49	0.48	0.52	0.52	0.59	....	....	0.74
6/ 2/1916	3.96	3.18	(15) 2.68	(10) 2.58	....	....	....	(9) 1.55	(8) 1.51	(5) 1.47	(5) 1.35	(6) 1.09	(10) 0.85	(11) 0.61	0.44	0.48	0.55	0.52	....	....	....	0.57
8/11/1916	(15) 4.32	(13) 3.72	(10) 2.92	(13) 2.31	....	....	....	(10) 1.52	(12) 1.37	(13) 1.15	(13) 0.98	0.74	0.60	....	0.35	0.37	0.39	0.42	....	....	....	0.43
8/12/1916	2.40	2.04	2.07	1.95	(12) 1.90	(11) 1.82	(11) 1.30	1.28	(15) 1.26	(12) 1.25	(9) 1.22	(7) 1.08	(8) 0.96	(8) 0.70	0.53	0.62	0.61	0.63	0.57	0.53	0.54	0.60
8/14/1916	2.52	2.22	2.00	1.74	1.62	1.27	0.89	(13) 1.48	(11) 1.41	(11) 1.28	(11) 1.13	(10) 0.94	(12) 0.80	(12) 0.60	0.59	0.64	0.64	0.65	0.58	0.63	0.68	0.59
8/15/1916	....	....	....	....	....	1.27	0.86	....	....	....	....	....	(11) 0.85	(7) 0.74	....	....	....	....	....	0.67	0.86	......
9/ 7/1916	(6) 5.28	(8) 4.14	(11) 2.90	....	....	1.27	0.86	(5) 1.67	(5) 1.57	(9) 1.35	(12) 1.12	(12) 0.83	0.65	0.42	0.32	0.38	0.47	....	....	0.51	0.49	0.51
7/27/1917	(3) 6.00	(5) 4.63	(5) 3.72	(4) 3.47	(6) 2.70	(5) 2.40	(5) 1.98	1.06	(16) 1.03	1.00	(15) 0.95	(15) 0.80	0.64	....	0.18	0.22	0.27	0.27	0.30	0.32	....	0.30
8/24/1918	2.40	2.22	2.04	1.89	(15) 1.70	(14) 1.45	(13) 1.12	0.89	0.87	0.84	0.78	0.67	0.60	0.45	0.37	0.39	0.41	0.41	0.39	0.41	0.40	0.39
10/27/1918	4.20	(16) 3.42	2.48	....	....	....	....	0.92	0.86	0.73	0.60	0.43	0.33	....	0.22	0.25	0.29	....	....	....	....	......
9/28/1919	(14) 4.56	(9) 4.14	(7) 3.50	(3) 3.48	(3) 3.01	(4) 2.61	(4) 2.34	1.10	1.07	1.01	(16) 0.94	(13) 0.83	(13) 0.75	(10) 0.67	0.34	0.26	0.29	0.27	0.28	0.29	0.29	0.29
9/28/1919	2.76	2.34	2.04	1.98	1.56	1.52	1.27	0.95	0.90	0.86	0.82	0.69	0.62	(15) 0.50	0.34	0.38	0.42	0.41	0.44	0.41	0.39	......
4/19/1920	(9) 5.04	(15) 3.48	(12) 2.80	(15) 2.23	(9) 2.06	(8) 2.03	(10) 1.41	(3) 1.74	(7) 1.55	(10) 1.34	(10) 1.14	(5) 1.10	(7) 1.08	(5) 0.80	0.34	0.45	0.48	0.49	0.53	0.53	0.57	0.62
9/ 8/1920	4.20	3.20	2.56	2.19	(14) 1.76	....	....	(6) 1.64	(6) 1.56	(8) 1.40	(8) 1.25	(9) 0.96	(14) 0.75	....	0.39	0.49	0.55	0.57	0.54	....	....	0.60
3/26/1921	....	....	....	....	....	(13) 1.53	....	....	....	....	....	....	(4) 1.17	....	....	....	....	....	....	0.77	....	......
4/25/1921	4.20	3.18	(16) 2.66	(14) 2.25	(10) 1.91	(12) 1.74	(12) 1.27	(2) 1.90	(2) 1.74	(3) 1.51	(6) 1.33	(8) 1.05	(9) 0.94	(9) 0.68	0.45	0.55	0.57	0.60	0.55	0.54	0.54	0.61
8/26/1921	1.98	1.74	1.48	1.32	1.03	....	....	1.24	1.13	1.03	0.93	(16) 0.75	0.64	....	0.63	0.65	0.70	0.70	0.73	....	....	0.79
8/27/1921	(13) 4.72	(10) 3.84	(6) 3.71	(6) 3.30	(4) 2.90	....	....	(7) 1.61	(4) 1.60	(2) 1.54	(2) 1.46	(2) 1.34	(3) 1.21	(4) 0.85	0.34	0.42	0.41	0.44	0.42	....	....	0.49

TABLE 4.—(*Continued*)

Date	Rainfall							Run-Off							$\frac{\text{Run-Off}}{\text{Rainfall}}$							$\frac{\Sigma Q}{\Sigma i}$
	Duration Period, in Minutes							Duration Period, in Minutes							Duration Period, in Minutes							
	5	10	15	20	30	40	60	5	10	15	20	30	40	60	5	10	15	20	30	40	60	
(*b*) STATION *B*, CITY BLOCK NO. 4841 (*Continued*)																						
7/16/1923	(4) 5.52	(4) 4.68	(2) 4.01	(2) 3.66	(2) 3.43	(3) 3.35	(2) 2.44	(11) 1.51	(10) 1.47	(7) 1.42	(4) 1.39	(3) 1.33	(2) 1.24	(2) 0.90	0.27	0.31	0.35	0.38	0.39	0.37	0.37	0.36
8/ 8/1923	(1) 8.76	(1) 8.34	(1) 7.88	(1) 7.11	(1) 6.36	(1) 4.71	(1) 3.34	(1) 2.43	(1) 2.36	(1) 2.30	(1) 2.23	(1) 2.15	(1) 1.95	(1) 1.40	0.27	0.27	0.29	0.31	0.34	0.41	0.42	0.40
6/23/1924	(16) 4.32	(12) 3.78	(9) 3.32	(5) 3.42	(5) 2.80	(10) 1.94	(9) 1.44	(12) 1.50	(9) 1.47	(6) 1.46	(3) 1.40	(4) 1.26	(6) 1.10	(6) 0.80	0.35	0.39	0.44	0.41	0.45	0.57	0.55	0.53
8/24/1924	(5) 5.44	(3) 4.74	(3) 3.92	(7) 2.91	2.00	1.14	(15) 0.96	(4) 1.70	(3) 1.65	(4) 1.50	(7) 1.27	(11) 0.87	....	....	0.31	0.35	0.38	0.44	0.43	....	....	0.44
9/12/1925	....	....	....	....	....	(2) 3.36	(3) 2.43	....	....	....	....	....	(5) 1.16	(3) 0.88	....	....	....	....	....	0.35	0.36	......
9/ 8/1926	(10) 5.04	(6) 4.62	(8) 3.48	(9) 2.89	(7) 2.46	(6) 2.15	(6) 1.75	1.02	0.99	0.93	0.89	(14) 0.81	(15) 0.73	(13) 0.53	0.20	0.21	0.27	0.31	0.33	0.34	0.30	0.31
6/17/1928	3.84	3.12	2.32	1.83	....	....	....	(8) 1.61	(13) 1.29	(15) 1.04	0.85	0.39	....	....	0.42	0.41	0.45	0.47	....	....	....	0.48
(*c*) STATION *C*, EWING AND WASHINGTON AVENUES																						
8/11/1916	3.48	3.06	(11) 2.91	(12) 2.36	(11) 1.60	(9) 1.37	....	1.98	1.86	1.66	1.37	1.02	0.82	....	0.57	0.61	0.57	0.58	0.64	0.60	....	0.69
6/ 2/1916	3.72	(10) 3.36	(10) 2.92	(8) 2.53	(7) 1.82	....	....	1.94	1.70	1.53	1.33	1.00	0.80	...	0.52	0.51	0.52	0.53	0.55	....	....	0.57
9/ 7/1916	3.72	3.30	(8) 2.98	(10) 2.40	(9) 1.72	....	....	1.98	1.64	1.50	1.30	0.97	....	....	0.53	0.50	0.50	0.54	0.56	....	....	0.56
7/27/1917	(8) 4.80	(5) 3.96	(5) 3.36	(5) 2.79	(4) 2.18	(5) 1.65	1.12	2.22	(12) 2.10	(10) 1.80	(10) 1.58	(11) 1.20	(9) 0.98	(8) 0.69	0.46	0.53	0.54	0.57	0.55	0.59	0.62	0.59
8/ 5/1917	3.00	2.34	1.96	1.62	1.20	1.19	0.82	1.55	1.37	1.16	0.99	0.76	0.70	0.53	0.52	0.59	0.59	0.61	0.63	0.59	0.65	0.39
9/17/1917	(11) 4.56	(8) 3.42	(7) 3.00	(9) 2.43	....	....	....	(11) 2.62	(10) 2.17	(8) 1.90	(9) 1.60	(12) 1.17	....	....	0.57	0.63	0.63	0.66	....	....	....	0.74
8/24/1918	(3) 5.52	(2) 5.10	(1) 4.76	(1) 4.83	(1) 4.46	(1) 4.35	3.60	(1) 5.55	(1) 5.37	(1) 5.07	(1) 4.88	(1) 4.32	(1) 3.94	(1) 3.26	1.00	1.05	1.06	1.11	0.97	0.91	0.91	0.97
10/27/1918	(7) 4.92	(11) 3.36	2.26	1.70	1.20	....	....	(9) 3.20	(8) 2.38	(11) 1.78	1.42	1.04	(11) 0.88	....	0.65	0.71	0.79	0.84	0.87	....	....	0.93
6/17/1919	3.60	2.88	2.84	(6) 2.61	(5) 2.12	(4) 1.67	1.14	1.89	1.81	(12) 1.77	(7) 1.76	(5) 1.42	(4) 1.17	(4) 0.82	0.53	0.63	0.62	0.68	0.67	0.70	0.72	0.43
7/11/1919	(12) 4.56	(9) 3.42	(12) 2.88	(11) 2.37	(10) 1.64	(11) 1.26	0.88	1.97	1.73	1.60	(12) 1.44	1.09	(12) 0.86	(11) 0.61	0.43	0.51	0.56	0.61	0.67	0.68	0.69	0.74
9/21/1919	....	....	....	...	1.26	(12) 1.20	0.97	1.29	1.15	1.06	1.01	0.91	0.82	(7) 0.71	....	....	....	....	0.72	0.68	0.73	0.41
9/28/1919	(4) 5.52	(1) 5.34	(2) 4.52	(2) 3.90	(2) 3.12	(2) 2.94	2.92	1.58	1.48	1.40	1.32	(9) 1.29	(3) 1.29	(2) 1.27	0.29	0.28	0.31	0.34	0.41	0.44	0.44	0.21
9/ 8/1920	(6) 5.04	(7) 3.48	(6) 3.20	(7) 2.61	....	....	....	(5) 3.50	(6) 2.67	(4) 2.39	(4) 2.02	(4) 1.43	(7) 1.09	....	0.69	0.77	0.75	0.77	....	....	....	0.83
9/11/1920	2.04	1.86	1.88	1.71	1.30	1.11	0.85	1.13	1.06	1.00	0.89	0.73	0.58	0.41	0.55	0.57	0.53	0.52	0.56	0.52	0.48	0.60
4/25/1921	(1) 6.10	(4) 4.62	(4) 3.52	(4) 2.88	(3) 2.40	(3) 2.10	1.46	(3) 4.13	(3) 3.03	(5) 2.33	(2) 2.55	(2) 2.13	(2) 1.77	(3) 1.23	0.68	0.66	0.66	0.89	0.89	0.82	0.84	0.74
6/23/1924	2.64	2.10	1.80	1.79	1.50	(10) 1.30	1.26	1.60	1.39	1.20	1.08	0.84	....	....	0.61	0.66	0.67	0.72	0.56	....	....	0.49
8/24/1924	(2) 5.76	(3) 4.86	(3) 3.92	(3) 3.00	(6) 2.06	(6) 1.61	1.10	(2) 4.43	(2) 3.67	(2) 2.86	(3) 2.30	(3) 1.55	(5) 1.17	(6) 0.78	0.77	0.75	0.73	0.77	0.75	0.73	0.71	0.78
6/17/1925	3.96	3.00	2.32	1.89	....	....	....	(8) 3.23	(9) 2.32	(9) 1.82	(11) 1.46	(7) 1.35	....	....	0.81	0.77	0.78	0.77	....	....	....	0.76
6/28/1925	(9) 4.68	(12) 3.36	1.30	....	....	....	....	(6) 3.44	(5) 2.77	(6) 2.23	(6) 1.80	(10) 1.26	....	....	0.74	0.83	1.72	....	....	....	....	0.94
8/11/1925	(5) 5.40	(6) 3.54	(9) 2.96	2.25	(12) 1.54	(8) 1.40	1.05	(4) 3.54	(4) 2.79	(3) 2.39	(5) 1.97	(6) 1.35	(8) 1.08	(5) 0.82	0.66	0.79	0.81	0.88	0.88	0.77	0.77	0.90
9/ 8/1926	2.40	2.22	2.00	1.92	(8) 1.74	(7) 1.54	....	1.52	1.48	1.39	1.39	(8) 1.29	(6) 1.14	....	0.63	0.67	0.69	0.72	0.74	0.74	....	0.79
9/30/1926	2.40	1.80	1.55	1.38	1.20	1.12	0.75	1.68	1.44	1.32	1.16	1.02	0.72	(12) 0.58	0.70	0.77	0.85	0.84	0.85	0.64	0.77	0.89
10/ 2/1926	(10) 4.68	2.52	....	....	....	....	....	(10) 2.66	1.95	1.48	1.17	....	....	....	0.57	0.77	....	....	....	....	....	0.91
10/ 2/1926	3.84	3.24	2.32	1.80	1.24	0.95	0.64	(7) 3.39	(7) 2.57	(7) 2.08	(8) 1.66	1.16	(10) 0.90	(10) 0.63	0.88	0.79	0.90	0.92	0.94	0.95	0.98	0.96
5/31/1927	2.88	2.34	1.85	1.46	....	....	....	(12) 2.40	(11) 2.13	1.68	1.40	1.04	0.82	(9) 0.68	0.84	0.91	0.68	0.96	....	....	....	More than 100%

Relation of the Final Value of P to Other Variables.—In Fig. 20 an attempt is made to correlate the value of $P_1 = \frac{\Sigma Q_t}{\Sigma i}$ with the various factors indicated, and the only apparent correlation is in the case of the maximum run-off rate for a 5-min period. This indicates that high rates of run-off are more likely to be the result of a combination of relatively low rainfall rates and high run-off factors than the result of high rates of rainfall. Similar graphs for the other locations show similar results.

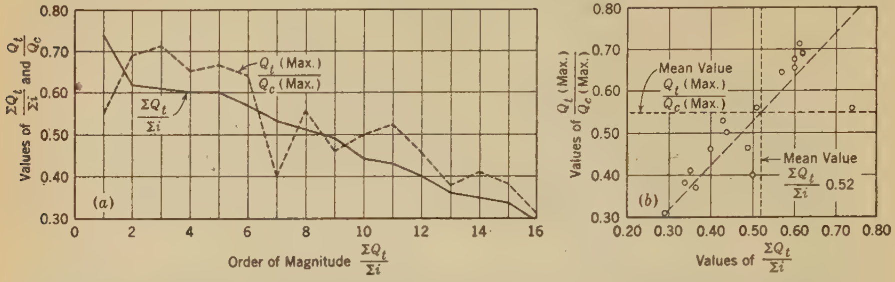


FIG. 19.—STATION *B*; CORRELATION DIAGRAM OF $\frac{\Sigma Q_t}{\Sigma i}$ AND $\frac{Q_t\,(\text{max})}{Q_c\,(\text{max})}$ FOR A 5-MINUTE DURATION PERIOD (BASED ON $j = 1$ AND $k = 1.2$).

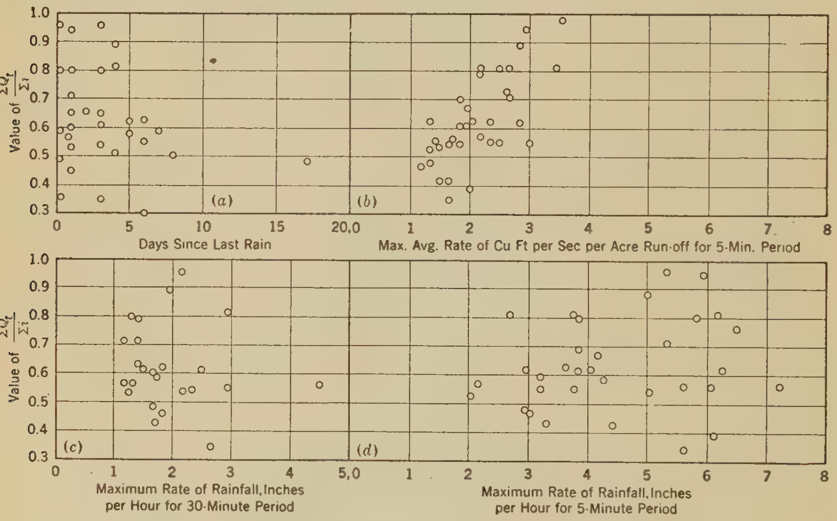


FIG. 20.—CORRELATION DIAGRAMS FOR $\frac{\Sigma Q_t}{\Sigma i}$; STATION *A*.

Ratio of Average Run-Off to Average Rainfall.—Pertinent information has been taken from the records and is given in Table 4. By means of these records it is possible to compare average rainfall rates with average run-off rates of the same duration, and also to compare these ratios with an end value of the run-off-to-rainfall ratio, P_1, as given in the last column.

In using Table 4, in the present case and in the frequency study that follows, it should be understood that the duration periods listed do not occur in the order listed for any particular storm; for example, at Station *A*, Table 4(*a*), for the rain of September 15, 1914, the entries under "20 minutes", for rainfall and run-off, which have numbers in parentheses of (11) and (17), are, respectively, the heaviest, average 20-min rainfall and the average rate of run-off for 20 min corresponding approximately with the particular rainfall occurrence. It is shown that the ratio of these average rates is 0.60 which corresponds closely to the ratio of the mean value for this storm. The numbers in parentheses correspond to the positions of these values in the frequency studies described subsequently.

Ratios of average run-off to average rainfall rates for 5-min duration and for 10-min duration have been taken from Table 4. As these values also vary over a wide range, they have been plotted as percentile curves, Fig. 18(*e*) and Fig. 18(*f*), and may be compared with other diagrams in Fig. 18. It should be noted that, while values on the various curves of Fig. 18 differ materially with the character of the ratio, the distribution of these values seems to follow about the same system for each diagram, as is indicated by a near parallelism of the lines. The close relationship of actual ratio values of curves in Fig. 18(*e*) and Fig. 18(*f*) with those of Fig. 18(*a*) is important.

General Conclusions to Part I

Certain general conclusions, which the writers are inclined to draw from these exhibits are, as follows:

(1) The value of all run-off ratios studied for individual storms varies between wide limits at each observation station.

(2) The values of "P_1 (final)" have been plotted as percentile curves for each of the areas under study. This curve, given as Fig. 18(*a*), is similar to Fig. 18(*c*), Fig. 18(*d*), Fig. 18(*e*), and Fig. 18(*f*), and shows that for each of the areas studied, P varies quite widely from the mean; but, as with the other factors, $\left(\frac{Q_{t\,(\max)}}{Q_{c\,(\max)}}\right.$ for absolute peaks; $\frac{Q_t}{Q_c}$ for 5-min averages at peaks; and $\frac{Q_t}{i}$ for 5-min and 10-min averages at peaks$\left.\right)$, the trend of the three curves is surprisingly uniform in character, indicating that the variables which cause this divergence enter into the occurrence for each of the drainage areas in about the same way. Straight lines have been drawn in Fig. 18(*a*) to represent an average of the percentile points. These lines indicate that the run-off at Station *A* throughout the full range of values is likely to exceed that at Station *B* by about 20%, while the rate for Station *C* will exceed that at Station *A* by about 10 per cent.

(3) For a given drainage area, the run-off factor, *P*, seems to be affected by the season of the year, general climatic conditions, previous precipitation, etc., but no co-ordination has been determined.

(4) No evidence of general correlation has been found between the "final value" of the factor, $P_1 = \frac{\Sigma Q_t}{\Sigma i}$, for individual storms, and the rainfall characteristics of the storms (such as maximum i or mean i). However, the variation of the run-off factor during the progress of a storm seems to depend, to some extent, upon the variation of the rainfall intensity. The effect is not pronounced except in extreme cases, such as double or multiple rains, when the rainfall intensity falls to low values for considerable periods between the peaks.

(5) The ratio, $\frac{\Sigma Q_t}{\Sigma i}$, or total measured volume of run-off to total measured volume of rainfall, has been found to have a close correlation with the ratio, $\frac{Q_{t\,(\max)}}{Q_{c\,(\max)}}$, based on 5-min duration periods (see Fig. 19, for example). The ratio, $\frac{\Sigma Q_t}{\Sigma i}$, can be found conveniently with great accuracy, and provides the best information as to the general relation between rainfall and consequent run-off rates for any given storm.

(6) No doubt, the value of the run-off factor, P, is greatly affected by the nature of the soil, but the local records can throw no light on this phase of the subject, since the soil is the same yellow clay at each of the observation stations. The sprinkling experiments were undertaken to fix the values given in this paper to a definite soil condition. (See Table 8.)

(7) The wide variation in the "final values" of P at each of the sites must be attributed to combinations of three variables: (*a*) Condition of the soil; (*b*) condition attributed to coverage; and (*c*) character of the distribution of the rainfall rates:

(*a*) Variation in Soil Condition. — This variation probably is closely similar to that of the soil-moisture content, and accordingly is affected by previous precipitation and by the seasons of the year.

(*b*) Variation in Condition of Coverage.—The effect of the character of the turf on the lawns and of the foliage on trees and shrubbery, also would appear to have some relation to the season of the year.

(*c*) Character of the Distribution of the Rainfall Rates.—The final value, P (which is the ratio of total run-off to total rainfall), is not subjected to the effect of distribution which the unit-graph study brings to the values of $\frac{Q_t}{Q_c}$ during the rainfall. Accordingly, it will vary seriously with the manner in which precipitation departs, in one way or another, from the condition of uniform intensity.

SUMMARY OF CONCLUSIONS TO PART I

In Part I, studies have been made of the run-off factor under the unit-graph method in three forms: First, as the ratio, $\frac{Q_t}{Q_c}$, or instantaneous ratios; second, as the ratio, $\frac{Q_t}{Q_c}$, for generally coincident peak values; and, third, as $P = \frac{\Sigma Q_t}{\Sigma Q_c}$, a moving ratio of mass values.

Studies have been made of the direct relation between run-off and rainfall: First, as ratios of average run-off rates to average rainfall rates for the same duration; and, second, as the final value, $P_1 = \frac{\Sigma Q_t}{\Sigma i}$. These experimentally determined ratios have been found to vary over a wide range, and a satisfactory correlation to other variables has not been found.

The unit-graph process is probably the best yet developed for analyzing rainfall and run-off data of the kind presented in this paper. If some modification of it can be applied generally to run-off data hereafter available, it will be possible to study the remaining variables entering into the relationship between rainfall and run-off in a much simpler form than heretofore has been undertaken.

Further prosecution of the investigation undertaken herein, might appear to involve a series of studies. First, the unit-graph method would yield a factor similar to the coefficient of retardation, representing the ratios between $Q_{c(\max)}$ as used in this section, and average rainfall intensity for particular duration periods classified as to frequency of occurrence. The second series would involve a determination of the values of $\frac{Q_t}{Q_c}$, a run-off factor, as has been attempted, with the proper relation to frequency of occurrence.

If such information were satisfactorily developed, the run-off rates would be determined by applying to average rainfall intensity of the proper frequency and duration, a coefficient of retardation, thus reducing it to a 100% run-off rate, and, thereafter, applying the value of $\frac{Q_t}{Q_c}$ to reduce to actual run-off rates.

Such an undertaking, however, becomes involved in numerous complications. It appears that the same results could be secured by a simpler means, although probably in a less scientific manner, if the information on rainfall and on run-off was made the subject of frequency-of-occurrence studies as if the two phenomena were independent.

PART II.—Frequency Study of Run-Off Rates

In Part II, run-off is considered as an independent phenomenon, and a frequency study is made of the average rates of run-off for various duration periods at each observation station, in order to determine the answer to the question, "How often in a given number of years will a given inlet area discharge storm water at a given rate for a given duration period?"

Frequency is expressed in years and has reference to the period of time during which an event of given magnitude will probably be equalled or exceeded one time. Table 4 shows the basic data used in the frequency study. A period of years was chosen for each location such that a continuous series of the best records for that location was available. At Station A, the nineteen years, from 1914 to 1932, inclusive; at Station C, the twelve years, from 1916 to 1927, inclusive; at Station B, the sixteen years, from 1914 to 1929, inclusive; and at the United States Weather Bureau Station, the twenty-two years, from 1907 to 1930, inclusive, were chosen as the most representative. (On September 29, 1925, the rain gauge at Station B was taken out of service. After that date, rainfall rates for this location were obtained by interpolation from three near-by gauges.)

These series although shorter than might be desired, are longer than the usually accepted minimum of about ten years. At each station all storms having significant average run-off rates (about 1.0 cu ft per sec per acre, or more) for the duration periods represented in Table 4, are listed in chronological order.

Under the heading, "Rainfall", in Table 4, is listed the highest average rate of rainfall, observed during the storm in question, for the various duration periods. Under the heading, "Run-Off," is listed the highest average rate of run-off observed during the storm in question, for the various duration periods of run-off.

The numbers in parentheses, shown with the rates, represent the order in magnitude of that rate for the duration period in which it is shown. For example, referring to Table 4(a), during the storm of September 15, 1914, as observed at Station A, the highest average rainfall rate for a period of 20 min is 2.47 in. per hr, and this rate for this duration period has been exceeded ten times in the nineteen years represented in the table. The highest average run-off rate, for a duration period of 20 min, is 1.47 cu ft per sec per acre, and this rate has been exceeded sixteen times during the nineteen years. Where no numbers in parentheses are shown, the rate has been exceeded more times than the number of years represented in the series.

The columns headed, $\frac{\text{Run-Off}}{\text{Rainfall}}$ (Table 4), contain the ratios of the run-off rates to the rainfall rates, as shown in the preceding columns. Of course, the duration periods shown for run-off rates are not identical in absolute time with the corresponding duration periods shown for rainfall rates, the difference being due to the lag. In one or two instances, the duration periods compared belong to entirely different phases of the rain.

The column headed, $\frac{\Sigma Q}{\Sigma i}$ (Table 4), shows the ratio of the total run-off to the total rainfall for the entire duration of the storm in question. This information is not given for a few storms of long duration where the rates are insignificant for most of the time, only the peak values being investigated.

Table 5, being taken from the records of the U. S. Weather Bureau, shows rainfall rates only. The information on which the frequency series was determined is complete and consecutive with the exception of one storm, that

TABLE 5.—DATA USED IN FREQUENCY STUDY OF RAINFALL (U. S. WEATHER BUREAU DATA)

Date	RAINFALL INTENSITIES, FOR THE FOLLOWING DURATION PERIODS (IN MINUTES):						
	5	10	15	20	30	40	60
August 7, 1907......	5.40 (6)	5.04 (1)	4.24 (2)	3.48 (5)	2.78 (4)	2.27 (6)	1.59 (7)
July 6, 1908.........	4.56	4.26 (8)	3.68 (6)	3.21 (6)	2.18 (10)	1.63 (13)	....
July 10, 1909........	....	....	....	....	1.60	1.44	1.37 (10)
September 4, 1910...	3.72	3.18	2.96	2.67 (14)	2.14 (12)	1.82 (8)	1.29 (12)
September 4, 1911...	3.12	2.40	2.24	2.04	1.68	1.47	1.29 (11)
June 5, 1912........	....	....	1.64	1.35	1.34	1.16	1.29 (13)
July 14, 1912.......	4.56	4.44 (6)	3.92 (5)	3.60 (3)	3.34 (1)	3.24 (1)	2.96 (1)
July 1, 1913........	4.44	4.02 (12)	3.12	2.49	1.82	1.50 (14)	1.08
August 19, 1914.....	3.12	2.88	2.68	2.64	1.92 (13)	1.49	....
September 1, 1914...	5.28 (7)	4.68 (4)	4.08 (3)	3.75 (2)	3.06 (3)	2.67 (4)	1.86 (5)
September 15, 1914..	....	....	2.08	2.13	1.90	1.79 (10)	1.49 (9)
June 20, 1915.......	3.24	3.18	2.64	2.16	1.52	1.67 (12)	1.55 (8)
August 11, 1916.....	4.32	4.08 (10)	3.52 (8)	2.67	1.82	1.30	....
August 12, 1916.....	5.28 (8)	4.74 (3)	4.32 (1)	3.81 (1)	3.10 (2)	2.79 (2)	1.87 (4)
August 14, 1916.....	5.64 (4)	4.38 (7)	3.36 (11)	2.97 (8)	2.74 (5)	2.70 (3)	2.03 (3)
September 27, 1916..	6.72 (1)	4.98 (2)	3.40 (9)	2.70 (13)	1.92 (14)	1.46	....
October 30, 1916....	4.92 (11)	3.06	....	....	....	....	....
July 23, 1917.......	6.12 (2)	4.56 (5)	4.00 (4)	3.60 (4)	2.58 (6)	....	....
September 7, 1917...	5.52 (5)	4.02 (13)	3.40 (10)	2.82 (9)	1.88	1.50	0.99
August 24, 1918.....	3.84	3.48	2.96	2.73 (11)	2.22 (9)	2.09 (7)	1.80 (6)
May 4, 1919.........	5.64 (3)	4.08 (11)	2.88	2.25	1.56	1.19	....
June 17, 1919.......	4.20	3.48	3.68 (7)	3.21 (7)	2.34 (8)	1.80 (9)	1.21 (14)
June 22, 1919.......	4.80 (12)	3.12	2.32	....	....	....	....
September 28, 1919..	3.48	3.42	3.12	2.82 (10)	2.50 (7)	2.58 (5)	2.40 (2)
August 15, 1920.....	4.80 (13)	3.96	3.20 (13)	2.55	1.78	1.36	....
September 8, 1920...	5.04 (10)	3.96 (14)	3.28 (10)	2.52	....	....	....
September 8, 1920...	5.28 (9)	4.26 (9)	3.20 (14)	2.70 (12)	2.14 (11)	1.71 (11)	....
December 13, 1920...	4.60 (14)	3.12	2.32	....	....	....	....

of August 8, 1923, for Station *C*, and is accurate within the limitations of the instruments used. The years, 1918 and 1932, are not represented in Table 4(*a*) for Station *A*, nor are the years, 1922 and 1929, in Table 4(*b*) for Station *B*, as no rains of sufficient magnitude occurred in these years to warrant inclusion. Several storms were included for which the data were open to suspicion on the grounds of non-conformity, such as that of September 12, 1925, at Station *A*, and the small rains showing more than 100% run-off, but for which no definite defect in the record could be found.

Derivation of Series.—A frequency series was made up for each vertical column under the headings, "Rainfall" and "Run-Off", in Table 4. The rates taken from each column were arranged in the order of magnitude beginning with the highest. Since the class of storms that occur, on an average, more than once each year are of little or no interest in design, the number of terms in each series was arbitrarily limited to the number of years represented in Table 4 from which it was taken. This makes the number of events equal to the number of years in the series, facilitating the computation of the time frequency.

The frequency percentages for each series were computed from the formula[4]:

$$F = \frac{2\,m - 1}{2\,n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

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[4] "Method of Least Squares", by Mansfield Merriman, M. Am. Soc. C. E.

in which, F = percentage of frequency; m = ordinal number of the item in the series (the maximum being No. 1); and, n = number of items in the series.

The rates of rainfall and run-off as selected for the various series were plotted against the corresponding frequency percentages on logarithmic probability paper (Fig. 21). Since direct comparisons were to be made between the different series, it was decided to derive the curves through the plotted points by computation, using statistical rather than graphical methods, in order to eliminate the personal equation and assure that each series would receive exactly the same treatment.

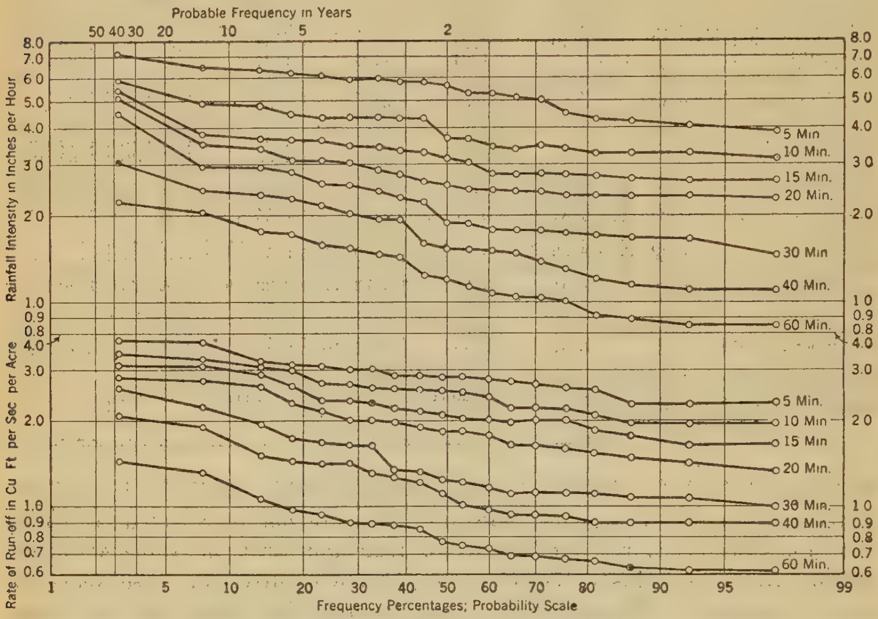

FIG. 21.—STATION *A*. RAINFALL AND RUN-OFF FREQUENCY SERIES, 1914 TO 1932, INCLUSIVE; RATE FREQUENCY CURVES.

The method used in computing the points on the smooth curves in Fig. 21 consists of calculating: First, the mean value of the terms in each series; and, then, the coefficients of variation and skew for the series. Knowing the value of these coefficients, the co-ordinates of points on the skew frequency curve were calculated from the information given in the table[5] of skew curve factors by H. Alden Foster, M. Am. Soc. C. E. Details of these computations and complete sets of curves are filed with the record manuscript in Engineering Societies Library.

The curves, such as those shown in Fig. 21, indicate the probable frequency of occurrence of given rates of rainfall and run-off, expressed as a percentage of the total number of events of the same class in a given array;

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[5] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), Table 2, p. 162.

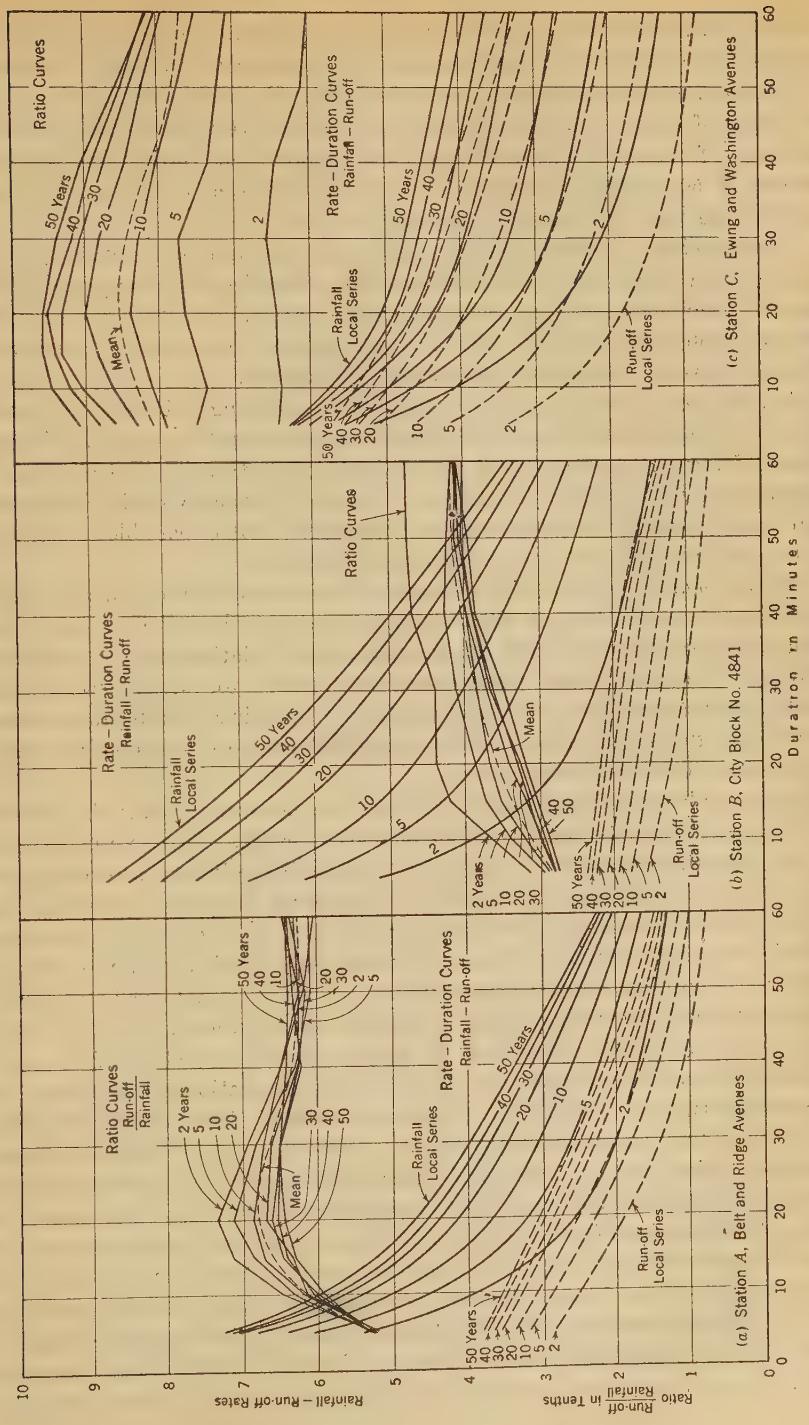


FIG. 22.—RELATION OF RATES OF RUN-OFF AND RAINFALL TO DURATION OF THESE RATES.

for example, a rainfall or run-off rate having an event frequency of 20%, as shown by the graph, will probably be equalled or exceeded once in five years.

Transposing the co-ordinate axes and plotting the data given by the curves of the type of Fig. 21, a series of curves are obtained which show the relation of the rates of run-off and rainfall to the duration of these rates for frequencies of 50, 40, 30, 20, 10, 5, and 2 years. These curves (Fig. 22) correspond to the familiar "rainfall-intensity" curve.

Correspondence Between Series.—There is no causal connection between corresponding points on the rainfall and the run-off curves for a given frequency. Such points do not represent actual storms, but only rates having the same probable frequency. The curves represent variations in concurrent, but not necessarily completely dependent, phenomena.

Fig. 22 shows also the ratios between corresponding rainfall and run-off rates for various frequencies taken from these rate-duration curves. The usefulness of these ratios lies in the fact that the data from several observation stations may be combined to provide a rainfall frequency series of greater length and possessing greater scope and accuracy than is possible in the case of the local rainfall and run-off frequency series.

Transformation to Long-Time Series.—The ratios between corresponding rainfall and run-off rates, as shown by the local rate-duration curves, reflect the local conditions that affect the rainfall-run-off relations at the station in question. Therefore, when these ratios are applied to rainfall rates given by the long-term rainfall frequency series, the resulting rate-duration curves should show the most probable values for run-off rates for various frequencies and duration periods at each station.

Another advantage of this method is that it takes into account the position factor of rainfall rates and its effect on run-off rates. A high rainfall rate at the beginning of a storm, may result in an insignificant rate of run-off while a much lower rate of rainfall of the same duration, occurring late in a storm, may result in a relatively high rate of run-off. In the long-term rainfall frequency series, all types of storms are included, and the position factor of the various rainfall rates is taken into account (at least indirectly) with the relative frequency of the various types of storms.

A long-time series, comprising 47 station-yr, may be obtained by combining the rainfall records of the three observation stations; but two of the stations (Stations *A* and *B*), are quite close together, while the third (Station *C*), is about four miles away (see Fig. 1). Since the U. S. Weather Bureau Station is fairly close to Station *C*, this fault may be minimized by combining the records from these four stations, thus obtaining a 61-station-yr series. It will be seen from Tables 1 to 5 that few critical storms are common to all four stations, and that the record at no one of the stations having a length of from 12 to 17 yr can be truly representative of the rainfall probabilities for the short storms of the St. Louis region. This 61-station-yr series, shown in Fig. 23(*b*), has been adopted as the basis of the final frequency study. It could have been improved somewhat by using all the rain-gauge sta-

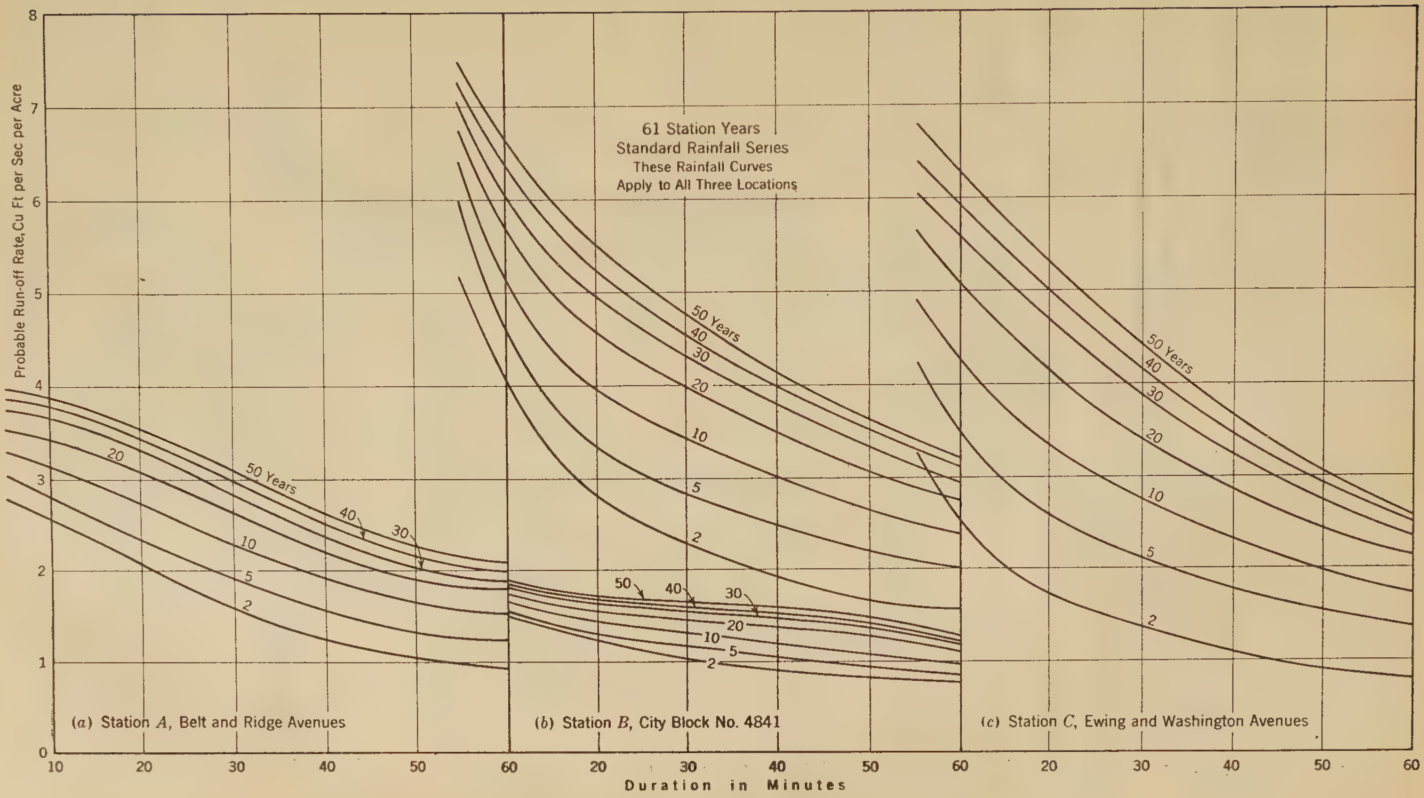


FIG. 23.—ADJUSTED RUN-OFF RATE CURVES BASED ON STANDARD RAINFALL SERIES.

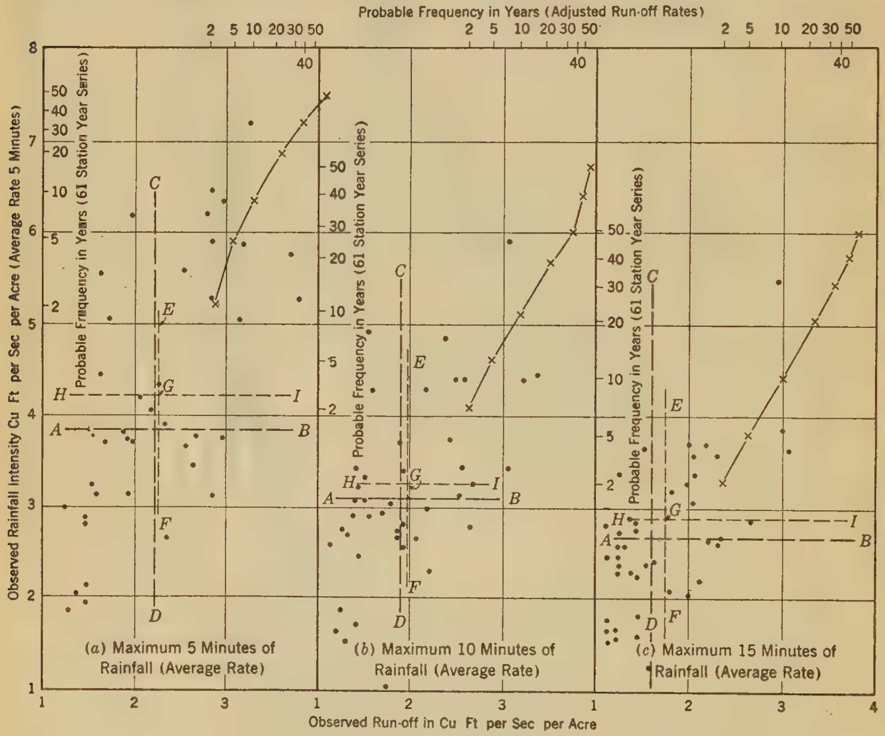
tions in St. Louis, but this would have had the disadvantage of not involving to so great an extent the localities or the rainfalls entering into the run-off study.

For each of the three blocks studied, frequency curves were developed by applying to the 61-yr rainfall series the ratio between rainfall and run-off shown graphically in Fig. 22. Thus, the modified run-off frequency series as given on the lower part of Fig. 23 is assumed to represent equivalent probabilities with the 61-yr rainfall series.

To determine whether or not the basic information had been in any way distorted in its manipulation through the frequency studies, a second study was made in the form indicated in Part III.

Correlation Diagrams—Rainfall and Run-Off Rates for Various Duration Periods

The graphs in the series of curves represented by Fig. 24, of which only three are published, are compiled from two independent sets of data. The plotted points on each graph represent the average rate of rainfall and run-off for the duration period for which it was prepared. Each point represents a single storm listed in the table from which the rainfall and run-off frequency series were compiled. Points above Line *A-B* (Fig. 24) represent storms included in the rainfall frequency series. Points to the right of Line *C-D*



(*a*) Maximum 5 Minutes of Rainfall (Average Rate)

(*b*) Maximum 10 Minutes of Rainfall (Average Rate)

(*c*) Maximum 15 Minutes of Rainfall (Average Rate)

Fig. 24.—Correlation Curves of Rainfall and Run-Off, Station *A*.

represent storms included in the run-off frequency series. Lines *E-F* and *H-I* define the mean values of the rainfall and run-off rates in relation to the vertical and horizontal axes, respectively, thus establishing Point *G* as the center of gravity of all the points.

Referring to Table 4(*a*), it is seen that for the first storm listed at Station *A* (September 15, 1914), the maximum average rainfall rate for a 5-min period is 3.12 in. per hr, and the maximum average run-off rate for the same period is 1.89 cu ft per sec per acre. The point representing this storm, therefore, is located from these co-ordinate values on Fig. 24(*a*).

The frequency scales shown on the rainfall and run-off rate axes are taken from the rate-duration curves (see Figs. 23(*a*) and 23(*b*)) where it is found that the rainfall rate having a probable frequency of 2 yr (for 5-min duration period) is 5.10 in. per hr, while the corresponding run-off rate is 2.85 cu ft per sec per acre, etc., for each frequency.

The points designated by crosses in Fig. 24, and connected by solid lines, are plotted opposite corresponding frequencies as shown on the rainfall and run-off axes on the right-hand top margins of each chart. These points do not represent actual storms, but only abstract rainfall and run-off rates having the same probable frequency, in years, as determined by the rate-frequency graphs for Station *A*. The relation of this line to the points shows that, although points on the frequency curve were extrapolated beyond the observed data, the general trend of the ratio of run-off rate to rainfall rate as observed in the original data has been preserved in the extrapolated values.

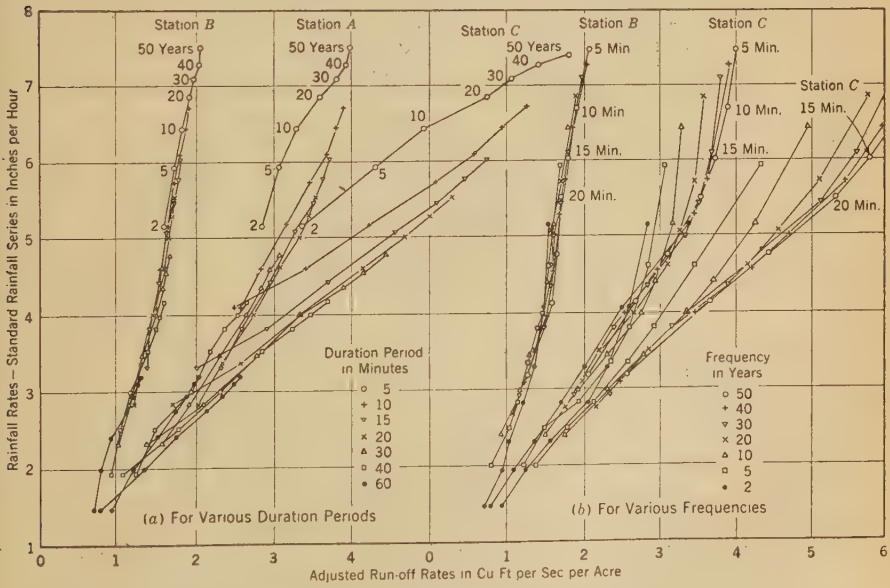


FIG. 25.—RELATION BETWEEN RAINFALL AND RUN-OFF RATES (SEE TABLE 6).

From this series of diagrams two composite charts (Fig. 25 and Table 6) were compiled, which gave identical information as to the plotted points,

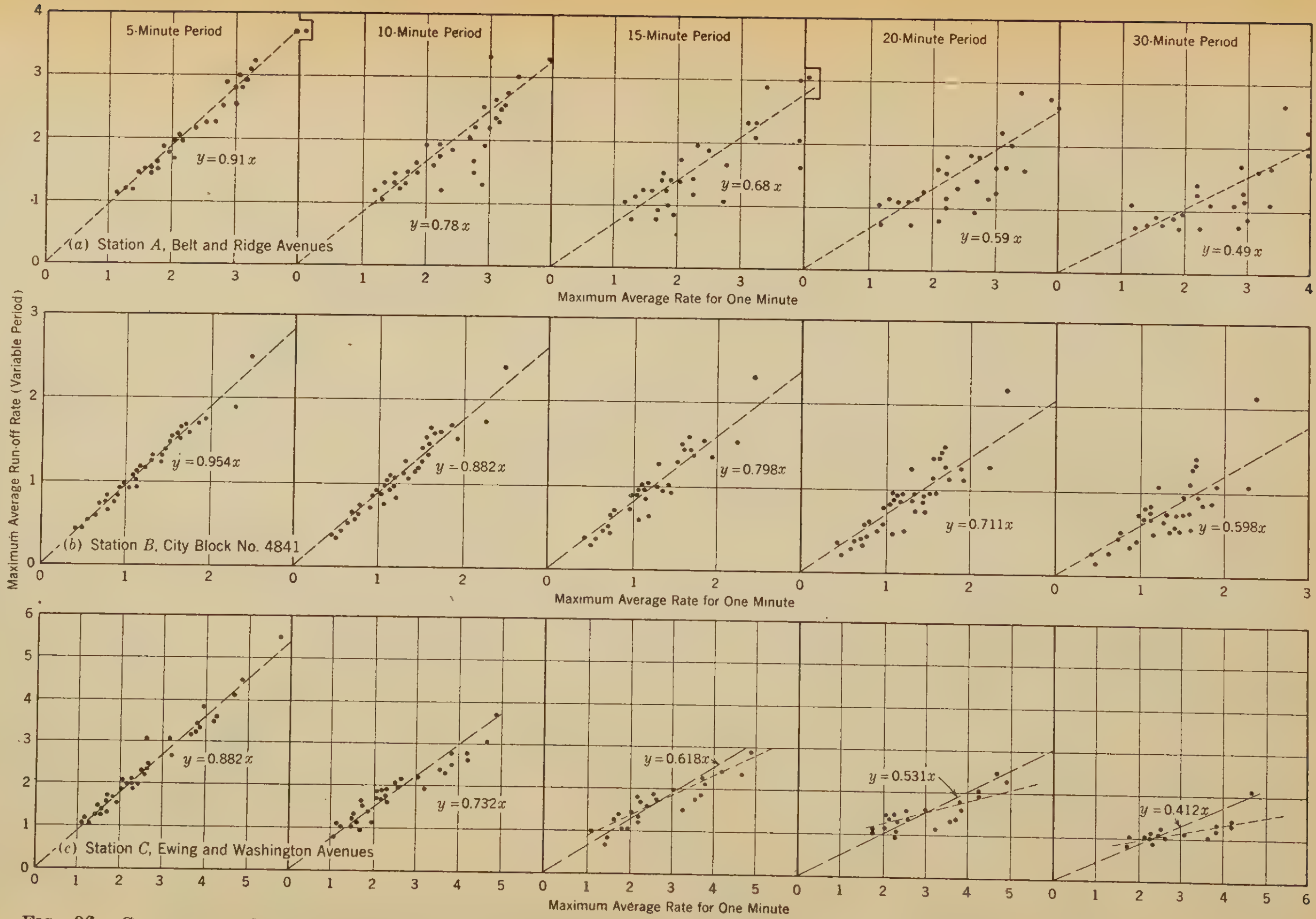


FIG. 26.—CORRELATION DIAGRAMS; MAXIMUM AVERAGE RUN-OFF RATES FOR VARIOUS DURATION PERIODS COMPARED WITH MAXIMUM AVERAGE RATE FOR 1 MINUTE (UNITS IN CUBIC FEET PER SECOND PER ACRE)

and differed only in the connecting lines shown between the points. They involve essentially the same information that is presented in Fig. 23. Each of the diagrams offers different aspects.

TABLE 6.—RELATION BETWEEN RAINFALL AND RUN-OFF RATES FOR VARIOUS DURATION PERIODS AND FREQUENCIES (SEE FIG. 25)

Description	Station *A*	Station *B*	Station *C*
Tributary area, in acres	2.25	3.25	4.33
Percentage of area, impervious	50	29	72
Percentage of area, pervious	50	71	28
Average slope (percentages)	3	0.7	2.0

Figs. 23 and 25 contain the results of the run-off frequency study. However, in Fig. 26, this material has been set up to present one other relationship. These diagrams were prepared by comparing the average run-off rates for particular duration periods, as, for example, 5 min and 10 min, with the maximum average run-off rate for 1 min for the same storm. The information as to the average 5-min and 10-min rates was taken from Table 4. The tabulated data for the 1-min rate are not presented, but they were taken directly from the hydrographs. The relationships indicated in these diagrams, as summarized on Fig. 27, are surprisingly consistent. It should be noted that the relationship for Station *C* (Ewing and Washington Avenues) for the 20, 25, and 30-min periods, does not seem to follow a slope line through the origin as clearly as is the case for other curves, and the line drawn is a mean used only for the purpose of preparing the curve on Fig. 27. While it is not expected that the relations given by the curves can be of immediate use in connection with sewer design, they are interesting and should be of real value to the hydrologist.

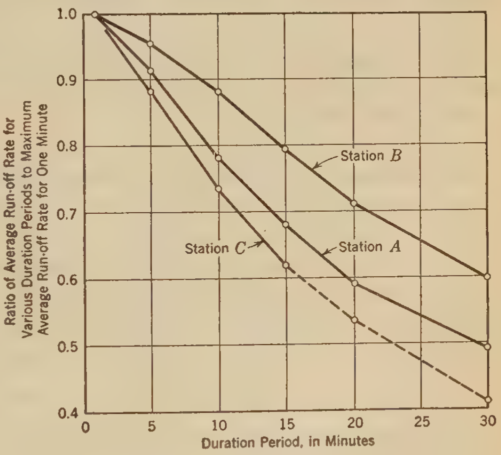


FIG. 27.

The run-off values shown on Fig. 22(*a*) for Station *A* and those for Station *B* in Fig. 22(*b*), seem to fit into a logical conception of the effect of topography and surface condition on run-off rates. The curve for Station *C* (Fig. 22(*c*)) giving the result for an entire city block, including many sub-areas such as roofs and yards that drain directly to the sewer, is not strictly comparable to the information for the other two blocks. The very high run-off rates shown in this case were surprising to the writers, even in view of their familiarity with conditions.

Figs. 22(*a*) and 22(*b*), show that for the two smaller locations, the greatest ratio between run-off and rainfall occurs for the lighter and more frequent rains; for Area *C* the reverse is true. Tracing the reversal phase back into the primary data, it is noted that the corresponding rainfall and run-off curves for Area *C* diverge to the right, whereas for the other sites these curves approach each other for the longer time.

This means that the range of run-off rates at Station *C* is greater than the range of rainfall rates; that is, the drainage system at Station *C* is more sensitive to changes in rainfall rates than the systems at the other locations, due probably to the manner of collecting the water. The mean velocity from points on Area *C* is greater than that from points on Area *A* at the same distance from the point of measurement.

Fig. 25 brings out in an interesting manner the fact that the run-off characteristics of the three sites are quite similar for the light rainfalls, and it is only in the field of the rarer occurrences and greater intensities that the marked differences appear. For Station *B*, the relationship of rainfall and run-off is affected, to a relatively small degree, by variation in duration or in frequency; for Station *A*, more important differences appear on this account; and, for Station *C*, the effects are distinctive.

CONCLUSIONS TO PART II

At various times during the period, 1924 to 1934, the writers and their associates have attempted to organize the information given herein in a manner that would justify publication. The present effort at analysis has extended over a period of nearly two years, and the writers now recognize, more fully than ever, the inadequacies of the results secured. They are more than ever impressed with the certainty that much of importance still lies buried in the available data.

To the hydrologists, they believe that they are presenting hitherto unpublished information susceptible of detailed study and interpretation.

For the engineer designer in the municipal drainage field, the information contained in Part II should be a valuable guide to judgment, although it is only a small fragment of the information desirable for reaching a sound basis for storm-sewer design. In Part III a method is suggested in the application of which the information presented will be usable in a broader field without departing too far from the available facts and figures.

PART III.—SUGGESTIONS FOR APPLICATION TO SEWER DESIGN

A study of the diagrams, Figs. 23 and 25, brings out the following crude relationships:

(1) For flat areas of relatively low percentage of imperviousness (such as, Area *B*), the relation between rainfall and run-off of equal probability does not vary greatly with frequencies or durations; the average ratio is slightly less than 0.4

(2) For steeper and less permeable areas (such as Area *A*), the correlation is fairly uniform except for the shorter 5-min rains, and the average ratio is about 0.65.

(3) For the closely built-up block (such as Area *C*), the correlation is not so simple, and the average values are greater than 0.8.

(4) Table 7 gives in round figures the principal characteristics of the three test sections.

TABLE 7.—PRINCIPAL CHARACTERISTICS OF AREAS STUDIED

Area (see Fig. 16) (1)	Average slope (percentages) (2)	Percentage of area, impervious (3)	General average ratio of run-off rates to rainfall rates (4)
B	0.7	30	0.4
A	3.0	50	0.65
C	2.0	72	0.8

(5) With only three sites sufficient sets of information are not available to permit of a proper attempt to isolate the effect on the run-off ratio (Column (4) Table 7), of the variable in Column (2), from that of the variable in Column (3). It has been interesting, however, to attempt to set up a series of ratios for the pervious and impervious areas, found by "cut-and-try", that would develop the actual run-off rates given in the final diagrams.

A typical, although incomplete, study of this type for a 15-yr rain is presented in Table 8.

TABLE 8.—EFFECT OF GROUND SURFACE SLOPE ON THE RUN-OFF RATIO (COLUMN (4), TABLE 7) (FIFTEEN-YEAR STORM DATA)

Item No.	Area (see Figs. 2, 3, and 4)	PERCENTAGES OF AREA:		Percentage of run-off, *P*	Rainfall rate, in inches per hour	RUN-OFF, IN UNITS, INCHES PER HOUR		
						Computed		Measured
		Impervious	Pervious			Product of Columns (3), (4), and (5)	Total	
(1)	(2)	(3a)	(3b)	(4)	(5)	(6)	(7)	(8)
(a) DURATION PERIOD, 50 MINUTES								
1...	*B*	30	....	80	2.80	0.67	1.16	1.15
2...	*B*	....	70	25	2.80	0.49		
3...	*A*	50	....	90	2.80	1.26	1.89	1.80
4...	*A*	....	50	45	2.80	0.63		
5...	*C*	72	....	90	2.80	1.81	2.28	2.20
6...	*C*	....	28	60	2.80	0.47		
(b) DURATION PERIOD, 30 MINUTES								
7...	*B*	30	....	80	3.70	0.89	1.46	1.40
8...	*B*	....	70	22	3.70	0.57		
9...	*A*	50	....	90	3.70	1.67	2.41	2.45
10...	*A*	....	50	40	3.70	0.74		
11...	*C*	72	....	90	3.70	2.40	3.02	3.10
12...	*C*	....	28	60	3.70	0.62		
(c) DURATION PERIOD, 20 MINUTES								
13...	*B*	30	....	75	4.3	0.97	1.57	1.50
14...	*B*	....	70	20	4.3	0.60		
15...	*A*	50	....	88	4.3	1.89	2.64	2.90
16...	*A*	....	50	35	4.3	0.75		
17...	*C*	72	....	88	4.3	2.72	3.39	3.70
18...	*C*	....	28	56	4.3	0.67		

Computation Table 8 is not offered as a good solution of this problem; the writers merely wish to leave it to practical sewer designers for further analysis. Note that the "pervious" surface on Areas *A* and *B* (Fig. 5(*a*)) is turf, and that on Area *C* (Fig. 5(*b*)), it is small areas of packed soil.

As an indication of the effect of slope on run-off from sodded areas such as lawns, a number of diagrams were drawn of the St. Louis sprinkling experiments of 1923 (filed with the record manuscript in Engineering Societies Library). Three runs each from Plot *B* (5% slope) and from Plot *C* (0.8% slope) are analyzed, using rainfall rates of 2½ in. (adjusted), and comparing run-off rates that seem to prevail for some time (30 to 40 min) after run-off begins.

A rough comparison of values in Table 9, in the absence of pondage, indicates that the run-off from good slopes may be 15% greater than from practically flat areas. Measured run-off rates from Area *A* (3% slope) are nearly two-thirds greater than from Area *B* (0.7% slope), but the former is also less pervious, while pondage is prevalent in the latter. These data indicate a difference in the water lost—that is, in water not appearing as run-off—of 11% as between the 0.5% slope and the 5% slope.

TABLE 9.—EFFECT OF SLOPE ON RUN-OFF FROM SODDED AREAS

(*a*) PLOT *B*, 5% SLOPE						(*b*) PLOT *C*, 0.8% SLOPE					
Date	Ground conditions	Run-Off Rate, in Units:		Loss		Date	Ground conditions	Run-Off Rate, in Units:		Loss	
		Each observation	Average	Rate, in units	Percentage			Each observation	Average	Rate, in units	Percentage
May 9, 1923	Damp	2.00	1.93	0.57	20	Aug. 31, 1923	Damp	1.90	1.73	0.77	31
July 26, 1923	Dry	2.00				May 12, 1923	Dry	1.70			
Aug. 1, 1923	Wet	1.80				July 26, 1923	Damp	1.60			

Within narrow limits, these values may be used to adjust, for slope, the coefficients in Table 8; for example, if the city block had all the characteristics of Area *A*, except that the slope is 2% instead of 3%, the coefficient of the pervious portion could be reduced from 0.45 to about 0.39. This should probably be charged against a slightly higher rate than the nominal run-off, to allow for water spilling from yard walks on to lawns, a condition equivalent to an increased rainfall rate on the lawns. In this instance, the revised run-off might be taken as 50% of 39% of 3 in. as compared with 50% of 45% of 3 in., a reduction in average run-off rate for the block from 1.89 to 1.80. Of course, modifications of this kind cannot be used too extensively or too far from the field of original figures. If used for very flat areas, such as Area *B*, allowance should be made for the effect of increased pondage.

The effect of soil characteristics on run-off rates, and on the probable frequency of occurrences of specific run-off rates, limits the direct value of the presented data to St. Louis conditions, or where all conditions are substantially equivalent to those of one of the blocks studied. The results can be modified. however, to be of value for other localities.

The volumetric ratio has proved to be the best index to the varying value of the ratio between run-off rates and rainfall rates during the progress of a storm. It provides a convenient means of comparison between different types of inlet areas (see percentile curves, Fig. 18(*a*)). It has given remarkably uniform and consistent results when applied to the sprinkling experiments of St. Louis and Dallas, Tex. (Complete supporting data are filed with the record manuscript, in Engineering Societies Library.) This ratio is suggested for use in adjusting run-off rates to other soil types, using the Dallas information, which is given in part in Fig. 28 and in Tables 10 and 11, as an example.

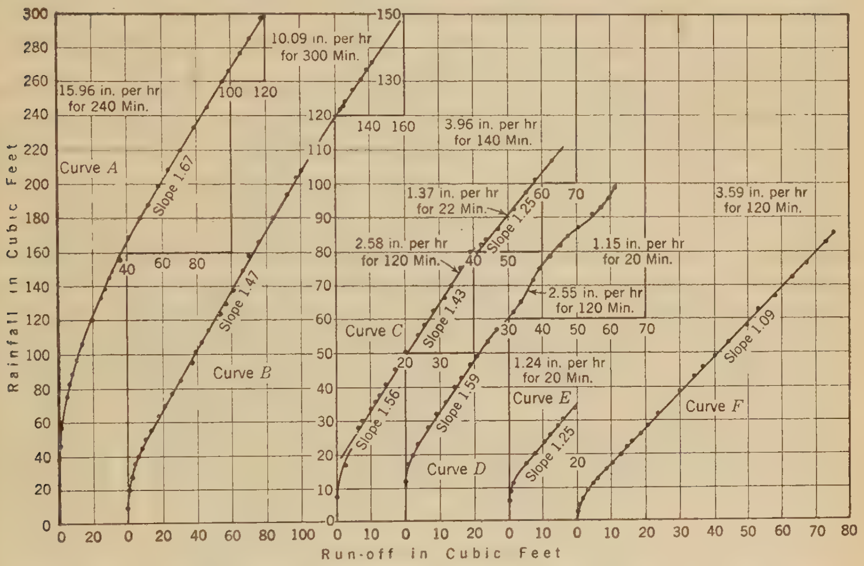


FIG. 28.—VOLUMETRIC RATIOS OF RAINFALL TO RUN-OFF, 1931–1932; SPRINKLING EXPERIMENTS, CITY PARK, DALLAS, TEX. (SEE TABLE 11).

TABLE 10.—VOLUMETRIC RATIOS OF RUN-OFF TO RAINFALL; SPRINKLING EXPERIMENTS (NO PONDAGE)

Item No.	(*a*) EXPERIMENTS AT ST. LOUIS, MO.					(*b*) EXPERIMENTS AT DALLAS, TEX.				
	Plot	Year	Soil condition	Ratio		Plot*	Season	Soil condition	Ratio	
				Range	Average				Range	Average
1...	*A*	1922	Bare	0.88–0.98	0.93	*A*	Spring	Sodded	0.62–0.80	0.74
2...	*A*	1923	Sodded	0.54–0.66	0.61	*A*	Fall	Sodded	0.52–0.66	0.60
3...	*B*	1922	Bare	0.70–0.95	0.86	*B*	Spring	Sodded	0.63–0.80	0.72
4...	*B*	1923	Sodded	0.60–0.86	0.76	*B*	Fall	Sodded	0.54–0.60	0.57
5...	*C*	1922	Bare	0.50–0.74	0.63	*C*	Spring	Sodded	0.82–0.95	0.89
6...	*C*	1923	Sodded	0.66–0.76	0.71	*C*	Fall	Sodded	0.52–0.63	0.58

* Plot *A* is Southern Methodist Univ.; Plot *B* = City Park Plot; and, Plot *C* = Exall Park Plot.

Plot *B*, Fig. 29, is comparable with the City Park Plot in Dallas, in that the slopes are about the same and both are sodded; and a direct comparison

TABLE 11.—OBSERVATIONS TO DETERMINE VOLUMETRIC RATIOS OF RAINFALL TO RUN-OFF; SPRINKLING EXPERIMENTS, CITY PARK, DALLAS, TEXAS (SEE FIG. 28).

Curve	Date	Time	Remarks
A	November 7, 1931....	9:30 A.M...	Dry; heavy dew on a good stand of thick, heavy, green grass.
B	November 9, 1931....	9:00 A.M...	Fair and cool; light wind; damp.
C	November 10, 1931....	8:30 A.M...	Cloudy; wet.
D	March 1, 1932........	12:30 P.M...	Cloudy; warm.
E	March 2, 1932........	....	Cloudy; cool; ground soft and damp; heavy dew.
F	March 5, 1923........	....	Ground saturated and frozen; fair, cold, and windy.

of the soil conditions on the two plots may be made by comparing their volumetric ratios between run-off and rainfall. In Table 10 it is seen that this ratio for Plot *A* in St. Louis (1923) is 0.76. For the City Park Plot in Dallas the ratio for spring rains is 0.72 while for fall rains it is 0.57.

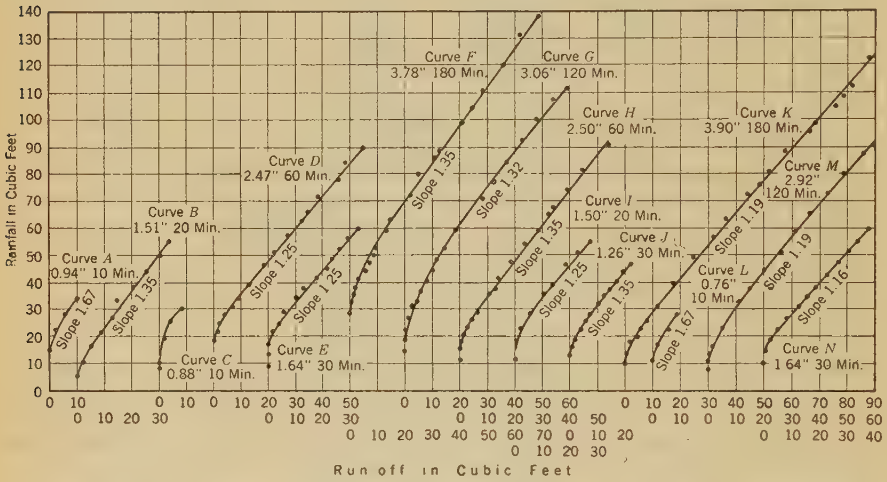


FIG. 29.—VOLUMETRIC RATIOS OF RAINFALL TO RUN-OFF, 1923; SPRINKLING EXPERIMENTS, PLOT *B* (SODDED), ST. LOUIS, MO. (SLOPE, 5.1 PER CENT).

Thus, the first important difference in conditions at the two localities becomes apparent; the correlation diagram, Fig. 30, shows that there is no marked difference in this ratio between spring and fall rains at St. Louis.

It is evident that if the Dallas plot is representative of general soil conditions in that city, the St. Louis run-off curves, Fig. 23 (ignoring frequencies, for Station *A* (Belt and Ridge Avenues), for example), could be modified to find the probable run-off from similar blocks in Dallas for given intensities and duration periods. For winter and spring rains at Dallas, little or no adjustment of St. Louis ratios would be necessary since the corresponding ratios are practically equal. For summer and fall rains, rates derived from the St. Louis data would need to be adjusted in accordance with the relative losses shown in the sprinkling experiments. For St. Louis, the water losses through surface film and absorption are given as $\frac{100 - 76}{100} = 24$ per cent. For Dallas, the corresponding loss is $\frac{100 - 57}{100} = 43$ per cent. The difference in water loss, therefore, is 19% of the rainfall. If all other conditions are equal, except that it is desired to alter Item No. 4, Table 8, for summer and fall soil conditions at Dallas, the computation for the pervious portion (Table 8, Item No. 4, Column (4)) would involve the reduction of the factor 45% by 19, or to 26%, and this, again, should probably be chargeable against a Dallas intensity of a frequency equivalent of the St. Louis rainfall rate of 3 in. per hr.

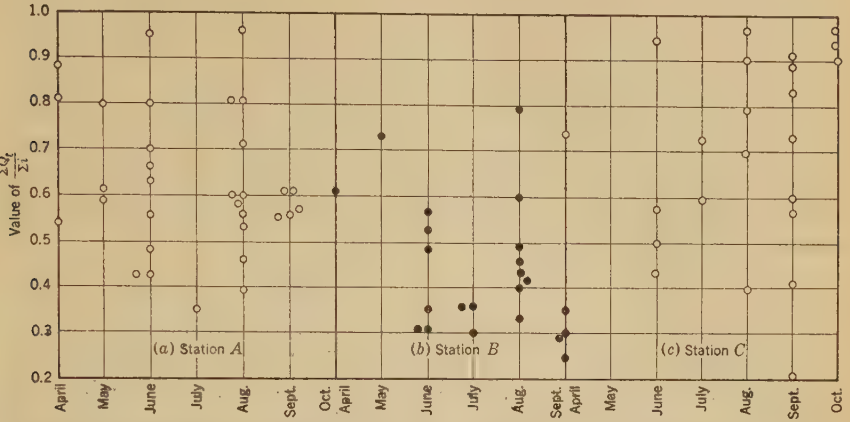


FIG 30.—CORRELATION DIAGRAMS; SEASONAL VARIATION OF VALUE $\frac{\Sigma Q_t}{\Sigma i}$, ST. LOUIS EXPERIMENTS

In this particular case, it would probably be advisable to draw two independent rainfall frequency curves for Dallas, one for winter and spring rains and one for summer and fall rains. For a given design frequency, it might be that for some duration periods the spring curve would govern. while for other periods the fall curves might be indicated.

CONCLUSIONS TO PART III

In general, it is suggested that the data presented may be made useful in localities other than St. Louis, by well-considered modifications in four respects:

(*a*) An adjustment of the run-off curves of Fig. 23 in the direction and proportion that rainfall curves of the new locality may bear to the 61-yr rainfall curve given in Fig. 23(*b*).

(*b*) With regard to varying percentages of impervious area, as suggested in Table 8.

(*c*) For slight differences of slope as outlined in Table 7.

(*d*) For other soil types as in the last paragraph.

Somewhat better information would be made available through such modifications than has generally been the case. It is highly desirable that additional tests and repetitions of these experiments be conducted in other localities in the near future.

DISCUSSION

Franklin F. Snyder,[6] Jun. Am. Soc C. E. (by letter).—Figs. 15 and 16 of the paper, giving basic data for typical storms on the three areas investigated, provide opportunity for various instructive studies. Computed run-off is obtained by applying the run-off factor to the 100% run-off rates as obtained by the unit-graph distribution of the rainfall. The writer used "net rainfall" and the author's unit graphs to study the same storms. "Net rainfall" (which, in amount, is equal to the run-off) is defined as precipitation minus losses, of which infiltration is much the largest and will be termed in this discussion "infiltration loss."

In accordance with the line of reasoning advanced by R. E. Horton, M. Am. Soc. C. E.,[7] a constant infiltration rate was determined for each storm. The constant infiltration rates for the three storms are, Area *A,* rain of September 8, 1926, 0.61 in. per hr; Area *B,* rain of August 27, 1921, 1.28 in. per hr; and Area *C,* rain of September 8, 1920, 0.45 in. per hr. Subtracting these rates of infiltration from the given precipitation rates minute by minute gave net rainfall rates, which were then distributed by the unit graph for the particular area to obtain computed run-off rates.

The computed run-off rates were then compared with the observed rates and the differences or errors obtained for each minute, from which, in turn, were computed the probable errors. For the storm of September 8, 1926, on Area *A,* the probable error for any one minute was 0.14 cu ft per sec, as compared with an approximate probable error, reported in the paper, of 0.21 cu ft per sec. This value and others given for the authors' method are only approximate, as the differences between computed and observed rates of run-off were scaled from Fig. 15. The probable error for the rain of August 27, 1921, on Area *B,* was found to be 0.19 cu ft per sec for any one minute, as against 0.08 cu ft per sec for the results given in the paper. For Area *C,* and the rain of September 8, 1920, the corresponding probable errors were 0.21 and 0.18 cu ft per sec, respectively.

This comparison indicates a close check of the degree of accuracy reported in the paper, although the run-off factor gives better results in two cases out of three. Possibly, improvements would be obtained by using a variable rate of infiltration loss and a variable run-off factor.

According to Mr. Horton,[7] the infiltration capacity of a natural soil is high when the rain begins, decreases rapidly at first, and approaches stability within an interval ranging commonly from 1 hr to 3 hr. If this is true a constant infiltration rate might not have been reached in the three storms studied, as they lasted less than 1 hr each. The writer's studies bear out the theory of a variable infiltration rate, although they do not show clearly the high rates of loss at the beginning of the rain with a graduated decline thereafter.

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[6] Care, T. V. A., Gen. Eng. Div., Knoxville, Tenn.

[7] "The Rôle of Infiltration in the Hydrologic Cycle," by R. E. Horton, M. Am. Soc. C. E., *Transactions*, Am. Geophysical Union, National Research Council of the National Academy of Science, June, 1933, p. 446.

A study of the algebraic sign and variation of the differences between computed and observed rates of run-off, and trials using a variable rate of infiltration, gave the following results for the three storms: The infiltration rate varied directly as the intensity of precipitation, but no attempt was made to determine the exact order of its variation. The approximate average rate of infiltration for 5-min periods for Area *A* ranged from about 1.0 in. per hr at the beginning of the storm (10:13 A.M.) to 0.6 in. per hr at 10:40 A.M., and decreased to about 0.5 in. per hr at 11:00 A.M. In Area *B* the beginning rates (average for 5-min intervals) were about 0.7 in. per hr, and these increased to 1.8 in. per hr at the end of the storm, when high intensity of rainfall occurred. In Area *C* an average infiltration rate of approximately 0.4 in. per hr continued for practically the entire storm, except for the two periods of high rainfall intensity, when it approached 0.5 to 0.8 in. per hr.

In all four peaks of the three storms the use of the average infiltration rates gave higher peak run-off rates than the observed rates, whereas the use of the run-off factor gave computed results lower than the observed rates for the higher peaks and higher than the observed rates for the two lesser peaks. The use of a rate of infiltration varying with precipitation, therefore, would bring the computed run-off rates nearer the observed rates in all four peaks, whereas a varying run-off factor would improve two of the computed results and add to the error of the other two. Accordingly, it appears that the use of a variable infiltration rate based on variations in precipitation intensity would give more consistent results than the use of a variable run-off factor.

Moreover, for all four peaks the computed run-off rates by the average infiltration method were larger than those obtained by the run-off factor—a difference which is inherent in the two methods. This is the result of subtracting a constant amount from the minute-by-minute rainfall, thus increasing the percentage of run-off from the higher rates.

Having an abridged copy of the original manuscript available, the writer compared average infiltration rates for the same storm (that of September 8, 1920) on the three areas. The results are given in Table 12. Column (5) shows the loss in inches per hour on an area that is 100% pervious, under the assumption of a straight-line relation between area of pervious ground and the quantity of infiltration. Comparison of Columns (3) and (5), Table 12, shows an apparent variation of average infiltration rate with intensity of rainfall.

TABLE 12.—RAIN OF SEPTEMBER 8, 1920

Area (1)	Percentage of area pervious (2)	Mean intensity of rainfall, in inches per hour (3)	Average loss, in inches per hour (4)	Ratio, Column (4) / Column (5) (5)
A	50	2.63	1.06	2.12
B	70	1.89	0.77	1.10
C	28	2.43	0.45	1.61

The authors have presented a comprehensive analysis of a valuable record of hydrologic data. The data and results are invaluable to the designer of sewers, and the basic data of the several storms provide an opportunity for the development and testing of methods of analysis of rainfall and run-off relations.

MERRILL M. BERNARD,[8] M. AM. SOC. C. E. (by letter).—Injecting modern thought into the science of storm-sewer design, the authors have given an interesting chronicle of an unusual experience in hydrologic research. It is particularly interesting to note the general applicability of the unit-hydrograph method to conditions ranging from those prevailing on small urban inlet areas of only 2 or 3 acres, with changes in rate of recorded rainfall at 1-min intervals, to those prevailing in stream basins covering several thousand square miles, utilizing U. S. Weather Bureau data in which rainfall is recorded at 24-hr intervals.

An absence of unit-time (1-min) records of rainfall in the studies recorded in this paper made it necessary for the authors to arrive at the factors fixing the shape and extent of their unit graphs "from a study of the records of a few short rains of fairly uniform intensities." That their method and the resulting equations are successful is demonstrated throughout the work and is one of the highlights of the paper. The resulting unit graph, however, is the same as that presented by LeRoy K. Sherman,[9] M. Am. Soc. C. E., being the hydrograph of 0.0167 in. of run-off depth (comparable to 1 in. of run-off depth from a 24-hr rainfall), applied throughout 1 min of rainfall at the rate of 1 in. per hr.

The writer has suggested the "distribution graph,[10] which shows the proportion of the run-off from any unit-time rainfall throughout the period of run-off, expressed as percentage of the total run-off. The conversion of the unit graph to the distribution graph, when rainfall is expressed in rate or in depth per unit of area, is accomplished by moving the decimal point two places to the right.

Table 13 constitutes a demonstration of the general application of the unit-hydrograph theory (on which Part I of the paper is based), applying methods developed on large basins to the rainfall of September 8, 1920, on Area *C*, shown in Fig. 15. In this case rainfall, expressed as rate in inches per hour, is reduced to inches of depth per minute and is recorded in Column (2), which makes it comparable to daily rainfall.

The unit graph for the area has been computed from Equations (1), (2), and (4) of the paper and the values for depth have been converted into the percentages of the distribution graph (see Column (3), Table 13). Rainfall (or, with the assumption of no loss, the theoretical 100% run-off) is distributed throughout the period of run-off by applying the 1-min figures of the distribution graph to rainfall depth, as shown in Columns (4) to (18), Table 13. Distributed rainfall (or 100% run-off) is now accumulated hori-

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[8] Hydr. Engr., Watershed and Hydrologic Studies, S. C. S., Washington, D. C.

[9] "Streamflow from Rainfall by Unit-Graph Method," *Engineering News-Record*, April 7, 1932.

[10] "An Approach to Determinate Stream Flow", *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 349.

TABLE 13.—APPLICATION OF DISTRIBUTION GRAPH FOR AREA *C* (FIG. 25)

Time, in minutes	Rainfall depth, in inches	Distribution graph (percentages)	RUN-OFF (100%) ACCUMULATION, IN INCHES															THEORETICAL 100% RUN-OFF, OR PLUVIAGRAPH		INCHES PER HOUR RE-ARRANGED IN:	
(1)	(2)	(3)	Rainfall (or 100% run-off) distribution (4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	In. inches (19)	In. inches per hour (20)	Order of magnitude (21)	Critical order (22)
5:11	0.010	0.8	0.0001	......	......	......	......	......	......	......	......	......	......	......	......	......	......	0.0001	0.006	0.060	0.012
12	0.010	3.1	0.0003	0.0001	......	......	......	......	......	......	......	......	......	......	......	......	......	0.0004	0.024	0.276	0.060
13	0.020	7.1	0.0007	0.0003	0.0002	......	......	......	......	......	......	......	......	......	......	......	......	0.0012	0.072	0.756	0.198
14	0.005	12.6	0.0013	0.0007	0.0006	0.0009	......	......	......	......	......	......	......	......	......	......	......	0.0026	0.156	1.578	0.486
15	0.005	19.7	0.0020	0.0013	0.0014	0.0002	0.0009	......	......	......	......	......	......	......	......	......	......	0.0049	0.294	2.796	0.936
16	0.020	28.3	0.0028	0.0020	0.0025	0.0004	0.0002	0.0002	......	......	......	......	......	......	......	......	......	0.0081	0.486	4.464	1.728
17	0.020	14.2	0.0014	0.0028	0.0039	0.0006	0.0004	0.0006	0.0002	......	......	......	......	......	......	......	......	0.0099	0.594	4.854*	2.574
18	0.060	7.0	0.0007	0.0007	0.0056	0.0010	0.0006	0.0014	0.0006	0.0005	......	......	......	......	......	......	......	0.0111	0.666	4.590	3.414
19	0.070	3.5	0.0004	0.0004	0.0028	0.0014	0.0010	0.0025	0.0014	0.0019	0.0006	......	......	......	......	......	......	0.0124	0.744	4.080	4.254
5:20	0.050	2.0	0.0002	0.0002	0.0014	0.0007	0.0014	0.0039	0.0025	0.0043	0.0022	0.0005	......	......	......	......	......	0.0173	1.038	3.630	5.046
21	0.030	0.9	0.0001	0.0001	0.0007	0.0003	0.0007	0.0056	0.0039	0.0076	0.0050	0.0016	0.0002	......	......	......	......	0.0258	1.548	3.240	5.370*
22	0.020	0.5	......	0.0001	0.0004	0.0002	0.0003	0.0028	0.0056	0.0118	0.0088	0.0035	0.0009	0.0002	......	......	......	0.0345	2.070	2.940	4.950
23	0.040	0.2	......	......	0.0002	0.0001	0.0002	0.0014	0.0028	0.0170	0.0138	0.0063	0.0021	0.0006	0.0004	......	......	0.0449	2.694	2.622	4.080
24	0.020	0.1	......	......	0.0001	0.0001	0.0001	0.0007	0.0014	0.0085	0.0198	0.0098	0.0038	0.0014	0.0012	0.0002	......	0.0471	2.766	2.220	3.270
25	0.050		......	......	0.0001	......	0.0001	0.0004	0.0007	0.0042	0.0099	0.0141	0.0060	0.0025	0.0028	0.0006	0.0005	0.0421	2.526	1.842	2.652
26	0.023	100.0	0.0002	......	0.0001	......	......	0.0002	0.0004	0.0020	0.0048	0.0070	0.0085	0.0039	0.0050	0.0014	0.0016	0.0351	2.106	1.554	2.046
27	0.090	....	0.0007	0.0007	......	......	......	0.0001	0.0002	0.0012	0.0025	0.0035	0.0043	0.0056	0.0078	0.0025	0.0035	0.0326	1.956	1.386	1.620
28	0.110	....	0.0016	0.0028	0.0009	......	......	0.0001	0.0001	0.0005	0.0014	0.0017	0.0021	0.0028	0.0112	0.0039	0.0063	0.0354	2.124	1.278	1.398
29	0.120	....	0.0029	0.0064	0.0034	0.0010	......	0.0001	0.0001	0.0003	0.0006	0.0010	0.0009	0.0014	0.0056	0.0056	0.0098	0.0391	2.346	1.236	1.284
5:30	0.050	....	0.0045	0.0113	0.0078	0.0037	0.0005	......	0.0001	0.0001	0.0004	0.0005	0.0006	0.0007	0.0028	0.0028	0.0141	0.0499	2.994	1.200	1.224
31	0.020	....	0.0065	0.0177	0.0139	0.0085	0.0016	0.0002	......	0.0001	0.0001	0.0003	0.0003	0.0004	0.0014	0.0014	0.0070	0.0594	3.564	1.146	1.152
32	0.020	....	0.0033	0.0255	0.0217	0.0151	0.0035	0.0006	0.0002	......	0.0001	0.0001	0.0002	0.0002	0.0008	0.0007	0.0035	0.0755	4.530	1.056	1.062
			0.0016	0.0128	0.0310	0.0236	0.0063	0.0014	0.0006	......	......	0.0001	0.0001	0.0001	0.0004	0.0004	0.0017	0.0801	4.806*	0.918	0.924
			0.0008	0.0063	0.0156	0.0340	0.0098	0.0025	0.0014	......	......	......	......	0.0001	0.0002	0.0002	0.0010	0.0719	4.314	0.696	0.696
			0.0005	0.0032	0.0077	0.0170	0.0141	0.0039	0.0025	......	......	......	......	0.0001	0.0002	0.0001	0.0005	0.0498	2.988	0.528	0.528
			0.0002	0.0018	0.0039	0.0084	0.0070	0.0056	0.0039	......	......	......	......	......	0.0001	0.0001	0.0003	0.0312	1.872	0.366	0.366
			0.0001	0.0008	0.0022	0.0012	0.0035	0.0028	0.0056	......	......	......	......	......	......	0.0001	0.0001	0.0194	1.164	0.240	0.240
			0.0001	0.0004	0.0010	0.0024	0.0017	0.0014	0.0028	......	......	......	......	......	......	......	0.0001	0.0099	0.594	0.120	0.120
			......	0.0002	0.0006	0.0012	0.0010	0.0007	0.0014	......	......	......	......	......	......	......	......	0.0051	0.306	0.060	0.060
			......	0.0001	0.0002	0.0006	0.0005	0.0004	0.0007	......	......	......	......	......	......	......	......	0.0023	0.138	0.030	0.012
			......	......	0.0001	0.0002	0.0003	0.0002	0.0004	......	......	......	......	......	......	......	......	0.0012	0.072	0.012	0.006
			......	......	......	0.0001	0.0301	0.0001	0.0002	......	......	......	......	......	......	......	......	0.0005	0.030	0.006	....
			......	......	......	......	0.0001	0.0001	0.0001	......	......	......	......	......	......	......	......	0.0003	0.018	....	....
			......	......	......	......	......	0.0001	0.0001	......	......	......	......	......	......	......	......	0.0002	0.012	....	....
			......	......	......	......	......	......	0.0001	......	......	......	......	......	......	......	......	0.0001	0.006	....	....

* Peak values.

zontally, giving the results for the author's theoretical 100% run-off graph, or the writer's pluviagraph (Columns (19) and (20), Table 13). The writer has distributed the figures computed for the unit graph beyond 14 min, throughout the distribution graph, accounting for the slight differences between the values in Column (22), Table 13, and those in Fig. 15.

As the storm of September 8, 1920 (Fig. 15), reached a peak in its later stages, it will be used to demonstrate the effect of the distribution of rainfall intensities throughout the period of rainfall on the 100% run-off or pluviagraph results. If the rainfall rates are arranged in order of magnitude, thereby assuming that the storm reached a peak in the first minute of rainfall, and that it was distributed as previously demonstrated (see Column (21), Table 13), the pluviagraph peak will shift from 5.33 ppm to 5.17 ppm without any appreciable change in value. It can be shown, however, that the most critical arrangement of intensities is that in which the high intensities are grouped around the minute that marks the peak of the distribution graph, which is, for the example given, the 6th minute of rainfall. Thus, on arranging rainfall and distributing it as previously explained (Column (22), Table 13), the pluviagraph peak shifts to 5.21 ppm and gives a result 12% greater than that developed by the storm itself.

The rate ratio, $P = \frac{\Sigma Q_t}{\Sigma Q_e}$, mentioned in connection with Fig. 17 of the paper, has been termed the "retention coefficient", by the writer in his recent paper,[11] but which should more properly be called the "storm, or flood, coefficient."

Under the heading "Suggestions for Application to Sewer Design," the authors place conservative limits upon the immediate use of their work in actual design, holding forth the hope that future research will make it possible to give proper values to such factors as j, k, and t_1 for conditions as they are combined in the typical city block. As sewer design is usually based on an estimate of the effect of future growth on surface conditions, these factors, no doubt, can be classified and standardized, as the now widely used "coefficient of imperviousness" and the coefficient, C, of the rational method.

The unit-hydrograph method has other advantages than those demonstrated by this paper. Assume, for example, that the means are at hand to produce acceptable values for the factors, j, k, and t_1, on an urban drainage area consisting of four inlet areas of 10 acres each. All are assumed to be alike in character, and, therefore, all have the same unit and distribution graphs. Other conditions are, as follows: The frequency to be met by design is, once in 10 yr; inlets are spaced at 1 000-ft intervals along a proposed sewer of circular section, beginning with Inlet No. 1; the slope ratio of the sewer gradient is 0.002; Kutter's n is 0.013; and, the 10-yr rainfall intensity equation for the locality is,

$$i = \frac{182}{t + 23} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

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[11] "An Approach to Determinate Stream Flow", *Transactions*, Am. Soc. C. E. Vol. 100 (1935), p. 358.

A synthetic storm is developed by computing average rainfall rates for the various duration periods from Equation (6). These rates have been reduced to an average rate throughout any minute, and are the minute differences in the product of i and t-values, representing typical deviations from the averages defined by the intensity equation. Sound design demands the assumption of limiting conditions, and, therefore, rainfall is arranged in critical order (see Column (2), Table 14). It is recognized that this special arrangement of rainfall intensities tends to modify the conception of the storm as having a 10-yr frequency.

Retardation, through surface pondage, is reflected and accounted for in the distribution graph of the area, leaving the principal direct loss, particularly for rainfalls of short duration, to be that of infiltration, which is well expressed as a deduction, as suggested by Robert E. Horton, M. Am. Soc. C. E.[12] The problem here considered assumes an average infiltration loss of 0.50 in. per hr (Column (3), Table 14). The average rainfall excess, or that portion which appears as run-off, is given in Column (4).

The distribution graph for any one of the inlet areas is given in Column (5), Table 14. As developed on large basins, this graph represents the flow created by surface run-off from any rainfall, confined to a unit-time interval, proportioned throughout the period of run-off, and expressed as percentage of the total flow. The peak of such a flood is not created by the coincidental arrival of waters from all parts of the basin, but by the arrival of a flood wave, which has received its impetus and is developed and built up by direct run-off and lateral stream flow contributing throughout the combined length of the principal channels. Consistent similarity in shape and dimension of distribution graphs for a particular basin would indicate that the phenomenon occurs in practically the same manner for all unit-time storm periods. This suggests that a grouping of certain area units may be assumed, which, because of their position relative to the collection system, can be conceived as being enveloped by a time contour representing a period at the end of which the group has contributed its collective effect to the creation and acceleration of the flood wave. Under this assumption the distribution graph can be applied to the area of the basin, giving its "area distribution graph," in acres (Column (6), Table 14). The conversion of the distribution graph (percentage of volume of flow) to the area distribution graph (percentage of area) in no way affects the computed results; that is, the computed results are the same as if there had been no resort to this assumption.

The application of the area distribution graph to rainfall excess (in inches per hour or in cubic feet per second per acre) distributes each minute of such rainfall excess throughout the period of run-off, in cubic feet per second. The horizontal accumulation of these 1-min increments of run-off produces the total run-off from the area at any minute, measured at the inlet (Column (24), Table 14).

As the rainfall is considered to be uniform over the entire area of 40 acres, and as each inlet area is contributing to the flow of the sewer, the flow, as it

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[12] "The Rôle of Infiltration in the Hydrologic Cycle," *Transactions*, National Research Council, 1933, p. 450.

TABLE 14.—COMPUTATION OF RUN-OFF FROM INLET AREA

Elapsed time, in minutes	Average rainfall intensity, in inches per hour	Average rate of infiltration, in inches per hour	Average rainfall excess, in inches per hour	DISTRIBUTION GRAPH		Run-off distribution	RUN-OFF ACCUMULATION, IN CUBIC FEET PER SECOND					
				Percentage	Acres							
(1)	(2)	(3)	(4)	(5)	(6)		(7)	(8)	(9)	(10)	(11)	(12)
1....	3.80	0.50	3.30	1	0.10		0.33	....	....	....	....	....
2....	4.25	0.50	3.75	2	0.20		0.66	0.38	....	....	....	....
3....	4.88	0.50	4.38	3	0.30		0.99	0.75	0.44	....	....	....
4....	5.57	0.50	5.07	5	0.50		1.65	1.13	0.88	0.51	....	....
5....	6.43	0.50	5.93	10	1.00		3.30	1.88	1.31	1.02	0.59	....
6....	7.58	0.50	7.08	17	1.70		5.61	3.75	2.19	1.52	1.19	0.71
7....	6.99	0.50	6.49	14	1.40		4.62	6.38	4.38	2.54	1.78	1.42
8....	5.93	0.50	5.43	11	1.10		3.63	5.25	7.45	5.07	2.97	2.13
9....	5.12	0.50	4.62	9	0.90		2.97	4.13	6.14	8.62	5.93	3.54
10....	4.45	0.50	3.95	7	0.70		2.31	3.38	4.82	7.10	10.08	7.08
11....	4.00	0.50	3.50	6	0.60		1.98	2.63	3.94	5.58	8.31	12.04
12....	3.50	0.50	3.00	5	0.50		1.65	2.25	3.17	4.56	6.53	9.92
13....	3.30	0.50	2.80	4	0.40		1.32	1.88	2.62	3.55	5.33	7.79
14....	3.20	0.50	2.70	3	0.30		0.99	1.50	2.18	3.04	4.15	6.37
15....	3.00	0.50	2.50	2	0.20		0.66	1.13	1.75	2.53	3.56	4.95
16....	2.80	0.50	2.30	1 100	0.10 10.00		0.33	0.75	1.31	2.03	2.96	4.25
17....	2.70	0.50	2.20				....	0.38	0.88	1.52	2.37	3.54

Exposed time, in minutes	RUN-OFF ACCUMULATION, IN CUBIC FEET PER SECOND — (*Continued*)											Run-off measured at inlet, in cubic feet per second
(1)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
1....	....	....	....	....	....	....	....	.....	.....	.....	.....	0.33
2....	....	....	....	....	....	....	....	.....	.....	.....	.....	1.04
3....	....	....	....	....	....	....	....	.....	.....	.....	.....	2.18
4....	....	....	....	....	....	....	....	.....	.....	.....	.....	4.17
5....	....	....	....	....	....	....	....	.....	.....	.....	.....	8.10
6....	....	....	....	....	....	....	....	.....	.....	.....	.....	14.97
7....	0.65	....	....	....	....	....	....	.....	.....	.....	.....	21.77
8....	1.30	0.54	....	....	....	....	....	.....	.....	.....	.....	28.34
9....	1.95	1.09	0.46	....	....	....	....	.....	.....	.....	.....	34.83
10....	3.25	1.63	0.92	0.40	....	....	....	.....	.....	.....	.....	40.97
11....	6.49	2.71	1.39	0.79	0.35	....	....	.....	.....	.....	.....	46.21
12....	11.03	5.43	2.31	1.19	0.70	0.30	....	.....	.....	.....	.....	49.04
13....	9.09	9.24	4.62	1.98	1.05	0.60	0.28	.....	.....	.....	.....	49.35
14....	7.14	7.61	7.85	3.95	1.75	0.90	0.56	0.27	.....	.....	.....	48.26
15....	5.84	5.97	6.47	6.72	3.50	1.50	0.84	0.56	0.25	.....	.....	46.23
16....	4.54	4.88	5.08	5.53	5.95	3.00	1.40	0.81	0.50	0.23	.....	43.55
17....	3.90	3.80	4.16	4.35	4.90	5.10	2.80	1.35	0.75	0.46	0.22	40.48

combines, will determine capacity. This synchronization is accomplished by determining the time taken by the run-off waters, as they combine, in passing through the various stretches of the sewer. A particularly satisfactory expression for average velocity in terms of the factors determinable in such a problem is that presented by R. L. Gregory and C. E. Arnold,[13] Associate Members, Am. Soc. C. E.

The sewer system in the problem begins with Inlet No. 1, Inlets Nos. 2, 3, and 4 contributing at 1 000-ft intervals progressing down the sewer. The purpose of Table 15 is to show how the flow from Inlets Nos. 1 and 2 may be combined with that of Inlet No. 3, the flow from Inlets Nos. 1 and 2 (Columns (2) and (3)) having been combined in the same manner. Column (4) gives the combined flow from Inlets Nos. 1 and 2 as such flow enters the stretch between Inlet No. 2 and Inlet No. 3. The average velocity, in cubic feet per second, computed by Equation (2), is shown in Column (5) and is converted to velocity, in feet per minute, in Column (6), from which the time in transit through the stretch is determined and listed in Column (7). Column (8) gives the period from the beginning of the rainfall to that minute at which the increment of flow enters the stretch, which, added to the time in transit through the stretch (Column (7)), determines the time of arrival at Inlet No. 3.

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[13] "Run-Off—Rational Run-Off Formulas," *Transactions*, Am. Soc. C. E., Vol. 96 (1932), Equation (50), p. 1094.

In the meantime Inlet No. 3 has been discharging into the sewer. For instance, at the end of 20 min this inlet will have passed its peak discharge of 49.35 cu ft per sec and will be discharging at the rate of 31.66 cu ft per sec, while Inlet No. 2 will be delivering 40.48 cu ft per sec and Inlet No. 1, 49.35 cu ft per sec to the same point in the sewer.

TABLE 15.—METHOD OF COMBINING FLOW FROM INLETS NOS. 1 AND 2 WITH THAT OF INLET NO. 3

	AVERAGE MINUTE DISCHARGE, IN CUBIC FEET PER SECOND			VELOCITY		TIME, IN MINUTES				AVERAGE MINUTE DISCHARGE, IN CUBIC FEET PER SECOND				
Elapsed time, in minutes (1)	From Inlet No. 2 (2)	From Inlet No. 1 (3)	From Inlets Nos. 1 and 2 (4)	In feet per second (5)	In feet per minute (6)	In transit from Inlet No. 2 to Inlet No. 3 (1 000 ft.) (7)	From beginning of rainfall (8)	Of arrival at Inlet No. 3 (9)	Coincident discharge from Inlet No. 3, in cubic feet per second (10)	Inlet No. 4 (11)	Inlet No. 3 (12)	Inlet No. 2 (13)	Inlet No. 1 (14)	Below Inlet No. 4 (15)
1	0.33	....	0.33	0.88	77	13.0	1	14.0	4.17	0.33	....	....	....	0.33
2	1.04	....	1.04	1.80	108	9.3	2	11.3	8.10	1.04	....	....	....	1.04
3	2.18	....	2.18	2.18	131	7.6	3	10.6	14.97	2.18	....	....	....	2.18
4	4.17	....	4.17	2.56	154	6.5	4	10.5	21.77	4.17	0.33	....	....	4.50
5	8.10	0.33	8.43	3.07	184	5.4	5	10.4	28.34	8.10	1.04	....	....	9.14
6	14.97	1.04	16.01	3.61	217	4.6	6	10.6	34.83	14.97	2.18	....	....	17.15
7	21.77	2.18	23.95	3.97	238	4.2	7	11.7	40.97	21.77	4.17	0.33	....	26.27
8	28.34	4.17	32.51	4.29	258	3.9	8	11.9	46.21	28.34	8.10	1.04	....	37.48
9	34.83	8.10	42.93	4.60	276	3.6	9	12.6	49.04	34.83	14.97	2.18	....	51.98
10	40.97	14.97	55.94	4.91	295	3.4	10	13.4	49.35	40.97	21.77	4.17	....	66.91
11	46.21	21.77	67.98	5.16	311	3.2	11	14.2	48.26	46.21	28.34	8.10	0.33	82.98
12	49.04	28.34	77.38	5.34	321	3.1	12	15.1	46.23	49.04	34.83	14.97	1.04	99.88
13	49.35	34.83	84.18	5.45	327	3.1	13	16.1	43.55	49.35	40.97	21.77	2.18	114.27
14	48.26	40.97	89.23	5.54	333	3.0	14	17.0	40.48	48.26	46.21	28.34	4.17	126.98
15	46.23	46.21	92.44	5.59	336	3.0	15	18.0	37.47	46.23	49.04	34.83	8.10	138.20
16	43.55	49.04	92.59	5.59	336	3.0	16	19.0	35.54	43.55	49.35	40.97	14.97	148.84
17	40.48	49.35	89.83	5.52	331	3.0	17	20.0	31.66	40.48	48.26	46.21	21.77	156.72
18	37.47	48.26	85.73	5.45	327	3.1	18	21.1	28.91	37.47	46.23	49.04	28.34	161.08
19	35.54	46.23	81.77	5.41	325	3.1	19	22.1	26.47	35.54	43.55	49.35	34.83	163.27
20	31.66	43.55	75.21	5.27	316	3.2	20	23.2	24.37	31.66	40.48	48.26	40.97	161.37
21	28.91	40.48	69.39	5.20	312	3.2	21	24.2	22.58	28.91	37.47	46.23	46.21	158.82

If, then, the hydrograph of flow from Inlet No. 3 is adjusted to this position relative to Inlets Nos. 2 and 1 (Column (10), Table 15), the total flow, minute by minute, through the stretch between Inlets Nos. 3 and 4 becomes available. Columns (11) to (15), Table 15, show the arrangement of computations for the synchronized flow from the four inlets entering the stretch below Inlet No. 4.

No attempt has been made to take into account the changes in flow volume within 1-min intervals throughout the period of flow adjustment, involving the first and last 5 or 6 min. The synchronization, therefore, is less accurate for the extremes of flow, but is in agreement to the minute for the period between the 12th and 30th minutes (not all included in the published table).

To continue on the premise that fixed time intervals divide the drainage area into zones of like contributing characteristics, the area distribution graphs of the four inlets can be combined as the foregoing analysis has shown their flows to combine. The complete computation, of which Table 14 is an example, showed that the flow from Inlet No. 4 enters the stretch in the first minute; that from Inlet No. 3 reaches the stretch 3 min later; that from Inlet No. 2 at the end of 7 min, or 3 min later than that of Inlet No. 3; and the flow from Inlet No. 1 is 4 min later than that of Inlet No. 2.

Table 16 gives the sub-area combinations for each minute throughout the duration of run-off, producing an area distribution graph for the entire 40 acres (see Column (6)), applicable to the point of design below Inlet No. 4. This area distribution graph is now applied to the rainfall excess produced by the 10-yr storm of the problem in the same manner as that suggested in determining the run-off at the various inlets. The distribution and accumulation are made in the same manner as for the inlet. The result is the total run-off from the area, minute by minute, as it reaches the point of design, and is found to be the same as those of Column (15), Table 15, any slight differences being due to the adjustment in position between the four area distribution graphs to the nearest minute.

TABLE 16.—Area Distribution Graphs (Units Are in Acres)

Elapsed time, in minutes (1)	Inlet No. 4 (2)	Inlet No. 3 (3)	Inlet No. 2 (4)	Inlet No. 1 (5)	Total (6)	Elapsed time, in minutes (1)	Inlet No. 4 (2)	Inlet No. 3 (3)	Inlet No. 2 (4)	Inlet No. 1 (5)	Total (6)
1.......	0.10	....	....	....	0.10	14...	0.30	0.60	1.10	0.50	2.50
2.......	0.20	....	....	....	0.20	15...	0.20	0.50	0.90	1.00	2.60
3.......	0.30	....	....	....	0.30	16...	0.10	0.40	0.70	1.70	2.90
4.......	0.50	0.10	....	....	0.60	17...		0.30	0.60	1.40	2.30
5.......	1.00	0.20	....	....	1.20	18...	10.00	0.20	0.50	1.10	1.80
6.......	1.70	0.30	....	....	2.00	19...		0.10	0.40	0.90	1.40
7.......	1.40	0.50	0.10	....	2.00	20...			0.30	0.70	1.00
8.......	1.10	1.00	0.20	....	2.30	21...		10.00	0.20	0.60	0.80
9.......	0.90	1.70	0.30	....	2.90	22...			0.10	0.50	0.60
10.......	0.70	1.40	0.50	....	2.60	23...				0.40	0.40
11.......	0.60	1.10	1.00	0.10	2.80	24...			10.00	0.30	0.30
12.......	0.50	0.90	1.70	0.20	3.30	25...				0.20	0.20
13.......	0.40	0.70	1.40	0.30	2.80	26...				0.10	0.10
										10.00	40.00

The advantages of having at any point under design an area distribution graph are obvious. With it various rainfall frequencies, rainfall intensities, infiltration rates, and run-off coefficients can be compared in terms of sewer dimension and cost. Another advantage of the hydrograph of flow, made available by the unit hydrograph, over a maximum only, as determined by other methods, lies in the ability to evaluate the effect of converting the several maxima of the hydrograph from sewer capacity to temporary backwater areas at the inlets. Particularly where the conditions produce distribution graphs, area distribution graphs, and hydrographs of flow of relatively short base and high peak, may this advantage become a factor in economical design.

It is possible, through the unit hydrograph, to analyze and correct what may be misconceptions in the "rational" method, a method which has enjoyed a wide use in the field of storm-sewer design and to the development of which the authors have contributed extensively. The method has back of it a ground of rationality that is entirely lacking in many of the older empirical formulas. Briefly, its theory is that a particular average rainfall intensity (of a given frequency) becomes the critical one for a drainage area when it is sustained throughout a period equal to the time of concentration for the area. The time of concentration is usually defined as that time necessary for the run-off from the "remote" portion of the area to reach the point of concentration. A further explanation of the theory is that even though higher rates will be reached for shorter "concentration" periods, run-off from them will be less than that from the critical intensity, because of the dissipating effect of surface pondage, together with the assumption that the rain will have stopped or so decreased in intensity before the end of each concentration period that the areas immediately adjacent to the outlet will have had time to relieve themselves of their run-off waters before the "remote" waters reach that point. Likewise, rainfalls of longer duration than the concentration period will be less effective, having lower average rates.

If the average rainfall intensities given by Equation (2), which are applicable to St. Louis, Mo., are distributed with the distribution graph reduced from the authors' unit graph for Area C, and each duration period is considered an independent rainfall, or synthetic storm, it will be found that, regardless of the coefficient or deductive factor used to reduce the theoretical 100% run-off to the actual run-off, average rates of rainfall for shorter periods than the period established by the base of the distribution graph, will produce potential run-off greater than that produced by the critical intensity.

The hypothetical problem used in the foregoing demonstration of the unit hydrograph method will here be solved by the rational method. The "inlet time" will first be taken as equal to the base of the distribution graph for the upper inlet, which is 16 min. By acceptable methods, the time in transit through the proposed sewer is found to be 11 min, giving a time of concentration of 27 min. From the rainfall-intensity formula, Equation (2), the corresponding critical intensity of 3.64 in. per hr is computed. The coefficient, C, is $\frac{3.14}{3.64} = 0.86$, and the rational equation becomes, $Q = C\ i\ A = 0.86 \times 3.64 \times 40 = 125.70$ cu ft per sec, producing a maximum flow which is 23% less than that developed by the unit hydrograph (Column (15), Table 15). Under the prevailing conception of concentration time, then, the rational method may give results that are too low, but the practical significance of this is lost in the wide range of values which have been given to the coefficient, C.

The foregoing conclusion, however, is predicated on the definition of concentration time as being equal to the base of the unit hydrograph of Inlet No. 1 plus time in transit, thus assuming that the "remote" area of the upper inlet is the last to contribute run-off to the point of concentration. The writer has found that, apparently, the unit hydrograph and rational methods may be brought into accord by defining concentration time as being equal to t_1, the time interval between the center of mass of rainfall and the flood peak or the "lag" interval, or the time position of the maximum ordinate of the unit hydrograph, plus any time in transit involved in reaching the point of concentration, and by converting average rainfall rates into average rate throughout any minute, arranged in critical order.

Following are the results of applying the rational method, so modified, to the Inlet No. 1 area, as shown in Table 14: Concentration time t_1 = 6 min (Column (5)); average of first 6 min of rainfall = 5.42 in. per hr (Column (2)); coefficient, $\frac{4.92}{5.42} = 0.91$; and (see Column (24) Table 14); $Q = C\ i\ A = 0.91 \times 5.42 \times 10 = 49.25$ cu. ft per sec.

The concentration time for the point of design below Inlet No. 4 is determined by adding to the 'inlet time," or concentration time for Inlet No. 1, the time in transit through the sewer. The maximum discharge, as determined by the rational method, is as follows: The inlet time = 6 min; time in transit = 11 min(see Table 15); concentration time = 17 min; average rainfall intensity for 17 min = 4.55 in. per hr; coefficient, $\frac{40.3}{4.55} = 0.89$; and, $Q = 0.89 \times 4.55 \times 40 = 162.20$ cu ft per sec (see Column (15) Table 15).

As it has been customary to estimate inlet time at 5 to 20 min, results from the rational method have probably not been greatly affected by the apparent inconsistency revealed in the foregoing comparison; rather has the analysis upheld the rational method as a quick and reliable means of estimating maximum flow in storm sewers.

LeRoy K. Sherman,[14] M. Am. Soc. C. E. (by letter).—An important analysis of the rainfall-run-off relation on urban areas is presented in this paper, which is a step beyond the usual procedure based on the assumption of a uniform rate of rainfall during the storm period. Recently, the writer examined the rainfall and stream-flow records, due to forty storms, on a combined sewer area of 4 000 acres, in the Rock Creek Basin of the District of Columbia. The records were from self-recording gauges. In all these storm hydrographs the effect of varying intensity of rainfall produced a pronounced peak, in spite of the fact that storm durations generally exceeded the concentration period. This indicates that the assumption of a uniform rate of rainfall is not in accord with Nature's procedure and that, for certain areas, it may not be satisfactory. By an application of the unit hydrograph, the authors give a procedure which utilizes the actual varying intensity of the rainfall.

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[14] Cons. Engr., Chicago, Ill.

In forecasting run-off from rainfall data by the unit hydrograph (or by any other method or formula), the chief problem is to determine the factor expressing loss in run-off due to infiltration or soil capacity. With few exceptions, Fig. 15 indicates a close agreement between observed and computed rates of run-off. The authors used the factor of average percentage of run-off applied to the 100% run-off graph.

Fig. 16 suggested the application of the storage equation to develop or apply the factor of infiltration loss. The storage equation is:

$$\text{Inflow Volume} = \text{Outflow Volume} \pm \text{Storage} \ldots\ldots\ldots\ldots (7)$$

which may be expressed:

$$\text{Rainfall} = \text{Run-Off} + \text{Infiltration} \pm \text{Storage} \ldots\ldots\ldots\ldots (8)$$

There are two unknown quantities in Equation (8). However, channel-storage depth is related to run-off rates and, likewise, storage depth on a plane surface is related to run-off rates. For the latter case the storage depth, d, is the same as the hydraulic radius, R. Therefore, the discharge rate is, $Q = C \sqrt{S\,d} \times d$, and,

$$\frac{Q}{Q_1} = \frac{d^{\frac{3}{2}}}{d^{\frac{3}{2}}_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

in which Q_1 = the 100% run-off rate; d_1 = the storage depth corresponding to Q; Q = any given run-off rate; and d = storage depth corresponding to Q.

With the flow, Q_1, there is no infiltration loss. Therefore, Equation (7) can be applied to the data in Fig. 16 and d_1 may be determined for any rate of Q_1. The ordinate between the lines for accumulated rain and accumulated 100% run-off gives the depth of storage, d_1, at any time from the beginning of the storm. Values of d can now be computed by Equation (9). The values of d can be plotted for any given time in Fig. 16. These values will be on a line above, and somewhat parallel to, the line of accumulated 100% run-off. The ordinates between this line for d and the line for accumulated actual run-off give, by Equation (8), the accumulated loss to any given time.

The writer has applied the foregoing procedure for determining rate of loss per hour, or infiltration capacity, to the examples in Fig. 15. He obtained results as good, but no better, than the authors except in the case of the peak on Area A (Fig. 15). The writer also applied this storage equation procedure to a hypothetical case wherein he assumed the rates of rainfall and loss, in inches, as shown in Table 17. For comparison, the computed losses are also shown. Evidently, the procedure indicated by Table 17 did not offer any improvement over the use of the authors' application of average percentage of run-off.

Recently, the writer received a paper[15] by Robert E. Horton, M. Am. Soc. C. E., who makes successful use of the storage equation in the rainfall-run-off relation by taking cognizance of factors which the writer has neglected in this discussion.

To the present time the writer has secured the best results with the unit hydrograph method by applying the percentage of run-off directly to rainfall without the use of the intervening 100%-run-off graph or Bernard's pluviograph.[11]

TABLE 17.—Hypothetical Case; Comparison of True Versus Computed Losses

Day	Rain, depth	Loss, depth	Computed loss, depth
1	1.4	0.4	0.29
2	2.4	0.4	0.45
3	3.4	0.4	0.45
4	1.4	0.4	0.50
	......	1.60	1.69

This paper represents considerable careful work. The authors have presented valuable data and study applicable to problems in storm-sewer design and to the science of hydrology. Parts of the complete paper, as filed in Engineering Societies Library, New York, N. Y., are fully as important as the subject-matter submitted in the printed part. The writer refers particularly to the author's experiments on rates of infiltration during continuous rains.

W. W. Horner,[16] M. Am. Soc. C. E., and F. L. Flynt,[17] Assoc. M. Am. Soc. C. E. (by letter).—The discussions by Messrs. Snyder and Sherman develop the possibilities of an alternate study of the rainfall-run-off relation in terms of losses. In Part I of the paper, the writers made a fairly exhaustive effort to determine the relationship as a ratio. They found the ratios, or percentage values, to vary over a wide range and, for a large part of the data, were unable to allocate, satisfactorily, the variation to other hydrologic factors. As was shown in Fig. 18, all these ratios, whether taken between various summation values, or between values for particular periods (as for 5 or 10 min), or between peak rates, resulted in percentile curves of about the same slope. Each of the discussers has emphasized the point that the percentage factor must necessarily be a varying one. This is a condition which is well recognized by hydrologists, of course. Sewer designers have adhered to the use of a run-off-rainfall factor and the formula of the rational method, however, because of its simplicity of application, and for the reason that the dearth of real basic information on run-off from urban areas appeared to make further refinement in method unjustifiable.

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[15] "Surface Runoff Phenomena," by Robert E. Horton, *Publication 101*, Horton Hydrological Laboratory, February 1, 1935, Edmonds Bros., Inc., Ann Arbor, Mich., Publisher.

[16] Cons. Engr., St. Louis, Mo.

[17] Civ. Engr., formerly with Sewer Design Dept., City of St. Louis, St. Louis, Mo.

The writers are impressed by the sample demonstration offered by Mr. Snyder, as well as by the reasonableness of Mr. Sherman's presentation. It is entirely possible that an alternate analysis of all the information used by the writers, involving the infiltration losses as determined by Mr. Snyder, and the further evaluation of storage as discussed by Mr. Sherman, might result in a set of relatively consistent values having a definite relationship to such basic characteristics as slope, shape of area, and character of surface coverage. Some of the relationships would probably be of the same general order as those suggested by Mr. Bernard in another paper.[18]

The writers agree with Mr. Snyder that a constant infiltration rate could not be used satisfactorily for rains of less than 1 hr in duration. A desirable approach to such a study, accordingly, would be along the lines of determining infiltration capacities for each of the particular areas (which capacities might be expected to vary on a seasonal basis), and also the probable rate of reduction of infiltration within the first hour or two of the storm. It would be necessary, however, to determine the proportionate part of the precipitation going into (temporary) surface storage during the rise of the hydrograph. This also is determinable, possibly, from the same data.

The recent studies by Robert E. Horton, M. Am. Soc. C. E., referred to by Mr. Sherman,[15] using in part the data presented in this paper (privately published and, unfortunately, not available to this discussion) seem also to indicate that satisfactory basic data of this type could be derived from the information originally analyzed by the writers. If such an analysis was found to produce reasonable values of infiltration and storage, some development of Mr. Sherman's Equation (8) might be substituted for the rational formula of sewer design.

Illustrative of how an analysis of basic rainfall and run-off data, such as are supplied by the St. Louis records, might throw some light upon the variation of infiltration rate and temporary storage or detention, the data for three storms at Area *A* are presented in Figs. 31, 32, and 33, together with a detailed analysis of the various curves derived from the basic data by various methods. Fig. 31 gives the data for the storm of July 23, 1931, which was chosen for its short duration and unusually uniform rate of precipitation. Fig. 31 (*b*) follows the method suggested by Mr. Horton and referred to by Messrs Snyder and Sherman. The basic data as to rainfall and run-off are represented by the $\Sigma\, i$-curve and the $\Sigma\, Q$-curve, respectively. For simplicity, infiltration is assumed to be at a constant rate and the mass infiltration curve is a straight line through the origin at the beginning of rainfall and with a final ordinate at the end of run-off equal to the difference between total rainfall and total run-off. (The value of "infiltration" thus determined, is in error in that it includes permanent retention and other losses.) The mass net rainfall curve is obtained by subtracting the ordinates to the mass infiltration curve from corresponding ordinates of the mass rainfall, or $\Sigma\, i$-curve. The intercept on a given time ordinate between the net rainfall curve and the mass run-off curve represents the depth of surface detention, including both sheet and channel storage, which, as pointed out by Mr. Sherman, bears a direct relation to the rate of run-off.

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[18] "An Approach to Determinate Stream Flow", *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 358.

On further consideration, the writers find that this rather unsatisfactory approach may be avoided entirely, and the desired item of mass loss and of detention may be determined directly from mass curves based on the unit graph, as originally presented in Fig. 15 of the paper.

Fig. 31 (*c*) illustrates the method of obtaining the depth of surface detention by the application of the unit graph formulas to the same basic data. In this case, the mass 100% run-off, or ΣQ_c-curve, is obtained by applying the unit graph formulas to the rainfall data. The surface detention is then determined by the intercept on the time ordinates between the Σi-curve and the ΣQ_c-curve.

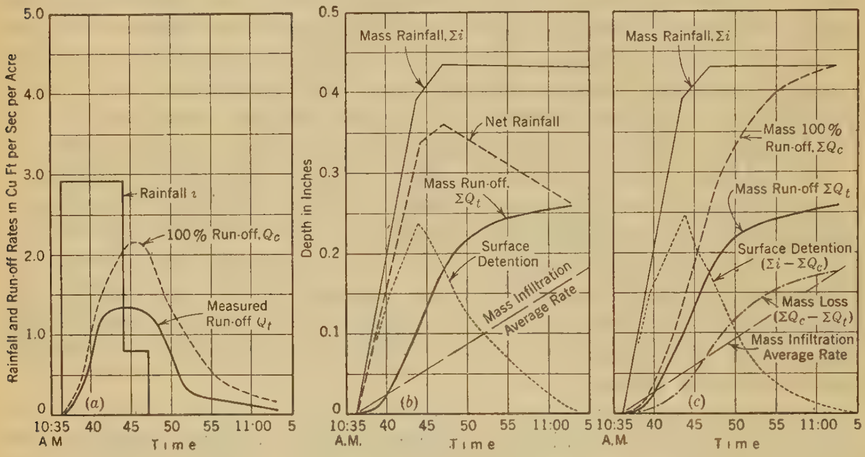

FIG. 31.—STORM OF JULY 23, 1931, AT AREA *A*.

The surface detention curves as determined by the two methods show a remarkable similarity in essential features despite the radical difference in the methods, which indicates that the unit graph formulas as developed by the writers may be close approximations to the true values. Such being the case, the procedure may be extended to determine the probable variation in the value of the rate of water loss (of which infiltration forms the greater part) during the progress of a storm. The accumulated water loss represented by the mass loss curve, Fig. 31 (*c*), is determined by the intercepts on the time ordinates between the ΣQ_c-curve and the ΣQ_t-curve.

At first glance, the comparatively low rate of water loss during the early minutes of the storm may seem surprising, but further study shows that this is quite reasonable when it is considered that the area in question is composed of both pervious and impervious surfaces and that the first water to reach the inlet (and, in fact, the only water to reach it for some time) is that which has fallen upon the adjacent paved streets, which are highly impervious surfaces. Naturally, then, the loss is low. As the pervious areas (in this case mostly sodded), begin to contribute water to the inlet, the percentage of loss becomes higher and the curve becomes steeper, and, for a considerable period (almost 10 min), the rate of loss is constant. This loss may well represent the average rate for the entire area, pervious and impervious. Soon after the cessation of rainfall, the rate begins to decline, which indicates that the run-off now consists mainly of gutter flow, in which case, again, the loss would naturally be low due to the impervious surface on which the water is flowing. It seems reasonable to infer that a straight line representing constant infiltration (such as in Fig. 31 (*b*)) would only apply to areas where the entire surface was of the same degree of permeability.

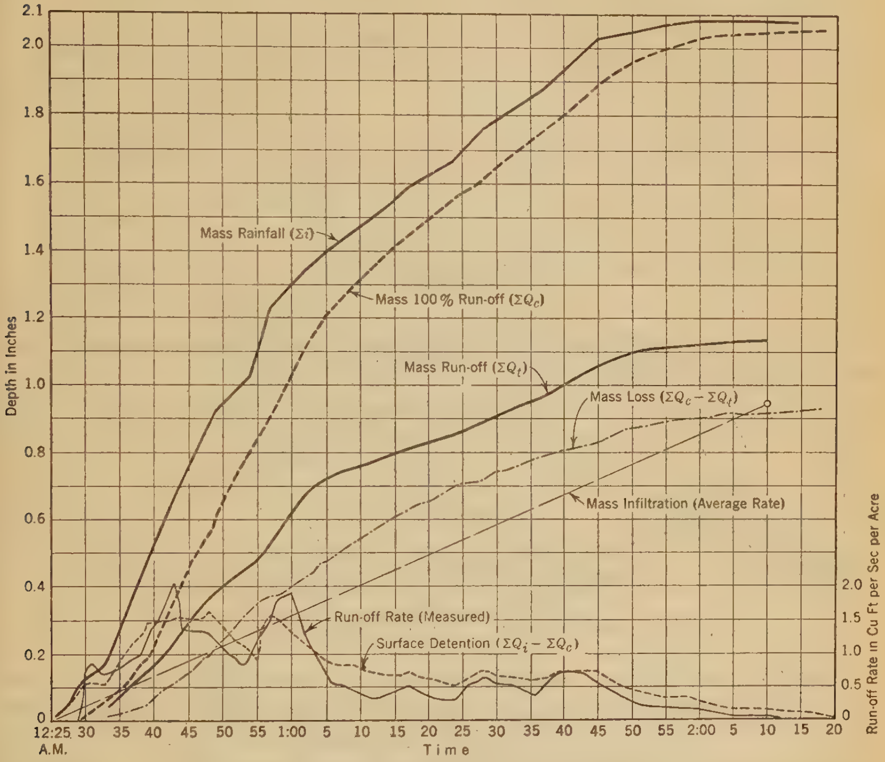

FIG. 32.—STORM OF SEPTEMBER 15, 1914, AT AREA *A*.

Fig. 32 gives the basic data and derived curves for the storm of September 15, 1914, which may be described as an "average rain" (Fig. 38, introduced subsequently, shows the rainfall pattern).

The loss curve in this case shows the same general characteristics as the one shown in Fig. 31 (*c*), the minor differences being explained by the different rainfall pattern. There is again the low initial loss and the high rate of loss during the period of high precipitation, when the pervious areas are contributing their full quota of water. After the period of heavy rainfall the gutter flow, which includes much of the water falling during the heavy rainfall period, tends to reduce the average rate of loss, and, as the rate of rainfall becomes less and less toward the end of the storm, the rate of loss declines due to the predominance of gutter flow.

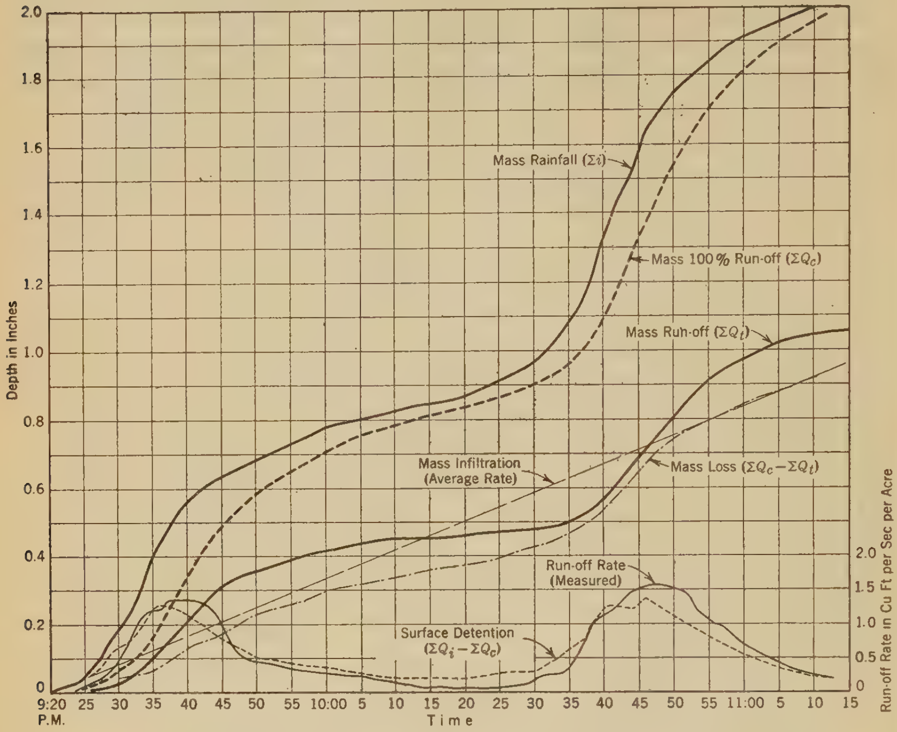

FIG. 33.—STORM OF JUNE 19, 1928, AT AREA *A*.

Fig. 33 shows the same analysis applied to the data for the storm of June 19, 1928, in which there were two periods of heavy rainfall about an hour apart, with continuous but very light precipitation between them. Here, again, is found the same variation in the rate of loss, which is susceptible to the same explanation given in the foregoing analysis. To the beginning of the second period of heavy rainfall, the variation follows the same general pattern as in Fig. 32. The sharp rise thereafter is due to the fact that the pervious areas are again contributing a large share of the water appearing as run-off, whereas during the lull between the storms the impervious areas were contributing most of the water due to the fact that the infiltration rate on the pervious area was nearly or quite as great as the rate of precipitation.

In the three storms (Figs. 31, 32, and 33), the close relation between the depth of surface detention and the rate of run-off is obvious. Fig. 34 shows an attempt to arrive at the mathematical correlation between these two factors by plotting the values for depth of surface detention for several peaks, in the aforementioned storms and two others, against corresponding

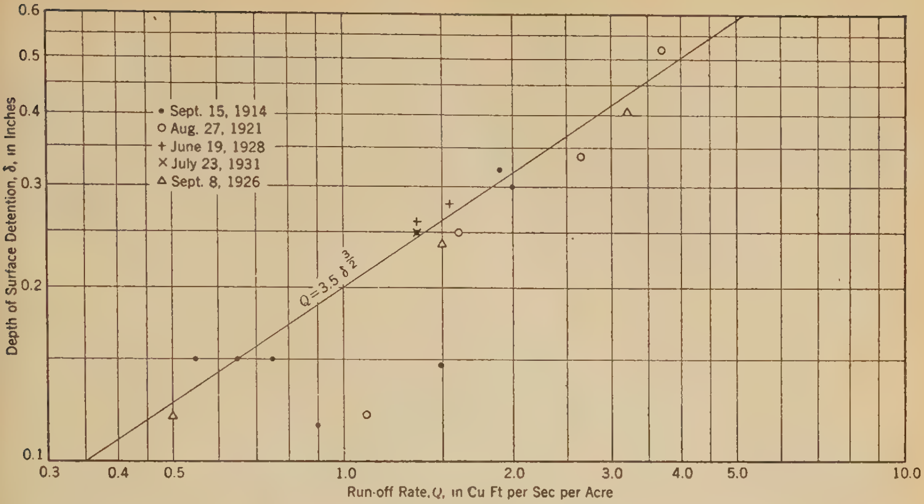

FIG. 34.—RELATION BETWEEN PEAK VALUES, SURFACE DETENTION, AND RUN-OFF RATE, AREA *A*.

run-off peaks in the same storms. The results are from a small number of storms, and both pervious and impervious areas are represented in varying proportions at different peaks. However, the general trend of the

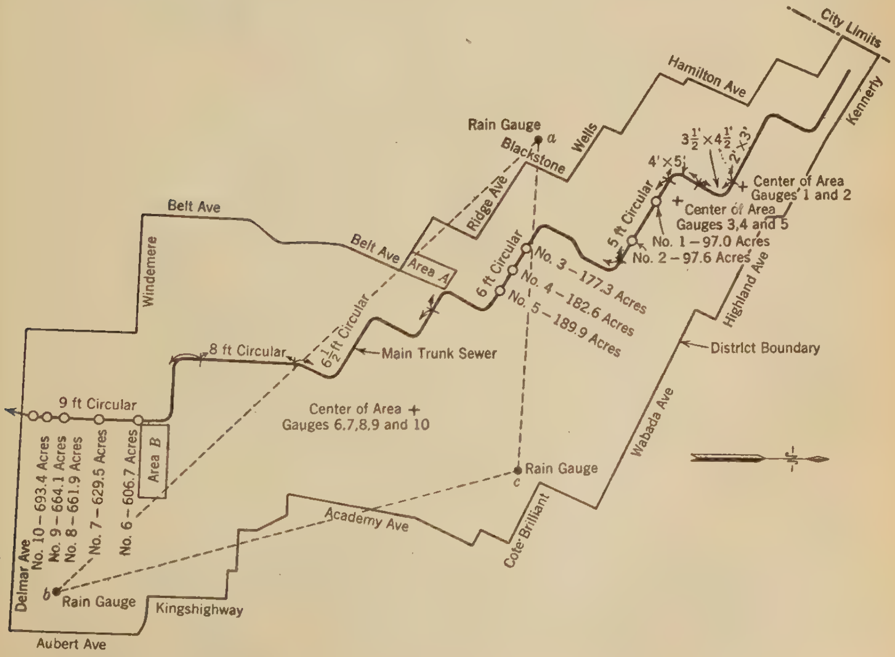
City Limits
Hamilton Ave
Kennerly
Rain Gauge
a
Blackstone
Wells
$3\frac{1}{2}' \times 4\frac{1}{2}'$
$2' \times 3'$
4'×5'
Center of Area
Gauges 1 and 2
Center of Area
Gauges 3,4 and 5
5 ft Circular
No. 1 – 97.0 Acres
No. 2 – 97.6 Acres
Ridge Ave
Belt Ave
Belt Ave
Area A
6 ft Circular
No. 3 – 177.3 Acres
No. 4 – 182.6 Acres
No. 5 – 189.9 Acres
Highland Ave
Windemere
$6\frac{1}{2}$ ft Circular
Main Trunk Sewer
District Boundary
8 ft Circular
9 ft Circular
Center of Area
Gauges 6,7,8,9 and 10
Wabada Ave
No. 10 – 693.4 Acres
No. 9 – 664.1 Acres
No. 8 – 661.9 Acres
No. 7 – 629.5 Acres
No. 6 – 606.7 Acres
Area B
Rain Gauge
c
Academy Ave
Cote Brilliant
Delmar Ave
b
Rain Gauge
Kingshighway
Aubert Ave

FIG. 35.—CLARENDON SEWER DISTRICT, SHOWING LOCATION OF GAUGES.

plotted points tends to agree with the equation given by Mr. Sherman: $Q = C\sqrt{Sd} \times d$, or Q varies as $d^{\frac{3}{2}}$.

It is unfortunate in its bearing on Mr. Bernard's demonstration, that it has not been possible to produce any considerable number of the sewer gaugings for the larger drainage areas in the Clarendon District, as this would have made possible a clearer appreciation of the manner in which run-off rates from individual city blocks are translated into rates applicable to the larger sewers. As an indication of the possibilites, the results for one particular rain are given herewith.

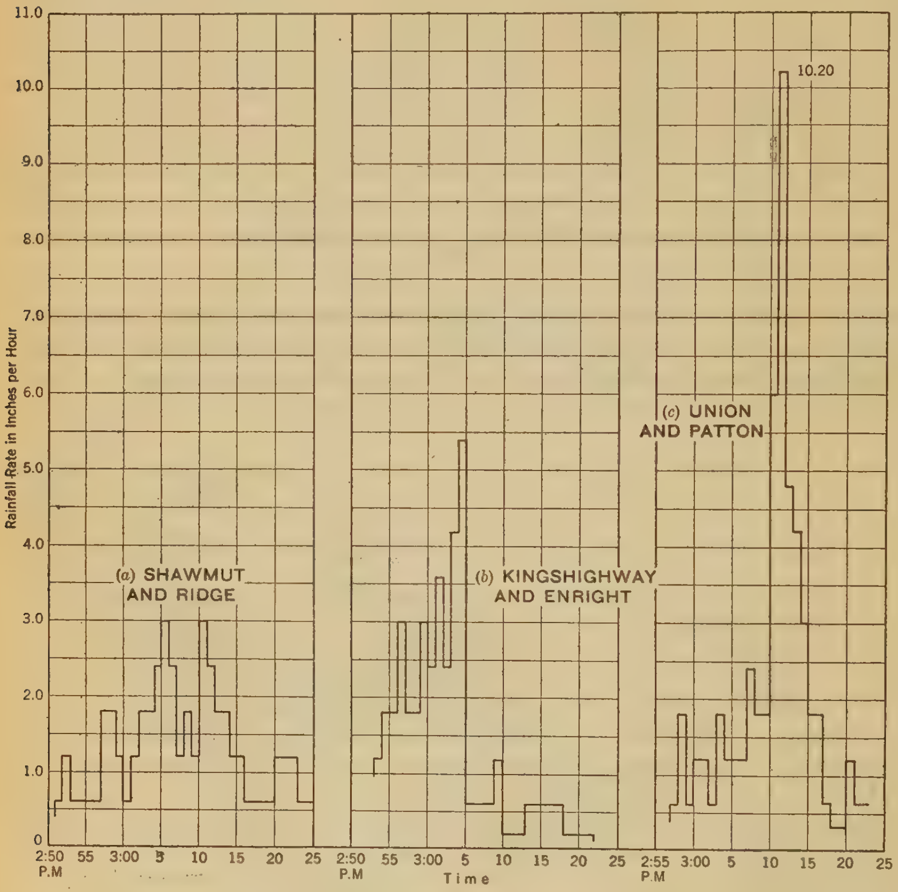

FIG. 36.

Fig. 35 is a plat of the Clarendon District, showing the location and sizes of the main sewers (the grades of which vary from one-third to two-thirds of 1%), the location of the principal pressure gauges in the main sewers by number, and the drainage area tributary to each. The location of the three tipping-bucket rain-gauges (marked *a*, *b*, and *c*) is also shown on this plat, as well as the location of the two city blocks from which the data for the original paper were collected.

Fig. 36 shows the rain record of July 10, 1920, at three rain-gauges, showing the wide variation in pattern possible even within a 600-acre district. This particular rain also showed a definite tendency to progress up the drainage area. Fig. 37 shows the interpolated rain diagram applicable to the centers of the drainage areas above each group of sewer gauges, and, on this same diagram, have been plotted the hydrographs of sewer flow for each gauge in the group. For comparison, the average rainfall rate for the approximate critical time of 20 min is shown in Fig. 37 (*a*) and Fig. 37 (*b*) and the average rate for 25 min is shown in Fig. 37 (*c*). Comparing these rain values with Fig. 23, it would appear that this rain would have a frequency of about 1½ yr.

The higher run-off rates, in cubic feet per second per acre, in Fig. 37 (*c*) may be explained in part by the longer and more compact rain diagram, and, in part, by the considerable width of the drainage basin in its lower sections. The lower rates, averaging 0.75 cu ft per sec per acre for the upper groups of gauges on Fig. 37(*a*) and Fig. 37 (*b*), give a good example of the effect of surface detention on run-off rates in short rains.

Comparing the rates at the three groups of gauges with the expected run-off rate from single city blocks, as given in Fig. 23 of the paper, it is seen that the run-off of 0.75 cu ft per sec per acre is less than one-half the expected rate for a 20-min storm of 1½-yr frequency for Area *A* and about 70% of the corresponding value for Area *B*. A rough comparison would seem to indicate that, for a composite area in which the Area *A* type of city block is predominant, the run-off rate in the sewer is 50% of the expected run-off of the single city block. For the lower gauges, shown on Fig. 37 (*c*), it would appear that the sewer-flow rate might be as much as 75% of that from an average city block. It is expected that a full analysis of this work will provide some excellent factors for use in calculations of the type developed by Mr. Bernard.

Throughout this entire study, it is apparent that the simplest method of expressing the relation between rainfall and run-off is by the volumetric ratio between the accumulated run-off and the accumulated rainfall. When this ratio is derived by comparing the accumulated run-off for a given time with the accumulated rainfall for a time earlier by the amount of the mean lag, for the location in question, it may be applied quite simply to approximate solutions of storm-flow problems. Figs. 38 and 39 show the results of this method as applied to the rain of September 15, 1914, at Area *A*.

The relation between the two factors seems to be remarkably constant in many of the storms that have been investigated, and there is reason to believe that, had the synchronism between the rainfall and run-off records been more reliable, this constant relation would have been more general. Fig. 38 shows how, by multiplying the rate of rainfall for each minute by the average ratio (0.54), as derived from Fig. 39, and plotting the result 4 min later, a graph is constructed which, in its peak values, and general configuration, closely approximates the measured run-off curve. This suggests a simple method for predicting the (approximate) probable run-off from a given rainfall when

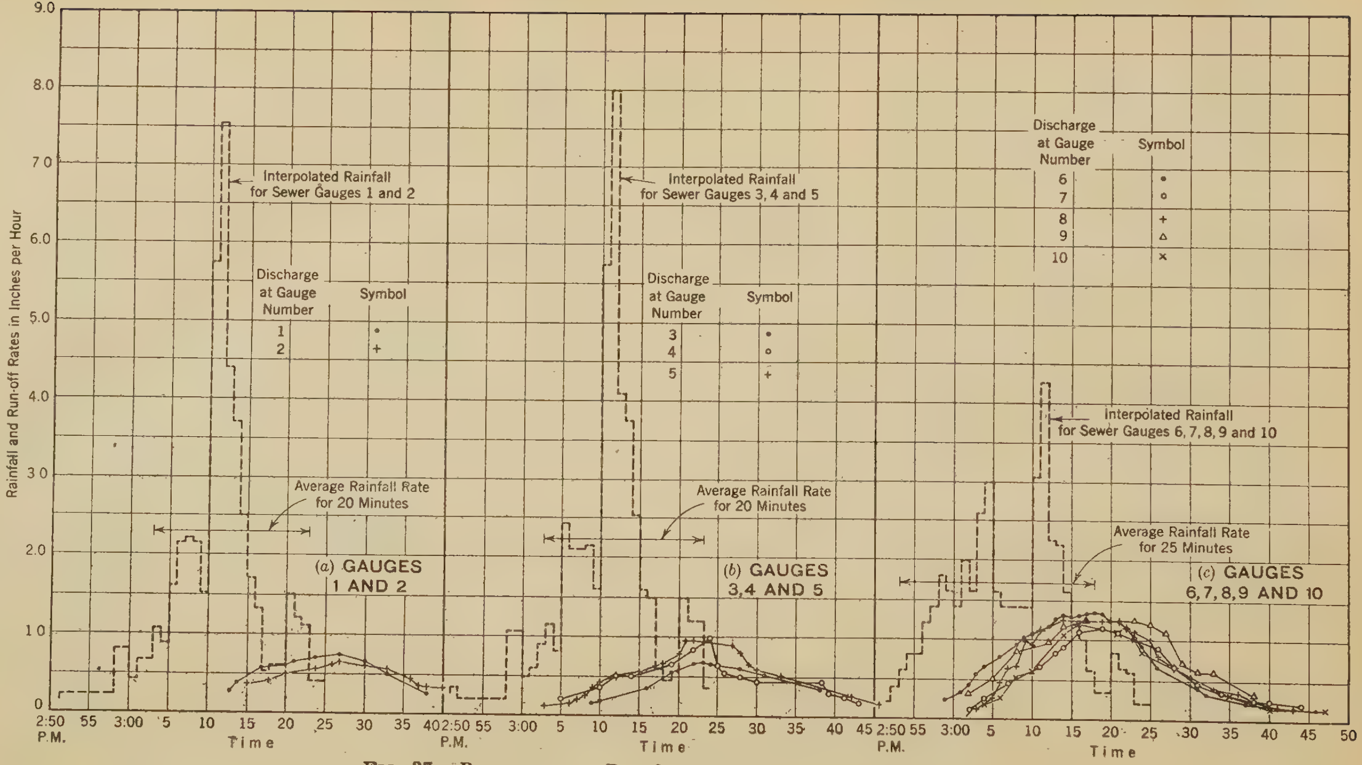

FIG. 37.—RAINFALL AND RUN-OFF DATA, STORM OF JULY 10, 1920.

the mean run-off factor and the lag are known for the location in question. Unfortunately, the run-off factor varies over wide limits from one storm to another, as shown in Fig. 18 of the paper.

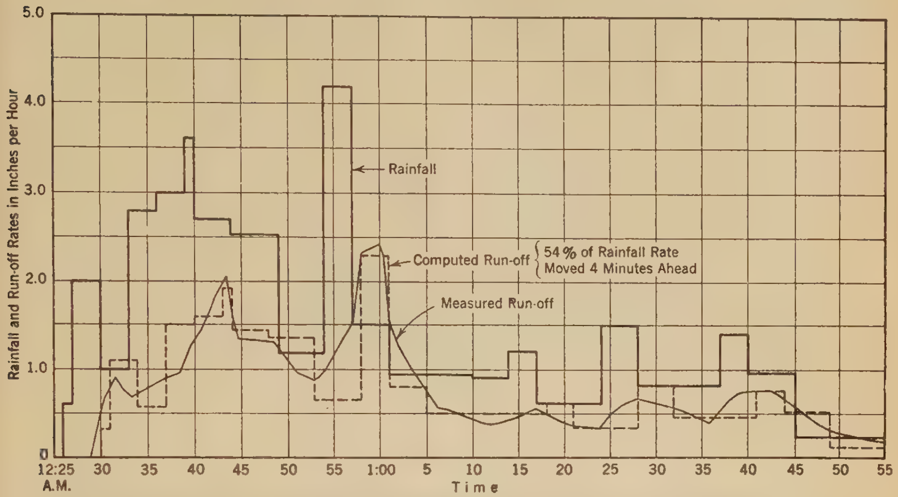

FIG. 38.—RAINFALL AND RUN-OFF, BELT AND RIDGE AVENUES, SEPTEMBER 15, 1914.

Little can be added to what Mr. Bernard has said about extending the unit graph method to large urban areas, except to commend the very constructive work he has done toward enlarging the scope of the data submitted in the paper and making it more useful to the sewer designer.

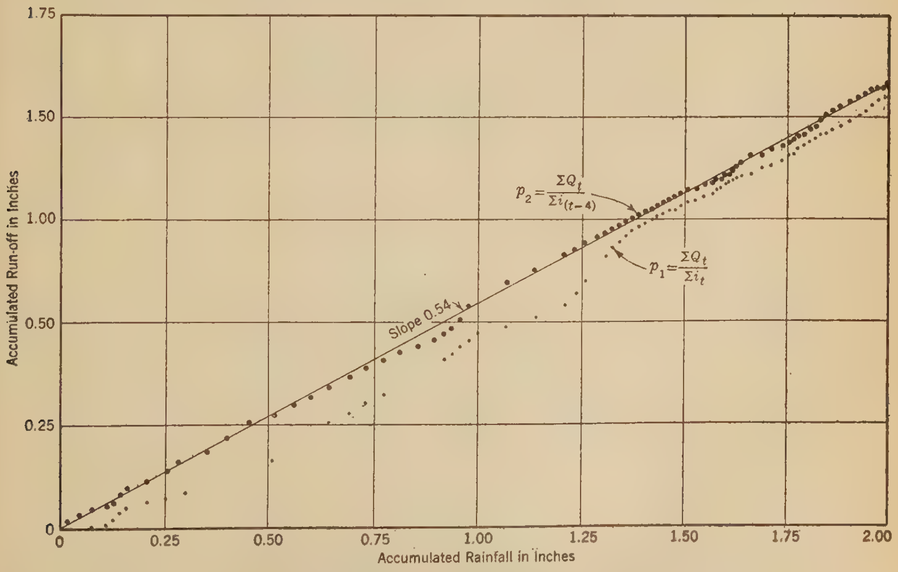

FIG. 39.—ACCUMULATED RAINFALL AND RUN-OFF, BELT AND RIDGE AVENUES, SEPTEMBER 15, 1914.

Acknowledgments.—The St. Louis Research Program, of which the data presented in the paper are a part, was developed by one of the writers, Mr. Horner, under the inspiration of the late James A. Hooke, formerly Sewer Commissioner of St. Louis, through whose energetic support funds were made available. The experimental work was continually under the supervision of Mr. Horner up to 1933; during a large part of this period the sympathetic interest of E. R. Kinsey, M. Am. Soc. C. E., at that time President of the Board of Public Service, was effective in providing continued financial support for the project. The publication of Departmental records is with the approval of Baxter L. Brown, M. Am. Soc. C. E., President of the Board of Public Service. The supervision of the instruments and records has been, at various times, in direct charge of Mr. Leland Chivvis, or Mr. Guy Brown, each of whom had some part in the analysis of the information. The present complete analysis of accumulated information for three city blocks has been the specific work of one of the writers, Mr. Flynt.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1927

THE SILT PROBLEM

By J. C. Stevens,[1] M. Am. Soc. C. E.

With Discussion by Messrs. Harry G. Nickle, E. W. Lane, Frank E. Bonner, Morrough P. O'Brien, Harry F. Blaney, W. W. Waggoner, Philip R. R. Bisschop, Herman Stabler, N. C. Grover, and J. C. Stevens.

Synopsis

All the basic data that the writer could secure on the silting of reservoirs, where actual capacity surveys have been made to determine the extent of silting, are contained in this paper. Remedial measures for silt elimination are presented and discussed. A table contains a brief of all data on the silt transported by the streams of the world. The physical laws of silt transportation are outlined, with pertinent discussion. The control of silt in canals, reservoirs, and on water-sheds is then considered. The paper closes with data and discussion on the origin of silt.

The original paper, containing additional data and more extended discussion, is on file in Engineering Societies Library, 29 West 39th Street, New York, N. Y.

Statement of the Problem

The word, "silt", as used in this paper, includes in its definition all material transported by flowing water whether carried in suspension or transported as bottom load. Silt originates from the disintegration of rocks by the climatic agencies of rain, wind, and frost, and by chemical agencies in water and air. The effect of such disintegration is evidenced by the wearing down of mountains, the gullying of their slopes, the filling of valleys, the extension of deltas into seas—in brief, the leveling of mountain ranges into plains.

Note.—Early in its existence the Special Committee on Irrigation Hydraulics selected the subject of "The Silt Problem" as one of ten for study and research. This paper was submitted to the Committee by its author, and the Committee has recommended its publication by the Society. Published in October, 1934, *Proceedings*.

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Geological history is marked by a succession of sedimentations and uplifts, in which varying climates, water supplies, vegetation, glaciation, and volcanism have played and are playing all-important parts. The processes of disintegration, erosion, transportation, sedimentation, mountain building, and plain leveling are still in progress and are as effectual as ever. Existing topography is a complex residual of these processes in which unnumbered eons have contributed to its present configuration.

Man can not hope to halt the processes of mountain erosion and plain building. The land he cultivates could not exist except for these forces. He must expect that rains will gully his fields, or cover them with mountain débris, and that the streams will continue to carry sediments that will fill the canals and reservoirs.

Man's problem lies in utilizing the agencies of sedimentation to his advantage where possible and in opposing the nullification of his endeavors by controlling them through whatever forces lie at his command.

Data on Silting of Reservoirs

The construction of large storage reservoirs on some of the silt-laden streams of the West has focused attention on the part sedimentation will play in the future history of Western civilization.

An empire exists below the storage reservoir that has been created by the Elephant Butte Dam, on the Rio Grande, in New Mexico. The land is phenomenally productive. This region embraces a substantial unit of civilization the very existence of which hangs upon the integrity of a storage reservoir to impound and deliver the water so vitally necessary to life.

The reservoir created by Elephant Butte Dam is slowly being deprived of its ability to store water. Silt is being deposited at an average rate of 20 000 acre-ft per yr. Its original capacity will be so depleted in two or three generations that the civilization now dependent upon it will have to seek other sources of water supply and storage.

Fortunately, sites are available where other reservoirs may be constructed; and, after these are gone, others will doubtless be found. Furthermore, the present dam could be raised to increase its storage capacity; but what of the ultimate future, when all available storage sites have been exhausted? Must those now fertile areas revert, ultimately, to the sage-brush and the cactus? Will sedimentation, that made possible this vibrant civilization, ultimately sound its death knell? How long can this episode in the annals of civilization continue and what can Man do to prolong its existence?

The Boulder Dam, in the Colorado River, will create the largest storage reservoir in the world. Upon it will depend not only the security of the entire Imperial Valley against damages from floods, and the irrigation of millions of acres of land, but also the domestic and irrigation supply for the metropolitan district of Southern California. The ability to furnish power to the people in seven States also hinges upon its functioning. Unless remedial measures are adopted, this reservoir will become virtually useless, by reason of silt deposits, before the passing of the fifth generation.

Fortunately, as on the Rio Grande, a number of other storage sites are available on the Colorado River. Ultimately, however, within some definite number of generations, the fact must be faced that all these reservoirs will have become useless for storage purposes. The dependent peoples must then be reduced to those that can subsist on the areas which the unconserved flow of the river will irrigate.

It is not the writer's intention to paint a dark picture, but rather to stimulate a more intensive study and an intelligent research that will ultimately effect a practical solution of this problem. The menace exists; it is real; and unless something constructive can be evolved, civilization in these regions must eventually decline. It is unfair to sit smugly complacent and to pass this problem flippantly on to future generations. The engineer should be equal to the task of finding a solution, but it will take many years of experimentation and study, and he should be at the task, amply financed.

The results of the sedimentation of reservoirs for which the extent of depositions have been determined by actual surveys, are given in Table 1. In order to make the data comparable, capacities and silt deposits have been referred to the controlled spillway level. On some reservoirs the silts deposited above this level may be a considerable part of the total, but this does not rob the reservoir of storage capacity. The following supplementary data pertaining to Table 1 are offered.

Boulder Reservoir (Under Construction).—Of the total capacity of this reservoir, 9 500 000 acre-ft. will be reserved for flood protection, 7 000 000 for silt, and 14 000 000 for irrigation. The maximum recorded discharge was 210 000 cu ft per sec on June 18 and 19, 1921. The maximum known flood occurred in 1884, for which the discharge was between 250 000 and 300 000 cu ft per sec (*6a*).[2] The spillway capacity is 400 000 cu ft per sec, exclusive of outlets for power and irrigation.

Elephant Butte Reservoir.—Capacity surveys for silt deposits in this reservoir were made in 1916, 1920, and 1925, with the results shown in Table 2(*a*). The maximum recorded mean daily discharge entering the reservoir area was 33 000 cu ft per sec on October 11, 1904. The overflow spillway capacity is 16 000 cu ft per sec. In addition, there are four regulating gates with a combined capacity of as much more.

Roosevelt Reservoir at Junction of Salt River and Tonto Creek.—This reservoir is 70 miles northeast of Phoenix, Ariz. Capacity surveys for silt content have been made as shown in Table 2(*b*). The maximum recorded inflow was 94 800 cu ft per sec on January 19, 1916. The spillway capacity is 150 000 cu ft per sec.

Keokuk Reservoir, at Keokuk, Iowa.—The Keokuk Reservoir was completed in 1913. It extends 42 miles up stream from the dam, and has a surface area of 25 200 acres and a capacity, at Elevation 518, of 370 300 acre-ft based on a survey made in 1891. The maximum recorded inflow

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[2] Figures in parentheses refer to numbered articles in the Bibliography (Appendix).

TABLE 1.—RESERVOIRS FOR WHICH AMOUNT OF SILTING HAS BEEN MEASURED
(Computations Carried to Three Significant Figures)

Reservoir	Stream	Drainage area, in square miles	Mean annual supply, in thousands of acre-feet	ORIGINAL CAPACITY*		YEAR OF CAPACITY SURVEY*		Period, in years	Water supply during the period, in thousands of acre-feet	SILT DEPOSITED				Reference No.†
				In thousands of acre-feet	Percentage of annual supply	First	Last			Total, in acre-feet	Annually, in acre-feet	Percentage of original capacity	Per thousand of water supply, by volume	
(a) WITHIN THE UNITED STATES														
Boulder	Colorado, River, Ariz.	140 000	15 000	30 500	203	....	....	.....	....	....	.....	......	......	(58), (6*a*)
Elephant Butte	Rio Grande, N. Mex.	30 000	1 200	2 640	224	1916	1925	8.67	10 800	178 000	20 500	6.7	16.3	(40*a*),
Roosevelt	Salt, Ariz.	5 670	840	1 370	164	1905	1925	20.0	....	101 000	5 050	7.4	......	(40*b*), (58)
Keokuk	Mississippi	119 000	46 000	370	0.80	1891	1928	15	640 000	112 000	7 460	30.2	0.18	(44)
Hales Bar	Tennessee, Tenn.	21 800	27 000	156.0	0.57	1913	1930	16.92	460 000	45 500	2 690	29.1	0.10	(70*a*)
Parksville	Ocoee River, Tenn.	600	950	97.0	10.2	1912	1930	18.75	17 100	20 800	1 110	21.5	1.21	(70*b*)
McMillan	Pecos, N. Mex.	......	300	90.0	30.0	1894	1932	39	12 000	50 000	1 280	55.5	4.20	(40*c*), (58)
Guernsey	North Platte, Wyo.	16 200	1 650	72.0	4.4	1927	1933	5.9	9 870	8 400	1 430	11.0	0.85	(52), (58)
Old Lake Austin	Colorado of Texas	38 200	2 000	49.3	2.7	1893	1900	6.75	16 000	23 600	3 500	48.0	1.50	(40*d*), (46)
Lake Worth	W. Fk., Trinity, Tex.	1 870	212	47.2	22.2	1915	1928	13.0	....	13 900	1 060	29.6	......	(40*e*), (46)
Cucharas	Cucharas, Colo.	......	......	43.0	......	1910	1932	22	....	7 850	357	18.3	......	(58)
Cheoah	Little Tennessee, Tenn.	1 620	3 030	41.6	1.38	1918	1930	11.83	35 900	4 350	367	10.4	0.12	(70*c*)
Sweetwater	Sweetwater, Calif.	181	17.2	36.3	205	1886	1927	39	740	6 170	158	17.0	0.84	(41), (58)
New Lake Austin	Colorado of Texas	38 200	1 910	32.0	1.7	1913	1926	13	25 600	30 600	2 350	95.6	1.20	(40*d*), (46)
Lake Chabot	San Leander, Calif.	42	21.7	17.0	78.0	1875	1923	48	854	3 700	77	21.7	4.30	(41), (58)
White Rock	White Rock Creek, Tex.	114	......	16.9	......	1923	1928	5	....	680	136	4.0	......	(40*f*)
Boysen	Bighorn, Wyo.	7 740	1 500	16.0	1.06	1911	1924	13	....	13 000	1 000	80.0	......	(46), (58)
Zuni	Zuni, N. Mex.	650	17.4	14.8	85.0	1906	1932	26.2	466	11 300	432	76.5	24.3	(40*g*), (58)
Gibralter	Santa Ynez, Calif.	220	18.6	14.6	78.0	1920	1931	11	206	2 100	190	14.4	10.0	(41), (58)
Lake Michie	Flat, N. C.	170	90.0	14.3	......	1906	1930	3	114	90	30	0.63	0.78	(11), (58)
Sterling Pool	Rock River, Ill.	8 760	3 500	13.7	0.39	1912	1930	18	....	2 010	111	14.7	......	(54)
Coon Rapids Pond	Mississippi, Minn.	19 000	4 400	8.0	0.20	1899	1931	16	....	1 380	86	17.0	......	(69)
Furnish	Umatilla, Ore.	1 200	485	5.50	1.16	1909	1931	22	10 200	4 500	204	82.0	0.44	(58)
Lake Penick	Clear Fk., Brazos, Tex.	2 250	111	3.09	2.70	1920	1927	7	....	965	138	32.2	......	(40*h*), (46)
LaGrange	Tuolumne, Calif.	1 500	1 970	2.33	0.12	1895	1931	36	70 000	1 940	54	83.0	0.028	(7*a*), (58)
Buckhorn	Buckhorn, Colo.	130	......	1.19	......	1907	1925	18	....	565	31.4	47.5	......	(40*b*), (58)
(b) OUTSIDE THE UNITED STATES														
Aswan	Nile, Egypt	620 000	66 000	4 400	5.7	1900	1927	25	....	0	0	0	0	(9),(24);(36*a*)
Burrinjuck	Murrumbidgee, Austral.	5 000	1 150	772	67.0	1910	1924	14	14 700	3 970	285	0.51	0.28	(58)
Dhukwan	Betwa, India	8 240	......	85.4	......	1910	1932	.....	....	....	.....	25.0	......	(58)
Pericha	Betwa, India	10 300	......	67.5	......	1885	1932	.....	....	....	.....	0	......	(58)
Habra	Habra, Algeria	......	......	24.3	......	1885	1907	22	....	14 200	650	58.0	......	(45), (5*a*)
Helena	Mundaring, W. Austral.	......	48	17.2	36.0	1900	1930	30	1 500	690	23	4.0	0.46	(58)
Hamiz	Hamiz, Algeria	......	6.0	11.4	......	1872	1901	29	....	1300	45	11.4	......	(45), (5*b*)

* To spillway level or to a level controlled by flash-boards or crest-gates. † See corresponding numbers in Appendix.

TABLE 2.—CAPACITY SURVEYS FOR SILT DEPOSITS
(Computations Carried to Three Significant Figures)

Period of record	Years	SILT DEPOSITED			RESERVOIR CAPACITY	
		In acre-feet	In acre-feet per year	Cumulative total, in acre-feet	In acre-feet	Percentage lost
(a) ELEPHANT BUTTE RESERVOIR						
1916	....	....	....	....	2 640 000	0
1916–1920	4.06	90 900	22 400	90 900	2 550 000	3.3
1920–1925	4.61	86 900	19 000	178 000	2 460 000	6.7
(b) ROOSEVELT RESERVOIR						
1905	....	....	....	....	1 370 000	0
1905–1914	9.0	27 000	3 000	27 000	1 340 000	2.2
1914–1916	2.0	35 000	17 500	62 000	1 310 000	4.4
1916–1919	3.0	0	0	62 000	1 310 000	4.4
1919–1925	6.0	39 000	6 500	101 000	1 270 000	7.4
(c) PARKSVILLE RESERVOIR (70d)						
1912	0	....	....	....	97 000	0
1912–1921	9	15 600	1 720	....	81 400	16.1
1921–1930	9.75	20 800	1 110	....	76 200	21.5
(d) McMILLAN RESERVOIR						
January, 1894	0	0	0	0	90 000	0
1894–1904	10.42	16 000	1 540	16 000	74 000	17.7
1904–1910	6.42	12 000	1 870	28 000	62 000	31.1
1910–1915	4.58	17 000	3 710	45 000	45 000	50.0
1915–1932	17.58	5 000	285	50 000	40 000	55.5
(e) AUSTIN RESERVOIR						
1913	....	....	....	....	32 000	0
1913–1922	9	26 700	2 970	26 700	5 300	83.5
1922–1924	2	2 460	1 230	29 200	2 800	91.2
1924–1926	2	1 420	710	30 600	1 400	95.6
(f) SWEETWATER RESERVOIR						
1895	....	....	....	....	36 300	0
1888–1895	7	1 200	172	1 200	35 100	3.4
1895–1916	21	4 190	220	5 390	30 900	4.9
1916	11	773	70	6 160	30 100	1.70
(g) LA GRANGE RESERVOIR						
1895	....	....	....	....	2 330	0
1895–1905	10	1 260	126	1 260	1 070	55.0
1905–1911	6	430	72	1 690	640	72.6
1911–1931	10	250	25	1 940	390	83.2

was 314 000 cu ft per sec in May, 1888. Since the dam was constructed, the maximum flood was 242 000 cu ft per sec on April 16, 1922.

Hales Bar Reservoir.—The dam is near Guild, Tenn., and was completed November 1, 1913. The normal pool level (crest of dam) is at Elevation 626.2. The original capacity of the reservoir to the crest of the dam was 142 000 acre-ft; 3 ft of flash-boards added later, increased this capacity to 156 000 acre-ft. Surveys were made of the Tennessee River in 1889–1891 by the United States Corps of Engineers. In 1905, the Power Company made a survey for flowage areas. In the fall of 1930, the U. S. Engineers made a capacity survey of the entire reservoir.

Parksville Reservoir (Also Called Ocoee No. 1).—The dam is 12 miles above the mouth of the Ocoee River, at Parksville, Tenn. About 70% of the drainage area is forest and wood lots. The Ducktown Mining District presents a striking example of the destruction of forest lands by the fumes from smelters, roast ovens, and incidental activities. About 20 sq miles have been completely denuded. Erosion is excessive as the poisonous gases will not permit the growth of any vegetable matter. This condition, and the clayey soils and excessive rainfall to which this basin is subject, offers a complete explanation of this most flagrant example of unnatural erosion.

Surveys to determine the silt content were made by The Tennessee Electric Power Company in 1917, 1921, and 1929, and by the U. S. Engineers in the fall of 1930. The early surveys included only the up-stream half of the reservoir, but that of 1930 covered the entire reservoir. The results of the capacity surveys (70*d*) are given in Table 2(*c*).

Lake McMillan North of Carlsbad, N. Mex.—The original dam was completed by the Pecos Irrigation District, in 1894, with a capacity of 32 500 acre-ft at the spillway (Elevation 3 258.9). The project was greatly damaged by floods in 1906 and thereupon was taken over by the United States Bureau of Reclamation. The spillway has been raised repeatedly, the last time in 1915 to Elevation 3 267.7, to which the capacity surveys in Table 2(*d*) have been referred. The great reduction in the rate of silting since 1915 has been due to a dense growth of tamarisk (salt cedar) on the flats at the upper end of the reservoir. This growth induces the river to drop most of its silt in the valley just above the reservoir. Fig. 1 shows

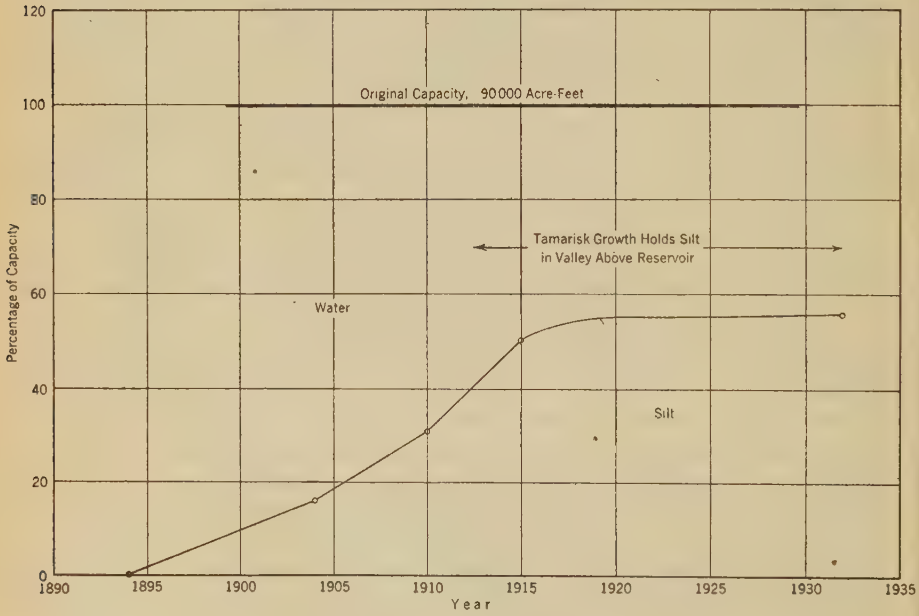

FIG. 1.—SILTING OF LAKE MCMILLAN.

the silting curve for this reservoir. Nearly one-third of the silt deposited during the last period resulted from the flood of April, 1915, during which the maximum recorded inflow of 42 000 cu ft per sec was observed. The nominal spillway capacity at the dam is 35 000 cu ft per sec.

Guernsey Reservoir.—This is a regulating reservoir on the North Platte River near Guernsey, Wyo. Storage began in March, 1927. Surveys for silt were made by soundings in February, 1929, showing 3 010 acre-ft of deposits; in January, 1931, 5 970 acre-ft; and in February, 1933, 8 400 acre-ft. Silt was found mostly in the old river channel in the up-stream half of the reservoir. The normal flow of 1 650 000 acre-ft is the average inflow for the period, 1909–1932. The maximum discharge was 21 000 cu ft per sec, on June 28, 1917.

Austin Reservoir, Austin, Tex.—The first dam was completed in May, 1893, and failed in April, 1900. It was rebuilt in 1911–1913. During the interval, 1900 to 1913, much of the silt deposited in the old reservoir was washed out. The new reservoir, with a spillway 9 ft lower than the old, had 65% of the capacity of the old reservoir when it was first filled in 1913. Capacity surveys were made in the late summer of the years indicated in Table 2(*e*). They refer to the new reservoir.

The spring flood of 1922 deposited more silt in two weeks than had accumulated in the previous five years. The maximum recorded discharge over the spillway was 151 000 cu ft per sec on April 7, 1900, just before the dam failed.

Lake Worth, Near Fort Worth, Tex.—The average water supply from October 16, 1923, to September 30, 1930, was 211 800 acre-ft per yr. This reservoir provides water for the City of Fort Worth. The maximum recorded discharge was 7 600 cu ft per sec, on November 18, 1923.

Cucharas Reservoir, Arkansas River Drainage East of Pueblo, Colo.—The drainage area is 6 000 to 12 000 ft high, composed of narrow valleys and eroded mesas sparsely timbered, except in the higher altitudes. No actual surveys of silt content have been made, but it is known to be silted to an average elevation of 68.0 ft (spillway level, 106.0 ft), for which the capacity is 7 850 acre-ft.

Cheoah Reservoir at Mouth of Cheoah River, Near Fairfax, Tenn.—The total drainage area of the Little Tennessee River is 2 650 sq miles, of which 60% is forest lands, most of which have been cut over. The Cheoah Dam and Hydro-Electric Plant were completed in 1919. The original capacity at normal pool level (Elevation 1 275.8) was 41 600 acre-ft. The Aluminum Company of America made a survey for silt content in the upper 2½ miles of the reservoir in 1922. The U. S. Engineers made a complete survey of the reservoir in the fall of 1930.

Sweetwater Reservoir.—This reservoir is in San Diego County, California. It was originally built in 1888, to a height of 70 ft above the outlet. The spillway was raised 5 ft in 1896. The height of the dam and spillway was again increased 15 ft in 1915. The present (1932) spillway is 90 ft above the outlet; capacity surveys were made in 1887, 1895, and 1917, and a partial survey in 1927, with the results shown in Table 2(*f*). Most of the silt was deposited in flood years. Stream-flow records cover 42 climatic years, from 1887 to 1929. The maximum inflow occurred in 1915–16 (160 000 acre-ft). The minimum was zero during the four years ending September 30, 1900, 1902, 1903, and 1904. During nine years (1895–1904), the total water supplied was only 9 290 acre-ft. The maximum recorded discharge was 45 500 cu ft per sec, on January 27, 1916.

Lake Chabot, Southwest of Oakland, Calif. — The drainage area is densely covered with brush and redwood. Capacity surveys referred to Spillway Gauge 83.5 are: For 1875, 17 000 acre-ft; for 1900, 15 500 acre-ft; for 1911, 13 800 acre-ft; and for 1923, 13 500 acre-ft.

White Rock Reservoir.—This reservoir is 4 miles east of Dallas, Tex. Of the drainage area, 75% is cultivated land. A survey in 1910 (not very accurate) gave a capacity of 21 500 acre-ft at Elevation 140.15 (local datum). The survey of 1923, which was made accurately, showed a capacity at Elevation 140.5 of 19 535 acre-ft, and 16 896 acre-ft, at Elevation 138.5 (spillway level).

Boysen Reseroir in Fremont County, Wyoming.—The dam was completed in 1911 to create head for a hydro-electric plant. Storage was of secondary consideration. Soundings were made through the ice on December 5, 1922, and again on January 23, 1924. On the latter date the reservoir was practically full of silt to the spillway level. The deposit of 13 000 acre-ft of silt is largely estimated. The power plant has carried no load since May, 1928, on account of silt accumulations.

Zuñi Reservoir.—The reservoir is at Blackrock, N. Mex., in the Zuñi Indian Reservation. The dam was completed in 1907. Capacity surveys have been made by sounding through the ice during each year when it was possible. When the ice did not cover the entire silt deposits, the reservoir

TABLE 3.—CAPACITY SURVEY OF ZUÑI RESERVOIR IN NEW MEXICO

Date of survey	Period, in years	Capacity, at Elevation 998.3, in acre-feet	Run-off for period, in acre-feet	SILT DEPOSITED DURING PERIOD		CAPACITY LOST	
				In acre-feet	Per thousand of inflow	Cumulative, in acre-feet	Percentage of original
1906...	....	14 800	......	......	......	......	......
June 26, 1910...	4.0	13 000	56 200	1 800	32	1 800	12.2
December 22, 1911...	1.5	11 800	24 600	1 200	48	3 000	20.2
January 1914...	2.1	10 600	14 500	1 200	83	4 200	21.4
January 1918...	4.0	9 240	141 000	1 360	9.6	5 560	37.6
January 1919...	1.0	8 560	5 430	680	125	6 240	42.1
January 1920...	1.0	7 310	46 400	1 250	27	7 490	50.6
January 1921...	1.0	6 500	9 640	810	84	8 300	56.0
February 10, 1922...	1.1	4 880	7 000	1 620	231	9 920	67.0
January 1924...	1.9	4 500	22 900	380	17	10 300	69.5
January 1925...	1.0	4 390	8 880	190	21	10 490	71.0
January 1926...	1.0	4 120	5 170	170	33	10 660	72.0
January 1927...	1.0	3 950	4 980	170	34	10 830	73.0
December 24, 1927...	0.9	3 500	16 000	450	28	11 280	76.0
December 29, 1928...	1.0	3 620	55 600	—120	Scour	11 160	75.0
January 29, 1930...	1.1	2 820	15 100	800	53	11 960	80.6
July 1932...	2.6	3 450	33 200	—630	Scour	11 330	76.5
Total..........	26.2	....	466 000	11 330	24.3	....	....

topography from the preceding survey was used for that part above the ice. The deposits between ice level and top of flash-board level (Elevation 998.3) for some of the years are, therefore, probably not included in the totals given in Table 3.

The survey of July, 1932, was a topographic survey of the entire reservoir taken on the surface of the silt deposits while the reservoir was practically empty. It is probably the most accurate survey that has been made.

Protective works to hold the silt on the water-shed were begun in 1923 on Rio de Los Nutrias, the principal silt-producing tributary. In July, 1931, a hole was blasted in the gate tower for the installation of a 4 by 6-ft sluice-gate, which was installed on October 15, 1931. Between these dates 500 acre-ft of silt were sluiced from the reservoir. Sluicing operations have continued each year whenever water was available. The gain in capacity for the surveys of 1928 and 1932 is partly due to sluicing and perhaps partly to inaccuracies in the preceding surveys. Fig. 2 shows the rate of silting. All capacities are referred to the top of the present flash-boards (Elevation 998.3, old datum, = 6 637.1, United States Geological Survey datum).

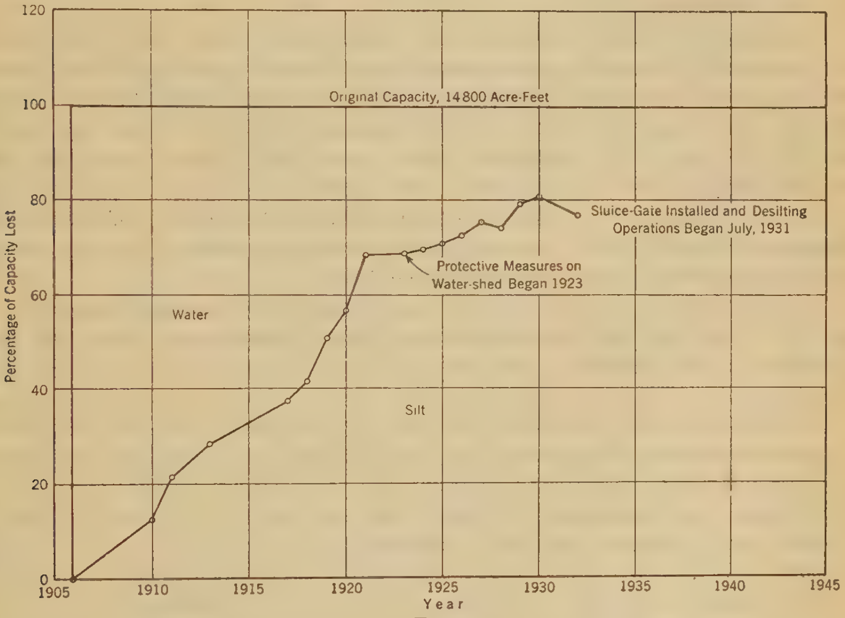

FIG. 2.—SILTING OF ZUÑI RESERVOIR, 1906–1930.

Gibraltar Reservoir.—Water for the City of Santa Barbara, Calif., is stored in this reservoir. The drainage area is sparsely covered with brush and small trees. Most of the silt came after the fire of 1923, which burned over the greater part of the water-shed. The water supply is very erratic. The year, 1921–22, yielded 65 500 acre-ft, whereas in 1923–24 the inflow was only 2 000 acre-ft. The maximum mean daily discharge (outflow) was 7 250 cu ft per sec, on April 8, 1926.

Lake Michie.—This reservoir stores water for the City of Durham, N. C. Ten permanently marked cross-sections of the reservoir were established just before filling it in the fall of 1926. Silting observations were made at these cross-sections in September, 1930, showing an average silt deposit of 8 to 10 in. Some of this deposit was due to wave action washing the top soil of the lake margin into the lake. The normal flow of 124 cu ft per sec is the average of the years, 1925–26 to 1929–30, inclusive.

Sterling Pool.—This is formed by the Government dam at Sterling, Ill. The dam and locks were constructed in connection with a feeder for the Illinois and Mississippi Canal. Soundings were made at a series of cross-sections above the dam in 1912–13.[3] The soundings in 1930 were made from the same base lines. During the 18-yr period a total of 3 360 000 cu yd of silts were deposited, while 123 000 cu yd of material were scoured. The maximum recorded flood was that of May 16, 1929 (29 200 cu ft per sec, 24-hr mean).

Coon Rapids Pond, Anoka, Minn.—This pond is used solely for power purposes, and its level is kept constant; no storage is utilized. The plant was completed in August, 1914, at which time the pond was first filled. Surveys with soundings of the area to be occupied by this pond were made in 1874 by the U. S. Engineer Office at St. Paul, Minn.; in 1899, by the Mississippi River Commission; and, again, in January, 1931, by the U. S. Engineers at St. Paul. Evidences of general scouring were detected between 1874 and 1899. General silting occurred between 1899 and 1931, which is assumed to have occurred since 1914.

There are dams at St. Cloud, Minn. (60 miles above Coon Rapids) on the Elk River, and on the Run River. These three control about 15 000 sq miles of the drainage area, but, obviously, they do not intercept all the silt.

Furnish Reservoir, Near Pendleton, Ore.—By 1914, this reservoir had silted so badly that it became necessary to put flash-boards on the spillway in order to store sufficient water for the lands under the Furnish Ditch. The use of flash-boards was discontinued in 1930. The sluice-gate in the dam is opened as soon as the storage has been exhausted, permitting the river to flow through the Furnish Reservoir. This cuts a channel through the silt to the original gravel bed. Thus, a certain amount of desilting is done each year. The survey of 1930 is little more than an estimate by the Water Master. In 1932, the writer estimated the remaining storage capacity at about 600 acre-ft.

Lake Penick.—This reservoir stores water for Stamford, Tex. Of the drainage area, 30% is farming and 70% grazing land. The dam was begun in 1918. The run-off at the U. S. Geological Survey gauging station at Nugent, with a drainage area of 2 220 sq. miles (1924 to 1930), showed 680 200 acre-ft, or 111 000 acre-ft per yr. The maximum discharge was 11 500 cu ft per sec on May 20, 1928.

LaGrange Reservoir.—The diversion dam is for the Turlock and Modesto Irrigation Districts, near LaGrange, Calif. Surveys for silt content have been made as shown in Table 2(*g*). The Don Pedro Dam completed in 1923 is 6 miles up stream. When its spillway was first used much river débris was washed into the LaGrange Reservoir. The maximum recorded discharge was 38 100 cu ft per sec on March 25, 1928.

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[3] H. R. Doc. No. 964, 63d Cong., 2d Session.

Buckhorn Reservoir, Near Loveland, Colo.—No records of stream flow have been kept. The reservoir had a capacity of 1 191 acre-ft upon completion in 1907. In 1925, surveys were made for increasing its capacity when it was found that the storage capacity had been reduced to 626 acre-ft by silt. The maximum flood of 10 500 cu ft per sec occurred on June 15, 1923 (61*a*).

Aswan Dam and Reservoir, Egypt.—The first dam was built to store water to a reservoir level (R. L.) of 106.0 m, with a capacity of 865 000 acre-ft. It was completed in 1902. No spillway was provided, but the dam has 180 sluice-gates to pass a maximum flood of 500 000 cu ft per sec. The dam was raised beginning in 1907 and completed in December, 1912, for storage to R. L. 113.0 m, providing a capacity of 1 970 000 acre-ft. A contract was let in 1929 to raise the dam again, this time to store water to R. L. 122.0 m, for which the capacity will be 4 400 000 acre-ft. The average annual flow of the Nile at Aswan (1912 to 1927) was 90 000 cu ft per sec, the maximum being 500 000 cu ft per sec.

Burrinjuck Reservoir.—Water for Murrumbidgee irrigation areas, in New South Wales, is stored in this reservoir. Construction was begun in 1908, and was completed by 1920. The spillway crest is at R. L. 1 180. The silt deposits were deduced from regular samplings above slack water, 1910 to 1916, and above slack water and below the dam, 1917 to 1924.

Dhukwan Reservoir.—This reservoir is above the mouth of Jamni River near Jhansi, United Provinces, India. The drainage area is without snow storage, and the silt deposits were estimated.

Pericha Reservoir.—This is 30 miles below the mouth of Jamni River and 35 miles below Dhukwan Reservoir. Silt deposits are reported to be negligible. The reason for silting in Dhukwan Reservoir and not in this reservoir could not be obtained. Silt interception by Dhukwan Reservoir does not fully explain this phenomenon, since it is 23 years old and has lost one-fourth of its capacity, while Pericha Reservoir has lost none in 48 years.

Habra Reservoir, in Algeria.—The dam broke in 1881 and all the silt was sluiced out. The reservoir was put in service again in 1885. During later years sluicing has been resorted to periodically. It is estimated that 4 900 acre-ft of silt have thus been sluiced out, so that the actual silt deposited should be increased by that amount. The river carries approximately 1% of silt by weight per annum.

Helena Reservoir.—The domestic water supply for Perth, Western Australia, is furnished by this reservoir. Surveys were made in 1913, 1920, and 1930. The stream flow into the reservoir from 1900 to 1930 registered a maximum per year of 151 000 acre-ft, and a minimum of 1 190 acre-ft.

Hamiz Reservoir, in Algeria.—Periodic sluicing was begun in 1901; 23% of the total water supply is used in sluicing out silt. No part of the silt deposited prior to 1901 has been removed but further loss of capacity has been prevented.

Sedimentation Processes

When a stream carrying a load of sediment into a reservoir meets the quiet waters of the reservoir its velocity is destroyed and its load of silt is deposited, forming a delta. In delta building the material is sorted as it is deposited. The finer material is carried far out into the reservoir and spread over the bottom. This part of the delta consists of the suspended load of the stream, sorted from coarse to fine and spread over a fan-shaped area the apex of which is at the mouth of the stream. For distinction this part of the delta is called the bottom-set bed.

Superimposed on this is the stream's bed load deposited over a fan-shaped area, and called the foreset bed. These deposits are also sorted, the coarser material settling first. The foreset beds are deposited in layers or strata normally inclined to the horizontal at the slope of repose of the material in its saturated condition ((81) (82)). These strata vary in thickness and in inclination with every variation of the stream's bed load and rate of flow.

The process of building the foreset beds is somewhat analogous to that of building road grades by continually dumping material on the advancing face. If the advancing face were fan-shaped and each carload was of different material in size, hue, and condition of saturation, a faint picture of the process of laying down a typical foreset bed may be had.

The top surface of the typical foreset bed is fairly level provided the water surface of the reservoir remains constant. On top of this is deposited the suspended load of the stream in the same manner that the original bottom-set beds were formed, except these deposits on top are of the coarser suspended material. This part of the delta is called the top-set bed. The finer suspended material is carried beyond the foreset bed and deposited as bottom-set bed in advance of the inclined strata of the foreset bed.

As the river flows over the top of the foreset bed, it may form ripples, dunes, and anti-dunes. Often the top-set beds are superimposed over these dunes and ripples, preserving them intact.

If the lake level varies periodically, but with sufficient time for deltaic deposits to be made at each level, the delta is formed in benches. Of such would be those in certain natural lakes the levels of which vary only with cyclic weather changes, as, for example, Great Salt Lake. When the level varies seasonally, such as is the case with most artificial reservoirs that may be emptied and filled one or more times each year, the delta soon loses its typical characteristics.

Variability of stream regimen, variability of reservoir levels, and variability of materials transported, introduce great complexities. Deposits at one level are cut through at a lower level and re-deposited farther out;

waves re-sort and flatten the slopes. The net result of these complex and interlocking processes may cause the deltaic shore to become a well-graded composite of coarse and fine material of considerably greater density than would be found in either of the delta beds in their idealistic state.

Where the reservoir capacity is small compared to the annual inflow, it may happen that a considerable part of the suspended load is carried through the reservoir and only the bed load is deposited. This is evidenced on many streams where the space above dams has been completely filled with coarse gravel and cobbles.

A sediment-carrying stream in a valley builds its bed and banks higher than the surrounding land. Alluvial valleys normally slope away from the stream channel. As the banks become higher the stream slope is reduced until finally a flood overtops and cuts through them and the river forms a new channel. The ends of the old channel soon become closed by silting and, thereafter, it stands as a lake until it is slowly filled by sediments from overflows and wind-borne materials.

A dam across a stream bed not only induces filling the slack-water space above it, but induces the deposition of river débris a long distance up stream. Fig. 3 is a profile of Bear Creek near Colfax, Calif., before and after the construction of a débris barrier. This comparatively low dam had the effect of withholding an enormous volume of coarse river débris from passing to the lower lands. This serves to illustrate an extreme case of aggrading a stream bed where only coarse sand, gravel, and boulders are concerned.

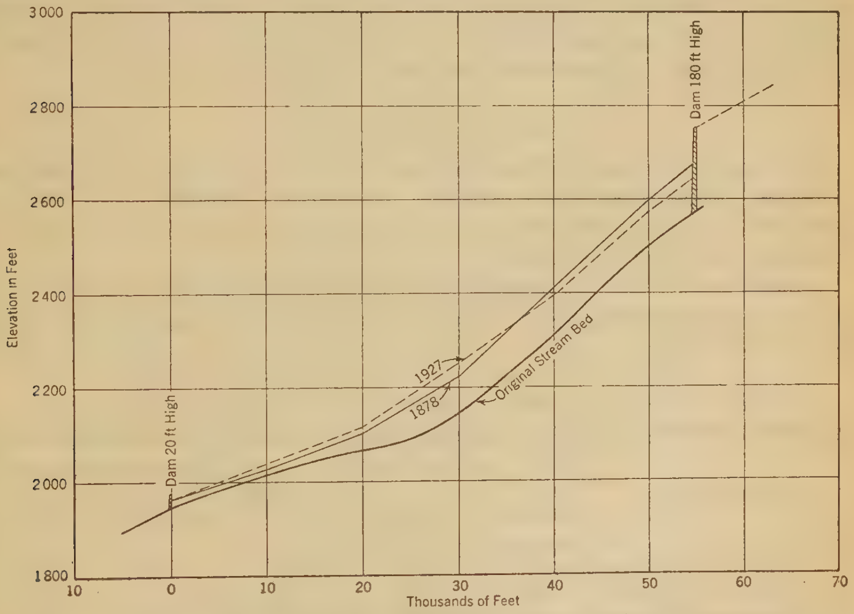

Fig. 3.—Débris Barrier on Bear Creek; California Débris Commission.

The Life of a Reservoir

A reservoir can only store water for subsequent use below its spillway level, or such other higher level as may be controlled by flash-boards or crest gates. The silt deposited below spillway level, therefore, robs the reservoir of that much water-holding capacity. Ultimately, a reservoir on a silt-laden stream will fill up to the spillway level, leaving only a meandering stream channel on the alluvial plane made by these deposits.

Table 1 shows the average rate of silt deposition per annum. The useful life of a reservoir is not the number of years obtained by dividing its original capacity by the deposition rate per year. Such generalizations are not permissible. The usefulness of a storage reservoir may be practically exhausted when its capacity has been reduced 50%, or some other substantial part of its original capacity. On the other hand, a reservoir for power purposes may still be useful after it is completely filled.

The rate of silt deposition diminishes as the reservoir fills. Deltas will form at the mouths of all tributaries. As the deposits fill the storage space a continually lessening volume of quiet water is available for deposition of the finer materials, with the result that a continually increasing part of the material entering is carried over the spillway. During extreme floods earlier deposits may be picked up and carried out of the reservoir. Such effects are more apparent in reservoirs the capacities of which are a relatively small part of the annual flow through them.

Space Occupied by Silt

The relation between weight and volume of silt is one of almost infinite variation. There is no fixed relation. The specific gravity of all silts is quite constant. The difference in unit weight of silt, therefore, is almost entirely a matter of the voids.

Assuming 2.6 as a fair average of the specific gravity of the silt particles, the weight per cubic foot of dry and saturated silt may be determined as shown in Table 4.

TABLE 4.—Computation: Weight of Dry and Saturated Silt

Percentage of voids	Weight per Cubic Foot		Percentage of voids	Weight per Cubic Foot	
	Dry silt	Saturated silt		Dry silt	Saturated silt
0	163	163	50	81	113
20	131	144	60	65	102
30	114	133	67.5	53	95
40	98	123	70	49	93
47.8	85	115	80	33	83

When silt is first deposited it is loose and flocculent, and the finer the silt the greater the volume it occupies. As it lies in place and more sediment is superimposed, it becomes more and more compact. Some have contended that the water pressure tends to compact the silt. Borings in silt deposits in California reservoirs (41) have shown: (*a*) That a definite water-table exists in the deposits commensurate with the water level of the reservoir; (*b*) that the deposits below the water-table are flocculent; and (*c*) that deposits do not shut off spring water inflow on the beds of reservoirs. Under such conditions the silt deposits are not consolidated by water pressure but by superimposed silt.

A most potent factor in compacting silt is exposure and consequent drying. The shrinkage of these deposits is evidenced by the cracks that form in a sun-dried exposure of fine silts.

Sands are not subject to such shrinkage. The amount of shrinkage, therefore, depends largely on the proportion of sand to clay and on the fineness of the clay. The more uniform the size of the particles the greater the voids and the less the specific weight. Deltaic deposits subject to wave action at all stages may become well graded from fine to coarse and, therefore, may be very dense.

These observations serve to explain why such disparity has been found in the volume-weight relation in various localities and by various investigators. Follett (48(*a*) (47) used a specific weight of 53 lb per cu ft for the Rio Grande silt. This was determined by selecting a single 3-in. cube from a sun-dried river bar. This sample was taken in 1904 and that weight has been used in all tables of the Rio Grande silt at San Marcial, N. Mex., since the beginning of the silt record in 1897. Humphreys and Abbot in 1861 used a weight of 120 lb per cu ft for the Mississippi River sediment.

Samples from exposed silt bars in various reservoirs in Texas showed the following (53):

Material	Dry weight, in pounds per cubic foot
Coarse silts at head of reservoir, probably consisted largely of bed load	92
Fine silts from much the same location	85
Deposits on surface of silt beds near middle of reservoirs	55
Finest material from submerged deposits in old river channels of reservoirs	31

Seventeen samples from exposed silt beds of Elephant Butte Reservoir in 1916 yielded the data presented in Table 5.

TABLE 5.—Samples from Exposed Silt Beds of Elephant Butte Reservoir

Description	Maximum	Minimum	Average
Weight as taken, in pounds per cubic foot	124.3	96.7	104.7
Moisture, percentage	20.9	4.4	11.6
Weight dry, in pounds per cubic foot	101.2	87.9	92.3
Specific gravity	2.66	2.59	2.64
Percentage of voids	46.6	39.0	44.0

In India (20*a*) engineers have found that the silts of the Indus River, when wet, averaged 95 lb per cu ft and that they contained 45 lb of water and 50 lb of dry matter. Many determinations have been made of the specific weight of the silts of the Colorado River with greatly varying results (42*a*). A summary of these determinations is, as follows:

Location	Number of samples, mean of:	Dry weight, in pounds per cubic foot
Yuma, Ariz.	20	85.4
Laguna Dam	10	81.6
Bed silts: Imperial Canals, 1925	17	102
Bed silts: Imperial Canals, 1917–18	12	97
Gila River: Silt bars (80)	15	74.2
Deposits in Settling Basins of Imperial Valley Municipal Water Systems:		
El Centro, Calif.	12	45.2
Imperial, Calif.	12	37.0
Calexico, Calif.	10	37.7
Experimental settling basin at Parker City	5	57.5

The writer found the dry weight of all the sediment filtered from the daily samples from Coeur d'Alene River, Idaho, for more than a year (consisting largely of tailings from ore-reducing works) to be 50 lb per cu ft.

If it were not for the uncertain quantity of silt carried into reservoirs as bed load, it would be quite a simple matter to determine the space occupied by the suspended silt in streams flowing into reservoirs where the volume of deposits have been determined by careful capacity surveys. The comparison between suspended silt and deposits in the Elephant Butte Reservoir for determining the probable bed load of the Rio Grande (see heading, "The Bed Load"), shows the futility of this method unless the bed load can be determined by direct measurement in the stream.

As a result of their study of Colorado silts, Fortier and Blaney (42*b*) recommended the use of a specific weight of 62.5 lb per cu ft for suspended silts. This was a convenient figure because the percentage of silt by weight and by volume then becomes equal. For the space occupied in reservoirs on the Lower Colorado River, on account of the probable mixture of coarser bed load and suspended silt and also to allow for compacting by superimposed silt, they suggested the use of a specific weight of silt *in situ* in the reservoir of 85 lb per cu ft (42*c*). As a general average of the silt-bearing streams of Southwestern United States, these figures are consistent with all available data and can be used without great error.

STREAM TRANSPORTATION OF SILT

It has been estimated that the surface of the United States is being removed at the rate of 1 in. in 760 yr (3). At some time or other, the material removed is transported by streams. The total of solids removed is estimated at 783 000 000 tons per annum, of which 513 000 000 tons is suspended matter and 270 000 000 tons is dissolved solids.

Flowing water has the power to transport large quantities of finely divided material as a suspended load and also to drag other materials along its bed.

The higher the velocity and the more turbulent the stream, the greater the proportion of suspended load it is capable of carrying. When velocities slacken this material settles and the bed-load movement is arrested. Material remains in suspension by reason of the vertical components of currents and eddies within the water prism.

The shape of the silt particle has an important bearing on the facility with which it remains in suspension or settles to the bottom. The finer the material the slower it settles; rounded particles will settle much faster than flat scale-like particles; and, disk-like particles settle in still water (flat side down) with an oscillating motion requiring a long time.

The velocity of the water, the degree of fineness of the material, and the predominating shape of the particles are three correlated factors that determine the variant between the suspended load and the bed load of a stream.

On mountain streams the bottom load consists of boulders, cobble-stones, and coarse sand. The transporting of stones along a stream bed grinds them, ultimately, into material sufficiently fine to permit their being carried to the ocean as suspended load or as bottom load of slow-moving rivers. By far the greater part of the sediment transportation by streams occurs at the time of a flood. "Cloudbursts" on mountain streams may cause the movement of millions of tons of débris, from fine sands to boulders as large as houses. The waters of great rivers, at their mouths, seldom acquire sufficient velocities to move anything but sand and the smaller gravels.

There is no line of demarcation between suspended load and bed load. In the same stream a given material may be carried in suspension in one reach and as bed load in another. A flood will put in suspension large volumes of material formerly deposited on the bed and banks of the stream and start rolling a new bottom load of much heavier material. The transportation of detritus from mountain top to ocean bed is thus accomplished in stages by a series of spasmodic expenditures of stream energy interspersed with periods of quiescence.

Suspended Silt.—Table 6 gives a summary of the results of a large number of measurements to determine the suspended silt content of the streams of the world. The streams in the United States are arranged alphabetically by the major drainage basins. Tributary streams within those basins are arranged in order from head to mouth.

The Bed Load.—Little information is available as to the quantity of material transported as bed load. Humphreys and Abbot estimated the bed load at the mouth of the Mississippi River to be 11% of the suspended load. Fortier and Blaney place that of the Colorado River, at Yuma, Ariz., at 20% of the total load (42*d*). Follett stated that the bed load of the Rio Grande at San Marcial, N. Mex., "may amount to 25% of the silt carried in suspension."

Comparing the silt content of the Rio Grande during the years between capacity surveys (1916–1925) of Elephant Butte Reservoir, it is found that:

Total suspended silt passing San Marcial. . = 217 000 000 tons
Total silt deposited in reservoir. . . . . = 178 000 acre-ft.

TABLE 6.—SUSPENDED SILT CARRIED BY STREAMS

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	SUSPENDED SILT		Reference number *
							Per thousand	Millions of tons (2000lb.) during period	
		(a) COLORADO RIVER SYSTEM							
1	Colorado.......	Kremmling, Colo.....	2 380	4/23/-12/30, 1905	157	1 030	0.18	0.256	(55)
2	Colorado.......	Palisade, Colo........	8 500	4/2/05-5/5/06	161	4 740	0.49	3.13	(55)
3	Colorado.......	Cisco, Utah..........	24 100	11/1/14-8/31/15	......	6 700	1.0	9.1	(42)
		Cisco, Utah..........	....	Oct-Sept. 1929-30	......	6 150	2.32	19.4	(31)
4	Colorado.......	Lees Ferry, Ariz......	....	Oct-Sept. 1929-30	......	13 200	6.1	110	(31)
				1925-6					
5	Colorado.......	Grand Canyon near Bright Angel Creek, Arizona..........	....	1926-7	93	14 400	11.5	225	(62), (58), (61)
			....	1927-8	106	17 300	16.9	396	(62), (58), (61)
			....		201	15 600	8.1	172	(62), (58), (61)
			....	1928-9	259	19 400	18.2	480	(62), (58), (61)
			....	1929-0	289	13 400	13.0	236	(62), (58), (61)
			....	1930-1	291	6 720	7.6	68.8	(62), (58), (61)
			....	Total mean...	1 239	86 800	13.4	1 580	
6	Colorado.......	Topock, Ariz.........	171 000	8/1/17-7/31/18	......	15 600	9.7	206	(42)
			....	Oct-Sept. 1925-6	98	14 300	7.2	140	(62), (58), (61)
			....	1926-7	104	17 000	14.9	345	(62), (58), (61)
			....	1927-8	103	15 400	10.0	209	(62), (58), (61)
			....	1928-9	98	18 900	17.1	437	(62), (58), (61)
			....	1929-0	102	13 200	13.6	245	(62), (58), (61)
			....	1930-1	94	6 770	7.0	65	(62), (58), (61)
			....	Total mean..	599	85 570	12.4	1 441	
7	Colorado.......	Yuma, Ariz..........	242 000	Jan-Dec., 1911	......	17 800	11.3	273	(50c)
			....	1912	......	18 400	6.5	163	(50c)
			....	1913	......	11 800	7.4	119	(50c)
			....	1914	......	20 700	9.3	261	(50)
			....	1915	......	14 600	11.6	230	(50)
			....	1916	......	22 900	14.2	442	(50)
			....	1917	......	20 600	5.9	165	(50)
			....	1918	......	13 100	6.3	112	(50)
			....	1919	......	10 700	10.0	145	(50)
			....	1920	......	21 400	8.6	251	(50)
			....	1921	......	19 400	8.6	228	(50)
			....	1922	......	17 000	8.2	190	(50)
			....	1923	......	17 800	10.4	252	(50)
			....	1924	......	11 400	8.3	129	(50)
			....	1925	......	12 400	9.6	162	(50)
			....	1926	......	12 200	7.6	126	(50)
			....	1927	......	17 100	10.6	245	(50)
			....	1928	......	12 800	6.6	116	(58)
			....	1929	......	17 500	12.4	292	(58)
			....	1930	......	10 600	12.9	186	(58)
			....	1931	......	4 800	8.1	53	(58)
			....	Total mean..	......	325 000	9.4	4 140	
8	Gunnison.......	Whitewater, Colo.....	7 870	4/2/-10/31, 1905	203	2 290	0.64	2.01	(55)
9	Green..........	Green River, Wyo....	7 450	5/1-11/1, 1905	145	935	0.10	0.132	(55)
10	Green..........	Green River, Utah....	40 600	8/1/14-8/31/15	......	4 700	1.3	8.3	(42)
11	San Juan.......	Bluff, Utah..........	24 000	11/1/14-8/31/15	......	3 000	4.1	16.6	(42)
		Bluff, Utah..........	....	Oct-Sept. 1929-30	......	1 740	19.3	45.6	(31)
12	Animas........	Durango, Colo........	810	3/19-12/18, 1905	188	850	0.18	0.210	(55)
13	Little Colorado..	Woodruff, Ariz.......	6 000	8/6/05-4/3/06	126	85.5	7.86	0.915	(55)
14	Gila...........	Florence Canal.......	....	11/28/99-3/7/1900	......	....	0.38	....	(42)
		Florence Canal.......	....	8/1-11/5, 1900	......	....	26.8	....	(42)
15	Gila...........	San Carlos, Ariz......	13 500	6/1-12/31, 1905	......	255	4.9	1.7	(42)
16	Gila...........	Yuma, Ariz..........	....	8/5-10/15, 1914	......	74.6	53.0	5.4	(42)
		Yuma, Ariz..........	....	11/11-12/3, 1916	......	54.5	5.4	0.4	(42)
17	San Francisco...	Alma, N. Mex........	1 800	4/14/05-4/22/06	161	165	1.76	0.396	(55)
18	Salt...........	Roosevelt, Ariz.......	5 760	4/9/05-4/23/06	155	4 280	2.01	11.7	(55)
19	Verde..........	McDowell, Ariz.......	6 000	4/5/05-3/10/06	143	615	1.23	1.03	(55)

* See Appendix.

TABLE 6—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	Suspended Silt		Reference number *
							Per thousand	Millions of tons (2000 lb.) during period	
				(*b*) Columbia River System					
20	No. Fork, Coeur d'Alene	Enaville, Idaho	....	5/13/21-6/30/22	414	1 440	0.074	0.146	(35)
21	So. Fork, Coeur d'Alene	Enaville, Idaho	....	5/13/21-6/30/22	414	466	0.97	0.615	(35)
22	Salmon	Malott, Wash.	150	5/23/05-1/13/06	139	25.5	0.067	0.0023	(55)
23	Malheur	Vale, Ore.	4 860	3/26-12/4, 1905	186	134	0.193	0.035	(55)
24	Payette	Horseshoe Bend, Idaho	2 240	5/15-9/13, 1906	75	1 030	0.031	0.043	(55)
25	Palouse	Hooper, Wash.	2 210	5/22-10/8, 1905	122	32.2	0.055	0.0024	(55)
				(*c*) Great Basin River Systems					
26	Truckee	Derby, Nev.	1 750	4/10/06-3/13/07	39	1 130	0.053	0.081	(55)
27	Owens	Round Valley, Calif.	400	5/13/06-3/30/07	290	232	0.026	0.0081	(55)
				(*d*) Mississippi River System					
28	Mississippi	Prescott, Wis.	....	4/26-7/30, 1881	14	12 800	0.161	2.79	(38)
29	Mississippi	Winona, Minn.	....	2/4-7/30, 1881	28	13 100	0.033	0.59	(38)
30	Mississippi	Winona, Minn.	....	May-Oct. 1932	.....	....	......	0.598	(69)
31	Mississippi	LaCrosse, Wis.	....	Apr.-Nov. 1932	.....	....	......	1.044	(69)
32	Mississippi	Clayton, Iowa	....	1/28-8/18, 1881	38	21 800	0.039	1.16	(38)
33	Mississippi	Clayton, Iowa	....	May-Oct. 1932	.....	....	......	1.673	(69)
34	Mississippi	Hannibal, Mo.	....	1/11-8/4, 1881	48	54 000	0.28	20.5	(38)
35	Mississippi	Grafton, Ill.	....	11/13/80-8/31/81	89	105 000	0.321	45.8	(38)
		Grafton, Ill.	....	3/22-6/6, 1929	12	41 000	0.256	14.2	(38)
		Grafton, Ill.	170 000	6/6/30-/2/28/31	36	20 900	0.098	2.79	(39)
36	Mississippi	St. Louis, Mo.	....	3/31-6/25, 1879	21	29 800	2.44	99.0	(38)
		St. Louis, Mo.	....	1/15-9/5, 1881	36	118 000	1.40	225	(38)
		St. Louis, Mo.	....	4/8-6/12, 1929	8	62 000	1.54	132	(38)
37	Mississippi	Columbus, Ky.	....	3/15-11/15, 1858	146	262 000	0.83	293	(38)
		Columbus, Ky.	....	3/4-7/2, 1879	79	376 000	1.43	177	(38)
38	Missssippi	Fulton, Tenn.	....	11/28/79-10/10/80	178	355 000	0.88	424	(38)
39	Mississippi	Hampton Ldg., Ark.	....	1/6-6/27, 1879	63	157 000	0.63	135	(38)
40	Mississippi	Helena, Ark.	....	12/13/78-6/18/79	37	212 000	0.65	188	(38)
		Helena, Ark.	938 000	9/2/30-2/28/31	61	55 000	0.35	26.4	(39)
41	Mississippi	Chicot, Ark.	....	4/2-6/25, 1929	25	240 000	0.44	142	(38)
		Chicot, Ark.	1 119 000	9/2/30-1/17/31	80	40 500	0.53	29.5	(39)
42	Mississippi	Lake Providence, La.	....	11/18/79-10/15/80	28	355 000	0.70	335	(38)
43	Mississippi	Kings Point, Miss.	....	1/17-5/30, 1879	49	195 000	0.57	152	(38)
44	Mississippi	Vicksburg, Miss.	....	3/13-6/6, 1929	16	235 000	0.55	177	(38)
		Vicksburg, Miss.	1 138 000	8/29/30-1/26/31	63	45 000	0.45	27.0	(39)
45	Mississippi	Tarberts Ldg., Miss.	....	3/19-6/21, 1929	25	263 000	0.39	140	(38)
46	Mississippi	Red River Ldg., Miss.	....	3/3-6/22, 1929	25	249 000	0.37	126	(38)
		Red River Ldg., Miss.	1 230 000	9/23/30-2/26/31	65	60 500	0.57	46.6	(39)
47	Mississippi	Carrollton, La.	....	2/17/51-2/15/52	52	452 000	0.63	380	(38)
		Carrollton, La.	....	2/16/52-2/20/53	32	530 000	0.81	582	(38)
		Carrollton, La.	....	12/19/79-10/8/80	29	304 000	0.71	293	(38)
		Carrollton, La.	....	3/12-6/25, 1929	39	227 000	0.80	246	(38)
		Carrolton, La.	1 238 000	9/16/30-2/27/31	65	63 000	0.26	22.0	(39)
48	Chippewa	Durand, Wis.	....	May-Oct., 1932	.....	....	......	0.050	(69)
49	LaCrosse	West Salem, Wis.	....	May-Oct., 1932	.....	....	......	0.071	(69)
50	Root	Huston, Minn.	....	May-Oct., 1932	.....	....	......	0.687	(69)
51	Wisconsin	Musooda, Wis.	....	May-Oct., 1932	.....	....	......	0.135	(69)
52	Illinois River	Lockport (Sta. 292)	217	Sept-Aug., 1921-22	.....	6 400	0.032	0.264	(59)
53	Illinois River	Joliet (Sta. 286)	1 120	Sept-Aug. 1921-22	.....	6 560	0.048	0.430	(59)
54	Illinois River	Morris (Sta. 263)	7 600	Sept-Aug. 1921-22	.....	10 800	0.082	1 20	(59)
55	Illinois River	Chillicothe (Sta. 179)	13 400	Sept-Aug. 1921-22	.....	13 900	0.039	0.750	(59)
56	Illinois River	Peoria (Sta. 166)	13 600	Sept-Aug. 1921-22	.....	14 000	0.030	0.565	(59)
57	Missouri	Ft. Benton, Mont.	24 600	July-June, 1929-30	†433	4 300	0.077	0.45	(43)
		Ft. Benton, Mont.	....	1930-31	.....	3 120	0.177	0.749	(43)
58	Missouri	Williston, N. Dak.	164 000	1929-30	†568	14 600	1.42	28.1	(43)
		Williston, N. Dak.	....	1930-31	.....	11 500	2.28	35.6	(43)
59	Missouri	Mobridge, S. Dak.	209 000	1929-30	†292	16 300	2.10	46.6	(43)
		Mobridge, S. Dak.	....	1930-31	.....	10 900	2.38	35.3	(43)
60	Missouri	Pierre, S. Dak.	244 000	1929-30	†220	17 400	3.37	79.7	(43)
		Pierre, S. Dak.	....	1930-31	.....	11 400	2.38	36.9	(43)
61	Missouri	Sioux City, Iowa	315 000	1929-30	†163	19 800	3.83	103	(43)
		Sioux City, Iowa	....	1930-31	.....	13 000	2.14	38.1	(43)
62	Missouri	Omaha, Nebr.	323 000	1929-30	†630	20 500	4.05	113	(43)
		Omaha Nebr.	....	1930-31	.....	13 100	2.48	44.1	(43)
		Omaha, Nebr.	....	1931-32	.....	16 800	4.82	110	(43)

* See Appendix.

† Number of observations for entire period.

TABLE 6—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	Suspended Silt		Reference number *
							Per thousand	Millions of tons (2000 lb.) during period	
				(*d*) Mississippi River System (*Continued*)					
63	Missouri	Plattsmouth, Nebr.	414 000	1929–30	†490	26 400	3.93	141	(43)
		Plattsmouth, Nebr.	....	1930–31	.....	18 300	2.07	51.4	(43)
		Plattsmouth, Nebr.	....	1931–32	.....	21 000	4.30	123	(43)
64	Missouri	Leavenworth, Kans.	425 000	1929–30	†648	28 900	4.20	170	(43)
		Leavenworth, Kans.	....	1930–31	.....	18 900	2.43	62.3	(43)
65	Missouri	Kansas City, Mo.	489 000	1929–30	†923	34 000	4.13	191	(43)
		Kansas City, Mo.	....	1930–31	.....	22 000	2.37	70.8	(43)
		Kansas City, Mo.	....	1931–32	.....	30 800	4.98	208	(43)
66	Missouri	Boonville, Mo.	506 000	1929–30	†499	39 400	3.57	191	(43)
		Boonville, Mo.	....	1930–31	.....	24 800	2.01	67.6	(43)
67	Missouri	Howard Bend, Mo.	529 000	1929–30	†988	43 800	3.24	193	(43)
		Howard Bend, Mo.	....	1930–31	.....	27 800	2.05	77.3	(43)
		Howard Bend, Mo.	....	1931–32	.....	44 900	3.00	183	(43)
68	Missouri	St. Charles, Mo.	....	2/1-10/31, 1879	273	45 500	4.10	253	(38)
		St. Charles, Mo.	528 000	8/28/30-2/28/31	44	13 500	2.56	47.0	(39)
69	Marias	Loma, Mont.	9 160	July-June, 1929–30	†324	589	0.467	0.374	(43)
		Loma, Mont.	....	1930–31	.....	355	0.578	0.279	(43)
70	Musselshell	Mosby, Mont.	9 570	1929–30	†157	75.2	1.13	0.115	(43)
		Mosby, Mont.	....	1930–31	.....	19.5	6.26	0.166	(43)
71	Milk	Nashau, Mont.	23 800	1929–30	†320	405	1.91	1.05	(43)
		Nashau, Mont.	....	1930–31	.....	62.9	0.171	0.0146	(43)
72	Yellowstone	Billings, Mont.	11 180	5/20–11/24, 1905	88	3 560	0.43	2.07	(55)
73	Yellowstone	Glendive, Mont.	66 100	3/28/05–4/21/06	172	7 570	1.14	11.8	(55)
		Glendive, Mont.	66 900	July-June, 1929–30	†380	8 480	2.16	24.9	(43)
		Glendive, Mont.	....	July-June, 1930–31	.....	7 240	3.64	35.8	(43)
74	Big Horn	Fort Custer, Mont.	20 700	6/10/05–6/9/06	72	4 360	1.23	7.3	(55)
75	Shoshone	Cody, Wyo.	1 480	4/1/05-3/31/06	287	1 020	0.121	0.168	(55)
76	Little Missouri	Medora, N. Dak.	6 320	July-June, 1929–30	†387	208	5.80	1.64	(43)
		Medora, N. Dak.	....	1930–31	.....	80.6	8.98	0.984	(43)
77	Cannonball	Timmer, N. Dak.	3 660	1929–30	†121	77.1	7.29	0.764	(43)
		Timmer, N. Dak.	....	1930–31	.....	16.8	2.07	0.0473	(43)
78	Grand	Wakpala, S. Dak.	5 660	1929–30	†150	105	5.76	0.822	(43)
		Wakpala, S. Dak.	....	1930–31	.....	44.1	3.62	0.217	(43)
79	Moreau	Promise, S. Dak.	5 200	1929–30	†121	87.0	6.17	0.730	(43)
		Promise, S. Dak.	....	1930–31	.....	42.8	6.24	0.363	(43)
80	Cheyenne	Carlin, S. Dak.	25 500	1929–30	†546	720	8.23	8.05	(43)
		Carlin, S. Dak.	....	1930–31	.....	410	10.86	6.05	(43)
81	Belle Fourche	Belle Fourche, S. Dak.	3 250	4/15–11/25, 1905	192	121	3.60	0.590	(55)
		Belle Fourche, S. Dak.	....	4/1–6/23, 1906	51	101	2.71	0.372	(55)
		Belle Fourche, S. Dak.	4 270	7/27–11/13, 1906	89	53.7	0.68	0.050	(55)
82	Redwater	Belle Fourche, S. Dak.	1 020	4/9–11/25, 1905	188	159	0.26	0.057	(55)
		Belle Fourche, S. Dak.	....	4/1–6/23. 1906	53	48.5	0.42	0.028	(55)
83	Bad	Fort Pierre, S. Dak.	3 110	July-June, 1929–30	†142	87.6	38.97	4.64	(43)
		Fort Pierre, S. Dak.	....	1930–31	.....	106	38.32	5.52	(43)
84	White	Oacoma, S. Dak.	10 200	1929–30	†227	432	22.99	13.5	(43)
		Oacoma, S. Dak.	....	1930–31	.....	278	18.21	6.88	(43)
85	Niobrara	Verdel, Nebr.	12 300	1929–30	†637	1 110	0.628	0.947	(43)
		Verdel, Nebr.	....	1930–31	.....	970	0.459	0.605	(43)
86	James	Scotland, S. Dak.	21 500	1929–30	†261	112	0.168	0.0255	(43)
		Scotland, S. Dak.	....	1930–31	.....	46.5	0.118	0.00747	(43)
87	Big Sioux	Akron, Iowa	9 420	1929–30	†374	358	0.742	0.361	(43)
		Akron, Iowa	....	1930–31	.....	92.2	0.173	0.0217	(43)
88	Little Sioux	Correctionville, Iowa	4 260	1929–30	†376	202	1.59	0.437	(43)
		Correctionville, Iowa	....	1930–31	.....	40.2	0.198	0.0108	(43)
89	Platte	Duncan, Nebr.	66 100	1929–30	†174	2 150	0.657	1.92	(43)
		Duncan, Nebr.	....	1930–31	.....	1 870	0.504	1.28	(43)
90	Platte	Plattsmouth, Nebr.	90 200	1929–30	†886	5 010	2.47	16.8	(43)
		Plattsmouth, Nebr.	....	1930–31	.....	4 440	1.55	9.34	(43)
		Plattsmouth, Nebr.	....	1931–32	.....	3 780	1.48	7.61	(43)
91	North Platte	Laramie, Wyo.	16 200	5/21–12/28, 1906	125	1 410	0.99	1.90	(55)
92	Loup	Genoa, Nebr.	13 600	July-June, 1929–30	†208	1 970	2.89	7.74	(43)
		Genoa, Nebr.	....	1930–31	.....	1 930	1.89	4.96	(43)
93	Elkhorn	Waterloo, Nebr.	6 560	1929–30	†191	744	3.19	3.32	(43)
		Waterloo, Nebr.	....	1930–31	.....	586	0.662	0.527	(43)
94	Kansas	Bonner Springs, Kans.	61 300	1929–30	†617	3 830	3.94	20.5	(43)
		Bonner Springs, Kans.	....	1930–31	.....	2 780	2.19	8.28	(43)
95	Kansas	Holliday, Kans.	62 000	Jan-Dec., 1907	365	3 820	0.592	4.48	(8a)
		Holliday, Kans.	....	1908	365	9 500	0.895	23.3	(8a)

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* See Appendix.

† Number of observations for entire period.

TABLE 6—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	SUSPENDED SILT		Reference number *
							Per thousand	Millions of tons (2000 lb.) during period	
	(*d*) MISSISSIPPI RIVER SYSTEM (*Continued*)								
96	Smoky Hill	Mentor, Kans.	8 420	July-June, 1929–30	†209	270	3.50	1.29	(43)
		Mentor, Kans.	....	1930–31	.....	224	1.79	0.544	(43)
97	Smoky Hill	Solomon, Kans.	19 200	1929–30	†368	794	2.52	2.72	(43)
		Solomon, Kans.	....	1930–31	.....	609	1.49	1.23	(43)
98	Saline	Tescott, Kans.	2 880	July-June, 1929–30	†244	79.8	1.76	0.191	(43)
		Tescott, Kans.	....	1930–31	.....	142	1.97	0.381	(43)
99	Solomon	Niles, Kans.	6 900	1929–30	†259	255	3.12	1.08	(43)
		Niles, Kans.	....	1930–31	.....	198	2.99	0.805	(43)
100	Republican	Wakefield, Kans.	25 300	July-June, 1929–30	†543	722	5.61	5.50	(43)
		Wakefield, Kans.	....	1930–31	.....	664	3.97	3.58	(43)
101	Big Blue	Randolph, Kans.	9 360	1929–30	†407	1 000	4.25	5.77	(43)
		Randolph, Kans	....	1930–31	.....	645	2.30	2.02	(43)
102	Grand	Gallatin, Mo.	2 250	1929–30	†53	611	3.93	3.26	(43)
		Gallatin, Mo.	....	1930–31	.....	166	2.34	0.527	(43)
103	Grand	Sumner, Mo.	6 880	1929–30	†484	1 960	3.05	8.12	(43)
		Sumner, Mo	....	1930–31	.....	959	2.63	3.43	(43)
104	Thompson	Trenton, Mo.	1 670	1929–30	†54	438	8.27	4.92	(43)
		Trenton, Mo.	....	1930–31	.....	134	4.87	0.887	(43)
105	Osage	Bagnell, Mo.	14 000	1929–30	†395	3 070	0.338	1.41	(43)
		Bagnell, Mo.	....	1930–31	.....	877	0.080	0.0955	(43)
106	Gasconade	Rich Fountain, Mo.	3 180	1929–30	†166	1 660	0.081	0.183	(43)
		Rich Fountain, Mo.	....	1930–31	.....	1 150	0.070	0.110	(43)
107	Ohio	Paducah, Ky.	....	12/16/78–12/30/79	76	174 000	0.32	76.0	(38)
108	Ohio	Mound City, Ill.	202 000	9/11/30– 2/27/31	66	23 000	0.12	3.75	(39)
109	White	Clarendon, Ark.	....	1/19–6/26, 1879	27	6 200	0.036	0.30	(38)
110	White	DeValls Bluff, Ark.	23 500	2/6–5/30,1931	36	7 100	0.113	1.09	(39)
111	Arkansas	Pine Bluff, Ark.	....	2/20–7/8, 1879	134	4 110	0.51	2.88	(38)
112	Arkansas	Tulsa, Okla.	74 700	10/18/30–9/8/31	73	2 400	4.15	13.55	(39)
113	Arkansas	Ozark, Ark.	152 000	10/22/30–9/2/31	94	13 200	1.81	32.4	(39)
114	Cimarron	Guthrie, Okla.	16 000	10/15/30–9/8/31	68	532	8.00	5.80	(39)
115	Verdigris	Okay, Okla.	8 140	10/28/30–9/7/31	74	1 840	1.12	2.80	(39)
116	Grand	Wagoner, Okla.	12 400	10/24/30–9/7/31	73	3 150	0.32	1.38	(39)
117	So. Canadian	Calvin, Okla.	29 700	10/30/30–9/2/31	71	450	5.70	3.46	(39)
118	Yazoo	Greenwood, Miss.	7 700	9/16/30–9/26/31	122	3 960	0.43	2.35	(39)
119	Ouachita	Monroe, La.	17 760	9/17/30–9/29/31	179	910	0.13	0.16	(39)
120	Red	Denison, Tex.	36 100	9/9/30–9/30/31	160	2 950	3.73	15.0	(39)
121	Red	Alexandria, La.	....	2/24–7/1, 1879	22	6 550	0.38	3.40	(38)
		Alexandria, La.	63 300	9/23/30–9/19/31	156	16 600	1.56	35.2	(39)
122	Salt Fk. of Red R.	Mangum, Okla.	1 220	4/11/05–6/28/06	253	63.5	2.36	0.204	(55)
123	North Fk. of Red R.	Granite, Okla.	2 210	4/12/05–3/16/07	482	425	4.21	2.30	(55)
124	North Fk. of Red R.	Headrick, Okla.	5 470	5/20/05–3/19/07	447	700	3.45	3.28	(55)
125	Elm Fk. of Red R.	Mangun, Okla.	750	5/13/05–3/22/07	509	350	7.02	3.34	(55)
126	Washita	Durwood, Okla.	....	9/8/30–9/23/31	109	785	4.40	4.70	(39)
127	Little River	Horatio, Ark	....	1/21–9/28, 1931	85	1 670	0.053	0.12	(39)
128	Sulphur	Darden, Tex.	....	9/10/30–3/24/31	125	1 010	0.65	0.885	(39)
129	Atchafalaya	Simmesport, La.	....	3/19–6/22, 1929	23	63 000	0.38	32.6	(38)
		Simmesport, La.	....	9/23/30–2/27/31	60	11 700	1.18	18.7	(38)
	(*e*) GULF OF MEXICO DRAINAGE SYSTEM								
130	Rio Grande	San Marcial, N. Mex.	30 000	Jan-Dec., 1897	.....	2 220	14.5	43.7	(47), (65)
			....	1898	.....	961	13.1	17.0	(48)
			....	1899	.....	239	18.1	5.9	(48)
			....	1900	.....	468	17.1	10.9	(48)
			....	1901	.....	656	23.8	21.3	(48)
			....	1902	.....	201	25.8	7.0	(48)
			....	1903	.....	1 270	8.2	14.2	(48)
			....	1904	.....	710	20.0	19.3	(48)
			....	1905	35	2 420	6.6	21.8	(48)
			....	1906	112	1 560	7.5	16.0	(48)
			....	1907	118	2 160	9.4	27.5	(48)
			....	1908	110	774	16.9	17.8	(48)
			....	1909	117	1 280	12.9	22.2	(48)
			....	1910	90	852	6.4	7.5	(48)
			....	1911	114	1 800	35.0	85.2	(48)
			....	1912	109	1 500	12.4	25.3	(48)

* See Appendix.

† Number of observations for entire period.

TABLE 6—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	Suspended Silt		Reference number *
							Per thousand	Millions of tons (2000 lb.) during period	
				(*e*) Gulf of Mexico Drainage System (*Continued*)					
130	Rio Grande	San Marcial, N. Mex.	30 000	Jan-Dec., 1913	.....	525	8.2	5.8	(58)
			....	1914	.....	1 180	25.3	40.5	(58)
			....	1915	.....	1 350	13.6	25.0	(58)
			....	1916	.....	1 650	13.2	30.0	(58)
			....	1917	.....	1 050	6.6	9.4	(58)
			....	1918	.....	410	6.6	3.7	(58)
			....	1919	.....	1 580	22.8	45.8	(58)
			....	1920	.....	2 220	10.4	31.5	(58)
			....	1921	.....	1 630	19.2	42.9	(58)
			....	1922	.....	964	10.2	13.3	(58)
			....	1923	.....	1 220	11.7	19.4	(58)
			....	1924	.....	1 440	7.5	14.6	(58)
			....	1925	.....	419	12.0	6.8	(58)
			....	1926	.....	1 050	7.2	10.2	(58)
			....	1927	.....	1 350	19.6	35.9	(58)
			....	1928	.....	590	5.7	4.6	(58)
			....	1929	.....	1 460	29.5	58.6	(58)
			....	1930	.....	731	5.4	5.4	(58)
			....	1931	.....	490	18.8	12.5	(58)
				Total mean, Item No. 130	.....	40 400	14.2	780.0	
131	Rio Grande	El Paso, Tex.	....	6/1/89-8/31/90	297	1 075	4.0	5.86	(48)
		El Paso, Tex.	38 600	1/8/05-4/30/07	248	2 750	8.1	30.3	(55)
		El Paso, Tex.	‡32 800	Jan-Dec., 1924	26	816	1.67	1.91	(14)
132	Rio Grande	Fort Quitman, Tex.	‡34 500	Jan-Dec., 1924	25	505	3.57	2 46	(14)
133	Rio Grande	Upper Presidio, Tex.	‡37 500	4/23-12/29, 1924	19	311	2.30	0.965	(14)
		Upper Presidio, Tex.	....	Jan-Dec., 1925	26	407	3.58	2.04	(14)
		Upper Presidio, Tex.	....	1926	26	570	4.06	3.26	(14)
134	Rio Grande	Lower Presidio, Tex.	‡60 100	4/24-12/30, 1924	19	780	3.03	2.18	(14)
		Lower Presidio, Tex.	....	Jan-Dec., 1925	26	2 720	3.50	13.1	(14)
		Lower Presidio, Tex.	....	1926	26	2 590	3.11	11.0	(14)
135	Rio Frande	Boquillas, Tex.	‡69 400	5/7-10/17, 1929	46	675	9.75	8.95	(14)
136	Rio Grande	Laredo, Tex.	‡133 000	6/13-12/16, 1924	17	3 360	3.85	17.4	(14)
		Laredo, Tex.	....	Jan-Dec., 1925	29	7 250	3.40	24.6	(14)
		Laredo, Tex.	....	1926	24	5 690	1.93	14.8	(14)
137	Rio Grande	Roma, Tex.	‡160 000	6/16-12/25, 1924	15	4 060	4.40	24.1	(14)
		Roma, Tex.	....	Jan-Dec., 1925	25	8 040	7.40	85.2	(14)
		Roma, Tex.	....	1/11-10/25, 1926	17	5 550	7.62	45.6	(14)
		Roma, Tex.	....	3/6-12/31, 1929	296	2 320	2.81	8.91	(14)
			....	Jan-Dec., 1930	257	3 400	3.76	17.3	(14)
		Roma, Tex.		1931	257	3 080	1.87	7.82	(14)
138	Rio Grande	Matamoras, Mex.	‡180 000	4/17-12/22, 1924	20	2 620	2.99	10.6	(14)
		Matamoras, Mex.	....	Jan-Dec., 1925	28	5 630	3.35	25.7	(14)
		Matamoras, Mex.	....	1926	27	5 740	3.68	28.8	(14)
139	Pecos	Santa Rosa, N. Mex.	2 900	7/7/05-12/27/07	380	191	5.5	1.43	(55)
140	Pecos	Dayton, N. Mex.	20 000	7/20/05-4/20/07	417	790	5.8	6.22	(55)
141	Pecos	Carlsbad, N. Mex.	22 000	5/22/05—4/30/07	366	1 110	0.53	0.80	(55)
142	Gallinas	Las Vegas, N. Mex.	90	5/19/05-4/31/06	262	28.3	0.05	0.0019	(55)
143	Hondo	Roswell, N. Mex.	1 040	4/26-8/4, 1905	96	78 0	9.80	1.04	(55)
144	Brazos	Mineral Wells, Tex.	23 100	Oct-Sept., 1924-27	.....	2 940	10.1	40.5	(53)
145	Brazos	Waco, Tex.	28 500	Oct-Sept., 1924-27	.....	4 890	8.6	57.2	(53)
146	Brazos	College Sta., Tex.	37 400	8/1-12/31, 1899	7	1 160	8.66	....	(34), (33)
		College Sta., Tex.	....	Jan-Dec., 1900	31	8 810	13.15	....	(34), (33)
		College Sta., Tex.	....	1901	60	977	12.62	....	(34)
147	Brazos	Rosenburg, Tex.	44 000	Oct-Sept., 1924-27	.....	15 600	4.3	106	(53)
148	Double Mt. Fk. of Brazos R.	Aspermont, Tex.	7 980	Oct-Sept., 1924-27	.....	510	21.3	13.9	(53)
149	Witchita	Witchita Falls, Tex.	3 050	2/10-12/31,1900	20	842	12.07	....	(34), (33)
		Witchita Falls, Tex.	....	Jan-Dec., 1901	52	298	15.57	....	(34)
				(*f*) Sacramento River System					
150	Sacramento	Red Bluff, Calif.	9 300	7/3/05-3/23/07	446	20 800	0.107	3.02	(55)
151	Pit	Bieber, Calif.	2 950	7/7/05-3/2/07	285	924	0.106	0.133	(55)
152	Feather	Oroville, Calif.	3 640	6/25/05-2/14/07	379	9 600	0.098	1.26	(55)
153	Puta Cr.	Winters, Calif.	805	1/2/06-3/1/07	371	873	0.43	0.507	(55)

* See Appendix. ‡ With all closed basins eliminated.

TABLE 6—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	Suspended Silt		Reference number *
							Per thousand	Millions of tons (2000 lb.) during period	
(g) Nile River System									
154	Nile	Sarras, Egypt	....	6/22–8/20, 1905	27	9 650	0.614	8.04	(23)
		Sarras, Egypt	....	7/7–8/20, 1906	19	25 300	1.97	67.6	(23)
155	Nile	Aswan, Egypt	....	Jan.	.....	3 500	0.25	1.20	(9)
		Aswan, Egypt	....	Feb.	.....	2 600	0.17	0.60	(9)
		Aswan, Egypt	....	Mar.	.....	1 960	0.10	0.27	(9)
		Aswan, Egypt	....	April	.....	1 500	0.08	0.16	(9)
		Aswan, Egypt	....	May	.....	1 280	0.07	0.12	(9)
		Aswan, Egypt	....	June	.....	1 600	0.10	0.22	(9)
		Aswan, Egypt	....	July	.....	4 850	0.15	1.00	(9)
		Aswan, Egypt	....	Aug.	.....	17 000	1.43	33.0	(9)
		Aswan, Egypt	....	Sept.	.....	19 200	1.32	34.0	(9)
		Aswan, Egypt	....	Oct.	.....	13 300	0.83	15.0	(9)
		Aswan, Egyp	....	Nov.	.....	7 000	0.57	5.4	(9)
		Aswan, Egypt	....	Dec.	.....	4 650	0.36	2.29	(9)
			....						
			....	Mean year...	.....	78 440	0.87	93.26	
		Aswan, Egypt	....	Jan-Dec., 1913	51	35 500	0.53	25.2	(36); (51)
		Aswan, Egypt	....	1914	52	63 900	1.20	104.0	(36); (51)
		Aswan, Egypt	....	1915	52	54 500	0.66	48.5	(36); (51)
			....	1916	52	85 600	0.85	98.7	(36); (51)
			....	1917	52	91 500	0.68	83.6	(36); (51)
			....	1918	52	67 600	0.52	48.2	(36); (51)
			....	1919	50	60 300	0.99	81.2	(36); (51)
			....	1920	52	67 400	0.85	78.0	(36); (51)
			....	1921	50	59 700	0.92	74.5	(36); (51)
			....	1922	51	72 000	1.02	100.0	(36); (51)
			....	1923	52	72 700	0.86	84.8	(36); (51)
			....	1924	52	75 500	0.87	89.3	(36); (51)
			....	1925	52	57 200	0.69	53.3	(36); (51)
			....	1926	52	69 000	0.90	84.2	(36); (51)
			....	14 years, total mean	722	932 000	0.83	1 050	
(h) Other Streams in Africa									
160	Orange	Orange R. Sta., C. P.	....	Jan-April, 1920	.....	5 990	6.9	534	(56*a*)
			....	3/17–5/31, 1921	.....	2 100	4.4	13.2	(56*a*)
161	Lower Orange	Kakamas	....	11/18/19–5/5/20	26	12 000	8.70	142	(56*b*)
162	Vaal	Kimberley	....	1/1/18–12/10/19	21	14 100	1.88	36	(56*c*)
(i) Tigris River, Asia									
163	Tigris	Amara, Irak	....	Jan., 1918	.....	0.60	4.1	3.00	(49)
			....	Feb.	.....	0.51	0.68	0.42	(49)
			....	March	.....	1.08	3.77	4.92	(49)
			....	April	.....	1.11	2.52	3.51	(49)
			....	May	.....	1.18	2.03	2.89	(49)
			....	June	.....	0.86	0.60	0.63	(49)
			....	July	.....	0.59	0.35	0.26	(49)
			....	Aug.	.....	0.43	0.21	0.11	(49)
			....	Sept.	.....	0.30	0.15	0.05	(49)
			....	Oct.	.....	0.28	0.12	0.04	(49)
			....	Nov.	.....	0.36	0.18	0.08	(49)
			....	Dec.	.....	0.62	0.77	0.59	(49)
			....	The year.....	.....	7.92	1.54	16.50	
(j) Streams in India									
164	Kistna	Bezwada, India	97 000	6/22–11/30, 1898	.....	51 800	2.78	196	(21)
			....	6/22–11/9, 1899	.....	19 300	5.26	138	(21)
			....	6/22–11/19, 1900	.....	56 500	3.41	262	(21)
			....	6/1–7/31, 1901	.....	15 000	5.14	105	(21)
			....	6/8–11/15, 1909	.....	42 100	3.08	176	(21)
			....	Total mean.....	.....	184 700	3.50	877	
165	Indus	Sukkur, India	....	Jan-Dec., 1902	.....	84 200	2.70	319	(20)
			....	1903	.....	116 000	3.14	495	(20)

* See Appendix.

TABLE 6—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	Suspended Silt		Reference number *
							Per thousand	Millions of tons (2000 lb.) during period	
(*j*) Streams in India (*Continued*)									
165	Indus..........	Sukkur, India........	....	1904	.....	102 000	2.69	373	(20)
			....	1905	.....	121 000	2.86	472	(20)
			....	1906	.....	113 000	3.68	566	(20)
			....	1907	.....	99 000	2.60	349	(20)
			....	1908	.....	136 000	3.04	562	(20)
			....	1909	.....	132 000	3.46	623	(20)
			....	1910	.....	129 000	4.02	708	(20)
			....	1911	.....	142 000	3.60	697	(20)
			....	1912	.....	127 000	3.56	615	(20)
			....	1913	.....	105 000	2.75	394	(20)
			....	1914	.....	159 000	2.93	635	(20)
			....	1915	.....	129 000	2.83	495	(20)
			....	1916	.....	118 000	2.80	448	(20)
			....	1917	.....	126 000	3.57	613	(20)
			....	1918	.....	116 000	2.86	450	(20)
			....	1919	.....	110 000	2.34	350	(20)
			....	1920	.....	106 000	2.77	400	(20)
			....	1921	.....	98 000	2.76	367	(17)
			....	1922	.....	119 000	2.80	454	(17)
			....	1923	.....	118 000	2.98	478	(17)
			....	1924	.....	127 000	2.66	459	(17)
			....	1925	.....	107 000	2.68	390	(17)
			24 years, total	mean Item No. 165	.....	2 840 000	3.03	11 712	
166	Indus..........	Kotri, India.........	....	Jan-Dec., 1902	.....	73 500	3.02	304	(20)
			....	1903	.....	98 000	3 56	474	(20)
			....	1904	.....	84 500	3.22	371	(20)
			....	1905	.....	106 000	3.12	450	(20)
			....	1906	.....	105 000	3.48	516	(20)
			....	1907	.....	206 000	1.27	356	(20)
			....	1908	.....	115 000	2.74	428	(20)
			....	1909	.....	106 000	3.08	444	(20)
			....	1910	.....	112 000	2.88	438	(20)
			....	1911	.....	122 000	3.38	563	(20)
			....	1912	.....	86 500	2.76	325	(20)
			....	1913	.....	84 700	2.76	318	(20)
			....	1914	.....	129 000	3.33	585	(20)
			....	1915	.....	108 000	3.48	512	(20)
			....	1916	.....	77 600	3.09	327	(20)
			....	1917	.....	115 000	3.66	573	(20)
			....	1918	.....	94 300	3.66	469	(20)
			....	1919	.....	111 000	3.13	473	(20)
			....	1920	.....	87 800	3.15	376	(20)
			....	1921	.....	93 500	3.00	382	(17)
			....	1922	.....	115 000	3.52	551	(17)
			....	1923	.....	109 000	3.19	473	(17)
			....	1924	.....	111 000	2.91	440	(17)
			....	1925	.....	89 500	3.16	384	(17)
			....	24 years, total mean	.....	2 540 000	3.05	10 500	
167	Sutlej..........	Head of Sirhind Canal, India............	....	Aug-Dec., 1893	47	....	1.15	....	(28)
			....	Jan-Dec., 1894	128	....	2.86	....	(28)
			....	Jan-Oct., 1895	85	....	2.00	....	(28)
			....	Apr-Oct., 1896	49	....	2.82	....	(28)
			....	June Aug., 1897	22	....	5.45	....	(28)
(*k*) Streams in China									
168	Yangtse........	Hankow.............	....	3/11-11/8, 1923	14	530 000	0.53	380	(68*a*)
			....	1/28-10/30, 1924	17	520 000	0.27	190	(68*b*)
169	Yangtse........	Wuhu...............	....	Jan.	.....	29 500	0.11	4.42	(67)
			....	Feb.	.....	26 300	0.08	2.87	
			....	March	.....	34 100	0.08	3.72	
			....	April	.....	47 800	0.16	10.4	
			....	May	.....	65 000	0.18	15.9	
			....	June	.....	78 000	0.34	36.0	

* See Appendix.

TABLE 6—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	Suspended Silt		Reference number *
							Per thousand	Millions of tons (2000 lb.) during period	
(*k*) Streams in China (*Continued*)									
169	Yangtse..	Wuhu	....	July	.....	101 000	0.74	101	
			....	August	.....	106 000	0.38	55.0	
			....	Sept.	.....	94 000	0.47	60.0	
			....	Oct.	.....	84 000	0.46	52.7	
			....	Nov.	.....	63 600	0.28	24.3	
			....	Dec.	.....	43 500	0.13	7.65	
				Mean year, Item No. 169	.....	773 000	0.36	374	
170	Huang Ho (Yellow R.)......	Chiang-Kou, China...	....	5/1–9/30, 1919	.....	23 300	39.5	1 250	(12)
171	Si Kiang (West).	Wuchow...........	....	7/6–10/13, 1915	9	....	0.44	....	(1); (20)
(*l*) Streams in Australia									
172	Murrumbidgee..	Burrinjuck..........	5 000	7/1–6/30, 1910–11	.....	810	0.15	0.162	(58)
			....	1911–12	.....	440	0.24	0 142	(58)
			....	1912–13	.....	840	0.40	0.450	(58)
			....	1913–14	.....	725	0.94	0.930	(58)
			....	1914–15	.....	386	0.37	0.195	(58)
			....	1915–16	.....	1 020	0.40	0.560	(58)
			....	Total mean....	.....	4 220	0.42	2.44	

* See Appendix.

Follett used a weight of 53 lb per cu ft for silt in the reservoir. If this specific weight is correct, 187 000 acre-ft of suspended silt passed San Marcial, which is more than was deposited in the reservoir. It is believed that only a negligible quantity of silt passed through the reservoir. On the basis of 65 lb per cu ft the volume of silt passing San Marcial amounted to 153 000 acre-ft of suspended silt, leaving 25 000 acre-ft for that brought in as bed load. This is 16% of the suspended load. It is thus seen that, in this case, the estimated volume of the bed load hinges on what specific weight is adopted for the reservoir deposits.

The writer investigated the silt carried and deposited by Coeur d'Alene River and from the resulting data made a rough estimate of the bed load. The investigation involved 9 miles of the river above Rose Lake, Idaho, in which twenty-six cross-sections were established and permanently marked. Surveys for silt content were made in May and November, 1921, and in March and July, 1922. Daily water samples were taken above and below this reach. Of the suspended load entering the reach, 75% was fine silt from ore-reduction works in the Coeur d'Alene Mining District and 25% was natural débris. The following data were obtained:

Period (May 13, 1921, to June 30, 1922), in days....	414
Suspended silt entering reach, in tons..............	444 600
Suspended silt leaving reach, in tons..............	363 100
Suspended silt deposited, in tons..................	81 500
Total deposits determined from cross-sections, in cubic yards	190 000

It is believed that 65 lb per cu ft will fairly represent the average specific weight of these deposits. Using this value, the bed load, in cubic yards, is found to be:

Total deposits, from soundings	190 000
Suspended load	93 000
Bed load	97 000

From this, it appears that the bed load and suspended load were each equal to about one-half the total. Although this work was done with care considerable uncertainty still remains. The volume assigned to bed load hinges on the specific weight assigned to the deposits. If instead of 65 lb per cu ft, a value of 80 lb is used, the bed load becomes 115 000 cu yd, or 60% of the deposits measured in place. If 50 lb is used, the bed load becomes 36% of the measured deposits.

Again, if the sampling gave results 10% too high at the upper end and 10% too low at the lower end, the entire volume deposited would have to be accounted for by bed load. If these errors were reversed the entire deposits would have come from the suspended load.

It would be better to express bed load as a percentage of the total load when possible. The foregoing data on bed load may be summarized in percentage of total débris transported, as follows:

	Suspended load	Bed load
Mississippi River, at mouth	90	10
Rio Grande, at San Marcial, N. Mex.	86	14
Colorado River, at Yuma, Ariz. (Fortier and Blaney)	80	20
Coeur d'Alene River, at Rose Lake, Idaho	49	51

The foregoing data are to be considered only as the roughest approximations; some are pure estimates, but they constitute the only data known to the writer. The difficulties of differentiating between bed load and suspended load are well illustrated in the case of the Coeur d'Alene River. Strictly, there is no line of demarcation—one merges indistinguishably into the other. As far as known no successful attempt has been made to measure the material transported in a river as bed load.

Laws of Silt Transportation.—Ordinary river flow is turbulent. The moving water prism contains many eddies and cross-currents. These serve to maintain the finer material in suspension and roll the coarser along the bed. The same material may alternate as bed load and suspended load. Bellasis (15*a*) defines suspended load as "silt", bed load as "drift"; and the weight of material per unit volume of water as the "charge".

There is an upper limit to the quantity of material a stream of given depth and velocity can transport. Such a charged stream will deposit silt at every reduction of velocity or decrease of turbulence and pick up more, if available, at every increase. A fully charged stream can not scour material from its channel, but its power of moving the drift or bed load is not impaired (15*b*).

About 1900, R. G. Kennedy, Executive Engineer, Public Works Department, Punjab, India, advanced the first formulas for transportation of silt by water as a result of his study of the Bari Doab Canal System. Kennedy presented the theory that there is a critical velocity for each depth that will neither scour nor permit deposits. His general formula is:

$$V_0 = C\, d^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which, V_0 is the critical velocity that will neither drop nor pick up sediment; d is the depth of water; and C is a coefficient depending on the kind of silt.

The values of the constants in Equation (1) have been established experimentally (see Table 7).

TABLE 7.—CONSTANTS FOR SUBSTITUTION IN EQUATION (1)

C	n	Authority	Applicable to:
0.84	0.64	Kennedy..................	Upper Bari Doab Canal, Punjab, India.
0.91	0.57	Kennedy..................	Shwebo Canal, Burma.
0.67	0.55	Kennedy..................	Godaveri, Western Delta, Madras, India.
0.93	0.52	Kennedy..................	Kistna, Western Delta, Madras, India.
0.95	0.57	Lindley....................	Lower Chenal Canal, Punjab, India.
0.39	0.73	Ghaleb.....................	Egypt.

Mr. Gerald Lacey presented the first general treatment of this complex subject in 1929 (57). Lacey[4] advances the theory that a stream flowing on its own alluvial plain possesses the following quite remarkable characteristics:

1.—The cross-section on straight reaches tends to become semi-elliptical.

2.—The parameter of the ellipse (ratio of the major to one-half the minor axis or surface width to maximum depth) depends solely on the character of the silt as regards fineness.

3.—For a given discharge, the wet perimeter is constant and independent of the character of silt.

4.—The silt factor bears a definite relation to the roughness coefficient in the Kutter and Manning formulas for discharge.

All attempts to apply the Kennedy formulas to the Imperial Valley conditions have proved unsuccessful (42*e*), and it appears that the Lacey formulas will prove equally disappointing.

It is scarcely possible that the complex laws of sediment transportation and deposit can be expressed for all kinds of river débris from dust to boulders as simply as Lacey has done it.

The writer is quite familiar with the Platte River in Nebraska. It flows on a deep bed of pure sand. The cross-section is anything but elliptical, and the channel conforms to no law except that of the mythical "Powder River" of the 91st Division of the American Expeditionary Force, whose slogan was: "It's a mile wide and an inch deep; we can swim it!"

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[4] The original paper on file in Engineering Societies Library presents the Lacey formulas with considerable data and discussion.

Effect of Clarifying a Silt-Laden Stream.—This promises to become a most important phase of the silt problem. The Boulder Reservoir will discharge clear water for the first time in untold ages into an alluvial river channel. Moreover, the discharge will be much more uniform than now obtains. What effect this will have on the regimen of the river, and how it will affect diversions into canals will be watched with great interest. Changes in the river channel of a silt-laden stream, after its regimen as regards flow and silt content has been radically altered, have been exemplified in the Rio Grande below Elephant Butte Dam. In the 125-mile stretch below this storage dam there are four dams diverting water into six irrigation canals (6*b*). The changes in river regimen that have occurred since the storage reservoir began operating in 1915, are listed in Table 8.

TABLE 8.—CHANGES IN REGIMEN OF RIO GRANDE

Item No. (1)	Point of measurement (2)	Period of observation (3)	Years (4)	Before construction of Elephant Butte Dam (5)	After construction of Elephant Butte Dam (6)
		(*a*) MEAN ANNUAL RUN-OFF, IN ACRE-FEET			
1.	San Marcial, N. Mex.	1897–1914	18	1 150 000	....
2.	San Marcial, N. Mex.	1915–1931	17	....	1 510 000
3.	E. Paso, Tex.	1897–1914	18	812 000	....
4.	El Paso, Tex.	1915–1950	16	....	638 000
		(*b*) MAXIMUM DISCHARGE, IN CUBIC FEET PER SECOND			
5.	San Marcial, N. Mex.	Oct. 11, 1904	....	33,000	....
6.	San Marcial, N. Mex.	Sept. 29, 1929	....	....	33 000
7.	E. Paso, Tex.	June 12, 1905	....	23 700	....
8.	El Paso, Tex.	Sept. 3, 1925	....	....	9 160
		(*c*) MEAN ANNUAL SILT CONTENT, IN TONS			
9.	San Marcial, N. Mex.	1897–1914	18	22 800 000	....
10.	San Marcial, N. Mex.	1915–1931	17	....	21 700 000
11.	El Paso, Tex.	1906–1909	....	17 500 000	....
12.	El Paso, Tex.	1916–1925	....	....	550 000

The general effect on the river channel has been to flatten the slope as indicated by cross-sections of the river taken since the storage began (1915 to 1925).

For the first 100 miles, the river has disposed of the débris from tributaries and, in addition, has taken about 480 acre-ft (2 100 000 cu yd) from its bed and banks and deposited them in the lower reaches. This degradation must continue until the river channel consists of a series of slopes between the diversion dams or other natural obstructions, just adequate to pass the mean-water and silt discharges under the new regimen that obtains under storage control.

On the Colorado River after the Boulder Reservoir is in operation a flattening of the river slope may be expected because large floods will be no more. Material will be picked up from the river channel by the clear water from the reservoir and deposited in the lower reaches. In time, the finer silts in the upper parts of the present channel may be carried entirely away, leaving only the coarser sands that may then move mostly as a bottom load in a fairly clear stream, except when it is muddied by floods on the tributaries.

Research on Silt Transportation.—The laws of silt transportation are known only imperfectly. They are now being made the subject of intensive research in both America and Europe. The first experimental study of the laws of sediment transportation with particular reference to bed load was undertaken by Gilbert (77) about 1914. His laboratory consisted of flumes with glass sides in which varying quantities and sizes of sands were introduced into streams of varying velocities. He observed transportation by saltation, the phenomena of sand ripples or dunes, and the conditions under which they migrated up stream or down stream. He developed equations for the tractive force to move sand mixtures. The work of Gilbert has not been followed up until recently.

Interest in this problem has again been stimulated and model experiments, together with mathematical studies of stream dynamics, are being made. MacDougall (71) is beginning where Gilbert left off, experimenting to determine the laws of bed-load transportation. Vogel (76), Matthes (75), and a staff of experimenters at the United States Waterways Laboratory at Vicksburg, Miss., are working extensively with models of particular reaches of the Mississippi and other rivers, studying the silt-transportation problems as regards shoaling, scouring, building of bars, effect of bends, etc.

The experimental and mathematical work by Rehbock, Prandtl (72), Hans Kramer, Assoc. M. Am. Soc. C. E., Bulle, and others in Germany, a mathematical review by O'Brien (74), and studies from the geological angle by Rubey (73) on the movement of débris as related to the conservation of energy in river systems, are all adding greatly to the sum total of the knowledge concerning the phenomena of sediment transportation and turbulent flow which are inseparably bound together.

These studies are incomplete, and discussion regarding the significance of observed phenomena is still rife.

CONTROL OF SILT

Except on certain small reservoirs for municipal or industrial purposes, it is generally impracticable to remove any substantial quantity of silt from reservoirs after it has been deposited. The most practicable remedy lies in preventing permanent deposits. Under certain circumstances this is quite possible. Some examples in which effective measures have been taken to prevent such permanent silt deposits will be cited.

The Aswan Dam is provided with sufficient sluice-gate capacity to pass the entire flood discharge of the Nile. These sluice-gates are opened at the beginning of the flood period, and the river flows through the reservoir practically as if no dam existed. During such flood periods, although the Nile is heavily charged with silt, no depositions occur in the reservoir. After the peak of the flood has passed, and the river begins to run clear, the sluice-gates are gradually closed and the reservoir is filled for use during the subsequent irrigation season. By this method of operation silting of the Aswan Reservoir has been so far, and probably will be, entirely avoided.

In Algeria, on the Habra and Hamiz Reservoirs, considerable success has been attained by opening sluice-gates of relatively small discharging capacity in the dams at the close of the irrigation season and allowing the stream to cut through the silt deposits, carrying out substantial quantities.

Bhatgurk Reservoir (5*c*) on the Yeluand River, in Bombay, India, completed in 1892, is provided with twenty under-sluices, each 80 sq ft in cross-section. The operations of these scouring sluices have no appreciable effect on silt already deposited, but having sufficient capacity to pass the average flood, the greater part of the silt is carried off while it is yet in suspension.

In the Zuñi Reservoir there were three 14-in. outlet gates in the tower, two of which became useless by silting. The outlet tunnel through the dam is 6 ft. in diameter. In 1931, a 4 by 6-ft sluice-gate was installed in the gate tower at Elevation 950. When first opened there immediately followed a flow of water and silt under a 40-ft head sufficient to fill the tunnel. From July 1, to October 15, 1931, 500 acre-ft (807 500 cu yd) of silt were sluiced out of this reservoir.

Silt Control on the Water-Shed.—Streams draining areas in arid climates, where fine sedimentary materials occur, are always silt laden. Many such streams are ephemeral; that is, they flow during storm periods only. Their valleys are of alluvial deposits that may be built up during long periods of years and then appear to suffer a fairly rapid degrading, the causes of which are complex and not fully understood. Among such streams are the Zuñi, Pecos, Puerco, Chaco, and Gila Rivers.

It is quite well established that a good growth of grasses is very effective in holding the soil. Where the precipitation is sufficient to maintain forests, extensive soil removal is prevented during ordinary amounts of precipitation. During heavy rains, causing extreme floods, neither grasses nor forests can prevent extensive erosion.

On the Zuñi River water-shed, silt control was begun in 1923 (40*i*). Nutria Creek was found to be supplying most of the silt, and systematic protective measures on this tributary were undertaken. Brush and rock checks were built across the main and tributary arroyos at critical points sufficiently close together to form a new and flatter gradient for the streams. At sharp bends and elsewhere at critical points rock and brush mattresses were constructed to prevent excessive cutting and consequent bank caving.

This work has been continued. Each year additional protective works have been built and the old structures maintained. Fig. 2 shows the silt deposits in Zuñi Reservoir. Note the diminution in the rate of silting after protective measures were undertaken in 1923.

The Rio Puerco is one of the main silt-bearing streams tributary to the Rio Grande above Elephant Butte Reservoir. This water-shed has been the subject of special study on behalf of the Middle Rio Grande Conservancy District (10). In former years, this river had a discontinuous channel, the valley floor having a goodly portion of broad, grassy or marshy areas over which moderate flood waters passed in thin sheets. In other places there was a definite channel, about 15 ft deep and 100 ft, or more, in width. About 1885, the river began a progressive degrading of its channel from its mouth up stream. An arroyo with many tributaries now exists well toward the head-waters. Surveys show that this main arroyo is 150 miles long and averages 28 ft in depth and 285 ft in width. The original channel had only about 12% of the volume of the present channel. These arroyos are being widened by caving of banks and transportation of the silt into the Rio Grande and on into Elephant Butte Reservoir. It has been estimated that during the 42 years prior to 1927 a total of 395 000 acre-ft of silt has been eroded from the valley floor of Rio Puerco and its tributary channels. This is an average of 9 400 acre-ft per yr and is nearly one-half the average deposits in Elephant Butte Reservoir. Protective measures have been planned, but to date have not been undertaken.

Excluding Silt from Canals.—The problem of preventing débris from entering canals diverting from silt-bearing streams has occupied the serious attention of engineers the world over.

On streams that run clear at ordinary stages, but carry a coarse bed load, the best solution has been found to construct the canal gates parallel to the shore of the stream and provide them with flash-board or over-flow gates so that the water taken into the canal is skimmed off the surface. If the water is raised by a dam, a sluice-gate is provided so that débris deposited in front of the gates can be sluiced away and passed on down the river whenever a surplus of water is available for sluicing.

On streams that carry considerable suspended silt in addition to bed load, settling basins have been used—located immediately below or in conjunction with the head-works, with provisions for periodically sluicing the deposits back into the river. One of the best examples of this method of silt control is afforded by the head-works of the Fort Laramie Canal, of the U. S. Bureau of Reclamation, at the Whalen Diversion Dam on the North Platte River, in Wyoming (6*d*).

All-American Canal.—On streams heavily laden with fine suspended silts, such as the Colorado River, desilting is a serious problem. On account of the slowness with which this silt settles even in still water, enormous settling basins are required. Studies are now (1934) under way for the head-works of the All-American Canal from the Lower Colorado River. The intake structure is planned to be located about 19 miles above Yuma, Ariz. Enormous desilting basins have been included in the preliminary designs.

S. L. Rothery, M. Am. Soc. C. E., has advanced (27) the theory that, in this stream, exclusion of the bed load from the canal up stream is of greater importance than exclusion of the suspended load.

The Laguna Dam, on the Colorado River, was completed in 1909. The head-works follow closely the design for silt-laden streams developed in India and Egypt. Three similar dams on the Nile River had been constructed during the fifteen years preceding the design of the Laguna Dam. The dam is 19 ft high, 4 800 ft long, and raises the water 10 ft at low water. It backs water about 10 miles up stream, creating a settling basin. This basin silted up soon after the dam was put in operation. At the ends of the dam a long sluice-way is provided which takes water from this basin. A long skimming weir with crest flash-boards in the land side of this sluice-way forms the head-gates proper, of the canal. The end of the sluice-way is provided with three large gates of sufficient capacity to pass 20 000 cu ft per sec. Fig. 4 is a general plan of the structure.

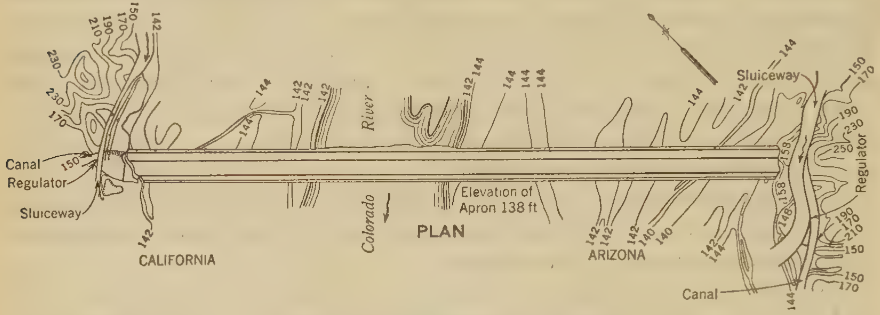


FIG. 4.—PLAN OF LAGUNA DAM.

Experiments on the efficiency of the desilting works at Laguna Dam during August and October, 1918, showed that the water in the main canal at the head contained 57% less silt than that of the river above the dam (42*f*).

Canals in India.—Much study and experimentation have been devoted to methods of preventing silts being taken into the large canal systems from the heavily silt-laden streams of India. In general, diversion is by means of a low dam or barrage. Several canals may head from one diversion dam and generally from both sides of the river.

A stream in alluvium consists of bends, straight reaches, and inflection points between bends. The latter are also called "crossings". At a bend the top flows toward the concave bank and the bottom filaments flow toward the convex bank. The effect of this phenomenon is to cause the bed load to be drawn to the convex side of the bend. A canal, therefore, that heads on the concave side of a bend will receive less bed load than one heading on the other side.

The Indian practice is to construct a dividing wall extending up stream from the barrage and reaching above high water. This wall forms an approach channel to the canal intake. Fig. 5 shows a plan of the Ferozepore Barrage, on Sutlej River. Two dividing walls are provided. Under-sluices are sometimes placed in the dividing wall opposite the canal gates in order to draw the bed load of sand away from the intake.

Passing Silt on to the Land.—Silt that passes into the canal at the head-works must be cleaned every year from the canal system unless the distributaries are designed and constructed to carry the silt through them to the land irrigated, or to discharge it through waste channels. The laws of silt transportation have already been discussed. The practice in India appears to be to pass as much silt as possible on to the lands. Suffice it to state, thus far, in the United States, the behavior of canals is so erratic and the laws of silt transportation are so imperfectly understood, that no one has succeeded in designing a canal system for the Western silt-laden rivers that makes it possible to pass the silt on to the lands. Certain reaches of certain canals have been observed to be practically non-silting and non-scouring for flows near the maximum. For lower flows that must be carried during a part of the season, however, extensive silting inevitably results.

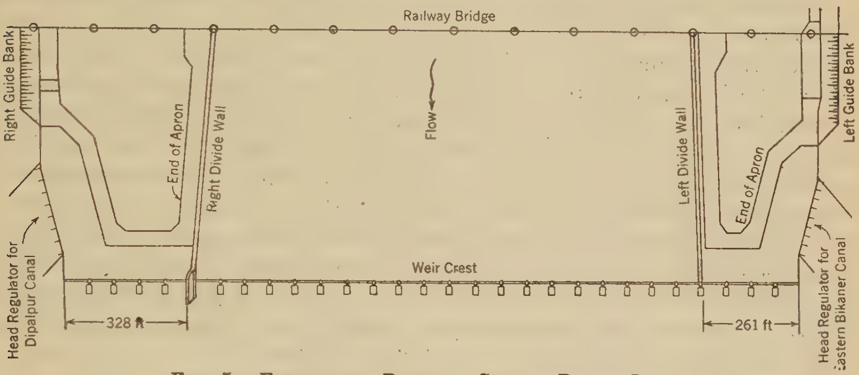


FIG. 5.—FEROZEPORE BARRAGE, SUTLEJ RIVER, INDIA.

The Imperial Irrigation District occupies a part of the delta of the Colorado River. The main canal diverts from the river at the Rockwood Heading, a short distance above the International Boundary. This structure is of concrete, built parallel to the river bank. It consists of seventy-five openings controlled by flash-boards, by which the river surface may be skimmed, thus excluding a part at least of the heavier material. There is no dam except a low brush-and-rock affair built during some years in the low-water period. Silt that enters the canal is removed by dredging. The canal diverts between 2 000 and 6 000 cu ft per sec from the river during the irrigation season. From the head-works of this canal 15 000 000 cu yd of silt were removed by dredging during the three seasons, 1918–1920 (50*b*).

The suspended silt found throughout this Canal System varies from 2 to 8 per 1 000 by weight during the height of the irrigation season. Practically all the suspended silt is so fine that it will pass a 200-mesh sieve, and the greater part passes a 300-mesh sieve (42*g*). A velocity of $\frac{2}{3}$ ft per sec will maintain this silt in suspension.

Experience in the Imperial Valley seems to prove that it would be cheaper to remove the silt from the Canal System by machinery than to pass it to, and care for it on, the cropped lands (42*g*). The quantity of silt removed from the Canal System by machinery is approximately 4 250 000 cu yd per annum (58), of which one-third is taken out by dredges at the head-works. This is a large quantity of material to handle each year, but it is a relatively small part of the amount taken into the system. In 1914, 30 000 000 cu yd of sediment were taken in at the Hanlon Head-Gate. About 4 000 000 cu yd were removed by machinery from the System, leaving 26 000 000 cu yd to be carried on to the land or passed through wasteways. It is impossible to hold the silt in the Canal System from which it could be dredged cheaply.

The disposal of silt in irrigation systems supplied by the Colorado River and similar silt-laden streams constitutes a serious problem, for which no adequate solution has as yet been found.

ORIGIN OF SILT

Exposed rocks disintegrate into soil by varied and complex mechanical and chemical processes. Rain water dissolves certain constituents; it enters crevices, freezes, and spalls off blocks and particles. Chemical agents in the air re-act with those of the rocks, causing them to break down and even to crumble into dust. Wind, rain, and streams transport the soils thus made to lakes and oceans, sorting and depositing them in layers or beds sometimes thousands of feet in thickness. Infiltration of limes or other cementing agents, great pressure from superimposed beds, intense heat, volcanic activity, and subsequent cooling re-convert these deposits into limestones, shales, sandstones, and various types of crystalline metamorphic rocks with infinite gradations and modulations as effected by a great complexity of forces and factors. Upheavals and erosion expose them again to disintegrating influences, and the cycle is repeated with infinite variety.

Sedimentation is that phase of the geological cycle involving water transportation and deposition. Every rill and river performs its allotted part in this process. Lakes and valleys are filled, great deltas extend fan-like into oceans, creating alluvial plains. The densely populous plains of the Nile, Ganges, Mississippi, and Yellow Rivers are the results of such sedimentation processes.

Most streams run fairly clear at low stages, but while so doing they may move quantities of sand and detritus along their beds. At high stages a suspended silt load develops and the stream is said to run "“muddy”. At every slackening of the velocity both suspended load and bed-carried material are deposited, forming bars, berms, deltas, valleys, and alluvial plains.

Some rivers carry a suspended load at all times. Generally, these rivers drain arid or semi-arid areas. It appears that in areas of abundant precipitation the streams are capable of carrying off all the detritus resulting from rock disintegration as fast as it accumulates. The result is that the streams run clear except at times of flood. On arid areas the débris from disintegration may remain in place for many years until an unusual rainfall (cloudburst) occurs, when great quantities of silts are transported. If the drainage area is large enough and has many tributaries, the “unusual” rainfall is almost continuously occurring successively in some places. This keeps the main stream continually supplied with an abundance of silt, which it must carry at all stages. Of such, are the water-sheds of the Rio Grande, the Colorado, Missouri, Colorado River of Texas, Yellow, and Indus Rivers.

The character of the drainage area and its vegetable covering are all-important factors. If the rocks of the area are sedimentaries, such as sandstone, clays, and shales, disintegration processes produce large quantities of fine soils. If forests exist this soil is effectively held in place against ordinary rains, but the forest rapidly loses its efficacy to prevent erosion in times of intense rainfall. If the area is too dry for forests, the soil may still be held effectively by substantial growths of grasses and small brush. If too arid for small growth, the soil lies at the mercy of every shower.

Bank cutting is a fruitful source of sediment. The process of moving soil from the mountains to the sea consists of an infinite number of starts and stops. A bar is formed during this flood that may not be moved for many years. An alluvial valley is built up during centuries of sedimentation and then deleted under a new set of cultural or climatic conditions. Alluvial streams build their beds and banks higher than the surrounding plain, a process which, however, can not continue indefinitely. A flood breaks the banks and the river finds a new channel, cutting out the deposits it had itself laid down in earlier years. The Yellow River built up its channel until the water surface was 25 ft above its plain. During the floods of 1851 to 1853 it broke its bank, inundated 50 000 sq miles of cultivated valley, snuffed out a million lives, and found a new mouth 500 miles to the north of its former outlet. This river of mud is often referred to as "China's Sorrow" (12*b*).

The Missouri River is always muddy in its lower reaches. Its Indian name means "Big Muddy". It receives much silt from the clay beds of the Bad Lands of South Dakota and from the enormous areas of shale in Montana and elsewhere in its upper valley, and, in addition, takes continuous toll from its bank through the alluvial plain through which it flows.

The Colorado River becomes a silt-laden stream after it passes into Utah. Much of its drainage area is so arid it is quite void of vegetation. Floods from local rains occur in great diversity on its many ephemeral tributaries that keep the main stream loaded with silt. These characteristics also apply in varying degrees to the Rio Grande, the Colorado River of Texas, the Brazos, Trinity, and many other Southwestern streams.

The Yellow River in Northern China carries nearly twice as much silt per annum as the Mississippi. Its drainage area is largely covered with loess, a yellowish, friable, wind-blown deposit of fine sands and dust-like soil that is carried away with every rainfall, keeping the streams continually surcharged. Its name and that of the Yellow Sea into which it flows is derived from the color this loess gives to the water.

In India, all the great rivers are silt-laden streams. Between the Himalayas and the Indian Peninsula lie the great alluvial plains watered by three great river systems. The Indus, with its principal tributary, the Sutlej, waters the western portion; the eastern portion is drained by the Brahmaputra; while between them lies the Ganges.

The Indus Valley is quite arid and is covered with fine alluvial silts and wind-blown sands. The Ganges and Brahmaputra areas have a seasonal climate. Parts of these areas are seasonally dry; irrigation is profitable. From these areas enormous quantities of silts are supplied to the rivers so that the streams flow on broad alluvial deposits, which although about 2 000 ft thick, are nevertheless geologically recent.

The Nile is perhaps the most famous river of history. The silts deposited on the lands at the time of its annual inundation have made the fertile plains of Lower Egypt. The Nile carries annually 95 000 000 tons of silt, of which 30 000 000 remains on the land and 65 000 000 is carried into the Mediterranean Sea. At ordinary stages, the Upper Nile is fairly clear, but in flood periods it is of a chocolate color from the brown soils of the Abyssinian plateaus brought in by the Sobat, Blue Nile, and Atbara, which streams are the chief sources of the Nile floods.

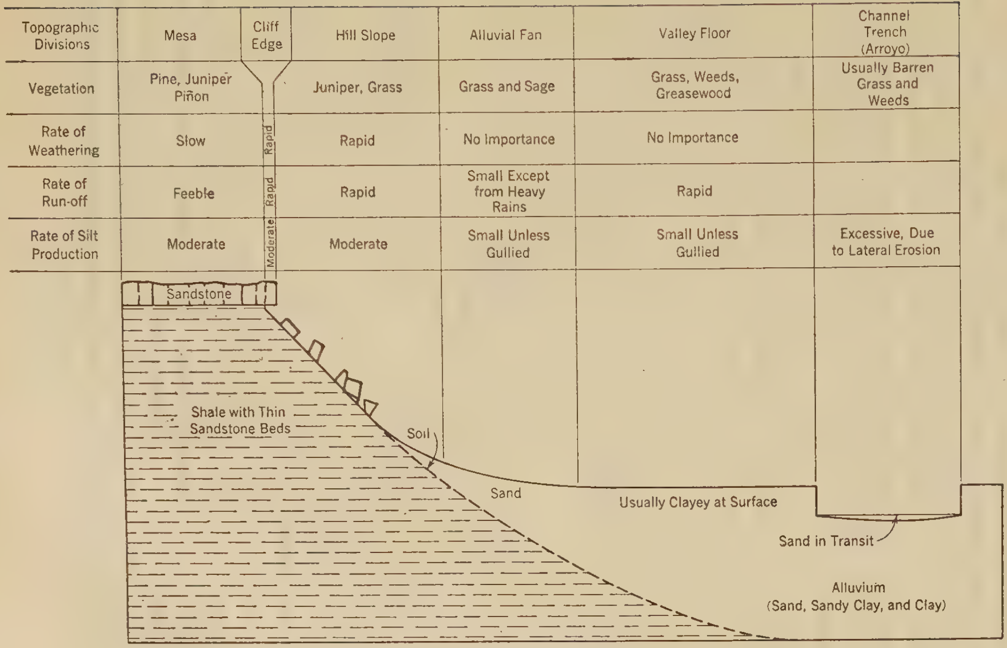
The Rio Puerco drains 5 700 sq miles of Western New Mexico. The name means "dirty river". It and its neighbouring stream, Rio Salado, are sources of abundant quantities of silt in the Rio Grande and pour these silts into that stream above Elephant Butte Reservoir. Because of this fact Rio Puerco has been the subject of special study from a geological standpoint by Kirk Bryan, Professor of Geology, Harvard University, and from an engineering point of view by Mr. George M. Post (10). Because their report contains data that are essentially applicable to many other streams, they will be presented as quite typical of the silt-bearing ephemeral streams of the arid Southwest.

The silts move only during floods— a quick rush of muddy water down a dry channel immediately following a rain. Rio Puerco has always carried large quantities of silt into the Rio Grande, but this quantity has been increased materially in recent years. The investigation was prompted by the hope and belief that the channel would be rehabilitated and the silt charge greatly reduced.

When first known, the main stream and its tributaries were in a process of alluviation. The valleys were plains over which muddy floods spread out or flowed in discontinuous channels. In places, these channels may have been 10 to 20 ft deep and 100 ft, or more, wide, but they were interspersed by broad, flat, marshy areas over which the water spread in sheets. About 1885, a new arroyo began to form at the mouth and to cut up stream, until the channel is now continuous for 150 miles and averages 28 ft in depth and 285 ft in width.

In order better to understand this process, Figs. 6 and 7 have been reproduced (10), showing how the alluvial valley was formed and how degradation now appears to be progressing in the Valley of Rio Puerco.

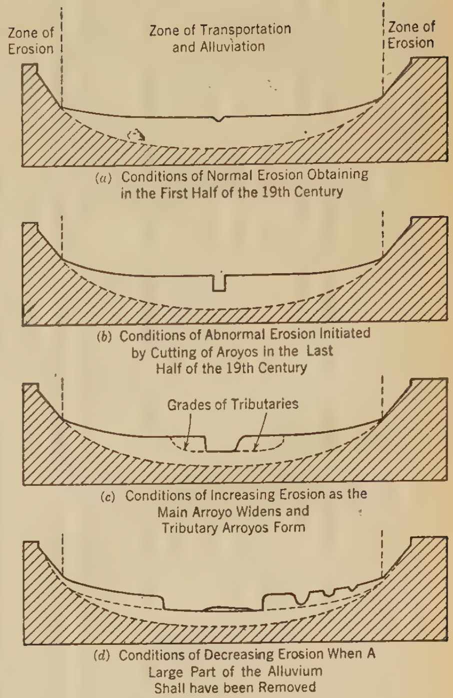
The silts come mainly from the disintegration of Cretaceous rocks that cover about one-half the drainage area. The characteristic topography of the silt-producing areas consists of mesas underlaid by sandstone lying between broad valleys eroded in the shales (Fig. 6). The sandstone cliff is undermined by the more rapidly eroding shale and breaks off along joint planes. These pieces weather into sands, while the shale base forms clays. The valleys become filled with fine argillaceous sands that are eroded easily unless held by grass sod or other vegetation. Gullies are formed on the steep hillsides that end in deltaic fans formed by silts deposited at the valley floor. These deposits form cracks when dried, creating vertical joint planes in the mass. When such deposits form the banks of a stream, they



Topographic Divisions	Mesa	Cliff Edge	Hill Slope	Alluvial Fan	Valley Floor	Channel Trench (Arroyo)
Vegetation	Pine, Juniper Piñon		Juniper, Grass	Grass and Sage	Grass, Weeds, Greasewood	Usually Barren Grass and Weeds
Rate of Weathering	Slow	Rapid	Rapid	No Importance	No Importance	
Rate of Run-off	Feeble	Rapid	Rapid	Small Except from Heavy Rains	Rapid	
Rate of Silt Production	Moderate	Moderate	Moderate	Small Unless Gullied	Small Unless Gullied	Excessive, Due to Lateral Erosion

Sandstone
Shale with Thin Sandstone Beds
Soil
Sand
Usually Clayey at Surface
Sand in Transit
Alluvium (Sand, Sandy Clay, and Clay)

FIG. 6.—DEMONSTRATING HOW ALLUVIAL VALLEYS ARE FORMED.



Zone of Erosion
Zone of Transportation and Alluviation
Zone of Erosion
Grades of Tributaries

(a) Conditions of Normal Erosion Obtaining in the First Half of the 19th Century

(b) Conditions of Abnormal Erosion Initiated by Cutting of Aroyos in the Last Half of the 19th Century

(c) Conditions of Increasing Erosion as the Main Arroyo Widens and Tributary Arroyos Form

(d) Conditions of Decreasing Erosion When A Large Part of the Alluvium Shall have been Removed

FIG. 7.—DEMONSTRATING HOW AN ALLUVIAL VALLEY MAY BE DEGRADED.

are easily undermined; the banks cave off in blocks, crumble, and become the suspended load of the stream.

Rio Puerco began degrading its valley floor about fifty years ago. Fig. 7 shows the successive events as they have been pieced together by historical inquiry and geological evidence. The silt derived from the valley walls was carried over the surface and on down stream through discontinuous channels, intermittently by the floods. This condition is shown in Fig. 7(*a*). With the cutting of the arroyos, the present period of abnormal erosion or degrading process began (Fig. 7(*b*)). The rate of silt production passes through a cycle. When the main arroyo is narrow and increasing in length up stream, the rate of silt production is increasing. When it becomes a continuous channel the widening process begins (Fig. 7(*c*)). At this stage, the rate of silt production has reached a maximum. When the channel becomes so wide that the floods only touch its sides lightly or intermittently, bank caving and consequent silt production begin to diminish.

A time will come when the entire slice of alluvium above the new grades of the main and tributary arroyos shall have been removed (Fig. 7(*d*)). The valley then reverts to the condition shown in Fig. 7(*a*), but at a lower level. A process of alluviation then sets in that tends to build up the valley floor to some higher level. After this has been accomplished, the cycle of degrading will be repeated.

Abundant evidences were found in the Valley of the Rio Puerco of earlier cycles of alluviation and subsequent degrading. Fig 8 is a map of the present channel superimposed over an ancient one. The present arroyo has nearly vertical banks, 25 ft in height and 180 to 500 ft apart. The outcrops of the buried ancient channel are readily distinguishable because its filling is lighter in color than that of the present valley alluvium. The bed of the ancient channel was about the same level as the present channel and varied from 75 to 220 ft in width.

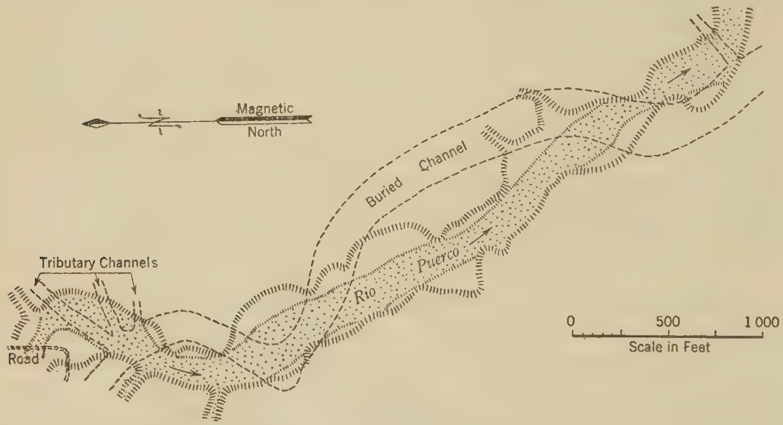


FIG. 8.—ANCIENT CHANNELS SHOW PREHISTORIC PERIODS OF ALLUVIATION AND DEGRADATION

Traces of other channels were found, but not mapped. These buried channels represent a period of erosion and degrading similar to that going on at present. Each was followed by a period of alluviation, that completely obliterated traces of old arroyos and in general made a plain of the valley floor over which the flood waters could spread and continue the upbuilding of the valley.

Another famous example of a cycle of alluviation and degrading has been found in the Rio Chaco (10*a*), (78). At Pueblo Bonito, an arroyo from 150 to 450 ft wide and 25 to 30 ft deep exists at present similar in all respects to the one on Rio Puerco. This arroyo has been cut since 1860. Pottery and charcoal, to depths of 20 ft, or more, are found in the valley alluvium showing that this alluviation occurred during its occupancy by pre-historic peoples. After Pueblo Bonito and other large pueblos were built and occupied, an arroyo was cut through the valley floor and later filled by alluviation to its original level. Where this buried channel outcrops in the walls of the present arroyo, pottery of very recent occupancy is found, while at similar depths in the main valley fill only pottery of much earlier periods has been discovered.

This ancient arroyo does not coincide with the present one, but is crossed, touched, and re-crossed by it. The ancient channel may be traced for about five miles. It tells the same story of periods of degrading followed by periods of alluviation.

These ancient peoples probably subsisted largely by "flood-water farming". When the ancient arroyo was cut in the valley about the end of the Great Pueblo period, it rendered such farming methods impossible. This has been assigned by Professor Bryan as a possible reason for the abandonment of these Indian villages (79*a*). A subsequent period of alluviation filled this arroyo and probably permitted re-occupancy. The present period constitutes another cycle of degradation.

The Zuñi River has been the subject of a study similar to that of the Rio Puerco (10*c*). It is an ephemeral stream, carrying only flood waters and large quantities of silt. The drainage basin consists mainly of plateaus and mesas underlaid by Cretaceous sandstone and shale. Like the Rio Puerco its silt is derived mainly from the erosion of the banks of arroyos cut in the valley fill. This stream is also undergoing a process of degrading its valley floor.

Ancient channels have been found in the Valley of Zuñi River, also, that attest to earlier periods of alluviation and degradation. Fig. 9 is a sketch of such an ancient channel exposed in the high banks of Nutria Creek.

Causes of Valley Degradation.—No complete explanation for these cycles of alluviation and degrading has as yet been found. The potent factors are deep seated and are quite certainly beyond Man's power to alleviate to any great extent, except in certain favored situations. The cause is probably to be found in climatic influences, and the physics of sedimentation as applied to the moulding of stream valleys. During a period of alluviation the slope of the valley is flattened. In time, it becomes too flat for the stream to carry its normal flood waters. A period of channel cutting ensues. This process of degradation is carried beyond the balance point, and a period of alluviation sets in. Thus, valley building appears to be an endless succession of periods of alluviation and degradation.

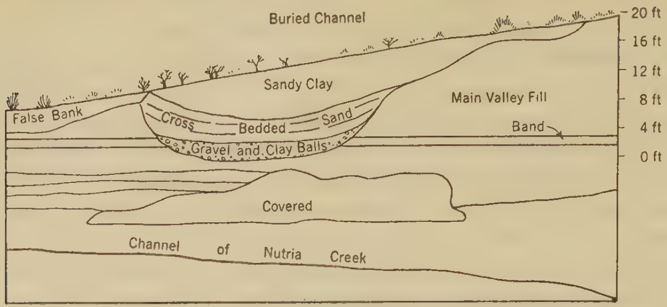


FIG. 9.—ANCIENT CHANNEL EXPOSED IN PRESENT ARROYO, ZUNI RIVER.

ACKNOWLEDGMENTS

The writer is deeply indebted to the following for suggestions, constructive criticisms, and valuable data furnished: Professor Kirk Bryan, Harvard University; the Silting Committee of the Power Division of the Society, of which, Andrew Weiss, J. E. S. Thorpe, C. W. Kutz, and Thorndike Saville, Members, Am. Soc. C. E., deserve special mention; L. J. Bevan, M. Am. Soc. C. E., Chairman of the Executive Committee of the Power Division of the Society; the Executive Committee of the Irrigation Division of the Society, with special mention of W. H. Code, M. Am. Soc. C. E.; the Special Committee on Irrigation Hydraulics, of the Society, of which Fred C. Scobey, Ivan Houk, and J. L. Savage, Members, Am. Soc. C. E., also, E. S. Lindley, M. Am. Soc. C. E., co-operating member, deserve special mention; L. M. Lawson, M. Am. Soc. C. E., International Boundary Commission, El Paso, Tex.; Herman Stabler, M. Am. Soc. C. E., Chief, Conservation Branch, U. S. Geological Survey; E. S. Howard, Chemist, U. S. Geological Survey; and E. W. Lane, M. Am. Soc. C. E., Research Engineer, U. S. Bureau of Reclamation, Denver, Colo., as well as the following Division and District Engineers, of the U. S. Corps of Engineers: Capt. Theodore Wyman, Jr., Kansas City, Mo.; Col. Wildurr Willing, St. Paul, Minn.; Lt.-Col. Glen E. Edgerton, M. Am. Soc. C. E., Rock Island, Ill., and Maj. R. R. Neyland, Nashville, Tenn.

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DISCUSSION

HARRY G. NICKLE,[6] JUN. AM. SOC. C. E. (by letter).—In gathering data from many sources all over the world, Mr. Stevens has shown clearly the many interrelated questions involved in the silt problem, and the need for continued research in these subjects. As the population continues to increase and it becomes necessary to conserve more fully the water resources of the United States, the problems discussed in this paper and the need for more data concerning them will become increasingly important.

Of immediate interest to those in charge of the construction of reservoirs along streams, especially in the more arid sections of the country, are the quantities of silt to be carried by streams into any proposed reservoirs and the conditions and volume of deposit of that silt in the reservoirs under various conditions of operation.

The quantity of suspended material carried by a stream can be determined with sufficient accuracy, by regular samplings, preferably taken daily, and such sampling stations should be established on all silt-laden streams which have possible reservoir sites so that as long a period of record may be obtained as possible before each reservoir is constructed. The quantity of bed-load rolled along the the bottom at any location is a more difficult problem to determine and much research remains in order to perfect a means of measuring this bed-load. The bed-load should be measured, if possible, at the same stations as the suspended silt and throughout the same period of time. The conditions and volume of silt deposits in reservoirs under different conditions of operation are known to vary widely, and much research is still necessary in this field—especially through studies and surveys of existing silt deposits—before complete information will be available. In this regard, in each reservoir investigated, a detailed study should be made of the contributing watershed, including its geology, topography, vegetal cover, and other pertinent characteristics.

Mr. Stevens states that the rate of silt deposition diminishes as the reservoir fills, due to a continually lessening volume of quiet water, with the result that more silt is carried over the spillway, and also that during extreme floods earlier deposits may be picked up and carried out of the reservoir. This, of course, is true ultimately of all reservoirs. However, in any large reservoir in which the capacity above the spillway or sluice-way is large compared with the inflow, it will generally be many years before such an effect will occur to any appreciable extent. O. A. Faris, M. Am. Soc. C. E., states[7]:

> "Suspended silt settles to the reservoir bottom soon after entering the slack water and, having a greater specific gravity than water, flows, in the form of liquid mud, down the slopes into depressions and along the main channel until blocked by the dam. Owing to its greater density, silt-charged water entering a reservoir partly filled with clear water does not mingle with

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[7] "The Silt Load of Texas Streams," by Orville A. Faris, M. Am. Soc. C. E., *Technical Bulletin No. 382*, U. S. Dept. of Agriculture, 1933.

the clear, but forces it down stream toward the dam. No suspended silt is carried through the reservoir and over the spillway until all of the clear water has been discharged."

On the other hand, in small reservoirs (those in which the capacity is relatively small compared to the inflow during floods and in which stream conditions prevail during floods rather than reservoir conditions), many different effects may occur. For instance, one flood may deposit large quantities of material and a later one may clean large volumes of this material from the reservoir; or in other cases a slope may be built on the upper side of the spillway so that heavy materials are rolled up and over it, as finer materials are carried over in suspension. The exact behavior of floods through these smaller reservoirs, the deposition and picking up of material in them, and the quantity of material carried over the spillways are subjects that could be investigated more thoroughly, because they are constantly being discussed when new small reservoirs are proposed or built.

When the water level is subject to fluctuations the rate at which silt is deposited in a reservoir also diminishes, in general. The reservoir becomes filled because a greater area of the deposited silt is exposed during the fluctuations of water surface, and, consequently, these later deposits are more dense. The increase in the density of these later deposits depends not only on the frequency of wetting and drying, but also on the nature of the particles of which the silt is composed.

Since the summer of 1924, the silt problem in Texas has been studied by the Bureau of Agricultural Engineering of the U. S. Department of Agriculture (the Division of Agricultural Engineering of the Bureau of Public Roads prior to July 1, 1934), in co-operation with the Texas Board of Water Engineers.

Tens of thousands of samples of the water have been taken from the principal streams of Texas at twenty-three regular sampling stations, and the quantity of silt has been determined at these stations over varying periods of time. Miscellaneous samples have also been taken at various other locations throughout the State. Conditions of silt deposits in several of the reservoirs of the State have been studied in detail, and other phases of the silt problem have been investigated, including the quantity of bed-load carried by streams, the distribution of the silt throughout the cross-section of the stream, the relationship of the quantity of silt to the velocity, and the size and character of the particles composing the silt deposits, both in the reservoirs and along the stream channels. This co-operative work was under the direction of the late Robert Grier Hemphill, Assoc. M. Am. Soc. C. E., until his death in 1930; then under that of Mr. Faris until the end of 1933; and, under the writer, since April, 1934.

The results of this silt investigation to the end of the year 1930 were published in September, 1933.[7] Only a small part of the results of this intensive silt investigation in Texas have been given in Mr. Stevens' paper.

In September, 1925, at a time of low stage, a silt survey was made of Medina Reservoir, on the Medina River, about 35 miles northwest of San Antonio, Tex. The drainage area of this reservoir is 587 sq miles, the larger part of which is brush-covered grazing land, with a range in elevation of from 1 000 to 2 500 ft above sea level, and with an average rainfall of 29 in. The storage capacity at the elevation of the spillway crest was 254 000 acre-ft when the reservoir was constructed. After thirteen years of operation, in 1925, this reservoir contained 2 692 acre-ft of deposited silt, or only 1.06% of the total capacity of the reservoir when constructed. This represents a yearly average of 207 acre-ft, or 0.35 acre-ft per sq mile of water-shed per year. This deposited material had an average dry weight of 30 lb per cu ft when the survey was made in 1925, but five years later, owing to exposure to the sun and atmosphere at various times during the five years, the average dry weight of the material in place was estimated to be 63.6 lb per cu ft. In other words, the 2 692 acre-ft of deposited silt measured in 1925 had shrunk to only 1 270 acre-ft in 1930, or only 0.50% of the total capacity of the reservoir when constructed.

Measurements of the average weight of dried silt per cubic foot of material in place, taken at several reservoirs, vary from 18 to 37 lb per cu ft, when the deposited material has been under water at all times, to 85 or 100 lb per cu ft, or even higher, at locations where the deposits were subject to alternate wetting and drying, these values differing in each case not only due to different conditions of operation of the reservoirs, but also to the varying gradations and sizes of the particles of which the deposits are composed. Due to varying characteristics of large drainage areas and the varying conditions under which reservoirs are operated, it has been impossible to set a definite value for the dry weight of the deposited silt. However, after considering all factors entering into the problems, including the fact that an indeterminable quantity of vegetable matter deposits also and lasts indefinitely, a value of 70 lb per cu ft of material in place was chosen as an average ultimate figure for reservoirs in which silt deposits are subject to alternate wetting and drying.

Samples of dried silt from reservoir deposits had an average specific gravity of about 2.65, whereas samples taken from suspension and from which vegetable matter was excluded, had an average specific gravity of 2.73. Mechanical analyses of various samples of suspended silt from the streams of Texas showed on the average more than 97% passing the No. 300 sieve. Of the liquid mud taken from the Medina Reservoir, 99.5% passed the No. 300 sieve; and of the silt samples taken of the deposited material at Lake Worth, only one showed greater than 0.2% retained on the No. 300 sieve.

No direct relation was found between the volume of suspended silt carried and the velocity of the stream. However, in the cases considered, capacity loads were not even approached and the quantity carried was a function of loading only.

Table 9 gives records (which are similar to, but which are not included in, Table 6 of Mr. Stevens' paper) of silt carried by the streams at the several silt-sampling stations to the end of 1930. At several of these stations, especially those in the upper reaches of the Brazos River water-shed, the flow is intermittent, sometimes many days or even several months elapsing without any flow past some of these stations.

TABLE 9.—SUSPENDED SILT CARRIED BY TEXAS STREAMS

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	Suspended Silt	
							Per thousand	Millions of tons (2 000 lb.) during period
1	Neches	Rockland	3 540	8/8-12/31/30	133	338.4	0.14	0.063
2	Double Mountain Fork, Brazos	Aspermont	7 980	6/4-12/31/24	....	10.9	21.6	0.32
				Jan.-Dec., 1925	....	130.5	28.3	5.02
				Jan.-Dec., 1926	....	314.9	19.5	8.34
				Jan.-Dec., 1927	....	64.6	18.3	1.61
				Jan.-Dec., 1928	....	119.8	25.6	4.17
				Jan.-Dec., 1929	....	113.0	22.4	3.44
				Jan.-Dec., 1930	....	176.1	31.5	7.56
Total Item No. 2	......	......	......	......	549*	929.8	24.1	30.46
3	Salt Fork, Brazos	Aspermont	4 990	6/4-12/31/24	130*	33.2	15.5	0.70
				1/1-8/29/25	....	104.4	32.5	4.62
4	Clear Fork, Brazos	Crystal Falls	4 320	Sept.-Dec., 1925	....	105.9	2.3	0.33
				Jan.-Dec. (except June), 1926	....	139.7	1.7	0.33
				Jan.-Dec., 1927	....	125.0	2.8	0.47
				Jan.-Dec., 1928	....	338.6	3.8	1.73
Total, Item No. 4	......	......	......	......	966*	709.1	3.0	2.86
5	Clear Fork, Brazos	Eliasville	5 740	6/3-12/31/24	291*	98.5	3.0	0.40
				Jan.-Aug., 1925	....	122.0	3.7	0.61
6	Brazos	Seymour	14 500	6/5-12/31/24	....	78.3	18.4	1.96
				Jan.-Dec., 1925	....	398.3	23.1	12.55
				Jan.-Dec., 1926	....	605.3	16.1	13.28
				Jan.-Dec., 1927	....	100.6	10.2	1.40
				Jan.-Dec., 1928	....	225.4	20.3	6.22
				Jan.-Dec., 1929	....	231.4	16.2	5.10
				1/1-7/13/30	....	423.5	17.7	10.23
Total, Item No. 6	......	......	......	......	584*	2 062.8	18.1	50.74
7	Brazos	Mineral Wells	23 100	6/2-12/31/24	....	201.0	6.8	1.87
				Jan.-Dec., 1925	....	1 149.5	12.7	19.93
				Jan.-Dec., 1926	....	1 368.3	9.5	17.71
				Jan.-Dec., 1927	....	443.6	4.4	2.66
				Jan.-Dec., 1928	....	964.7	7.9	10.43
				Jan.-Dec., 1929	....	756.1	8.9	9.14
				Jan.-Dec., 1930	....	1 697.6	7.8	18.02
Total, Item No. 7	......	......	......	......	2 241*	6 580.8	8.9	79.76
8	Brazos	Glenrose	24 800	June-Dec., 1924	....	228.0	6.3	1.95
				Jan.-Dec., 1925	....	1 119.8	11.1	16.85
				Jan.-Dec., 1926	....	1 772.0	10.2	24.60
				Jan.-Dec., 1927	....	653.6	6.7	5.99
				July-Oct., 1928	....	452.3	5.7	3.49
				Jan.-Aug., 1929	....	516.9	8.1	5.73
Total, Item No. 8	......	......	......	......	858*	4 742.6	9.1	58.61
9	Brazos	Waco	28 500	June-Dec., 1924	....	293.3	4.9	1.96
				Jan.-Dec., 1925	....	1 268.8	10.9	18.86
				Jan.-Dec., 1926	....	2 307.1	9.1	28.55
				Jan.-Dec., 1927	....	1 445.9	5.5	10.83
				Jan.-Dec., 1928	....	1 375.2	8.3	15.60
				Jan.-Dec., 1929	....	1 330.3	7.6	13.70
				Jan.-Dec., 1930	....	2 460.1	7.2	24.15
Total, Item No. 9	......	......	......	......	1 882*	10 480.7	8.0	113.65
10	Brazos	Rosenberg	44 000	6/11-12/31/24	....	664.8	1.0	0.93
				Jan.-Dec., 1925	....	3 274.2	5.1	22.74
				Jan.-Dec., 1926	....	7 843.0	4.2	44.40
				Jan.-Dec., 1927	....	5 038.5	4.3	29.63
				Jan.-Dec., 1928	....	2 864.9	6.1	23.88
				Jan.-Dec., 1929	....	6 429.5	3.8	33.61
				Jan.-Dec., 1930	....	6 543.0	5.8	51.82
Total, Item No. 10	......	......	......	......	2 209*	32 657.9	4.7	207.01

* Number of observations for entire period.

TABLE 9.—(*Continued*)

Item No.	Stream	Locality	Drainage area, in square miles	Period	Number of observations in period	Quantity of water, in thousands of acre-feet	SUSPENDED SILT	
							Per thousand	Millions of tons (2 000 lb. during period
11	Little	Little River	5 250	6/8-12/31/24	....	64.3	0.3	0.024
				Jan.-Dec., 1925	....	227.0	3.5	1.07
				Jan.-Dec., 1926	....	773.7	2.1	2.21
				Jan.-Dec. 1927	....	660.4	1.8	1.59
				Jan.-Dec., 1928	....	243.8	1.6	0.53
				1/1-5/27/29	....	114.1	1.7	0.26
Total, Item No. 11	....	....	......	....	1 754*	2 083.3	2.0	5.68
12	San Gabriel	Circleville	602	6/7-12/31/24	....	22.5	0.08	0.0024
				Jan.-Dec., 1925	....	62.0	2.7	0.23
				Jan.-Dec., 1926	....	198.0	2.6	0.70
				Jan.-Dec., 1927	....	166.5	1.6	0.37
				Jan.-Dec., 1928	....	36.5	0.3	0.015
				Jan.-Oct., 1929	....	112.8	3.3	0.51
Total, Item No. 12	....	....	......	....	1 891*	598.3	2.2	1.83
13	Colorado of Texas	San Saba	30 600	9/11-12/31/30	107	1 071.1	3.2	4.69
14	Colorado of Texas	Tow	31 100	Oct.-Dec., 1927	....	218.2	2.7	0.81
				Jan.-Dec., 1928	....	896.0	3.6	4.42
				Jan.-Dec., 1929	....	762.0	3.7	3.88
				Jan.-Dec., 1930	....	1 763.5	3.1	7.46
Total, Item No. 14	....	....	......	....	824*	3 639.7	3.3	16.57
15	Colorado of Texas*	Columbus	40 800	8/3-12/31/30	126	1 671.5	3.8	8.64
16	San Antonio	Falls City	2 070	9/13-12/31/27	....	24.9	0.5	0.016
				Jan.-Dec., 1928	....	117.3	1.8	0.29
				Jan.-Dec., 1929	....	181.2	2.1	0.51
				Jan.-Dec., 1930	....	87.2	0.5	0.056
Total, Item No. 16	....	....	......	....	1 202*	410.6	1.6	0.88
17	Nueces	Three Rivers	15 600	Oct.-Dec., 1927	....	119.1	1.1	0.18
				Jan.-Dec., 1928	....	245.1	1.5	0.51
				Jan.-Dec., 1929	....	770.4	1.2	1.29
				Jan.-Dec., 1930	....	572.4	0.9	0.73
Total, Item No. 17	....	....	......	....	1 135*	1 707.0	1.2	2.71

* Number of observations for entire period.

E. W. LANE,[8] M. AM. SOC. C. E. (by letter).—As the development of water resources for water supply, water power, irrigation, and navigation is becoming more thorough, and the control of floods of streams in the United States is growing more urgent, the silt problem which Mr. Stevens so interestingly describes, is increasing in importance yearly; in the future, no doubt, it will be even more vital. In order to find adequate solutions for the present and prospective silt problems, a more complete development of the governing laws is necessary. At present, this science is in an undeveloped state. Although much progress has been made, the problem is very complex, due to the large number of variables involved, and much remains to be accomplished before a satisfactory science will have been developed.

Like other phases of hydraulic engineering, present knowledge of the mechanics of the solids load of streams has been a gradual growth, with its beginning many years ago. Probably the first scientific approach was made by Guglielmini near the end of the Seventeenth Century. He developed laws governing fluvial phenomena, in connection with the control of the Po River in Italy. The problem first assumed importance in the United States in connection with the controversies which arose over the control of the floods of the Mississippi River and the construction of jetties for navigation improvement at its mouth. Extensive observations of suspended load were made in the studies of Humphreys and Abbot near the middle of the Nineteenth Century, and the conclusions they reached were at variance with current opinion. These results led to a violent controversy with the late James B. Eads, F. Am. Soc. C. E., who about 1875 proposed to deepen the river mouths by jetties, and this discussion created considerable interest in solids load phenomena. Upon the formation of the Mississippi River Commission in 1879, extensive silt measurements were begun on the Mississippi and its principal tributaries, but were continued only two or three years. Except for the excellent summary of the studies of engineers all over the world published by Dr. E. H. Hooker in 1896,[9] little progress was made in this country on this problem until the work of G. K. Gilbert about 1914.[10] More recently, valuable work on the filling of reservoirs has been done by T. U. Taylor,[11] M. Am. Soc. C. E. A comprehensive description of the silt problem of the Lower Colorado was published by the late Carl Ewald Grunsky, Past-President, Am. Soc. C. E.,[12] in 1929. Another excellent report[13] on the Colorado River is that by the late Samuel Fortier, M. Am. Soc. C. E., and by H. F. Blaney, Assoc. M. Am. Soc. C. E.

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[8] Prof., Hydr. Eng., Univ. of Iowa, Iowa City, Iowa.

The last few years, however, have seen an active interest in this subject in this country due largely to the return of the Freeman Scholars from Europe, where active work has been going on for many years, and to the importance which the problem has assumed in connection with the control of the Mississippi, Missouri, and Colorado Rivers, and in a number of beach and harbor problems. The subject is now, or has recently been, under active study at the U. S. Waterways Experiment Station at Vicksburg, Miss., the National Hydraulic Laboratory, the Bureau of Reclamation Laboratory, and at the laboratories of the Universities of California, Iowa, Minnesota, and the Massachusetts Institute of Technology. The result of this activity has been a considerable increase of knowledge in the science of transportation and control of the solids load.

Because of the great range of conditions, the solids load problems of the United States have had many aspects. The principal ones were the silting of reservoirs and the control of rivers, generally in fine material, for navigation improvement. The problems in other countries have also depended on local conditions. Formerly, the studies in Europe dealt principally with the control of rivers for navigation, and were concerned with bed-load movement, generally of relatively coarse material. More recently there has been considerable interest in the bed-load movement of finer material, as an outgrowth of efforts to represent the relatively coarse river bed material by finer material in model tests. The control of mountain torrents has also assumed considerable importance in recent years, and excellent studies are now (1935) being made of the suspended load problem.

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[9] "The Suspension of Solids in Flowing Water," *Transactions*, Am. Soc. C. E. Vol. XXXVI (1896), p. 239.

[10] "Transportation of Débris by Running Water," U. S. Geological Survey, *Professional Paper No. 86*; and "Hydraulic Mining Débris in the Sierra Nevada," U. S. Geological Survey, *Professional Paper No. 105*.

[11] "Silting of Lake Austin, Texas," *Transactions*, Am. Soc. C. E., Vol. 93 (1929), p. 1681; and "Silting of Reservoirs," *Bulletin 3025*, Univ. of Texas, 1930.

[12] "Silt Transportation by the Sacramento and Colorado Rivers and by the Imperial Canal," *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 1104.

[13] "Silt in the Colorado River and Its Relation to Irrigation," *Technical Bulletin 67*, U. S. Dept. of Agriculture.

Another large field for these studies has been the irrigation systems of India. There the investigations have been confined almost entirely to designing stable channel sections and the elimination of silt from canal intakes. The first of these was the classical paper by R. G. Kennedy,[14] published in 1895. This was followed by papers by Messrs. Lindley, Woods, Bottomley, and Griffith and, finally, the paper by Gerald Lacey.[15] The latter paper has stirred considerable interest in the subject and thorough research is in progress in several of the Provinces to determine the various factors controlling stable channel shapes.

In recent years, the silt problem of the Nile has been thoroughly studied in its relation to the silting of the irrigation ditches leading from it, but little has been published regarding the results. In the last few years considerable attention has been given to the silt in the rivers of China. In that country the problem is probably more vital than in any other. The floods of the rivers of North China are largely due to the tremendous loads of sediment brought down from their loess-covered drainage areas. The famous floods and channel changes of the Yellow River[16] are due to this cause. The silt concentrations in these rivers sometimes attain surprising values. S. Eliassen, Assoc. M. Am. Soc. C. E.,[17] reports that on the Ching River, "silt contents of 48% by weight were observed, and at one time 51%. The silt content often went above 30% and would stay above 30% for several days." He also states that in the 1933 flood the Yellow River at Sanchow Honan carried 39% by weight. O. J. Todd, M. Am. Soc. C. E., has reported 23.09% by weight on the Fen River,[18] and a maximum of 38% by weight has been reported in the Yung Ting River.[19] The only records in the United States comparable with these, as far as the writer knows, are those on the San Juan River, a tributary of the Colorado. A maximum of 40.8% by weight has been reported for this river by Howard and Love.[20]

It may seem queer that so little progress toward the development of a definite science should be made in a subject to which so much study has been given, but when one reflects on how many factors are involved, it is not surprising. The material transported by any section of a stream is the result of the interaction of two groups of factors. In the first group are those which influence the quantity and quality of the load brought down to that section of the stream. In the second group are factors which influence the capacity of the stream to transport load. A tentative list of these factors is as follows (these are not all independent and, in some cases, their effect is not definitely known):

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[14] "The Prevention of Silting in Irrigation Canals," *Minutes of Proceedings*, Inst. C. E., Vol. 119, p. 281, 1895.

[15] "Stable Channels in Alluvium," *Minutes of Proceedings*, Inst. C. E., Vol. 229, p. 259.

[16] "Flood Problems in China," by the late John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 1436.

[17] *Journal*, Assoc. of Chinese and American Engrs., September, 1935, pp. 26 and 29.

[18] Fen Ho Rept., 1933, Plate XXVI.

[19] The Abridged Rept. on the Radical Improvement Scheme for the Yung Ting Ho. 1934, p. 9.

[20] *Engineering News-Record*, Vol. 105, p. 620.

Group 1.—Solids Brought Down to the Stream.—

(*a*) Quality: Size; specific gravity; shape; and dispersion.
(*b*) Quantity: Geology of water-shed; magnitude and distribution of rainfall; vegetal cover; cultivation and grazing; and erosion.

Group 2.—Capacity of Stream to Transport Solids.—

(*a*) Shape of Stream Prism: Depth; width; form; and alignment.
(*b*) Hydraulic Properties of the Stream: Slope; roughness, hydraulic radius; velocity; velocity distribution; turbulence; tractive force; temperature; and uniformity of flow.

Factors in Group 1 are subject to so many variations, not only between streams, but also at a given point of one stream, that analysis of any case in a quantitative way is impracticable. They resemble closely the factors controlling stream run-off, and, as in the case of run-off, about the only way they can be treated quantitatively is to observe them over a long period and record the results. Like records of stream run-off, solids load records are very valuable. In gathering so large a number of records from the scattered sources, Mr. Stevens has performed a valuable service. Too often a decision as to the construction of a project must be made before it is possible to collect sufficient data on the load of that stream, and recourse must be had to comparisons with loads observed on other streams. For this purpose, Mr. Stevens' data will be found to be invaluable. The present situation in regard to the solids load is much the same as that which existed in the case of stream run-off in the Nineties. In the future engineers may expect more solids load records to be collected until data will be available on all the large streams in the United States.

Although the prospect of a quantitative determination of the load brought down, derived from the various factors involved, does not seem practical, the study of the other phase, the capacity of a stream to transport material of a given quality, is making considerable progress. The relations between a number of the factors involved from the science of hydraulics is already known and the remaining ones are being subjected to close study. It is possible to study many of the these relations in the laboratory where the conditions can be controlled and the effect of varying only one factor at a time observed.

In general, the solids load is transported both in suspension and by rolling along the bottom as bed-load. The intermediate stage of saltation is usually unimportant. Apparently, the laws governing these two forms of transportation are different. The most recent work on the science of bed-load movement has been along the line of the relation between load and tractive force, and a solution of this problem seems nearer than that of the suspended load. Considerable study is also being given to the determination of laws of the transportation of solids in suspension. For many years this has been a matter of controversy among engineers. In Hooker's[9] thorough review of the pertinent literature first published about 1896, he set forth the two principal theories: (1) That the solid material is held in suspension by the difference in velocities between adjacent filaments of water; and (2) that it is held in suspension by the upward movement of currents or eddies in the flowing water. Both these theories are still widely held to-day and progress will be accelerated when the false one is abandoned. The first of these theories was developed by Dupuit[21] who rotated a glass of water containing sand grains and from the action of the sand concluded that the particles tended to move toward the fastest moving water, and, therefore, were lifted toward the surface in a stream.

The writer has never been able to accept this theory for several reasons One is that according to it there could be no material in suspension above the point of maximum velocity in a stream, since above this point the velocity gradient would usually be downward and, therefore, there would be no force lifting the particles against the force of gravity. It is well known, however, that sediment does occur in suspension above the point of maximum velocity. Furthermore, if Dupuit's theory were valid, in a dredge pipe through which water and sand were flowing at high velocity, because of the high velocity gradients which would occur toward the center of the pipe, there would be great forces tending to move the material toward the center of the pipe and hold it there. This would leave the remainder of the pipe filled with practically clear water. No such action has been observed. A third reason is that in large, deep channels, the differences in velocity at the two sides of a small particle of silt are so minute that it seems unreasonable to believe they could generate enough force to hold the particle up against the attraction of gravity.

In order to inquire more fully into the claims of the Dupuit theory, the writer has recently experimented with sand in a revolving glass vessel, as Dupuit did, and finds as much evidence to support the turbulence theory as that of Dupuit. As a result of these experiments and the three reasons mentioned previously the writer is convinced of the fallacy of the Dupuit explanation of the suspension of solids in flowing water and confirmed in the belief that solids are carried in flowing water by the upward currents and eddies.

It is not difficult, however, to account for the persistence of the Dupuit theory through many years, in spite of the unanswerable objections which have been raised against it. Although the velocity gradient does not produce the forces causing the solid particles in suspension to move against the force of gravity, a high velocity gradient usually indicates high turbulence, and, therefore, a region of high suspending ability. The velocity gradient and suspending forces, therefore, are related but not, as Dupuit supposed, in a cause and effect relation.

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[21] "Etudes," par. J. Dupuit, p. 220.

The most recent studies in the suspended load problem from the theoretical standpoint in this country have been along the line of relating suspended material and turbulence. The work of Professors O'Brien and Leighly has already been mentioned in the paper. Since the paper was written, an additional paper by Professor Leighly,[22] and one by Professor von Kármán[23] have added much to an understanding of the problem.

Related to the problem of turbulence and suspended load transportation is a striking phenomenon which one often sees on the surface of a canal or stream of very muddy water. This is a streak of practically clear water, surrounded by very turbid water. As this phenomenon throws some light on forces keeping particles in suspension it was closely observed by the writer. The cause of this phenomenon seems to be, as follows: When the water at the surface of a flowing stream is undisturbed for a short period the silt settles down, leaving at the surface a thin unnoticed layer of desilted water. Throughout the stream, however, there are eddy currents rising from within the water prism to the surface. In moving through the mass of water, they encounter comparatively little resistance on their upward course, the action of the force of gravity being balanced by a downward movement of an equal quantity of water in another part of the stream. When they reach the surface and attempt to rise higher, their motion is not balanced by an equivalent downward motion and their upward momentum, which is relatively small, is quickly overcome by the force of gravity. These upward currents then spread out in an umbrella shape in all directions near the surface of the stream. If two of these upward eddies come up on opposite sides of any area where there is a slight layer of clear water, as previously described, the eddies spreading out on the surface squeeze the clear water layer between them together, until it forms a narrow streak of clear water which has sufficient depth to be apparent to any observer.

In addition to being a noticeable phenomenon, this action probably is an important factor in the mechanics of the transportation of solids in suspension, as it seems to explain the reason why the maximum velocity in a stream occurs below the surface, which must be taken into account in any complete and rational treatment of the suspension of silt in streams. The mass of water in these eddies in passing up from the bottom movés through the main body of the water and retards its longitudinal motion somewhat, because having originated at the bottom, the longitudinal motion of the eddy is slower than the average of the mass. The greatest effect, however, is at the surface, where the eddies spread out and join together to form the top layer, with a lower longitudual velocity than the water immediately below. The eddies which cause the visible clĕar-water streaks previously mentioned are of considerable magnitude, but smaller eddies no doubt similarly contribute to the upper, slow-moving layer. There seems to be no reason why the limitation to motion imposed by the surface of the stream should not have a similar effect on the intimate internal motion which makes up the turbulences in the flowing water. As the observation on the silt transported in a stream led the writer to a better understanding of an idea of the internal mechanics of the flow of water in an open channel, it seems probable that knowledge of the internal mechanics of flowing water and its turbulence may perhaps be reached more efficiently by the study of the transportation of solids in suspension than by observations on clear water flowing in channels.

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[22] "Turbulence and the Transportation of Rock Débris by Streams," *The Geographical Review*, Vol. 24, No. 3, July, 1934.

[23] "Some Aspects of the Turbulence Problem," Aeronautics and Hydraulics Div., Am. Soc. M. E.

FRANK E. BONNER,[24] M. AM. SOC. C. E. (by letter).—Contributions to technical literature on the important subject of silt movement and control are increasing in desirable degree. The comprehensive paper by Mr. Stevens provides a notable addition, and the author deserves particular commendation for the large amount of effort obviously expended in the correlation of the statistical data presented by Table 6. The inclusion of the large number of determinations made by the Corps of Engineers, U. S. Army, during the past few years throughout the Mississippi Basin, bridges a gap existing in previous tabulations. It is evident, however, that a number of the recorded observations cover too short a period for the determination of dependable normals, and some caution will have to be exercised in the use of the data for such purposes. The same is true in regard to part of the voluminous data abstracted from *Water Supply Paper 274* of the U. S. Geological Survey. For example, Item No. 2, Table 6, is based on measurements made at Palisade, Colo., on 160 days of an elapsed period of 399 days in 1905 and 1906. The silt load value is given as 0.49 part per thousand. However, analysis of the same data reported in *Water Supply Paper 617* gives 4 000 000 acre-ft and 1 758 700 tons (950 acre-ft at 85 lb) as the estimated annual water discharge and silt load, respectively. These data give a silt load value of 0.323 part per thousand which is less than the tabular value by about 34 per cent. Similar discrepancy between different interpretations of results is illustrated by Item No. 3, Table 6, and others. It is thus apparent that, for detailed uses, reference to the original sources of data may be desirable.

The paper refers to the reservoir silting problem with considerable emphasis on the hazards of ultimate depletion. The writer prefers to view this problem with less alarm concerning the possible misfortunes of the remote future. Somehow, these threatened disasters due to "impending exhaustion" of different elements of natural resources never seem to materialize with the serious results anticipated. Imminent shortages automatically set in motion the ingenuity of mankind toward the provision of substitutes which generally serve better than the materials displaced. Consider the lumber situation for example: Decreasing per capita consumption due to rapid increase in the use of wood substitutes has relieved the pressure on the timber resources of the United States. Experts concede that "the threat of a timber famine which loomed so large some 25 years ago has somewhat worn off."[25] Likewise, in the case of the water power resources; several decades ago extensive public concern arose over their adequacy to meet future requirements. To-day, water power development is in the discard, due to the amazing increase in the efficiency of steam-electric generation and the widespread production of cheap fuels. Past bugaboos relative to impending scarcity of fuels, farm lands, fertilizers, and other resources have gone through the same cycles. Present fashion tends toward "saving the soil from erosion" and this movement will probably have to run its course.

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[24] Cons. Engr., Piedmont, Calif.

The accumulation of silt, of course, is one of the elements to be considered in the case of every proposed reservoir. Rapid depletion of capacity may affect the economic plan of the project and, perhaps, even the design of the dam. Fortunately, however, the problem is not serious in many cases. Rapid depletion of capacity is rather well restricted to the reservoirs on the silt-laden streams of the arid Southwest and to those with capacities disproportionately small in relation to the area of their tributary water-sheds.

Elephant Butte and Boulder are frequently cited to illustrate the transitory character of reservoirs. Silt conditions at both structures were fully recognized by the builders. It is probable that no streams in the United States have been studied in regard to silt transportation more thoroughly than the Rio Grande and the Colorado River. The excessive débris-producing capacity of these water-sheds will be continuous. Herman Stabler, M. Am. Soc. C. E., has showed[26] the impracticability of arresting erosion in these arid regions by so-called "water-shed protection" methods.

Actual accumulations of silt in Elephant Butte Reservoir, since its completion in 1916, correspond closely with the forecasts based on studies made prior to its construction. Unless serious errors have been made in the economic programs of these projects, it is reasonable to assume that the capital outlays will be amortized long before sedimentation of the reservoirs seriously impairs their usefulness. When that time eventually comes, any desirable restoration of capacity will be a problem for the engineer of that distant day. If engineers attempt to provide a solution now, it would propably be found entirely wrong long before the time arrived for its application.

On the basis of present conditions, ultimate construction of new storage would doubtless be considered the proper remedy; but under the unpredictable conditions of the dim future dredging, sluicing, or some other means may be found more economic. Moreover, considering the amazing march of science in the recent past, it is conceivable that water storage requirements 50 yr or 100 yr hence may be on an entirely different basis than the present. History teaches the futility of trying to peer very far into the future.

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[25] "Place of Forestry in Land Utilization Program", by Raphael Zon, *United States Daily*, July 1, 1932.

[26] "Rise and Fall of the Public Domain," by Herman Stabler, *Civil Engineering*, September, 1932, p. 541.

Various estimates have been made relative to the average rate of surface leveling in the United States based on silt movement. Mr. Stevens cites an estimate published twenty-five years ago by the U. S. Geological Survey giving the annual total of suspended matter at approximately 500 000 000 tons. The accumulation of more recent data supports the writer's estimate of about 933 000 000 tons.[27] The frequently quoted figure of 3 000 000 000 tons, attributed to the Federal Soil Erosion Service, is clearly excessive.

MORROUGH P. O'BRIEN,[28] ASSOC. M. AM. SOC. C. E. (by letter).—The compilation on the silting of reservoirs which is contained in this paper shows clearly the economic importance of the silt problem and the necessity for a continuation of investigations in this field. A paper by Dr. Fritz Orth[29] dealing with the same subject and published almost simultaneously contains much additional information, including the data given in Table 10. All the numerical values have been converted to the same units as those used by the author.

The author mentions several estimates of the percentage of the total silt transportation which moves as bed-load. Any such estimate must be made on the basis of some arbitrary quantitative definition of what is meant by bed-load. As the tractive force moving the material increases, the region of bed transportation gradually expands until the concentration at or near the surface becomes appreciable and a condition of suspension is said to exist. Measurements show, and the theory of turbulent flow indicates, that the concentration of material should increase downward, the rate of increase becoming greater for larger materials. When materials in suspension are being transported over a mobile bed containing the same material, it is doubtful whether the bed has a well-defined surface along which material could be said to move as bed-load. In the laboratory, bed-load is defined as that material caught in a properly designed trap, whereas in studies of sedimentation in reservoirs or lakes it is the difference between the volume of material deposited and that accounted for by measurements of suspended sediment and discharge. Whatever the definition may be, the percentage of bed-load rises abruptly to 100 as movement starts and then decreases as material is thrown far enough into the stream to be classed as suspended material. Therefore, it is different for every stage of the river.

Some progress has been made in the application of the theory of turbulent flow to the distribution of suspended sediment[30] and the greatest obstacle at present is the lack of precise data. W. Schmidt[31] has developed the basic equation and some data are available for checking it, but much more will be needed before its validity can be fully established. The basic data needed are: Measurements of the vertical distributions of velocity and sediment; and the

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[27] *Transactions*, Am. Soc. C. E., Vol. 100 (1936), p. 315.

[28] Associate Prof., Mech. Eng., Univ. of California, Berkeley, Calif.

[29] "Der Verlandung von Staubecken," von Fritz Orth, *Die Bautechnik*, June 19, 1934, Vol. 12, No. 26.

[30] "Review of the Theory of Turbulent Flow and Its Relation to Sediment Transportation," by Morrough P. O'Brien, Assoc. M. Am. Soc. C. E., Am. Geophysical Union, Section of Hydrology, 14th Annual Rept., p. 487 (1933).

[31] "Die Massenaustauch," von W. Schmidt, H. Grand, Hamburg, Germany, 1925.

TABLE 10.—SILTING OF RESERVOIRS REPORTED BY F. ORTH

Reservoir	Stream	Drainage area, in square miles	Mean annual supply, in thousands of acre-feet	ORIGINAL CAPACITY		Period, in years	SILT DEPOSITED		
				Thousands of acre-feet	Percentage of annual supply		Total, in acre-feet	Percentage of original capacity	Annually, in acre-feet
Faal	Drau	5 140.	....	....	......	6.5	2 420	....	373.
Jettenbach	Inn	4 730.	9 090.	....	......	6	1 750	....	292.
(Genfer See)	Rhone	2 860.	....	72 100.	......	.....	......	....	2 400.
Pernegg	Mur	2 420.	....	4.0	......	1.5	284	70.0	203.
(Bodensee)	Rhine	2 360.	776.	....	......	20	8 010	....	401.
St. Denis du Sig	Sig	1 350	....	2.8	......	8	608	17.4	76.
Cheurfas	Meckerra	1 160.	23.3	10.1	41.7	50	4 950	48.8	99.
Gokak	....	1 080.	....	20.8	......	33	4 910	29.1	149.
Avignonnet	Drac	772.	890.	0.8	0.09	8	810	100.	101.
Quinson	Verdun	695.	....	1.1	......	5	730	67.9	146.
Medina Lake	Medina River	602.	....	253.	......	13	2 680	1.1	207.
Rosshaupten	Lech	550.	1 590.	....	......	7	1 780	....	255.
(Bieler See)	Aare	532.	1 710.	1 010.	58.6	20	5 430	0.5	272.
						16	2 030	0.2	126.
Kallnach	Aare	525.	....	1.5	......	6	810	55.6	138.
Perolles	Sarine	487.	....	0.8	......	14	810	100.	57.6
(Thuner See)	Kander	414.	994.	5 270.	531.	152	46 000	0.87	303.
(Chiem See)	Tiroler Ache	392.	1 090.	1 790.	164.	34	2 630	0.15	68.9
						1	......	....	160.
Saalach	Saalach	386.	1 010.	2.8	0.28	17	2 340	82.2	138.
Avisia	Avisia	369.	....	1.6	......	8	1 620	100.	203.
Djidovia	....	328.	....	1.6	......	.....	......	....	203.
(Vierwaldstadter See)	Reuss	321.	608.	9 560.	1 570.	27	3 200	....	118.
(Bodensee)	Begrenzer Ache	321.	776.	39 400.	5 070.	24	2 510	....	105.
Loch Raven	Gunpowder River	306.	300.	1.6	0.52	20	1 330	85.0	66.5
Pont du Loup	Drac	290.	....	2.4	......	1.1	1 220	50.0	1 110.
Wallensee	Linth	240.	....	2 030.	......	51	3 030	0.15	60.1
Steyrdurchbruch	Steyr	222.	510.	6.7	0.12	22.5	575	84.	25.6
Cismon	Cismon	192.	446.	10.	0.64	10	1 420	14.2	142.
Monte Reale	Celina	168.	567.	1.	......	1	681	....	681.
Urfttalsperre	Urft	146.	146.	36.9	25.3	16	16	....	1.0
Wetzmann	Gail	125.	303.	0.5	0.16	1	486	100.	486.
Marklissa	Queis	120.	138.	12.2	8.82	25	152	1.25	6.1
Goldenstraum		108.		8.5	6.17	9	62	0.73	7.0
Pont	Armonoon	106.	373.	4.3	11.5	50	61	1.41	1.2
Lake Wichita	Holliday Creek	69.	....	14.0	......	25	490	3.48	19.5
Breitenhain	Weistritz	56.	51.1	6.5	12.7	15	112	1.72	7.5
Tlétat	....	50.	....	0.6	......	.....	......	....	17.8
Dambal	....	43.	....	2.6	......	48	462	17.7	9.6
Muchkundi	....	26.	....	16.2	......	41	2 590	16.0	63.2
Tarcento	Torre	24.	90.9	0.12	0.13	12.5	122	100.	9.7
(Wolfgansee)	Zinkenbach	22.	52.4	501.	957.	18	108	....	6.0
Grosbois	LaBronne	11.	8.9	7.5	83.7	150	24	0.33	0.16
Lete	Lete	11.	....	0.8	......	17	11	1.4	0.66
Laragnina	Gorzento	9.8	....	0.8	......	20	324	38.1	16.2
Marinkop	....	6.3	....	1.0	......	46	169	17.0	3.6
Pontebba	Vogelbach	3.9	7.8	....	......	18	227	....	12.6
Brux	Einsiedlerbach	3.2	2.8	1.3	47.1	.....	......	....	0.8
Tillet	Tillet	2.1	0.73	0.42	57.8	.....	24	5.77	0.02
Saifnitz	Luscharibach	1.7	3.39	0.02	0.7	1	24	100.	24.3
McKinney	....	1.4	....	0.01	......	10	12	100.	1.2
Camperdown	Umlassfluss	.....	....	1.86	......	16	1 000	53.5	62.4
						8	316	17.0	39.7
Holtwood	Susquehenna	.....	28 400.	54.8	0.2	18	13 000	23.7	721.
Bhatodi	Mehekari	.....	6.77	3.50	51.7	50	2 610	74.6	51.9

fall-velocity of the material in suspension. Useful supplementary data are: The size of the bed material; the hydraulic radius or depth; and the energy slope. This need is mentioned in the hope that engineers taking sediment samples will also obtain the other data mentioned.

In Table 10 the variation in silt deposited per thousand parts of water supply, by volume, is from 0.028 to 16.3. Interesting as these data are, they do not provide a means of predicting the life of a proposed reservoir and this is the principal problem involved. Most of the factors causing this wide variation are mentioned by the author. Further progress in this field appears to require a breaking down of the problem into two phases, of which one is the volume of sediment transported by the stream and the other, the characteristics of reservoirs as a silt trap. Laboratory and field studies can contribute much valuable information on these points, but any prediction of the rate at which sediment will be transported to a reservoir is necessarily subject to all the uncertainty involved in forecasting stream flow.

On the assumption that a given project will continue to need the volume of effective storage originally provided for it, the monetary damage caused by silting should not be computed from the cost of the original reservoir but from the cost of the necessary additions. As the sites which are less expensive per unit of storage are developed first, silting involves an increasing rate of damage as it proceeds and it may ultimately be impossible to provide sufficient storage capacity at any cost. At present, construction of additional reservoirs appears to be the practical solution, but the time may come when by-pass channels and other silt-controlling works, costing perhaps more than the dam itself, will be found to be economically justified. Such structures are now used successfully at diversion dams, and it seems possible that conditions at some storage dams might warrant the construction of works for diverting a portion of the silt.

Harry F. Blaney,[32] M. Am. Soc. C. E. (by letter).—The author calls attention to facts which have often been overlooked in the past. The paper is a valuable contribution to the literature on the subject of silt. Since the writer is more familiar with conditions along the Lower Colorado River and the Middle Rio Grande, this discussion will be confined to the areas served by these streams through the reaches mentioned.

As indicated by Mr. Stevens, silt is a menace to irrigated agriculture in some sections. This problem is one of the most serious confronting the people of New Mexico to-day, especially in the Rio Grande Valley. This valley supports about one-third of the people of the State, as well as a large population in Texas. The silt problem of the Middle Rio Grande may be divided into three phases: (*a*) The silting of the river channel above Elephant Butte Reservoir; (*b*) the deposition of silt in the Elephant Butte Reservoir; and (*c*) the accumulation of coarser silt in the main stream channel below the Elephant Butte Dam, due to lack of seasonal flood flushing. The Federal Governments of the United States and Mexico have started a project to rectify the conditions below the dam. Recently, the State Planning Board of New Mexico has recommended that a relief project be undertaken to control erosion on the Rio Puerco, which is the principal source of silt in the Upper Rio Grande water-shed, and reports[33] that:

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[32] Irrig. Engr., Bureau of Agri. Eng., U. S. Dept. of Agriculture, Los Angeles, Calif.

> "The Rio Grande Valley is in great danger of destruction as a habitation for man, due to the rapid erosion of its tributaries and silting of the river itself. In approximately 50 years the great Elephant Butte Dam and irrigation works connected with it will lose their usefulness, due to the silting up of the reservoir and the destruction of its storage capacity."

It is estimated in this report that silt endangers an investment of $100 000 000 in land and improvements. The river above the Elephant Butte Dam for a distance of 150 miles has such a moderate fall that in many places a large quantity of silt is deposited. This building up of the river channel is increasing flood hazards and water-logging the bordering lands. Surveys made in 1934 show a rise of the river channel at La Joya (just below the mouth of the Rio Puerco) of 7 ft since 1918; at Albuquerque, 2 to 4 ft; and at Alameda, 1 to 3 ft. The most important stream that enters the Rio Grande above Elephant Butte Reservoir is the Rio Puerco, and its control would go a long way toward solving the silt problem.

As indicated by the author, the Boulder Reservoir will not completely solve the silt problem on the Lower Colorado River. On March 1, 1935, about 110 000 acre-ft of water had accumulated in Boulder Reservoir, and the water leaving it was clear. However, an additional silt load will undoubtedly be picked up by the river below the dam for some time to come. If the experience on the Rio Grande below Elephant Butte Dam is repeated on the Colorado River, the clear water released from either Boulder Reservoir or from Parker Reservoir will pick up a load of silt and scour the bed, progressively lessening in depth but extending in distance. With an average discharge of 15 000 to 20 000 cu ft per sec, the bed of the river will probably be washed in time so that most of the fine silts will be removed and only the bed sands left to line the channel. Occasional floods entering the river below the Boulder Dam or the Parker Dam will carry large quantities of silt and will cause a temporary increase of suspended silt in the river at the Imperial Valley Diversion.

The character of the silt will be changed materially, since it will be a mixture of silt picked up from the bed of the river. This will consist of silts from different tributaries of the river, no one of which will predominate.

There is a difference of opinion among engineers as to how many years it will take the regulated river to clean its channel of the finer silts and cease to carry any appreciable quantity of material in suspension.

With the river unregulated, a low percentage of suspended silt usually occurs in the early summer at the peak of the annual flood resulting from the melting of the snows in the upper water-sheds, while the river water generally carries the highest percentages of silt in the late summer months or fall when erratic floods are caused by rains in areas drained by its lower tributaries.

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[33] "Silt Control on the Rio Puerco," by S. R. DeBoer, Mimeographed Rept., New Mexico State Planning Board, November, 1934.

The writer believes that the low percentage of silt carried in suspension by the river in past years is a good index of what may be expected immediately after Boulder Reservoir becomes effective in desilting the water. The annual quantity of silt carried in suspension in the river will be reduced by at least one-half at the Imperial Valley Intake, and for the most part, the large quantities of objectionable fine silt which have been present in the water during the peak of the irrigation season will be eliminated.

When the new Imperial Diversion dam and heading are completed about five miles above Laguna, a portion of the present main canal serving the Yuma Project will probably be abandoned, and its water brought through the upper end of the All-American Canal. Laguna Dam will remain, to serve as a drop structure in the river, controlling recession of grade below Imperial Dam. It has been estimated that it will take from three to five years to complete this project. Meanwhile, Imperial Valley will receive water through the present system, and silt conditions will be about the same as heretofore.

The effect of silt on the cost of operation and maintenance of the irrigation system in Imperial Valley has been a serious one. The use of silty water for irrigation as it comes from the river means the handling of an enormous quantity of silt, which, of necessity, must be deposited in the various canals of the system, or on the land itself. In some sections it is necessary to keep dredgers continually at work in order to maintain the canals and laterals in condition to carry the required quantity of water. The disposal of this silt dredged from the canals is also fast becoming a perplexing problem. The banks of the canals are constantly being raised and widened. The Imperial District spends approximately $500 000 per yr for silt disposal. In addition, there is the cost of cleaning farm ditches and re-leveling of land because of silting. A survey by the U. S. Bureau of Agricultural Engineering in 1933 indicated that the total cost of cleaning ditches and re-leveling land as a result of silt deposits in Imperial Valley ranged from 55 cents to $4.63 per acre per yr and averaged $1.80. This is in fair harmony with the usual estimate of $2.00. Applying the latter figure to the 400 000 acres under irrigation in the Valley, the cost would reach $800 000 per yr. Adding to this the amount spent by the Imperial Irrigation District gives a total annual cost of silt disposal of more than $1 250 000.

Several different plans for the exclusion of silt from the All-American Canal, both bed and suspended load, have been considered. Research investigations being conducted by the U. S. Bureau of Reclamation and the Bureau of Agricultural Engineering are expected to furnish the fundamental data for the preparation of the final designs for desilting works at the new Imperial Dam. The Bureau of Reclamation has been successful in eliminating about 50% of the suspended silt and most of the bed silt in the desilting operations at Laguna Dam.[34] The writer feels confident that when the new

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[34] "Silt in the Colorado River and Its Relation to Irrigation," by the late Samuel Fortier, M. Am. Soc. C. E., and Harry F. Blaney, Assoc. M. Am. Soc. C. E., *Technical Bulletin 67*, U. S. Dept. of Agriculture, 1928, p. 58.

Imperial Heading is finally built it may be expected to eliminate practically all the bed silt and at least 50% of the suspended silt carried by the river water. With the regulated river carrying 50% of its former quantity of suspended silt only 25% of the former suspended load would enter the irrigation system. However, it will be advisable to retain some silt in the water so as to prevent the growth of moss. Under these conditions the annual expenses to the individual farmer of cleaning farm ditches and re-leveling land, and to the Imperial Irrigation District for silt disposal, would be reduced considerably.

W. W. WAGGONER,[35] M. AM. SOC. C. E. (by letter).—In an exhaustive and comprehensive manner Mr. Stevens has presented "the silt problem" for discussion. His description of the silting presents a dark and important picture. The tabulation of the quantities of silt transported by the various rivers is indeed startling. It shows how much greater the mud flow is from the sedimentary and fragmentary formations than from the crystalline rocks.

The answer to the question of preventing the silting of a reservoir is to build another one above it, to impound the silt, and also to encourage the deposition of silt above the reservoir as in the case of Lake McMillan which the author mentions. It would be interesting and constructive to the discussion of the paper if a detailed description of the silt impounded at Lake McMillan could be given.

The gold mines of California are drained by the Yuba River, a tributary of the Sacramento River. This area has been the scene of the most important operations in hydraulic mining. Due to his long contact with the gold-mining industry, the writer has made a detailed study of the débris problem in its various phases and desires to record the débris deposit in the Yuba River, and its relation to the subject under discussion.

Denudation is more than the eroding of the mountains, and the filling of the rivers with sand, or the forming of deltas at the river's mouth. It is one of the grandest features of the Creator's plan for maintaining the fertility of the soil for the benefit of mankind throughout the ages. Consider the nature of the soil: Professor E. W. Hilgard[36] has stated that the soil is composed of 3 to 5% of organic matter, and 97 to 95% of inorganic matter from which the plants obtain the nine mineral elements that enter into their growth, namely, phosphorus, potassium, calcium, magnesium, sodium, iron, silicon, chlorin, and sulfur. These elements combine to form the common minerals such as feldspar, mica, and hornblende, which compose the crystalline rocks.

It may be stated that the valleys are the granaries from which the people obtain the major portion of their food, and that the mountains are the storehouses from which the depleted soils receive a new supply of fertilizer. The action of the elements is to break up the cementing power that holds the minerals in a solid mass, and to form individual grains that compose the fine, inorganic part of the soil. These, in turn, are also oxidized, and the elements are slowly dissolved, and carried into the soil for plant food. It is remarkable how a thin soil, with an adequate precipitation can produce a great forest growth, due to the abundance of plant food.

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[35] Hydr. and Min. Engr., Nevada City, Calif.

[36] "Geochemistry," by E. W. Hilgard, *Bulletin 491*, U. S. Geological Survey.

For example, consider Central California, on the western slope of the Sierras, with its great forests and its abundance of precipitation. During the dry seasons the erosion is negligible, but during the wet, or flood, seasons it becomes of great volume. The storms beat against the mountain slopes where they dissolve the plant food and carry it away with the particles of sand into the ravines and creeks. These waters concentrate into streams which debouch into the valley. By so doing, great canyons are formed and mountains leveled. With the heavy grades, and high velocities, the streams are capable of carrying all the silt brought to them. When the valley has reached the flatter grades, sand and gravel are deposited upon their beds. The finer material is carried onward. During floods the water spreads over the valley; some of it soaks into the ground where, at least, a part of the dissolved plant food is left behind to act as a fertilizer.

With the occupation of the valley these flooded lands are very attractive for farming. Levees are constructed to protect these lands from the floods and to get rid of the so-called nuisance. The finer sands, together with the dissolved matter, of necessity are transported to the bays and the ocean.

As an illustration of the fertility of such land may be cited the case of a farm with 270 acres in wheat, 10 acres of which were measured and shown to produce 50 470 lb. In 1858, the wheat from this tract was so rich in gluten that it was in great demand in the markets of New York, N. Y., and Liverpool, England. To-day, 1 000 lb per acre is an average yield, but the wheat is so poor in gluten that it has to be mixed with other wheat from new lands to make a marketable flour, due to the depletion of new plant food.

Consider next the history of a great débris deposit upon the Yuba River, which is comparable to that at Lake McMillan. After a long period of industrial depression in the United States, gold was discovered in far-away, unknown California, on January 24, 1848. This discovery electrified the world. A man could go to California, accumulate a fortune by gathering gold from its sands, and then return home to live in peace and comfort for the remainder of his days!

Never was there such an army of active, energetic, educated, and resourceful men as gathered in California. While many followed their original plan, most of them remained to found a State. They had to be housed and fed. This led to the opening of farms, and the setting out of orchards. All the best lands were brought into cultivation. Ten years after the discovery of gold there was a greater acreage of farms on the Yuba drainage area than there is to-day. These industries needed lumber to carry on their operations and for the towns that grew up not only in the mountains but in the valley as well. In 1859, there were ninety-nine saw-mills in the three counties covering the Yuba drainage area. The long, dry seasons were ideal for lumbering.

It can well be imagined how the ground was plowed up by the ox teams that were used for snaking the logs to the mills. This continued for about thirty years, or until the wonderful forests (which would be hard to find elsewhere) were exhausted.

In November, 1861, a cycle of wet winters began, which continued for twenty years, and produced the worst floods in the later history of the State. During December, 1861, and January, 1862, four great cyclonic disturbances occurred that enveloped the entire Pacific Coast, and produced floods, from the Columbia River to and including Los Angeles, Calif. In the Sierras it was commonly called "The Deluge." Upon the middle belt of the Sierras during these months 75 in. of rain fell, and near the summit 42 in. of rain and 50 ft of snow fell; 5 and 6 in. per day fell during a 4-day period. A seasonal total of 109 in. occurred in the gold belt.

Combining the stories of the historians of that period the water flowed into the valley at a greater rate than could be discharged through the Carquinez Straits, and the Golden Gate. From the foothills of Mt. Shasta to those leading up to the Tehon Pass the great California Valley was a vast lake not unlike the shape and size of Lake Michigan. The tides ceased to act, and a stream of thick, muddy water flowed through the Golden Gate, which discolored the ocean for a distance of 40 miles from the land. The bays became fresh, killing the oysters which were planted near Oakland.

The Yuba River discharged its quota coming from its various tributaries surcharged with mud and sand gathered from the mines, fields, and forests. The Lower Yuba River bottoms was a narrow valley, 1 mile to 3 miles wide, carved in the peneplain of the valley. The mud and sand buried the farms and orchards 15 ft deep; with the succeeding years came more floods and with them the deposit increased in depth until it was raised about 10 ft above the general valley plain. Levees were constructed to confine the deposit. One of the results that occurred as soon as the deposit reached the level of the valley floor was to raise the water-table on the adjacent lands. It was brought out in the litigation which followed that they were known as "redlands" and were poor even for grazing. After the raising of the water-table they were used for general farming.

In the early days the gold was recovered from the gravel by washing in sluice-boxes, the gravel being moved principally by picks and shovels. In 1870, mechanical equipment was invented by which means a considerable volume of water could be used under high heads, and a large volume of gravel could then be moved at a reasonable cost. These giant machines are now used in the construction of hydraulic-fill dams and similar structures. This equipment led to the consolidation of small holdings and the formation of large companies financed in San Francisco, New York, and London. Reservoirs, the highest in the United States, were built, and long lines of canals were constructed to the mines. These structures were completed about 1876.

At the same time the agitation against the débris flow that began after the floods of 1862 and (were charged against the mines) took the form of litigation which culminated in the injunction that closed the mines in 1884. During the litigation, the State Engineering Department of California was created. Elaborate surveys of the navigable rivers and the Yuba River were made. In 1894, the Army engineers made a series of borings to determine the depth of the deposit. These borings were supplemented by those of the gold dredgers.

Combining the various surveys it was found that the deposit which was 20 miles long, had a maximum width of 3 miles, covering 16 000 acres and containing 600 000 000 cu yd. It was 20 ft deep at the river's mouth, 35 ft deep at the edge of the foothills, and 80 ft deep 5 miles higher up the Yuba River. The grade of the original bed was 5 ft per mile. After the fill was made, the grade per mile was 2½ ft at the mouth, 10 ft at the middle zone, and 20 ft on the upper reaches.

The deposit soon became covered with a dense growth of willows. It remained intact with a few, low-water channels coursing through it. About 25 yr ago the Government enlarged the main low-water channel until now (1935) it will carry a moderately large flood. Much of the adjacent land that was supposed to be destroyed forever was cleared and planted to pears, the uncleared land being held at more than $250 per acre.

In making the deposit the law of transporting material is probably the same as that which obtains in the sluices of the hydraulic mines. It is known that with water flowing 10 to 12 in. deep the transporting capacity varies directly as the slope. Furthermore, with water flowing 2 ft deep, more material will be carried per cubic foot per second than with water flowing 10 in. deep. Mathematically, the transporting capacity of force, F, varies as V^2.

It is interesting to record that fields which were denuded of their soil are now covered with second growth pine and brush. One variety of brush known as "sweet birch" is especially valuable as a cattle food. One enthusiastic cattle man claimed that it was nearly as good as alfalfa, and made it a practice to gather and scatter the seed over his range; while another farmer claimed that a good soil could be formed by it in 25 yr.

The lesson of the débris deposit on the Yuba River is of far-reaching application. By spreading the water over a wide area by low barriers, and by encouraging brush growth, the silt will be deposited and can be raised to any height. The material thus impounded will be prevented from moving down stream, filling reservoirs, or from entering the navigable rivers where it would have to be dredged at a cost of 6 or 7 cents per cu yd. Below the barriers the stream will then be able to scour its bed and increase its capacity. The cost of such impounding will be less than 1 cent per cu yd. It is conservation in the highest degree.

PHILIP R. R. BISSCHOP,[37] ASSOC. M. AM. SOC. C. E. (by letter).—In South Africa the silt problem is no less severe than that encountered in the southwestern part of the United States. Of late years particularly, the rapid rate of silting recorded in many of the irrigation reservoirs has caused considerable concern. In a number of cases the very existence of productive centers of population is threatened, and for what period of time it will be possible to preserve the present water supply is a moot point. Indeed, it is the definite policy of the present Government to limit the extensive further program of irrigation development to those rivers that are least subject to silt encroachment.

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[37] Cons. Irrig. Engr., African Consolidated Investment Corporation, Johannesburg, Union of South Africa.

Typical of some of the conditions encountered in this country is the case of the Sundays River, in the Cape Province. Rising at elevations of 6 000 to 7 000 ft, this river drains about 6 000 sq miles on the south escarpment of the Sneeuwberg Range which divides the Great Karroo Plateau of 4 000 to 5 000 ft elevation from the Small Karroo Plateau of 1 500 to 2 000 ft. In the foothills of this inclined plane, on the main branch of the river, is situated the Van Ryneveld's Pass Reservoir with a catchment of 1 500 sq miles. At the southern edge of the Small Karroo Plateau, where the river starts breaking through the coastal ranges, is situated Lake Mentz Reservoir with a catchment area of 4 500 sq miles.

The entire drainage area can be divided roughly into the "Mountain" Section and the "Flats" Section, the first-named reservoir taking on part of the mountain topography whereas the second takes care of the remainder of the mountain area and all of the "Flats" Section.

The average annual rainfall varies from 14 to 18 in. in the mountains to from 10 to 14 in. in the flats. Approximately 80% of this rain is concentrated in the summer months, particularly February and March. Consequently, the river flows are essentially ephemeral in character and are liable to exceedingly high flood peaks in very short periods of time, falling again with only slightly less suddenness.

Previous to the building of the reservoirs, intermittent irrigation was practised extensively, by what is known as "flood furrow irrigation"; that is, individual small systems diverting water into fairly large canals in order to obtain a maximum of water in a minimum of time whenever available. Such irrigation was necessarily of an exceedingly up-and-down character, incapable of sustaining much more than "snatch" crops and occasional cuttings of alfalfa. With the completion of the reservoirs, land development and settlement occurred along lines quite similar to those experienced in the United States. To-day, both reservoirs sustain intensively developed irrigated areas, subdivided in small holdings on which a struggling population is obtaining a living from citrus, deciduous fruits, alfalfa, and cereal crops.

Below the reservoirs, therefore, there is a cropped area totally dependent on conserved water, and above them, a pastoral area subject to severe droughts and high floods. In the past, the latter area has been subjected to extensive over-grazing and although to-day the more progressive farmers are fully alive to the decreasing grazing capacity of their farms and are actively endeavoring to rehabilitate their veldt, the damage already done by soil erosion is such that it has become a national problem against which the incumbent Government has organized an intensive campaign of research, subsidization of anti-erosion work and rehabilitation of the original vegetation. The benefits of such work can necessarily only become apparent over a considerable period of time and cannot be other than partial in overcoming all erosion. In the meantime, the lower lying reservoirs are gathering the accumulated burden. For example, Lake Mentz, completed in 1922, has silted up to 42.3% of its original capacity (see Table 11), say, in twelve years of life, and Van Ryneveld's Pass is estimated to have lost 16.4% of its capacity between 1925 and July 1, 1933.

TABLE 11.—DEPOSITION OF SILT IN THE LAKE MENTZ STORAGE DAM (ORIGINAL STORAGE CAPACITY, 94 000 ACRE-FEET)

PERIOD BETWEEN SURVEYS;		CAPACITY, IN ACRE-FEET		PERCENTAGE LOSS OF STORAGE	
From: (1)	To: (2)	Inflow (3)	Silt deposited (4)	By volume; $\frac{\text{Column (4)}}{\text{Column (3)}} \times 100$ (5)	Cumulative total (6)
November 30, 1922*	April 30, 1924	95 250	2 200	2.3	2.3†
May 1, 1924	March 30, 1926	111 770	2 900	2.6	5.2‡
April 1, 1926	November 30, 1927	24 630	620	2.5	6.0‡
December 1, 1927	March 31, 1929	443 910	10 000	2.25	16.6‡
April 1, 1929	June 30, 1930	175 070	4 030	2.3	20.8‡
July 1, 1930	December 27, 1931	116 140	2 670	2.3	23.7§
December 28, 1931	January 4, 1932	440 000	9 320	2.1	33.6§
January 5, 1932	June 30, 1933	131 840	3 030	2.3	336.75§
July 1, 1933	December 31, 1934	227 440	5 230	2.3	42.3§

* Date of closure. † Based on actual survey ‡ Mean of Column (5). § Based on previous survey findings.

In the case of Lake Mentz, the raising of the dam has already begun and the levels of other reservoirs (such as Lake Arthur on a tributary of the adjoining Great Fish River) will have to be raised in the near future. Such periodic raising of the dam walls is necessarily limited, if not by physical features, certainly by financial considerations. Already, in a manner very parallel to that experienced in the United States, the administrators of many of the South African irrigation projects have found it impossible to meet their redemption and interest rates, and the Government, which financed the projects, has found it necessary and expedient to write off approximately $25 000 000—the entire capital loans and arrear interest payments of a large number of projects. In agreeing to these "write-offs," the Government stipulated an annual assessment of approximately $0.30 per acre for the special purpose of establishing in each project a reserve fund to cover the cost of periodical heightening of the dam walls. The amount of this assessment cannot possibly be expected to cover the increasing cost of preserving the water supply. What then is to become of the endangered projects? Where further reservoir sites are available, or where the existing reservoir sites are not as yet fully developed, the problem resolves itself to an extension of the already widely held viewpoint that irrigation development is essentially a national function and that, as such, the Government should not only finance and subsidize—either directly or indirectly—the original cost, but also that of preserving the supply.

Further supplies are not always available, however, and existing reservoir sites must ultimately become fully developed if they are not developed already. Furthermore, public opinion may not support the conception that it is the function of the Government to preserve a water supply for an unlimited period. There appears to be only one solution, namely, to recognize frankly that some irrigation projects are doomed to a definite span of life and will have to revert back to the original type and manner of flood irrigation, to prepare the irrigators against such time and contingency, and, in the meantime, to delay the end as long as possible by actively combating sheet and river erosion within each catchment area.

The detailed technical data pertaining to the deposition of silt in the Van Ryneveld's Pass Dam are:

Drainage area, in square miles	1 475
Mean (estimated) annual supply, in acre-feet	38 000
Original full supply capacity, is acre feet	64 200
Percentage of annual supply, $\left(\frac{64.2}{38.0} \times 100 =\right)$	169
Date of reservoir completion	1925
Elapsed time, in years, between completion of reservoir and last capacity survey	6
Water supply, in acre-feet, during the six-year period	134 600
Silt Deposited, in Acre-Feet:	
Total	5 244
Annual	874
Percentage of original capacity	7.4
Percentage of original capacity, by volume $\left(\frac{5\,244}{134\,600} \times 100 =\right)$	3.9

Since the date of the last capacity survey (February, 1931), a flood of 100 000 acre-ft passed through this reservoir, on January 1 and 2, 1932. By July 1, 1933, it was estimated that, in all, 10 500 acre-ft of silt had been deposited with a reduction in capacity of 16.4 per cent.

Lake Mentz is a non-overflow, gravity dam provided with five Stoney gates, 30 ft wide and 25 in. high. In March, 1928, a flood of 356 000 acre-ft entered the reservoir, of which 310 000 acre-ft passed through the gates. During the flood of January 1 and 2, 1932, furthermore, 421 000 acre-ft were by-passed through the gates. At Van Ryneveld's Pass, the dam is of the overflow, gravity section, type and has only been in action once when 46 000 acre-ft were spilled.

It will be noticed from Table 11 that the percentage, by volume, of silt retained at Lake Mentz is only 2.3 as against that of 3.9 at Van Ryneveld's Pass. This may be accounted for, possibly, by the lesser run-off and lesser erosion from the flatter and, at the same time, lower rainfall portion of the catchment area as against the higher run-off and greater erosion in the

Mountain Section. It may also be accounted for by the probability that a certain quantity of the suspended and bed-load is scoured out by the gates as against the skimming action of an overflow type of dam. The two reservoirs also provide an interesting illustration of the effect of the size of the mean annual run-off and the storage capacity on the life of the reservoir. Although Lake Mentz has a lower volume percentage of silt and a 50% larger storage capacity, the annual run-off from the larger catchment is such that its life, theoretically, is 30 yr as against 50 yr for Van Ryneveld's Pass.

Considerable work has been done in the United States in estimating the life of reservoirs by silt sampling. In this connection, the writer would like to recall the work and method of approach adopted by Mr. C. H. Warren, while he was Circle Engineer of the South African Irrigation Department. Silt samples were taken in bottles and analyzed both for specific gravity and silt percentage by weight, and a curve was obtained expressing the relationship. From samples taken from the bed of the Great Fish River, Mr. Warren obtained a weight of 52.8 lb per cu ft of dry silt, or 84.6% of the weight of 1 cu ft of water. A hundred cubic feet of water, containing 1% by weight of silt, therefore, will deposit $\frac{100}{84.6}$, or 1.18 cu ft of silt. As the silt percentage increases, so also will the weight per cubic foot of water laden with that silt increase. Therefore, he calculated a ratio, *R*, expressing the relationship between the percentage by volume and the percentage by weight for increasing loads of silt in the water (see Table 12).

TABLE 12.—RATIO OF PERCENTAGE BY VOLUME TO PERCENTAGE BY WEIGHT FOR INCREASING LOADS OF SILT

Specific gravity (1)	Weight of 1 cubic foot, in pounds (2)	Percentage of silt by weight (3)	Ratio, $R = \frac{\text{Percentage by volume}}{\text{Percentage by weight}}$ (4)	Percentage of silt by volume (5)	Specific gravity (1)	Weight of 1 cubic foot, in pounds (2)	Percentage of silt by weight (3)	Ratio, $R = \frac{\text{Percentage by volume}}{\text{Percentage by weight}}$ (4)	Percentage of silt by volume (5)
1.000	62.42	0.00	1.182	0.00	1.040	64.92	5.86	1.230	7.21
1.005	62.74	0.73	1.188	0.87	1.050	65.55	7.32	1.241	9.09
1.010	63.05	1.46	1.194	1.74	1.060	66.17	8.79	1.253	11.03
1.015	63.36	2.20	1.200	2.64	1.070	66.79	10.25	1.265	12.97
1.020	63.67	2.93	1.206	3.53	1.080	67.42	11.72	1.277	14.97
1.025	63.99	3.66	1.212	4.44	1.090	68.04	13.18	1.289	16.99
1.030	64.30	4.40	1.218	5.36	1.100	68.67	14.65	1.301	19.05

The factor, *R*, is obtained as follows: Assume a quantity of water loaded with 4.40% of silt by weight. From Table 12, 1 cu ft of water thus laden will weigh 64.30 lb and, hence, 100 cu ft will deposit 64.30 × 4.40, or 282.92 lb., or $\frac{282.92}{52.8}$, or 5.358 cu ft, and $R = \frac{5.358}{440} = 1.218$.

With the foregoing information, Mr. Warren traced the silt load of the Tarka River, discharging into Lake Arthur Reservoir. Over a 3-yr period an average of 2.76% by weight, or an equivalent of 3.31% by volume, was found.

From the published records of the United States Reclamation Service, Mr. Warren assumed that the rolling bed-load was 25% of the suspended load, thus making a total silt load of 4.14% by volume. A silt survey of the reservoir at the end of the 3-yr period showed a silt content of 4.34% by volume, a difference of less than 5 per cent. Although the foregoing result may be fortuitous in some small degree, it constitutes a logical attempt, nevertheless, to take into account the varying degrees of suspended silt carried at different stages during the same flood as well as the varying silt load in different floods resulting from different rates of intensities of rainfall.

The writer has recently had an opportunity of viewing the anti-erosion work conducted both in South Africa and in the United States, and has formed the opinion that this work, judiciously planned and co-ordinated to local circumstances and supplemented by further work in the river channels proper, can reduce the silting of reservoirs materially.

Although it is axiomatic, of course, that a certain degree of erosion will occur, and that silt will always be present, due to ever-present dynamic geological processes, these factors, nevertheless, have been considerably augmented by the careless and haphazard processes of mankind and, in particular, by his agricultural activities. With the opening of new territories, for example, areas have been put under the plow that should never have been plowed. To-day, washes and gullies (or "dongas," as they are known in South Africa), are in evidence, that have widened from 50 to 300 ft "within the living memory of man." The writer has often asked himself the question whether, if the white man had not entered upon his activties in these areas, these selfsame gullies and "dongas" would not still to-day be nearer to the 50-ft width instead of the 300-ft width.

Obviously, it will be impossible both physically and economically to overcome entirely that portion of erosion which can be attributed to Man's activities. A large part, however, can be overcome. The results obtained at Lake McMillan and Zuni Reservoirs, discussed by Mr. Steven, can be considered at least as definite and established indications of what can be accomplished. In the former case, the accidental propagation of tamarasks provided an effective silt trap. In the latter, anti-erosion work of a limited character was conducted on only a portion of the catchment area. In neither of these cases can it be said that the catchments were subjected to a systematic, judicious, and complete program of anti-erosion work, yet in both instances the rate of silting during the past 10 to 15 yr has been reduced to less than 40% of that experienced formerly.

It was with great interest, therefore, that the writer acquainted himself with the principles and aims of the United States Soil Erosion Service and the manner and methods in which these are executed and adapted to each local circumstance. In his opinion, they deserve the serious attention of the profession.

With the careful study that soil erosion is receiving on each of the Soil Erosion Service projects, chiefly from the more agricultural aspects, and the judicious execution of methods to combat these phases, benefits on the resulting silting must be inevitable. If these are next followed up by the more purely engineering aspect of combating erosion in the water channels proper —such as the maintenance of river alignments, the prevention of under-cutting the banks, the formation of vegetative silt traps, the construction of débris basins, as well as the further work of fire protection, the writer does not feel that he is over-optimistic in anticipating a reduction in the silt percentage by volume on many streams to 50% of that occurring at present.

Where engineering works can thus profit, it behooves the profession to pay sincere and serious attention to, and offer constructive co-operation in, the planning and execution of such work. Both in the United States and in South Africa, the writer feels, there is considerable scope for the Civil Engineer to participate beneficially in the soil erosion aspect of the silt problem.

The writer wishes gratefully to acknowledge the data supplied him by Mr. A. S. Bridgman, Maintenance Engineer of the Sundays River Irrigation Board, in regard to Lake Mentz Reservoir.

HERMAN STABLER,[38] M. AM. SOC. C. E. (by letter).—The author has performed a real service to water-supply engineers by making available for ready reference such a mass of quantitative information on the silting of reservoirs and the suspended matter carried by streams. His statement of the silt problem and his discussion of the origin, transportation, and control of silt and of sedimentation processes include a summarization of facts and principles, largely devoid of expressions of opinion, that serves to round out a general picture of the silt problem in a manner both interesting and illuminating. In his discussion of the origin of silt the author touches on geology but, in the writer's opinion, gives it too little emphasis. As defined by the author, silt is stream-borne material derived from the disintegration of rocks. Two processes, weathering (disintegration in place) and corrasion (the tearing away or placing in motion of disintegrated material) are involved in the origin of silt, and the speed of the first of these processes is materially affected by the nature of the country rock. Igneous rocks, in general, are hard, dense, and crystalline; and they weather slowly. The older sedimentary rocks are likewise relatively dense and offer resistance to weathering. The younger sedimentary deposits, particularly recently laid alluvium, are especially susceptible to rapid weathering. Consequently, other things being equal, regions of igneous and precarboniferous rocks are regions of clear-water streams, and regions of recent sedimentaries and valleys filled with alluvial deposits are most likely to give rise to streams heavily laden with silt.

Among the younger sedimentary rocks, differences of texture and cementation are reflected by differences in speed of weathering. Loosely cemented sandstones and friable sandy shales are readily susceptible to weathering and corrasion and are prolific producers of silt.

The chemical composition of rock material is likewise a factor in the silt problem. Rocks that contain high percentages of aluminum silicate weather into clays with fine flaky particles readily susceptible to corrasion and water transportation. Regions of such rocks are likely to be regions of cloudy although not necessarily heavily silt-laden streams.

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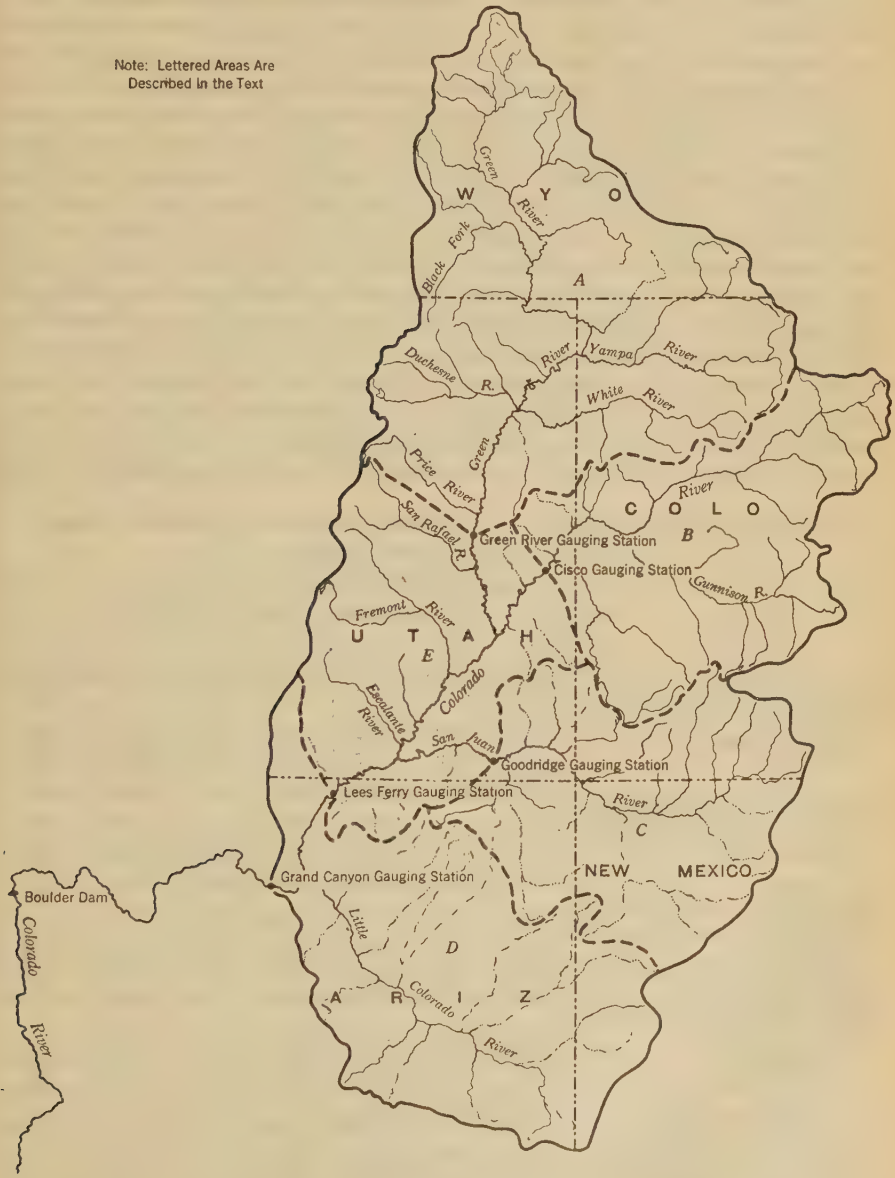
The geologic mission of rain and other forms of precipitation, from the moment they strike the earth, is to wear the land surface down to the level of the sea. This is accomplished mainly by weathering and corrasion, by solution, and by water transportation. In some regions of the United States more material is carried by streams in solution than as silt. In other sections the opposite is true. By and large the silt load of the streams of the United States is several times as great as the load carried in solution, but the latter is by no means an unimportant factor.

The relative speed of weathering and of removal and transportation of the weathered material, depending primarily on geology, topography, and climate, is a factor in the silt problem. In some regions streams run clear because little transportable material is available for water to move and only extraordinary storms and floods can muddy the waters. In other regions relatively gentle rains are sufficient to place in motion a wealth of weathered material awaiting only the means of transportation.

Vegetation, an incident of climate, is an important factor in the weathering of rocks and in the availability of weathered material for water transportation. By disruptive action of roots and chemical action of organic solvents vegetation aids weathering and adds to the volume of material suitable for transportation as silt and to the dissolved load of streams. By protection of the land surface from action of wind and water it retards or prevents silty material from corrasion and thus tends to avoid the overloading of streams with suspended matter. Cultivation of field crops artificially aids weathering and accelerates the speed of the natural process. Road building, the herding of stock, lumbering, and a multitude of other activities of Man that scar the earth's surface also speed up the natural process of weathering. Most of these activities are essential to human occupation of the land but some of them may be modified so as to minimize their effect on weathering and still retain their beneficial character. On the other hand, works of Man, such as buildings, streets, and paved roads, substantially prevent weathering whereas his river-regulating works and consumptive uses of water tend to decrease the power and quantity of water that may serve to transport silt and other rock débris. After all, Man's activities, although by no means a negligible factor in the silt problem, must be considered as incidental. Wisely guided, Man will undertake, when practicable, to diminish the removal of material valuable *in situ* to places where it will be deleterious, and to direct the deposition of transported silt at places where it may serve a useful purpose. In attempting to abolish the silt problem Man can be but a Don Quixote tilting at the windmills of Nature.

In attacking the silt problem, should the energies of Man be directed toward prevention or cure? Rather obviously good productive soil in a farming region should be maintained in place for agricultural use so far as practicable. In this case prevention pays, regardless of the silt problem; but it is not so clear that great effort should be made to retain in place the surface material of non-agricultural regions. Regulation of grazing to maintain a maximum permanent useful vegetable cover for forage purposes is itself a profitable conservation measure, and, incidentally, will tend toward silt prevention. This is a practicable measure of silt prevention that may


Note: Lettered Areas Are Described In the Text

Green
River
W
Y
O
Black Fork
A
Duchesne R.
River
Yampa
River
White
River
Green
Price
River
San Rafael R.
Green River Gauging Station
Cisco Gauging Station
River
C
O
L
O
B
Gunnison R.
Fremont
River
U
T
A
H
E
Escalante
River
Colorado
San Juan
Goodridge Gauging Station
Lees Ferry Gauging Station
River
C
NEW
MEXICO
Grand Canyon Gauging Station
Boulder Dam
Colorado
River
Little
D
Colorado
A
R
I
Z
River

FIG. 10.—SUBDIVISIONS OF THE COLORADO RIVER BASIN ABOVE THE PRINCIPAL GAUGING STATIONS

reasonably be undertaken in a relatively arid or non-agricultural region. To be practicable, however, it should be undertaken primarily in the interests of the stock industry rather than for the purpose of limiting the silt load of streams. Otherwise, costs are likely to exceed the values conserved.

The Colorado River occupies one of the major stream basins in the United States and has silt problems perhaps in greater degree than any other major stream of the country. Its problems of water supply, water utilization, land use, and silt have interested the writer for many years. Most of the extensive surveys of its stream channels, reservoir sites, and dam sites made by the U. S. Geological Survey in the two decades, 1915–1935, were under his general direction. The quality-of-water surveys (determination of daily silt load and dissolved mineral load of the Colorado and of its more important tributaries for a period continuous from 1929, which were made largely by C. S. Howard and S. K. Love under the direction of W. D. Collins, of the Geological Survey) were instigated by him and he has studied their results with interest. These researches, basic to land and water planning, although still inadequate to supply full information, afford probably the best available basis for the study of the silt problems of a large river in the United States. A brief consideration of some of their results will serve by example to throw light on some of the problems discussed by Mr. Stevens. Fig. 10 is a map showing the subdivisions above the principal gauging stations of the Colorado River Basin and Table 13 shows some of the characteristics of these

TABLE 13.—DATA RELATED TO THE SILT PROBLEM ON COLORADO RIVER ABOVE GRAND CANYON, ARIZONA.

Description	WATER-SHED SUBDIVISIONS (SEE FIG. 10)						
	A	*B*	*C*	*D*	*E*	*F*	*G*
Area, in square miles	40 600	24 100	24 000	31 000	19 300	50 300	139 000
Area, in percentage of total	29.2	17.3	17.3	22.3	13.9	36.2	100.0
Surface Geology, Percentage of Area:							
Pre-Triassic	14.3	38.5	10.8	30.8	16.0	24.5	21.6
Triassic-Jurassic	3.0	7.3	21.5	37.5	55.3	45.0	22.2
Post-Jurassic	82.7	54.2	67.7	31.7	28.7	30.5	56.2
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Vegetable Cover in Percentage of Area:							
Timber (pine, spruce, lodgepole)	23.5	70.2	30.0	26.1	11.0	19.8	30.9
Woodland (juniper, piñon)	16.9	17.0	28.6	36.0	26.3	31.8	24.4
Brush and grass	59.6	12.8	41.4	37.9	62.7	48.4	44.7
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Palatable cover, animal-unit-years per square mile	4.8	6.25	4.4	4.25	3.4	3.9	4.5
Stock population, animal units per square mile	12.9	16.6	12.0	8.3	7.8	8.0	11.6
Area irrigated, in acres per square mile	13.7	25.8	7.6	0.6	4.8	2.2	10.6
Human population per square mile	2.1	3.3	2.3	1.9	0.6	1.4	2.1
Rainfall, 1929–1934, in inches	10.30	14.05	12.55	11.20	7.75	9.72	11.12
Rainfall, 1929–1934, in Percentage of Area:							
More than 10 in	46.2	59.4	58.8	51.6	13.2	35.3	46.6
6 to 10 in	51.1	40.6	41.2	41.6	60.7	49.7	47.0
Less than 6 in	2.7	0.0	0.0	6.8	26.1	15.0	6.4
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Run-off, 1929–1934, in inches	2.00	4.46	1.55	0.21	0.62	0.38	1.78
Run-off, 1929–1934, in percentage of total	33.2	43.9	15.1	2.8	5.0	7.8	100.0
Silt, 1929-1934, in tons per square mile	696	737	2 440	2 230	3 770	2 810	1 770
Silt, 1929–1934, in percentage of total	12.1	7.6	19.7	29.5	31.1	60.6	100.0

subdivisions above Grand Canyon, with special reference to the silt load. The lettered subdivisions are identified as follows:

A = the basin of Green River above Green River, Utah.
B = the basin of the Colorado River above Cisco, Utah.
C = the basin of San Juan River above Goodridge, Utah.
D = the basin of the Little Colorado, Paria, and other tributaries of the Colorado River between Lee's Ferry and Grand Canyon, Ariz.
E = the basin of the Colorado River and tributaries between Lee's Ferry, Ariz., and Green River, Cisco, and Goodridge, Utah, including San Rafael, Fremont, and Escalante Rivers.
F = the basin of the Colorado River from Grand Canyon, Ariz., to Green River, Cisco, and Goodridge, Utah.
G = the basin of the Colorado River above Grand Canyon, Ariz.

In Table 13, the information on area, surface geology, palatable cover, run-off, and silt are from records of the Geological Survey; that on rainfall, from the records of the Weather Bureau; that on population, stock population, and area irrigated from the Bureau of the Census, and that on vegetable cover from the vegetation map of the United States by H. L. Shantz, of the Bureau of Plant Industry. The 5-yr period considered is from October, 1929, to September, 1934, inclusive. Acknowledgment is made of assistance by J. C. Miller, C. E. Nordeen, Assoc. M. Am. Soc. C. E., and Depue Falck, all of the Conservation Branch of the Geological Survey, in the compilation of the information.

Inspection of Table 13 shows that, for the period considered (which includes a year of high rainfall and run-off, a year of low rainfall and run-off, and, as a whole, is not far from an average 5-yr period) about 77% of the water and 20% of the silt at Grand Canyon came from the basin above Green River and Cisco, Utah, which comprises 46.2% of the entire area of the basin above Grand Canyon (see Subdivisions *A* and *B*, Table 13); and that less than 8% of the water and more than 60% of the silt came from 36% of the basin's area, situated below Green River, Cisco, and Goodridge, Utah, and above Grand Canyon, Ariz. (see Subdivision *F*). Properly belonging to this region of low run-off and heavy silt load is about one-half the area related to Subdivision *C*, or approximately that part of the San Juan Basin above Goodridge that is situated in Arizona, New Mexico, and Utah. Including this part of the San Juan Basin it is probable that what is commonly known as the plateau region of the Colorado, comprising an area of 60 000 to 65 000 sq miles, contributes less than 10% of the water and more than 75% of the silt load recorded at the Grand Canyon gauging station. Clearly, if silt prevention on the Colorado is a worthy objective, intensive study should be made of the origin, and the possibilities of the prevention of silt derived from this plateau region of the basin. Referring to Table 13, it is evident that this is, relatively, a region of Triassic and Jurassic rocks—loosely cemented sandstones and sandy friable shales—and to these rocks must be attributed the origin of most of the silt.

The plateau region has a low rainfall. One-half the area has an annual rainfall of 6 to 10 in. per yr, 15% has a rainfall of less than 6 in.; and the average for the 5-yr period considered is 9.72 in. The distribution of the rainfall is of importance, and the records show that, substantially, it all occurs in one to six storms during the year. These desert storms are torrential in character, well adapted to corrasion of the rather finely divided weathered material that abounds in the region.

To the arid climate may be attributed the excess of weathered material awaiting transportation and to the torrential character of the storms, the heavy though sporadic flows of silt.

The plateau region has a cover of brush, grass, junipers, and piñon, there being little timber and that is confined to the mountainous outer rim of the basin. The palatable cover is light, being confined to brush interspersed with grass and weeds, occurring as single plants, and the stock and human population is low. In small areas there is a grassy vegetational aspect but in the main the region has the appearance of a desert with vast expanses of bare rock and sand. Probably 90% of the surface of the region is devoid of vegetation. The irrigated area is small and confined almost entirely to the head-waters of streams. It is a land of stock ranges that will furnish year-round feed supply for less than 225 000 cattle, or the equivalent in other stock. The sparseness of vegetation permits the torrential rains to do their work with a minimum of hindrance.

What measures of silt prevention can be undertaken to advantage in such a region? Artifical stimulation of vegetable growth on 60 000 sq miles of surface now devoid of cover and more than one-half of which has rainfall of less than 10 in., is a stupendous, and a hopeless, task. Could it be accomplished to an appreciable degree there is no doubt that the results would be beneficial. There is no lack of fertility, no lack of seeds in the ground. After every soaking rain grass and weeds spring up in abundance, wither, and die. Moisture alone is needed, and moisture it is impracticable for Man to supply. Already 96% of the scant precipitation is devoted to evaporation and transpiration. Only by decreasing the former can much additional moisture be conserved for beneficial use. Stimulation of vegetable cover by regulation or grazing operations is worthy of consideration. Total exclusion of stock from the region for a time would encourage stronger root growth, heavier crowns, and a full opportunity for natural reseeding. This would be beneficial, but how much of the 190 000 000 tons of silt per yr contributed by the plateau region would it keep from the river and for how long and at what cost? The principal cost would be the annual loss of forage valued at about $1 500 000. To this would be added the administrative expense of excluding stock. An annual cost of $2 000 000 might be justified if 10% of the silt could be prevented from reaching Boulder Reservoir and it were worth $150 to $200 per yr to maintain an acre-foot of reservoir capacity. Total exclusion of stock would be impossible because of the human relations involved. Over considerable parts of the area, in the Navajo Indian reservations, for example, stock raising is practically the entire source of livelihood for the people. Nevertheless, the principle of reducing stock population is worthy of consideration, and it is reasonably assured that, where possible, limitation of grazing to that compatible with maintenance of normal cover, would prove profitable to the local stock industry and to some slight degree lengthen the life of reservoirs on the streams below. Increased vegetable growth, of course, would deplete the water supply.

Since the transporting power of flowing water is required to convey it to perennial streams, a second method of preventing silt, would be to conserve the water supply of the plateau region so far as practicable for irrigation or other consumptive use within that region. If the value of the water of the plateau region is $2 per acre-ft for irrigation in the Lower Colorado Basin and 25 cents per acre-ft for the development of power in the canyons, a total value of the order of magnitude of $3 000 000 per yr is indicated. Exclusion of this water from the main river, even without beneficial use in the plateau region on the basis of such an assumed value, would cost at the rate of less than 2 cents per ton of silt excluded with it—far less than the cost of maintaining reservoir capacity by operations at the side under any known method. Reservoir capacity could thus be maintained at a cost of only $20 to $30 per acre-ft per yr. It seems rather clear that the development of beneficial consumptive uses of water in the plateau region should be encouraged to the utmost for whatever effects in silt prevention may result.

A third method of holding silt in the plateau region would be through the construction of detention reservoirs and spreading works. There are many good and some very large reservoir sites in the region. Eventually, many of them will be developed. They could serve to retard the flow of silt to the Colorado although its eventual movement could not be thus prevented. The effective life of such reservoirs would be rather short but the cost of many could doubtless be justified by their beneficial effects in conservation of water and retardation of silt movement. Theoretically, at least, spreading works have merit. If 1 in. of torrential rainfall on 1 000 acres could be spread to 0.01 in. on 100 000 acres, serious silt movement could be prevented. In the process, local vegetable growth would be encouraged and flow of water to the Colorado prevented. Doubtless there are favorable localities in the plateau region for works of such character.

Summarizing, regulation of grazing, development of beneficial consumptive use of water, and construction of reservoirs and spreading works in the plateau region all seem to the writer to merit consideration as means of withholding silt from the Colorado River. He would urge that before expending large sums of money in any such undertaking economic benefits commensurate with the costs be reasonably well assured, full consideration being given to the limitations imposed by the value of storage capacity in reservoirs below and the capitalized cost of maintaining such capacity by operations at the reservoirs. He ventures the opinion, based on personal knowledge of the physiography of the plateaus herein considered, that no practicable means will be found for holding back permanently from the Colorado more than a minor percentage of the silt load derived from the sandstones and shales of the region. It is his firm belief that after all reasonable preventive measures have been exhausted, a now unknown cure for the silt problem will have to be developed, or the inexorable geologic processes that give it rise will deplete seriously, within relatively few generations, and will finally destroy the effectiveness, of the water supply systems of the arid Southwest. It should be recognized, however, that the processes of silt accumulation and deposition have been generally beneficial to mankind. The best agricultural regions of the country are the result of Nature's solution of former silt problems. In the course of geologic time, in the arid Southwest, as elsewhere, the relatively useless sands and clays of the uplands may be expected to contribute their share to the building up of the fertile loams of rich agricultural valleys. Man must adjust his activities to Nature.

N. C. GROVER,[39] M. AM. SOC. C. E. (by letter).—The silt problems of the Colorado River at the Boulder Dam are treated in this paper. Some up-to-date information has been obtained as to the effect of this dam on the silt regimen of the river, in connection with the regular activities of the U. S. Geological Survey in its program of measuring and recording the flow of surface streams. Among the 3 000 gauging stations operated, the program includes, of course, a group of stations in the Colorado River Basin. Because of the importance of silt in connection with the utility of the Colorado River, detailed silt studies were undertaken in 1925. Continuous records of the silt content of the water are kept at several of the gauging stations, including those at Grand Canyon, Willow Beach, and Topock, in Arizona.

Fig. 11(*a*) shows the quantities of silt, in millions of tons, carried by the river during each month, from January to June, 1935, at these three gauging stations. Fig. 11(*b*) shows the quantities of water, in acre-feet, carried during the same months. Grand Canyon station is situated well above the reservoir created by the Boulder Dam; Willow Beach station is 10 miles below the dam; and Topock station is about 115 miles below the dam. The Parker Dam is about 40 miles below Topock and about 155 miles below the Boulder Dam.

The construction and utilization of the Boulder Dam create a situation in the river below that dam that has great interest for engineers. The reservoir which is more than 100 miles long desilts the river so that the discharge through the dam is essentially clear water. This clear water, however, is discharged into, and flows through, a channel of silt and picks up a new load, thereby deepening the channel below the reservoir. The deposition of silt in the Boulder Reservoir and the deepening of the channel below it, of course, were anticipated. The speed with which the new load is acquired has general interest as has also the deposition of the new load in the reservoir formed by the Parker Dam which will rapidly lose its storage capacity. The ultimate silt content of the water as it leaves the Parker Dam will affect the problems related to the diversion into Southern California by the Metropolitan Water District of Southern California.

The Colorado River above the Boulder Dam carries apparently an annual load of, say, 200 000 000 to 300 000 000 tons of silt. This load, which prior to the construction of the dam was carried through to Yuma, essentially unchanged, is now deposited in the reservoir. In the five months between February 1, 1935, when storage in the reservoir began to be effective, and June 30, 1935, the load of silt that was carried past the Willow Beach gauging station amounted to about 4 000 000 tons, derived in part from operations at the dam. At the Topock gauging station the total load of sediment carried in the five months was more than 12 000 000 tons. During the same period the quantity of silt carried into the reservoir above the dam was more than 100 000 000 tons.

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[39] Chf. Hydr. Engr., U. S. Geological Survey, Washington, D. C.

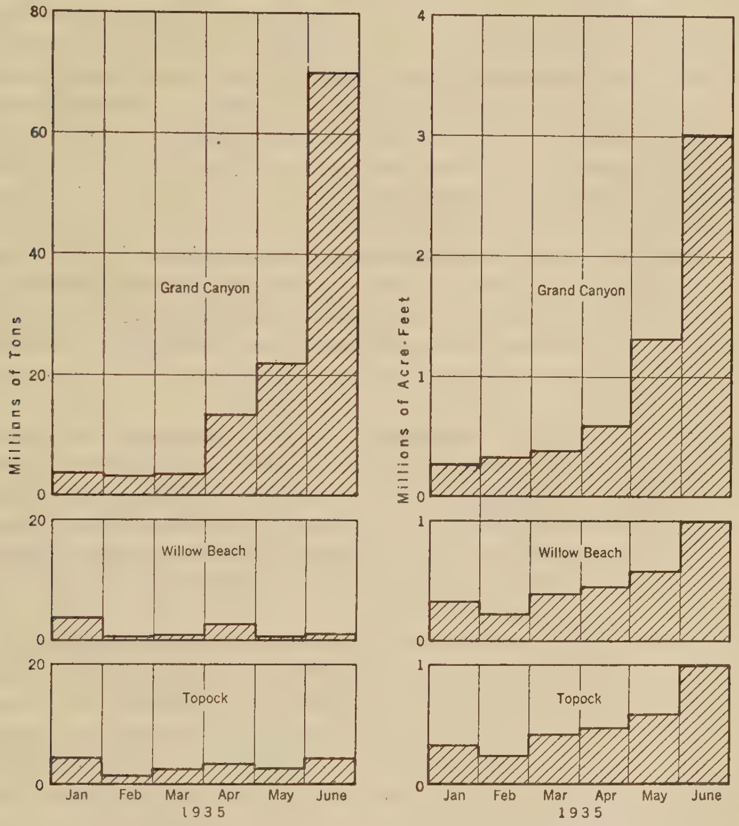

FIG. 11.—SUSPENDED MATTER, IN TONS, AND DISCHARGE, IN ACRE-FEET, AT GAUGING STATIONS ON COLORADO RIVER, IN ARIZONA.

As a result of the pick-up of silt below the dam the channel has already been deepened. At the Willow Beach gauging station the bed was appreciably lower in September, 1935, than it was a few months previously. It is interesting to note also that the particles carried past Willow Beach and Topock are on the average much larger than the particles carried past Grand Canyon and into the reservoir.

Engineers will be interested in following the history of the new and controlled Colorado River, including not only the rates of loss of capacity of the

Boulder and Parker Reservoirs, but also the rate of erosion of the river bed and banks below the Boulder Reservoir. Many years will doubtless pass before the bed, banks, and silt load are stable.

J. C. Stevens,[40] M. Am. Soc. C. E. (by letter).—A great mass of additional data, particularly on the silting of reservoirs, has appeared in the discussion of the writer's paper. In one paper will now be gathered practically all the basic data regarding the silting of the important reservoirs of the world, as well as summarized data as to the quantity of sediment carried by most of its rivers.

Mr. Nickle mildly chides the writer for not including the excellent data gathered in recent years by the Texas Board of Water Engineers, when as a matter of fact every attempt was made to secure them. Special thanks are due to him for Table 9 containing a summary of the sediment transported at seventeen stations on ten Texas streams, mostly over a 6-yr period, and arranged to conform to the writer's Table 6.

It is interesting to note that of all the stations given in Tables 6 and 9, the maximum concentration of suspended sediment is found in Bad River, at Pierre, S. Dak., where for two consecutive years the sediment averaged more than 38 per thousand. None of the Texas streams listed reached this total even for one year, although the 6-yr average for Double Mountain Fork of Brazos River exceeded 24 per thousand, which shows the next highest concentration of silt.

Mr. Nickle also advances additional evidence of the effect of drying silt in reducing its volume. Medina Reservoir silt shrunk to one-half its original volume and doubled its specific weight in five years of intermittent exposure. The specific weight of 70 lb per cu ft of silt in place for Texas reservoirs, when subject to alternate wetting and drying, is not far different from the writer's average value of 65 lb per cu ft, nor that of 62.5 lb per cu ft adopted by Messrs. Fortier and Blaney.[34]

Professor Lane has given a very interesting historical summary of silt investigations, and a clear statement of the several factors affecting sediment transportation, first to the stream and second by the stream. He also calls attention to the fallacy of the Dupuit theory, namely, that silt is carried in suspension as the result of differences in velocity of adjacent filaments. Turbulence is the real cause, and theory and evidence are now in accord that sediment is held in suspension by the upward velocity components of turbulent flow; that is, by eddy currents.

Mr. Bonner's statement that the capital outlays for reservoirs will be amortized long before they may be rendered useless by silting, and that the solution of the problem may be safely left to posterity, is scarcely a fitting answer to the questions raised. Because of such a major reservoir a civilization comes into being, a virile, complex, pulsating social system. The capital investment in such a civilization may be wiped out and the system destroyed but it can never be amortized in a financial sense.

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[40] Cons. Hydr. Engr, (Stevens & Koon), Portland, Ore.

A policy of *laissez-faire* is entirely unsuited to the exigencies of the situation. The problem should be squarely faced now and research undertaken with a view to its ultimate solution. Mr. Bisschop cites conditions in South Africa that are now threatening the "very existence of productive centers of population." He states that further extensive developments are being limited to those streams least subject to silting. Lake Mentz on Sundays River (1935) has lost 42% of its capacity in 12 yr. Raising the dam has already begun in order to preserve the civilization that is dependent upon it. Amortization of the capital investment is surely not the answer to this problem. Human institutions must continue.

In Table 10, Professor O'Brien has added a wealth of data on silting of reservoirs from the paper by Dr. Fritz Orth, of which the writer was wholly unaware. This table, however, does not give the silt deposited in terms of the inflow, doubtless because the data were not available. Seven reservoirs are cited that have been completely filled with silt, and fifteen that have lost 50% or more of their original capacity. All of them, however, had original capacities of less than 1% of the annual water supply which means that they were little more than diversion dams. The writer doubts whether any such reservoir could actually lose 100% of its capacity. Table 1 does not show any reservoir as having been entirely filled. As the reservoir fills, more and more of the silt is carried through so that there must be left at least a river channel in the new surface of the reservoir, for which condition all silt passes through.

Mr. Blaney calls attention to the fact that a silty stream or canal is quite free from moss and aquatic plants, and that clarifying such a stream will result in a new source of annoyance, that of moss growth. He states that even if it were possible to clarify the Imperial Valley canals some silt should be left in the water to minimize operation troubles from moss growth.

Mr. Waggoner calls attention to the great fall flood of 1861 that affected the entire Pacific Coast from the Willamette Valley, in Oregon, to Los Angeles, Calif. On December 8, 1861, occurred the largest flood known on the Willamette River. That period was marked by a series of wet years from which the present cycle of equally pronounced dryness has emerged. Obviously, silt depositions and sediment transportation respond with marked fluctuations to these climatic and flood variations.

Mr. Stabler asks whether Man's efforts should be directed toward prevention or cure. It seems quite evident that to show any measure of success the labors of Man must be confined to preventative measures. Of course, he can desilt streams and canals at considerable expense and domestic supplies are necessarily desilted, but these are of secondary importance in the great silt problem. To preserve intact the large agricultural areas he must prevent soil loss. He must learn to prepare the land and cultivate it in such a way as to maintain the soil in place. In the European and Asiatic countries it has been the practice, for centuries, to terrace sloping lands. America has not as yet adopted this practice.

In non-agricultural areas, such as the Western grazing lands, it is doubtful whether extensive structures for soil maintenance are practicable or even desirable except on certain areas that contribute large quantities of silt to reservoired streams.

It is not sufficient merely to construct soil-holding structures. Provision must be made for their continued maintenance; otherwise, the structures deteriorate, and the entire volume of silt accumulated behind them may be loosed by a single storm with much more serious consequences than if it had been allowed to pass in smaller yearly quantities.

Once silt has found its way into a reservoir—except those for domestic supplies—no practicable method has yet been devised of removing it. All Man can hope to do is to reduce the quantity of silt flowing into the reservoir by such means of holding the soil on the area as he may devise. Any structures such as silt dams, bank protection, etc., built for this purpose, should be of permanent type and ample provision should be made for intelligent annual maintenance.

Considerable can be done in stimulating a protective vegetable growth in favorable areas. In unfavorable areas in which the West abounds such efforts are certain to prove futile. Grazing control will have a favorable effect if properly administered although, by and large, the results are likely to prove disappointing. The writer agrees with Mr. Stabler that if the Western grazing lands are managed primarily in the interests of permanency of forage crops for stock production, little more can be hoped for in the way of soil conservation from grazing control.

The data on the Colorado River drainage are most instructive. It would seem that the plateau region deserves extensive study. About 65 000 sq miles contribute 75% of the silt and only 10% of the water supply; yet all that silt is moved by the 10 per cent. If it were practicable to impound that 10% on the area and use it for irrigation instead of allowing it to carry silt into the Colorado, such an irrigated section would pay handsome dividends even if it never marketed a crop. There would be some justification for subsidizing such a project.

Mr. Grover illuminates, with some new data, the effect of desilting the Colorado River at Boulder Reservoir. In 10 miles of canyon the clear waters issuing from the control gates picked up, and carried in suspension, approximately 4 000 000 tons in 5 months. In 115 miles farther this quantity was trebled, and yet during the same period 100 000 000 tons were left in the reservoir.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1928

EFFECT OF SECONDARY STRESSES UPON ULTIMATE STRENGTH

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WITH DISCUSSION BY MESSRS. C. H. SANDBERG, J. D. GEDO, L. E. GRINTER, L. T. EVANS, F. E. FAHY, LAMOTTE GROVER, A. A. EREMIN, AND JOHN I. PARCEL AND ELDRED B. MURER.

SYNOPSIS

An analysis of the action of members of a bridge truss subjected to axial stress and secondary bending that arises from the deflection of the truss in its own plane, is given in this paper. The study is devoted primarily to the question of the effect of secondary stress in reducing the ultimate strength of such members, and, therefore, particular attention is given to the redistribution that occurs when the outer fiber stresses approach the yield point. As a supplementary study, a series of laboratory tests was made to determine the actual behavior of compression members with thin walls subjected to high secondary stresses and loaded to failure.

From the general analysis, and from these tests, it is concluded that for types of members and loading conditions investigated (which are believed to simulate closely the essential conditions for most bridge members), the ultimate strength is practically unaffected, even by high secondary stresses, if, in the case of compression members, the relative wall thickness is maintained at the ratio ordinarily required by the leading standard specifications.

INTRODUCTION

The problem of secondary stresses has occupied a prominent place in the theory of structures since the original investigations of the subject by Winkler,

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NOTE.—Published in November, 1934, *Proceedings*.

[1] Cons. Engr. (Sverdrup & Parcel), St. Louis, Mo.

[2] Wayland, Mo.

Asimont, Engesser, and Manderla in the late Seventies. A correct and complete analysis was presented by Manderla in 1879.[3] Throughout the half century following, the problem has been the subject of many researches, both analytical and experimental, directed in the main toward clarifying and simplifying the analysis and to its experimental verification. To a considerable extent this effort has been successful. The theory of secondary stress analysis is now generally accepted by most authorities on the same footing as other phases of statically indeterminate stress analysis, and although the numerical calculations involved are tedious, various permissible simplifications have rendered the method quite workable for office design.

American engineers became generally interested in the secondary stress problem much later than European engineers, probably due in some degree to the predominance of the pin-connected truss (for which, if all connections are true hinges, the secondary stresses are negligible) in American practice, and in part to a general distrust of the refinements of statically indeterminate stress analysis which at an earlier period was rather widespread. However, the last two decades (1914 to 1934) have witnessed a striking change in bridge engineering practice. The pin truss is no longer the dominant type; all spans from the smallest to the largest are being built as partly or fully riveted trusses, and the problem of secondary stresses has become a leading question in structural design. Most specifications now require such stresses to be computed for all sub-paneled trusses and for other cases in which there is reason to suspect that they may be large. That for many riveted bridge trusses of the massive type the secondary stresses are high is well established, theoretically and experimentally. In many cases the unit stresses due to secondary bending will reach 60 to 100% of the primary unit stresses, and in certain extreme cases they may exceed these limits.

The present method of providing for secondary stresses in design is subject to rather wide variations. Usually, a considerably increased unit stress (25 to 35% above the normal) is allowed for combined secondary and primary stresses and the member is proportioned by,

$$f = \frac{S}{A} + \frac{Mc}{I} \quad \text{.......................} (1)$$

in which, f = unit stress; S = primary tension or compression; A = cross-section area; M = bending moment; c = distance from the neutral axis to the extreme fiber; and I = moment of inertia. More commonly, perhaps, a blanket allowance, applied alike to all members, is provided in the prescribed unit stresses.

In a number of monumental riveted bridges (notably the Quebec (cantilever), the Sciotoville (continuous truss), and the Hell Gate (arch) Bridges), elaborate and more or less expensive special devices in fabrication and erection were used to reduce the secondary stresses to a point at which little or no excess material was required.

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[3] "Die Berechnung der Sekündarspannung welche im einfachen Fachwerke infolge starrer Knotenverbindungen auftreten", *Allgemeine Bauzeitung*, 1880.

While it is thus clear that the possible occurrence of high secondary stresses has been widely recognized and the best standards of practice have required that these shall either be provided for in the design or largely eliminated in fabrication and erection, it does not appear that any considerable attempt has been made to evaluate the effect of secondary stress upon the actual ultimate strength of a member.

It is clear that the action of combined primary and secondary stress is a different phenomenon from that arising from direct stress and flexure due to applied loads. This will be elaborated later in the paper; it may be noted here that (*a*) secondary stresses are not required to maintain the equilibrium of a bridge truss; and (*b*) that they are induced by the relative joint displacements of the structure, which, in turn, are conditioned by the distortion of the truss as a whole; and when these displacements have occurred, there is no further tendency for the stresses to increase. This clearly is quite a different condition from that which obtains in, say, an eccentrically loaded column in which the deflection and moment, at high stresses, increase much faster than the load. This peculiarity of secondary stress action has been noted.[4] Indeed, the opinion has been advanced[5] that even a high percentage of secondary stress, resulting in extreme fiber stresses beyond the proportional limit, will have little effect in reducing the ultimate strength of the truss, since the increased fiber deformations occurring in the neighborhood of the yield point tend largely to relieve the secondary stress. While this extreme view has not been generally accepted by the profession, the subject is believed to merit more attention than it has thus far received.

It is a practically universal principle in structural design that all fiber stresses shall be kept well within the yield point of the material. This, however, is subject to certain exceptions; for example, bearing stresses on rivets and pins are generally permitted to run 50% in excess of the stresses on the main sections. If this specification is sound and if the structure is designed consistently, it means that such local stresses may pass the yield point without endangering the safety of the structure. It is also common practice to permit unit stresses in the stiffening trusses of suspension bridges greatly in excess of the stresses in the towers or in the cable, if comparable material is used in the latter. The logic of this practice is that the stiffening truss is not a main carrying member and that if loaded beyond the yield point, the bridge may still be in no danger of actual failure.

It is scarcely to be presumed that stresses beyond the yield point in either of these cases would ever be regarded as other than undesirable, but since such a condition would not produce structural collapse, it is felt that a smaller margin of safety is permissible than would be the case for the main carrying members.

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[4] See, for example, "Theorie und Berechnungen der Eisernen Brücken", von F. Bleich, pp. 424–486; discussion by Edward Godfrey, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 193; and Second Progress Report of Special Committee on Steel Column Research, *Transactions*, Am. Soc. C. E., Vol. 95 (1931), p. 1220 *et seq.* A clear statement of the limitations of secondary stresses will be found in "Modern Framed Structures", by Johnson, Bryan, and Turneaure, Pt. III, p. 14. No use is made of this, however, in the later discussion of the reduction in ultimate strength of columns due to secondary bending (see pp. 57–59).

[5] "Theorie und Berechnungen der Eisernen Brücken", von F. Bleich.

A somewhat similar argument might well be advanced in regard to secondary stresses if it can be shown that, under the normal range of conditions, such stresses do not actually reduce the ultimate carrying power of the members. It may still be regarded as undesirable to have any considerable portion of the section stressed beyond the yield point, but it may be permissible to tolerate much higher limits for such stresses if this in no way endangers the safety of the structure. The remainder of the paper is devoted to a consideration, in some detail, of the relation of secondary stress to ultimate strength.

ANALYSIS OF PROBLEM

General.—Secondary stresses, as considered in this paper, arise from the displacement of the joints in the plane of the truss when the latter is subjected to external loads. If the members are connected by perfectly smooth hinges (and are non-continuous at all joints) no secondary stresses can develop. If, however, the members are connected by riveting (or welding) to gussets so as to form a practically rigid joint, there will be, in general, some degree of restraint at the ends of each member, and corresponding bending moments will be developed. The secondary end moments in any truss member, *m-n*, may be expressed by the well-known slope-deflection equation:[6]

$$M_{mn} = \frac{2\,EI}{L}\,(2\,\theta_m + \theta_n - 3\,R) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

in which, θ represents the angular displacements of the joints referred to their original positions, and $R = \left(\frac{\Delta}{L}\right)$, the angular displacement of the line, *m-n*, each in radians. If τ_m and τ_n represent the angles between the end tangents and the line, *m-n*, at *m* and *n*, respectively, the equation for the secondary moment becomes,[7]

$$M_{mn} = \frac{2\,EI}{L}\,(2\,\tau_m + \tau_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

which is the form originally proposed by Manderla.

It may be well to call attention here to two basic characteristics of secondary stress action:

(*a*) Even when the secondary unit stress is a high percentage of the primary, the secondary moments offer no appreciable assistance in carrying the loads, and the members are always designed on the basis of full hinge action at the ends.

(*b*) The quantities, θ and R, in Equation (2) are computed from the linear displacements, Δ, of the truss joints, and for all cases of any practical importance they are directly dependent upon the axial distortions of the various truss members and, therefore, are linear functions of the loads.

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[6] "Abhandlungen aus dem Gebiete der Technische Mechanik", von O. Mohr, 1906, p. 422 *et seq;* see, also, "Statically Indeterminate Stresses", by John I. Parcel and George A. Maney, Members, Am. Soc. C. E., Chapter VII.

[7] See "Modern Framed Structures", by Johnson, Bryan, and Turneaure, Pt. II, pp. 427–428.

The influence of secondary bending upon the ultimate strength of members depends importantly upon the type of stress acting on the member concerned. The several classes are considered separately in the following discussion.

Tension Members.—In Fig. 1 is shown a tension member, m-n, acted upon at the ends by the moments, M_m and M_n. At any point, x, distant from the left end, the moment is (considering clockwise moments positive),

$$M_{(x)} = M_m - \frac{M_m + M_n}{L} x - S y \quad \text{.............. (4)}$$

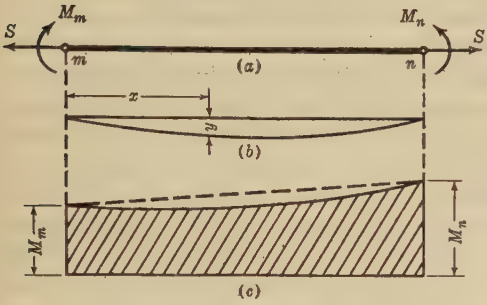

FIG. 1.

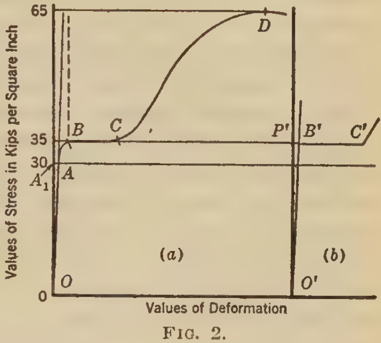

FIG. 2.

For the particular case in which the end moments are equal, and oppositely directed,

$$M = M_m - S y \quad \text{........................ (5)}$$

It is clear that the maximum fiber stress occurs at the end of the member where the larger moment is applied, and (if this is the m-end) is equal to,

$$f = f_{\text{axial}} + f_{\text{bending}} = \frac{S}{A} + \frac{M_m c}{I} \quad \text{................ (6)}$$

Since the relation, $f_b = \frac{Mc}{I}$, is derived on the assumption of linear stress variation, Equation (6) is not strictly correct beyond the proportional limit. When the yield point is reached and a large plastic flow takes place, the formula becomes quite inapplicable. The general tendency of the re-adjustment is toward an equalization of the tensile and compressive fiber stresses, respectively, such that a given maximum extreme fiber stress corresponds to a larger resisting moment.

Assume that Equation (3) is valid, then, as long as no stresses exceed the proportional limit, the behavior of the member is practically the same whether the end moments are due to eccentricity of axial loading ($M_m = S e_m$; $M_n = S e_n$), or to secondary bending. When the region of plastic flow is reached, however, the behavior is fundamentally different. In the first case, while the extreme fiber stress is relieved by the stress-strain re-adjustment, a total resisting moment equal to $S e$ must be developed regardless of the state of strain. In the latter case, since it is the tangential displacements that are invariably proportional to the loads (as long as $\frac{S}{A}$ remains within the proportional limit), and since a given angular displacement corresponds to a smaller bending moment when the stresses in the outer fibers approach the yield point, it is clear that within this region the secondary moments increase more slowly than the loads. Some elaboration of this point may be desirable.

Fig. 2(a) is a typical stress deformation curve for mild steel. Point A marks the limit of proportionality; Point B, the yield point; Segment B-C, the region of marked plastic flow; and Point D, the ultimate strength. Of course, Point A cannot be fixed accurately since there is no clearly defined limit of linear elasticity; slight deviations will frequently be observed at fairly low stress. However, it is not usual to find variations from proportionality of any considerable magnitude until the stress reaches the neighborhood of the yield point, as indicated in Fig. 2(a). For purposes of approximate calculation, a conventionalized stress-strain diagram, as shown in Fig. 2(b), is sometimes used.[8] This graph shows, roughly, that when the stress reaches the yield-point value at Point B', no further increase in stress can occur until a deformation equivalent to the length, $B'C'$, has taken place. This deformation may be as much as ten to fifteen times the length, $P'B'$, the value corresponding to linear elasticity.

It is evident from the foregoing relations that in the case of a beam in flexure, if plane sections are assumed to remain plane, the outer fiber cannot be stressed beyond the yield point until the fiber stress, f'_b, over the greater part of the depth of the section has reached this stress limit. Referring to Fig. 3, a beam of symmetrical cross-section is assumed to be loaded so that the outer fiber stress, QP, is just up to the yield point value (Line $O'P'$ in Fig. 2(b)). If the load is further increased, the stress curve takes the form of Area $OPQT$ in Fig. 3. It is particularly to be noted that the contribution to the beam deflection of a vertical slice of thickness, dx,

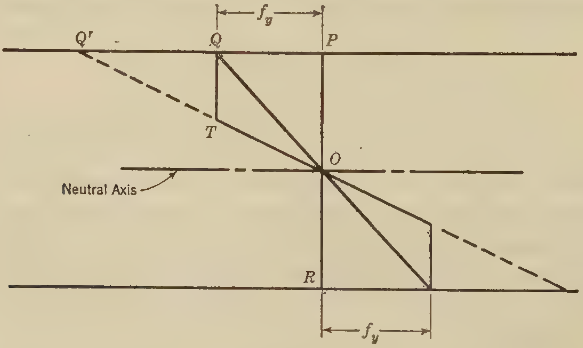

FIG. 3.

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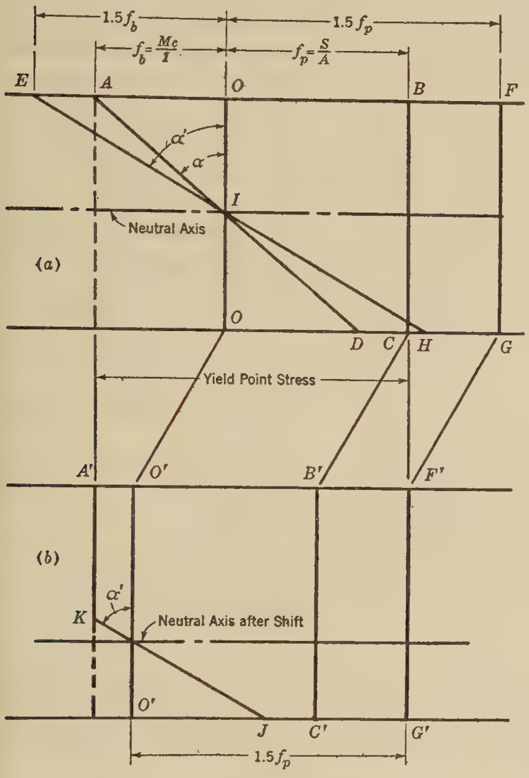
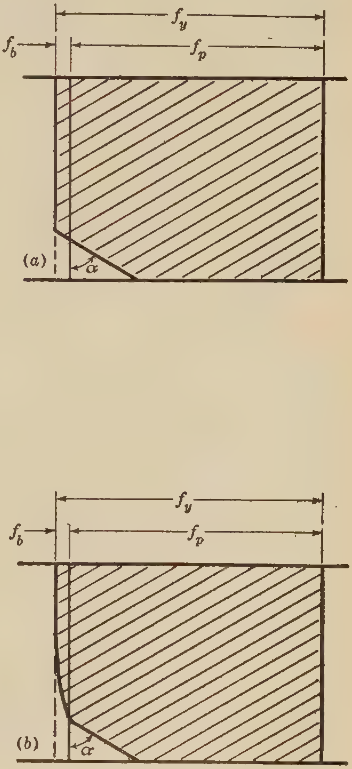
[8] "Excentrische beanspruchte Säulen", von A. Ostenfeld, *Mitteilungen No. 5*, Lab. für Baustatik, Technische Hochschule, Copenhagen, p. 20.

at Section *POR*, in the two cases will be in the proportion of PQ' to PQ, while the moments will have the proportion of $PQTO$ to PQO. It is clear from this illustration that when the outer fibers reach the yield point, a wide range of beam deflection is possible without any appreciable increase in the extreme fiber stress. Since the angular rotations at the ends of a member can increase no more rapidly than the axial loads, it will be impossible even with high secondary stresses for the extreme fiber stress to exceed the yield-point value without the distortion increasing far beyond any value reasonably to be expected in a truss in service. A simple example may be taken to illustrate the point.



1.5f_b
1.5 f_p
$f_b = \frac{Mc}{I}$
$f_p = \frac{S}{A}$
E
A
O
B
F
α'
α
I
Neutral Axis
(a)
O
D
C
H
G
Yield Point Stress
A'
O'
B'
F'
(b)
α'
K
Neutral Axis after Shift
O'
J
C'
G'
1.5f_p

FIG. 4.



f_y
f_b
f_p
(a)
α
f_y
f_b
f_p
(b)
α

FIG. 5.

It will be assumed that a truss member, *m-n*, of symmetrical cross-section, is subjected at the end, *m*, to an average axial tensile stress $= f_p = \dfrac{S}{A}$, and a secondary stress $= f_b = \dfrac{M_m c}{I}$, in which, M_m is the larger of the secondary end moments. Then, the maximum stress in the member will be at the end, *m*, on the side of the tensile flexural stress and will be given by Equation (6).

The stress graph is indicated by Curve *ABCD* in Fig. 4(*a*). Let it be assumed as a particular case that $f_p = 20\,000$ lb per sq in.; $f_b = 0.75\ f_p = 15\,000$ lb per sq in.; and that the yield-point stress, f_y, is 35 000 lb per sq in. The maximum fiber stress, *AB*, is then just at the limit beyond which plastic flow takes place.

Let it now be assumed that f_p receives an increment of 50%, bringing it to 30 000 lb per sq in. If the linear relation of stress to strain was to hold, f_b would become $1.5 \times 15\,000 = 22\,500$; the total maximum extreme fiber stress would be 52 500; and the stress graph would be the curve, *EFGH*, as shown in Fig. 4. On the other hand, assuming a yield-point stress of 35 000 lb per sq in., it is clear that this value cannot be exceeded until the extreme tensile fiber has suffered a stretch of approximately ten times that corresponding to 35 000 lb per sq in. It is obviously impossible, in a section of ordinary depth, for any such extreme deformation to take place unless the entire section is stressed to the yield point. Therefore, a redistribution of stress must be expected on the tensile side similar to that shown in Fig. 3. Again, it is noted that, as long as the average axial stresses are below the yield point, the tangential deflection angles (τ of Equation (3)) are strictly proportional to the primary unit stresses. In this case (for the small angles involved in the elastic deflections of steel structures), the angle, τ, should be increased in the proportion of $\frac{EO}{AO} = \frac{\alpha'}{\alpha}$. The actual stress graph as modified by the redistribution is the curve, *A′ F′ G′ JK*, as indicated in Fig. 4(*b*). Even if a much higher ratio of $\frac{f_b}{f_p}$ is assumed, the modified stress graph is quite similar. As f_p approaches the yield point still closer, the secondary stress becomes negligibly small as indicated in Fig. 5(*a*), the neutral axis shifting toward the side of compressive flexural stress.

The preceding discussion is based upon the conventionalized stress-strain diagram of Fig. 2(*b*). If the more correct relationships of Fig. 2(*a*) are used, a stress graph similar to that shown in Fig. 5(*b*) results (for f_p slightly below the yield-point stress).

It may be concluded from this analysis that the redistribution of stress which occurs in the neighborhood of the yield point of a ductile material, such as structural steel, will practically nullify any effect of secondary bending.

Fig. 6 shows the elastic curve and moment diagram for a bar under a primary tension, *S*, and equal secondary end moments, *M*. Within the proportional limit the moment diagram appears as Area *abcd* in Fig. 6(*b*). As the extreme fiber stress (which is maximum at the end) passes the proportional limit, and the material begins to flow, a much smaller moment corresponds to a given rotation. Since it is the latter which increases in proportion to the primary stress, when the strain reaches the plastic range, the moment will tend to fall off and the diagram will approach the form, *a′ b′ c′ d′* (Fig. 6(*d*)). The elastic curve tends to become flat; the curvature (and the region of over-strain) is confined to a short distance at each end (p and q, Fig. 6(c)).

There is precedent for ignoring over-stress of this type when strictly localized. Mention has been made of higher unit stresses allowed in bearing on rivets and pins (see "Introduction"). It may also be noted that most riveted end connections are productive of considerable local over-stress. Reference may also be made to the fact that the presence of a hole in a plate, otherwise uniformly stressed, results in a heavy stress concentration at the edge of the hole. If the diameter of the hole is so small that the width of the plate may be assumed infinite in comparison, analysis shows that the stress at the edge of the hole is three times the average. For normal ratios of diameter to widths, tests have shown stresses 2.3 times the average. However, the re-adjustment that occurs when the highly stressed region reaches the yield point, practically nullifies any effect on the ultimate strength. The two stress graphs are shown in Fig. 7. Below the elastic limit relatively high local stress concentrations must occur in all plates composing riveted members; but, for reasons just stated, these are properly ignored in all normal cases.[9]

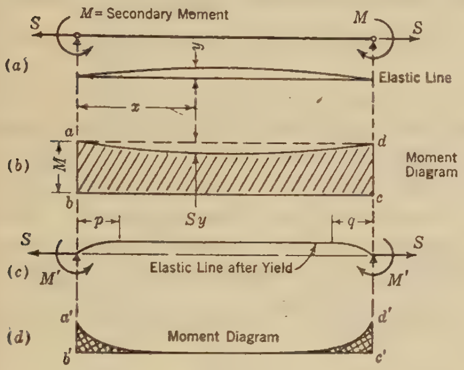


FIG. 6.

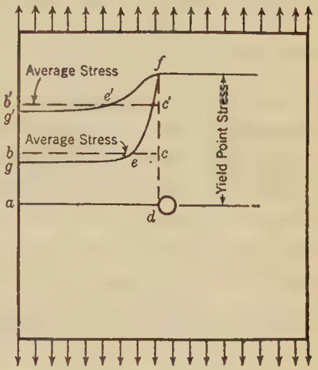


FIG. 7.

If the ultimate strength of a tension bar is understood to be its extreme effective strength as a bridge member, this value is limited to an average stress, $\frac{S}{A}$, equal to the yield point of the material. The preceding discussion tends to show that this effective strength is unchanged by any degree of secondary bending that may be expected in a well-designed structure.

Compression Members.—It is well known that flexure combined with compression produces a different behavior in a member from that which occurs under flexure and tension. The axial stress in the latter case tends to decrease the bending where in the former case it increases it; the moment represented by the axial stress times the deflection adds to the normal flexural moment, and in moderately flexible members at high stresses the total moment builds up rapidly. When the more highly stressed fibers pass the proportional limit, and the deflection begins to increase faster than the stress, the member proceeds rapidly to failure.

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[9] Reference may be made to "Drang and Zwang", von A. and L. Föppl, pp. 303–305, for presentation of the theory (originally due to Kirsch); see. also, "Applied Elasticity", by Timoshenko and Lessells, p. 9; and "Theorie und Berechnungen der Eisernen Brücken", von F. Bleich, pp. 249–252.

Most of the compressive chords (and frequently other compression members) in a bridge truss are of a stocky type which, even when subjected to bending stresses of from 50 to 100% of the axial stresses, shows relatively slight deflections from a straight line. The ultimate strength of such members is reached when the average stress over the section reaches, or approaches, the yield point of the material.

In considering the strength of compression members under primary and secondary stresses, the secondary end moments may be either of the same or of opposite signs. When the moments are of the same sign and of nearly the same magnitude, the member is bent into an S-shaped curve, the moment near the center is zero, and the deflections are so small that in any ordinary case buckling action for the member as a whole (giving rise to an $\frac{L}{r}$ failure) is practically negligible. When the moments are oppositely directed, the member is bent in single curvature, the maximum deflection occurs near the center, and the buckling tendency may be considerable.

Compression Member Bent in Single Curvature.—For simplicity consider the case in which the end moments are equal in magnitude. One analysis of the effect of secondary bending on the ultimate strength of a column bent in single curvature proceeds on the assumption that the action is analogous to the case of a virtual eccentricity $= \frac{M}{S}$ at each end.[10] By this method it is found that a considerable reduction in column strength results from the secondary bending. For example, 40% secondary stress reduces the strength of a short column about 28%, assuming the limit to be determined by the maximum fiber stress, $\frac{S}{A} + \frac{Mc}{I}$. A column with a slenderness ratio,

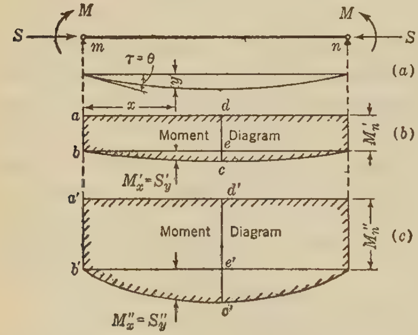

FIG. 8.

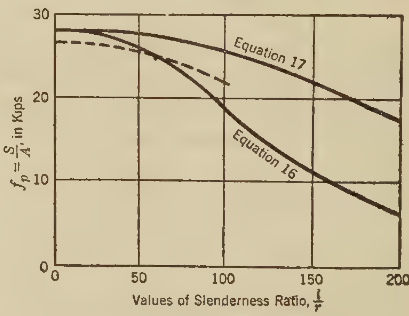

FIG. 9.

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[10] See "Modern Framed Structures", by Johnson, Bryan, and Turneaure, Pt. II, pp. 57–59.

$\frac{L}{r} = 75$, however, is reduced 35%, due to the added buckling tendency from the end moments.

This theory is open to serious criticism as applied to members that develop any considerable curvature of the central line before failure, because, in such cases, the effect of the moment, $S\,y$, is such as to cause the angle, τ, to increase more rapidly than the loading. This effect is ignored in the common (as distinguished from the exact) theory of secondary stresses.

Moment diagrams for two successive loading stages are shown in Figs. 8(*b*) and 8(*c*). It is clear that: (1) $\frac{\tau_1}{\tau_2} = \frac{\text{Area } abcd}{\text{Area } a'b'c'd'}$; and (2), that these ratios, due to the effect of the moment, $S\,y$, cannot be in the proportion of $\frac{M'_n}{M''_n}$. A more thorough analysis presented elsewhere[11] shows that if $q = \sqrt{\frac{S}{EI}}$, $\phi = \frac{qL}{2}$, and α = percentage of secondary stress,

$$f_b = \alpha\, f_p \frac{\phi}{\sin\phi} \qquad \text{(7)}$$

This remarkably simple form admits of a ready comparison with the case in which $M = S\,e$; thus:

$$f_b = \frac{e\,c}{r^2} f_p \sec\phi \qquad \text{(8)}$$

from which is derived the ordinary "secant" formula for columns:

$$f_p = \frac{f}{1 + \frac{e\,c}{r^2} \sec\phi} \qquad \text{(9)}$$

Similarly, by means of Equation (7):

$$f_p = \frac{f}{1 + \frac{\alpha\,\phi}{\sin\phi}} \qquad \text{(10)}$$

It will be found that, since $\frac{\phi}{\sin\phi}$ increases much more slowly than $\sec\phi$, the effect of secondary bending on the buckling of a column is relatively much less than for end moments that increase in direct proportion to the load (as would be the case for eccentric loading). Since $\alpha = \frac{f_b}{f_p}$ in the first case, and $\frac{e\,c}{r^2} = \frac{f_b}{f_p}$ in the latter case, if the same values are taken for these terms (say, α = 25% and f = 36 000) the curves[12] for Equations (9) and (10) will be as shown in Fig. 9.

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[11] *Transactions*, Am. Soc. C. E., Vol. 95 (1931), p. 1221 *et seq.*

[12] *Loc. cit.*, p. 1223.

This analysis is not strictly correct in that, while it seeks to make the tangential angle, and not the end moment, proportional to the load, it assumes the correctness of Equation (3). The correct relation, however, is:[13]

$$M_{mn} = \frac{2\,EI}{L}\,(2\,a\,\tau_{mn} + b\,\tau_{nm}) \quad \ldots\ldots\ldots\ldots\ldots(11)$$

in which, a and b are converging infinite series in $q\,L$. Thus,

$$a = 1 + \frac{(q\,L)^2}{30} - \frac{11}{25\,000}\,(qL)^4 + \quad \ldots\ldots\ldots\ldots\ldots(12)$$

A rigorous analysis of the problem is possible, but the resulting equations are exceedingly involved and are not readily applied. Their interest is largely academic, since a glance at the graph of the approximate equation in Fig. 9 will show that, for all values of $\frac{L}{r}$ less than 70, the effect of secondary stress on buckling action, measured in this case by the ratio, $\dfrac{1 + 0.25}{1 + \dfrac{0.25\,\phi}{\sin\phi}}$, is slight, and for values less than 50, it is quite negligible. This holds true regardless of the value of α (herein assumed as equal to 25 per cent). Two further facts are to be noted:

(1) Extremely high secondary stresses nearly always occur in members bent in an S-curve. The lower section of the end post in sub-paneled trusses, or trusses with a collision strut, is practically the only exception to this rule.

(2) From a consideration of the properties of bridge trusses and the manner in which secondary stresses arise, it is virtually impossible for a high secondary stress to be developed in a member bent in single curvature which, at the same time, is slender enough to develop any considerable buckling action. The exceptions to this rule are too few to be of importance in ordinary design. With a percentage of secondary stress of 35, or less, the dotted curve on Fig. 9 shows that, as far as buckling is concerned, columns with values of $\frac{L}{r}$ ranging from 50 to 80, have practically the same strength as similar pin-ended columns with $\frac{e\,c}{r^2} = 0.25$.

Compression Members Bent in Double Curvature.—When the secondary end moments are in the same direction, the bar will be bent into an S-curve for which the moment diagram is as shown in Fig. 10. (It is again assumed for simplicity that the moments are equal.)

The deflections in this case will be much smaller than for bending in single curvature, and the points of maximum deflection will fall outside the quarter-points. Obviously, in such a case, the effect of secondary bending on the buckling action of the column as a whole is distinctly favorable since it forces the member to bend in double curvature, and the added bending moment due to axial stress is practically negligible. The high flexural stresses are near the ends of the member, and, in general, the effect of the secondary stresses on ultimate strength is the same as for tension members previously discussed; that is, a high extreme fiber stress occurs over a limited region, and in the neighborhood of the yield point the re-adjustment of stress due to plastic flow reduces the effect to negligible proportions before the average stress reaches the yield point.

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[13] "Modern Framed Structures", by Johnson, Bryan, and Turneaure, Pt. II, p. 512.

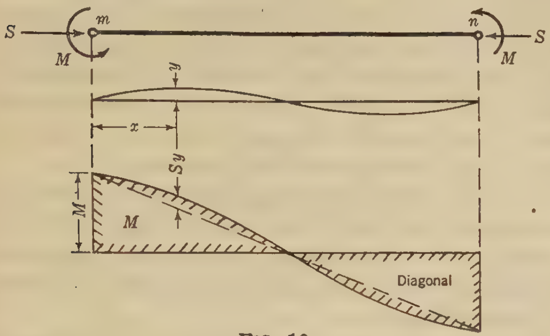

FIG. 10.

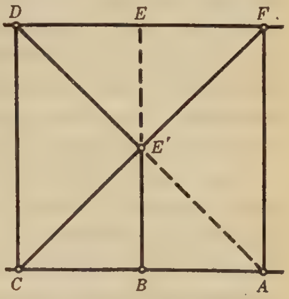

FIG. 11.

Local Buckling in Thin-Walled Compression Members.—Most bridge trusses of the massive type for which the secondary stresses reach high percentages will have relatively stocky chord members $\left(\frac{L}{r} \leqq 40\right)$. Built-up members of such proportions show little or no transverse deflection until the average unit stress passes the proportional limit, and their ultimate strength is ordinarily from 80 to 100% of the yield-point stress of the material. It is generally considered that, for columns of this type, failure is imminent when any considerable part of the section is stressed to the yield point (although the average stress may be considerably less). More often than not the failure occurs through the local buckling of one of the comparatively wide, thin plates that make up the section.

Consider the top chord of a bridge of the conventional box type of section. Assume the material to have a yield point of approximately 35 000 lb per sq in. and the maximum permissible primary unit stress to be 17 000 lb per sq in. The secondary stress will be assumed as 100% of the primary. Under maximum working loads, then, the material on the most stressed side is loaded approximately to the yield point. If the bridge should be subjected to a 50% over-load, and the secondary moments are taken as proportional to the angular changes (that is, if Equation (3) were still to hold), the extreme fiber stress would be 51 000 lb per sq in., although the average would be only one-half this value. It has already been shown that beyond the proportional limit important re-adjustment of stress and strain takes place, resulting in a large percentage decrease in the actual secondary stress. None the less, if the member is assumed to be bent so as to throw the upper side into compression, the entire cover-plate and a portion of the top angles and web (amounting perhaps to 25% of the entire section) will be stressed to the neighborhood of the yield point. It would appear reasonable to suppose that the cover-plate, considered as an independent piece, would be on the verge of buckling as soon as the yield point is reached, and since the high secondary bending causes this state of stress to be reached much sooner than otherwise, it is pertinent to ask whether this may not hasten failure and thus reduce the ultimate strength of the member.

The analysis of the buckling of plates due to loads in their own plane is now well developed. The theory was initiated by Professor G. H. Bryan[14] many years ago; it has been further developed and its applications (particularly to structural problems) greatly extended by Professor S. Timoshenko[15] in his classic memoir on elastic stability. Further important contributions to the subject have been made by Professor H. Reissner,[16] the late Hans H. Rode,[17] M. Am. Soc. C. E., and H. M. Westergaard,[18] M. Am. Soc. C. E.

The most comprehensive analysis of the buckling of plates as parts of bridge members that has yet appeared, so far as the writers know, has been presented by Professor F. Bleich.[19] Professor Bleich attempts to include in his analysis the effects of partial restraint at the edges of the plate and also to extend the results into the "plastic" range.

Some results of these analyses may be summarized briefly. If a plate of length, a, width, b, and thickness, t, is considered to be loaded in the direction of a, with a uniformly distributed compression, $P = f_p\ t\ b$, and restrained against linear (but not angular) displacement along the edges parallel to the loading, Bryan's formula for buckling in a single wave is,

$$f_p = \frac{\pi^2 E}{12\,(1 - m^2)} \left(\frac{t}{b}\right)^2 \left[\frac{a}{b} + \frac{b}{a}\right]^2 \dots\dots\dots\dots\dots(13)$$

in which, m = Poisson's ratio (usually 0.25 to 0.3 for steel).

Bleich proposes using a variable factor, τ, a function of the stress, f_p, to modify the value of E for stresses beyond the proportional limit. For conditions assumed in Equation (13), he derives,

$$f_p = \frac{\pi^2 E \sqrt{\tau}}{12\,(1 - m^2)} \left(\frac{t}{b}\right)^2 \left[\frac{a}{b\sqrt[4]{\tau}} + \frac{b\sqrt[4]{\tau}}{a}\right]^2 \dots\dots\dots(14)$$

For $\tau = 1$, this reduces to Bryan's formula.

From Equation (13), for a fixed ratio of $\frac{t}{b}$, f_p will vary with the quantity in the square bracket and will be a maximum for $a = b$, giving,

$$f_p = \frac{\pi^2 E}{3\,(1 - m^2)} \left(\frac{t}{b}\right)^2 \dots\dots\dots\dots\dots\dots\dots(15)$$

Taking $m = 0.3$, Equation (15) gives 67 900 lb per sq in., and 43 400 lb per sq in., respectively, for $\frac{t}{b} = \frac{1}{40}$ and $\frac{t}{b} = \frac{1}{50}$

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[14] *Proceedings*, London Math. Soc., 1891, p. 54.

[15] "Sur la Stabilité des Systémes Elastiques", *Annales des Ponts et Chaussées*, 1913, Vol. 3, p. 496 *et seq.*

[16] "Uber die Knicksicherkeit ebener Bleche", *Zentralblatt der Bauverwaltung*, 1909, p. 93.

[17] 'Beitrag zur Theorie der Knickerscheinungen", *Der Eisenbau*, 1916, p. 281 *et seq.*

[18] "Buckling of Elastic Structures", *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 576.

[19] "Theorie und Berechnungen der Eisernen Brücken", pp. 216–239, Berlin, Julius Springer, 1924.

If the edges are fully fixed, the foregoing buckling loads are practically doubled. Undoubtedly, the true condition is intermediate between these extremes. These results tend to show that for normal proportions of a compression chord, buckling of the cover-plate will not take place within the elastic limit of the material. Beyond this point, of course, the formula does not apply.

In evaluating his expression for the buckling load in the more general case in which some of the material may be strained beyond the proportional limit, Bleich, after a lengthy analysis, obtains for f_p (in tons per square centimeter),

$$f_p = \frac{\phi}{2} - \sqrt{\frac{\phi^2}{4} - 9.61} \qquad (16)$$

In which, $\phi = \dfrac{\left(\dfrac{b}{t}\right)^4}{E \times 10^4} + 6.2.$

For $\dfrac{t}{b} = \dfrac{1}{40}$, $f_p = 36\,600$ lb per sq in.; and, for $\dfrac{t}{b} = \dfrac{1}{50}$, $f_p = 32\,800$ lb per sq in. Again, it should be noted that these values are for the case of no edge restraint.

Any attempt to take into account the variation in E beyond the proportional limit is attended with much difficulty; hence, Equation (16) cannot be regarded as more than roughly approximate, and is scarcely applicable in the region of large plastic flow.

In applying the buckling formulas to the secondary stress problem, consideration must be given to the important difference between a plate with freely supported edges, loaded in its plane, and, say, a cover-plate of a chord member under combined primary and secondary stress. In the former case, when the stress reaches, or closely approaches, that corresponding to the point, B, on the stress-strain curve of Fig. 2, any further increase will result in distortions many times as great as those previously sustained. Such distortion has the effect of grossly exaggerating any slight imperfection in material or manufacture, and since no plate is perfectly straight or homogeneous, relatively large transverse deflections are developed which lead to immediate collapse.

In the case of a cover-plate subjected to combined primary and secondary stress causing the extreme fiber to reach the yield-point value, it is equally true that any material increase in the stress will probably result in a buckling failure. As has been shown, however, this increase cannot take place as long as the average stress on the section is well below the yield point, since it must be accompanied by a large plastic flow, and, due to the restraint of the adjacent material, such large flow cannot possibly occur in one part of the section until practically the entire area is stressed to the yield-point region. It appears reasonable to suppose that what actually happens is that when the yield-point stress is reached in the cover-plate, a practically constant state of stress is maintained under increasing load, while the deformation (which in this region of large plastic flow may vary widely with almost no change in the stress) varies only as is required to produce the necessary secondary displacements. As long as the average primary stresses are within the proportional limit, these secondary displacements will increase no faster than the applied loads. In other words, the free flow (accompanying a stress reaching the yield point) which would take place in an independent plate, is restrained to comparatively small limits by the action of the neighboring material in the case of axial stress and secondary bending, and this, it may be expected, will have a large effect in reducing the buckling tendency in the cover-plate.

When the average stress over the section passes the proportional limit, the cover-plate distortion begins to increase rapidly, and somewhere between this and the yield point of the column as a whole, local buckling of the cover may be anticipated.

One further point deserves some emphasis. When the common theory of secondary stresses is applied, it is assumed, of course, that the law of linear elasticity applies and that the effect of bending due to $S\,y$ is negligible. When, therefore, as discussed previously, an average primary stress of 26 000 lb per sq in., and a secondary stress equal to 100% of the primary, are computed for any member, what really happens is that the member is distorted at the end so that the tangential angle is such as would correspond to 26 000 lb per sq in. flexural extreme fiber stress, assuming E to be constant and the effect of axial stress on bending negligibly small. If the stress-strain diagram of Fig. 2(*a*), is examined, however, it will be noted after the limit of strict proportionality is passed (that is, when E becomes variable and a function of the load), that there is a considerable deviation from linearity before the region of free plastic flow is reached. The actual deformation at Point B (the beginning of marked plastic flow) will ordinarily be equivalent to that which would be produced by a stress of 40 000 to 50 000 lb per sq in., if the material followed the linear stress-strain law. In other words, the variation in E, before the actual yield point is reached, is sufficient to take care of a nominal extreme fiber stress due to combined primary and secondary effects considerably greater than the yield point of the material, and probably equal to any value to be found in well-designed trusses, even of the heaviest and most rigid type.

The possible effect of stresses beyond the elastic limit upon local buckling, especially of the cover-plates in the standard type chord sections, is the one point upon which the deductions from rational analysis are rather indecisive, and, if the preceding analysis is sound, the only manner in which there is any reason to suspect that the ultimate strength of a compression member may be affected appreciably by secondary stress. These facts seemed to justify an experimental study of the phenomena of failure in a built compression member of the box type subjected to a high percentage of secondary stress, with particular emphasis on the behavior of the cover-plate in the region of high local stress. In the following section, an account is presented of a limited group of laboratory tests undertaken to throw some light on this particular phase of the secondary stress problem.

Experimental Study

Model Simulation of Conditions.—The experimental study consisted of the testing of four columns of symmetrical section of standard types, giving in all eight separate tests of columns under the action of secondary stress and direct axial, or primary, stress. Column No. 1 was made up in the shop of the Experimental Engineering Laboratory, University of Minnesota, Minneapolis, Minn., and the remaining sections were fabricated in the Minneapolis Plant of the American Bridge Company. All specimens were of relatively small cross-section as the testing machine available was not of sufficient capacity to handle a full-sized normal bridge chord.

Because of the difficulty of simulating the partial fixity of the ends of a member in a riveted truss, introducing as it would the problem of controlling and measuring the induced rotations, all specimens were tested with hinged ends. Such a member is closely analogous to the top chord member in a pin-connected sub-paneled truss with the members continuous over the sub-panel points (see Fig. 11). A truss of this type, while nominally classed as pin-connected, may show high secondary stresses at such places as Point *E*. However, it should be emphasized again, the object of the experiments was to study the behavior of built compression members under high local stress due to combined primary and secondary action. As long as the essential features of such action are maintained, it is immaterial for the purposes of the test whether or not all details correspond to those which obtain in the case of an actual truss member under high secondary stress.

It is true, of course, that ordinarily the highest percentages of secondary stress occur in trusses with fully riveted chords. In Fig. 11, Joint E' will deflect about the same amount regardless of the end conditions of Member *D F*. If the end joints are fully riveted, the added stiffness of the member due to the partial restraints at Points *D* and *F* will cause (for approximately constant center deflection) considerably higher secondary stress at Point *E* than would be the case for pin connections. As long as the fundamental characteristics of secondary stress behavior are preserved, however, high stresses, artificially produced, will have the same local effect on a member of given cross-section as if these same stresses were brought about by exact simulation of all the details of riveted truss action. This remark applies also to the method of loading which was followed in the test.

Again referring to Fig. 11, it will be noted that if Member EE' is removed, there will be no secondary stress at Point *E*, since this latter is caused by the displacement of Point E' relative to Line *DF*. This displacement tends to be transmitted to Point *E*, through Member EE' (the blank strut), developing a considerable tension in the latter. The action is identically that of a beam, *DEF*, with a transverse load at Point *E*, with this exception: The deflection of Point E' (and, therefore, of Point *E*) is strictly limited by the relative displacement arising from axial distortions, and which may be computed analytically or by means of a Williot diagram. Quite contrary to the case of a free transverse load at Point *E*, when the displacements have reached the prescribed value, Point *E* is rigidly held in position, no further deflection being possible. This is the unique characteristic of secondary stress action. Clearly, it can be simulated accurately in the case of a column by applying the axial loads, together with a certain controlled deflection. If, in such case, one wishes to study the behavior of the member at a given stage of combined primary and secondary stress, it is immaterial whether (*a*) a given primary stress is applied, and then the desired deflection is induced; (*b*) the desired deflection is induced and then the direct stress is applied; or (*c*), the two are developed simultaneously. The second method was followed in this investigation.

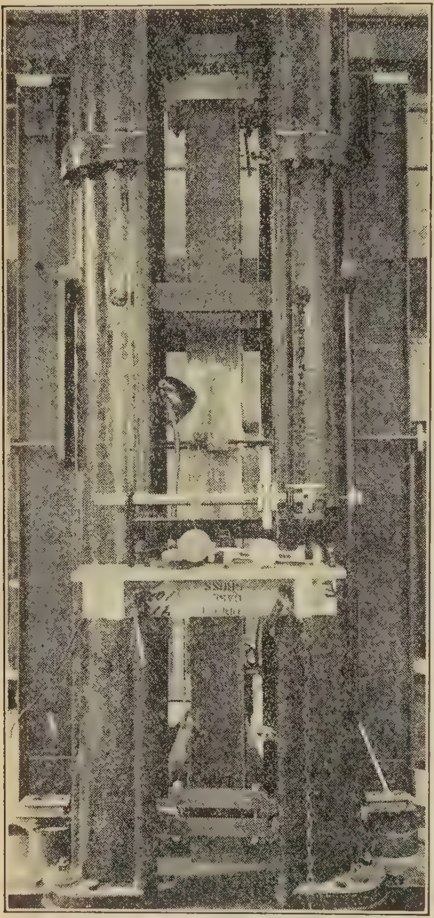

FIG. 12.—SET-UP OF EXPERIMENTS ON SECONDARY STRESSES.

The columns tested were made to conform to the foregoing condition by being subjected to a constant bending deflection and consequent secondary stress as the axial stress was increased. This combination gave maximum effects due to combined secondary and primary stresses at the center line, with primary stresses only acting at the ends. In order to realize this condition the columns tested were placed inside a rectangular frame of rigid I-beam sides with thin flexible ends. The bending deflection of the column was then maintained at a constant value by the use of a toggle extending about the sides of the frame. The flexible end plates were sufficient to transfer the transverse thrust, but not stiff enough to cause appreciable moment on the end of the member or to interfere with the axial loading of the column by the testing machine. In this manner a combination of stresses was maintained in the cover-plate of the column analogous to the combined primary and secondary stresses at the stay connection of Member *DEF* in Fig. 11. The actual amount of the secondary stress was thus dependent upon the transverse deflection given the specimen and was varied somewhat for different sections, depending upon their properties. Reference to the general photograph of the set-up (Fig. 12), and to the detail drawing of Column No. 2 (Fig. 13), will help to clarify the description of the specimen and testing frame.

Column Dimensions and Properties.—Column No. 1, Table 1, was similar to the standard top chord section of a bridge, consisting of plates and angles with a top cover and open bottom (see Fig. 14).[20] Because of the limited capacity of the testing machine and the difficulty of providing the necessary transverse bending force in the toggle, the model was made one-fifth the area of the chord member referred to, as closely as commercial material would allow. (The diameter of the rivets was 0.17 in., the rivet holes being drilled. The rivet spacing in the angles is the same as in the cover-plate.) For Test

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[20] The proportions follow closely the top chord of the typical truss given in "Modern Framed Structures", by Johnson, Bryan, and Turneaure, Pt. III, Pl. III.

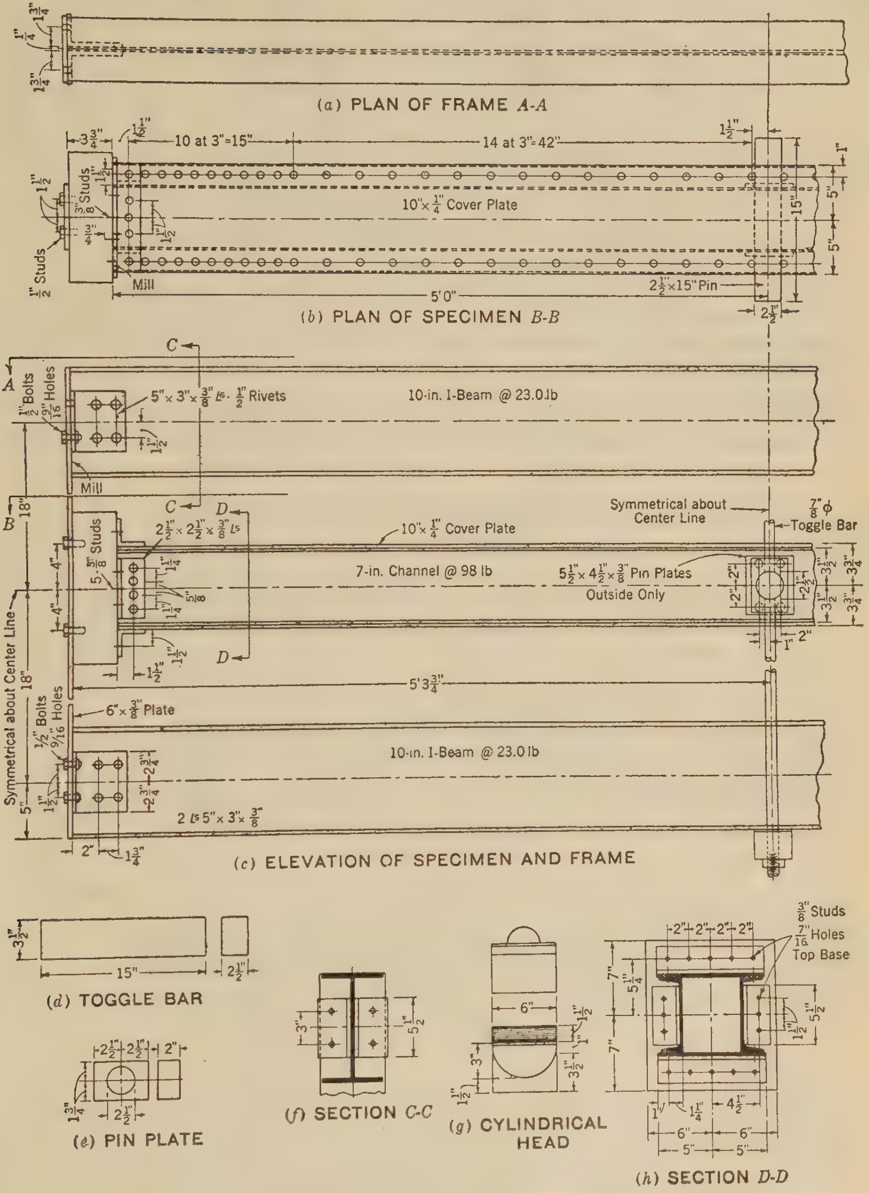

(a) PLAN OF FRAME A-A
(b) PLAN OF SPECIMEN B-B
10 at 3"=15"
14 at 3"=42"
10"×1/4" Cover Plate
Studs
Mill
5'0"
2 1/2"×15" Pin
(c) ELEVATION OF SPECIMEN AND FRAME
5"×3"×3/8" Ls - 1/2" Rivets
10-in. I-Beam @ 23.0 lb
1/2" Bolts
9/16" Holes
Symmetrical about Center Line
7/8" φ Toggle Bar
2 1/2"×2 1/2"×3/8" Ls
10"×1/4" Cover Plate
7-in. Channel @ 9.8 lb
5 1/2"×4 1/2"×3/8" Pin Plates Outside Only
5'3 3/4"
6"×3/8" Plate
2 Ls 5"×3"×3/8"
(d) TOGGLE BAR
15"
(e) PIN PLATE
(f) SECTION C-C
(g) CYLINDRICAL HEAD
6"
3/8" Studs
7/16" Holes
Top Base
(h) SECTION D-D

FIG. 13.—DETAILS AND DIMENSIONS OF COLUMN NO. 2.

TABLE 1.—CHARACTERISTICS OF TEST COLUMNS

Column No.	Test	Area, A, in square inches	Length, L, in inches	Moment of inertia, I, in inches *	Distance, c, from neutral axis to extreme fiber, in inches	Radius of gyration, r, in inches $= \sqrt{\frac{I}{A}}$	Slenderness ratio, $\frac{L}{r}$	Thickness, t, of cover-plates, in inches	Distance, b, between center lines of rivet rows, in inches	Ratio, $\frac{b}{t}$	Thickness, w, of web, in inches	Rivet spacing, p, in inches at center line	Diameter of rivets, D, in inches
1	1	3.91	86.3	23.5	3.04	2.45	35.2	⅛	5⅛	41	⅛	2	0.17
	2	.....	80.6				33.0						
2	.....	10.70	128.2	108.2	3.75	3.18	40.3	¼	8	32	0.24	3	⅜
3	.....	9.82	128.2	91.2	3.69	3.13	41.0	$\frac{3}{16}$	9	48	0.24	2¼	⅜
4	End	9.82	.....	.....	......	....	....	....	....	....	....	.....	.....
	Center	.....	128.2	91.2	3.69	3.13	41.0	$\frac{3}{16}$	9	48*	0.24	2¼	⅜
	Line	10.99	.....	.....	......	....	....	....	....	....	....	.....	.....

* Reinforced.

2, Column No. 1 was the same as for Test 1, except that it was shorter. The only change in the riveting was in the cover (see Fig. 14(*d*)) for which a new

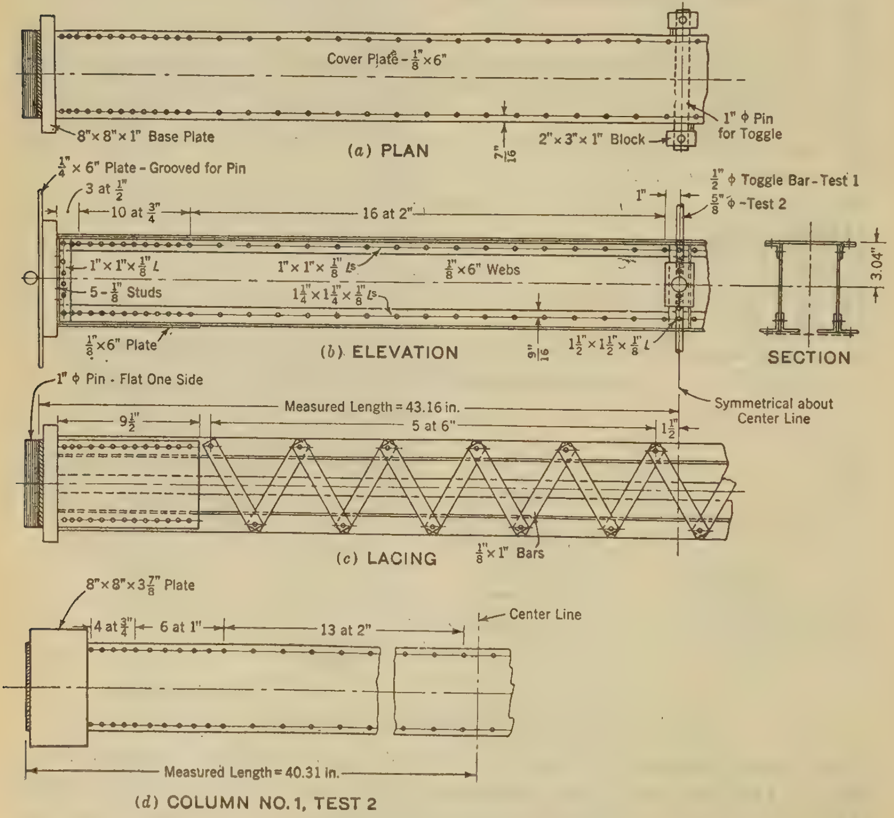
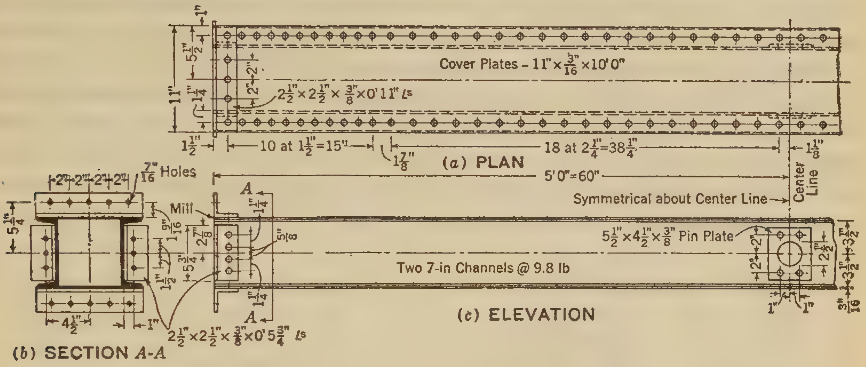

Cover Plate - 1/8"×6"
8"×8"×1" Base Plate
(a) PLAN
7/16"
2"×3"×1" Block
1" ϕ Pin for Toggle
1/4" × 6" Plate - Grooved for Pin
3 at 1/2"
10 at 3/4"
16 at 2"
1"
1/2" ϕ Toggle Bar - Test 1
5/8" ϕ - Test 2
1"×1"×1/8" L
1"×1"×1/8" Ls
1/8"×6" Webs
5 - 1/8" Studs
1 1/4"×1 1/4"×1/8" Ls
1/8"×6" Plate
(b) ELEVATION
9/16"
1 1/2"×1 1/2"×1/8" L
SECTION
3.04"
1" ϕ Pin - Flat One Side
Symmetrical about Center Line
Measured Length = 43.16 in.
9 1/2"
5 at 6"
1 1/2"
(c) LACING
1/8"×1" Bars
8"×8"×3 7/8" Plate
4 at 3/4"
6 at 1"
13 at 2"
Center Line
Measured Length = 40.31 in.
(d) COLUMN NO. 1, TEST 2

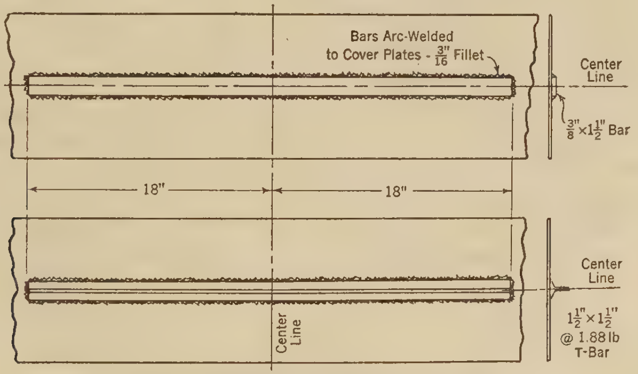
FIG. 14.—DETAILS AND DIMENSIONS, COLUMN NO. 1.

base plate was provided; otherwise, the details at the center line and ends were the same in both cases. Columns Nos. 2, 3, and 4 were of slightly different section, being made of two standard channels with two cover-plates (closed). Because of the symmetry of the section, two tests could be made from each specimen. They were fabricated in the standard manner using punched holes and $\frac{3}{8}$-in., hot-driven rivets, while Column No. 1 had small cold-driven rivets in drilled holes. The ends of the members were milled to provide full bearing. The properties and detail drawings of the various column sections are shown in Figs. 13 to 16. Column No. 4 was the same as Column No. 3 (Fig. 15) except for the welded bars shown in Fig. 16.


(a) PLAN
(b) SECTION A-A
(c) ELEVATION

1"
5½"
11"
1¼"
2"
2"
Cover Plates - 11" × 3/16" × 10' 0"
2½" × 2½" × ⅜" × 0' 11" ∠s
1½"
10 at 1½" = 15"
1⅞"
18 at 2¼" = 38¼"
1⅛"
5' 0" = 60"
Symmetrical about Center Line
Center Line
2" 2" 2" 2"
7/16" Holes
Mill
5¼"
1 9/16"
2⅞"
5¾"
1½"
A
1¼"
⅝"
A
4½"
1"
2½" × 2½" × ⅜" × 0' 5¾" ∠s
Two 7-in Channels @ 9.8 lb
5½" × 4½" × ⅜" Pin Plate
2"
2"
2½"
3½"
3½"
3/16"
1"
1"

FIG. 15.—DETAILS AND DIMENSIONS, COLUMN NO. 3.

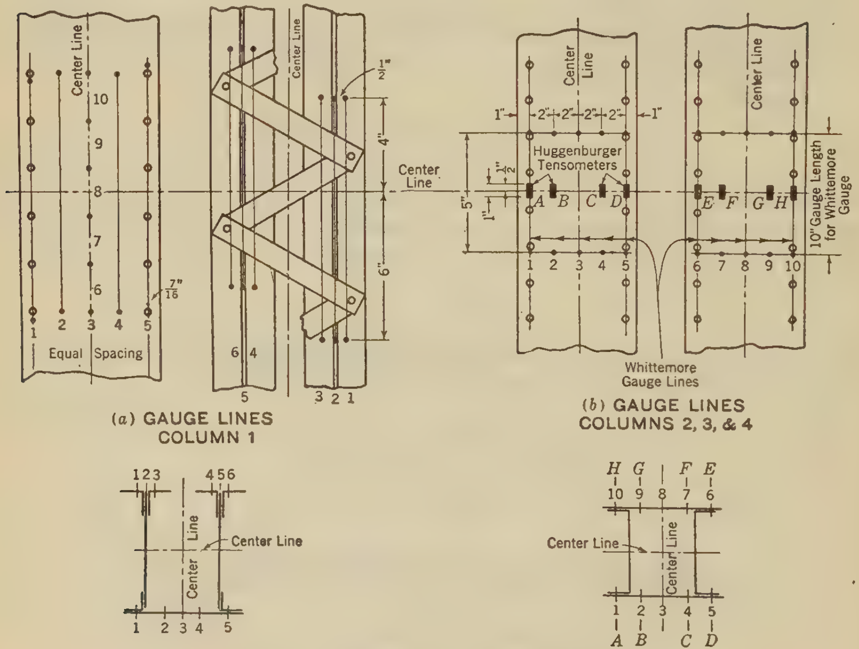


Bars Arc-Welded to Cover Plates - 3/16" Fillet
Center Line
⅜" × 1½" Bar
18"
18"
Center Line
1½" × 1½" @ 1.88 lb T-Bar
Center Line

FIG. 16.—BARS WELDED TO COVER-PLATES, COLUMN NO. 4.

Test Procedure.—The axis of the member was set as nearly vertical in the testing machine as could be done with a hair wire and weight. The pins were placed at the centroid of the section as closely as this could be measured and transferred to the base plate. Pins were flattened on one side to fix the point of rotation and to give a definite value of L in the plane considered. They were supported on cylindrical heads in the other plane to fit them to any angularity between the column ends and the testing machine. The detail drawing of Column No. 2 (Fig. 13) will make the test set-up clear.

The initial loading used in all cases was 1 000 lb and the increments of axial loading were equivalent to 2 500 lb per sq in. on the section for the first test, and 5 000 lb per sq in. for the second test, on each specimen. Strain readings were taken at all increments of loading and, in many cases, loadings were repeated several times to establish the behavior of the member more definitely. Strains were obtained on Column No. 1 by the Whittemore strain-gauge, and on Columns Nos. 2, 3, and 4 by both Huggenberger tensometers and the Whittemore gauge. Consistent deformations were obtained to 0.00001 in. per in., or equivalent stresses of approximately 300 lb per sq in., with the aforementioned apparatus. Gauge lines on Column No. 1 consisted of five 10-in. lines on the compressive face, and six 10-in. lines on the tensile face placed symmetrically about the center line. Gauge lines on Columns Nos. 2, 3, and 4 consisted of one 10-in. line and four 1-in. lines on each face symmetrically placed. (See Fig. 17.)


Center Line
10
9
8
7
6
1
2
3
4
5
Equal Spacing
7"/16
Center Line
1"/2
4"
6"
Center Line
6
4
5
3 2 1
1"
2"
2"
2"
2"
1"
Huggenburger Tensometers
5"
1/2"
1"
A B C D
E F G H
Center Line
10" Gauge Length for Whittemore Gauge
1 2 3 4 5
6 7 8 9 10
Whittemore Gauge Lines
123
456
Center Line
Center Line
1 2 3 4 5
H G F E
10 9 8 7 6
Center Line
Center Line
1 2 3 4 5
A B C D

(a) GAUGE LINES COLUMN 1

(b) GAUGE LINES COLUMNS 2, 3, & 4

FIG. 17.—GAUGE LINE DIMENSIONS.

As the deflection of the column was to be maintained at a constant value such that the nominal secondary stress remained constant throughout the range of primary loadings below the material yield point, it was necessary to measure the deflection value accurately. This was accomplished by means of an optical micrometer in the case of Column No. 1 and an electrical contact screw micrometer for Columns Nos. 2, 3, and 4, measuring the deflection between a hair wire and a point on the specimen center line in the web. It must be noted that the actual column deflection was not measured in any case, as the hair wire was not fastened to the point of rotation of the column ends, but to some point on the deflecting structure. This had no effect on the problem because it was only necessary to have a relative value that could be measured and maintained. The deflection was not used directly for stress calculations, but was kept at a constant relative value in order to give a certain nominal secondary stress. In the case of Columns Nos. 2, 3 and 4, the wire was fastened so closely to the point of rotation that approximately the actual deflection values were read, while in the case of Column No. 1 the wire was fastened to the one-tenth points on the specimen. This fastening necessitated a calculation of the elastic curve of the column so that an approximate value of the relative deflection between the one-tenth points and the center would be available. The elastic curve was calculated carefully, and it was noted that little change occurred in the deflection values for a large range of axial loadings. The equation for the elastic curve based on a constant center deflection and variable axial loading takes the following form:

$$y = \frac{\Delta}{\frac{\mathrm{t}\cdot\left(\frac{\cdot}{}\right)}{q} - \frac{L}{2}}\left[\frac{1}{q}\,\frac{\sin(q\,x)}{\cos\left(\frac{}{2}\right)} - x\right] \qquad \text{..........(17)}$$

in which, Δ = center deflection and $q = \sqrt{\frac{}{EI}}$.

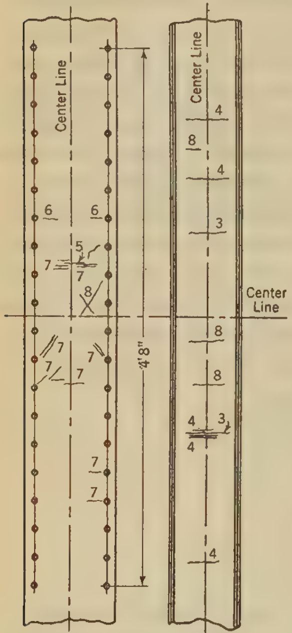
It was decided to use a secondary stress of approximately 13 000 lb per sq in. for Column No. 1, Test 1, giving roughly 50% secondary stress at the material yield point. Since this test indicated that such a value was insufficient to cause local failure to develop before the normal ultimate strength of the columns was reached, in all later tests a value of 20 000 lb per sq in. was used in an attempt to produce a more marked local effect. Using approximately a secondary stress of 20 000 lb per sq in., the required deflection of the column was calculated and this value used as a basis for the approximate measurement in the tests. Table 2 gives calculated deflections and lateral loads corresponding to required secondary stress values. (The action of the column under lateral load was that of a simple beam prior to the application of axial loadings.)

TABLE 2.—CALCULATED LATERAL LOADS AND DEFLECTIONS FOR ASSUMED SECONDARY STRESS

Column No.	Lateral load, P, in pounds	Displacement, Δ, in inches	Stress, f', in pounds per square inch
1......................	4 750	0.0900	13 250
2......................	19 200	0.2685	21 300
3......................	16 200	0.2556	20 000
4......................	16 200+	0.2556	20 000

During the testing operations the deflection was checked and corrected if necessary at each increment of axial loading, and when the axial stress reached a value sufficient to reduce appreciably the lateral force necessary to maintain the deflection, the stability of the columns was assured by stressing the toggle on the compressive face, thereby preventing sudden $\frac{L}{r}$-failure in the direction of the imposed deflection. This additional toggle was also used in adjusting the deflection after the bending resistance of the column had been reduced due to plastic flow, and the original transverse force would have caused more than the required deflection.

Before testing, the specimen was painted with a dilute solution of plaster of Paris and water to make the strain lines visible. These are clearly shown in Fig. 19. The strain lines were recorded and sketched as they appeared during the test and a typical set is shown in Fig 18, with Table 3. Determination of the transverse force necessary to maintain the required deflection was accomplished by strain measurements on the toggle-bars and subsequent calculation of the force transmitted. As an indication of the reduction in moment and consequent reduction in the transverse force, P, as the material in the column cover-plate approached the yield point, and stress re-adjustment took place, the measured P-values are given in Table 4 for Columns Nos. 2 and 3.



(a) COVER PLATE (b) WEB

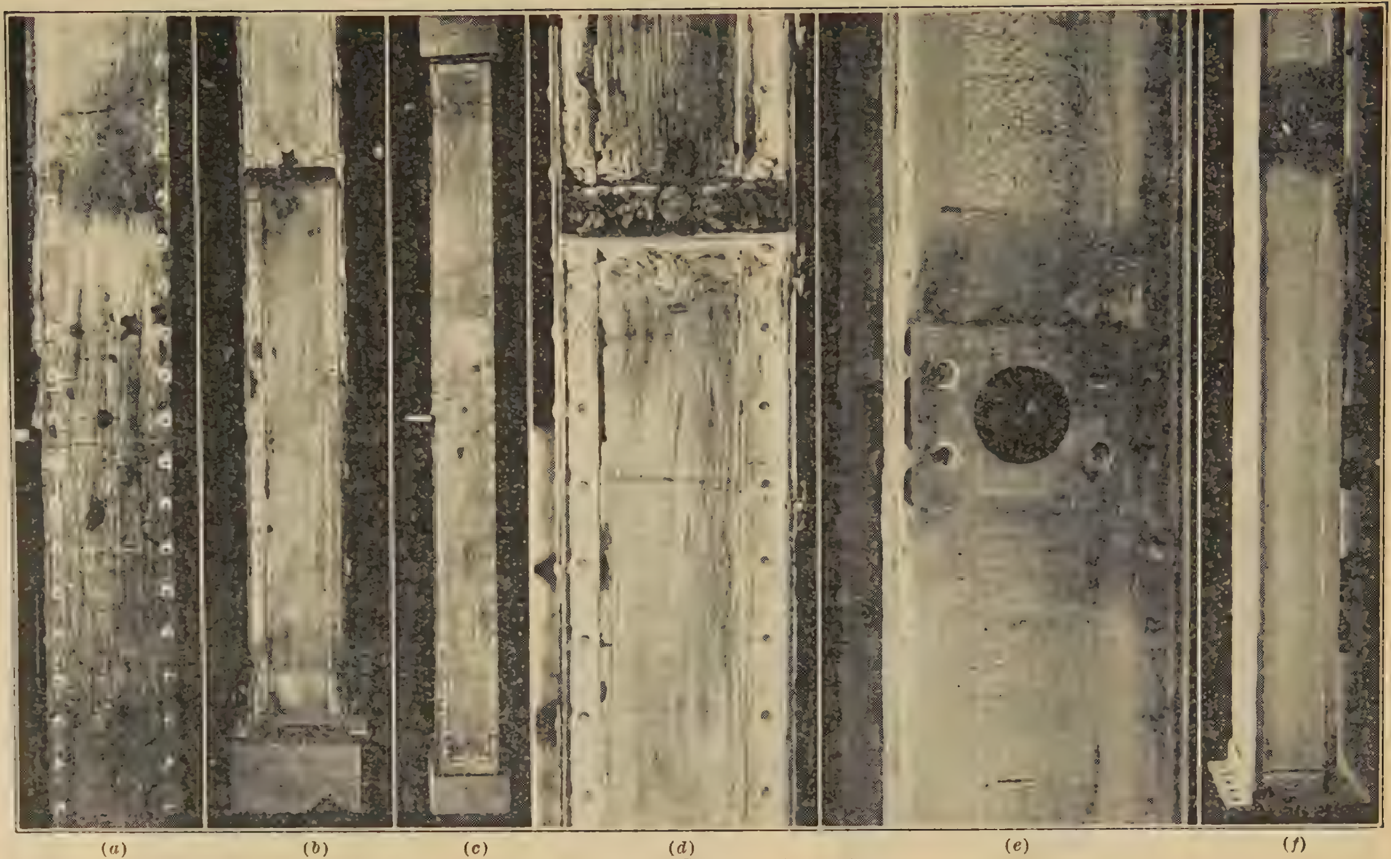
FIG. 18.

TABLE 3.—OBSERVATIONS OF STRAIN LINES, TESTS 3 AND 4

TEST 3		TEST 4	
Strain Line No. (see Fig. 18):	Load, in pounds	Strain Line No. (see Fig. 18):	Load, in pounds
1........................	161 500*	5........................	157 000
2........................	188 250*	6........................	215 000
3........................	301 000	7........................	268 500
4........................	339 000	8........................	322 000
....	....	....	354 000†

* Lines at Loads 1 and 2 for face not shown in Fig. 18.
† Innumerable lines appeared at this load.

For the purpose of this paper, it was not considered desirable to reproduce all the test data in detail. Tables 5 and 6 present the data from Column No. 1, Test 2, and Column No. 2, Test 3, showing deformations, in millionths of an inch per inch, for various axial loadings. These data are presented as typical examples of all eight tests, there being remarkably close agreement in all cases.



(a) (b) (c) (d) (e) (f)

FIG. 19.—VIEWS OF COLUMNS AFTER TESTING.

TABLE 4.—OBSERVED VALUES OF LATERAL LOAD, P.

($E = 30.0 \times 10^6$)

Primary load, S, in pounds	LOADING RODS ($\frac{7}{8}$-INCH ROUND; $A = 0.60$ SQUARE INCH)			STAY-RODS ($\frac{5}{8}$-INCH ROUND, $A = 0.30$ SQUARE INCH)			Total load, P, in pounds
	Deformation, in millionths of an inch per inch	Stress, f', in pounds per square inch	Load, P, in pounds	Deformation, in millionths of an inch per inch	Stress, f', in pounds per square inch	Load, P, in pounds	
			(a) COLUMN NO. 2, TEST 3				
1 000*	564	16 900	20 300	....	....	....	20 300
54 500*	551	16 550	19 800	....	....	....	19 800
108 000*	538	16 150	19 400	....	....	....	19 400
161 500*	493	14 800	17 800	....	....	....	17 800
188 250*	497	14 900	17 900	171	5 130	3 080	14 820
215 000*†	495	14 850	17 800	264	7 930	4 760	13 040
			(b) COLUMN NO. 3, TEST 5				
1 000*	515	15 500	18 600	....	....	....	18 600
50 100*	....	....	....	18.5	550	330	18 270
123 800*	463.5	13 900	16 700	....	....	....	16 700
148 300*	....	....	....	40	1 200	720	15 980
197 400†	....	....	....	167	5 010	3 010	13 600

* Column is in deflected position.
† Readings discontinued at this point.

The ultimate loads for Tests 2 and 3 were 167 800 lb and 381 800 lb, respectively, corresponding to a stress of 42 800 lb per sq in. and 35 700 lb per sq in. In Column No. 1 (see Table 5), failure occurred first in the web, near the base, after which the cover-plate buckled. A set of 80 millionths of an in. per in, is shown on the tension side, which indicates that even this side was stressed beyond the elastic limit (compression forces).

TABLE 5.—OBSERVED DEFORMATIONS, COLUMN NO. 1 (TEST 2), IN MILLIONTHS OF AN INCH PER INCH, FOR VARIOUS AXIAL LOADINGS

(+ = Lengthening (Tension))

Loads, S, in pounds	READINGS AT GAUGE LINES (SEE FIG. 17):											No. of sets	AVERAGES	
	1	2	3	4	5	1	2	3	4	5	6		Cover-plate	Bottom angles
	Cover-Plate					Bottom Angles and Web								
1 000 to 140 000*	−1 112	−1 113	−1 139	−1 156	−1 164	−1 061	−1 094	−1 080	−1 112	−1 147	−1 132	2	......	......
1 000†	−624	−581	−590	−579	−586	+674	+831	+701	+672	+947	+700	2	−592	+686
40 000†	−973	−944	−936	−934	−941	+392	+557	+424	+393	+660	+412	2	−945	+405
80 000†	−1 298	−1 273	−1 272	−1 276	−1 285	+85	+239	+120	+60	+313	+74	2	−1 281	+85
100 000†	−1 728	−1 758	−1 800	−1 814	−1 721	−98	+55	−75	−115	+141	−101	2	−1 764	−97
120 000†	−2 666	−2 617	−2 819	−2 784	−2 662	−325	−174	−321	−359	−106	−344	2	−2 710	−337
140 000†	−3 530	−3 579	−3 713	−3 693	−3 427	−676	−559	−672	−728	−505	−718	1	−3 588	−690
1 000	......	......	......	......	......	−61	−7	−107	−133	......	−9	2	......	−89

* Column condition O. K. † Column is in deflected position.

Properties of the Column Material.—The modulus of elasticity of the material was determined from the load-strain relations of the column under axial loadings only, and checked in several cases by similar tests on coupons

TABLE 6.—OBSERVED DEFORMATIONS, COLUMN NO. 2 (TEST 3), IN MILLIONTHS OF AN INCH PER INCH, FOR VARIOUS AXIAL LOADINGS (+ = Lengthening (Tension))

Load, S, in pounds	READINGS AT GAUGE LINES (SEE FIG. 17 (*b*)):										No. of sets	AVERAGES			
	3	*A*	*B*	*C*	*D*	8	*E*	*F*	*G*	*H*		*A* and *D*	*B* and *C*	*E* and *H*	*F* and *G*
	Compression Cover-Plate					Tension Cover-Plate									
1 000*	—743	—703	—755	—756	—709	+742	+687	+782	+842	+745	13	—706	—756	+716	+812
27 750*	—829	....	....	....	....	+665	.....	.....	.....	.....	4	......	......	......	....
54 500*	—915	—873	—917	—923	—863	+564	+527	+595	+650	+563	8	—868	—920	+545	+622
81 250*	—1 003	—958	—1 002	—1 008	—940	+458	+443	+502	+543	+468	2	—949	—1 005	+455	+522
108 000*	—1 104	—1 076	—1 122	—1 100	—1 008	+374	+359	+408	+446	+383	2	—1 042	—1 111	+371	+427
134 750*	—1 241	—1 170	—1 275	—1 175	—1 093	+269	+284	+306	+342	+297	2	—1 132	—1 225	+290	+324
161 500*	—1 436	—1 272	—1 267	—1 275	—1 195	+171	+200	+204	+243	+210	2	—1 233	—1 271	+205	+224
188 250*	—2 088	—1 316	—1 232	—1 200	—1 880	+ 11	+ 99	+ 60	+ 93	+ 81	2	—1 598	—1 216	+ 90	+ 76
215 000*	—2 557	—1 520	—1 745	—1 530	—4 560	—109	+ 15	— 51	— 12	+ 3	2	—3 040	(Buckle)	+ 9	— 31
251 000*	—3 796	—2 410	—2 475	....	....	—326	—136	—255	—233	—195	2	......	......	—166	—244
301 000*	—5 631	....	....	....	....	—671	—379	—578	—594	—479	2	......	......	—429	—586

* Column in deflected position.

cut from the column itself. Table 7 gives the moduli for each column as determined by the tests on the columns themselves. These values were used in determining stresses.

TABLE 7.—MODULI OF ELASTICITY, E, IN POUNDS PER SQUARE INCH, AS DETERMINED BY TESTS

Column No.	Instruments used	Range of loading stress, in pounds per square inch	Average modulus of elasticity	No. of sets
1.............	Whittemore.........	20 000	30.7×10^6	14
2.............	Whittemore......... Huggenberger.......	20 000 20 000	27.7×10^6 28.8×10^6	12 24
3.............	Whittemore......... Huggenberger.......	20 000 20 000	28.8×10^6 28.6×10^6	10 32
4.............	Whittemore......... Huggenberger.......	10 000 10 000	29.0×10^6 29.0×10^6	6 6

As it was necessary to conduct a part of the test with some of the material stressed beyond the elastic limit, the characteristic stress-strain curves for the material were obtained in order that an estimate of the stress increase after passing the elastic limit could be made. A piece of material from the same rolling was used to determine the relations for Column No. 1 (see Fig. 20, Curve *A*). For Columns Nos. 2, 3, and 4, a coupon was cut from the cover-plate of Column No. 2 (see Fig. 20, Curves *B* and *C*). The modulus of elasticity is seen to check closely that determined from the columns. It is noted that the curve is nearly horizontal beyond the proportional limit even though the deformation was several times that of the yield-point stress. On this basis the stress carried by any material beyond the yield point was assumed to be a constant and of intensity equal to the yield-point value.

An inspection of Fig. 20 shows that the increase in stress after the proportional limit is reached and before the curve becomes so flat as to make the stress nearly constant is approximately 3 500 lb per sq in. for Column No. 1 and from 2 500 to 3 000 lb per sq in. for the remaining columns. The proportional limit for the material of Column No. 1 was taken to be 39 000 lb per sq in., which gave a value of 42 500 lb per sq in. for maximum stress, a close check on the ultimate strength of the column. Similarly, the proportional limit of the materials of Columns Nos. 2 to 4, inclusive, was found to be 32 500 lb per sq in., with a maximum value of approximately 35 000 lb per sq in., which is also a good check on the ultimate strength of the columns.

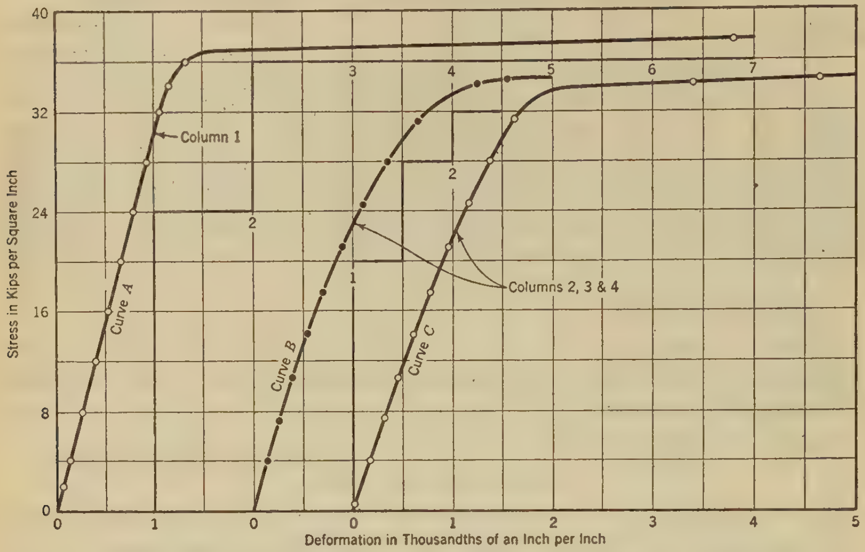


FIG. 20.—STRESS-STRAIN CURVES FOR MATERIAL IN TEST COLUMNS.

In computing stresses in the cover-plates from the measured strains, it must be remembered that the deformations obtained by the Whittemore gauge are average values over a 10-in. gauge length. (The Huggenberger gauges have 1-in. gauge lengths.) When the column is deflected under transverse moments these deformations do not represent the maximum strains, inasmuch as the strains due to bending vary from a maximum at mid-length to zero at the end of the column. This results in an appreciable change in the resultant unit deformation within the gauge length of the instrument when long gauge lengths are used. An approximation to the difference between the maximum and the average reading may be estimated on the basis of the moment change. Since the moment is nearly proportional to the distance from the end of the column (concentrated load), the moment diagram approximates a triangle and the deformation decrease is proportional to the average decrease in length from the end. In this case (L = 128.2 in. for Columns Nos. 2. 3. and 4), the percentage of change equals

$$\frac{\frac{5}{2} \times 100}{\frac{128.2}{2}} = 3.9.$$

Since the deformation from bending was approximately one-half the total, the error involved in neglecting this factor was approximately 2 per cent. As this value is approximately the probable error in reading, loading, and sectional area, it has been neglected in the curves and stress calculations.

TABLE 8.—STRESSES DETERMINED FROM STRAINS IN TABLE 5
(+ = Tension)

Load, S, in pounds	Deformations, in Millionths of an Inch per Inch		Stresses, in Kips per Square Inch	
	Cover-plate	Bottom angles	Cover-plate	Bottom angles
1 000*............	— 592	+686	—18.2	+21.0
40 000*............	— 945	+405	—29.0	+12.4
80 000*............	—1 281	+ 85	—39.4	+ 2.6
100 000*............	—1 764	— 97	—54.2	— 3.0
120 000*............	—2 710	—337	—83.3	—10.3
140 000*............	—3 588	—699	....	—21.5
167 800*............	....	....	....	—40.0†
1 000............	....	— 80	....	....

* Column in deflected position.
† Estimated from a set of 80 millionths of an inch per inch at 1 000 lb. with specimen not deflected.

Tables 8 and 9 give the stresses for the two tests as determined from the strains given in Tables 5 and 6 and the moduli of elasticity given in Table 7. After the proportional limit has been passed these stresses are incorrect, and the values are only given to show the nominal stresses that would occur in the plate were Hooke's law still operative. The actual values after the elastic limit has been passed, have been discussed previously.

TABLE 9.—DEFORMATIONS AND STRESSES DETERMINED FROM STRAINS IN TABLE 6
(+ = Tension)

Load, S, in pounds, with column in deflected position	Whittemore Gauge				Huggenberger Gauges; Cover-Plate							
	Compression		Tension		Compression				Tension			
	Strain, in millionths of an inch per inch, Gauge Line 3	Stress, f, in kips per square inch	Strain, in millionths of an inch per inch, Gauge Line 8	Stress, f, in kips per square inch	Strain, in millionths of an inch per inch, Gauge Lines A and D	Stress, f, in kips per square inch	Strain, in millionths of an inch per inch, Gauge Lines B and C	Stress, f, in kips per square inch	Strain, in millionths of an inch per inch, Gauge Lines E and H	Stress, f, in kips per square inch	Strain, in millionths of an inch per inch, Gauge Lines F and G	Stress, f, in kips per square inch
1 000	— 743	—20.6	+742	+20.55	— 706	—20.3	— 756	—21.8	+716	+20.6	+812	+23.4
54 500	— 915	—25.3	+564	+15.6	— 868	—25.0	— 920	—26.5	+545	+15.7	+622	+17.9
81 250	—1 003	—27.8	+458	+12.7	— 949	—27.2	—1 005	—29.0	+455	+13.1	+522	+15.1
108 000	—1 104	—30.6	+374	+10.4	—1 042	—29.4	—1 111	—32.0	+371	+10.7	+427	+12.3
134 750	—1 241	—34.4	+269	+ 7.45	—1 132	—32.6	—1 225	—35.3	+290	+ 8.35	+324	+ 9.3
161 500	—1 436	—39.8	+171	+ 4.7	—1 233	—35.6	—1 271	—36.6	+205	+ 5.9	+224	+ 6.45
188 250	—2 088	—57.8	+ 11	+ 0.3	—1 598	—46.0	—1 216	—35.0	+ 90	+ 2.6	+ 76	+ 2.2
							(Buckle)					
215 000	—2 557	......	—109	— 3.0	—3 040	......	—1 637	......	+ 9	+ 0.26	— 31	— 0.89
251 000	—3 796	......	—326	— 9.0	....	......	—2 475	......	—166	— 4.78	—244	— 7.02
301 000	—5 631	......	—671	—18.6	....	......	....	......	—429	—12.4	—586	—20.4

Re-Adjustment in Stress-Strain Relations.— By plotting the measured stresses on the compressive and tensile faces of the column with respect to a fixed line, a graph of stress distribution across the section was obtained. After the proportional limit is passed on the compressive side this graph is no longer a straight line, and, finally, the part of the curve beyond the material yield point becomes asymptotic to the line representing the ultimate unit strength of the column. The gauge lines used in determining the stresses plotted are given on the graphs shown, so that by reference to Fig. 17, the section referred to will be readily identified. Fig. 21 contains examples of these curves for the data of Tables 8 and 9. It should be emphasized again that the curves presented are nearly identical with those of the other tests.

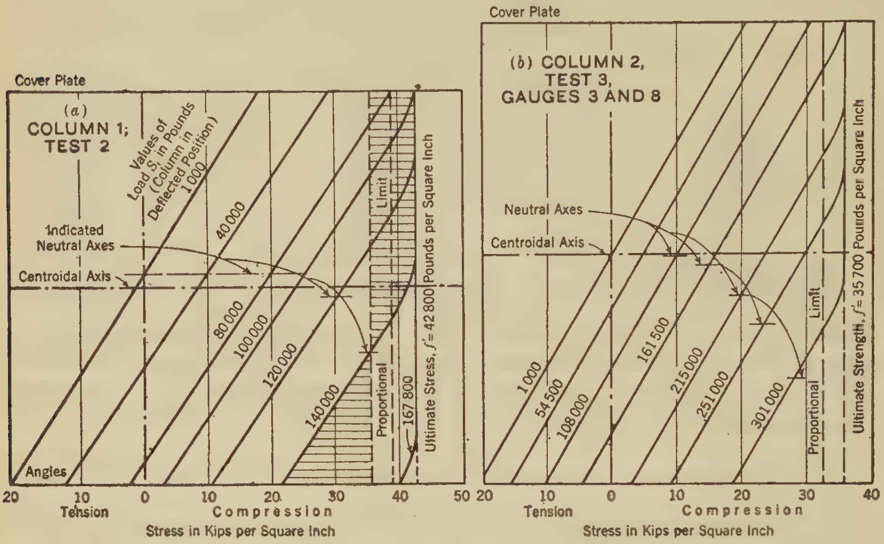


FIG. 21.—STRESS DISTRIBUTION (AVERAGE READINGS), COVER-PLATES AND BOTTOM ANGLES.

In Fig. 21(*a*), the curve for the 140 000-lb load has been shaded for the purpose of illustrating the re-adjustment more clearly. This curve shows a stress on the tensile face of 22 000 lb per sq in. in compression, and if the straight portion were extended, would give a stress of approximately 57 000 lb per sq. in. on the compressive face. Since the yield point was determined as 39 000 lb per sq in., and the ultimate testing strength as 42 800 lb per sq in., it is evident that such a high nominal stress cannot occur and that a re-adjustment consequent upon plastic flow must take place as indicated by curvature of the graph. The direction of the straight portion was made parallel, in all cases, to the last curve on which the end points were within the elastic limit. The shaded area represents the bending stress over the section, while the heavy dashed line (which intersects the curve at the indicated neutral axis) shows the axial stress. Fig. 21(*a*) shows this section to be almost completely under the action of axial stress at failing load.

Cover-Plate Stiffeners.—The views of Column No. 3, in Figs. 19(*e*) and 19(*f*) show a buckle occurring at the point of critical stress. (This column had a cover-plate relatively thinner than permitted by standard specifications.) The significance of this action will be discussed later. It will suffice here to note that it seemed worth while to investigate the feasibility of the application of a simple stiffening device to the plate. This was done in the case of Column No. 4 which (see Fig. 16) is a duplicate of Column No. 3, except for the plate stiffeners. No special attempt was made to develop the most practical device for commercial use; nor was any attempt made to follow a welding schedule and to reduce heating effects. The object in view was to test the effectiveness of the simplest type of stiffeners in supporting the wide, thin cover-plate when buckling was impending, so that it might suffer a certain amount of plastic deformation without "dishing" out of its original shape. Since little force is necessary to restrain such a plate from bending, small sections were welded to the cover-plate to give the necessary support against displacement normal to its plane, and thus, in effect, to cut down the free width. It was felt that if a simple device could be applied which would make the thin plate as resistant to buckling as a thicker one, this would be a desirable result regardless of the fact that in the specimens tested the plate buckling did not materially reduce the ultimate strength.

Discussion of Results

For specimens of the dimensions and character used in this study, it would be reasonable to expect that a high degree of integrity of section would be maintained under an intensity of stress corresponding to ordinary working values. On Column No. 1, however, a noticeable slip was observed between the web and the flange. The deformation difference amounted to approximately 30% of the bending strain. For the plate and channel sections used in Columns Nos. 2, 3, and 4, no direct measurement of the slip was made as in Column No. 1. However, since the stress-deflection values checked rather closely, it was concluded that no appreciable slip had occurred.

Fig. 19 shows strain lines occurring on both flanges of the specimen and across the web joining the two flanges. Since the nominal secondary stress due to bending was approximately 20 000 lb per sq in., such strain lines could not have occurred on the tensile side, under elastic conditions, until the average axial loading had reached 52 000 lb per sq in. (the yield point being approximately 32 000 lb per sq in.). As the ultimate axial load was approximately 35 000 lb per sq in., it is evident that the secondary bending was relieved by the plastic condition on the compressive face, which permitted a given angular change to take place with a much lower extreme fiber stress. This angular change, in turn, reduced the tensile flexural stress, with the result that the stresses due to the axial load of 35 000 lb per sq in. were sufficient to cause appreciable yield. That a complete re-adjustment resulting in a nearly uniform distribution over the section was the final state of stress, is evident from an examination of the strain lines on the tensile cover-plate, which, while nominally holding a stress of 15 000 lb per sq in., passed the elastic limit despite the fact that the ultimate average axial stress was only a little greater than the value of the elastic limit as measured.

The stress distribution curves (of which Fig. 21 is characteristic) bear out further the action indicated by Fig. 19. They show a redistribution of stress within the plastic range in complete agreement with the theoretical analysis presented previously. As the loading is increased, the neutral axis is progressively lowered until it lies near the edge of the section; the bending stresses become negligible and the distribution is nearly uniform over the entire section.

Referring to Fig. 2 and recalling that the plastic deformation in the neighborhood of the yield point is approximately fifteen times the elastic deformation at the proportional limit, it is obvious that it would require far greater secondary bending than the value developed in the test to cause the extreme fiber stress to exceed the yield point; that is, to reach Point *C*, Fig. 2. Assuming elastic behavior, the test loading gave a nominal maximum extreme fiber stress of 55 000 lb per sq in. The corresponding deformation, however, was less than twice the value occurring at the proportional limit; for the stress to reach the value, *C*, would require six to eight times this deformation.

Another point to be emphasized is the fact that if the stress is assumed to remain at an approximately constant value in the compressive face while the yield is traversing the flat part of the curve of Fig. 2(*b*), even though the material flows freely without appreciable increase in stress, it can deform only to a limited extent because of the action of the remainder of the section; and, until such time as the entire section is approaching the yield point, it will deform only enough to get relief from additional stress, while continuing to hold its yield-point value. This condition is radically different from that of a specimen subjected to a stress that cannot be relieved in intensity by a slight yield.

In examining the distribution curves it is of interest to note that when yield on the compressive face is shown by a curvature of the graph, the increment of stress received by the tensile plate for uniform increments of loading is constantly increasing. A measurement of this change gives an idea of the "relief" of the bending stresses as plastic flow permits the column to assume the deflected position that originally caused the secondary stresses. A further estimate of this relief is obtained from Table 4, which gives the measured transverse force, *P*, necessary to cause the original deflection and its value during the test as plastic flow was reached. The reduction shown is quite marked. The amount of bending stress remaining (as shown by the stress diagrams of Fig. 21) may be checked by comparison between the permanent set in the column after test and the deflection used for secondary stress, the difference in deflection being used in an approximate calculation of the bending stresses necessary to cause this small deflection. The values of the permanent set are given in Table 10.

An examination of the data presented in Tables 8 and 9, and in the plotted graphs, shows a deformation along the center line of the column somewhat greater than that at the rivet lines. Since the bending stresses are transmitted to the plate at the rivet lines, such a distribution would appear, at first glance, to be contrary to what should be expected. As a possible partial explanation, it was noted that on all specimens a marked transverse curvature of the cover-plate (concave outward) developed under bending action alone, somewhat analogous to that commonly observed in wide flanged I-sections. This dishing inward was largely confined to the region of high flexural stress, and apparently resulted in an appreciable accentuation, locally, of the longitudinal curvature of the plate, which would be reflected in a somewhat higher deformation in the outside fibers along the center line of the column, where the gauge readings were taken.

TABLE 10.—PERMANENT SETS AND MAXIMUM LOADS

Column No.	Test	Deflection produced, in inches	Maximum load, S, in pounds (column in deflected position)	Permanent set, in inches
1..........	1	0.0900	140 000	0.0796
1..........	2	......	167 800	0.2033
2..........	3	0.2685	339 000	0.2055
2..........	4	0.2685	381 800	......
3..........	5	0.2556	271 000	0.1530
3..........	6	0.2556	344 000	0.2360
4..........	7	0.2556	295 500	0.1945
4..........	8	0.2556	305 000	0.2314

It is worthy of note that this dishing action has one distinctly favorable effect on the ultimate buckling strength of the cover in that, by directing the buckling inwardly, it reduces the unsupported width of the plate from the distance between the rivet lines to a value considerably less—probably almost the clear distance between the backs of angles or channels. It will also be observed that the bending of the column as a whole has a tendency to direct the buckling of the cover-plate inward. These facts are of considerable significance for cases in which local buckling is likely to be the criterion of ultimate strength.

In the general analysis the results of buckling formulas for ratios of thickness to unsupported width of one-fortieth and one-fiftieth were presented. Reference to Table 1 shows the $\frac{b}{t}$-ratios of the specimens tested to have varied from 32 to 48. The aforementioned results are quite applicable to these cases and clearly indicate that buckling is improbable in such cover-plates at stresses under the yield-point limit. The preceding discussion shows that stresses in execess of this limit are not realized under the most extreme conditions herein considered. In connection with this point it must be noted that the general buckling formulas become invalid as soon as the stresses pass the limit of proportionality, and furnish only roughly approximate values. For such cases the yield-point stress is usually considered the buckling stress. It is with regard to the latter point that the tests are significant. It might be reasoned that the wide thin cover is in an unstable condition when the stress reaches the yield point (Point B', Fig. 2(b)), and that local buckling of this cover should cause collapse of the entire section at a value slightly greater than the yield-point stress of the cover. However, in all cases except that of Column No. 3, despite the fact that a deformation twice that corresponding to the yield-point (Point B', Fig. 2(b)) stress was realized, there was no sign of buckling until the average stress over the entire section reached the yield point, and unlimited distortion became possible. In Column No. 3 where the $\frac{t}{b}$-ratio was $\frac{1}{48}$ (well below the limit permitted by most specifications), incipient buckling action was noticeable shortly after the extreme fiber had passed the proportional limit. However, the development of the buckle was so slow that the column as a whole maintained its integrity until, as in cases of other specimens, the average stress over the entire section attained a value slightly above the yield point.

It is believed that the reason for this action lies in the previously discussed fact that the distortion corresponding to entirely free plastic flow cannot take place in the cover, until the average stress is of sufficient intensity to cause the entire section to behave plastically. It is reasoned, therefore, that as soon as the cover reaches the yield point, and plastic flow takes place, the member assumes the deflected form with a greatly decreased resistance, the secondary stresses largely disappear, and the stress approaches uniform distribution across the section. Simultaneously, the distortion and direction of impending buckle of the cover-plate are controlled so as to permit the necessary deformations without actual buckling, thereby giving approximately the full yield-point stress as the ultimate load-carrying capacity of the section.

It has been noted that indications of buckling of the thin cover-plate in Column No. 3 were observed at an earlier stage of loading than for Columns Nos. 1 and 2, and that for an identical section in Column No. 4 simple stiffening devices were provided. The effectiveness of these stiffeners in holding the thin cover-plate in line in the region of high stress is clearly demonstrated. A comparison of Test 5, Column No. 3, with Test 7, Column No. 4 (the columns being identical except for the cover-plate stiffeners on the latter) shows that at the average unit stress causing a marked incipient buckling in Column No. 3, Column No. 4 showed no sign of such effects. While Test 8 (Column No. 4) showed a lower final value than Test 6 (Column No. 3), this was due to the fact that the former was not loaded to the point of complete collapse. At a load of 305 000 lb axial stress, however, the cover-plate was in better condition than that of Column No. 3 at the same load.

The results of the test indicated some serious defects in the particular stiffening device used. The sudden change in section at the end of the stiffener created a marked local field of stress concentration, the occurrence of which was indicated by the appearance of strain lines at a load well under the proportional limit of the column as a whole. It is clear that if such a stiffener is used, it should be extended farther toward the end of the column, or its section should be gradually tapered off to avoid such a marked change in cross-sectional area of the plate, since in the present case the unit stress in the plate adjoining the stiffener was practically equal to the maximum stress (stiffener and plate) at the center of the column. Further study and tests are required to determine the most effective form of stiffener, but the indications of this test are that a simple and inexpensive device of this type could readily be developed should it appear desirable to do so.

SUMMARY AND CONCLUSIONS

From the general analysis presented, the following significant features of secondary stress action may be summarized:

1.—Secondary stresses are limited and controlled by certain deflection quantities (see Equation (3)). For any particular loading condition, these deflections are (for primary unit stresses within the elastic limit) approximately proportional to the axial distortions of the truss members, considered as pin-connected, and when, for a given loading, these deflections have been attained, there is no further tendency for the stresses to increase.

2.—As long as both axial and bending stresses remain within the elastic limit and the transverse deflection is small, the behavior of a member under primary stress and secondary bending is closely analogous to a similar member under eccentric axial loading.

3.—If the maximum extreme fiber stresses pass the proportional limit, or if the member is of such type that the transverse deflection is large, the preceding analogy ceases to hold, since, in the case of secondary action, it is the deflection and not the moments that are proportional to the load. In particular, when the extreme fiber stresses reach the neighborhood of the yield point, a radical re-adjustment takes place, greatly relieving the flexural stresses.

4.—As a result of the re-adjustment in stress-strain relations, it may be inferred:

(*a*) That any tension member tends, as the load increases, to a condition of uniformity of stress over the cross-section, except for a local over-strain in a short section near each end.

(*b*) Compression members bent in single curvature may be seriously affected as regards $\frac{L}{r}$-failure if the transverse deflections due to secondary bending become large. This can only occur, however, in the case of large secondary moments and flexible members, a combination that is not ordinarily realizable. It was noted previously (see heading "Compression Member Bent in Single Curvature") that for values of $\frac{L}{r} \leqq 70$, the effect of secondary action in inducing $\frac{L}{r}$-failure is small — well under that corresponding to a pin-ended column with normal "equivalent eccentricities" at the ends. In such a case the rigid joint action which gives rise to secondary stress acts as a brake on the long column deflection before the point of ultimate column strength is reached.

(*c*) For a column bent in double curvature, the secondary action, by forcing the curvature into two "waves", may actually have a beneficial effect as regards $\frac{L}{r}$-failure.

(*d*) For "stocky" columns $\left(\text{say, } \frac{L}{r} \overline{\overline{<}}\ 40\right)$, which are ordinarily the only members that develop high secondary stresses, failure is nearly always due to local over-strain. Until the average stress approaches the yield point, the transverse deflection is negligible, and column action in the ordinary sense cannot occur. For such columns the secondary stresses, whether resulting in single or double curvature (almost invariably the latter will be the case for high secondary stresses) will merely result in high stresses on the compressive face, which are rapidly relieved by plastic flow of the material as the yield point is approached.

5.—It might be expected that a combination of primary and secondary stresses beyond the proportional limit would adversely affect the stability against local buckling of the wide, thin cover-plates commonly used in the compression chords of bridges. An analysis of the problem indicates, however, that local buckling is unlikely to occur, in plates having a $\frac{t}{b}$-ratio consistent with the requirements of standard specifications, until the average stress over the column approaches the yield point of the material, which, in all compression members, marks the ultimate strength of the member.

The tests made primarily for the purpose of exhibiting the behavior of the box type of compression member with respect to local failure under primary stress and a high percentage of secondary stress, have been discussed rather fully in the previous pages. The test results may be summarized briefly, as follows:

6.—Re-distribution of stress beyond the proportional limit was found to take place in a manner closely agreeing with theoretical analysis.

7.—For combinations producing nominal values of extreme fiber stress as great as 55 000 lb per sq in. ($f_p = 35\,000$ and $f_s = 20\,000$ lb per sq in.), it was found: (*a*) That while strain lines began to appear in the cover-plates as the proportional limit was passed, no indications of local buckling appeared until the average stress over the section approached the yield point of the material, for plates with $\frac{t}{b} = \frac{1}{32}$ and $\frac{1}{42}$; but, (*b*) for plates with $\frac{t}{b} = \frac{1}{48}$, some evidence of incipient local wrinkling was noticed at a load somewhat less than the ultimate capacity of the column, although this had apparently no appreciable effect on that value.

8.—When stiffeners were applied to the cover-plates of Column No. 4 (see heading, "Cover-Plate Stiffeners"), the indications of incipient buckling noted in Column No. 3 (Summary Item 7(*b*)) were entirely absent.

Although the general analysis of the behavior of bridge members under combined primary and secondary action resulting in local stresses beyond the proportional limit is believed to apply without restriction, it is obviously impossible to draw valid general conclusions from the test results on four specimens. A much more comprehensive program of experimentation would have to be undertaken before a final pronouncement could be made. Subject to these limitations, the following significant conclusions are indicated:

(*a*) The ultimate practically utilizable strength of a tension member is not affected by any secondary stress action within reasonable practical limits.

(*b*) Compression members sufficiently flexible to develop $\frac{L}{r}$-failure before the yield point is reached, will usually exhibit too low a value for the secondary bending to reduce the ultimate carrying capacity of the member materially.

(*c*) The rigid type of built column, in which alone high secondary stresses are to be expected, almost invariably fails by local over-stress, usually at a point at which local buckling can readily take place. High secondary stresses have no effect in hastening such local failure, and, therefore, have no effect in reducing the actual ultimate strength of such compression members, providing the section proportions are governed by the limits of present standard specifications.

The results of the investigation do not justify the conclusion that secondary stresses are of no importance. The fundamental principle which has so largely governed structural design (that no fiber stresses should exceed the yield point of the material), is too firmly established and is supported by too many sound reasons to be lightly ignored. Such a principle, however, cannot be applied blindly; it is of the utmost importance for the designer to know the consequence of overstressing any part of the structure. This may mean general failure, a limited, localized failure not endangering the structure as a whole, or merely undesirable local distortion and permanent set. The practical significance of this investigation lies in the fact that it indicates that overstressing due to secondary bending falls in the latter class. This is of vital importance in fixing the unit stresses and otherwise determining the margin of safety for cases in which high secondary stresses are involved.

The investigation does not cover certain important by-products of secondary bending as, for example, the effect on riveted and welded connections of alternating secondary moments. For certain types of joints, this effect may be quite important. Due to this and perhaps other effects, and the general desirability of avoiding large permanent set, it is highly desirable that the designer should know if and where high secondary stresses occur in a structure. Under certain conditions, he may deem it desirable to modify the design to reduce such stresses or to provide for them specially in the detail design. In any case, it is a matter of great practical designing significance to be assured that an unforeseen and unexpected overload, producing a combination of primary and secondary stress nominally far above the yield point of the material, will not endanger the safety of the member as long as the average stress over the section is maintained safely below the yield limit.

ACKNOWLEDGMENTS

All the experimental work was conducted in the Experimental Engineering Laboratory of the University of Minnesota, by aid of funds jointly furnished by the Graduate School, of which Guy Stanton Ford is Dean, and the Engineering Experiment Station, of which O. M. Leland, M. Am. Soc. C. E., is Director. Acknowledgment is also made of the co-operation of R. W. Robinson, Engineer of the Minneapolis Plant of the American Bridge Company, who furnished the test specimens at cost. The grateful thanks of the writers are also due to Professors C. A. Hughes and G. C. Priester for valuable criticisms and suggestions, freely given throughout the progress of the work; and to others who rendered valuable assistance during the conduct of the tests.

DISCUSSION

C. H. SANDBERG,[21] ASSOC. M. AM. SOC. C. E. (by letter).—Until recent years the analyses of the problems concerning secondary stresses and indeterminate stresses have been treated in a rather theoretical and academic manner. It is gratifying, therefore, to read a paper that discusses this problem of secondary stresses from a practical standpoint.

One point not emphasized by the authors is the advantage of knowing the effects of secondary stresses in determining the necessity for the replacement of old truss spans. Many of the old, riveted, railroad truss spans, for example, were fabricated with absolute disregard of eccentricity of connections. When the theoretical stresses are computed for the members of such trusses the extreme fiber stresses are often found to be at, or beyond, the yield point.

In addition to secondary stresses and those caused by seriously eccentric connections, some old truss spans carry the live load directly on either the top or the bottom chords. This causes extremely high combined stresses, in some cases; and yet the members are apparently carrying these theoretically excessive loads without any dangerous over-stress. Thus, engineers acquainted with old riveted truss spans can verify from actual experience, the conclusions advanced in this paper. Of course, it is agreed that secondary stresses and similar stress effects cannot be ignored indiscriminately.

It is of interest to note the beneficial effect of the stiffener bar that was welded to the cover-plate of Test Column No. 4. This method of stiffening could be applied, readily and cheaply, to compression members of existing old truss spans. It is hoped that some further investigations and experimental research will be made along these lines.

J. D. GEDO,[22] M. AM. SOC. C. E. (by letter).—The importance of this lucid paper is that it evaluates the effect of secondary stresses upon the ultimate strength and thus furnishes considerable information in regard to the fixing of rational unit stresses.

The only disturbing element in the exposition is that the secondary stresses sometimes are ascribed to the deflection of the truss (see under headings, "Synopsis", and "Analysis of Problem"), and sometimes to the gussets. If the first version were correct, one would conclude that there are no bending moments in the truss represented in Fig. 22, which is carved from a single piece of steel and in which every joint is supported.

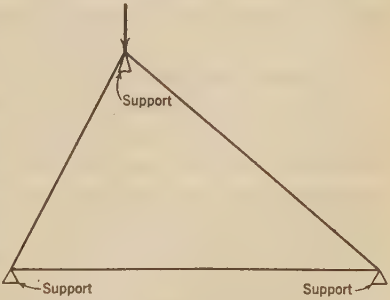


FIG. 22.

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L. E. GRINTER,[23] ASSOC. M. AM. SOC. C. E. (by letter).—The attempt of the authors to investigate the effect of secondary stresses upon the ultimate strengths of truss members deserves sincere applause. For a period of more than half a century the question of the importance of secondary stresses has been widely argued without any successful effort to reach even a compromise agreement. Practice ranges from the entire disregard of all secondary stresses to their full inclusion at standard working stresses. The designers of a few important structures have taken the intermediate path either of attempting to eliminate secondary stresses by the introduction of opposite erection stresses, or of taking the secondary stresses into consideration at increased working stresses.

There should be no disagreement with the authors' premise that secondary stresses should be disregarded if it can be shown that they do not hasten ultimate failure of the structure. However, there are many capable designers who will not agree with this statement because it seems in conflict with the conception of a factor of safety based upon the elastic limit of the material. After all, that is certain to be the major point of disagreement. The old conception of a factor of safety based upon the ultimate strength has been shown to be fallacious and has been replaced by a factor of safety (or better, a factor of ignorance) based upon the elastic limit. Now, it is suggested that the range of plasticity that exists barely above the elastic limit should be exploited to reduce the effect of the secondary stresses and add to the ultimate theoretical capacity of the structure.

The authors' suggestion will immediately bring up the subject of proper working stresses. It has been shown repeatedly that the maximum secondary stresses in an ordinary truss approach one-third the values of the corresponding primary stresses. Hence, a member designed at 18 000 lb per sq in. might actually be stressed to 24 000 lb per sq in. The range from 24 000 lb per sq in. to be the minimum elastic limit (between 30 000 and 35 000 lb per sq in.) has been considered sufficient to cover fabrication stresses, erection stresses, exceptional overload or unusual impact, and an allowance for reduction of area by corrosion. If the authors are prepared to suggest the entire neglect of secondary stresses in design, it would seem that this would permit an increase in the basic working stress of about 33 per cent. Probably most readers will be unwilling to agree to such an increase as being reasonable although they may feel that the authors have established their contention.

Although the authors point out that the "best standards of practice" require that provision for secondary stresses shall be made in design, it is well known that the great majority of designers have not had the necessary theoretical training to compute the values of the secondary stresses and, therefore, have depended upon the allowance in the working stress. Only with the introduction of the Cross method[24] and the writer's simplified method[25] has the designer had available a simple, rapid, and understandable method of secondary stress analysis. Based upon the assumption that the "best standards of practice" will continue to require the consideration of secondary stresses in design, there is reason to believe that "standard practice" will be developed to include the introduction of secondary stresses into bridge design at increased working stresses. In the past, the question as to whether secondary stresses should be introduced into a design has been largely academic, because of the lack of a convenient method of secondary stress analysis.

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[24] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), pp. 1–156.

[25] *Loc. cit.*, Vol. 99 (1934), pp. 610–669.

The entire argument of the paper is based upon the premise that secondary stresses increase in direct proportion to primary stresses until the yield point of the material is reached at one face of the member, after which an increase in primary stress is not accompanied by an increase in secondary stress. Two reasons are offered to explain this phenomenon. The first (with which the writer agrees) is that local plastic deformation will occur, which will redistribute the stress over the cross-section and maintain (temporarily) the extreme fiber stress at the yield point. The second (with which the writer disagrees), is made clear by the following quotation in reference to Fig. 3:

> "Since the angular rotations at the ends of a member can increase no more rapidly than the axial loads, it will be impossible even with high secondary stresses for the extreme fiber stress to exceed the yield-point value without the distortion increasing far beyond any value reasonably to be expected in a truss in service."

The title of this paper, however, indicates that the study relates to ultimate strength, and the deflections, the angular rotations, and the proportional secondary stresses will be several times as large at failure as those presupposed by the authors in the foregoing quotation.

Evidently, this entire matter needs to be clarified by an unqualified statement as to whether the structure is to be designed for working conditions, for an intermediate condition in which stresses will remain at the yield point during a considerable increase of load, or for those conditions existing at failure. If the authors select the intermediate condition unqualifiedly, they will approach more nearly to a justification of their hypothesis that secondary stresses are of little importance in design. However, they will then face the criticism that their structures designed with a standard factor of safety upon the yield point have far less reserve strength above the yield point than the designer has been accustomed to expect. This follows because joint rotations and secondary deformations increase more rapidly than the loads as soon as the direct stresses exceed the elastic limit of the material, and the authors have suggested no provision for secondary stresses in the design.

Considerable emphasis is given to the fact that stresses above the yield point are not unusual elsewhere in structural design. Of course they are not, although there is still a considerable group of engineers who object to plasticity as a basis for structural design. Nevertheless, when the authors declare that stress concentrations above the elastic limit probably exist around rivet holes, they should not forget that their suggestion of the entire neglect of secondaries assumes that this severely deformed material around the rivet holes will be able to continue to flow in order to reduce secondary stresses without developing incipient cracking. The reference to increased allowable bearing stresses on pins and rivets must be considered together with the obvious fact that the highly stressed material is restrained against flow by low stressed adjoining material. Professor Swain made this point clear by stating[26] that a cannon ball cannot possibly fail under hydraulic pressure.

An important factor not considered by the authors would seem to be the possibility of reversal upon accumulative plastic deformation. If reversal can be shown to be sufficiently important to over-stress, alternately, the opposite sides of a member, such accumulative distortion might be serious. Another obviously important consideration is the instantaneous stresses produced by live load impact and vibration. Since there would be no possibility of reducing such stresses by plastic flow, they could not be placed in the same category as the secondary stresses caused by dead load. Although it is difficult to visualize impact as producing deflection and joint rotation, there is reason to believe that the shock effect would increase both secondaries and primaries. In fact, since the secondary stresses do not support the load, it is easy to visualize the possibility that they might be doubled by the individual vibration of the member which could act as an instantaneous application.

In summarizing, the writer would like to express the opinion that this paper will be of great value in opening the entire field of secondary stresses and proper working stresses for discussion. However, he cannot at present agree that the paper justifies, entirely, its hypothesis that secondary stresses should be completely neglected. The probability of high secondaries near the ultimate load, the danger of over-dependence upon plastic deformation where other stress concentrations exist, the possibility of cumulative plastic deformation, and the impossibility of any reduction in secondary stresses caused by rapid advance of live load, or by impact and vibration, point to the danger that might be involved in a complete neglect of secondary stresses. Probably all could agree, however, that, if specifications are to be rewritten so as to make it obligatory to introduce secondary stresses into the design, a slightly increased basic working stress, further increased one-third as an allowance for secondary effects, is permissible.

This paper marks the authors as pioneers in pointing to the great possibilities that lie in the future for exploiting the advantages of steel as a plastic material.

L. T. Evans,[27] Assoc. M. Am. Soc. C. E. (by letter).—While the subject of secondary stresses has been popular with many writers, all papers prior to this one have left the impression that the analogous eccentric-load relation existed for all increments of load.

For some time the writer has felt that some relief must accrue to highly stressed parts of structural members when a large part of that stress was due to secondary moments. Actual structures substantiate this belief. Several years ago the writer was employed on the design of a large, steel, rigid-frame structure. The girders were trusses with web members that had secondary stresses as great as four times the primary stresses. Since these members were short and "stocky," with an indeterminate length (due to large gusset-plates), it was decided to design the section for the full primary, and one-half the calculated secondary, stress with the elastic limit as the working stress. After erection, the structure was encased in concrete and has been in daily service for several years with no signs of distress.

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[26] "Structural Engineering—Strength of Materials", by George F. Swain, Past-President and Hon. M. Am. Soc. C. E., McGraw-Hill Book Co., N. Y., 1924, p. 551.

[27] Cons. Structural Engr., Long Beach, Calif.

This paper will be appreciated by engineers who have given secondary stresses considerable thought, but it is hoped that the careless reader will not get the impression that such stresses will "take care of themselves" and can be ignored.

F. E. Fahy,[28] Assoc. M. Am. Soc. C. E. (by letter).—The analysis presented in this paper defines clearly the effect of the relieving local deformations which have been said at times[29] to account, at least in part, for discrepancies (generally on the side of safety) between observed and computed stresses in riveted trusses. The writer has long felt that the calculation of secondary stresses by any of the usual methods involves too many rather arbitrary assumptions to be quantitatively accurate, and that the actual effect of secondary stresses on the members of a heavy riveted truss is closely akin, as the authors suggest, to the many cases of theoretically high localized stresses which are of little practical significance.

A rather interesting analogy might be drawn between the action of a riveted truss joint at which secondary moments cause deformations beyond the yield point, and that of a multiple-pin joint. Where a multiple-pin joint does not materially affect the static stability of the framework (at the intermediate support panel points of a continuous or cantilever truss, for instance) it can be used advantageously; it permits the connected members to rotate about their respective pins without incurring any flexural deformations except those due to pin friction. In such a case the stresses remain essentially axial. After the extreme fiber stresses in the connected members pass the yield point of the material, secondary stress action under rigid joint conditions will tend toward the same effect, except, as the authors indicate, in the case of slender compression members bent in single curvature. The points of yielding might be considered as approaching pin-end conditions, about which the members involved "rock" to a final state of static equilibrium.

The foregoing analogy will not apply beyond the first application of load because of strain hardening, but it is suggested as a convenient method of visualizing the action of a truss in which flexural overstress occurs due to secondary bending. (As a matter of convenience the writer will use the term, "overstress," to mean stress beyond the elastic range.) In many cases, incidentally, it is the first loading stage that is final and most important; heavy building trusses, perhaps the worst offenders in the matter of high secondary stresses, are essentially dead load structures, and after the first application of full load has imparted to them a certain deformed condition, will hold that condition under the subsequent steady loads to which they are subjected. The truss outlined in Fig. 23 (*a*) can assume the position shown in Fig. 23 (*b*), in which deformations have been idealized and exaggerated for the sake of clearness; and, under the small deformations which actually occur, it can hold that position without harm. It may be noted, also, that the inherent instability of a multiple-pin joint does not appear in an overstressed rigid joint because of the fact that the overstressed members can always offer flexural resistances (until, of course, failure is imminent) even after the extreme fiber stresses have passed the yield point.

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[28] Research Engr., Bethlehem Steel Co., Bethlehem, Pa.

[29] See, for example, "Stress Measurements on the Hell Gate Arch Bridge," by D. B. Steinman, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXII (1918), p. 1071; and "Investigation of Secondary Stresses in the Kenova Bridge," by G. A. Maney and J. I. Parcel, Members, Am. Soc. C. E., University of Minnesota, *Studies in Engineering, No. 4*, p. 3.

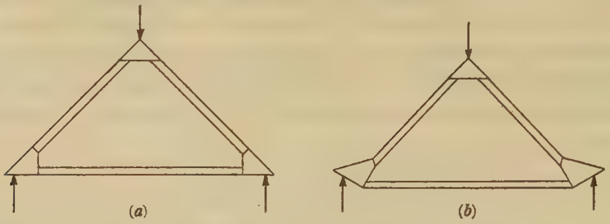
(*a*) (*b*)

FIG. 23.

It might be suggested that in small bridges, especially railway bridges, which are essentially "live load structures," fatigue might play a fairly important part in causing failure if the combined effect of dead load, live load, and impact stresses runs as high as the yield point of the material. Current practice properly tends to neglect fatigue as a serious consideration in discussing the utilizable strength of such structures. In the first place, the number of full-load applications which the ordinary bridge receives during its normal life span is so small as to preclude the necessity for considering fatigue either in design or review. Assume, for the sake of argument, however, that this is not the case. All the evidence with which the writer is familiar at the present time seems to indicate that for most ferrous metals the endurance limit for a stress ranging from zero to a maximum value of one sign only is well above the static yield point of the material. The Goodman value is 1.5 times the endurance limit for completely reversed stress, which endurance limit, for the ferrous metals, seems to approach roughly a value equal to one-half the ultimate strength.[30] (In general, for the types of steels commonly used in bridges the endurance limits for completely reversed stress appear to be somewhat less than the yield points, but not enough less than the usual specified minimum yield points[31] to invalidate the comments which follow.) For the ordinary carbon steel commonly used in short-span and medium-span bridges the yield point is also, roughly, equal to one-half the ultimate strength. As the authors have indicated, stresses in bridge members, even with low length-depth ratios, will scarcely exceed the yield point, which means that such stresses can never reach the endurance limit for a range ratio of zero. Furthermore, bridge members which are short enough to be under high secondary moments will almost always be under some dead load stress, in which case the stress range decreases and the endurance limit increases. An exception might be the case of comparatively heavy web members in the center panel or panels of a shallow deck truss. In these members, stress reversals due to secondary bending are quite likely to occur, and reversed primary stresses actually do occur. It would be a rare case indeed in which the resultant primary and secondary stresses in such members in a well-designed structure, would approach anything like the yield point or the endurance limit of the material. The writer simply introduces the subject of fatigue, and the foregoing discussion, as further substantiating the authors' conclusions that, in well-designed structures, overstressing due to secondary bending generally means either a "limited, localized failure not endangering the structure as a whole, or merely undesirable local distortion and permanent set."

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[30] "The Fatigue of Metals", by Moore and Kommers, pp. 160–165.

[31] *Loc. cit.*, pp. 128–146; also, A. S. T. M. Standards.

The writer does not mean to imply that consideration of secondary stresses should be omitted entirely from the design process. No conditions under which stresses reach the yield point of the material can be considered as other than most undesirable and to be avoided if at all practicable. As far as the members themselves are concerned, however, the writer concurs with the authors in their conclusion that the presence of secondary bending stresses even as high as the yield point are not necessarily damaging; nor are they to be considered as rendering the structure unsafe or poorly designed.

A brief discussion of the effects of secondary bending on the joints of a structure seems pertinent, especially in considering secondary stresses due to live load. The presence of secondary moments at a truss joint means that the connection between member and gusset-plate ceases to be a direct tension or compression joint, but must resist flexure as well. Since the intensity of this flexure varies considerably, in the average structure, between the dead load and full live load plus impact conditions, there may be considerable danger of loosening the rivets and generally overstressing the entire group, in a riveted joint, or of causing progressive fracture of a weld.

Consider a tension member made up of two 15-in. 55-lb channels laced or battened together, each channel being riveted to the gusset-plate by means of thirty $\frac{7}{8}$-in. rivets, as in Fig. 24. Assume that the full dead load, live load, and impact stress in the member is 16 000 lb per sq in. of net section, and that the minimum live load stress is zero. (The writer has made this assumption partly for the sake of simplicity and partly because of the fact that under most current specifications members and connections which are sub-

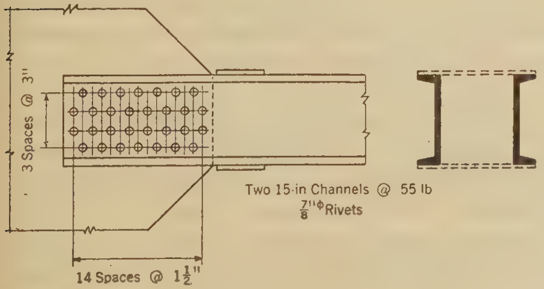

FIG. 24.

ject to reversals of primary stress are quite conservatively designed. Furthermore, such members are usually relatively slender and not likely to be subjected to high secondary stresses. Reversals of secondary bending may heighten the effects noted herein, but the writer has made no attempt to make the examples which follow especially severe, except in so far as secondary bending causes yield-point stresses.)

Basing the allowance for rivet holes on the 1923 specifications of the American Railway Engineering Association, this gives a total primary stress due to dead load, live load, and impact of 431 000 lb. The total primary stress per rivet is then $\frac{431\,000}{60} = 7\,180$ lb. The gross area of the member is 32.22 sq in., giving a primary stress of 13 400 lb per sq in. on the gross section. The section modulus of the member, based on gross section, is 114.4 in.³ Assuming a yield point of 32 000 lb per sq in., the secondary bending moment required to produce yielding in the extreme fiber will be $(32\,000 - 13\,400) \times 114.4 = 2\,130\,000$ in.-lb. The moment of inertia of the rivet group (Σd^2) is 3 160 in.,⁴ and the distance from the centroid of the group to the extreme rivet is 10.6 in. Then the total stress in the extreme rivet due to the secondary moment which will produce yielding in the member is $\frac{2\,130\,000 \times 10.6}{3\,160} = 7\,130$ lb. Combining this with the primary stress of 7 180 lb and neglecting the probably small transverse shear which is caused by reversed bending or unequal end moments, gives a resultant total stress of 10 900 lb on the extreme rivet, or 18 100 lb per sq in. in single shear. This is approximately the shearing yield strength of the rivet. If 25% of the total stress is due to dead load, the stress in two of the extreme rivets of the group will vary from 4 500 lb per sq in. to 18 100 lb per sq in. If the rivet has slipped at the first full-load application (as it probably has), it would appear that a relatively small number of live load applications would suffice to loosen it beyond any useful resistance, and thus produce progressive failure of the joint—especially when the highly non-uniform distribution of primary stress among the rivets comprising the group is considered.

Here again, however, the nature of secondary stress action must be remembered. It is essentially due, as the authors state, to angular displacements of the ends of the member. These angular displacements are small, and when the rivets themselves slip, the joint is no longer rigid. Slight rotation of the member is permitted, which has the effect of reducing secondary bending and the rivet stresses caused by it. The extreme rivets will slip at low stresses, once the effect of friction is broken down, producing at once a more uniform distribution of primary stresses throughout the group and a reduction in secondary bending. If the connected member has been deformed to the yield point, but not through or beyond it (say, to a point slightly to the right of Point *B*, Fig. 2(*a*)), the effect on the rivets may not be damaging, although the sharp rotation occurring at the overstressed section will introduce some eccentricity of primary stress. If it has not been so stressed, slip will not reduce the loosening effect of rotation and counter-rotation of the joint. Field and laboratory investigation of the effect of variable bending on groups of rivets in shear would be of value in connection with the subject of secondary stress effects.

Rivet slip may play an appreciable part in reducing secondary bending, which accounts to some extent for discrepancies between observed and calculated stresses. The relieving effect of rivet slip is not to be found in welded joints, although it has a counterpart in the plastic flow which occurs at the ends of such joints even under low loads. Due to the fact that welds are usually located along the edges of the member, the moment of resistance of a welded joint will be relatively higher than in a web-riveted joint, on the usual design assumption of uniform stress distribution through the length of the joint. Even so, however, the stresses in a welded joint at which the connected members are stressed to the yield point by secondary bending, will be high.

The damaging effect of high secondary stresses will likely be most severe on the joints at the ends of non-continuous members, such as the web members, end posts, and end-panel chords of the ordinary types of bridge and building trusses, and on splices in chord members, where such splices are near the panel points. This will be especially true for members in which the ratio of dead load stress to total dead load plus live load plus impact stress is low or negative. Severe joint stresses are not likely to occur at panel points at which the chord is continuous, except in the web-member connections. In Fig. 25(a), for example, $M_1 = M_2 + M_3 + M_4$, and if the chord is continuous only $M_3 + M_4$ must be resisted by the rivets between the chord and the gusset-plates. These rivets are usually considerably in excess of those required for primary stress transmission, and, due to the length of the joint, the group has a high moment of resistance against torsional loads.

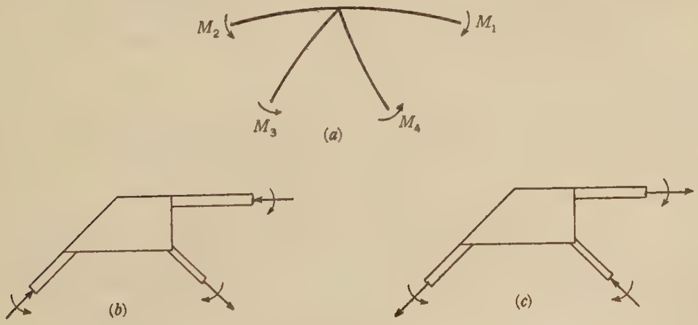

FIG. 25.

Another point which may be worth mentioning is the possible effect of high joint moments in causing severe stresses in the gusset-plates. If the stresses in a gusset-plate due to joint moments are opposite in sign to those due to primary stresses in the members the net effect will be a relief and lowering of the gusset-plate stresses. Such would be the case for the joint of Fig. 25(*b*), under the stresses and moments there shown. If, however, the primary stresses act as shown in Fig. 25(*c*), the secondary stresses add directly to the primary stresses in the gusset-plate. If the result is a tensile stress at a rolled edge of the plate equal to the yield point of the material no serious damage will result, either statically or from fatigue. At a sheared edge there may be some danger of progression of an incipient crack. If the resulting stress at the edge of the plate is compressive and equal to the yield point of the material, buckling may occur unless the unsupported edge of the plate is very short or unless it is stiffened by angles or a diaphragm connecting the usual pair of plates to each other. In the case of fairly large plates under compressive stress along a long unsupported edge, diaphragms or stiffener angles provide an easy and effective means of insuring, to a certain extent, against failure of the plate by buckling. In any design where high secondary stresses are likely to occur gusset-plates stresses should receive some consideration.

LaMotte Grover,[32] Assoc. M. Am. Soc. C. E. (by letter).—The practical application of the results of secondary stress analysis to the design of bridge members is discussed in this paper. This subject is most timely in view of the recent tendency toward the use of more comprehensive design methods in order to effect the most favorable distribution of material.

The increased demand for comparatively shallow deck truss construction resulting from higher standards in grade and alignment for both highway and railway structures, together with the trend away from movable spans to the use of fixed spans which require shallow deck construction in many cases, results in types of design in which the effect of secondary stresses is quite appreciable.

With the rapidly increasing use of structural welding, advantage is being taken of the better economy and reliability thus afforded for making rigid joints in which behavior can be predicted with more certainty. The effect of large gusset-plates upon the stiffness and stress distribution in members is a feature that can not be evaluated closely. These large gusset-plates are either reduced in size or eliminated entirely in properly designed welded connections, thereby reducing some of the annoying uncertainties that exist with riveted construction. Such developments challenge the designer to keep in pace by devoting more attention to such refinements as statically indeterminate stress analysis and the investigation of secondary stresses, and they also make it possible for the interpretation of such analysis to be evaluated more accurately and, therefore, better justified.

Certain other problems of the structural engineer, such as residual stresses caused either by rolling steel shapes or by welding, are inherently self-limiting and these stresses will be better understood as a result of the studies described in this paper. One of the ways in which structural welding will have an influence upon behavior under secondary stress is that welded cover-plates will be virtually fixed along their edges. This will contribute further toward minimizing the tendency for relatively thin plates to fail locally by buckling. Several limitations of the study have been acknowledged by the authors and these might be extended to include a number of others that would indicate the need for further and broader studies in this field.

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[32] Bridge Engr., Dept. of Design, State Highway Comm., Topeka, Kans.

Further experiments should be made to determine the effect of heavy gusset-plates and the stress concentrations that they cause at the ends of members in double curvature, where the secondary stresses are most serious. Studies might well be made to demonstrate the effect of elaborate provisions which have been made during the construction of some of the larger bridges for the purpose of relieving secondary stresses.

Another condition that would be worthy of investigation is that of an occasional severe strain accompanied by plastic flow, with intermittent strains of lesser magnitude and of a frequency that approaches that which is recognized to be dangerous from the standpoint of "fatigue" failure. It is quite possible to obtain a sufficient number of repeated or alternating stresses to cause apprehension in this regard, if the life and service of modern bridge structures are to be as great as is estimated by many.

The authors have mentioned the possibility of unfavorable effect upon riveted or welded joints due to repeated or alternating stresses, but it should be emphasized that the effect of such stresses might be very severe upon the members themselves, especially in those connected with heavy gusset-plates and bent into double curvature as a result of secondary stresses; and likewise the effect of suddenly applied loads in causing combined stresses of serious magnitude before the metal has had time to creep and permit the adjustment of the stresses over the section by plastic flow. These remarks are especially pertinent in the case of smaller, lighter bridges.

It should also be stressed that the authors' "Summary and Conclusions" pertain exclusively to structures for which an occasional severe overload may be anticipated. They pertain to the stresses that would be caused by such an occasional load as might be expected in the case of a highway structure. Perhaps the results of the studies and experiments are less pertinent in the case of railroad bridges where the loads are subject to more careful control and are of at least daily occurrence.

No explanation is given as to the reason for the inward "dishing" of cover-plates as a result of local buckling, except the general tendency of the bending of a compression member as a whole. It might be considered questionable whether this tendency is decided enough to justify relying upon it, especially if other influences such as rust due to neglect in maintenance, tend to cause outward "dishing."

A. A. Eremin,[33] Assoc. M. Am. Soc. C. E. (by letter).—The analysis of secondary stresses at the yield point in a steel truss has considerable economic importance. The authors state that the ultimate strength of a steel truss is not affected by the secondary stresses at the yield point. This is true in respect to a truss in which the number of members, n, and number of joints, N, satisfy the equation,[34]

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[33] Associate Bridge Designing Engr., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

$$n = 2N - 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

If n is less than $2N - 3$, the fiber stresses at the yield point in the joints will permit excessive displacements and evidently will influence the ultimate strength of the truss. This may occur in a Vierendeel truss in which the secondary stresses at the yield point will affect the ultimate strength of the truss.

If n is greater than $2N - 3$, the secondary fiber stresses at joints as well as the axial stresses in some members of a steel truss, may reach the yield point without materially affecting the ultimate strength of the truss.

The secondary stresses at the yield point may also influence the distribution of the axial stresses in the truss members when the stiffness of the members or the eccentricity of the joints is considered in computing the axial stresses.[35] A method of computing axial stresses, taking the stiffness of the members into consideration, has been presented by Charles A. Ellis, M. Am. Soc. C. E.

The authors are to be congratulated for their valuable contribution to the theory and practice of diagnosing the stresses at the yield point.

JOHN I. PARCEL,[36] M. AM. SOC. C. E., AND ELDRED B. MURER,[37] JUN. AM. SOC. C. E. (by letter).—The discussions submitted have raised a number of additional points of interest not mentioned in the paper or at most touched upon very lightly, and they have also called attention to certain apparent obscurities of statement. The writers are much gratified that, for the most part, the discussers appear to agree with the main conclusions of the paper.

Mr. Sandberg makes a very good point with reference to the importance of knowing the effect of high secondaries upon the ultimate strength of members in the case of determining the strength of old structures still in service, where it is proper to use a much lower margin of safety than would be the case for a new structure. He mentions the common occurrence of eccentric connections in such structures. It is important to note that secondary stresses arising from this cause are of a character quite different from those contemplated in the paper, as will be further noted in the consideration of Mr. Eremin's discussion.

The point of Mr. Gedo's criticism is not quite clear. He notes as a "disturbing element" that secondary stresses "sometimes are ascribed to the deflection of the truss * * * and sometimes to the gussets". This certainly was not the writers' intention. Secondary stresses as herein considered, of course, arise from a combination of the deflection of the structure and the stiffness of the joints as stated in the first paragraph under the heading, "Analysis of Problem". The frame-work shown by Mr. Gedo in Fig. 22, if the supports are assumed as rigid, would exhibit neither primary nor secondary stresses.

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[34] "Structural Engineering", by George Fillmore Swain, Past-President, and Hon. M. Am. Soc. C. E., p. 81.

[35] "Williot Equations for Statically Indeterminate Structures in Combination with Moment Equations in Terms of Angular Displacement," by Charles A. Ellis, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 580.

[36] Cons. Engr. (Sverdrup & Parcel), St. Louis, Mo.

[37] Wayland, Mo.

The first three paragraphs of Professor Grinter's very able discussion contain a particularly clear and forceful statement of the salient points involved in estimating the importance of secondary stress upon ultimate strength. Professor Grinter disagrees with the statement in the paper that the joint rotations upon which the secondary stresses are conditioned increase in direct proportion to the load. He regards this as inconsistent with the title of the paper, which would indicate that the conditions to be considered are those in force when the structure is stressed to its ultimate limit, in which case he states that the displacements would be several times as great as those computed on the basis of elastic behavior.

It is believed that this criticism rests upon a misconception. It is definitely stated in the paper that the effective ultimate strength of a tension member is marked by the yield point of the material, and that ordinarily a compression member will fail at a point slightly above (and, on occasion, at a point slightly below) this stress. The most important conclusion of the paper is that secondary stresses, even when they result in local stress combinations nominally in excess of the yield limit, do not apparently hasten the development of an average stress over the entire section equal to the yield-point stress. So long as the average axial stress remains within the elastic limit, the joint rotations cannot increase faster than the loads, since they are direct functions of the changes of length of the truss members, and the latter, so long as Hooke's law prevails, are directly proportional to the loads. It was not the writers' intention to consider the distortions arising after the average primary unit stresses had reached the yield point.

Professor Grinter apparently feels that one should not place too much dependence upon the analogy between local over-stress due to secondary bending and that due to a number of other causes mentioned in the paper. With this opinion the writers are in entire agreement as should be clear from the statement made in next to the final paragraph of the paper. The point to be made was merely that standard practice at present permits certain violations of the general rule that all stresses should be kept within the yield-point limit, and that the mere fact that secondary stresses exceed this limit does not, without further consideration, establish them as dangerous.

The suggestion that due consideration should be given to the fact that high secondary bending places added strain on the material around rivet holes, which is already strained considerably beyond the surrounding material, is worthy of special consideration. The importance of this effect is considered in some detail in Professor Fahy's discussion. Undoubtedly, further experimental work upon the entire problem of the detailed behavior of riveted and welded joints subjected to a combination of high primary and secondary stresses is much to be desired.

Professor Grinter raises several interesting questions regarding reversal of stress, repetitive stresses, and impact, although the writers are inclined to feel that this interest is largely academic. As regards reversal and repeated stress, the following should be noted:

(1) In any ordinary bridge truss, the only members in which a marked reversal of both primary and secondary stresses takes place are the comparatively slender flexible web members near the center (of a simple truss) for which the secondary stresses are very small.

(2) Relatively stocky members which normally exhibit high secondary stresses—chords, end posts, short web members in end panels—ordinarily do not reverse either primary or secondary stress in a truss with simple webbing such as is commonly used for bridge trusses of spans up to about 300 ft. For longer spans, in which sub-paneling is used, some of this group of members may show a very large reversal[38] in live load secondary bending (although not, of course, in primary stress) when the secondaries are computed in the conventional manner. When a more accurate analysis is made, taking into account the induced distortions,[39] the secondary moments are found to be greatly reduced, and, when dead load effect is considered, they will almost never be sufficient to reverse the total extreme fiber stress appreciably.

(3) Structural steel with a yield-point value of 35 000 lb per sq in., may be expected to show a safe "endurance range" of from, say, − 20 000 lb per sq in. to + 35 000 lb per sq in.; or, when no reversal takes place, from 0 to 45 000 lb per sq in. It is believed that the present study shows conclusively that in no case can an extreme fiber stress be developed to exceed the yield-point value (35 000 lb per sq in. in this case) until the primary stress approaches this limit; that is, until failure is impending, regardless of secondary bending. It is also believed that only a "freak" truss will ever show a reversal of total extreme fiber stress ranging from − 20 000 lb per sq in. to + 35 000 lb per sq in. The extraordinary case where such action may take place, of course, will require special treatment.

Professor Grinter argues that impact stresses must be treated separately, since they may be practically instantaneous in character, allowing no time for relief to develop through plastic flow. This is a very important point and requires some further development of the conception of "relief".

If a designer is working to specifications requiring a minimum yield point of, say, 33 000 lb per sq in., and if he finds that in a certain compression member, for example, the computed maximum primary unit stress is 20 000 lb per sq in. and the corresponding secondary stress on the extreme fiber is 18 000 lb per sq in., he will naturally infer that his material at the face of the member is stressed 5 000 lb per sq in. beyond the yield point and, therefore, in danger of collapse due to local buckling. A principal conclusion of the paper was that, due to relief from plastic action, a stress greater than 33 000 lb per sq in. in such a case cannot possibly develop under any degree of secondary bending practically realizable. Remembering that in secondary stress action it is distortions and not stresses that are fixed, it is pertinent to ask: What will be the effect on the material if a given deformation is imposed on a member so rapidly that the material has no time to flow? It may be presumed that in this case a much higher stress will develop, but since it is strain and not stress which, repeatedly applied, tends to break down the material, it is not clear that, as long as the strain is the same, whether imposed instantaneously or gradually, there is any serious effect from the added stress required in the former case. So far as is known there are no experimental data bearing directly on this point.

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[38] "Secondary Stresses in the Kenova Bridge", by G. A. Maney and J. I. Parcel, Members, Am. Soc. C. E., Univ. of Minnesota, *Studies in Engineering, No. 4*, p. 29.

[39] *Loc. cit.*, p. 47 *et seq.*

Professor Grinter states that "it is difficult to visualize impact as producing deflection and joint rotation". It would appear that since such displacements are the direct result of the axial deformations of truss members, as previously noted, they should be as clear and definite as distortions arising from other sources. Any calculation of secondary stresses should always include impact effects as well as those due to static live load, and recent researches appear to indicate that the allowance for impact required in standard railroad bridge specifications is probably considerably greater than is ever actually realized.

Professor Grinter refers to "the authors' suggestion of the entire neglect of secondaries". No such suggestion was made; on the contrary, the writers specifically state "the results of the investigation do not justify the conclusion that secondary stresses are of no importance". The exceptional cases cited by Professor Grinter as possibilities are among the reasons for this conclusion.

He states further that "in the past the question as to whether secondary stresses should be introduced into a design has been largely academic, because of the lack of a convenient method of secondary stress analysis" and that, "only with the introduction of the Cross method and the writer's simplified method has the designer had available a simple, rapid and understandable method of secondary stress analysis". With this statement, the writers are in complete disagreement. They have the greatest respect for the brilliant method of analysis introduced by Professor Cross and extended by Professor Grinter, and they recognize, of course, that there is always a wide difference of opinion among individuals as to the convenience and effectiveness of different analytical methods. So far as secondary stress computation is concerned, however, an extended experience has convinced them that the slope-deflection method, when the equations are solved by the method of successive substitutions, is a more rapid and convenient method than moment distribution. This method of solving slope-deflection equations has been in use for at least a quarter of a century.

Mr. Evans agrees with the main conclusions of the paper and notes that he has for some time held similar views. In this connection the writers would like to emphasize the statement made in the paper that the idea of large stress relief in the case of high nominal secondaries is by no means new and has been proposed by a number of authorities here and abroad during the past twenty years.

Mr. Evans cites a most interesting case arising in his practice a number of years ago in which a design solution was evolved, which embodied essentially the conclusions of this paper, and the subsequent results of which afford a confirmation of the soundness of the design.

The writers consider Professor Fahy's discussion a most important contribution to the subject. It is so thorough and clearly stated as to offer little room for comment. They have found his detailed study of rivet stresses and comments thereon especially illuminating. As regards the last point made in his discussion—the effect upon gusset-plates of a combination of high secondary and primary stresses—they feel that some further experimental investigation is needed before any final conclusions can be drawn. Until such information is available, they are inclined to agree with him that in all doubtful cases the edges of gusset-plates should be reinforced.

Mr. Grover raises a number of interesting questions, particularly as regards the effect of secondaries on riveted and welded connections, some of which are partly answered in Professor Fahy's discussion. His questions on the subject of fatigue have been answered in the comment on Professor Grinter's discussion. He refers particularly to the stress concentration arising at the edge of gusset-plates as a point which should receive special consideration in further investigations. If the writers understand correctly what he has in mind, they are inclined to feel that there is no essential difference in the stresses to which he refers and any other secondary stress of equal intensity occurring in the body of a member.

The inward dishing of the cover-plate to which Mr. Grover refers was not offered as a phenomenon which could be depended upon invariably, but rather as one that served to explain in part certain experimental data obtained. Of course, serious rusting or injury causing a marked incipient outward dish in the member would tend to offset the apparent natural tendency of the plate to bend inward.

Mr. Eremin raises two important points which perhaps should have been emphasized more in the paper. It is true that the term, "secondary stress", has been used rather loosely in the literature of structural engineering and is sometimes thought of as any bending stress developing at the ends of a member in a framework with rigid joints. This, of course, is not the type of stress with which the present investigation deals. It is noted in the "Introduction" that secondary stresses are not required to maintain the equilibrium of the structure, which clearly excludes such types as the Vierendeel truss; such structures are unstable without the resistance offered by the rigid joints. Under the heading, "Analysis of Problem", it is stated that,

> "Even when the secondary unit stress is a high percentage of the primary, the secondary moments offer no appreciable assistance in carrying the loads, and the members are always designed on the basis of full hinge action at the ends."

Perhaps this statement is too sweeping a generalization, although the writers know of no case in which a bridge or building truss (using the term, truss, to mean a framework with $m \geqq 2n - 3$) has been designed so as to include the secondary bending in the load-carrying capacity of the truss.

If a structure were so designed the bending moments in the members would become primary effects and as such are outside the scope of the present treatment.

For the most part, the same remarks apply to so-called secondary stresses arising from joint eccentricity. Such stresses are necessary to maintain the stability of the structure; in general, they will increase substantially in proportion to the primary stresses and, since a certain definite applied moment is to be balanced, no relief will result from local stressing beyond the yield point.

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Paper No. 1929

THE SPRINGWELLS FILTRATION PLANT, DETROIT, MICHIGAN

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With Discussion by Messrs. F. H. Stephenson, Robert Spurr Weston, and Eugene A. Hardin

Synopsis

The design and construction of a filtration plant and filtered water reservoir at Springwells Station is described in this paper. These units form a part of the additional works constructed during 1929-1932 for the Board of Water Supply of the Metropolitan Area of Detroit, Mich. The filtration plant is of the rapid sand, gravity type, having an ultimate capacity of 300 000 000 gal daily. The reservoir is in two sections each of which has a capacity of 20 000 000 gal, with provision for a third section to be added later. The general basis of design, the data pertaining to the various parts of the plant, the hydraulics of plant flow, the construction program, and the tabulation of both construction costs and engineering costs are outlined herein.

The water supply for the City of Detroit is the largest to be completely filtered by rapid sand filtration, and includes the two largest rapid sand filtration plants in the world—the Water-Works Park Filtration Plant, having a maximum daily capacity of 360 000 000 gal, and the Springwells Filtration Plant, having a maximum daily capacity of 300 000 000 gal (capacities based on a filtration rate of 180 000 000 gal daily per acre).

History of Water Treatment at Detroit, Michigan

The City of Detroit has taken its water supply from the Detroit River for more than 100 years. Although privately owned and operated for the first eight years of their existence, the water system and supply works have been owned by the municipality since 1836. Since that time, from a small city of

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Note.—Published in November, 1934, *Proceedings*.

[1] Detroit, Mich. (Formerly Designing Engr., Dept. of Water Supply.)

6 900 inhabitants, the district served has increased in population to more than 2 000 000 and in area to more than 100 sq miles, extending about 10 miles along the river and about 10 miles inland from the river. Improvements for increasing the quantity of the supply, and its distribution, were the main considerations, except in the location of intakes, until, in 1912, when purification of the supply was first considered and permanent provision was made for chlorination. The quality of the water in the Detroit River at the intake built in 1905 (named 1905 Intake) has been comparatively good, but with increasing pollution and the growing consciousness of the consumers of the value and need of improvement in the quality of the water, the desire for a purer and more palatable supply increased until, in 1917, an experimental filtration plant with a capacity of 200 000 gal per day was constructed and operated for more than a year, establishing the feasibility of filtering and developing popular approval by dispensing filtered water to those who came for it. One year of operation (from October 1, 1917, to September 30, 1918) was used as the basis of a report by the late R. Winthrop Pratt, M. Am. Soc. C. E., dated March 1, 1919, outlining the general design of a filtration plant, to treat the entire then existing supply. In 1921, contracts were let for filtration works at Water-Works Park Station (see Fig. 1) and the filtration plant was completed and put in service on December 22, 1923.

While it improved the quality of the water greatly, the filtration plant at Water-Works Park did not add to the quantity. Due to the rapid increase in size of the city in both population and area, it was evident that a far-sighted and comprehensive plan of additional supply for the Detroit Metropolitan Area was necessary immediately. Accordingly, a Commission of Consulting Engineers, composed of Messrs. George H. Fenkell, W. C. Hoad, Clarence W. Hubbell, Theodore A. Leisen, and the late Gardner S. Williams, Members, Am. Soc. C. E., was appointed, and reported jointly on January 5, 1924. This Commission recommended an extensive program, involving the construction of additional supply works, beginning immediately with (*a*) the building of a new intake in the river; (*b*) a raw-water tunnel from intakes to the west side of the city; (*c*) a complete new supply station at the site selected on the west side; (*d*) additional distribution mains; and (*e*) future construction of a third complete supply station at a site to be selected in the northeastern part of the city. Item (*c*) included recommendations for a low-lift pumping plant, a filtration plant, a filtered water reservoir, and a distribution pumping plant.

General Description of Flow

The new system provides for the distribution of additional water to the Metropolitan Area of Detroit from three separate stations, each having a filtration plant and the necessary pumping plants (see Fig. 1): (1) The old Water-Works Park Station on East Jefferson Avenue; (2) the Springwells Station on West Warren Avenue; and (3) a future Northeast Station (see Fig. 1). The water is taken from a new, lagoon type, intake built in the Detroit River just northeast of the up-stream end of Belle Isle and not far from the 1905 Intake crib, which will remain in service as an auxiliary intake, and will be

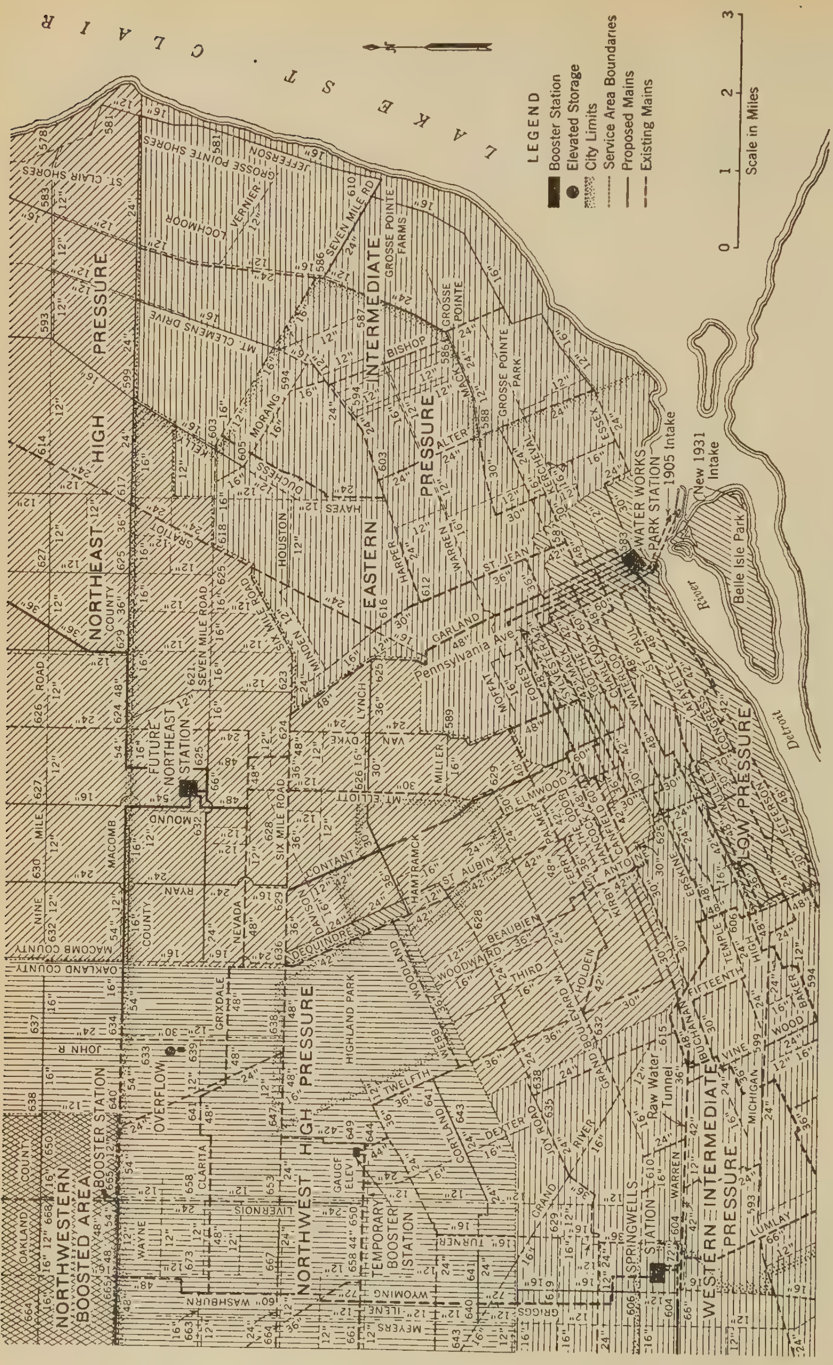
LAKE ST. CLAIR
LEGEND
Booster Station
Elevated Storage
City Limits
Service Area Boundaries
Proposed Mains
Existing Mains
Scale in Miles
0
1
2
3
NORTHEAST HIGH PRESSURE
EASTERN INTERMEDIATE PRESSURE
NORTHWEST HIGH PRESSURE
NORTHWESTERN BOOSTED AREA
WESTERN INTERMEDIATE PRESSURE
LOW PRESSURE
FUTURE NORTHEAST STATION
BOOSTER STATION
OVERFLOW
TEMPORARY BOOSTER STATION
GAUGE ELEV.
SPRINGWELLS STATION
Raw Water Tunnel
WATER WORKS PARK STATION
1905 Intake
New 1931 Intake
Belle Isle Park
Detroit River
GROSSE POINTE SHORES
GROSSE POINTE FARMS
GROSSE POINTE
GROSSE POINTE PARK
ST. CLAIR SHORES
HAMTRAMCK
HIGHLAND PARK
MACOMB COUNTY
OAKLAND COUNTY
Pennsylvania Ave.

FIG. 1.—LOCATION OF PRINCIPAL WATER SUPPLY WORKS, METROPOLITAN AREA, DETROIT, MICHIGAN

carried through a system of gravity tunnels to the old station and across the city to the new stations. The intake, the Springwells Station, and the raw-water tunnel connections to Water-Works Park Station and Springwells Station are part of the present project. Northeast Station and the branch tunnel to it will be built at a future date.

The water to Springwells Station flows from the intake through 3 568 ft of concrete-lined tunnel, 15.5 ft in diameter, built in rock under the river, to a riser well and screen chamber on the shore at Water-Works Park; thence through 10 633 ft of concrete tunnel, 14.0 ft in diameter, in clay, north to the intersection of Pennsylvania and Forest Avenues; thence west through 44 705 ft of concrete tunnel, 12.0 ft in diameter, in clay, to the Springwells Station site, where it turns and enters the suction well of the raw-water (low-lift) pumping plant.

At this point the pumps elevate the raw water into the mixing chamber of the filtration plant through two 10 by 8-ft (or equivalent section) concrete conduits running behind the pumping plant and under the generator plant floor. The flow through the filtration plant is shown in Fig. 2. (The superstructures of the plant are not shown.) Just before entering the mixing chamber the water is metered through Venturi tubes cast in the concrete conduits. Coagulants or chlorine may be applied to the raw water at the entrance to the low-lift plant, at the entrance or exit of the raw-water meters, or at the entrance to the mixing chamber. Thorough and rapid dispersion of the chemicals through the water is accomplished by the turbulence in the pumps, meters, or entrance gates, as the case may be. In the mixing chamber the water is stirred by mechanical agitators and brick baffles.

The mixing chamber discharges into a conduit running the full width of the four settling basins, from which conduit the water may rise at two junction and gate wells which admit the water into the distributing channels of each basin. Entrance to the basins is through vertical slots spaced to give uniform distribution. After passing through these basins at settling velocities the clarified water is decanted from the surface by weirs, and flows into the filter building at two points through main conduits. At the filter operating galleries these conduits branch into the filter influent mains from which the inlets are connected to the sixty-eight rapid sand-filter units. The water flows on to the filter sand beds from the inlets and passes through 20 in. of sand and 18 in. of gravel to the perforated-pipe grid collector system below.

The filtered water that is collected is discharged through automatic rate-of-flow controllers into the main filtered-water conduits in each of the two pipe galleries between the four rows of filters. The two filtered-water conduits discharge into a weir chamber at their eastern ends where the water level is controlled to maintain submergence of the filter-effluent piping as well as the high suction level on the distribution pumps when operated according to the shunt system, described subsequently. From the weir chamber the water may flow either directly to the pumping plant or into the filtered-water reservoir, as flow conditions demand.

Circulation in the filtered-water reservoir is induced by a center baffle-wall in each section and inlets and outlets controlled by flap check-gates. Water

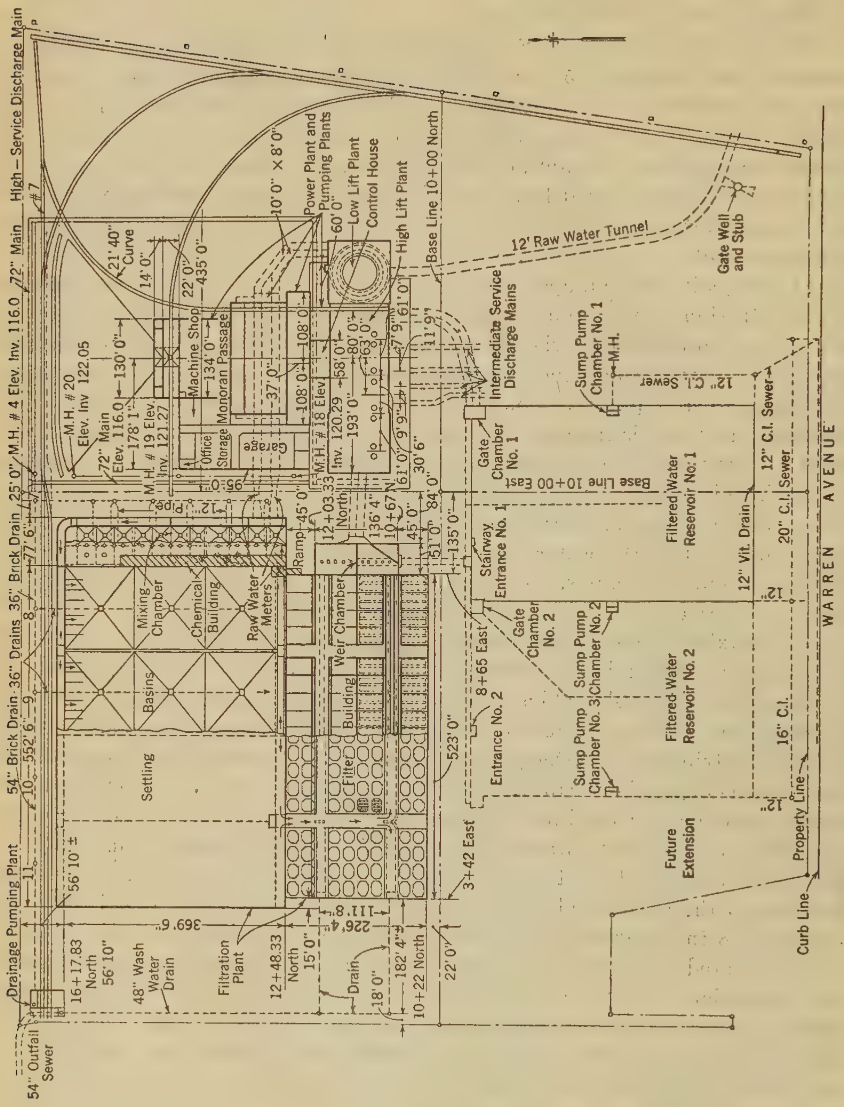
High – Service Discharge Main
72" Main
M.H. # 7
M.H. # 4 Elev. Inv. 116.0
25' 0"
36" Brick Drain
36" Drains
54" Brick Drain
Drainage Pumping Plant
54" Outfall Sewer
16+17.83 North
56' 10"
48" Wash Water Drain
Filtration Plant
12+48.33 North
15' 0"
Drain
18' 0"
182' 4"
10+22 North
22' 0"
369' 6"
226' 4"
111' 8"
56' 10' ±
552' 6"
77' 6"
Settling
Basins
Mixing Chamber
Chemical Building
Raw Water Meters
Pipe
Weir Chamber
Building
Filter
Ramp
45' 0"
12+03.33 North
136' 4"
10+67 N
45' 0"
51' 0"
135' 0"
84' 0"
523' 0"
3+42 East
8+65 East
Entrance No. 2
Stairway Entrance No. 1
Gate Chamber No. 1
Gate Chamber No. 2
M.H. # 20 Elev. Inv 122.05
72" Main Elev. 116.0
178' 1"
130' 0"
M.H. # 19 Elev. Inv. 121.27
95' 0"
12"
Office
Storage
Garage
Machine Shop
Monoran Passage
134' 0"
37' 0"
108' 0"
M.H. # 18 Elev. Inv. 120.29
193' 0"
58' 0"
80' 0"
63' 0"
61' 0"
9' 9"
7' 9"
11' 9"
30' 6"
21' 40" Curve
14' 0"
22' 0"
435' 0"
10' 0" × 8' 0"
60' 0"
Power Plant and Pumping Plants
Low Lift Plant
Control House
High Lift Plant
Base Line 10+00 North
12' Raw Water Tunnel
Intermediate Service Discharge Mains
Gate Well and Stub
Sump Pump Chamber No. 1
M.H.
12" C.I. Sewer
Base Line 10+00 East
Filtered Water Reservoir No. 1
12" C.I. Sewer
20" C.I. Sewer
12" Vit. Drain
12"
Sump Pump Chamber No. 2
Sump Pump Chamber No. 3
Filtered Water Reservoir No. 2
16" C.I.
Future Extension
Property Line
Curb Line
WARREN AVENUE

FIG. 2.—GENERAL PLAN, FILTRATION PLANT, SPRINGWELLS STATION

is drawn from the reservoir through conduits entering the opposite end of the pumping plant to that connected directly to the weir chamber. The distribution pumps discharge through steel mains connected to each of two service districts (a high-pressure area and an intermediate pressure area, see Fig. 1), which are served jointly by the Water-Works Park Station and the Springwells Station. The detailed description which follows covers only the design and construction of the filtration plant and reservoir at Springwells Station.

Experimental Work and Studies Preliminary to Design

Experimental Filter Plant.—In the summer of 1925 an experimental filter plant with a capacity of 150 000 gal per day was designed and built for the express purpose of studying the phenomena of mixing and settling. This plant was composed of rectangular wooden tanks supported on a concrete mat foundation and enclosed in a light frame structure. The plant was arranged so that a reasonable range of times and velocities of mixing and a wide range of times and velocities in coagulation basins could easily be obtainable by making slight alterations. Two identical filter units were provided for use as a measure of results.

After about six months' operation in parallel with the large filter plant, the experimental plant was placed in charge of Mr. Roberts Hulbert and Frank W. Herring, Assoc. M. Am. Soc. C. E., who began on the program of settling-basin studies and obtained some very good results; but certain peculiarities in these results led to an intensive search to determine the reason for incon-

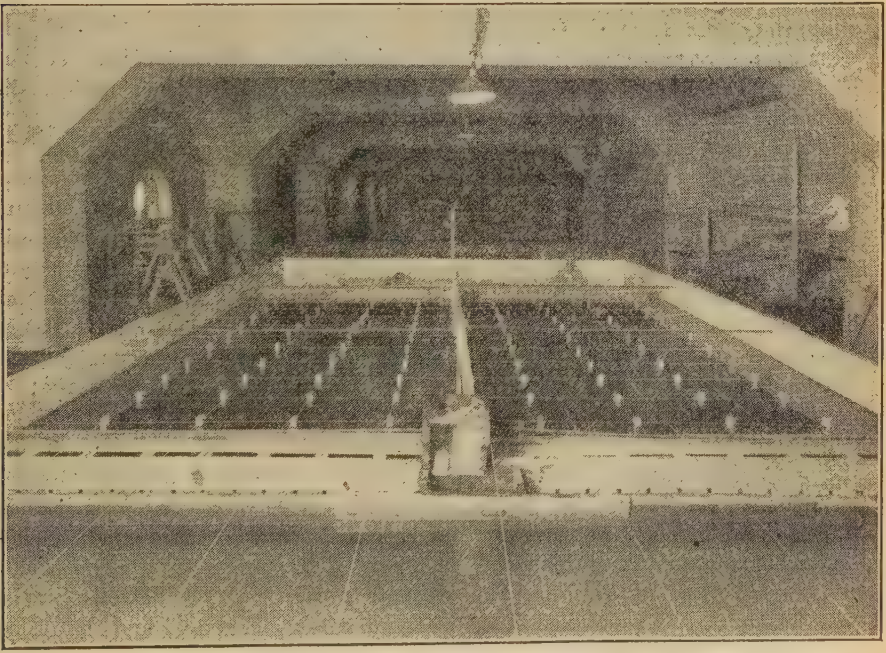

Fig. 3.—View of Settling Basin Model

sistencies. Finally, the trouble was discovered in the fact that the two filter units used to measure the results were not washed uniformly. Having become absorbed in the development of better methods of washing filters, the remainder of the time available for the experimental filter plant operation was spent on this subject and a technique was developed of washing them to a certain definite sand expansion, instead of at the usual constant rates of wash-water flow.[2]

Scale Model Tests of Settling Basin Inlets.—Concurrently with the operation of the experimental filter plant, investigations of flow in settling basins were made by the use of a 1 : 25 scale model replica of the basin proposed to be used at the Springwells Plant. This model (see Fig. 3) was the means of determining the detail of inlet and outlet construction required to give a uniform distribution of flow across the full width of the basin and the longest actual retention period, as determined by floats, dyes, and salt solutions. Potassium permanganate dye was most satisfactory in determining the condition and distribution of flow by observing the advancing cloud of colored water. The time element was denoted by plotting the front edge of the color cloud at the end of each minute after the application of the color until it flowed out at the outlet. This plotting gave a record of each run, such as the typical form shown in Fig. 4.

The outlet conditions were early found to have little effect on the flow distribution. Therefore, attention was concentrated on the inlet detail. From the test results on the model, a vertically slotted wall inlet was adopted, with training walls between pairs of slots, The arrangement of the entrance baffles is shown in Fig. 4, with seventeen slots of the size and spacing indicated. In the entrance chamber the openings are beveled on the down-stream side (see Fig. 4(*b*) and Fig. 4(*c*)). Guide-vanes were of galvanized iron. The basins have "straight-through" flow.

Filter Under-Drain Lateral Experiments.—To determine how uniform the distribution of wash-water flow would be from a tentatively adopted design of filter under-drain lateral and to develop a low-cost brass, or bronze, bushing for the perforations in the under-drain laterals, a testing apparatus was built adequate for one typical full-sized lateral, almost identical in dimensions with those that would go into the actual filters. Several types of pipe materials in this apparatus, were tested, as well as types of perforations and bushings. Incidental to this test, a series of submerged orifices was calibrated and discharge coefficients were obtained.

Analytical Investigations and Studies.—Analytical studies were made to determine the economic size of filter units, economic filtering head, the design of the wash-water supply system (with particular regard to the relationship of the size of wash-water tanks and wash-water pumps supplying those tanks), and the costs of types of filter bottoms, wash-water trough materials, and chlorine-handling equipment.

General Basis of Design.—The ultimate capacity of Springwells Station was determined by a detailed study of the distribution system and the proper division of load among the three plants designated by the consultant's report of 1924.[3] This report and the results of experimental work and preliminary studies, together with previous experience, led to adoption of values given in Table 1, as the basis of design for the Springwells Filtration Plant. The design required, furthermore, that coagulants should be applied to the initial mix at the pumps or the meters and that the mixing be by mechanical agitators. The design of the details of the settling basins was determined by scale model studies (see Fig. 4). The filter bottom specified was the perforated-pipe grid type with a center manifold.

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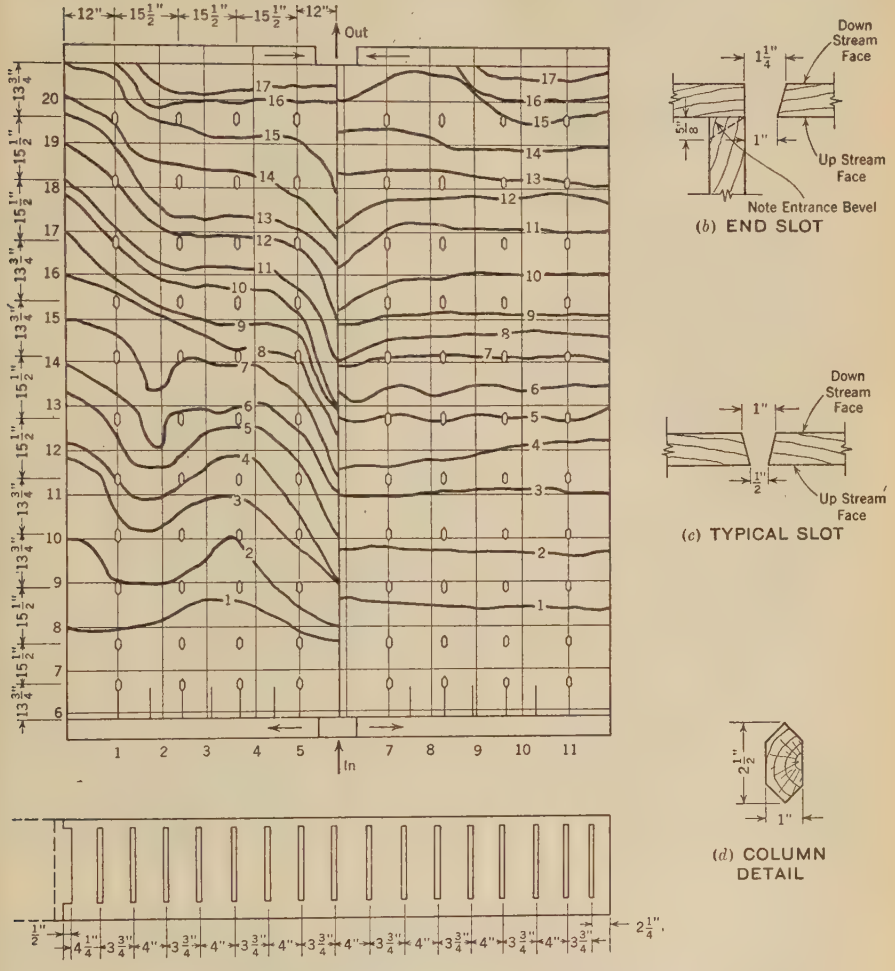
[2] "Studies on the Washing of Rapid Filters," by R. Hulbert and F. W. Herring, *Journal*, Am. Water Works Assoc., Vol. 21, November, 1929, p. 1445.


12"
15½"
15½"
15½"
12"
Out
In
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
13¾"
15½"
Down Stream Face
Up Stream Face
1¼"
5/8"
1"
Note Entrance Bevel
1"
½"
2½"
1"
½"
4¼"
3¾"
4"
2¼"

(b) END SLOT

(c) TYPICAL SLOT

(d) COLUMN DETAIL

(a) HOLE SPACING

FIG. 4.—DETAILS OF MODEL AND TYPICAL RECORD OF TEST RUN

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[3] "Trunk Main Water Distribution System for Greater Detroit," by L. E. Ayres, M. Am. Soc. C. E., *Engineering News-Record*, Vol. 100, No. 22, May 31, 1928, p. 855.

TABLE 1.—DESIGN FACTORS: SPRINGWELLS FILTRATION PLANT

Item No.	Description	Value
	(*a*) ULTIMATE CAPACITY (ESTIMATED AS OF 1955)	
	Net Output, in Million Gallons Daily:	
1	(*a*) Average day	206
2	(*b*) Maximum day	278
3	Six filters supplying wash water, in million gallons daily	27
4	Required gross capacity, in million gallons daily	305
	Filtered-Water Reservoir, in Million Gallons:	
5	(*a*) Ultimate	60
6	(*b*) Present	40
	(*b*) MIXING AND COAGULATION	
7	Time of mixing contact at maximum capacity, in minutes	15
8	Number of mixing chamber units	3
	(*c*) SETTLING BASINS	
9	Time of retention at maximum capacity (300 mgd), in hours	2
10	Velocity in basins, in feet per minute	2 to 3
	Maximum Velocity, in Feet per Second:	
11	(*a*) Inlets	1
12	(*b*) Outlets	1
13	Number of basin units	4
	(*d*) FILTERS	
	Rate of Filtration, in Million Gallons Daily per Acre:	
14	(*a*) Normal	160
15	(*b*) Maximum	180
	Rate of Washing, in Inches of Rise per Minute:	
16	(*a*) Normal	36
17	(*b*) Maximum	39
	(*d*) FILTERS (*Continued*)	
	Size of Each Unit:	
18	(1) Area of sand, in acres	$\frac{1}{40}$
	(2) Filtering rate, in million gallons daily:	
19	(*a*) Normal	4.0
20	(*b*) Maximum	4.5
21	Number of filter units	68
22	Maximum velocity of influent, in feet per second	1
23	Operating head, in feet	9.5
24	Capacity of wash-water tank, in million gallons	0.1
25	Capacity of wash-water pumps, in gallons per minute	28 000
	(*e*) STRUCTURAL LOADS AND STRESSES	
	Weight, in Pounds per Cubic Foot:	
26	Water	62.5
27	Earth	100.0
28	Concrete	150.0
	Maximum Allowable Unit Stress, in Pounds per Square Inch:	
	(1) Tension in reinforcement, for:	
29	(*a*) Floors, walls, and footings	16 000
30	(*b*) Roofs and columns	18 000
	(2) Compression in concrete, for:	
31	(*a*) Floors, walls, and footings	650
32	(*b*) Roofs and tied columns	750
33	(*c*) Circular columns with spirals	1 000
34	Maximum allowable load on a bearing pile, in tons	25
	Maximum Allowable Bearing Value of Earth Foundations, in Tons per Square Foot:	
35	(*a*) Filtration plant structures	0
36	(*b*) Reservoir	1

Horizontal earth pressures were assumed to be equivalent to hydrostatic pressures, plus surcharge when imposed. Live loads were determined by conditions in substructures; in superstructures, live loads were determined by conditions, or by reference to the Detroit Building Code, whichever was the higher.

DESCRIPTION OF PLANT

General.—The keynote of the entire design is practical simplicity. Attention was given to each detail to avoid complex situations which would be costly to build, or which later might cause trouble in operation and maintenance. Where auxiliaries have been used, such as master control for filters, summation devices for meters, float-controlled water-level indicators, float controls for pumps, etc., they have been installed so that their discontinuance from service will not interrupt the operation of the plant or affect the functioning of the standard equipment with which they are used. As a rule, automatic devices, such as the differential chemical feed, the automatic filter shut-off, the proportional chlorine feed, etc., have been avoided, preference being given to manual control where practicable.

The construction of the conduits, chambers, basins, filters, and reservoirs, is of reinforced concrete. The superstructures have structural steel frames encased in concrete (with the exception of the steel frame of the Chemical Building which is only partly encased), reinforced concrete roofs insulated with cork and tar and gravel roofing, and hollow-tile curtain-walls faced on the outside with Indiana limestone and on the inside with terra cotta, glazed brick tile, or glazed hollow tile. The operating floors of the filters are finished with 6 by 6-in., red, quarry tile with dark green terrazzo borders. The floors in the Chemical Building, storage rooms, etc., are finished with concrete hardened with granite chips and ground smooth.

The entire structure is supported on a foundation of wooden bearing piles driven to hardpan through a subsoil of plastic and soft blue clay. The total load averages 20 tons per pile with the piles arranged so that, when the plant is entirely filled and loaded, there is a variation of not more than 5 tons per pile. On account of the unfavorable subsoil conditions, care was taken to avoid excessive concentrations of loads in the plant. Some variation in loading is occasioned by fluctuating live loads, such as in storage bins, basins, filters, and wash-water tanks.

Chemical Plant.—The Chemical Building is over a part of the mixing chambers, and contains the chemical storage tanks, chemical unloading and conveying equipment, the dry-feed machines, and hoppers for dosing ground sulfate of aluminum and ammonium sulfate. The arrangement of chemical equipment is shown in Fig. 5. This equipment is arranged so that, by a minimum of revision, the manufacture of aluminum sulfate may be provided for, at the same time preserving the dry alum equipment for stand-by service. The provision for alum manufacture contemplates making the alum syrup and feeding it as a solution.

The design of the Chemical Plant was based on the following capacities:

(1) At maximum demand, a storage capacity of 30 days' supply of raw materials.
(2) At normal demand, a storage capacity in each alum feeder bin, of 1 day.
(3) At maximum demand, a storage capacity in solution feed tanks, of 1 day.
(4) At maximum demand, a storage capacity for alum syrup, of 3 days.
(5) Unloading equipment to handle a 40-ton car in 8 hr.
(6) Reclaiming conveyors sufficient to handle 1 day's supply in 3 hr.

The equipment installed in the Chemical Plant consisted of the following:

(1) Suspended steel storage tanks of 390-tons total capacity consisting of:

(*a*) Four storage tanks at 60 tons capacity each.
(*b*) Three storage tanks at 30 tons capacity each.
(*c*) Five feeder hoppers at 12 tons capacity each.

(2) Pneumatic conveyor for unloading from railroad cars and delivering to storage (capacity 5 tons per hr).

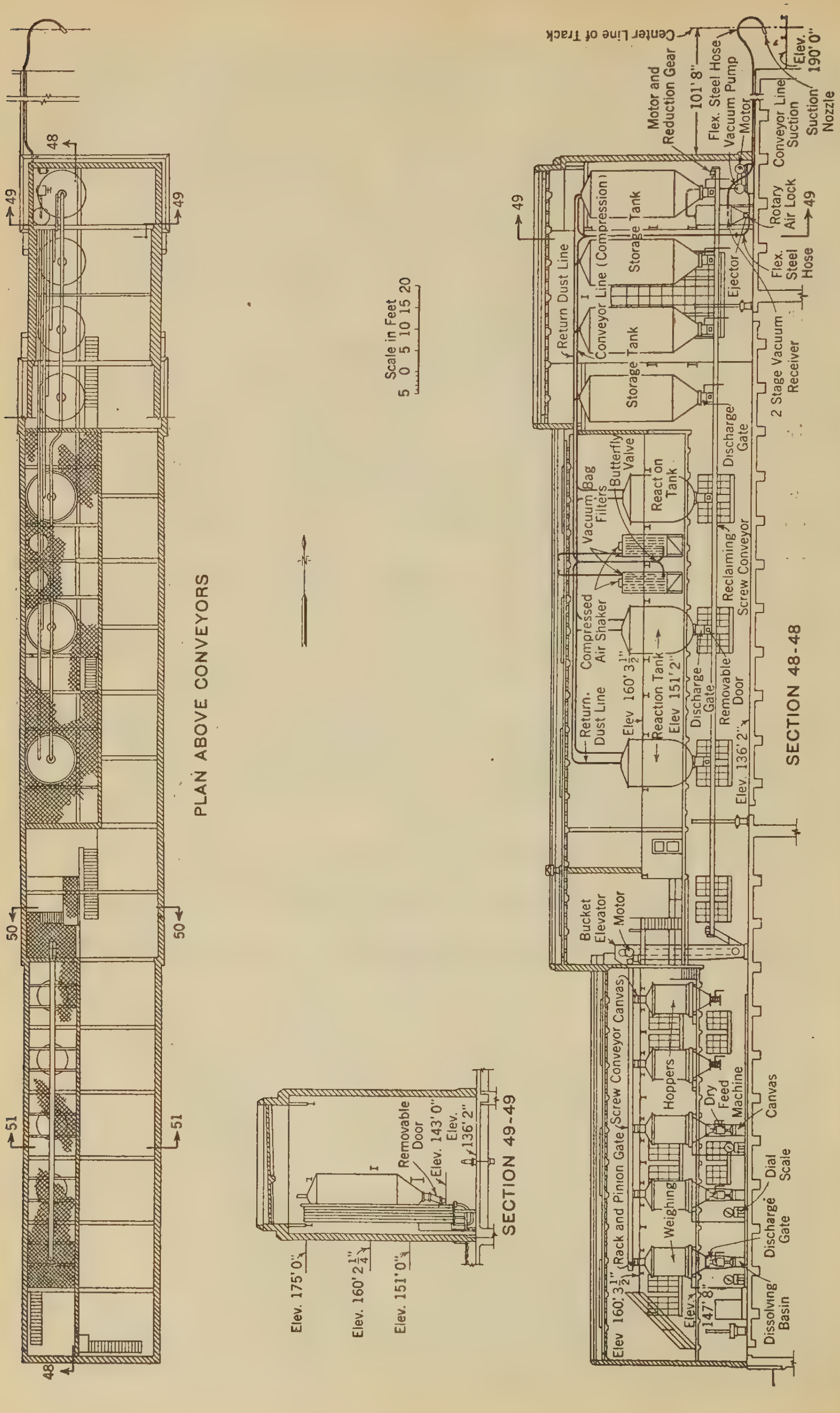

PLAN ABOVE CONVEYORS
48
49
50
51
Scale in Feet
5 0 5 10 15 20
SECTION 49-49
Elev. 175'0"
Elev. 160'2¼"
Elev. 151'0"
Removable Door
Elev. 143'0"
Elev. 136'2"
SECTION 48-48
Center Line of Track
Motor and Reduction Gear
101'8"
Flex. Steel Hose
Vacuum Pump
Motor
Conveyor Line Suction
Elev. 190'0"
Suction Nozzle
Rotary Air Lock
Flex. Steel Hose
Ejector
2 Stage Vacuum Receiver
Storage Tank
Storage Tank
Conveyor Line (Compression)
Return Dust Line
Discharge Gate
Butterfly Valve
Reaction Tank
Vacuum Bag Filters
Reclaiming Screw Conveyor
Compressed Air Shaker
Return Dust Line
Elev. 160'3½"
Reaction Tank
Elev. 151'2"
Discharge Gate
Removable Door
Elev. 136'2"
Bucket Elevator
Motor
Screw Conveyor Canvas
Rack and Pinion Gate
Hoppers
Dry Feed Machine
Canvas
Weighing
Dial Scale
Discharge Gate
Elev. 160'3½"
Elev. 147'8"
Dissolving Basin

FIG. 5.—ARRANGEMENT OF CHEMICAL EQUIPMENT

(3) Screw conveyor, 200 ft long, for reclaiming material from storage tanks (capacity 5 tons per hr).

(4) Bucket elevator for lifting material from the reclaiming conveyor to the conveyor above the feeder hoppers (capacity 6 tons per hr).

(5) Screw conveyor for distributing material to the feeder hoppers (capacity 6.6 tons per hr).

(6) Five dry-feed machines suspended from the feeder hoppers.

(7) A weighing scale for each feeder hopper and dry feed machine, with 30-in. indicator dial and recorder.

(8) A dissolving basin for each dry-feed machine, delivering to the chemical dosing lines.

(9) Three 3-in. dosing line headers of 99% pure copper extra heavy tubing, with stream-line fittings and valves of bronze composition resistant to the action of the alum solution.

Raw-Water Conduits and Meters.—The raw water is delivered to the mixing chamber through two 10 by 8-ft concrete conduits which run under the floor of the generator room in the power plant. At the west side of the power plant the conduits change shape to 9-ft square conduits and slope upward on a grade of about 12 per cent. In this section, a distance of about 60 ft, a raw-water meter is installed in each conduit. These meters are of the Venturi type with 9-ft square inlet and outlet ends and 4-ft square throats. The meters are cast of concrete with bronze-lined iron throat and pressure-ring castings and bronze-lined cast-iron inlet pressure rings set in the concrete. The raw-water meter registers are in a passageway of the building convenient to both alum and chlorine feeders. Dosing points for coagulating chemicals and chlorine are provided at both inlet and outlet ends of the meter tubes. In the design of the meter tubes advantage was taken of the upward slope in the conduits so as to compensate for their convergence in such a way that no air is trapped, on filling the conduits, and all water may be drained out, on emptying the conduits.

Mixing Chambers.—The dosed raw water enters the mixing chambers from the down-stream ends of the two parallel raw-water meters. The mixing chambers are in three units of approximately 5-min retention period, each with a by-passing channel. The general arrangement and details are shown in Fig. 6. The water enters the by-passing channel on the east side which is equipped with gates to shut off or admit water to any of the three chambers. Normally, the water enters through the three 6 by 10-ft sluice-gate inlets to the south chamber, flows northward through the three chambers (the dividing wall-gates of which are normally open) and leaves through the three 6 by 10-ft gates at the northwest corner into the coagulated water conduit to the settling basins.

All the sluice-gates in the mixing chambers are of the rising stem type, and are hand-operated, with worm-geared floor stands. While it is in the mixing chambers the water is stirred by mechanically driven paddles rotated at a peripheral speed of 0.67, 1.0, 1.33, or 2 ft per sec, the speed being regulated

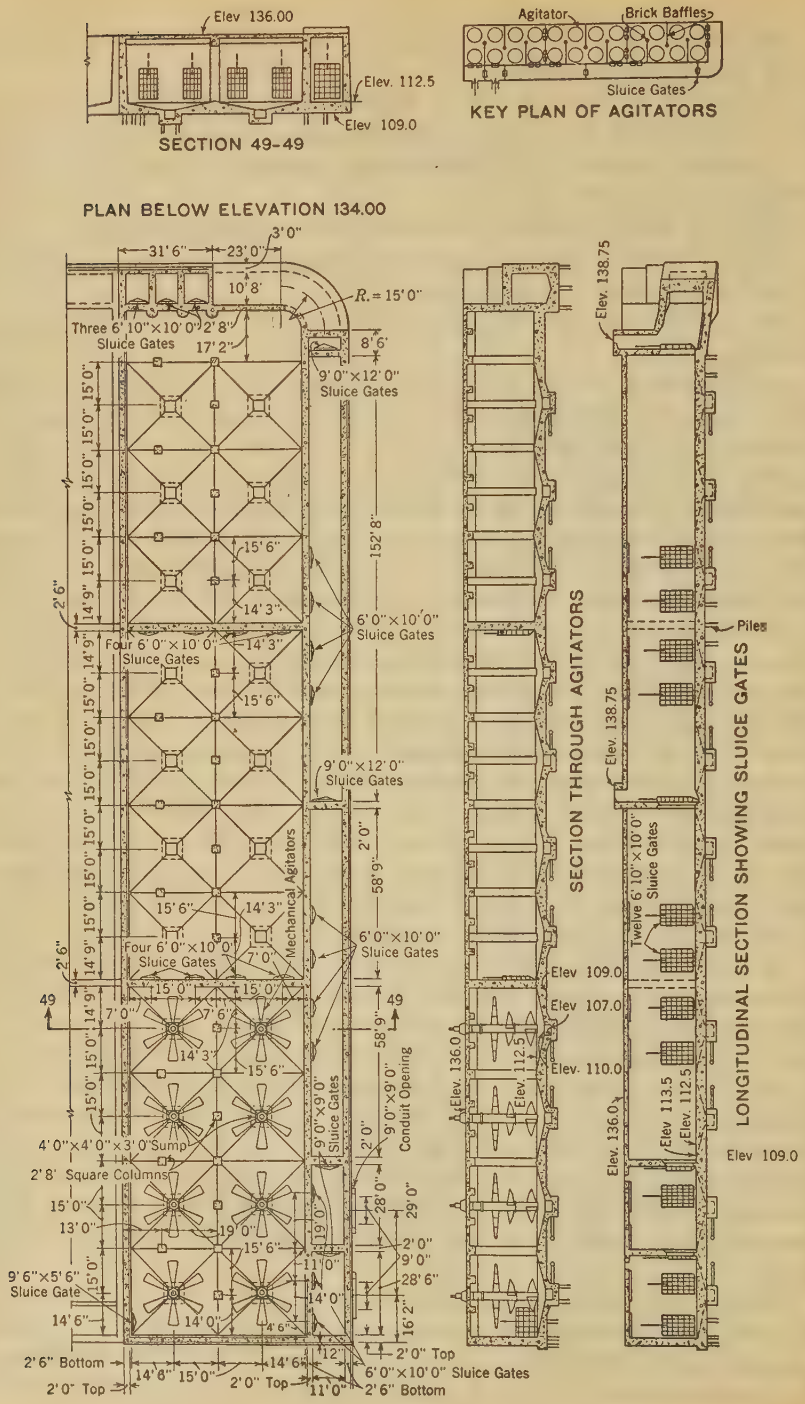

Elev 136.00
Elev. 112.5
Elev 109.0
SECTION 49-49
Agitator
Brick Baffles
Sluice Gates
KEY PLAN OF AGITATORS
PLAN BELOW ELEVATION 134.00
31' 6"
23' 0"
3' 0"
10' 8"
R. = 15' 0"
Three 6' 10"×10' 0" Sluice Gates
2' 8"
17' 2"
8' 6"
9' 0"×12' 0" Sluice Gates
152' 8"
15' 6"
14' 3"
6' 0"×10' 0" Sluice Gates
Four 6' 0"×10' 0" Sluice Gates
9' 0"×12' 0" Sluice Gates
2' 0"
58' 9"
Mechanical Agitators
7' 0"
15' 0"
7' 6"
49
9' 0"×9' 0" Sluice Gates
9' 0"×9' 0" Conduit Opening
4' 0"×4' 0"×3' 0" Sump
2' 8' Square Columns
13' 0"
19' 0"
28' 0"
29' 0"
11' 0"
9' 0"
28' 6"
14' 0"
16' 2"
9' 6"×5' 6" Sluice Gate
14' 6"
4' 6"
12"
2' 0" Top
2' 6" Bottom
14' 6"
6' 0"×10' 0" Sluice Gates
11' 0"
2' 6" Bottom
2' 0" Top
2' 6"
14' 9"
SECTION THROUGH AGITATORS
Elev. 138.75
Elev 109.0
Elev 107.0
Elev. 110.0
Elev. 136.0
Elev. 112.5
Piles
Twelve 6' 10"×10' 0" Sluice Gates
Elev 113.5
Elev. 112.5
Elev. 136.0
Elev 109.0
LONGITUDINAL SECTION SHOWING SLUICE GATES

Fig. 6.—Plan and Details of Mixing Chambers

to give the best conditioning of the water. At present (1934), only the first mixing chamber is equipped with agitators, and brick baffle-walls have been built in all three chambers. As the plant load is increased the other chambers will be equipped with agitators. The location of the mechanical agitator units for the south section of the mixing chamber is indicated in Fig. 6. Steel-plate paddles, 10 ft long, are mounted spirally about a shaft of 12-in. steel pipe hung from a thrust-bearing in a base casting mounted on the cover slab of the chamber. The lower end of the shaft is steadied in a sleeve-bearing attached to the bottom of the chamber sump. This paddle assembly is driven through a 900:1 vertical reduction gear by a four-speed squirrel-cage vertical motor mounted on the gear housing.

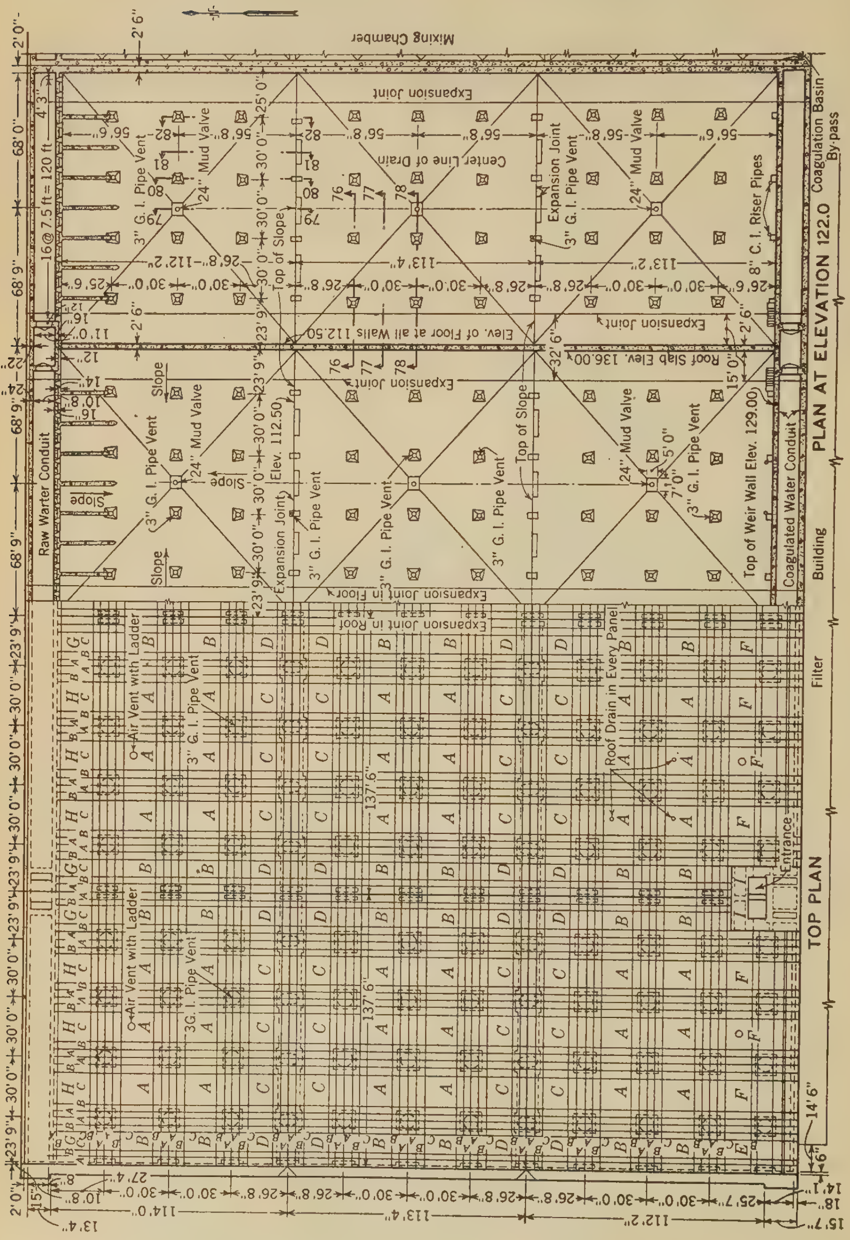
The following data will serve as a summarized description of the mixing chamber:

Elevations, in Feet, Above:	
High water	131.6
Normal water	131.1
Bottom Slab:	
At wall line	112.5
At edge of sumps	110.0
Volume of water, in million gallons	3.201
Average dose of alum, in grains per gallon	0.6
Time of mixing, in minutes, at 300 mgd capacity	15
Length of horizontal travel, with baffles, in feet (approximately)	1 000
Velocities, in Feet per Second:	
Of horizontal flow, at 300 mgd capacity	1.11
Of agitators (peripheral)	0.67 to 2.0
In gates	2.5 to 4.0

Settling Basins.—The general features of the settling basins are shown in Fig. 7. Referring to Fig. 8, column vanes are 12 in. thick at Elevation 112.5 and 6 in. thick at the top, from Point *A* to the entrance baffle. Intermediate vanes are 6 in. thick at the top and 12 in. at Elevation 112.5 for their full length and the batter continues to the top of the vane.

The settling basins are in four units each about 135 by 340 ft in plan, with about 18 ft of water depth and 5 ft of free-board from the water surface to the top of the roof slab. The entrance wall, or inlet structure, consists of a lower conduit, 10 ft 8 in. wide by 12 ft 3 in. deep, which extends the full width of the four basins, and an upper channel which distributes the water across the width of each basin by means of vertical slot openings in its inside wall. At two points, one between each pair of basins, there is a junction chamber which allows the water to flow up from the lower conduit into the upper channel or inlet of each respective basin. Each inlet channel entrance is provided with a 9 by 12-ft. sluice-gate. The inlet channel and slotted openings were constructed as nearly as possible similar to the most favorable inlet design obtained in the scale model tests previously mentioned.

The water distributed across the north end of each coagulation basin by the slotted openings in the upper entrance channel is directed straight



Mixing Chamber
Expansion Joint
Center Line of Drain
Expansion Joint
3" G. I. Pipe Vent
24" Mud Valve
Top of Slope
Elev. of Floor at all Walls 112.50
Roof Slab Elev. 136.00
Top of Weir Wall Elev. 129.00
8" C. I. Riser Pipes
Coagulation Basin By-pass
Coagulated Water Conduit
Raw Warter Conduit
16@7.5 ft = 120 ft
Slope
Elev. 112.50
Expansion Joint in Floor
Expansion Joint in Roof
Air Vent with Ladder
3" G. I. Pipe Vent
3G. I. Pipe Vent
Roof Drain in Every Panel
Entrance
137'6"
PLAN AT ELEVATION 122.0
TOP PLAN
Building
Filter
113'4"
113'2"
112'2"
114'0"
56'6"
56'8"
30'0"
26'8"
25'6"
23'9"
68'0"
68'9"
2'0"
2'6"
32'6"
15'0"
14'6"
13'4"
15'7"

FIG. 7.—GENERAL PLAN OF SETTLING BASIN

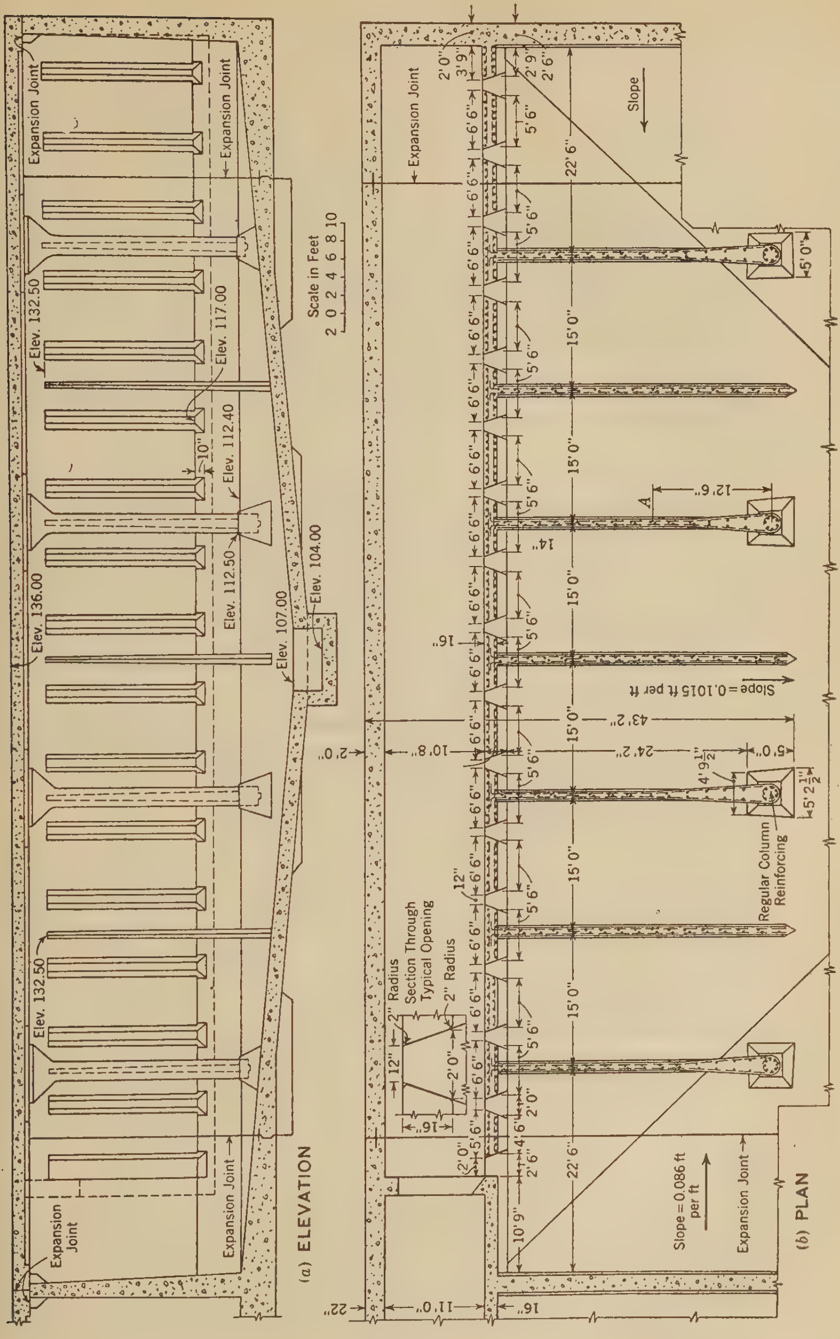

(a) ELEVATION
Expansion Joint
Elev. 132.50
Elev. 117.00
Elev. 112.40
Elev. 112.50
Elev. 136.00
Elev. 107.00
Elev. 104.00
10"
Scale in Feet
2 0 2 4 6 8 10
(b) PLAN
Expansion Joint
Slope
Slope = 0.1015 ft per ft
Slope = 0.086 ft per ft
Section Through Typical Opening
2" Radius
Regular Column Reinforcing
A
2'0"
3'9"
2'9"
2'6"
6'6"
5'6"
22'6"
15'0"
5'0"
12'6"
14"
16"
43'2"
24'2"
10'8"
4'9½"
5'2½"
12"
2'0"
4'6"
10'9"
11'0"
22"

Fig. 8.—Plan and Elevation of Settling Basin Inlet

through the basin by straight vanes, or by training-walls about 29 ft long. The water flows at a velocity of 2 to 3 ft per min for the 340 ft to the south end of the basins, where it is taken off over a weir. The weir is normally submerged about 1½ ft, into a collecting conduit, 11 ft 6 in. wide by about 10 ft deep. The collecting conduits flow together at two points into junction chambers which admit the water to the Filter Building. Each collecting channel outlet is provided with a 9 by 12-ft sluice-gate for shut-off purposes. Beneath the collecting conduits is a by-pass conduit, 11 ft wide by 6 ft 6 in. deep, extending the full width of the basins, through which water may be by-passed directly from the mixing chamber to the filters. This conduit is also an equalizer of the flow from basins to the two filter influent mains. At the outlet weir there is a skimmer baffle and trough of somewhat unusual design, so constructed that, during short periods of high-water level in the basins, the collected scum will flush off into the drain. The skimmer trough acts also as an overflow.

To facilitate cleaning, the bottom of each basin is shaped into three shallow hoppers, with a sump at the center of each. A hydraulically operated 24-in. mud valve is operated on the basin drain line in this sump. The valves on the pressure lines for operating the mud valves are grouped in the two, outlet, junction-chamber, gate-houses. One basin is equipped with a perforated belt line of small pipe for flushing the floor during cleaning. If this proves efficient the other basins will be so equipped. A water main with hose connections is provided in each basin for use in cleaning.

Structurally, the basins are of simple reinforced concrete design; the side walls and division walls are of the cantilever type; the floor is of the flat-slab type; and the roof is a flat slab supported by cylindrical columns 30 in. in diameter, spaced 30 ft from center to center both ways.

The roof-slab spans are 30 ft square between intermediate panels; the end panels have reduced spans in the direction in which the continuity is broken by walls and expansion joints. Since flat slabs of 30-ft span are quite close to the maximum in common use, the analytical design was checked by a mechanical instrument simulating a model of the structure.

The inlet and outlet wall structures are of somewhat special construction, as typical of such features: Inlet and by-pass conduits are designed as rigid monolithic structures; the slotted inlet wall, subject to little horizontal load, is designed as a curtain-wall or baffle, tied at the top, bottom, and ends; the outlet weir wall is a cantilever; and the junction chambers have horizontally spanned walls.

The roof of the basins is covered with 2 ft of fill, consisting of 6 in. of pea gravel and 18 in. of earth. Over each junction chamber is a gate-house superstructure in which the 9 by 12-ft sluice-gates are hoisted by electricity and stored. The part of the basins above the general finished grade of the site (Elevation 130.0) is faced with limestone to match the building group, and the curb wall retaining the basin roof fill is surmounted by an ornamental iron fence with stone posts.

The following data will serve as a summarized description of the settling basin:

Elevations, in Feet, above Detroit City Datum:	
High water	131.0
Normal water	130.6
Bottom Slab:	
At wall line	112.5
At edge of sumps	107.0
Crest of outlet weir	129.0
Volumes, in Cubic Feet:	
Water in each basin (Elevation 112.5 to Elevation 131.0)	844 770
Sludge hoppers, below Elevation 112.5	86 675
Total capacity in four basins:	
With water at Elevation 131.0	3 725 780
With water at Elevation 130.5	3 634 460
Number of basins	4
Time of detention at 306 mgd capacity, in hours	2
Velocity of flow through the basin, in feet per minute, at 306 mgd capacity	2.8
Velocities, in Feet per Second, at 306 mgd Capacity:	
Entrance velocity (normal water level)	1.0
Outlet velocity over weir (normal water level)	0.6
Maximum velocity in basin outlet channel	1.0
Turbidity of settled water, in parts per million.	10 to 15
Dimensions, in Feet:	
Length of flow in basin	340.0
Width of one basin	135.5
Effective water depth, one basin	18.0 to 18.5

Slotted entrances and 29-ft guide-vanes constitute the baffling system; the outlet of the basin is operated as a submerged weir; and the basins are covered.

Filters.—The group of sixty-eight filter units is adjacent to and centers on the south wall (which is the outlet end) of the settling basins. The filters are arranged in four rows along two parallel pipe galleries with concrete influent conduits connected by cross-conduits to the outlet chambers of the settling basins. The top slab of the influent conduits forms the operating floor between the rows of filters.

The interior of a filter unit showing the wash troughs and the under-drain system before the filtering material is placed, is shown in Fig. 9. The filters are of the front gullet type, with longitudinal wash troughs of cast iron and perforated cast-iron pipe under-drains. The under-drain laterals are connected to a central manifold cast in the concrete bottom of the filter. The ends of the laterals at the side walls of the filter unit are connected by 4-in. cast-iron headers parallel to the walls to eliminate dead ends and to assure more uniform distribution of flow in the under-drain system. These wall headers are perforated also to give a slight excess of wash water along the filter walls, during washing.

The filtering medium consists of an 18-in. depth of specially selected gravel and 20 in. of silica sand. The gravel grades in size from 3 in., maximum diameter at the bottom, to $\frac{1}{16}$-in. particles at the top of the gravel layer. The sand is 95% pure silica having an effective size of 0.5 mm and a uniformity coefficient of 1.3; 2% of it is composed of particles larger than 1 mm. The sand finer than 0.3 mm was washed out and is disregarded in computing the effective size and uniformity coefficient.

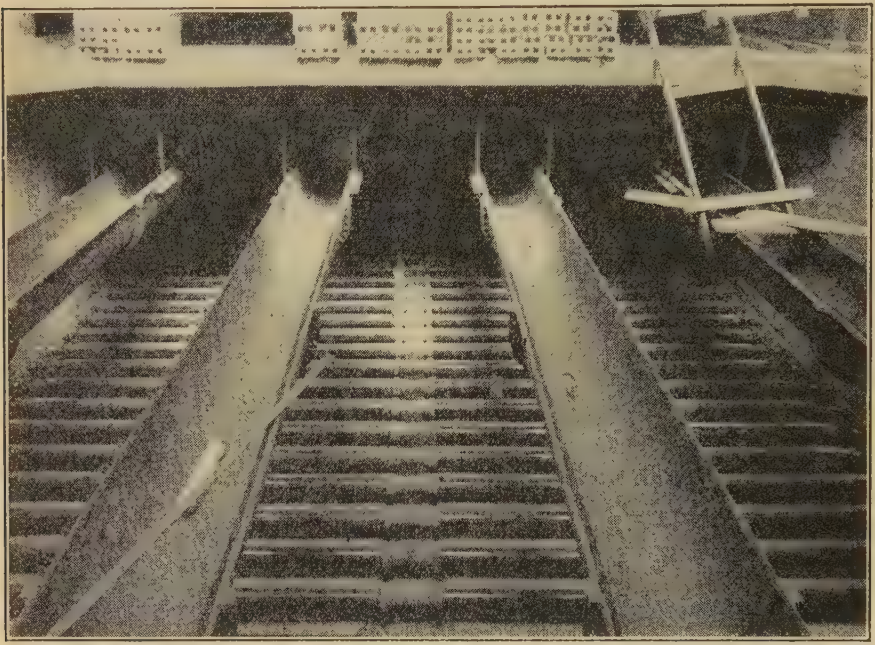


FIG. 9.—VIEW OF INTERIOR OF FILTER, SHOWING UNDER-DRAIN SYSTEM AND WASH TROUGHS

While it is filtering, water normally stands within 6 in. of the top of the filter walls, making about 7 ft of water depth on the surface of the sand bed. In washing, the sand expands 50% and rises to the bottom of the lower ends of the wash troughs.

A typical section of the pipe galleries and the piping for a typical filter unit is shown in Fig. 10. All header conduits are of concrete and are arranged to form the floor of the pipe gallery as well as the filter operating floor above. The filter connections are made with cast-iron fittings and flanged valves of the following sizes (in inches):

Influent	36
Effluent	24
Wash water	24
Sewer	30
Re-wash	8

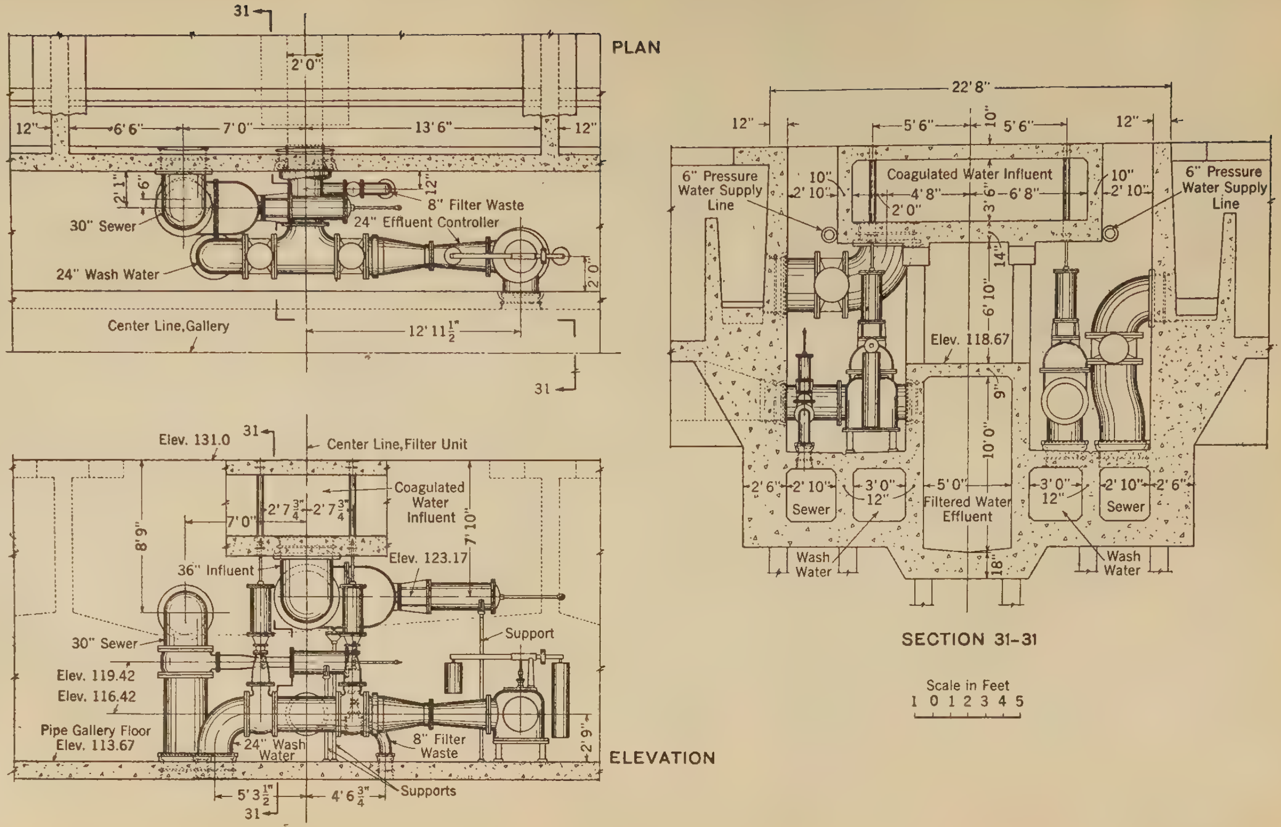

PLAN
31
2'0"
12"
6'6"
7'0"
13'6"
12"
2'1"
6"
12"
30" Sewer
8" Filter Waste
24" Effluent Controller
24" Wash Water
2'0"
Center Line,Gallery
12'11½"
31
ELEVATION
Elev. 131.0
31
Center Line, Filter Unit
Coagulated Water Influent
8'9"
7'0"
2'7¾"
2'7¾"
7'10"
Elev. 123.17
36" Influent
30" Sewer
Support
Elev. 119.42
Elev. 116.42
Pipe Gallery Floor Elev. 113.67
24" Wash Water
8" Filter Waste
2'9"
5'3½"
4'6¾"
Supports
31
22'8"
12"
5'6"
10"
5'6"
12"
6" Pressure Water Supply Line
10"
2'10"
Coagulated Water Influent
4'8"
3'6"
6'8"
2'0"
10"
2'10"
6" Pressure Water Supply Line
14"
6'10"
Elev. 118.67
9"
10'0"
2'6"
2'10"
3'0"
12"
5'0"
3'0"
12"
2'10"
2'6"
Sewer
Filtered Water Effluent
Sewer
18"
Wash Water
Wash Water
SECTION 31-31
Scale in Feet
1 0 1 2 3 4 5

FIG. 10.—PIPE GALLERY DETAILS, FILTRATION PLANT

All valves are of the cast-iron, bronze-mounted, double-disk type, actuated by hydraulic cylinders operated from a marble-faced operating table or cabinet located on the filter operating floor at the front of each filter. All hydraulic cylinders are bronze-lined throughout, including the cylinder heads and pistons, and all pressure piping to the valve cylinders is of seamless copper tubing.

Each filter is equipped with a rate-of-flow controller of the Venturi type and a loss-of-head gauge, both actuating an indicating and recording gauge head mounted on the filter operating table above. The rate controllers are arranged for master-control setting or for individual setting as desired. The pipe galleries are heated by steam radiators and ventilated by a line of steel grating along each side of the operating floor of the filter.

The wash-water supply system consists of two pumps with capacities of 6 000 gal per min each, two pumps with capacities of 8 000 gal per min, located in recesses off one pipe gallery, and two 50 000-gal tanks in the cross-monitors of the Filter Building. The pumping capacity is sufficient for washing filters continuously, one at a time, while the tank capacity is sufficient for one single-filter wash. The depth of the wash-water tank and the rate of filter washing are indicated on a large illuminated gauge near the center of each operating gallery.

The following data will serve as a summarized description of the filters:

Rate of Filtration, in Million Gallons per Day, per Acre:	
Maximum	180
Normal	160
Number of filter units	68
Capacity of a filter unit (maximum rate), in million gallons daily	4.5
General Dimensions, in Feet:	
Width of sand bed	27.00
Length of sand bed	40.33
Depth of filter box	11.0
Clear distance between troughs	4.44
Height, bottom of wash-water tank above sand surface	35.5
Detail Dimensions, in Inches:	
Diameter of connections:	
Influent	36
Effluent	24
Wash water	24
Drain	30
Re-wash	8
Thickness of sand bed	20
Gravel:	
Thickness of layer	18
Maximum size	3
Minimum size	$\frac{1}{16}$
Height, surface of sand to crest of wash troughs	40

Elevations, in Feet, above Detroit City Datum:	
Bottom of filter	120.0
Water surface on filters	130.5
Effective size of sand, in millimeters	0.5
Uniformity coefficient of sand	1.3
Under-Drains (Dimensions, in Inches):	
Laterals, size	4
Laterals, spacing, center to center	12
Orifices, size	0.504
Orifices, spacing, center to center	6
Ratio of orifice area to sand area (× 100)	0.277
Rate of washing, in inches per minute	30 to 39
Tank capacity for wash water, in gallons (net)	95 000
Pump capacity for wash water, in gallons per minute	28 000
Percentage of wash water used (average)	1.3

The strainer system is composed of perforated cast-iron pipe, with perforations brass bushed. The wash gullets are at the front of the filters, and the sand is not agitated during the washing process except by the high velocity of the wash water.

Filter Building.—The filter group housed in a single-story building with two longitudinal monitors, over the filter operating galleries, and two cross-monitors in which the wash-water tanks and air-conditioning and heating equipment are placed. This building is of concrete-encased, structural-steel frame construction, with reinforced concrete roof of the ribbed-slab type. The curtain-walls are of hollow tile faced with buff terra cotta on the interior and chat-sown Indiana limestone on the exterior.

The concrete ceiling and columns are painted a light cream color. The operating floor is finished with 6-in. square, red, quarry tile, laid in black mortar, with borders and spill-rail curbs of dark green terrazzo. A railing of ornamental iron with an oak top rail is along each side of the operating galleries. All ornamental iron work, steel floor gratings, ventilator grilles, and steel sash and door framing are painted a dark green. The roof is insulated with a 1½-in. layer of pressed cork on top of which is standard 4-ply tar and gravel roofing. Large skylights of the vault type in the roof over the filters augment the light from the gallery monitors.

Heating System.—Considerable attention was given to the heating and ventilating of the Filter Building. Tests made in the existing Water-Works Park Plant indicated considerable variation in temperatures throughout the building and a large consumption of heat to maintain an average of 50° F during zero weather. Condensate was seen to collect on the steel work, roof slabs, and window sash in the spring and fall when the building was not heated. With this example to observe and test, it was expected that an improved heating and ventilating system could be devised for the Springwells Filtration Plant. The uncertain heat loss to the open water surface of the filters complicated the problem considerably. Attempts to measure this loss failed, due to lack of instruments for measuring such a small temperature change. A heat transfer coefficient of 2.00 was taken as the most probable for heat loss to an open water surface. In collaboration with experienced heating and ventilating consultants, the following outline of a heating system for the Springwells Filter Building was developed.

Heating.—The pipe galleries were to be heated by direct radiation; and the main building (except one room), by re-circulated air. The controlling temperatures, in degrees Fahrenheit, were:

Maximum average, controlled to	70
Minimum, average	50
Maximum allowable variation in working area	20
Maximum temperature of heated air	90

Ventilation.—The specifications required four changes of air per hour, re-circulated, with one change of fresh air admitted and tempered. Stale air was to be exhausted from the floor, at the ends of the pipe galleries, at the rate of ½ air change per hr. A probable air leakage of ½ air change per hr from the building was expected, due to maintaining inside the building, by means of the fans, a slightly higher pressure than normal atmospheric pressure. Re-circulated air was to be taken from the main roof at the center of the filter group and returned at the four intersections of the operating galleries, and at the cross-galleries. Maximum permissible air velocities, in feet per minute, were established as follows:

In air ducts	600
At outlets of fans	1 600
Through heaters	1 000

The total heat required to maintain the main part of the building at an average temperature of 50° F, with zero temperature outside, and with water at 32° F in the filters, was computed as 6 846 900 Btu per hr. This heat loss was apportioned as shown in Table 2. The loss in the pipe galleries could be estimated only approximately. Considering the exposed area of pipes as condenser surface and estimating 1 air change per hr due to convection currents, a total of 587 600 Btu per hr was computed as necessary to keep the pipe galleries above 50° F, with water at 32° F in the plant, and the main building at 50° F.

TABLE 2.—Heat Required to Maintain the Filter Building at 50 Degrees Fahrenheit, with 0 Degrees Fahrenheit, Outdoors

Surface	Area, in square feet	Heat, in British thermal units per hour	Surface	Area, in square feet	Heat, in British thermal units per hour
Glass	7 163	526 690	Skylights	9 360	598 000
Walls	33 681	450 510	Water	80 784	2 586 000
Roof	101 744	874 800	Floor over conduits	21 960	131 400
For 1 air change (1 608 000 cu ft) per hr					1 679 500

The heating surface required was computed on the basis of 5-lb steam pressure in the heaters and radiators. An allowance of a 2-lb pressure drop in the steam lines between the source of the steam (the low-pressure side of the turbines in the generator room of the power plant) and the farthest heater, governed the sizes of steam pipes.

The heating equipment consisted of the following principal units, complete with accessories:

Four blower fans of 33 500 cu ft per min capacity of air at 70° F against a back pressure of $\frac{7}{8}$ in. of water.
Four batteries of cast-iron heaters having 2 064 sq ft of heating surface each, for re-circulated air.
Four batteries of cast-iron heaters having 408 sq ft of heating surface each, for tempering fresh air.
Four dampered louvers for fresh-air intakes.
Four distributing louvers or diffusers for re-circulated air.
Four exhaust fans of 4 000 cu ft per min capacity of air at 70° F against a back pressure of 0.3 in. of water.
Twenty-eight radiators in pipe galleries, each of 126-sq ft heating surface.
Two vacuum return pumps for pipe-gallery radiation, each of capacity ample for 5 000 sq ft of radiation.

The re-circulating fans and air heaters are installed in the space beneath the wash-water tanks in the cross-monitors of the building. Condensed steam returns by gravity from the air heaters to a vacuum return pump in the basement of the plant office, which is on the route back to the boilers.

The steam main to the air heaters in the monitors is installed on the roof of the Filter Building with the air-recirculating duct. The steam supply lines to the pipe galleries are placed along the walls of the galleries. All steam lines and return lines are insulated with 85% magnesia pipe covering, re-canvassed. An exposed part of the steam main on the roof is also covered with a steel sheath. The heating system was installed during the time of interior finishing and was completed soon after the building was constructed.

Office and Laboratory Building.—This building was constructed complete under one general contract, including heating, ventilating, plumbing, electrical work, and all other trades. Its construction was typical of buildings of this nature.

The first floor contains offices in front for the filter plant superintendent, the power and pumping plant superintendent, the clerical force, and a conference or receiving room. The rear of the first floor contains locker rooms, shower rooms, and toilets for the plant workmen and operators, as well as a time office and storage rooms. The second floor contains the filter plant laboratory, with offices for the chief chemist and bacteriologist.

Weir Chamber.—It was required that the effluent water level be maintained at a height sufficient to submerge the filter-effluent connections and effluent-main conduits and to operate the shunt system.[4] For this purpose, a chamber is provided, at the outlet ends of the filtered-water, effluent-main, conduits, which contains a filter-seal weir with crest at Elevation 118.00 and a reservoir weir, in two sections, with crest at Elevation 119.50, as shown in Fig. 11. This chamber is of reinforced concrete construction. In addition to its operating function it serves as a foundation for the plant office as well as providing storage space for stop-logs, spare parts, castings, etc. The two sections of reservoir weir are built diagonally to gain length and also to converge the flow from the filter-seal weir to the entrance of the conduits leading directly to the high-lift pumping plant. A central division wall is provided for use in shutting off one-half the plant. Openings in this division wall and in the reservoir-weir walls (filled by stop-logs) are provided for flexibility in by-passing or running under special operating conditions.

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[4] "Shunt System of Operating Filtered Water Reservoirs," by E. A. Hardin, *Engineering News-Record,* Vol. 103, No. 26, December 26, 1929, p. 1011.

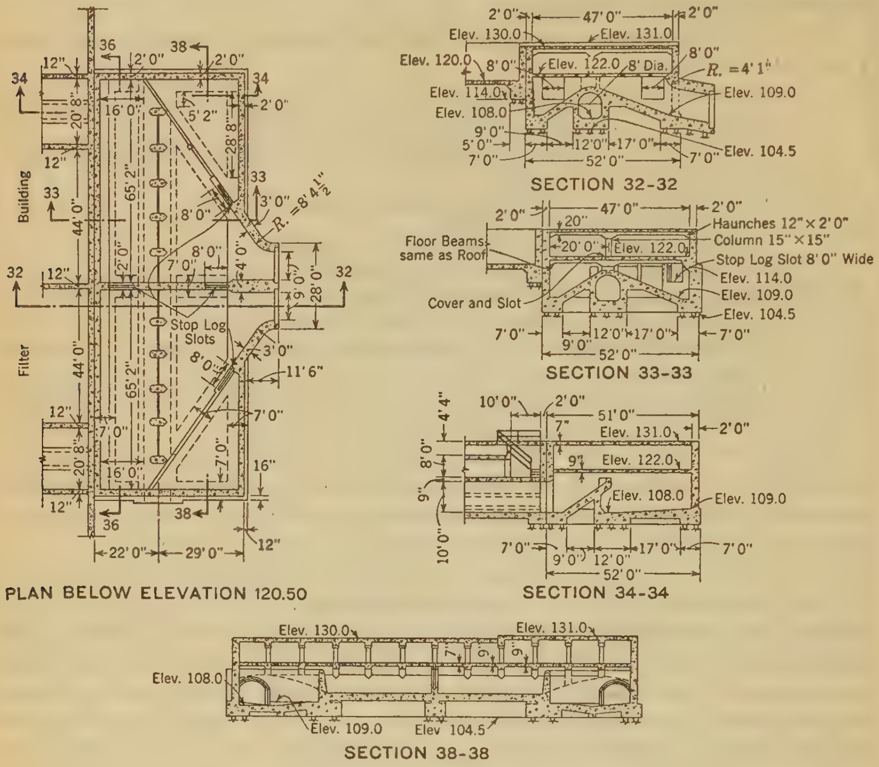

PLAN BELOW ELEVATION 120.50
Building
Filter
Stop Log Slots
R. =8'4½"
22'0"
29'0"
SECTION 32-32
Elev. 130.0
Elev. 131.0
Elev. 120.0
Elev. 122.0 8' Dia.
R. =4'1½"
Elev. 114.0
Elev. 108.0
Elev. 109.0
Elev. 104.5
52'0"
47'0"
SECTION 33-33
Floor Beams same as Roof
Haunches 12"×2'0"
Column 15"×15"
Stop Log Slot 8'0" Wide
Cover and Slot
SECTION 34-34
SECTION 38-38

FIG. 11.—SECTIONS OF WEIR CHAMBER

Two 9 by 9-ft conduits which run from the weir chamber to the west end of the high-lift pumping plant, are designed to carry water direct to the pumping plant. An 8 by 8-ft conduit conveys water from the south end of the weir chamber to the filtered-water reservoir. The conduits to the pumping plant take the water that overflows the filter-seal weir, while the reservoir conduit takes the water that overflows the higher reservoir weir.

The Shunt System.—The shunt system is a method of operating the high-lift pumping plant so that about three-fourths of the total pumpage is drawn directly from the filter plant at about 1½ ft above the level of the effluent-seal weir, while the remainder, which is taken on the daily peak, is drawn from the filtered-water reservoir at a level that may vary with the reservoir stage, from Elevation 120.0 to Elevation 108.0 each day. Thus, considerable head on the suction side of the pumps is conserved for most of the water pumped. This is accomplished by dividing the pumping units into two groups with a division wall in the suction gallery of the pumping plant. A small group at the remote end from the filter plant, comprising about one-fourth of the high-lift pumping capacity, draws from the reservoir through Gate Chamber No. 1 and reservoir-outlet conduits connected to the east end of the pump suction gallery, while the remainder of the pumps draw directly from the filter plant, *via* the weir chamber.

The weir chamber effects the division of the water at the filter plant by a secondary weir, called the reservoir weir, the crest of which is high enough above the seal weir so that the entire flow of filter effluent may pass over it without overtopping the reservoir weir. The direct flow to the high-lift pumping plant is taken off between these weirs. Thus, the entire capacity output of the filter plant may pass directly to the high-lift pumps without affecting the reservoir levels at all. If the pumps taking suction direct from the filter plant do not require the full filter output, the effluent level rises until it overtops the reservoir-weir crest at Elevation 119.5, and the excess is discharged to the reservoir. During peak pumpage hours, when the output of the high-lift plant is more than that of the filter plant, pumps at the remote end are placed in service as required to meet the demand, and the pumpage in excess of the filter output rate is taken from the reservoir.

For convenience in operating the pumps, a water-level gauge is provided in the weir chamber between the two weirs and a control level of about Elevation 119.50 is maintained at this point. If this water level rises above Elevation 119.50, it is known that water is flowing into the reservoir and that, if possible, additional work by the pumps west of the division wall is permissible and less on the east side of the division wall is in order. If the control water level falls to Elevation 119.0, or below, less pumping west of the division wall, and more, east of the division wall, is desirable. To safeguard the operation of the pumps and to insure suction water from the reservoir to all pumps in the high-lift plant, flap-gates are provided in the suction-gallery division wall which are opened by the pressure from the reservoir-water side at any time the water level on the filter-plant side is below that on the reservoir side of this wall.

From the foregoing it is seen that by the shunt system the main flow of water from the filter plant is diverted directly to the high-lift pumps at a normally constant high level, and the reservoir is on a secondary loop, or shunt, feeding a small group of pumps, normally isolated as to suction from the remainder of the station. Experience with the shunt system at the Water-Works Park Station (see Fig. 1) has shown that it causes no inconvenience whatever in the pump operation. The pumps in the reservoir group, as a rule, are put in service only in peak hours and little concern is given the shunt system, which works automatically.

Hydraulics of Flow Through Plant.—The computed velocities, head losses, and water-surface elevations at governing points throughout the plant are given in Table 3 for both a maximum-capacity flow of 306 000 000 gal daily, and an average-capacity flow of 210 000 000 gal daily. For these computations it is assumed that the water-surface elevations in the filtered-water reservoir are less than 119.5. When the reservoir is at higher stages, up to its normal high-water level of Elevation 121.00, the filter effluent levels will be raised correspondingly (except for the slight effect of the greater submergence of the effluent weir), thus reducing the available operating head of the filters at times of high water in the reservoir.

Under-Drainage System and Main Drains for Plant.—The entire sub-grade area of the filtration plant and reservoir is provided with drains for controlling the elevation of the ground-water under the plant. The drains are of 6-in. vitrified sewer pipe laid with open joints in shallow trenches at the surface of the sub-grade and surrounded by open gravel. They are spaced approximately 7 ft from center to center, located so as to clear the piling. Under the coagulation basins and mixing chamber and under the floor-slabs of the filters there is also a layer of open gravel about 4 in. thick for distributing the water between the drains. There is no gravel layer between the drains under the reservoir. There are no drains under the wall footings and the pipe galleries. In addition to the under-drains there is also a belt drain of perforated, 12-in., cast-iron pipe laid completely around each structure at the top of the wall footings to intercept ground-water from the surrounding area.

The drainage system under the structures constituting the filtration plant is connected to the main drain from the plant by an overflow connection which discharges water when the level is above Elevation 118.5. It is expected that, usually, there will be a flow from the drains due to leakage and ground-water; but to assure a ground-water level at sufficient height to cover the timber piling a water-supply connection is provided with a float-operated valve that admits water to the drainage system when the level in it becomes less than Elevation 118.5 (which is 4 ft above the top of the highest pile).

The level in the reservoir drainage system is maintained at desired levels by float-controlled sump pumps which discharge into the sewer.

Filtered-Water Reservoir.—The reservoir for filtered water consists of two sections, each with a capacity of 20 000 000 gal, and each a rectangle, 455 ft long by 313 ft wide (interior dimensions). There will be a third section, eventually, of about the same size. Water flowing to the reservoir is distributed to the sections by a conduit, built along the north wall, which is connected to the sections through a gate-chamber in each one and also through an additional 66-in. inlet valve in the east section. In each gate-chamber are four 48-in. double-disk gate-valves, hand-operated by geared floor stands with cranks. The water may enter or leave through these valves. Flap check-valves are placed on the gate-valve openings in such a manner that the water flows in through the valves on one side of the center baffle-wall and out through the valves on the other side. By this means, water may be circulated through the reservoir sections, if desired. In the east section, the 66-in. valve serves as the inlet, and two of the 48-in. valves equipped with

TABLE 3.—Computed Velocities, Head Losses, and Water-Surface Elevations

Item No.	Point	Average Capacity (210 Mgd)			Maximum Capacity (306 Mgd)		
		Velocity, in feet per second	Loss of head, in feet	Elevation, in feet	Velocity, in feet per second	Loss of head, in feet	Elevation, in feet
	(a) Entrance to Mixing Chambers						
1	Elevation above Detroit City Datum, in feet	....	......	131.13	....	......	131.60
2	Entrance gates (two, 6 by 10 ft)	2.70	0.11	......	3.95	0.25	......
3	Mixing chamber	0.29	0.01	......	0.43	0.02	......
4	Intermediate gates (four, 6 by 10 ft)	1.35	0.04	......	1.97	0.09	......
5	Outlet gates (three, 6 by 10 ft)	1.80	0.05	......	2.63	0.11	......
6	Conduit to coagulation basins	1.70	0.12	......	2.48	0.28	......
7	Entrance to coagulation basins	0.75	0.02	......	1.09	0.04	......
	(b) Inlet End of Coagulation Basins						
8	Elevation above Detroit City Datum, in feet	....	......	130.59	....	......	130.72
9	Coagulation basins	0.03	0.00	......	0.05	0.01	......
10	Basin outlet weir	0.65	0.03	......	0.95	0.06	......
11	Outlet channel and 9 by 12-ft gate	0.75	0.03	......	1.09	0.05	......
12	Junction chamber and settled water conduit to gallery (12 by 20 ft)	0.68	0.01	......	0.99	0.03	......
13	Settled water conduit to Gallery 2 (6 by 19 ft)	0.7	0.00	......	1.05	0.01—	......
14	Filter influent header	0.7 to 0	Slight regain	......	1.0 to 0	Slight regain	......
15	Filter inlets	0.77	0.02	......	0.99	0.04	......
	(c) Filters						
16	Elevation above Detroit City Datum, in feet	....	......	130.50	....	......	130.50
17	Filters (available operating head)	....	9.95	......	....	9.11	......
18	Filter effluent piping and wide-open rate controller	1.72	0.63	......	2.22	1.03	......
	(d) Normal Gradient in Up-Stream End of Effluent Header						
19	Elevation above Detroit City Datum, in feet	....	......	119.92	....	......	120.36
20	Filter effluent main	0.23 to 3.25	0.26	......	0.36 to 4.83	0.55	......
	(e) Above Effluent Weir in Weir Chamber						
21	Elevation above Detroit City Datum, in feet	....	......	119.66	....	......	119.81
22	Effluent weir	1.63	0.16	......	3.09	0.31	......
	(f) Below Effluent Weir in Weir Chamber						
23	Elevation above Detroit City Datum, in feet	....	......	119.50	....	......	119.50
24	Conduits to high-lift pumping plant (two, 9 by 9 ft)	2.0	0.10	......	2.9	0.21	......
25	Entrance to 50-mgd pump suction chamber (6 by 6-ft sluice-gate)	2.15	0.20	......	2.15	0.20	......
	(g) High-Lift Pumps on Direct Draft						
26	Elevation above Detroit City Datum, in feet	....	......	119.20	....	......	119.09
	(h) Normal High-Water Level in Reservoir						
27	Elevation above Detroit City Datum, in feet	....	......	119.00	....	......	121.00
	(i) Low-Water Level in Reservoir						
28	Elevation above Detroit City Datum, in feet	....	......	103.00	....	......	103.00

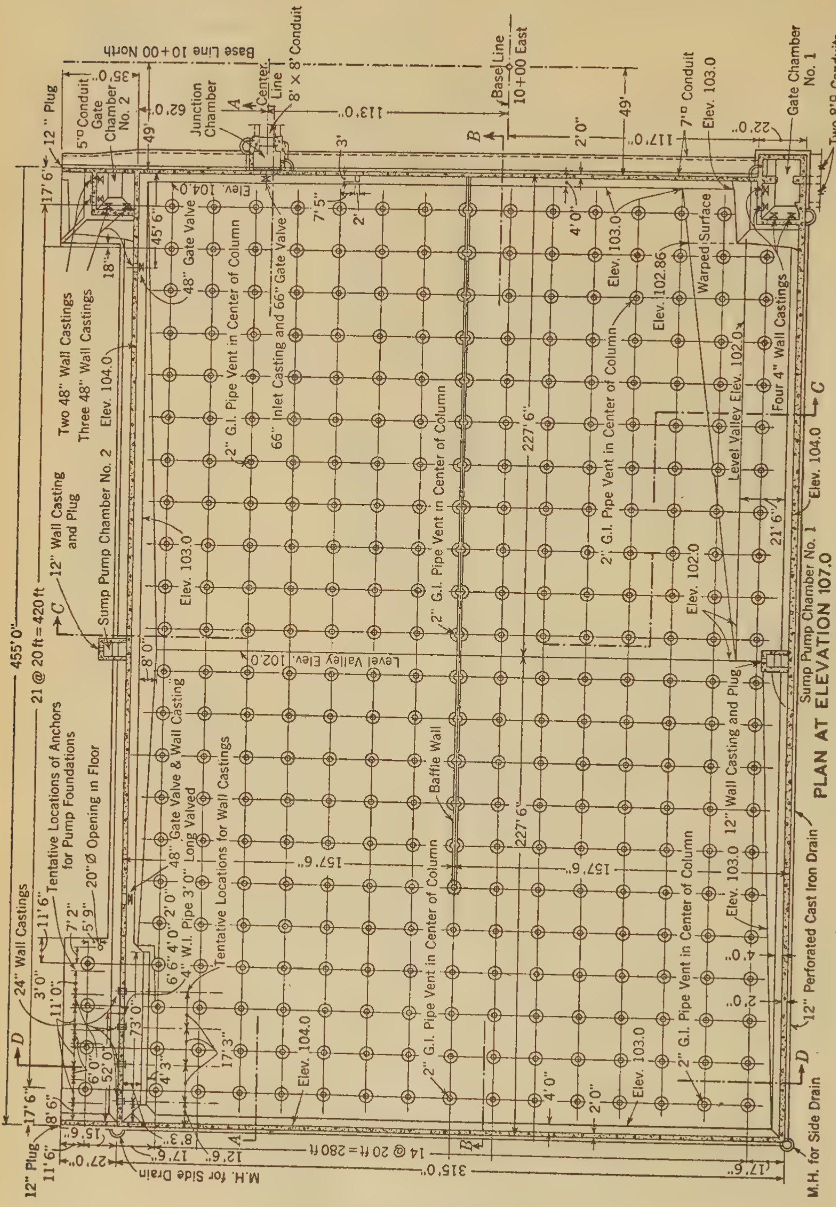
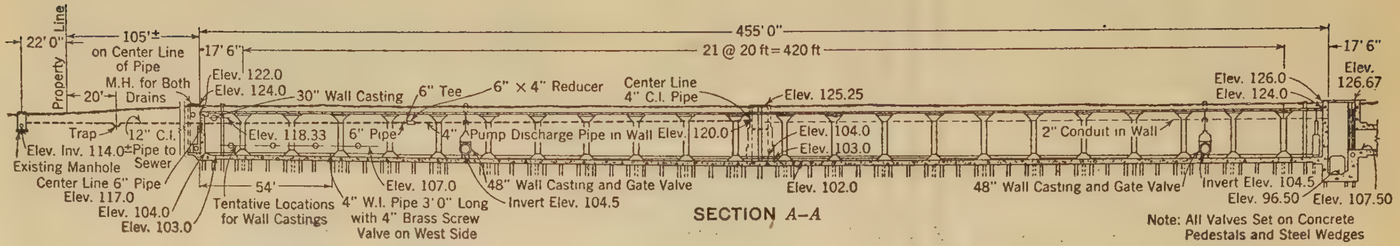
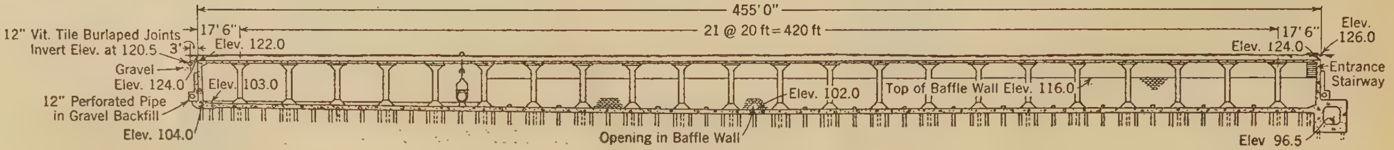

PLAN AT ELEVATION 107.0
Base Line 10+00 North
Base Line 10+00 East
35'0"
62'0"
113'0"
117'0"
22'0"
49'
5'□ Conduit
Gate Chamber No. 2
Junction Chamber
Center Line
8' × 8' Conduit
7'□ Conduit
Elev. 103.0
Gate Chamber No. 1
Two 8'□ Conduits
12" Plug
17'6"
455'0"
21 @ 20 ft = 420 ft
12" Wall Casting and Plug
Two 48" Wall Castings
Three 48" Wall Castings
Elev. 104.0
Sump Pump Chamber No. 2
18'
45'6"
48" Gate Valve
2" G.I. Pipe Vent in Center of Column
66" Inlet Casting and 66" Gate Valve
7'5"
2'
3'
2'0"
4'0"
Elev. 103.0
Elev. 102.86
Warped Surface
Four 4" Wall Castings
Level Valley Elev. 102.0
227'6"
Elev. 102.0
21'6"
Elev. 104.0
Sump Pump Chamber No. 1
Level Valley Elev. 102.0
8'0"
Baffle Wall
Tentative Locations of Anchors for Pump Foundations
20"Ø Opening in Floor
48" Gate Valve & Wall Casting
48" Long Valved
4" W.I. Pipe 3'0"
Tentative Locations for Wall Castings
157'6"
12" Wall Casting and Plug
12" Perforated Cast Iron Drain
24" Wall Castings
11'6"
7'2"
5'9"
3'0"
11'0"
6'6"
4'0"
2'0"
73'0"
52'0"
6'0"
4'3"
17'3"
8'6"
15'6"
8'3"
17'6"
12'6"
27'0"
11'6"
14 @ 20 ft = 280 ft
315'0"
17'6"
M.H. for Side Drain
Elev. 104.0
Elev. 103.0
A
B
C
D

FIG. 12.—PLAN OF FLOOR-SLAB, FILTERED WATER RESERVOIR NO. 2


22' 0"
Property Line
105'± on Center Line of Pipe
20'
M.H. for Both Drains
Trap
12" C.I. Pipe to Sewer
Elev. Inv. 114.0±
Existing Manhole
Center Line 6" Pipe Elev. 117.0
Elev. 104.0
Elev. 103.0
17' 6"
Elev. 122.0
Elev. 124.0
30" Wall Casting
6" Tee
6" × 4" Reducer
Center Line 4" C.I. Pipe
Elev. 125.25
455' 0"
21 @ 20 ft = 420 ft
17' 6"
Elev. 126.0
Elev. 124.0
Elev. 126.67
Elev. 118.33
6" Pipe
4"
Pump Discharge Pipe in Wall Elev. 120.0
Elev. 104.0
Elev. 103.0
2" Conduit in Wall
54'
Tentative Locations for Wall Castings
4" W.I. Pipe 3' 0" Long with 4" Brass Screw Valve on West Side
Elev. 107.0
48" Wall Casting and Gate Valve
Invert Elev. 104.5
Elev. 102.0
48" Wall Casting and Gate Valve
Invert Elev. 104.5
Elev. 96.50
Elev. 107.50
Note: All Valves Set on Concrete Pedestals and Steel Wedges

SECTION *A–A*


455' 0"
21 @ 20 ft = 420 ft
17' 6"
17' 6"
12" Vit. Tile Burlaped Joints
Invert Elev. at 120.5
3'
Elev. 122.0
Gravel
Elev. 124.0
12" Perforated Pipe in Gravel Backfill
Elev. 104.0
Elev. 103.0
Elev. 102.0
Opening in Baffle Wall
Top of Baffle Wall Elev. 116.0
Elev. 124.0
Elev. 126.0
Entrance Stairway
Elev 96.5

SECTION *B–B*

FIG. 13.—GENERAL SECTIONS, FILTERED WATER RESERVOIR NO. 2

flap-gates serve as the outlet, to produce circulation around the baffle-wall. Figs. 12 and 13 give the general plan and sections of the existing sections of the reservoir.

The reservoir is constructed of reinforced concrete of standard design. The walls are one-way slabs spanned vertically between the base and the roof. The roof and floor are of typical two-way flat-slab construction with cylindrical columns spaced 20 ft center to center both ways.

The reservoir is completely back-filled and its roof is covered with 6 in. of gravel and 18 in. of earth. It thus forms the front-yard area of the station and will be landscaped and planted. Water is drawn from the reservoir to the pumps in the east end of the high-lift pumping plant through two, 8-ft square, concrete conduits running from Gate Chamber No. 1 to the east end of the suction gallery in this pumping plant.

CONSTRUCTION

General Plan and Administration.—In constructing the Springwells Filtration Plant and Reservoir, the Board of Water Commissioners acted somewhat in the capacity of a general contractor. Most of the equipment, valves, sluice-gates, piping, and castings were purchased separately, and contracts for the

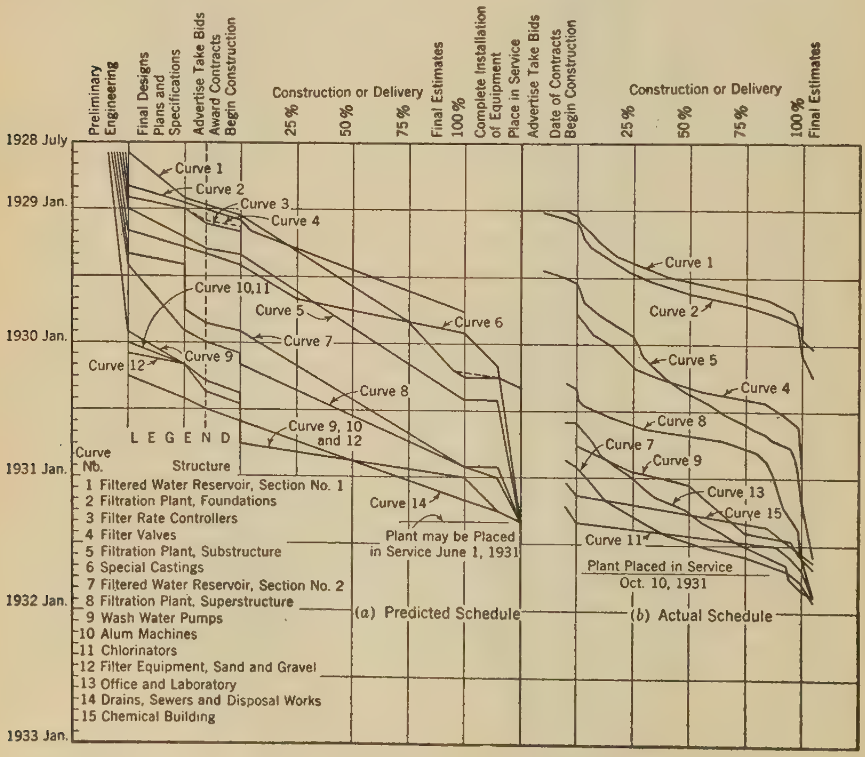


FIG. 14.—PROGRESS OF CONSTRUCTION

construction work and installation were let to building contractors and trades, this equipment, piping, castings, etc., being furnished. The reservoir was constructed under two contracts. The filtration plant was constructed under eight separate major contracts: (1) Excavation and piling; (2) substructure (mostly heavy concrete construction); (3) superstructure of Filter Building; (4) electrical work in Filter Building; (5) plumbing, heating, and ventilating work in Filter Building; (6) Chemical Building, including alum-handling equipment; (7) masonry facings of coagulation basins and mixing chamber and the construction of the superstructures of the drainage pump house and two gate-houses on the coagulation basins; and (8) complete construction of the Office and Laboratory Building.

The operating tables in the filtration plant, the chlorinators and chlorine piping, the chlorine scales, chlorine hoist, pressure water pumps and sump pumps, drainage pumps, filtered water meters, wash-water pumps, wash-water meters, and the wash-water pipes were furnished and installed under separate equipment contracts, let directly to the various equipment manufacturers.

The subdivision of the work resulted in smaller contracts and the letting of work directly to the proper trades. This eliminated considerable administration cost from the actual contracts, but required considerable administrative work on the part of the Engineering Division of the Board (which handled the work) with attendant expense reflected in the engineering cost figures, as shown in Table 4 discussed subsequently. It is believed that the savings in contract costs effected by this procedure considerably more than offset the increased administrative and engineering expense involved. Considerable saving in contract costs was obtained also by fully and completely detailing the contract drawings upon which bids were based, thus eliminating guesswork on the part of the bidders and resulting in low and close bidding. Dividing the work in this manner expedited the construction since construction was begun on the foundations while the plans were being completed for the superstructures and subsequent work. To schedule and expedite the various purchases and contracts properly, in order to prevent delays and to preserve harmonious working conditions where a number of different contracts were under way on the limited area of the site, involved careful attention both to the preliminary planning and to the progress of the work. It also added greatly to the duties of the engineering force.

Progress Schedule.—Before any purchase was made or contract prepared, a preliminary progress schedule of the complete project was made as shown in Fig. 14(*a*) by which it was estimated that the plant might be placed in operation by June, 1931. This schedule, of course, could not be followed exactly, but served as a guide in preparing the work. The actual construction schedule is similarly shown in Fig. 14(*b*).

The construction was completed to the point at which operation of the filter plant would have been possible July 1, 1931, if the water demand had required it. However, since the drop in water demand resulting from the general recession in business activity made the operation of the plant less urgent at the time, and since several miscellaneous items of work were desirable and more

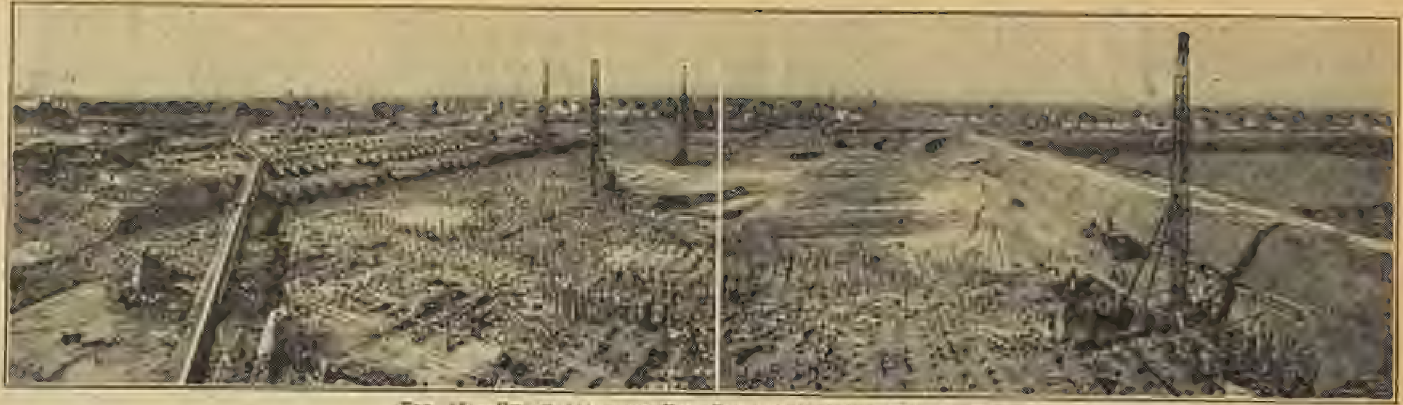


FIG. 15.—EXCAVATION AND PILE-DRIVING NEARING COMPLETION

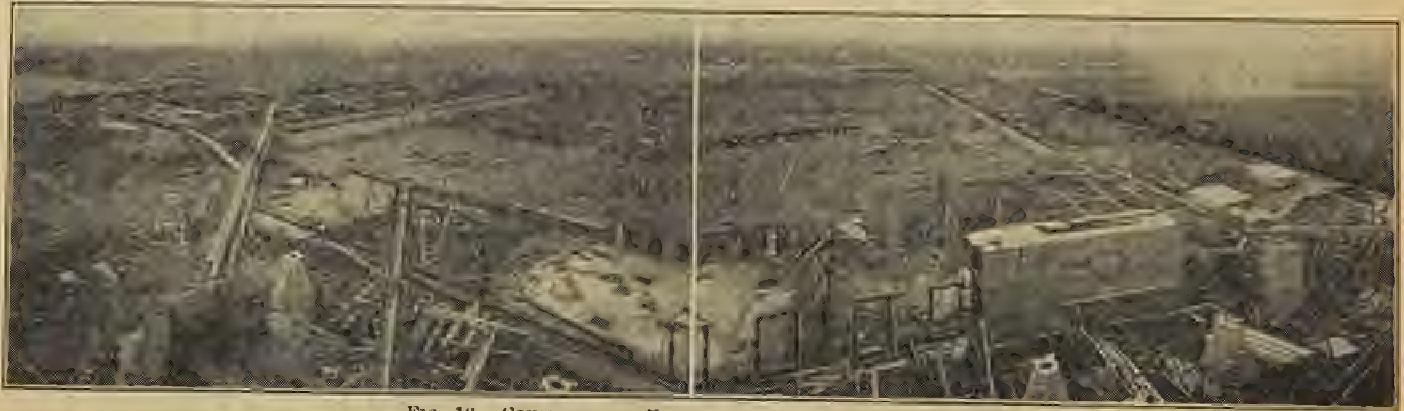


FIG. 16.—CONSTRUCTING FILTER WALLS AND MIXING CHAMBER

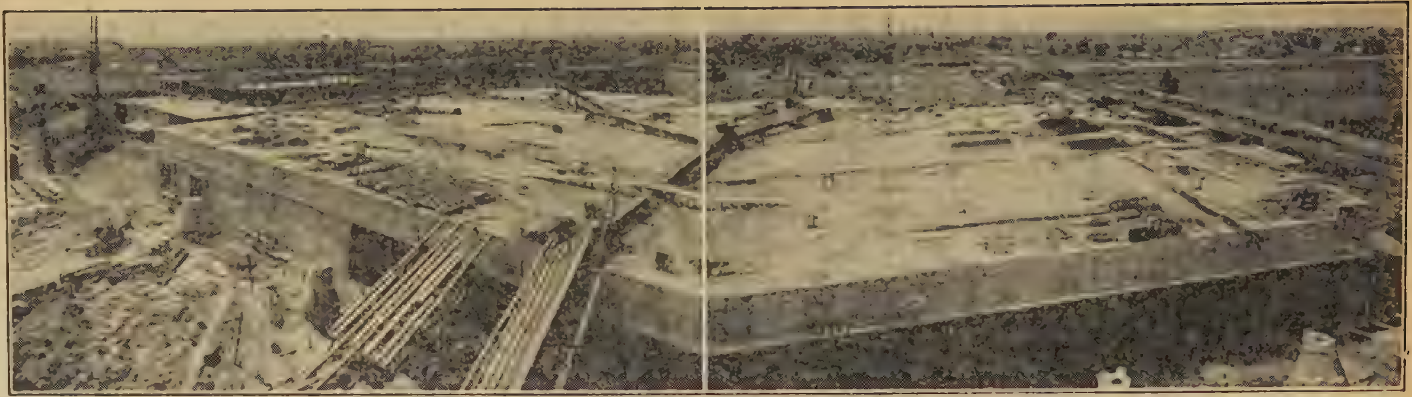


Fig. 17.– Extending Settling Basin; Filter Building Steel Work Erected

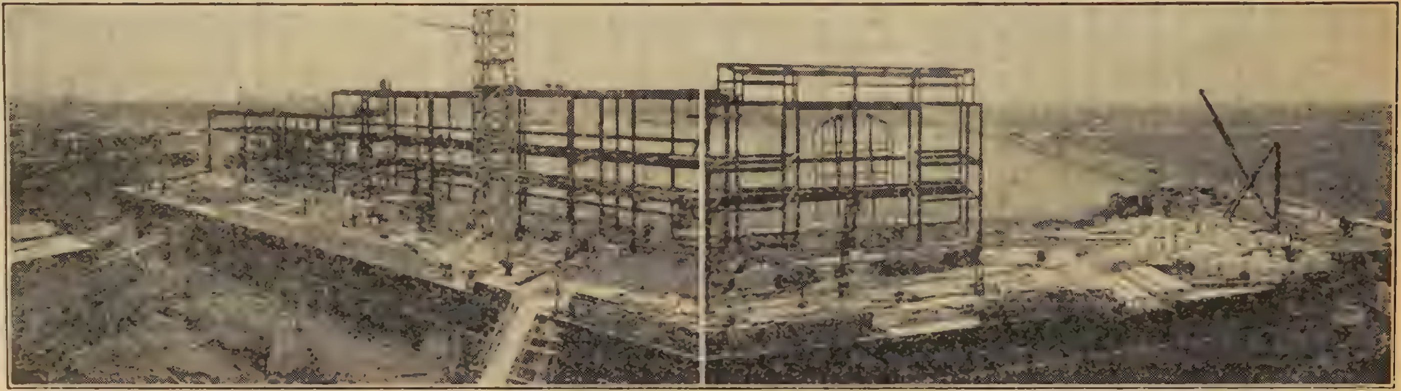


Fig. 18.—Chemical Building Steel Work Erected

economical to be done before the plant was put in service, the actual operation was begun on October 10, 1931, about four months later than scheduled.

Construction Methods.—There was nothing particularly special or peculiar about the construction of this project. Standard first-class construction methods were used throughout. Most of the excavation was by drag-line excavators and shovels. Piles were driven by ordinary wooden skid drivers. Concrete was mixed in a central mixing plant for each job and transported to the forms by chutes, belt conveyors, and industrial train, each contract making use of a different transporting method. The mixers were all of about 1 cu yd capacity. The superstructure work was typical of such construction. The major part of the structural work, except the excavation and pile-driving for the filter plant, was done by two contracting firms, the W. E. Wood Company and the Bryant and Detwiler Company, of Detroit, Mich. The Whitney Brothers Company, of Duluth, Minn., did the excavation and piling work for the filtration plant.

The construction photographs, Figs. 15 to 18, inclusive, show the magnitude and type of the construction work. They were taken from the same point (the top of the concrete mixing plant) and show the filtration plant in successive stages of construction.

The principal problems of construction arose from the magnitude of the project and the extent of the area covered by the plant structures, and were mainly problems of transportation. Approximately 18 acres were occupied by the actual structures themselves and large volumes of materials had to be handled for considerable distances over this construction area while many large pipe castings, valves, and pieces of equipment had to be installed as the work progressed.

A general idea of the magnitude of the work may be obtained from the following summary of the approximate quantities of the principal construction materials that went into the Springwells Filtration Plant and Reservoir:

Excavation, in cubic yards	420 000
Timber bearing piling, in linear feet	1 500 000
Concrete, in cubic yards	120 000
Reinforcing steel, in tons	7 200
Concrete forms, in square feet	2 000 000
Structural steel, in tons	1 120
Ornamental iron, in tons	70
Iron pipe and castings, in tons	2 000
Gravel fill and drainage material, in cubic yards	15 000
Tile under-drains, in linear feet	56 000
Stone masonry, in cubic feet	42 000
Brick and tile masonry, in cubic feet	77 000
Terra cotta, in square feet	33 000
Steel sash, in square feet	11 000
Metal doors, in square feet	1 000
Tar and gravel roofing (squares)	1 250
Skylights, (vault type), in square feet	11 000
Cork insulation, in square feet	134 000
Copper roofing and flashing, in square feet	35 000
Terrazzo and ground-concrete floors, in square feet	30 000
Quarry tile floors, in square feet	22 000

TABLE 4.—Cost of Springwells Filtration Plant

Item No.	Units constructed	Construction Cost		Date of receipt of bids	Cost index based on the year 1913 as 100
		Actual	Adjust to 1913 costs by Index Number		
	(*a*) Filtered-Water Reservoir				
1	Section 1: Contract	$541 478	$258 585	January 16, 1929	209.40
2	Section 1: Force account work	14 055	6 789		207.02
3	Section 2: Contract	347 226	176 382	December 2, 1930	196.86
4	Section 2: Force account work	16 612	9 160		181.35
5	Section 3: Force account work	125	74		170
6	Section 3: Future construction (estimated cost, including grading, planting, and roadways)	400 000	235 300		170
7	Sections 1, 2, and 3: Equipment, including valves, pumps, and level gauges (installation of valves included in Items Nos. 1 and 3)	56 971	27 770	May 13, 1929	205.15
	Cost of Engineering:				
8	Total, exclusive of supervision on Section 3	74 577	36 028		207
9	Estimated for supervision on Section 3	20 000	11 765		170
10	Total cost of reservoir (60 mg capacity)	$1 471 044	$761 853		
11	Cost of engineering ($94 577), in percentage of total cost	6.43			
12	Total cost per million gallons capacity (60), including engineering, incidental expense, and contingencies	$24 517	$12 698		
	(*b*) Filtration Plant Structures				
	Foundation Work (Excavation and Pile-Driving):				
13	Contract	$607 035	$288 515	February 6, 1929	210.40
14	Force account labor and miscellaneous expense on foundation work	19 692	9 512		207.02
	Substructure Construction:				
15	Contract	1 518 574	741 600	July 10, 1929	204.77
16	Force account labor and miscellaneous expense on substructure construction	55 680	27 449		202.85
	Superstructure of Filter Building:				
17	Contracts	604 754	291 982	April 30, 1930	207.12
18	Force account labor and miscellaneous expense	25 910	12 773		202.85
	Chemical Building, Including Mechanical Equipment, Chemical-Handling Equipment, Heating, Ventilating, Plumbing, and Electrical Work:				
19	Contract	178 100	91 577	January 27, 1931	194.48
20	Force account labor and miscellaneous expense	11 713	6 459		181.35
	Basin Facings and Gate-Houses:				
21	Contract	46 381	27 318	October 20, 1931	169.78
22	Force account labor	1 746	1 032		169.28
23	Water conduits to and from pumping plant and reservoirs	90 710	43 796	April 30, 1930	207.12
	Drainage Pump House:				
24	Substructure by force account labor	45 382	25 088	April–September, 1931	180.89
25	Superstructure by contract	18 734	11 034	October 20, 1931	169.78
	Plant Drains and Sewers:				
26	By force account labor	60 721	33 568	April–September, 1931	180.89
27	By contract	13 365	6 527	July 10, 1929	204.77
28	Total cost of Filtration Plant (structures)	$3 298 497	$1 618 230		
	(*c*) Filtration Plant Equipment				
29	Drainage pumps, electric controls, valves and piping, installed	$12 335	$7 214	September 8, 1931	171.40

TABLE 4.—(*Continued*)

Item No.	Units constructed	CONSTRUCTION COST		Date of receipt of bids	Cost index based on the year 1913 as 100
		Actual	Adjust to 1913 costs by Index Number		
	(c) FILTRATION PLANT EQUIPMENT—(*Continued*)				
	Sluice-Gates and Valves:				
30	In coagulation basins	$53 197	$25 784	October 21, 1929 January 6, 1930	206.32
31	In mixing chamber	135 970	65 903	October 15, 1929	206.32
32	Mechanical mixing equipment, installed (estimated)	30 000	17 647	(Future)	170
33	Raw-water meters and gauges, installed	9 326	4 697	November 11, 1930	198.54
34	Filtered water meters and gauges, installed	4 096	2 137	April, 1931	191.63
35	Water-level gauges and signal system, installed	15 545	8 303	June, 1931	187.23
36	Pressure piping in coagulation basins, installed	4 338	2 291	May, 1931	189.33
37	Stop-logs and flash-boards, installed	1 451	766	May, 1931	189.33
38	Auxiliary equipment, including pressure pumps and sump pumps in pipe galleries, installed	5 947	3 104	April 14, 1931	191.63
39	Sample pumps and piping	868	453	....	191.63
40	Wash-water pumps and piping, including meters and valves, installed	79 921	39 772	July 15, 1930	200.95
41	Chlorinating equipment and piping, including feeders, scales, and all dosing lines, installed	33 257	17 098	March 31, 1931	194.51
42	Total cost of Filtration Plant equipment, installed	$386 251	$195 169	....	......
	(d) FILTER EQUIPMENT				
43	Filter operating tables, including pressure tubing, installed	$55 547	$27 952	October 7, 1930	198.72
	Filter Rate Controllers and Gauges:				
44	Delivered	93 958	45 220	March 6, 1929	207.78
45	Installed by contract	16 377	7 998	July 10, 1929	204.77
	Filter Valves:				
46	Delivered	214 562	103 095	March 2, 1929 November 1, 1929	208.12
47	Installed by contract	5 297	2 587	July 10, 1929	204.77
	Filter Piping and Special Castings:				
48	Delivered	76 614	37 345	May 16, 1929	205.15
49	Installed by contract	10 603	5 178	July 10, 1929	204.77
	Wash-Water Troughs:				
50	Delivered	86 553	42 190	May 16, 1929	205.15
51	Installed by contract	10 773	5 261	July 10, 1929	204.77
	Filter Under-Drain Piping and Supports:				
52	Delivered	46 627	23 975	January 16, 1931	194.48
53	Installed by force account labor	8 241	4 726	July, 1931	174.37
	Filter Gravel:				
54	Delivered	17 116	9 576	May–September, 1931	178.74
55	Installed by force account labor	15 668	8 766	May–September, 1931	178.74
	Filter Sand for Twenty-Two Filters:				
56	Delivered	12 264	6 305	March 24, 1931	194.51
57	Placed by force account labor	6 891	3 855	May–September, 1931	178.74
58	Miscellaneous general expense	2 783	1 392	....	200
59	Filter sand in place for forty-six filters (estimated)	37 000	21 765	(Future)	170
60	Total cost of filter equipment, installed	$716 874	$357 186	....	......
61	Total cost of Filtration Plant construction, fully equipped	$4 401 622	$2 170 585	....	......
62	Total cost of engineering, including design, supervision, and administration	409 518	197 835	....	207
63	Total cost of Filtration Plant (300 mgd capacity)	$4 811 140	$2 368 420	....	......
64	Cost of experimental investigations (including basin-model tests, experimental filter plant, filter under-drain, sand-washing studies, pile testing, soil testing, and roof load tests)	$80 898	$39 081	....	207

TABLE 4—*Continued.*

Item No.	Units constructed	Construction Cost		Date of receipt of bids	Cost index based on the year 1913 as 100
		Actual	Adjust to 1913 costs by Index Number		
	(*d*) Filter Equipment—(*Continued*)				
65	Engineering cost (percentage of total cost)	8.51	....	....	....
66	Experimental work cost in percentage of total cost	1.68	....	....	....
67	Filtration Plant cost per million gallons daily capacity (exclusive of experimental work)	$16 037	$7 895	....	....
	(*e*) Office and Laboratory Building				
68	Foundation work	$31 928	$15 997	September 2, 1930	199.58
69	Building construction	91 370	45 781	September 2, 1930	199.58
70	Heating, ventilating, and plumbing	39 595	19 839	September 2, 1930	199 58
71	Electrical work, including clocks and gauges	19 598	9 820	September 2, 1930	199.58
72	Laboratory equipment	5 018	2 423	April 1, 1930	207.12
73	Engineering cost, including design, supervision, and administration	60 041	29 577	....	203
74	Total cost of Office and Laboratory Building	$247 550	$123 437	....	....
	(*f*) Summary				
75	Reservoir (60 mg capacity)	$1 471 044	$761 853	....	....
76	Filtration plant (300 mgd capacity)	4 811 140	2 368 420	....	....
77	Experimental work	80 898	39 081	....	....
78	Office and Laboratory Building	247 550	123 437	....	....
79	Total	$6 610 632	$3 292 791	....	....
80	Total engineering cost (included in Items Nos. 75 to 78)	$564 136	....	....	....
81	Engineering cost in percentage of total cost	8.53	....	....	....

Cost Analysis

Before the beginning of any construction work on this plant, a Cost Accounting Bureau was organized with competent clerks and bookkeepers to keep complete records of the actual cost of the various features of the work as nearly as could be obtained by independently taking the time and material used in the work and checking where possible with the contractors' records. The costs of the main units of the plant as compiled from the account ledger are given in Table 4. These costs represent not only the contract costs, but also all costs incidental to the work, including force account work done by direct labor. The engineering costs include all design work, checking, preparation of plans and specifications, consulting architects' and mechancial engineers' fees, printing of plans and specifications, supervision of construction, time-keeping, clerical work, purchasing expense, checking of shop details, inspection of materials, shop inspection, and general administration of the work. The cost of experimental work is listed separately. For ease in comparing with other projects the costs have also been reduced to the basis as of the year 1913 by using the construction cost index figures of the *Engineering News-Record.*

ACKNOWLEDGMENTS

The design and supervision of construction of the Springwells Filtration Plant was done by the Filtration Bureau of the Division of Engineering formed by the Board of Water Commissioners of Detroit for this purpose. This Division worked under the general superintendence of George H. Fenkell, M. Am. Soc. C. E., General Manager and Chief Engineer, and F. H. Stephenson, M. Am. Soc. C. E., Engineer of Water System, of the Department of Water Supply. The Filtration Bureau was in the general charge of A. B. Morrill, M. Am. Soc. C. E., Assistant Engineer (Filtration), with the writer in charge of design, and J. W. Orton, Assoc. M. Am. Soc. C. E., Assistant Civil Engineer in charge of construction. Mr. J. C. Thornton, Architect, was responsible for the architectural work, and the E. R. Little Company, Mechanical Engineers, consulted on the heating and ventilating design. Mr. E. W. Frey, Accountant, had charge of the keeping of costs and records.

CONCLUSIONS

The execution of this project has contributed to the progress of water purification practice: (1) In the development of knowledge of the flow in coagulation basins and distributing inlet details by experimental model investigation (2) in the development of higher rates of filter washing and their control by sand expansion; (3) in the conception and development of the shunt system of operating filtered-water reservoirs; and (4) in the improvement of details, such as low loss-of-head rate controllers, summation of Venturi meters, perforated pipe filter under-drains, and chemical handling equipment.

DISCUSSION

F. H. STEPHENSON,[5] M. AM. SOC. C. E. (by letter).—Adequate and easily comprehended records of design and construction of large municipal projects, are not presented as frequently as they should be, and the author should be commended and congratulated for his contribution to this class of literature.

The writer's direct connection with the water supply of Detroit began with studies for the preparation of the Pratt report of 1919 mentioned by the author, and continued until after the beginning of operation of the Springwells Filtration Plant. With this background, the writer submits a few observations regarding the filtration of the water supply of Detroit.

The experimental filtration plant built in 1917 was designed primarily to give visual proof of the feasibility of filtering the existing water supply, and the improvement in appearance, taste, and purity which would be brought about by filtration. Its successful operation demonstrated that a satisfactory effluent could be produced with 2 hr. sedimentation and filtration at a normal rate of 160 000 000 gal daily per acre. The experimental plant constructed in 1925, was designed, as the author states, primarily to obtain information relating to the phenomena of mixing of chemical solutions with raw water, the formation of floc, and the settling of the treated water before it is conveyed to the filters.

The Springwells Plant, as built, retained the normal rate of filtration (160 000 000 gal daily per acre) and the 2-hr settling period found satisfactory in the older plant. Improved methods of introduction to, and guidance of, mixed water through the settling basins came as one of the results of lessons learned from the operation of the experimental settling basin. The designers of the new plant set their goal as efficient mixing of chemicals and raw water, with optimum floc and settlement of the treated water, in order to lighten the work of the filters.

The addition, to the filter under-drain system, of a line of perforated pipe connecting the ends of the lateral pipes, should result in improved distribution of wash water at or near the filter walls. The arrangement for operating the pumps supplying filtered water to the distribution mains, called the shunt system, should result in reduced power costs. Where conditions are favorable, this method of operation should find favor.

The methods of unloading, storing, mixing, and delivering of chemicals or chemical solutions are modern and labor-saving. Filter and wash-water control are effective and efficient. The general appearance and architectural treatment of the plant are pleasing, and the municipality possesses a filtration system of which it may well be proud.

ROBERT SPURR WESTON,[6] M. AM. SOC. C. E. (by letter).—The account of the construction of the new Filtration Plant at Detroit, Mich., by its designer, which has followed the "Studies on the Washing of Rapid Filters",[7] by Messrs. Roberts Hulbert and Frank W. Herring, is of great importance to those interested in the progress of the art of water purification.

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[5] Associate Engr., Specifications and Contract Section, T. V. A., Knoxville, Tenn.

[6] Cons. Engr. (Weston & Sampson), Boston, Mass.

The plant at Detroit exemplifies the great improvement which has been evolved in the method of acquiring a water purification plant in the past forty years. In the Nineties one bought a stock filter and tried to make it work with the water supplied to it. Some good ready-made fits were obtained, and some of the purchases—Biddeford, Me.; East Providence, R. I.; Elmira, N. Y.; Atlanta, Ga., and others—are still in use, although some have been "re-tailored" in order to fit their waters better. Many others of these ready-made plants have been replaced.

Now, as exemplified at Detroit, one studies the water to be treated and builds a plant to fit it. As the author has shown, the first requisite is proper treatment — coagulation. With proper treatment, almost any sand bed produces good results; without it, the excellent one designed by the author might fail even if it is true that the proper treatment of the Detroit River water is not as difficult as that of unstored, colored or turbid waters, such as those of the Dismal Swamp, the Ohio River, or the Mississippi River.

For example, with inadequately treated water, it probably would be impracticable, even at Detroit, to use sand as coarse as that having an effective diameter of 0.55 mm, or beds only 20 in. thick; also, it would probably be impracticable to use a system of pipes for under-drains.

Naturally, the ideal system in mixers and basins for chemical treatment and coagulation is turbulent mixing after the addition of chemicals, to be followed, in turn, by slow mixing and subsidence, the velocity of flow in basins roughly conforming to a decreasing parabolic curve which reaches its minimum just before the water is applied to filters. In practice, this ideal is approached "step-wise", but invariably there are pipes and channels between mixers and basins and between basins and filters in which velocities are increased even to a degree that damages flocs formed previously under favorable conditions.

In this connection, the designers are to be congratulated in maintaining in so large a plant, influent and effluent velocities of 1.0 ft per sec, which, while high in comparison with the velocity of 2.8 ft per min through the basins, are lower than those obtaining in many other plants. Especially praiseworthy is the simple system of distributing the basin influent, although in cases where silt-bearing waters are treated there is something to be said in favor of diverting the inflowing water toward the bottom of the basin; also, where colored waters are treated, there is something in favor of the parabolic deflectors near the outlet weir, so successfully used at Baltimore, Md.

In common with some others, the practice of the writer has tended toward the use of filters with false bottoms because they provide a better distribution of wash water. This is because the conditions more nearly approach the ideal of multiple orifices discharging from a large tank. Experience shows that great damage is done to sand beds for which the wash-water distribution is uneven, and unbalanced hydraulic conditions seem to arise in certain designs of pipe manifolds which no reasonable thickness of gravel beneath the sand seems to correct, although Gore, in Canada, has used layers of cemented gravel for this correction, with reported success.

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[7] *Journal* Am. Water Works Assoc., Vol. 21 (1929), p. 1445.

The recent design of the Mahoning Sanitary District filters,[8] using false bottoms supporting Wheeler pyramids and balls, and providing a large accessible water space beneath, seems to be superior to any pipe system. The writer's firm has recently used it at Braintree, Mass., with entire satisfaction.

The selection of wash-water rates of from 30 to 39 in. per min for a sand having an effective size of 0.50 to 0.55 mm, is interesting to the writer because, in 1913, he made experiments at Concord, Mass.,[9] using various types of sand. From these experiments there was derived an optimum velocity of 30 in. for a sand having an effective size of 0.55 mm, at which velocity it was estimated the sand would expand 39 per cent. In these experiments, however, no attention was given to variations in viscosity of wash-water due to variations in temperature.

In conclusion, the writer expresses his admiration, not only of the simplicity of the design, but of the fullness of the data presented in the paper, data truly useful to designers and users of other plants.

EUGENE A. HARDIN,[10] M. AM. SOC. C. E. (by letter).—The favorable attitude of the discussers leaves the writer with very little excuse for a closure, except to express his gratitude. To complete the description of this plant it appears that a summary of operating results should be included. Therefore, the data in Table 5, taken from the March, April, and May, 1935, reports of

TABLE 5.—OPERATING DATA, DETROIT FILTRATION PLANTS: MONTHLY AVERAGES

	MARCH, 1935		APRIL, 1935		MAY, 1935	
	Water-Works Park Plant	Springwells Plant	Water-Works Park Plant	Springwells Plant	Water-Works Park Plant	Springwells Plant
(a) PLANT OPERATION						
Plant Output:						
Water filtered, in million gallons daily	128.819	97.105	126.787	97.080	128.129	101.654
Water pumped to mains, in million gallons daily	125.389	95.687	125.068	95.933	125.953	99.290
Loss or use in plant, in million gallons daily	3.402	1.418	1.719	2.147	2.176	2.364
Percentage of loss or use in plant	2.66	1.46	1.36	2.15	1.7	2.32
Wash water, in million gallons daily	3.045	....	2.046	....	1.973	....
Percentage of wash water	2.63	....	1.62	....	1.53	....
Filters:						
Filters in service	66	29	60	29	61	29
Filters washed	28	25	19	24	18	32
Average filter run, in hours	38.5	28	60.8	29.1	72.3	22.2
Filtering rate, in million gallons daily	119	137	113	139	111	144
Rate of head loss, increase in feet per hour	0.14	0.20	0.10	0.21	0.09	0.28
Gallons per square foot per foot of head loss	826	643	1 082	666	1 173	511

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[8] *Engineering News-Record*, Vol. 111 (1933), pp. 317–321.

[9] *Engineering News*, Vol 72, p. 22.

[10] Detroit, Mich.

TABLE 5.—(*Continued*)

	MARCH, 1935		APRIL, 1935		MAY, 1935	
	Water-Works Park Plant	Spring-wells Plant	Water-Works Park Plant	Spring-wells Plant	Water-Works Park Plant	Spring-wells Plant
(*a*) PLANT OPERATION (*Continued*)						
Chemical Treatment:						
Alum used, in pounds per million gallons	83.5	61.8	107.1	69.0	78.8	81.8
Alum used, in grains per gallon	0.58	0.43	0.75	0.48	0.55	0.57
Ammonium Sulfate Used:						
Sulfate, in pounds per day	469	211.2	460	213.3	471	228.7
NH_3, in pounds per day	117	52.8	115	53.3	118	57.2
NH_3, in pounds per million gallons	0.91	0.54	0.91	0.55	0.92	0.56
NH_3, in parts per million	0.11	0.06	0.11	0.07	0.11	0.07
Chlorine Used:						
In raw water, in pounds per million gallons	1.96	1.78	1.87	1.50	1.86	1.50
In raw water, in parts per million	0.23	0.21	0.22	0.18	0.22	0.18
In filtered water, in pounds per million gallons	1.06	0.99	1.44	1.00	1.24	1.00
In filtered water, in parts per million	0.13	0.12	0.17	0.12	0.15	0.12
(*b*) LABORATORY RESULTS						
Agar Plate Counts:						
20°C, raw water	15 628	3 000	1 432	300	1 439	255
20°C, applied water	61	86	22	15	21	14
20°C, filtered water	40	38	10	57	6	23
20°C, tap water	10	4	2	7	1	1
37°C, raw water	33	38	19	21	8	8
37°C, applied water	24	18	11	5	4	2
37°C, filtered water	5	3	2	2	1	0
37°C, tap water	4	2	2	1	1	0
Confirmed *B coli* Out of 900 Tubes:						
10 cu cm raw water	291	340	201	172	314	205
10 cu cm applied water	0	0	1	5	5	0
10 cu cm filtered water	0	0	1	0	1	0
10 cu cm tap water	0	0	0	0	0	0
Turbidity, in Parts per Million:	Ave. Max.	Ave. Max.	Ave. Max.	Ave. Max.	Ave. Max.	Ave. Max.
Raw water	14 39	12 32	38 131	32 112	31 102	30 94
Applied water	9 31	8 17	16 25	9 15	15 22	10 13
Filtered water	0.16 0.77	0.04 0.3	0.26 1.65	0 0.1	0.21 0.73	0.03 0.5
Plankton Count:						
Raw water	261 422	159 228	333 418	272 364	312 343	419 493
Applied water	170 222	89 139	122 187	125 146	113 128	145 229
Chemical:						
Chlorine Residual, in Parts per Million:						
Applied water	0.07	0.10	0.07	0.09	0.10	0.10
Filtered water	0.04	0.08	0.03	0.06	0.02	0.04
Weir water	0.11	0.19	0.10	0.19	0.08	0.19
Tap water	0.13	0.15	0.16	0.12	0.13	0.13
pH-value, raw water	0.1	7.9	8.1	7.9	8.2	8.0
pH-value, tap water	7.5	7.4	7.5	7.4	7.7	7.4
Alkalinity, in Parts per Million:						
CO_2, raw water	2.4	2.4	2.4	3.6	2.8	4.0
CO_2, tap water	0	0	0	0	0	0
Total, raw water	81.9	82.3	84.4	85.0	84.4	85.9
Total, tap water	77.0	77.7	78.3	80.1	80.2	81.1

W. M. Wallace, Superintendent of Filtration, Department of Water Supply, Detroit, Mich., on the operation of both the Water-Works Park and the Springwells Filtration Plants are submitted.

Mr. Stephenson's reference to the architectural treatment may be appreciated from Fig. 19, a view of the plant interior. It should be explained, however, that the architectural features were placed secondary to the functional design features. The architect developed his design from a general layout and plant skeleton already planned and given to him for architectural treatment. With this handicap, architecturally, he succeeded commendably in obtaining a very pleasing appearance from the straight lines and mass of the structures with a minimum of embellishment and with the use of plain, and not unduly expensive, materials.

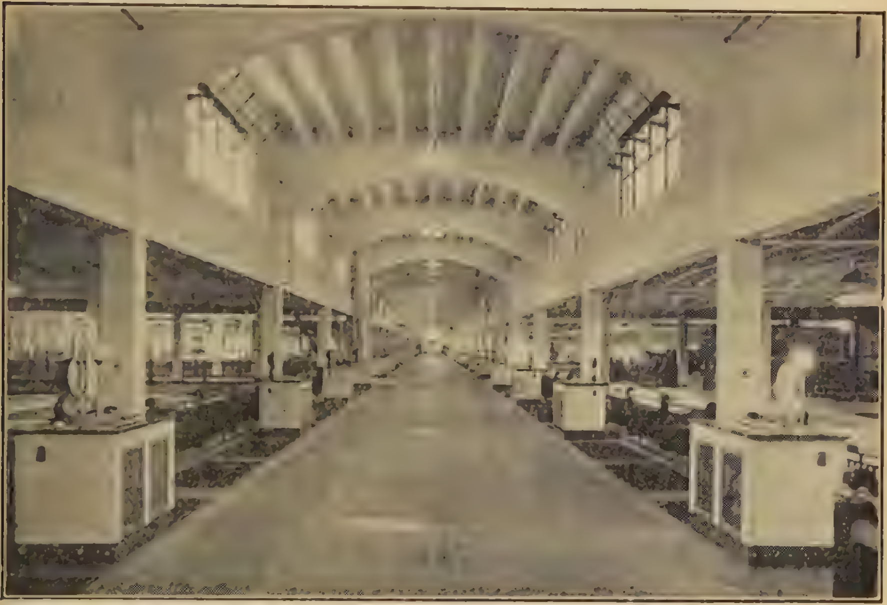

FIG. 19. INTERIOR VIEW OF OPERATING FLOOR, SPRINGWELLS FILTRATION PLANT, DETROIT, MICHIGAN.

The writer agrees with Mr. Weston that the Wheeler bottom is a very desirable type of filter under-drain system. It was one of a number of types considered for the Springwells Plant, but was not adopted on account of its considerably greater cost than the perforated pipe under-drain system.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1930

ANALYSIS OF MULTIPLE ARCHES

BY ALEXANDER HRENNIKOFF[1], ESQ.

WITH DISCUSSION BY MESSRS. L. E. GRINTER, N. M. NEWMARK, T. Y. LIN, A. H. FINLAY, A. W. FISCHER, A. A. EREMIN, AND ALEXANDER HRENNIKOFF.

SYNOPSIS

A continuous system of arches on elastic piers is analyzed in this paper, and a method of finding moments and horizontal thrusts at the ends of individual arch spans and piers, is presented. The subject of designing multiple arches, in the wide sense of the term, "design," is not treated; the paper is restricted to the discussion of a problem in engineering mechanics of determining stresses under given conditions of loading when all dimensions of the structure are known.

The method is based on the well-known principle of moment distribution originated by Hardy Cross, M. Am. Soc. C. E.,[2] but the manner of applying that principle, and various details of the method are original.

OUTLINE OF THE METHOD

The proposed method may be divided into the following operations:

(*a*) Analyze the individual spans on the assumption that they are fixed-ended and, on this assumption, determine the end moments and thrusts for each arch.

(*b*) Find the unbalanced moments and thrusts at each pier-head by adding, algebraically, the fixed-ended moments and thrusts coming from two adjacent arch spans (referring to a specific example, it will be assumed in Fig. 1, that the unbalanced joint functions are present only at the pier-head, *C*, none being present at the joint, *B*).

(*c*) Determine a series of quantities that may be appropriately termed, "the end distribution factors;" they are similar to distribution factors and carry-over factors in the method of moment distribution, and their nature will be explained subsequently.

NOTE.—Published in December, 1934, *Proceedings*.

[1] Instr., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[2] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

(*d*) Determine the "joint distribution factors," four in number, which represent moments and thrusts at the joint with unbalanced forces, *C*, when it is given a unit rotation or a unit horizontal displacement by some outside agency. These joint factors are equal to algebraic sums of the respective end distribution factors (Operation (*c*)) for the three members meeting at Joint *C*.

(*e*) Distribute the unbalanced moment and thrust at Point *C*. This requires finding the rotation and horizontal displacement of Joint *C*, necessary to balance the forces at the joint, an operation involving the solution of two simple equations, or the construction of a diagram.

(*f*) Determine the resultant forces at the ends of all members.

Referring to Operation (*c*), there are four end distribution factors at each end of each member when the joint, *C*, moves (by member is meant each individual arch span or pier, so that, in all, there are five members in Fig. 1): (1) The rotation moment factor, m_α; (2) the rotation thrust factor, h_α; (3) the displacement moment factor, m_Δ; and (4) the displacement thrust factor, h_Δ. The first two factors are, respectively, the moment and the thrust at the end of any member such as Point *A* in Fig. 1, when Joint *C*, with an unbalanced fixed-ended moment and thrust, is given a unit rotation without any linear displacement. Similarly, m_Δ and h_Δ are moments and thrusts that occur when Joint *C* is given a unit horizontal displacement, without any rotation or any vertical displacement.

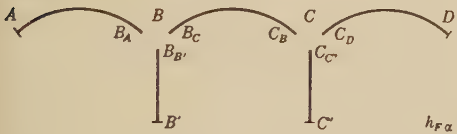

FIG. 1.

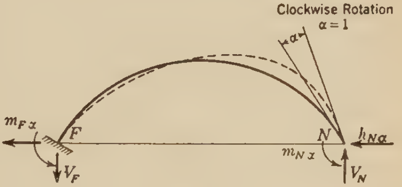

FIG. 2.

In the method of this paper unlike the method of moment distribution, the pier-head, *B*, adjacent to the one undergoing the movements, is not considered fixed, but is allowed to move its proper amount. This fact explains the difference in the method of determining distribution factors of two groups of members in Fig. 1. Members of the first group, *CC′* and *CD*, have one end fixed, the other end undergoing a known movement (unit rotation or unit translation). Their distribution factors will be found by formulas, derived subsequently. The other three members have unknown movements of one end, and their distribution factors will be found by means of a special algebraic process.

In concluding this brief outline it may be mentioned that the method suggested consists largely in following a certain simple arithmetical procedure, without recourse to higher mathematics. Formulas are used only in evaluating the end distribution factors of single members, and the constants involved are those commonly used in expressing the familiar elastic properties of arches and piers.

END DISTRIBUTION FACTORS OF SINGLE ARCHES AND PIERS

Rotation Factors of a Symmetrical Arch Rib.—Let FN (Fig. 2) be a single symmetrical span of a multiple arch. It is required to find expressions for the moments and thrusts at the ends, N and F, when F is kept fixed, and N is made to rotate, without any linear displacement, through an angle, $\alpha = 1$ radian, in positive direction (which will be assumed to be clockwise).

The end, N, which moves, will be referred to as the "near" end, and the fixed end, F, will be termed the "far" end. Then, the rotation thrust factor and the rotation moment factor at the near end will be designated, respectively, $h_{N\alpha}$ and $m_{N\alpha}$, and similar quantities at the end, F, will be denoted by $h_{F\alpha}$ and $m_{F\alpha}$.

A definite agreement as to the exact meaning and signs of these symbols is most important. In line with a common convention in the method of moment distribution, it will be assumed that m and h are the moments and thrusts with which the arch acts on the joint; and the positive directions for these actions will be clockwise for the moment, and to the right for the thrust.

Since each joint acts on the arch with forces equal and opposite to those with which the arch acts on the joint, a free-body diagram of the arch with Joint N rotated clockwise through an angle, $\alpha = 1$ radian, will appear as shown in Fig. 2. Directions for arrows, m and h, in this diagram are determined by the aforementioned convention concerning meaning and signs of the end distribution factors. The actual forces may have directions opposite the arrows shown, in which case the corresponding distribution factors will be found to be negative. Thus, it is quite evident that a clockwise moment is required at the end, N, to effect clockwise rotation of this end; consequently, the moment factor, $m_{N\alpha}$, will ultimately be negative. In addition to m and h, vertical reactions, V_N and V_F, will be required to keep the span in static equilibrium.

The derivation of formulas for the moment factors, m, and the thrust factors, h, is not a special feature of this paper, but is given for completeness. The necessary expressions can be found easily by the neutral point method. Fig. 3 is drawn for the same conditions of deformation as Fig. 2. The arch has a rigid arm extending from the end, N, to the neutral point, O, and is loaded at Point O with the forces, $h_{N\alpha}$, V_N, and M_o, which have been moved there from Point N. Since systems of forces in Figs. 2 and 3 are equivalent to each other, the following relation is evident from statics:

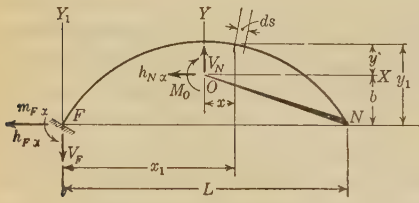

FIG. 3.

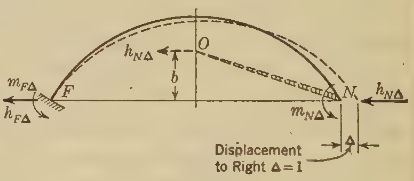

FIG. 4.

$$m_{N\alpha} = -\left[M_o + V_N \frac{L}{2} - h_{N\alpha}\, b \right] \quad \ldots\ldots\ldots\ldots\ldots (1)$$

By assumption, the only movement at Point N is rotation through an angle, $\alpha = 1$ (see Fig. 2); consequently, a composite movement occurs at the neutral point, O, as follows: (1) Clockwise rotation, $\alpha = 1$; (2) vertical displacement, $\frac{L}{2}$, upward; and (3) a horizontal displacement, b, to the right, in which, b is the height of the neutral point above the springing line.

In order to effect these displacements, the forces must be exerted at Point O such as to satisfy the following conditions (derived from the well-known properties of the neutral point):

$$M_o \int \frac{ds}{EI} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$V_N \int \frac{x^2\,ds}{EI} = \frac{L}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and,

$$-h_{N\alpha}\left[\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}\right] = b \quad \ldots\ldots\ldots\ldots (4)$$

The unknowns, M_o, V_N, and $h_{N\alpha}$, are easily found from these relations; and, then, $m_{N\alpha}$, $m_{F\alpha}$, and $h_{F\alpha}$ are determined from Equation (1), and from the conditions of equilibrium of the arch. The resultant expressions for the rotation distribution factors of a single arch are:

$$m_{N\alpha} = -\left[\frac{1}{\int \frac{ds}{EI}} + \frac{\left(\frac{L}{2}\right)^2}{\int \frac{x^2\,ds}{EI}} + \frac{b^2}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}}\right] \quad \ldots\ldots (5)$$

$$m_{F\alpha} = \frac{1}{\int \frac{ds}{EI}} - \frac{\left(\frac{L}{2}\right)^2}{\int \frac{x^2\,ds}{EI}} + \frac{b^2}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots (6)$$

and,

$$h_{F\alpha} = -h_{N\alpha} = \frac{b}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots\ldots\ldots (7)$$

It should be pointed out that the axes of the co-ordinates, X and Y, to which values of x and y are referred in Equations (5), (6), and (7), pass through the neutral point, O (Fig. 3), and are directed horizontally and vertically. The distance, b, determining the location of the neutral point, is found from the relation:

$$b = \frac{\int \frac{y_1\,ds}{EI}}{\int \frac{ds}{EI}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The following will be found useful in evaluating the integrals:

$$\int \frac{x^2\,ds}{EI} = \int \frac{x_1^2\,ds}{EI} - \left(\frac{L}{2}\right)^2 \int \frac{ds}{EI} \quad \ldots\ldots\ldots\ldots\ldots (9)$$

and,

$$\int \frac{y^2\,ds}{EI} = \int \frac{y_1^2\,ds}{EI} - b^2 \int \frac{ds}{EI} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

In Equations (8), (9) and (10), x_1 and y_1 are co-ordinates of the arch axis referred to Axes X_1 and Y_1 with the origin, F, at the springing.

Displacement Factors of a Symmetrical Arch Rib.—Following the same convention as in the case of the rotation factors, the displacement factors are defined as the moments and horizontal thrusts with which the arch acts on the joints when the far end, F, remains fixed, and the near end, N, is displaced horizontally to the right (with no rotation) a distance of one unit of length (see Fig. 4). Positive directions for these actions of the arch on the joints (not of the joints on the arch) will be again assumed as to the right for thrusts, and clockwise for moments.

The movement of the neutral point under these circumstances will evidently be the same as that of the point, N, namely, one unit of length to the right; and this may be effected by a single horizontal force, $h_{N\Delta}$, applied at Point O, as shown by the broken line in Fig. 4, thus:

$$h_{N\Delta} = -\frac{1}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Expressions for the other factors will be found from the conditions of equilibrium:

$$h_{F\Delta} = -h_{N\Delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and,

$$m_{F\Delta} = -m_{N\Delta} = \frac{b}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots\ldots\ldots (13)$$

Equations (5) to (13), inclusive, developed on the condition that the right end of the arch moves and the left end remains fixed, hold true in exactly the same form (and with the same signs) when the left end moves, and the right end stands still.

For investigating the influence of movements of the abutments, it is necessary to write expressions for terminal forces when one end of the arch settles vertically without rotation through a distance of one unit of length, and the other end remains fixed. It is easy to prove that no horizontal thrusts occur in this case, and that the end moments are equal to,

$$m_V = \frac{\frac{L}{2}}{\int \frac{x^2\,ds}{EI}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

the sign being plus on both ends if the right end of the arch moves down.

Rotation and Displacement Factors for a Pier with Fixed Base.—Fig 5 represents an elastic pier, fixed at the base, *B*, with a unit clockwise rotation at the top, *T*, with no linear displacement, and subjected to the action of the forces, $m_{T\alpha}$ and $h_{T\alpha}$ (reversed rotation factors for the pier). In Fig. 6 these forces have been moved as M_0 and $h_{T\alpha}$ to the extremity of a rigid arm terminating at the neutral point of the pier, a distance, *a*, below its top. Evidently,

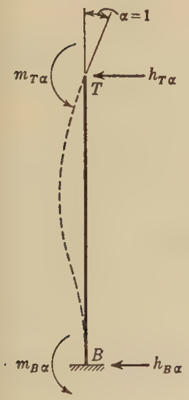

FIG. 5.

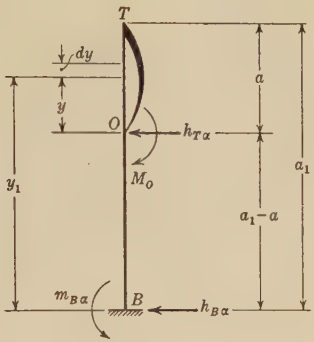

FIG. 6.

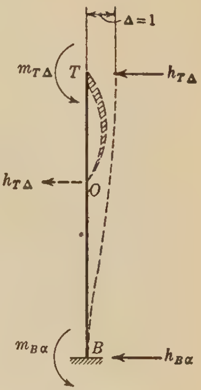

FIG. 7.

$$m_{T\alpha} = -\left[M_o + h_{T\alpha}\, a\right] \dotfill (15)$$

The neutral point, *O*, has the following movements: Clockwise rotation, $\alpha = 1$, and linear displacement to the left, *a*. These are produced by,

$$h_{T\alpha} = \frac{a}{\int \frac{y^2\,dy}{E\,I}} \quad \dots\dots\dots\dots\dots\dots (16)$$

and, $M_o = \dfrac{1}{\int \frac{dy}{E\,I}}$; then, from Equation (15),

$$m_{T\alpha} = -\left[\frac{1}{\int \frac{dy}{E\,I}} + \frac{a^2}{\int \frac{y^2\,dy}{E\,I}}\right] \quad \dots\dots\dots\dots (17)$$

The *y* - co-ordinates of the pier axis, under the integral signs in Equations (16) and (17), are measured from the origin at the neutral point. In order to simplify the evaluation of the integrals, the following relation may be used:

$$\int \frac{y^2\,dy}{E\,I} = \int \frac{y_1^2\,dy_1}{E\,I} - (a_1 - a)^2 \int \frac{dy_1}{E\,I} \quad \dots\dots\dots (18)$$

in which, y_1 and a_1 are measured from the base of the pier (see Fig. 6).

Similar argument leads to the following expressions for the displacement factors of a pier with fixed base (Fig. 7):

$$h_{T\Delta} = -\frac{1}{\int \frac{y^2\,dy}{EI}} \qquad \text{(19)}$$

and,

$$m_{T\Delta} = \frac{a}{\int \frac{y^2\,dy}{EI}} \qquad \text{(20)}$$

Rotation and Displacement Factors for a Pier That Can Move at the Base.—When investigating the influence of yielding foundations under a pier it is necessary to write expressions for the rotation and displacement factors of a pier that can rotate or move horizontally at the base, while the head remains fixed. Using the foregoing notation and convention as to the signs and directions of motions, expressions for thrusts and moments at the pier base (h_{Ba}, m_{Ba}, $h_{B\Delta}$, and $m_{B\Delta}$) can be obtained from corresponding expressions in Equations (16), (17), (19), and (20) at the pier-head by substituting $-(a_1 - a)$ for a. The force functions at the top of the pier, while the pier-head still remains locked, can then be found by statics.

NUMERICAL EXAMPLES

Application of this method will be demonstrated by two examples of arch systems with ribs and piers of the proportions used by Charles S. Whitney, M. Am. Soc. C. E., in his fundamental papers on arch design.[3] All ribs and piers (see Fig. 8) will be assumed to be 1 ft thick, and will have the properties listed in Table 1.[4]

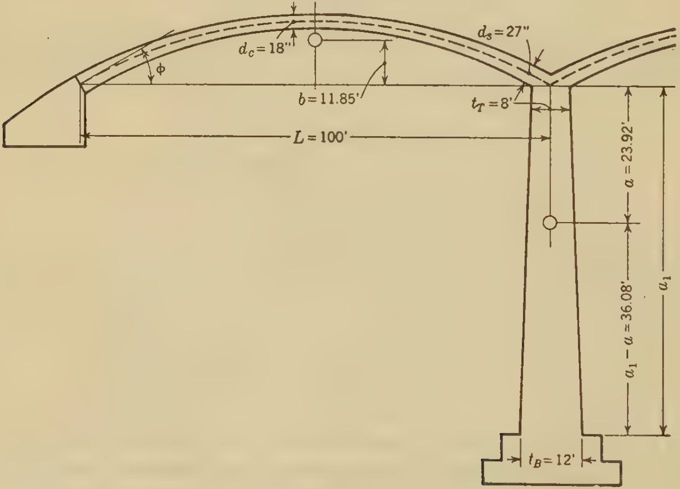

FIG. 8.

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[3] *Transactions*, Am. Soc. C. E., Vol. 88 (1925), p. 931; and Vol. 90 (1927), p. 1094.

[4] *Loc. cit.*, Vol. 90 (1927), pp. 1112 and 1120.

TABLE 1.—PROPERTIES OF RIBS AND PIERS IN NUMERICAL EXAMPLES (SEE FIG. 8)

Length of span, L, in inches	1 200	Distance, y_c, at the crown from the arch rib axis to the neutral point, in feet	3.15
Rise of arch axis, r, in feet	15	Thickness of Pier, in Feet:	
Ratio, w, of unit loads, w_S, at springing to w_c, at crown	4.70	(a) At the top, t_T	8.0
Ratio, n, of $\frac{I_c}{I_S \cos \phi}$	0.339	(b) At the bottom, t_B	12.0
Coefficient, k, in $W = \cosh k$	1.5	Moment of Inertia of Arch Barrel, in Inches^4:	
Height of pier, a_1, in feet	60	(a) At the crown, I_c	6 399
Moment of inertia, I_T, of pier at the top $\left(= 1.1 \times \frac{8^3}{12}\right)$, in feet^4	46.85	(b) At the springing, I_s	24 560
Thickness of Arch Barrel, in Inches:		Vertical distance, b, of neutral point above springing, in inches	142.2
(a) At the crown, d_c	18	Total height, $a_1 + b$, in feet	71.85
(b) At the springing, d_S	27	Distance, a, in inches (see Fig. 8)	287
Cosine ϕ,	0.769	Distance, $a_1 - a$, in inches (see Fig. 8)	433

The following values have been found from the formulas developed by Mr. Whitney:

For an Arch Rib:

$$E \int \frac{ds}{EI} = 0.1252 \text{ in.}^{-3} \quad \text{.......................(21)}$$

$$E \int \frac{x^2\,ds}{EI} = 11\,354 \text{ in.}^{-1} \quad \text{.......................(22)}$$

and,

$$E \int \frac{y^2\,ds}{EI} = 247.9 \text{ in.}^{-1} \quad \text{.......................(23)}$$

For a Pier:

$$E \int \frac{dy}{EI} = 0.0004106 \text{ in.}^{-3} \quad \text{.......................(24)}$$

and,

$$E \int \frac{y^2\,dy}{EI} = 16.7 \text{ in.}^{-1} \quad \text{.......................(25)}$$

The influence of rib-shortening, expressed by the integral, $\int \frac{ds}{EA}$, is slight and will be disregarded in the following examples. Numerical values for the end distribution factors are obtained by substituting the quantities in Equations (21) to (25) in Expressions (5) to (20). The results are listed in Table 2, the factors being divided by E. For example, the rotation factor, h_{Fa}, in Table 2, is actually $\frac{h_{Fa}}{E}$.

It is easy to understand that the absolute values of distribution factors are not essential, and that quantities proportional to them will serve as well in distributing unbalanced joint moments and thrusts. Moreover, the coefficient of proportionality for rotation factors may be made different from that for the displacement factors. A change in this coefficient may be advisable in order to raise or to lower the values of the factors, thus avoiding the inconvenience of using numbers that are too large or too small.

In the following numerical examples, the quantities, $\frac{h_a}{E}$ and $\frac{m_a}{E}$, will be used as the rotation factors, and the quantities, $\frac{100\,h_\Delta}{E}$ and $\frac{100\,m_\Delta}{E}$, as the displacement factors, $\frac{1}{E}$ and $\frac{100}{E}$, being the coefficients of proportionality.

It is evident from the physical meaning of the distribution factors that: m_a is measured in force length units; h_a, in force units; m_Δ, in $\frac{\text{force length}}{\text{length}}$ units; and, h_Δ, in $\frac{\text{force}}{\text{length}}$ units.

TABLE 2.—END DISTRIBUTION FACTORS OF RIBS AND PIERS IN NUMERICAL EXAMPLES (DIVIDED BY E; SEE TABLE 1)

ROTATION FACTORS			DISPLACEMENT FACTORS		
Equation No.	Factor	Value	Equation No.	Factor	Value
(7)	h_{Fa}	0.574 in.^{2}	(12)	$h_{F\Delta}$	0.00404 in.
(7)	h_{Na}	—0.574 in.^{2}	(11)	$h_{N\Delta}$	—0.00404 in.
(5)	m_{Na}	—121.2 in.^{3} = — 10.1 in^2-ft.	(13)	$m_{N\Delta}$	—0.574 in.^{2} = — 0.0478 in-ft.
(6)	m_{Fa}	57.8 in.^{3} = 4.82 in^2-ft	(13)	$m_{F\Delta}$	0.574 in.^{2} = 0.0478 in-ft.
(16)	h_{Ta}	17.19 in.^{2}	(19)	$h_{T\Delta}$	—0.0599 in.
(17)	m_{Ta}	—7371 in.^{3} = — 614.2 in^2-ft.	(20)	$m_{T\Delta}$	17.19 in.^{2} = 1.433 in-ft.
(14)	m_s	0.0528 in^2 = 0.0044 in-ft	.....	......	....

The important feature is that the moment factors have length dimensions of a power one degree higher than that of the corresponding thrust factors. The same characteristic is preserved in the units of the proportional values (see Table 2). This extra length dimension (an exponent of 3 in m_{Na}, for example, as compared with 2 in h_{Na}) is different from the remaining dimensions, and although the others may be expressed either in inches or in feet, the units of this particular dimension must agree with the length units of the fixed-ended moments.

This is the reason for the seemingly peculiar combinations of inches and feet in the units of the moment factors in Table 2. These combinations evidently result from the fact that the fixed-ended moments are expressed in kip-feet. Of course, it would be quite correct, although inconvenient because of small values, to have the proportional distribution factors expressed in feet only. Aside from this single qualification, the units of distribution factors are quite immaterial, and may be disregarded completely, unless the absolute values of terminal deformations are considered, as in the case of yielding foundations.

Example 1.—Table 3 contains the complete solution of a 2-span arch. Moments are expressed in kip-feet, and thrusts, in kips. The ribs and piers have the dimensions and elastic properties listed in Tables 1 and 2. The structure is loaded to the middle of the left span, as shown in Fig. 9, with a uniform load of $w = 0.1$ kip per lin ft. Pier Base B' and the abutments, A and C, are considered as absolutely fixed.

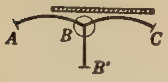


FIG. 9.—EXAMPLE 1.

TABLE 3.—END MOMENTS AND THRUSTS, TWO-SPAN ARCH BRIDGE (SEE FIG. 9)

(Thrusts, *h*, are in kips; and moments, *m*, are in kip-feet)

Item No.	Conditions	End Moments and Thrusts for Member:									
		A_B	Joint B								C_B
			B_A		$B_{B'}$		B_C		Total, Joint B		
	(1)	m^* (2)	h (3)	m (4)	h (5)	m (6)	h (7)	m (8)	h (Columns (3), (5), and (7)) (9)	m (Columns (4), (6), and (8)) (10)	$m^\dagger$ (11)
1	$\alpha = \frac{1}{E}$	+4.82	−0.574	−10.10	+17.19	−614.2	−0.574	−10.10	+16.04	−634.4	+4.82
2	$\Delta = \frac{100}{E}$	+4.78	−0.404	−4.78	−5.99	−143.3	−0.404	−4.78	−6.80	+133.7	+4.78
3	Fixed-ended forces....	−25.02	+4.482	−12.40	....	....	−8.964	−12.62	−4.482	−25.02	+12.62
4	$\alpha = \frac{0.3541}{E}$ ..	−1.71	+0.203	+3.58	−6.09	+217.5	+0.203	+3.58	....	....	−1.71
4	$\Delta = \frac{149.4}{E}$ ..	−7.15	+0.603	+7.15	+8.95	−214.0	+0.603	+7.15	....	....	−7.15
6	Two-span forces....	−33.88	+5.288	−1.67	+2.87	+3.56	−8.158	−1.89	0.0	0.0	+3.76

* Values of *h* are the same as in Column (3), with opposite signs. † Value of *h* are the same as in Column (7), with opposite signs.

The vertical columns in Table 3 contain moments and thrusts. The values refer to the ends of the various members, thus: Columns (3) and (4) (Table 3), headed "B_A", give forces at End B of Member BA, Fig. 9. Columns (9) and (10) contain the values of forces acting on Joint B. They are computed by adding, algebraically, the quantities of the three preceding columns, B_A, $B_B{'}$, and B_C.

The conditions or causes that produce the forces are listed in Column (1), Table 3. Items Nos. 1 and 2 contain forces caused by the rotation, $\alpha = \frac{1}{E}$, and the displacement, $\Delta = \frac{100}{E}$, of Joint B; for this reason, Joint B is shown in a circle in Fig. 9; in other words, the quantities listed in these two items are the distribution factors for movements of Joint B.

Item No. 3, Table 3, contains the fixed-ended forces of the two single spans under the action of given uniform load. These values have been computed by means of diagrams and formulas given in the paper by Mr. Whitney, previously cited. If there was any known movement of the supports its influence would be calculated by Equations (5) to (20), and included with the fixed-ended forces from the load.

It follows from Item No. 3, Table 3, that there are at Joint B, an unbalanced thrust of $-$ 4.482 kips and an unbalanced moment of $-$ 25.02 kip-ft, which cause this joint to rotate through an unknown angle, α, and to move horizontally an unknown distance, Δ, so that the forces at this joint become balanced. If the rotation, α, is measured in angular units containing $\frac{1}{E}$ radians, and the displacement, Δ, in linear units of the magnitude, $\frac{100}{E}$, the numerical values of these unknowns are found from the equations of equilibrium of the joint, B: $\Sigma H_B = 0$ and $\Sigma M_B = 0$, which give:

$$h_\alpha \alpha + h_\Delta \Delta + h_{B\,(\text{fixed})} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

and,

$$m_\alpha \alpha + m_\Delta \Delta + m_{B\,(\text{fixed})} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

Substituting values of h and m (Joint B) from Items Nos. 1, 2, and 3, Table 3, Equations (26) and (27) become, respectively:

$$16.04\ \alpha - 6.80\ \Delta - 4.482 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

and,

$$-\ 634.4\ \alpha + 133.7\ \Delta - 25.02 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

in which the coefficients before the α and Δ - terms are the distribution factors of Joint B. From Equations (28) and (29): $\alpha = -\ 0.3541$; and, $\Delta = -\ 1.494$.

Items Nos. 4 and 5, Table 3, give forces caused by the rotation, $0.3541\ \frac{1}{E}$, and the displacement, $1.494\ \frac{100}{E}$, of Joint B. Item No. 6 is obtained by adding, algebraically, Items Nos. 3, 4, and 5, and gives the total terminal moments and thrusts in various members of the structure under the loading considered.

In Item No. 6, values of h in Columns (3), (5), and (7), and values of m in Columns (4), (6), and (8), should add to zero. This constitutes a good check on the solution of the equations. A small discrepancy found as a result of this check has been thrown into the pier, the rigidity and distribution factors of which exceed, greatly, the corresponding values for the other members of this joint. The amounts of rotation and translation of Joint B under the action of unbalanced forces can also be found graphically. The graphical solution, however, will not be given in this paper.

Example 2.—A 4-span arch is analyzed in this more difficult example (see Fig. 10). The arch ribs and piers are alike and have the same dimensions and elastic properties as those in Example 1 (see Tables 2 and 3). The abutments and pier bases are again considered as absolutely fixed.

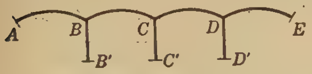


FIG. 10.—EXAMPLE 2.

Several methods of procedure can be adopted, the simplest one being, perhaps, to consider the structure as a combination of two 2-span arches, *ABC* and *CDE*, with Joint *C* fixed; and then to unlock this joint and to introduce the influence of its movements. This procedure necessitates, first, the determination of distribution factors for the movements of Joint *C*, which differ from those given by Equations (5) to (13), because the structure is extended two spans from Joint *C* instead of one.

Fig. 11 is arranged on the same basis as Table 3; the diagrams of structures in Column (2) being inserted to assist in forming mental pictures of the various steps taken, before the final solution is reached. As is evident from the diagrams, the necessary characteristics of the 4-span arch are determined after a preliminary study of 1-span and 2-span arches. The joint undergoing movement at each step is indicated by a circle around it on the diagram.

Fig. 11, Items Nos. 1 and 2, contains the already familiar distribution factors of a 2-span arch, *ABC*, with Points *A*, *B′*, and *C* fixed and Point *B* (in a circle) free to move. Item No. 3 gives rotation factors for a single arch, *BC*, when Joint *B* is fixed, and Joint *C* is permitted to move. Of course, these quantities are the same as those in Item No. 1, Columns (3), (4), and (5).

If Joint *B*, acted upon by forces in Item No. 3, is unlocked, it rotates and moves horizontally, as a center joint of a 2-span arch, so that the forces become balanced. These movements are found from formulas similar to Equations (26) and (27) by substituting the proper thrust and moment factors from Fig. 11, thus:

$$16.04\ \alpha - 6.80\ \Delta + 0.574 = 0 \quad \ldots\ldots\ldots\ldots (30)$$

and,

$$-634.4\ \alpha + 133.7\ \Delta + 4.82 = 0 \quad \ldots\ldots\ldots\ldots (31)$$

For clearness, Equations (30) and (31) are stated also in Fig. 11. From these equations, $\alpha = 0.0505$; and, $\Delta = 0.2035$.

The effect of these movements is expressed by the forces calculated in Items Nos. 4 and 5. When these values are added to the original forces in Arch *BC* before Joint *B* has moved (Item No. 3), the result is the final forces (Item No. 9), in various members of the double-span arch, *ABC*, when Joint *C* undergoes the specified pure rotation, $\frac{1}{E}$, and Joint *B* moves in a manner suitable to the occasion. In other words, Item No. 9 contains rotation distribution factors of the 4-span arch corresponding to movements of the center joint, *C*. The displacement factors are found similarly in Items Nos. 6, 7, 8, and 10, Fig. 11.

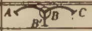
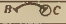
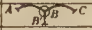
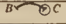
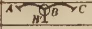
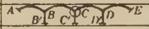
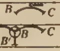
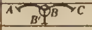
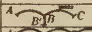
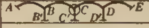
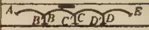
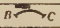
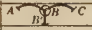
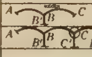
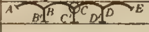
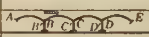
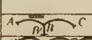
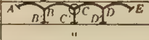
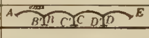
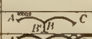
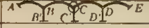
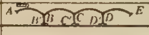
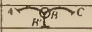
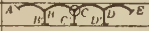
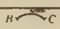
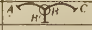
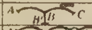
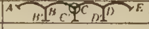
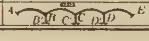
		Conditions	Structure Considered	A_B	B_A		$B_{B'}$		B_C		Joint B	
				m	h	m	h	m	h	m	h	m
	1	$\alpha = \frac{1}{E}$		+4.82	-0.574	-10.10	+17.19	-614.2	-0.574	-10.10	+16.04	-634.4
	2	$\Delta = \frac{100}{E}$	"	+4.78	-0.404	-4.78	-5.99	+143.3	-0.404	-4.78	-6.80	+133.7
α and Δ Factors · Joint C Moves												
	3	$\alpha = \frac{1}{E}$							+0.574	+4.82		
	4	$\alpha = 0.0505 \frac{1}{E}$		+0.244	-0.029	-0.510	+0.868	-31.03	-0.029	-0.510		
	5	$\Delta = 0.2035 \frac{100}{E}$	"	+0.973	-0.082	-0.973	-1.218	+29.16	-0.082	-0.973		
	6	$\Delta = \frac{100}{E}$							+0.404	+4.78		
	7	$\alpha = 0.0399 \frac{1}{E}$		+0.192	-0.0229	-0.403	+0.686	-24.52	-0.0229	-0.403		
	8	$\Delta = 0.1533 \frac{100}{E}$	"	+0.734	-0.0619	-0.734	-0.920	+21.98	-0.0619	-0.734		
	9	$\alpha = \frac{1}{E}$ (3)+(4)+(5)		+1.22	-0.111	-1.48	-0.352	-1.86	+0.463	+3.34	0.00	0.00
	10	$\Delta = \frac{100}{E}$ (6)+(7)+(8)	"	+0.93	-0.085	-1.14	-0.234	-2.50	+0.319	+3.64	0.00	0.00
Loading Case *I*												
	11	Fixed-Ended Forces							-4.482	-25.02		
	12	$\alpha = -0.3542 \frac{1}{E}$		-1.708	+0.2032	+3.577	-6.09	+217.7	+0.2032	+3.577		
	13	$\Delta = -1.496 \frac{100}{E}$	"	-7.15	+0.604	+7.15	+8.96	-214.3	+0.604	+7.15		
	14	2-Span Forces		-8.86	+0.807	+10.73	+2.868	+3.56	-3.675	-14.29	0.00	0.00
	15	$\alpha = 0.1800 \frac{1}{E}$		+0.219	-0.0200	-0.266	-0.063	-0.34	+0.083	+0.601		
	16	$\Delta = 0.9956 \frac{100}{E}$	"	+0.926	-0.085	-1.135	-0.229	-2.49	+0.318	+3.623		
	17	4-SPAN FORCES (14)+(15)+(16)		-7.71	+0.702	+9.33	+2.572	+0.74	-3.274	-10.07	0.00	0.00
Loading Case *II*												
	18	Fixed-Ended Forces							-4.482	+12.40		
	19	$\alpha = -0.2371 \frac{1}{E}$		-1.143	+0.1360	+2.395	-4.074	+145.5	+0.1360	+2.395		
	20	$\Delta = -1.220 \frac{100}{E}$	"	-5.84	+0.492	+5.84	+7.31	-174.8	+0.492	+5.84		
	21	2-SPAN FORCES (18)+(19)+(20)		-6.98	+0.628	+8.24	+3.226	-28.88	-3.854	+20.64	0.00	0.00
	22	$\alpha = 0.3232 \frac{1}{E}$		+0.394	-0.0359	-0.479	-0.113	-0.615	+0.1497	+1.08		
	23	$\Delta = 1.375 \frac{100}{E}$	"	+1.277	-0.1168	-1.566	-0.316	-3.44	+0.4386	+5.00		
	24	4-SPAN FORCES (21)+(22)+(23)		-5.31	+0.475	+6.20	+2.791	-32.92	-3.266	+26.72	0.00	0.00
Loading Case *III*												
	25	2-Span Forces		-18.04	+3.854	-20.64	-3.226	+28.88	-0.628	-8.24	0.00	0.00
	26	$\alpha = 0.06612 \frac{1}{E}$		+0.081	-0.0073	-0.098	-0.0233	-0.123	+0.0306	+0.221		
	27	$\Delta = 0.2568 \frac{100}{E}$	"	+0.239	-0.0218	-0.293	-0.0601	-0.642	+0.082	+0.935		
	28	4-SPAN FORCES (25)+(26)+(27)		-17.72	+3.825	-21.03	-3.310	+28.11	-0.515	-7.08	0.00	0.00
Loading Case *IV*												
	29	2-Span Forces		+21.26	+3.675	+14.29	-2.868	-3.56	-0.807	-10.73	0.00	0.00
	30	$\alpha = 0.0847 \frac{1}{E}$		+0.103	-0.0094	-0.125	-0.0298	-0.157	+0.0392	+0.282		
	31	$\Delta = 0.3292 \frac{100}{E}$	"	+0.306	-0.028	-0.375	-0.0771	-0.824	+0.105	+1.198		
	32	4 SPAN FORCES (29)+(30)+(31)		+21.67	+3.638	+13.79	-2.975	-4.54	-0.663	-9.25	0.00	0.00
m and h Factors Joint B Moves								ALTERNATIVE				
	33	$\alpha = 0.313 \frac{1}{E}$		+1.510	-0.1796	-3.161	+5.38	-192.2	-0.1796	-3.161		
	34	$\Delta = 0.739 \frac{100}{E}$	"	+3.530	-0.2984	-3.530	-4.427	+105.95	-0.2984	-3.530		
	35	$\alpha = 0.0616 \frac{1}{E}$	"	+0.297	-0.0353	-0.6222	+1.0595	-37.82	-0.0353	-0.6222		
	36	$\Delta = 0.2921 \frac{100}{E}$	"	+1.396	-0.1180	-1.3965	-1.750	+41.9	-0.1180	-1.396		
	37	$m = 100$ (33)+(34)	"	+5.04	-0.478	-6.69	+0.956	-86.62	-0.478	-6.69	0.00	-100.00
	38	$h = 1$ (35)+(36)	"	+1.693	-0.1533	-2.019	-0.6934	+4.038	-0.1533	-2.019	-1.000	0.00
m and h Factors Joint C Moves												
	39	$\alpha = 0.3339 \frac{1}{E}$		+0.407	-0.0371	-0.494	-0.1175	-0.621	+0.1546	+1.115		
	40	$\Delta = 0.819 \frac{100}{E}$	"	+0.761	-0.0696	-0.933	-0.1916	-2.048	+0.2613	+2.981		
	41	$\alpha = 0.0683 \frac{1}{E}$	"	+0.083	-0.0076	-0.101	-0.0240	-0.127	+0.0316	+0.228		
	42	$\Delta = 0.3182 \frac{100}{E}$	"	+0.296	-0.0271	-0.363	-0.0745	-0.796	+0.1015	+1.159		
	43	$m = 100$ (39)+(40)	"	+1.17	-0.107	-1.43	-0.309	-2.67	+0.416	+4.10	0.00	0.00
	44	$h = 1$ (41)+(42)	"	+0.379	-0.0347	-0.464	-0.0984	-0.923	+0.1331	+1.387	0.00	0.00
Loading Case *I*	45	Fixed-Ended Forces							-4.482	-25.02		
	46	$m = -25.02$		-1.260	+0.1196	+1.674	-0.2392	+21.67	+0.1196	+1.674		
	47	$h = -4.482$	"	-7.60	+0.6878	+9.053	+3.107	-18.09	+0.6878	+9.053		
	48	2 SPAN FORCES (45)+(46)+(47)		-8.86	+0.808	+10.73	+2.866	-3.56	-3.674	-14.29	0.00	0.00
	49	$m = -21.26$		-0.249	+0.0228	+0.304	+0.0657	+0.568	-0.0885	-0.872		
	50	$h = +3.674$	"	+1.394	-0.1275	-1.703	-0.3615	-3.392	+0.4895	+5.100		
	51	4 SPAN FORCES (48)+(49)+(50)		-7.71	+0.703	+9.33	+2.570	+0.73	-3.273	-10.06	0.00	0.00

FIG. 11.—END MOMENTS AND THRUSTS OF A FOUR-

C_B		$C_{C'}$		C_D		Joint C		D_C		$D_{D'}$		D_E		Joint D		E_D
h	m	h	m	h	m	h	m	h	m	h	m	h	m	h	m	m
+0.574	+4.82															
+0.404	+4.78															
-0.574	-10.10							$16.04\alpha - 6.80\Delta + 0.574 = 0$ $-634.4\alpha + 133.7\Delta + 4.82 = 0$								
+0.029	+0.244															
+0.082	+0.973															
-0.404	-4.78							$16.04\alpha - 6.80\Delta + 0.404 = 0$ $-634.4\alpha + 133.7\Delta + 4.78 = 0$								
+0.0229	+0.192															
+0.0619	+0.734															
-0.463	-8.88	+17.19	-614.2	-0.463	-8.88	+16.26	-632.0									
-0.319	-3.85	-5.99	+143.3	-0.319	-3.85	-6.63	+135.6									
+4.482	-12.40							$16.04\alpha - 6.80\Delta - 4.482 = 0$ $-634.4\alpha + 133.7\Delta - 25.02 = 0$								
-0.2032	-1.708															
-0.604	-7.15															
+3.675	-21.26							$16.26\alpha - 6.63\Delta + 3.675 = 0$ $-632.0\alpha + 135.6\Delta - 21.26 = 0$								
-0.083	-1.60	+3.094	-110.6	-0.083	-1.60			+0.083	+0.601	-0.063	-0.34	-0.020	-0.266			+0.219
-0.318	-3.83	-5.96	+142.6	-0.318	-3.83			+0.318	+3.623	-0.229	-2.49	-0.085	-1.135			+0.926
+3.274	-26.69	-2.873	+32.12	-0.401	-5.43	0.00	0.00	+0.401	+4.22	-0.296	-2.82	-0.105	-1.40	0.00	0.00	+1.15
+4.482	+25.02							$16.04\alpha - 6.80\Delta - 4.482 = 0$ $-634.4\alpha + 133.7\Delta + 12.40 = 0$								
-0.1360	-1.143															
-0.492	-5.84															
+3.854	+18.04							$16.26\alpha - 6.63\Delta + 3.854 = 0$ $-632.0\alpha + 135.6\Delta + 18.04 = 0$								
-0.1497	-2.87	+5.56	-198.4	-0.1497	-2.87			+0.1497	+1.08	-0.113	-0.615	-0.0359	-0.479			+0.394
-0.4386	-5.29	-8.23	+196.9	-0.4386	-5.29			+0.4386	+5.00	-0.316	-3.44	-0.1168	-1.566			+1.277
+3.266	+9.88	-2.678	-1.72	-0.588	-8.16	0.00	0.00	+0.588	+6.08	-0.435	-4.04	-0.153	-2.04	0.00	0.00	+1.67
+0.628	+6.98							$16.26\alpha - 6.63\Delta + 0.628 = 0$ $-632.0\alpha + 135.6\Delta + 6.98 = 0$								
-0.0306	-0.587	+1.137	-40.6	-0.0306	-0.587			+0.0306	+0.221	-0.0233	-0.123	-0.0073	-0.098			+0.081
-0.082	-0.989	-1.539	+36.8	-0.082	-0.989			+0.082	+0.935	-0.0601	-0.642	-0.0218	-0.293			+0.239
+0.515	+5.40	-0.402	-3.82	-0.113	-1.58			+0.113	+1.16	-0.084	-0.77	-0.029	-0.39			+0.32
+0.807	+8.86							$16.26\alpha - 6.63\Delta + 0.807 = 0$ $-632.0\alpha + 135.6\Delta + 8.86 = 0$								
-0.0392	-0.752	+1.455	-52.0	-0.0392	-0.752			+0.0392	+0.282	-0.0298	-0.157	-0.0094	-0.125			+0.103
-0.105	-1.267	-1.97	+47.15	-0.105	-1.267			+0.105	+1.198	-0.0771	-0.824	-0.028	-0.375			+0.306
+0.663	+6.84	-0.519	-4.82	-0.144	-2.02	0.00	0.00	+0.144	+1.48	-0.107	-0.98	-0.037	-0.50	0.00	0.00	+0.41
		METHOD														
+0.1796	+1.510							$16.04\alpha - 6.80\Delta = 0$ $-634.4\alpha + 133.7\Delta + 100 = 0$								
+0.2984	+3.530															
+0.0353	+0.297							$16.04\alpha - 6.80\Delta + 1 = 0$ $-634.4\alpha + 133.7\Delta = 0$								
+0.1180	+1.396															
+0.478	+5.04															
+0.1533	+1.693															
-0 1546	-2.964	+5.74	-205.0	-0.1546	-2.964			$16.26\alpha - 6.63\Delta = 0$ $-632.0\alpha + 135.6\Delta + 100 = 0$								
-0.2613	-3.152	-4.908	+117.4	-0.2613	-3.152											
-0.0316	-0.606	+1.173	-41.92	-0.0316	-0.606			$16.26\alpha - 6.63\Delta + 1 = 0$ $-632.0\alpha + 135.6\Delta = 0$								
-0.1015	-1 225	-1.905	+45.6	-0.1015	-1.225											
-0.416	-6.12	+0.832	-87.76	-0.416	-6.12	0.00	-100.00									
-0.1331	-1.831	-0.7338	+3.662	-0.1331	-1.831	-1.000	0.00									
+4.482	-12.40															
-0.1196	-1 260															
-0.6878	-7.60															
+3.674	-21.26															
+0.0885	+1.301	-0.1768	+18.65	+0.0885	+1.301			-0.0885	-0.872	+0.0657	+0.568	+0.0228	+0.304			-0.249
-0.4895	-6.740	-2.695	+13.45	-0.4895	-6.740			+0.4895	+5.100	-0.3615	-3.392	-0.1275	-1.703			+1.394
+3.273	-26.70	-2.872	+32.14	-0.401	-5.44	0.00	0.00	+0.401	+4.23	-0.296	-2.83	-0.105	-1.40	0.00	0.00	+1.15

Span Arch Under Uniform Load, $w = 0.1$ Kip per Foot.

Items Nos. 9 and 10, Fig. 11, are completed by filling Columns C_C and C_D, and by adding, algebraically, the three C-columns to find the joint distribution factors. There is no need to fill in the various D and E-columns, since their factors are identical with those of the symmetrically opposite members, B and A. This concludes the first stage, preliminary to investigation of four cases of loading. For clearness, it may be re-emphasized that this first stage has resulted in the determination of:

(1) Terminal forces in the 2-span arch, *ABC*, caused by unit rotation and unit translation of Joint B (Items Nos. 1 and 2, Fig. 11).

(2) Terminal forces in the 4-span arch, *ABCDE*, caused by unit rotation and unit translation of Joint C at the center (Items Nos. 9 and 10, Fig. 11).

Item No. 11 contains the fixed-ended forces of a single arch, *BC*, under a uniform load (Case 1), on the right half of Span *BC*. Item No. 14, the sum of Items Nos. 11, 12, and 13, gives the terminal forces of the double-span arch, *ABC*, under the same loading, after the unbalanced forces at Joint B have been distributed, using factors in Items Nos. 1 and 2 and solving the equations given on the right side of Fig. 11 opposite Item No. 11. The only remaining step is to distribute the unbalanced forces at Joint C, using the distribution factors in Items Nos. 9 and 10. The following equations of the form of Equations (26) and (27) are used:

$$16.26\,\alpha - 6.63\,\Delta + 3.675 = 0 \quad \text{.................}(32)$$

and,

$$-632.00\,\alpha + 135.60\,\Delta - 21.26 = 0 \quad \text{.................}(33)$$

from which the movements of the joint, C, are found to be: $\alpha = 0.1800\,\frac{1}{E}$ and $\Delta = 0.9956\,\frac{100}{E}$. The effects of movements of the joint, C, on the members in various D and E-columns are the same as on the members symmetrically located on the left side of the structure. The resultant terminal forces are entered in Item No. 17.

Case 2, in which a uniform load occupies the left half of Span *BC*, is investigated and the results are entered as Items Nos. 18 to 24, Fig. 11. Only two distributions of unbalanced forces are needed for its solution, since distribution factors of Joint C are already known. The equations used are all stated in Fig. 11.

It is interesting to notice that only one distribution is required for either Case 3 or Case 4, since their 2-span values can be written down directly by comparison with Cases 2 and 1. Investigation of all four cases thus requires less than twice the work on one case.

General Case of a Structure with Any Number of Spans.—Example 2 presents a good illustration of the use of the method, not only for a 4-span arch, but also for a series with any number of spans. Probably, the best procedure is to divide the structure into two parts by fixing a joint at the center (or near the center for an odd number of spans), to analyze each part separately, and then, having determined distribution factors for movement of the temporarily fixed joint, to distribute the unbalanced forces at that joint.

After the preliminary work of calculating distribution factors, the necessary distributions for the load require relatively little time. In designing a multiple arch there are several cases of loading under which the structure should be investigated, and when using this method, the more cases are analyzed the less work on the average is involved in each, as has been already demonstrated in connection with Example 2.

The method can also be applied to the construction of influence lines. An influence line for moment or thrust at any point can be found by the use of Maxwell's reciprocal theorem, as a deflection curve of a suitably deformed structure after the unbalanced terminal forces caused by the deformation have been distributed in the usual manner. This procedure is cumbersome, because it requires the translation of moments into deflections.

However, a complete set of influence lines of all terminal forces (and this is what is generally required in the final analysis of the structure, if influence lines are used at all) can be found comparatively easily, following closely the procedure of Example 2, by placing a unit load, successively, at different points of the arch. In Example 2 this method requires only twenty-seven distributions to construct influence lines for all ten terminal forces, having ordinates at the tenth-points of the spans. This amount of work is not unreasonable in view of the great number of force functions.

Alternative Method.—In cases requiring a large number of distributions, a change in the procedure is recommended which does away with the equations determining the necessary movements of the joints. This modified procedure is presented, as applied to Case 1 under the sub-title, "Alternative Method," in Fig. 11. The idea is to replace the rotation and displacement factors with, what may be termed, m-factors and h-factors. There are again four of these factors at each end of each member: (1) The m-moment factor, m_m; (2) the m-thrust factor, h_m; (3) the h-moment factor, m_h; and (4) the h-thrust factor, h_h.

The first two are, respectively, the moment and the thrust that occur at the end of any member (as, for example, A_B), when the joint under consideration (say, B) is acted upon by a moment of 1, or 100, created by some external agency. As a result of this moment, Joint B undergoes both rotation, α, and horizontal translation, Δ, the values of which can be found easily, and from these movements the moments and thrusts at the ends of all members (the m-factors) can be computed. Similarly, the h-factors are the moments and thrusts at different terminals when the joint in question is acted upon by a thrust, $h = 1$.

The m-factors and h-factors for a 2-span arch, ABC, with movement at Joint B are determined in Items Nos. 33 to 38, Fig. 11, by the use of rotation and displacement factors in Items Nos. 1 and 2. Equations necessary for finding the movements of Joint B, when it is acted upon by a moment, 100, are as follows:

$$16.04\,\alpha - 6.80\,\Delta = 0 \qquad (34)$$

and,

$$-634.40\,\alpha + 133.70\,\Delta + 100.00 = 0 \qquad (35)$$

whence $\alpha = 0.313$ and $\Delta = 0.739$.

Forces caused by these movements are added in Item No. 37, Fig. 11, which thus presents the *m*-factors of the 2-span arch, *ABC*. The *h*-factors, found in a similar manner, are recorded in Item No. 38. Items Nos. 43 and 44, Fig. 11, contain the *m*-factors and *h*-factors for a 4-span arch with motion at the center joint, *C*; they are found by following a similar procedure.

These four additional distributions involved in determining *m*-factors and *h*-factors eliminate the equations when investigating the influence of unbalanced fixed-ended forces. Thus, in the 2-span arch, *ABC*, under the loading of Case 1 (Items Nos. 45 to 48, Fig. 11), the influence of the fixed-ended thrust, − 4.482 kips, at Joint *B*, is found at Item No. 47 as a product of this quantity and the corresponding *h*-factors, without resort to equations. The same is true with regard to the unbalanced moment, − 25.02 kip-ft.

CONCLUSIONS

The following conclusions are advanced as to the value and limitations of the method presented in this paper:

(1) With the exception of evaluation of distribution factors for single members (which is not considered a requisite part of this analysis), the method does not involve mathematics higher than elementary algebra.

(2) The idea behind the method is simple, and the various steps in the solution are quite easy to visualize, especially when accompanied by small supplementary diagrams.

(3) The nature of the computations, involving multiplication of a string of numbers by a single factor, makes the method particularly suitable for slide-rule work. (All computations in Table 3 and Fig. 11 have been made with an ordinary 10-in. slide-rule.)

(4) The orderly tabulation diminishes a possibility of errors.

(5) The algebraic sum of the terminal forces for three members meeting at a joint must equal zero. This provides an effective check on the results, the only exception being the moments at the abutments. In fact, there is little chance for mistake, after the fixed-ended forces and the single-span distribution factors have been determined correctly.

(6) Perhaps the part of the method most likely to cause confusion is the convention pertaining to the meaning and signs of distribution factors. The signs of some of the single-member distribution factors, when observed casually (see Items Nos. 1 and 2, Fig. 11), may seem quite odd. However, once the convention is mastered, and one visualizes the movement of the end of the member in relation to its neutral point, a good reason becomes apparent for each sign.

(7) The examples solved have dealt with a series of equal symmetrical arches, but there is no change when individual arches are different but sym-

metrical. The method still applies to asymmetrical arches, although Expressions (1) to (14), for distribution factors, are no longer valid.

(8) Vertical settlement of the joints due to pier-shortening has been disregarded; it is not important, but can be taken care of independently afterward, if desired. With the exception of this theoretical limitation, the method is "exact" in a sense that it is not based on assumptions other than the ordinary ones of the theory of elasticity.

(9) While, perhaps, this method may prove more laborious than some others when investigating one load condition, the writer believes that time will be saved when applying it to several load conditions, or to the construction of influence lines. It may be mentioned in this connection that a casual inspection of Fig. 11 is likely to convey an exaggerated idea of the labor involved in the solution. The fact is that many of the values are merely copied from one column into another, and any one acquainted with the procedure will find simplifications and "short-cuts" that will lessen the work.

Acknowledgment

The writer wishes to acknowledge his indebtedness to A. H. Finlay, Assoc. M. Am. Soc. C. E., for reading the manuscript and for most valuable criticism and suggestions.

DISCUSSION

L. E. Grinter,[5] Assoc. M. Am. Soc. C. E. (by letter).—Another exposition of the application of the Cross method of moment distribution to the analysis of multiple arches, is contained in this paper. The first was a discussion[6] of Professor Cross' paper on the "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Donald E. Larson, Jun. Am. Soc. C. E. Mr. Hrennikoff's presentation is highly satisfactory and should be read widely by those who have become familiar with the Cross method.

The extension of the conception of balancing moments and shears to the complex problem of the analysis of a series of arches on elastic piers has intrigued the interest of numerous investigators. The writer made his own extension in 1927 and has taught the theory of multiple-arch analysis by the general method of successive corrections[7] to his graduate students since 1929. It is interesting that the general method of successive corrections, devised to apply to any frame in which joint translation occurs, becomes essentially the same as the method described by Mr. Hrennikoff when it is applied to a multiple-arch system.

It is perhaps unfortunate that the author does not generalize his method of attack and explain its wide application, for the reader is more than likely to gain the impression that the paper deals with a group of special devices, of use only in the analysis of multiple arches. The simple explanation that follows makes the method generally applicable to all multiple-bay or multiple-story structures:

1.—With joints fixed against both rotation and translation, determine the fixed-end moments and the reactions at the ends of each loaded span.

2.—Balance moments with joints held against translation and determine the changes in the restraining forces that exist at the joints.

3.—Eliminate all artificial joint forces by permitting the joints to translate without rotation under the action of an equal and opposite set of forces. Determine the new set of fixed-end moments introduced by this translation.

4.—Repeat Steps 2 and 3 as many times as necessary (seldom more than twice) to reduce the unbalanced moments and artificial joint forces to negligibly small factors.

The importance of placing emphasis upon these simple steps is that they attach a physical significance to the procedure which is so likely to be obscured when use is made of a table, such as Fig. 11. The method of successive corrections can be arranged into a mechanical procedure, but it is doubtful whether any important advantage is obtained. Of course, an orderly arrangement of calculations will reduce the labor of analysis and the probability of error; but over-standardization will make any procedure so highly mechanical that the various steps lose their physical significance and the method is then likely to become unintelligible to most readers. The primary reason why the Cross method has become widely popular is that each step of the procedure has a physical significance, and there is no reason why the general method of successive corrections should not retain this important characteristic.

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[5] Prof. of Structural Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

[6] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 127.

[7] *Loc. cit.*, p. 18: and Vol. 98 (1934), p. 611.

The only difficulty experienced by the writer's students in the use of this method for continuous arch analysis has been in regard to the signs of the thrusts at the tops of the piers. This difficulty is largely eliminated by the use of arrows to indicate the direction of thrust, instead of the conventional signs. The suggestion is in line with the desire to maintain the greatest possible physical significance for each step of the process. The convention that a positive bending moment at the end of a member tends to rotate the adjacent joint clockwise has been found most satisfactory.

The detailed procedure by which the constants for moment or thrust distribution are obtained, is of no great importance. Mr. Hrennikoff seems to prefer formularization by means of the neutral point conception, whereas the writer would commonly choose the simple direct application of the column analogy.[8] A knowledge of the column analogy for obtaining distribution constants and a full understanding of the four listed steps of successive corrections make totally unnecessary the remembrance of any special terminology or special devices for the analysis of continuous arch systems.

The determination of influence lines for multiple arches is reasonably simple if the shape of the deflected load line corresponding to the proper unit displacement is found by the aid of the conjugate-beam theory. In this case it is particularly convenient to produce the proper unit distortions, successively, by the use of the column analogy. There is no particular reason for choosing between the determination of influence lines for arch forces or pier reactions. In either case one must obtain three key influence lines for each span after which these may be combined by statics to obtain any influence line desired.

The writer is hopeful that Mr. Hrennikoff's paper may help to convince designers that the proper analysis and design of a system of continuous arches is no longer an impossibly difficult task, or, for that matter, not even a particularly tedious task. Certainly, there is no real justification for neglecting the elastic properties of a pier when such neglect might conceivably endanger the structure. Much "water has passed under the bridge" in the last few years in regard to the theory of structures, and this paper is clearly indicative of the changes that are taking place.

N. M. NEWMARK,[9] JUN. AM. SOC. C. E. (by letter).—A method of analyzing multiple arches is presented in this paper which Mr. Hrennikoff states "is based on the well-known principle of moment distribution." A discussion of similar methods that have appeared in previous American literature may be of some interest.

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[8] "The Column Analogy," by Hardy Cross, M. Am. Soc. C. E., *Bulletin No. 215*, Univ. of Illinois, Eng. Experiment Station, Urbana, Ill.

[9] Research Asst. in Civ. Eng., Univ. of Illinois, Urbana, Ill.

It appears that the author's analysis of multiple-span arches is based upon a solution for the movements of the pier tops due to a given loading. These movements are then converted to forces and moments in the spans and piers. The method is similar, therefore, to the slope deflection method. Equations may be written for the forces (moments, thrusts, and shears) at the end of a member, either arched or straight, in terms of the displacements of that end, and of the far end, of the member.[10] Summing the forces at a joint (for all the members meeting at that point) in terms of the displacements and equating this sum to the unbalanced force at that joint yields a group of relations between the unbalanced forces and the joint displacements necessary to balance these forces. These are the equations used by Mr. Hrennikoff. When only two arch spans and a pier are considered (the far ends of the arch spans being fixed and vertical deflection of the pier being neglected), there are only two degrees of freedom of movement of the joint; hence, there are two unknowns and two equations to solve. When the displacements are found, the forces may be determined directly. When there are more spans, each equation will contain more unknowns, but the work of solution may be performed in the manner indicated by the author.

The general case has been treated by Mr. A. P. Hjort[11] who solves the equations by successive elimination of the unknowns, which is rather a tedious process when the number of unknowns is large; but for some types of structure, the coefficients of the unknowns are such that an accurate solution is possible without unwarranted precision in intermediate computations.

Professor Cross has suggested that these equations may be solved by successive convergence or successive approximations.[12] In an unpublished thesis submitted to the University of Illinois in partial fulfillment of the requirements for the degree of Master of Science in 1930, D. E. Larson, Jun. Am. Soc. C. E., solved the equations in the manner suggested by Professor Cross.

Certain tricks facilitate the solution of the equations, and the author's paper demonstrates some of these tricks.

The concept of moment distribution and force distribution lends itself readily to the solution of this problem. The distribution of a moment at a given joint amounts to allowing that joint to rotate until the sum of the moments in the members meeting at the joint is equal to the unbalanced moment that existed before rotation. Thus, in the two-span, single-pier, arch system the unbalanced moment and the horizontal force at the pier top, due to any loading on the arches or piers, can be distributed by a rotation and a translation of the pier top. It will be found, however, that in general such a rotation balancing the moment, will introduce an unbalanced thrust at the pier top. Furthermore, a horizontal movement of the top of the pier balancing the thrust, will introduce an unbalanced moment at that point.

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[10] See, for example, "The Column Analogy," by Hardy Cross, M. Am. Soc. C. E., *Bulletin 215*, p. 72, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

[11] "Design of Continuous Arches on Elastic Piers," by A. P. Hjort, *Proceedings*, Am. Concrete Inst., Vol. XXIX, 1933, p. 143.

[12] "Statically Indeterminate Structures," by Hardy Cross, pp. 115–118, 1926.

However, the joints may be balanced successively until convergence in some particular cases where convergence does obtain; for example, see the discussion by Mr. Larson[13] of Professor Cross' paper on moment distribution.

The foregoing explanation may be extended, of course, to a multiple-span structure and, in terms of the moment distribution concept, moments and thrusts may be distributed at all the joints and carried over; then at each joint the thrust due to the moment distribution and the moment due to the thrust distribution are added to the carry-over moments and thrusts to obtain the new unbalanced moments and thrusts at that joint. In a number of cases, however, the unbalanced moments and thrusts are almost as great as the original forces, and convergence may be very slow. It is possible to obtain more rapid convergence by use of a procedure such as that given by the author when a movement of the pier top is found to balance both moment and thrust at the given joint.

Mr. Hrennikoff does not use the procedure of carry-over and redistribution, but prefers to take into account the actual degree of fixation of the far end of the members. However, for the example which he gives, the convergence will be so rapid that it is scarcely necessary to go through the exact computations. Note that in the author's method two equations in two unknowns must be solved to distribute the forces at each joint.

Professor Cross has recommended[14] a moment-distribution procedure to be used for arches on elastic piers which eliminates the solution of simultaneous equations. If, instead of distributing moments and forces at the joint, these are distributed at a point away from the joint so located that a unit rotation of this point will produce no unbalanced thrust at the point and a unit translation of the point will produce no unbalanced moment at the point, then moments and forces can be distributed directly. It might be noted that this is equivalent to solving the two equations in two unknowns of Mr. Hrennikoff's method by substituting for them two equations which contain one unknown each and hence may be solved directly. Mr. Hrennikoff's equations might have been set up in terms of the "neutral point" with this simplification. Note that the neutral point will change in position if the far ends of the members are not considered fixed. Professor Cross proposes carrying over the changes in thrust along certain thrust lines in the different members to shorten the computations.

The essential difference, then, between Professor Cross' method[15] and that of Mr. Hrennikoff is the use of the neutral point and of successive convergence for solving the equations.

The rapid convergence of the Cross procedure is surprising. The writer has had occasion to use it in the analysis of a number of three-span arches on elastic piers. In this problem, when the structure happened to be symmetrical, considerable simplification was possible; by using the scheme of symmetry and anti-symmetry no carry-overs were necessary and all distributions were made directly.

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[13] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 127.

[14] *Loc cit.*, pp. 152–154.

[15] For formulas for the location of the neutral point and for the distribution and carry-over factors, as well as a complete description of the moment-distribution concept applied to multiple-arch analysis, see "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, Members, Am. Soc. C. E., pp. 316–330, 1932.

It is gratifying to note the distinction the author makes between analysis and design. So far as design is concerned refinements in analysis are not warranted. It is almost impossible to determine accurately the stresses in a multiple-span structure, because of the variation in elastic properties of the different spans and between different parts of each span. Furthermore, the results are always complicated by the effect of the superstructure. The elastic constants for arch spans may be modified as much as 100%, or even more, by the effect of the deck.

It is worse than futile to compute influence lines for a multiple-span arch neglecting the effect of the deck. It is possible, of course, to compute such influence lines for total moment, thrust, or shear at a section through both rib and deck when, by some means or other, fixed-end influence lines and elastic constants for the composite structure are obtained.

T. Y. LIN,[16] JUN. AM. SOC. C. E. (by letter).—The principle of moment distribution has been applied to the analysis of multiple arches in many ways, but the procedure suggested herein is among the very simplest.

It would probably be better to use the "kip-inch" unit instead of the "kip-foot" unit for the fixed-end moments. Not only will this eliminate much confusion and misunderstanding, but it will enable the application of the reciprocal theorem[17] to check many of the calculated values. For example, in Fig. 11, Items Nos. 1 and 2, in the column headed "B_A", applying that theorem, it is known that h due to $\alpha = \frac{1}{E}$ must equal m due to $\Delta = \frac{1}{E}$, and, therefore, $- 0.574 = - 4.78 \times 12 \div 100$. The same is true in the column headed "B_B", and the column headed "Joint B".

For Items Nos. 9 and 10, in the column headed "C_B": $- 0.463 = - 3.85 \times 12 \div 100$. Furthermore, in the column headed "Joint C": $16.26 = 135.6 \div 100$. For Items Nos. 40 and 41, in the column headed "Conditions", Δ due to $m = 1$ must equal α due to $h = 1$ and, therefore, $0.819 \div 12 = 0.0683$. Evidently, when more joints are considered, more such checks will be available.

A. H. FINLAY,[18] ASSOC. M. AM. SOC. C. E. (by letter).—The problem of analyzing multiple arches on elastic piers is one in which structural engineers are evidencing an increasing interest and Mr. Hrennikoff's paper, presenting as it does a novel and very direct method of analysis, is happily timed. The author's method presents a nice blending of two points of view. Starting with the basic idea of distribution factors and fixed-ended force functions he finds, in a simple and original manner, the joint movements necessary for equilibrium and, from these movements, the force functions to be combined with the original ones in order to get the final values.

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[16] With Ministry of Railways, Chinese National Govt., Nanking, China.

[17] "Elastic Arch Bridges", by McCullough and Thayer, pp. 271 and 306.

[18] Asst. Prof. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

Professor Cross has presented[19] a most ingenious solution of this problem, using his principle of force distribution in its most general form throughout. A description of this process will be of assistance in appraising the value of Mr. Hrennikoff's solution. For convenience of reference, the structure, symbolism, and sign conventions in this discussion are the same as those chosen by Mr. Hrennikoff.

If unbalanced force functions (that is, thrust and moment) are distributed in the usual manner many cycles will be necessary before a suitable balance is obtained. This is due, in part, to the fact that when, say, Joint *B* (Fig. 12), with adjacent joints locked, is translated horizontally without permitting rotation (in order to distribute horizontal thrusts), the thrusts ($h_{N\Delta}$ and $h_{T\Delta}$) produced in each adjoining arch and the pier by the moment, will produce in general, an unbalanced moment at Joint *B*. In other words during the translation the joint will have to be held against rotation. Similarly, when the joint is rotated without permitting translation (as when distributing moments) the horizontal components ($h_{N\alpha}$ and $h_{T\alpha}$) of the thrusts produced in each adjoining arch and the pier (Fig. 13) by the movement will produce, in general, an unbalanced thrust at Joint *B*. In other words, during the rotation of the joint it will have to be held against translation.

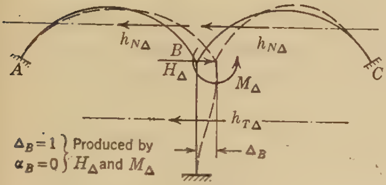

FIG. 12.

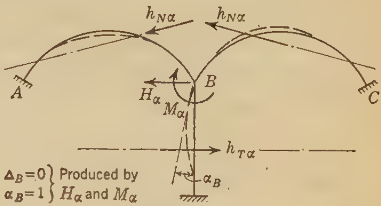

FIG. 13.

The lines of action and the magnitudes of the forces shown in Figs. 12 and 13 may be found in many ways. For convenience of comparison with the author's paper his h_Δ-forces evidently act along the thrust lines of Fig. 12 and his h_α-forces evidently represent the horizontal components of the forces acting along the thrust lines in Fig. 13, each of which passes through the neutral point of its respective member. The thrust lines in Fig. 13 are best located perhaps by finding their vertical distances from Points *A* and *B*. To facilitate comparison these distances, in the author's terminology, are, respectively, $\frac{m_{F\alpha}}{h_{F\alpha}}$, $\frac{m_{N\alpha}}{h_{N\alpha}}$, for an arch, and $\frac{m_{T\alpha}}{h_{T\alpha}}$ for the pier.

The foregoing comments have been intended to emphasize two points: First, the nature of the forces set up in each member of the assembly by each movement of the joint; and, second, the annoying unbalancing of one set of force functions caused by distributing the other set, which is inevitably associated with any distribution of force functions as long as such distribution is effected by pure translation and pure rotation of the joints themselves.

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[19] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 152 *et seq.*; and "Continuous Frames of Reinforced Concrete," by Messrs. Cross and Morgan, p. 316 *et seq.*

Such a method of distribution, of course, will eventually produce a solution, and an interesting example of such a solution has been presented by Donald E. Larson, Jun. Am. Soc. C. E.[20]

Professor Cross has discussed this difficulty of slow convergence and has suggested how, in large measure, it may be avoided. He distributes force functions by pure translation and pure rotation of a point so chosen that it translates only under the action of a suitably directed force applied to the point and rotates only under the action of a couple applied to the point. He well styles this point the neutral point of the joint. In all cases this is a point on the vertical through the joint if the pier be incompressible. It may be defined in general as that point in which the resultant of the centroidal forces in each member of the assembly, accompanying pure translation of the joint, intersects this vertical. Since a force, such as H_Δ (Fig. 14), applied at this point causes only translation, the reciprocal theorem (of which that of Maxwell is a special case) shows that a couple, such as M'_α (Fig. 15), applied at the same point will cause it to rotate only.

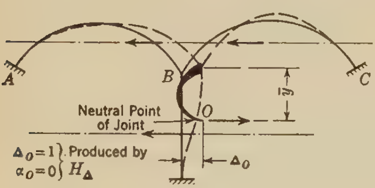

FIG. 14.

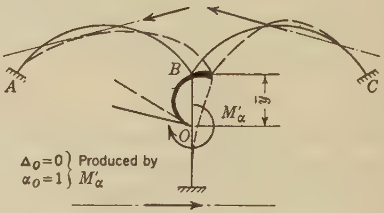

FIG. 15.

It will be interesting to note the lines of action of the forces produced in each member of the assembly by pure rotation and pure translation of such a point. These lines are shown in Figs. 14 and 15. As before, the magnitudes of the various forces can be found in different ways. One way is by considering the deflection of the neutral point of each member accompanying each specified movement of the point, O, and from these known deflections the forces accompanying them may be computed. For purposes of comparison it may be noted that the forces set up in the arches and pier in Fig. 14 are the same, of course, as those in Fig. 12 whereas, in Fig. 15, since the joint, B, has moved a distance, $\bar{y}$, to the right and has rotated through a unit angle, the forces set up are those in Fig. 13 plus $\bar{y}$ times those in Fig. 12. These relations are of no special interest, of course, except as they afford a convenient means, in this specific example, of checking numerical quantities used against those given by the author.

The distribution and carry-over factors may be found from the forces shown in Fig. 14 and Fig. 15, but it is more convenient to find them directly from consideration of the forces produced in each member of the assembly by a unit force and a unit couple, respectively, applied to the point, O. For the structure under consideration these forces are shown in Figs. 16 and 17.

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[20] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 127.

The magnitudes of the horizontal components of the thrusts are written on their lines of action. Since the sign conventions follow those of the author the forces considered are those which the members exert on their terminal points. Positive directions are to the right for thrust and clockwise for moment. In both diagrams, therefore, Arch *A-B* is pulling to the left on Point *B* and to the right on Point *A*.

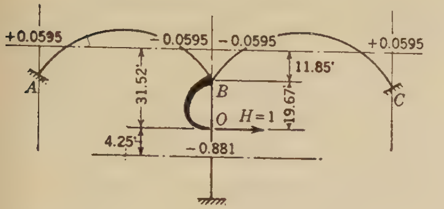


FIG. 16.

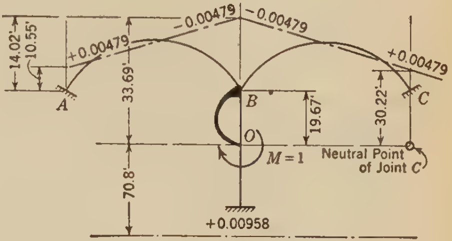


FIG. 17.

Distribution factors at Point *O* for thrust are −5.95% for each arch and −88.1% for the pier. Carry-over factors for thrust are evidently equal to the foregoing values and, for equilibrium, of opposite sign. Distribution factors for moment at Point *O* are, evidently, −0.00479 (33.69) 100 = −16.1% for each arch and 0.00958 (−70.8) 100 = −67.8% for the pier. The carry-over factors for moment may be obtained in the same manner by multiplying the thrust by its appropriate lever arm, or they may be found from the distribution factors by multiplying by the appropriate ratio of distances. For example, the part of any unbalanced moment at Point *O* that is carried over to the neutral point of an adjacent joint, such as Point *C*, is $+16.1 \frac{(30\ 22)}{(33.69)} = +14.5$ per cent. The plus sign merely means that, in this example, the carried-over thrust due to rotation is exerting a clockwise moment on the neutral point of Joint *C*. Summarized briefly, the foregoing means that unbalanced force functions at the neutral point of any joint are distributed among the members at such a joint with signs opposite to that of the unbalanced force function while they are carried over to the neutral points of adjacent joints with signs the same as that of the original unbalanced force function.

It should be realized clearly that, when distributing unbalanced thrusts by translating the neutral point, the distributed thrusts will cause no unbalanced moment at the neutral point. This follows, of course, from the definition of the neutral point, but a clear physical picture of this phenomenon can be obtained by realizing that the moments of the arch and pier thrusts in Fig. 14, or in Fig. 16 must balance about the neutral point of the joint. Similarly, when distributing moments by rotation of the neutral point, the distributed moments in the arches and pier will cause no translation of that point; that is, the horizontal components of the thrusts in the arches and pier accompanying such distribution must total zero as may be seen in Fig. 15, or in Fig. 17. This is the central idea in Professor Cross' ingenious method of distributing force functions. It follows from this that, during the actual distributing, no account need be kept of distributed functions but only of those parts of the unbalanced functions which are carried over to the neutral points of adjacent joints, since it is only these latter quantities that can unbalance any previously balanced neutral point.

The writer cannot emphasize too strongly the need of visualizing with the aid of Figs. 16 and 17 just what happens when the force functions are distributed. When thrusts are distributed, the distributed thrusts act, as has been stated, along the lines of action shown in Fig. 16. When moments are distributed the distributed moments set up thrusts which act along the lines of action shown in Fig. 17. If these two physical aspects are clearly understood the method of translating changes in thrusts into changes in moment, and *vice versa*, will be obvious.

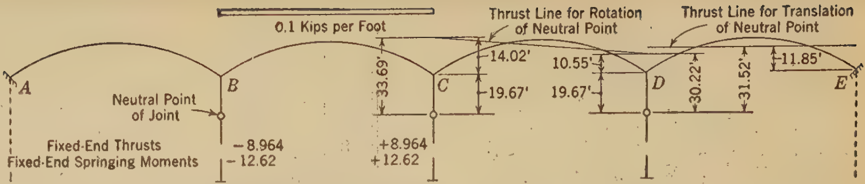
Fig. 18 contains the full solution of all terminal forces for one load position. It should be noted that about one-half the tabular matter is devoted to the distribution proper, the remainder being given over to the tiresome but necessary translation of the various changes in thrust into final thrusts and moments. This latter is only simple arithmetic combined with the laws of statics, but, of course, it must be done. It is in this last step of the work that the distributed thrusts which, heretofore, have been ignored, enter the calculations. Since these distributed thrusts are equal numerically and opposite in sign to their respective carried-over thrusts they may be written by inspection. The end arches and the piers need not appear in the original distribution. This reduces the work materially but, of course, eliminates any check of the arithmetic by statics.

If many loading conditions are to be investigated the part of the work devoted to distribution can be reduced materially (as Professor Cross has demonstrated) by first distributing a unit horizontal force and a unit couple at each neutral point, beginning near the center of the structure. In this way a set of distribution factors is built up which eventually allows of direct distribution at any neutral point. If this modification is used it will be seen that Mr. Hrennikoff's alternative method is similar to it in general idea.

The writer, familiar with both Mr. Hrennikoff's and Professor Cross' analysis of this problem, is inclined to favor the former if for no other reason than that it gives springing thrusts and moments directly without the arithmetic of Lines 24 to 33 in Fig. 18. On the other hand, with Mr. Hrennikoff's method there are the equations to be solved, although with his alternative method the number of these equations is greatly reduced. Both methods of course are powerful tools, each well adapted to the task, and the writer earnestly commends a study of both of them to those concerned with this type of analysis.

Carry-over +5.95% of Thrust
and +14.5% of Moment

All Thrusts, in Kips
All Moments, in Kip-Feet


0.1 Kips per Foot
Thrust Line for Rotation of Neutral Point
Thrust Line for Translation of Neutral Point
A
B
C
D
E
Neutral Point of Joint
33.69'
14.02'
19.67'
10.55'
19.67'
30.22'
31.52'
11.85'
Fixed-End Thrusts
Fixed-End Springing Moments
−8.964
−12.62
+8.964
+12.62

Item No.		A	B	B	C	C	D	D	E
1	Fixed-ended thrusts			−8.964	+8.964				
2	Distribute and carry-over thrusts. Record			+0.533	−0.533		+0.533		
3	Only carried-over thrusts			−0.032	+0.032	+0.032	−0.032		
4				+0.004	−0.002	−0.002	+0.004		
5	Sum of carried-over thrusts (Lines 2+3+4)			+0.505	−0.503	+0.030	+0.505		
6	Reduce to moments at neutral point (multiply by 31.52)			+ 15.9	− 15.9	+0.9	+15.9		
7	Fixed-ended moments at neutral point (±12.62 ±8.964 (19.67))			−188.0	+188.0				
8	Total unbalanced moment at neutral point			−172.1	+172.1	+0.9	+15.9		
9	Distribute and carry-over moments to neutral point			+25.0	−24.9	+2.3	+25.0		
10	Record only moments carried over			− 3.3	+ 3.6	+3.6	− 3.3		
11				+ 1.0	− 0.5	−0.5	+ 1.0		
12					+ 0.1	+0.1			
13	Sum of carried-over moments (Lines 9+10+11+12)			+22.7	−21.7	+5.5	+22.7		
14	Reduce to thrusts (divide by 30.22)			+0.750	−0.718	+0.182	+0.750		
15	Distribute and carry-over thrusts			−0.032	+0.045	+0.045	−0.032		
16				+0.005	−0.002	−0.002	+0.005		
17	Sum of carried-over thrusts (Lines 15+16)			−0.027	+0.043	+0.043	−0.027		
	Thrusts Caused by Translation of Neutral Points:								
18	(*a*) Due to carried-over thrusts (Lines 5+17)	−0.460	0	+0.478	−0.460	+0.073	+0.478	0	+0.073
19	(*b*) Due to distributed thrusts (not shown)	0	+0.460	+0.460	−0.478	−0.478	−0.073	−0.073	0
20	Total thrust caused by translation (Lines 18+19)	−0.460	+0.460	+0.938	−0.938	−0.405	+0.405	−0.073	+0.073
	Thrusts Caused by Rotation of Neutral Points:								
21	(*a*) Due to carried-over moments (Line 14)	−0.718	0	+0.750	−0.718	+0.182	+0.750	0	+0.182
22	(*b*) Due to distributed moments (not shown)	0	+0.718	+0.718	−0.750	−0.750	−0.182	−0.182	0
23	Total thrust caused by rotation (Lines 21+22)	−0.718	+0.718	+1.468	−1.468	−0.568	+0.568	−0.182	+0.182
24	Thrust due to translation and rotation (Lines 20+23)	−1.178	+1.178	−2.406	−2.406	−0.973	+0.973	−0.255	+0.255
25	Fixed-ended thrusts			−8.964	+8.964				
26	Actual thrusts in arches	−1.178	+1.178	−6.558	+6.558	−0.973	+0.973	−0.255	+0.255
	Moments of Springings Caused by:								
27	(*a*) Thrusts due to translation (Line 20)(11.85)	− 5.5	+ 5.5	+11.1	−11.1	− 4.8	+ 4.8	−0.9	+0.9
28	(*b*) Thrusts due to carried-over moments (Line 21) (10.55)	− 7.6	0	+ 7.9	− 7.6	+ 1.9	+ 7.9	0	+1.9
29	(*c*) Thrusts due to distributed moments (Line 22) (14.02)	0	+10.1	+10.1	−10.5	−10.5	− 2.5	−2.5	0
30	(*d*) Fixed-ended moments			−12.62	+12.62				
31	Actual springing moments (Lines 27+28+29+30)	−13.1	+15.6	+16.48	−16.58	−13.4	+10.2	−3.4	+2.8
32	Thrusts at pier tops (by subtraction)		+5.380		−5.585		−0.718		
33	Moments at pier tops (by subtraction)		−32.08		+29.98		−6.8		

FIG. 18.

A. W. FISCHER,[21] ESQ. (by letter).—The author is to be commended for his efforts to simplify the solution of one of the widely discussed problems arising in structural engineering. His method is short and simple when the spans are all the same and symmetrical and the arch has been analyzed for fixed-end moments and thrusts; but when the fixed-end moments and thrusts are not given and the axis does not follow the axis as given in the papers[3] by Charles S. Whitney, M. Am. Soc. C. E., the method is still quite long especially for the analysis of three spans or more. In the latter case it seems that a simple algebraic solution developed by Mr. H. Yu[22] is at least as rapid, and the results will agree very closely since both methods are based on the deflection of curved beams.

When applied to the two-span arch, Example I (which is one of the cases treated by Mr. Whitney[3]), the results obtained by the author's method practically agree with those of Mr. Whitney. In order to demonstrate the simplicity and brevity of the author's method when the fixed-end moments and thrusts are given, and to show how the results compare with other methods, it seems desirable to present the calculations in detail for a unit vertical load at a definite point. (All computations in this discussion were made with a 20-in. slide-rule.)

TABLE 4.—END DISTRIBUTION FACTORS OF RIBS AND PIER FOR NUMERICAL EXAMPLE I (DIVIDED BY E)

ROTATION FACTORS			DISPLACEMENT FACTORS		
Equation No.	Factor	Value	Equation No.	Factor	Value
(7)	$h_{F\alpha}$	0.5736 in.²	(12)	$h_{F\Delta}$	0.004034 in.
(7)	$h_{N\alpha}$	—0.5736 in.²	(11)	$h_{N\Delta}$	—0.004034 in.
(5)	$m_{N\alpha}$	—10.11 in²–ft	(13)	$m_{N\Delta}$	—0.04780 in-ft.
(6)	$m_{F\alpha}$	4.815 in²–ft	(13)	$m_{F\Delta}$	0.04780 in-ft.
(16)	$h_{T\alpha}$	2.151 in²	(19)	$h_{T\Delta}$	—0.007493 in.
(17)	$m_{T\alpha}$	—76.93 in²–ft	(20)	$m_{T\Delta}$	0.1794 in-ft.

The values in Table 4 have been calculated by using those given in Mr. Whitney's papers[3] and substituting in Equations (5) to (20), inclusive, of this paper. Table 5 is arranged in a manner similar to Table 3 of the paper and gives the values for a unit vertical load at the 0.9th point[23] of the two-span arch on an elastic pier as given in Example I in Mr. Whitney's papers.[3] Moments are expressed in foot-pounds and thrusts in pounds. (For other values of the two-span forces for unit vertical loads at Points 0.8 to 0.1, as calculated by the author's method, see Table 6.)

The structure is analyzed for a unit vertical load at the tenth points,[23] and it is assumed that the base of the pier and the abutments are considered absolutely fixed. In Tables 4 and 5 the sign conventions adopted by the author were used since they are very simple to follow; but in Table 6 the Whitney sign convention was used. On comparing values in Table 6 with what Mr. Whitney[24] gives, it is seen that the two methods agree rather well, and for the two-span arch on an elastic pier the Hrennikoff method is a quick solution for the moments and thrusts.

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[21] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[22] "Reinforced Concrete Multiple-Arch Bridge on Elastic Piers," by H. Yu, 5a Si-Siao Shaun-Ma-Chuang, North end of Pei-Hsin-Hwa-Chieh, Peiping, China.

[23] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 1127, Fig. 18.

TABLE 5.—END MOMENTS AND THRUSTS, TWO-SPAN ARCH BRIDGE.
(Thrusts, *h*, are in pounds; and moments, *m*, are in foot-pounds)

Item No.	Conditions	End Moments and Thrusts for Member:										
			Joint *B*									
		A_B	B_A		$B_{B'}$		B_C		Total, Joint *B*		C_B
	(1)	*m* (2)	*h* (3)	*m* (4)	*h* (5)	*m* (6)	*h* (7)	*m* (8)	*h* (Columns (3), (5), and (7)) (9)	(Columns (4), (6), and (8)) (10)	*m* (11)
1	$a = \frac{1}{E}$	+4.815	−0.5736	−10.11	+2.151	−76.93	−0.5737	−10.11	+1.0038	−97.15	+4.815
2	$\Delta = \frac{100}{E}$	+4.780	−0.4034	−4.780	−0.7493	+17.94	−0.4034	−4.780	−1.5561	+ 8.38	+4.780
3	Fixed-ended forces at 0.9 point.	+6.38	+0.191	+1.52	0.0	0.0	0.0	0.0	+0.191	+1.52	0.0
4	$a = \frac{0.02778}{E}$	+0.13	−0.016	−0.28	+0.060	−2.14	−0.016	−0.28	......	......	+0.13
5	$\Delta = \frac{14.07}{E}$ ..	+0.67	−0.057	−0.67	−0.105	+2.52	−0.057	−0.67	......	......	+0.67
6	Two-span forces...	+7.18	+0.118	+0.57	−0.045	+0.38	−0.073	−0.95	0.0	0.0	+0.80

As reinforced concrete arches, with either fixed or multiple spans on elastic piers will be used more in the future, it seems that some short exact analysis such as that offered in this paper should be applied to multiple arches.

TABLE 6.—END MOMENTS, IN FOOT-POUNDS, AND THRUSTS, IN POUNDS, FOR UNIT VERTICAL LOADS AT THE TENTH POINTS BY THE AUTHOR'S METHOD.

Ratio of span, length, $\frac{x}{l}$ (1)	Horizontal thrust, H_A (2)	Bending Moment		Ratio of span, length, $\frac{x}{l}$ (1)	Horizontal thrust, H_A (2)	Bending Moment		Ratio of span, length, $\frac{x}{l}$ (1)	Horizontal thrust, H_A (2)	Bending Moment	
		M_A (3)	M_B (4)			M_A (3)	M_B (4)			M_A (3)	M_B (4)
1.0	0.000	0.00	0.00	0.3	0.841	+3.51	−7.24	0.4′	0.507	+5.79	+6.39
0.9	0.118	−7.18	+0.57	0.2	0.509	+2.91	−8.24	0.5′	0.596	+6.72	+7.62
0.8	0.406	−9.41	+1.47	0.1	0.190	+1.39	−6.20	0.6′	0.563	+6.32	+7.28
0.7	0.745	−7.97	+1.80	0.0	0.000	0.00	0.00	0.7′	0.429	+4.79	+5.58
0.6	1.027	−4.45	+0.86	0.1′	0.001	+0.13	−0.18	0.8′	0.242	+2.69	+3.15
0.5	1.155	−0.53	−1.43	0.2′	0.139	+1.71	+1.52	0.9′	0.073	+0.80	+0.95
0.4	1.083	+2.35	−4.52	0.3′	0.333	+3.77	+4.06	1.0′	0.000	0.00	0.00

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[24] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), Table 11, Example I, pp. 1118–1119.

A. A. EREMIN,[25] ASSOC. M. AM. SOC. C. E. (by letter).—A simplified method of computing stresses in a system of multiple-arch spans on elastic piers has been developed in this paper. The table of moments and thrusts in Fig. 11 is a useful guide in the practical application of the method.

In using the alternative method developed by the author in Fig. 11, time may be saved by releasing points starting from the end span of a system. This is true especially in reference to a series of variable lengths of arch spans. The advantage is twofold: First, the error in computing distribution factors for moments and thrusts may be traced easily because the effect of one additional span on distribution factors is more evident than in the case of a series of spans in which the structure is divided into two or more series of spans as suggested by the author; and, second, the step of balancing distribution factors in far spans may be omitted because they are negligible.

Mr. Hrennikoff assumed that joints in a system of multiple-arch spans do not move vertically. If the support is compressible, forces at the joint are balanced after it has rotated through an angle, α, moved horizontally for a distance, Δ, and vertically for a distance, δ. The angle of rotation, and the horizontal and vertical displacements of the joint may be determined from the equations of equilibrium of the joint: $\Sigma H_B = 0$; $\Sigma V_B = 0$; $\Sigma M_B = 0$. Thus,

$$h_\alpha \alpha + h_\Delta \Delta + h_\delta \delta + h_B \text{ (fixed)} = 0 \ldots\ldots\ldots\ldots\ldots (36)$$

$$v_\alpha \alpha + v_\Delta \Delta + v_\delta \delta + v_B \text{ (fixed)} = 0 \ldots\ldots\ldots\ldots\ldots (37)$$

and,

$$m_\alpha \alpha + m_\Delta \Delta + m_\delta \delta + m_B \text{ (fixed)} = 0 \ldots\ldots\ldots\ldots\ldots (38)$$

in which, h_δ, m_δ = the horizontal force and moment factors, respectively, corresponding to a vertical displacement of a joint; v_α = vertical force factor with reference to an angle of rotation; v_Δ = vertical force factor with reference to a horizontal displacement; v_δ = vertical force factor with reference to a vertical displacement; and v_B = the sum of the fixed-end vertical forces at a joint from the given loading.

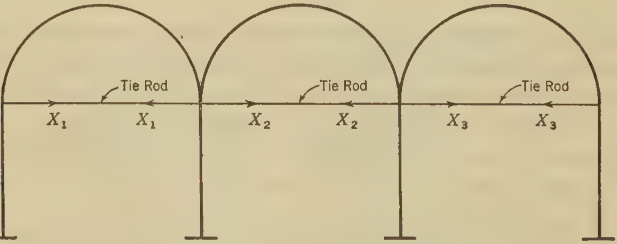

FIG. 19.—MULTIPLE-SPAN ARCH ON ELASTIC PIERS WITH TIE-RODS AT JOINTS.

Mr. Hrennikoff's method may be extended so that stresses can be computed in a system of multiple-arch spans on elastic piers with flexible tie-rods in arches, such as that in Fig. 19. Equations for computing stresses in tie-rods may be written according to Maxwell's theorem:

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$$X_1 (\Delta_{11} + \Delta_1) + X_2 \Delta_{21} + X_3 \Delta_{31} = \Delta_{01} \ldots\ldots\ldots\ldots (39)$$

$$X_1 \Delta_{12} + X_2 (\Delta_{22} + \Delta_2) + X_3 \Delta_{32} = \Delta_{02} \ldots\ldots\ldots\ldots (40)$$

and,

$$X_1 \Delta_{13} + X_2 \Delta_{23} + X_3 (\Delta_{33} + \Delta_3) = \Delta_{03} \ldots\ldots\ldots\ldots (41)$$

in which, X_1, X_2, X_3 = stresses in the tie-rods (subscripts indicate number of the span); Δ_{01}, Δ_{02}, Δ_{03} = changes in lengths of spans at the tie-rods of the system (Fig. 19 with tie-rods removed and loaded with the given loading); Δ_{11}, Δ_{12}, Δ_{13} = changes in span lengths at the tie-rods when the system of arch spans is sustaining a unit force, $X_1 = 1$, acting along the axis of the tie-rod in the first span; Δ_{21}, Δ_{22}, Δ_{23} = changes in span lengths at the tie-rods when the system of arch spans is sustaining a unit force, $X_2 = 1$, acting along the axis of the tie-rod in the second span; Δ_{31}, Δ_{32}, Δ_{33} = changes in span lengths at the tie-rods when the system of arch spans is sustaining a unit force, $X_3 = 1$, acting along the axis of the tie-rod in the third span; and Δ_1, Δ_2, Δ_3 = changes in the lengths of tie-rods corresponding to a unit load acting along the axis of the tie.

Then: $\Delta_1 = \dfrac{L_1}{A_1 E_s}$; $\Delta_2 = \dfrac{L_2}{A_2 E_s}$; and $\Delta_3 = \dfrac{L_3}{A_3 E_s}$, in which L_1, L_2, and L_3 = lengths of tie-rods; A_1, A_2, and A_3 = sectional area of tie-rods; and, E_s = modulus of elasticity in steel.

Stresses in a system of multiple-arch spans on elastic piers with removed ties (Fig. 19), sustaining external loading, or unit forces acting along the axes of the ties, may be determined by Mr. Hrennikoff's method. The horizontal displacements at tie-rods (Fig. 19) may be determined either graphically, or analytically. By solving simultaneously, Equations (39) to (41), stresses in the tie-rods (Fig. 19), X_1, X_2, and X_3, are determined. Equations similar to Equations (39) to (41) may be written for a system with any number of spans.

Computation of the stresses in a multiple-span arch on elastic piers has been generally recognized as an involved problem, and in simplifying it the author deserves the highest credit.

ALEXANDER HRENNIKOFF,[26] ESQ. (by letter).—The favorable attitude of discussers toward the method presented in this paper, and their general agreement with the writer leave no necessity for an extensive closure.

To Professor Grinter and Mr. Newmark the writer extends his thanks for further elucidation of the principles underlying the method and of its relation to Professor Cross' method of moment distribution. Professor Grinter prefers solution of the multiple-arch system by Larson's method of successive approximations in which the joints are allowed to rotate and to translate in succession, with the adjacent joints kept fixed temporarily. However, a very slow convergence of the series which results from such a procedure presents an obstacle to the successful application of the method—a fact which is mentioned in the discussions by Professor Finlay and Mr. Newmark.

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[26] Instr., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

It is this objection that led Professor Cross to devise his neutral-point method of analyzing multiple arches, in which the movements of the pierhead are replaced by the movements of the neutral point of the joint. Although it also deals with successive approximations this method is free from the disadvantage of slow convergence, and presents a powerful tool in multiple-arch analysis. It is very original and ingenious, but not as simple as appears on the surface, and the brevity of its presentation by Professor Cross[19] requires a novice to do considerable thinking before grasping its essence thoroughly.

Undoubtedly, this feature has been a hindrance to assimilation of this valuable method by the profession, and the discussion by Professor Finlay, popularizing Professor Cross' analysis, is most timely and appropriate. He takes great pains to explain the neutral-point method and to improve the form of computations. The writer agrees with Professor Finlay that a proper form, such as that of Fig. 18, is very necessary for orderly calculations of the results, especially if the method is intended for use in a designing office.

The same statement applies, of course, to the writer's tabulation of the solution for the four-span arch in Fig. 11. It is rather peculiar to notice in this connection that Professor Grinter does not approve of the table (Fig. 11), and believes that it obscures the physical side of the picture. This is exactly the opposite of the writer's opinion, and the suggested form of the table with small diagrams of arches on the left-hand side has been especially devised for the purpose of clarifying the successive physical steps taken. This tabular form, therefore, is an integral and important part of the method. It is practically self-explanatory if the reader only keeps in mind the following few simple facts:

1.—The given multiple arch is solved after considering arches of fewer spans;

2.—Each line contains moments and thrusts at various terminals in the structure drawn on the diagram when the encircled joint undergoes a rotation, α, or a horizontal displacement, Δ, of the magnitude stated in the column headed "Conditions";

3.—The necessary movements are determined by the solution of two simultaneous equations stated on the right-hand side and representing the conditions of equilibrium of the joint;

4.—The "joint distribution factors" utilized as coefficients before α and Δ in these equations are themselves determined along the same lines in the earlier part of the analysis (see the first ten lines of Fig. 11.).

The foregoing facts, as well as the supporting theory, seem quite simple to the writer; but of course he realizes that the advantage of familiarity may easily blind him to the difficulties inherent in his method. The judgment of disinterested engineers should be more reliable from this point of view; but most of the discussions are disappointingly general in nature, avoiding, for the most part, direct specific comments.

Mr. Fischer is the only discusser who made a detailed study of the writer's method as applied to a two-span arch. It is to be regretted that he did not extend his thorough investigation to the case of the four-span arch in which the advantages of the method become more pronounced. A two-span arch can be analyzed quite easily even by the otherwise cumbersome method developed by D. E. Larson, Jun. Am. Soc. C. E.

Mr. Lin's suggestion of using kip-inch units for moments instead of kip-feet units is not without merit. If inches and kips are used throughout for moments as well as for *E*, *I*, and *A*, the units of distribution factors cannot be in error. However, Mr. Lin is mistaken in claiming that such a choice of units provides a check on the solution. The equality of numerical values of certain distribution factors, discovered by him, follows from the identity of general expressions from which these numerical values are determined, and, consequently, does not check anything.

The problem of analyzing a multiple-arch system on elastic piers with flexible tie-rods in the arches, suggested by Mr. Eremin, suits the writer's method admirably. The tie-rods should be treated as separate members. Columns should be added in the computation table at each joint for each tie-rod. The single-span, displacement, thrust factor for a rod can be expressed by the formula:

$$h = \pm \frac{AE}{l} \quad \text{.........................(42)}$$

the sign being plus for the far end and minus for the near end. The three other distribution factors for the rod are each equal to zero. The factors for the rods, of course, are included with the others in the calculation of the joint distribution factors. Introduction of tie-rods, therefore, does not increase the complexity of the problem, and causes only a slight increase in the numerical work. The solution outlined by Mr. Eremin, consisting in removing the restraints caused by tie-rods, and in equating the deformations of the arches to the changes in length of the tie-rods, is very long, and has the disadvantages of other algebraic solutions based on the so-called "first principles."

To summarize the preceding discussion, the following methods have been proposed to date for the solution of open-rib multiple arches on elastic piers: (1) The algebraic method involving a number of simultaneous equations for the determination of the movements at the pier-heads; (2) the Larson method; (3) the neutral-point method by Professor Cross; and (4) the writer's method.

Method (1) is generally a long process. Method (2) is quite satisfactory for two-span arches; but is unwieldly for a greater number of spans, except when the piers are very rigid. With more than two spans the choice should be between Methods (3) and (4), the data so far available being insufficient to enable one to make a final selection.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 1931

RATIONAL DESIGN OF STEEL COLUMNS

By D. H. Young,[1] Jun. Am. Soc. C. E.

With Discussion by Messrs. William R. Osgood, Alfred S. Niles, J. F. Baker, K. L. De Blois, Marvin A. Gray, R. G. Sturm and Marshall Holt, F. E. Turneaure, N. J. Durant, E. C. Hartmann, Edward Godfrey, L. T. Wyly, and D. H. Young.

Synopsis

A basis for the design of steel columns is offered in this paper. The effect of general imperfections is represented by an assumed form of initial curvature of the axis of the column, or by an eccentricity in the application of the load; the loading first producing yielding in the most stressed fibers, due to the assumed curvature or eccentricity, is then used as the criterion for the selection of the working load.

The proposed method is applied to pin-ended columns under several conditions of loading, design formulas being developed for each case. In addition, an approximate solution is made for the case of columns in rigid frame construction.

The question of shear in built-up steel columns is treated on the same basis and for the same conditions of loading, formulas being developed for the design of lacing or batten-bars in such columns.

The paper represents, in somewhat condensed form, work done by the writer in preparing his thesis[2] entitled "Rational Design of Steel Columns."

Introduction

Since the time of Euler, the question of the resistance and stability of compression members has been controversial. The inherent difficulty with the column is that slight imperfections have a pronounced effect upon its

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Note.—Presented at the Joint Meeting of the Structural Division, Am. Soc. C. E., and Applied Mechanics Division, Am. Soc. Mech. Engrs., Chicago, Ill., June 29, 1933, and published in December, 1934, *Proceedings*.

[1] Instr., Eng. Mechanics, Univ. of Michigan, Ann Arbor, Mich.

[2] Submitted to the Univ. of Michigan in 1934, in partial fulfillment of the requirements for the degree of Doctor of Science in Civil Engineering.

behavior under load. The chief factors that affect the behavior of the column may be listed, as follows: (1) Imperfect elasticity of the material; (2) initial crookedness of the axis; (3) accidental eccentricity in the application of load; and (4) uncertainty of conditions at the ends of the column. Theories that do not take into account the extent of the effect of such imperfections are of little practical value to the designer.

As a result of such influences as Factors (1) to (4), it is present-day practice to design columns by empirical formulas. Because such formulas are backed by considerable experience, and because they generally use a liberal factor of safety, they have proved satisfactory, in most cases. The designer, however, has at his disposal such a variety of formulas that widely different results can be obtained for the same column. Curves are available[3] showing the relation between the allowable average compressive stress and the slenderness ratio, as prescribed for steel columns by various building codes and specifications in the United States. Taken altogether, such curves represent a band across the range of slenderness ratios fully 10 000 lb per sq in. wide. Of course, in all justification, it must be admitted that they represent a wide range of conditions, and each of them carries with it a number of restrictions upon its use, so that, finally, there is not the utter state of confusion that might be apparent at first glance. However, such a collection of curves is far from representing any unified agreement regarding the strength of steel compression members.

Granting the sufficiency of such empirical formulas for ordinary cases, it seems desirable to have a more general theoretical basis for design, which will take into account as far as possible all necessary factors. A rational formula has the advantage of being consistent over a much wider range of conditions than the empirical formula. Furthermore, it may show how Factors (1) to (4) affect the strength of the column, and thus the designer can make some saving in material by a careful control of workmanship or material; again, experiments conducted in the light of some kind of a rational theory, even if it is far from perfect, can yield far more valuable results than experiments conducted more or less blindly.

NOTATION

The symbols used throughout the paper are given in the Appendix. An effort has been made to conform as nearly as practicable with the "Symbols for Mechanics, Structural Engineering, and Testing Materials" advanced by the American Standards Association.[4]

PART I.—COLUMNS OF SOLID CROSS-SECTION

1.—BASIS FOR DESIGN

Nature of the Material.—A typical stress-strain diagram for ordinary mild steel is shown in Fig. 1. The curve has three significant points: (1) The proportional limit; (2) the yield point; and (3) the ultimate strength. One of these three points is usually taken as a basis when considering allowable working stresses.

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[3] For example, see "Steel Construction," Manual of the Am. Inst. of Steel Construction, First Edition, 1933, p. 176.

[4] A S A—Z10*a*—1932.

At one time working stresses used in design were based on the ultimate strength of the material. This was illogical because, at the same time, stresses were computed on the assumption that Hooke's law applied. On such a basis the designer had no such factor of safety against failure as he assumed.

At present, the yield point of the steel is quite generally recognized as the limit of usefulness. The deformation which takes place during yielding is generally from ten to fifteen times the elastic deformation at the proportional limit. Such yielding, while not properly representing complete failure, results in structural damage which cannot be allowed in ordinary structures.

To analyze a structure, the ordinary equations of elasticity based upon Hooke's law can be used, strictly speaking, only for stresses within the proportional limit. For stresses beyond this the true shape of the stress-strain diagram (Fig. 1) must be taken into account, and the problem frequently becomes very complex. In general, the effect of a slight deviation from a straight line between the proportional limit and the yield point will be small and can be safely neglected.

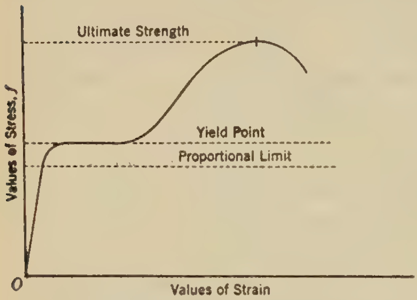

FIG. 1.—TYPICAL STRESS-STRAIN DIAGRAM FOR MILD STEEL.

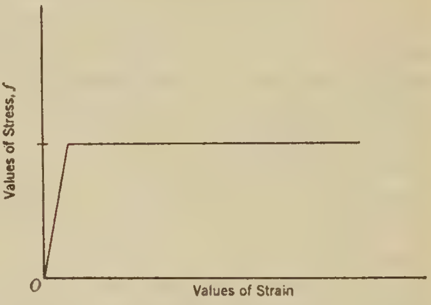

FIG. 2.—IDEAL STRESS-STRAIN DIAGRAM.

Assuming that the material is perfectly elastic up to the yield point, and interpreting yielding as failure, amounts to assuming a stress-strain diagram such as that shown in Fig. 2. It is, therefore, expedient and justifiable, in every way, to work with such an ideal stress-strain curve, and this is the usual practice. Furthermore, it is present-day practice to select the allowable loads for beams, subjected to bending, on the basis of the load that first produces a stress in the extreme fibers equal to the yield point, notwithstanding the fact that the beam will not collapse completely until all the inner fibers have begun to yield or until the extreme fibers have actually ruptured. It is well known that a beam of I-section is much nearer complete collapse when the extreme fibers first begin to yield than a beam of circular or rectangular cross-section. It is not usual practice in structural design, however, to take any account of this reserve strength against complete collapse in the case of the circular or rectangular cross-section; but rather to take the beginning of yielding in the extreme fibers as the criterion for design. In other words, this kind of structural damage is interpreted as failure.

Thus, all difficulty with Factor (1) (imperfect elasticity of the material) is eliminated. It will be logical to approach the problem of column design in the same way. To establish the logic of such a treatment, it will be convenient to consider qualitatively the general behavior of columns under load before going into more detail.

A Perfectly Straight, Axially Loaded, Pin-Ended Steel Column.—If the perfectly straight, pin-ended, steel column shown solid in Fig. 3(*a*), is slender, so that the Euler load may be reached before the fiber stresses become equal to the proportional limit, the load-deflection diagram is similar to Curve *OABC* in Fig. 3(*b*). At first, as the load increases, there is no lateral deflection. At the Euler load, P_e, the column becomes elastically unstable and may have any small deflection. A very slight increase in load above the Euler value produces a large lateral deflection. (For an increase of load 1% above the Euler value the lateral deflection at the center becomes in the order of 9% of the length of the bar.) As the load increases there finally comes a point beyond which the load necessary to maintain further deflection falls off rapidly. Before this occurs, however, the stresses in the extreme fibers will have passed the yield point. Point *B*, corresponding to the maximum load which the column can carry, represents the buckling load and is slightly greater than the Euler load.

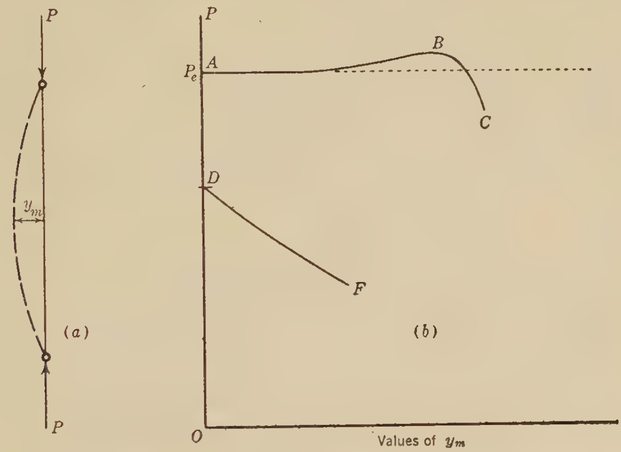

Fig. 3.—Behavior of an Initially Straight, Axially Loaded Column.

If the column is short enough so that the average compressive stress becomes equal to the yield-point stress before the Euler load is reached, the load-deflection diagram will be similar to Curve *ODF* in Fig. 3(*b*). The column remains straight for loads up to the maximum load at Point *D*, which corresponds approximately to the yield point, and then buckles suddenly, the load necessary to maintain any deflection falling off rapidly as the deflection is increased. The theory of such inelastic buckling was first developed by F. Engesser[5] in 1891.

Initially Curved or Eccentrically Loaded, Pin-Ended Steel Columns.— When a slender column is initially curved, or straight and eccentrically loaded, as represented in Fig. 4(*a*) and Fig. 4(*b*), the load deflection diagram will be similar to Curve *OABC* in Fig. 4(*c*). For any value of the load there is a definite lateral deflection. This deflection increases very slowly until the load approaches the Euler value, after which it begins to increase rapidly to some point, beyond which the load necessary to maintain further deflection falls off. Point *B*, corresponding to the maximum load which the column can carry, represents the buckling load. Before this load is reached the maximum fiber stresses will have passed the yield point.

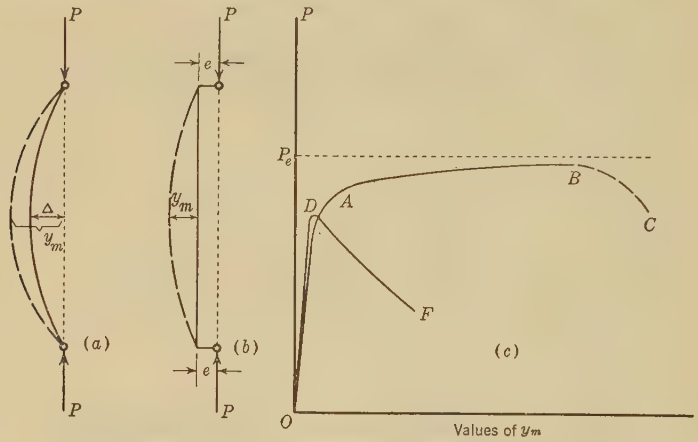

FIG. 4.—BEHAVIOR OF AN INITIALLY CURVED OR ECCENTRICALLY LOADED COLUMN.

If the column is short, the load-deflection diagram will be similar to Curve *ODF* in Fig. 4(*c*). Deflection increases slowly to Point *D*, and then the column buckles suddenly, the load necessary to maintain further deflection falling off rapidly as the deflection is increased. The theory of such inelastic buckling for initially curved or eccentrically loaded steel columns was first developed by Theodor von Kármán, M. Am. Soc. C. E., in 1910.[6] Professor von Kármán also obtained such curves as those in Fig. 4(*c*), experimentally, and succeeded in getting remarkable agreement with the theory.

The determination of the curves, as shown in Fig. 4(*c*), must take account of the slenderness ratio of the column, the shape of the cross-section, the true characteristics of the stress-strain diagram (Fig. 1), and the manner of loading. The labor involved in making these calculations is great, and also a definite stress-strain diagram must be assumed. Since the variation in the yield point of steel for various specimens is probably as much as ± 10%, this limitation would not seem to warrant such refinement in calculating buckling loads, for purposes of design, as has been outlined.

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[5] "Die Knickfestigkeit gerader Stäbe," von F. Engesser, *Zentralblatt der Bauverwaltung*, Berlin, 1891.

[6] "Untersuchungen über Knickfestigkeit," *Forschungsarbeiten*, Nr. 81, 1910; see, also, "Strength of Steel Columns," by H. M. Westergaard, M. Am. Soc. C. E., and William R. Osgood, Assoc. M. Am. Soc. C. E., *Transactions*, A. S. M. E., Vol. 49–50 (Paper APM 50–9), 1927–28.

Furthermore, due to the sudden nature of inelastic buckling, it is obvious that the fiber stresses have gone past the yield point before complete buckling occurs. It is doubtful, therefore, whether the buckling load should be used as a criterion for the design of columns any more than the loads that produce complete collapse of a beam should be used as the criterion for the selection of design loads. It would seem more consistent to design columns on a basis of the loading that first produces extreme fiber stresses equal to the yield point, as is done in the case of beams.

Finally, the introduction of initial curvature or eccentricity of load makes the column problem always one of combined bending and direct stress. Consider, for example, the eccentrically loaded column shown in Fig. 4(*b*). As the eccentricity, e, varies from zero to infinity, the complete range from pure compression to pure bending is covered. Since most compression members in structures are subjected to bending as well as to compression, it seems desirable to have one basis of design consistent throughout this range. The loading first producing a maximum fiber stress equal to the yield point will furnish such a basis.

Basis for Design.—From the foregoing discussion the following basis for the design of pin-ended steel columns is proposed: Take a column with some definite initial curvature of the axis or eccentricity of load to represent the effect of all imperfections, and then use the loading that first produces yielding in the extreme fibers as a basis for the selection of the working load, using any desired factor of safety; that is, if P_y denotes the load that first produces yielding due to assumed initial curvature or eccentricity, and n, the desired factor of safety, then the allowable working load, P_w, is to be obtained by the formula:

$$P_w = \frac{P_y}{n} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

2.—IMPERFECTIONS REPRESENTED BY INITIAL CURVATURE

The purpose of studying the behavior of an initially curved bar under compressive loads will be to learn the effect of possible accidental initial curvature on the strength of the column. Such accidental initial curvature of the bar may be of almost any shape. For a given initial deflection the most serious condition for the bar (as far as bending stresses are concerned) will be a curve, such that all points on the axis of the bar are displaced on the same side of a straight line through the ends. Various types of curvature satisfying this condition are the arc of a circle, a parabola, a half sine curve, etc. Since actual accidental initial curvature is as likely to be approximated by one of these curves as another, it seems logical to assume the form that can be most simply handled mathematically. For this reason the half wave of a sine curve will be chosen.

Consider a bar initially curved as shown in Fig. 5. The maximum deviation from a straight line at the center of the bar is Δ, and the initial deflection at any other point distant, x, from the left end, will be given by the expression:

$$y_0 = \Delta \sin \frac{\pi x}{L} \quad \text{.........................} (2)$$

When such an initially curved bar is acted upon by compressive end loads, P, as shown, the bar assumes a definite shape for each value of the load.

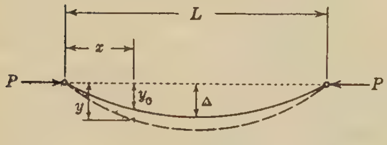

FIG. 5.—INITIALLY CURVED PIN-ENDED COLUMN.

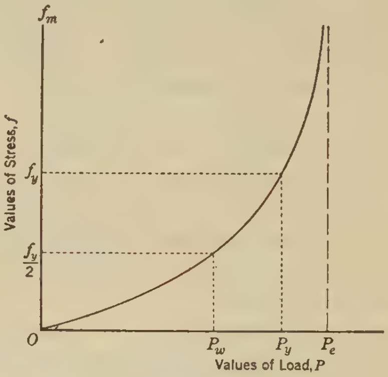

FIG. 6.—NON-LINEAR RELATIONSHIP BETWEEN AXIAL LOAD AND MAXIMUM FIBER STRESS.

Choosing the original of co-ordinates at the left end of the bar, the general relation between curvature and bending moment at any section will be,

$$EI\left(\frac{d^2y}{dx^2} - \frac{d^2y_0}{dx^2}\right) = -Py \quad \text{..................} (3)$$

Equation (3), of course, holds only for a fairly flat deflection curve; also it is based on the assumption that Hooke's law applies. The possibility of buckling perpendicular to the plane of bending is not considered. Substituting the value of $\frac{d^2y_0}{dx^2}$ from Equation (2) and letting the quantity, $\frac{P}{EI} = q^2$, gives:

$$\frac{d^2y}{dx^2} + q^2 y = -\frac{\pi^2}{L^2} \Delta \sin \frac{\pi x}{L} \quad \text{..................} (4)$$

Equation (4) represents the differential equation of the elastic line of the bar and its solution is,

$$y = C_1 \sin q x + C_2 \cos q x + \frac{\Delta \pi^2}{\pi^2 - q^2 L^2} \sin \frac{\pi x}{L} \quad \text{........} (5)$$

in which, C_1 and C_2 are constants to be evaluated from the end conditions of the bar. These constants will be: $C_2 = 0$; and $C_1 \sin q L = 0$. From the latter of these two conditions either $C_1 = 0$, or $\sin q L = 0$. If $\sin q L = 0$

then either $q\,L = 0$, or $q\,L = i\,\pi$, in which, i is any integer. From these two conditions either $P = 0$, or $P = \frac{i^2\,\pi^2\,E\,I}{L^2}$. The condition, $P = 0$, obviously holds no significance. For the second condition, P is seen to represent the Euler load for the bar, in which case the deflection becomes infinite. It follows, then, that for finite deflections, $C_1 = 0$ must be taken. Equation (5) then reduces to,

$$y = \frac{\Delta\,\pi^2}{\pi^2 - q^2\,L^2}\,\sin\frac{\pi\,x}{L} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Equation (6) defines the axis of the bar for any value of the load between $P = 0$ and $P = P_e$. The deflection will be a maximum at the center of the bar. Making $x = \frac{L}{2}$, in Equation (6), gives,

$$y_m = \frac{\Delta\,\pi^2}{\pi^2 - q^2\,L^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Substituting $q^2 = \frac{P}{E\,I}$, and $\frac{\pi^2\,E\,I}{L^2} = P_e$, Equation (7) may be written:

$$y_m = \Delta\left(\frac{1}{1 - \frac{P}{P_e}}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The maximum bending moment, at the center of the bar, will be $P \times y_m$, or,

$$M_m = \Delta\left(\frac{P}{1 - \frac{P}{P_e}}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

The maximum fiber stress, in the outside fiber, on the concave side of the bar will be,

$$f_m = \frac{P}{A} + \frac{M_m}{S} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

in which, S is the section modulus of the cross-section. Substituting the value of M_m into Equation (10) gives,

$$f_m = s\left[1 + \frac{\Delta}{k}\left(\frac{1}{1 - \frac{s}{s_e}}\right)\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

in which, $k = \frac{S}{A}$ is the core radius of the cross-section; $s = \frac{P}{A}$ is the load per unit area or average compressive stress; and $s_e = \frac{P_e}{A} = \frac{\pi^2\,E}{\left(\frac{l}{r}\right)^2}$. The use of the core radius, k, makes possible a simple interpretation of Equation (11). If a short block is compressed by a load, P, applied with an eccentricity, $e = \Delta$, the maximum compressive stress from Equation (10) becomes, $f_m = \frac{P}{A}\left(1 + \frac{\Delta}{k}\right)$. The term, $\frac{\Delta}{k}$, then represents the ratio of the bending stress to the direct compressive stress. The term, $\left(\frac{1}{1 - \frac{s}{s_e}}\right)$, in Equation (11), may now be considered as a magnification factor which shows the effect, on the bending stress, of taking into account the deflections of the bar. The ratio, $\frac{\Delta}{k}$, will be called the eccentricity ratio. Equation (11) gives the maximum fiber stress for any given value of the load, P, and any amount of initial curvature as represented by Δ.

Making $f_m = f_y$ and, at the same time, denoting the corresponding average compressive stress by s_y, Equation (11) may be written:

$$s_y = \frac{f_y}{1 + \frac{\Delta}{k}\left(\frac{1}{1 - \frac{s_y}{s_e}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

which gives the load per unit area first producing a maximum fiber stress equal to the yield point.

If values of f_m computed from Equation (11) are plotted against the load, P, a curve such as that shown in Fig. 6 is obtained. It is seen from this curve that no proportionality exists between load and maximum fiber stress. This means that merely to assume a safe working stress, say, one-half the yield point (Fig. 6), and to use this in Equation (11) to compute the safe working load, is not permissible. Any desired factor of safety, however, may be incorporated directly into Equation (11) as follows: Assume that P_w is the allowable load and n is a factor of safety, such that the maximum stress becomes equal to the yield point when the load becomes equal to $n\,P_w$. If, in Equation (11), the load per unit area, s, is replaced by $n\,s_w$, and, at the same time, f_y is substituted for f_m, Equation (11) becomes:

$$s_w = \frac{f_y}{n + \frac{n\,\Delta}{k}\left(\frac{1}{1 - \frac{n\,s_w}{s_e}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

The value of s_w computed from Equation (13) will always be such that n times this value will produce a maximum fiber stress equal to the yield point, and, hence, the desired factor of safety is realized. The appearance of n in the magnification factor then takes care of the lack of proportionality between load and fiber stress. This manner of incorporating a factor of safety, n, in formulas similar to Equation (13), was proposed by K. S. Zavriev,[7] who also calculated tables of allowable average stresses for various kinds of combined bending and compression of bars.

Curves may also be plotted from Equation (12) for any given value of f_y and various values of the eccentricity ratio, $\frac{\Delta}{k}$. These curves will show the average compressive stress, s_y, as a function of the slenderness-ratio, $\frac{L}{r}$, at which yielding will first begin. Any desired factor of safety, n, may also be obtained by simply dividing the value of s_y, read from the curve, by the desired factor of safety. Such a set of curves for a steel having a yield point of 36 000 lb per sq in., and values of $\frac{\Delta}{k}$ ranging from 0 to 1.0, is shown in Fig. 7.

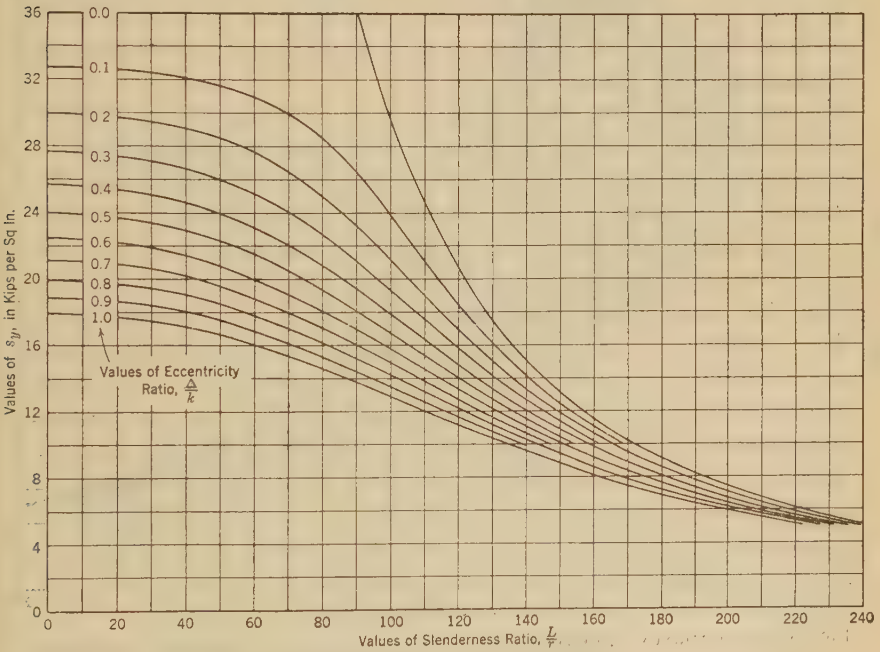

Fig. 7. — Yield-Point Load for Various Eccentricity Ratios; Initially Curved Column.

It will be interesting to compare the values of the average compressive stress that first produce yielding with those[6] that cause buckling (complete failure) of the column, as discussed in Section 1. In Fig. 8, this comparison is made for the same steel, having a yield point of 36 000 lb per sq in., a proportional limit of 30 000 lb per sq in., and values of the eccentricity ratio ranging from 0.0 to 0.9. The curves for buckling apply only for columns of rectangular cross-section. For columns of I-section, the difference between corresponding curves will be less. The solid line curves show the value of average compressive stress at which yielding first begins, whereas the broken lines represent complete buckling. It will be seen from a study of these curves that for large values of $\frac{\Delta}{k}$ (more bending in proportion to direct stress), the buckling load is considerably greater for medium long columns than the yield-point load. As $\frac{\Delta}{k}$ decreases, the difference between corresponding curves becomes less. For values of $\frac{\Delta}{k}$ less than about 0.1, Equation (12) gives values of the load greater than the buckling load. This discrepancy on the side of danger, for nearly perfect columns, is of no consequence, since in the practical design of columns allowance will have to be made for values of $\frac{\Delta}{k}$ greater than 0.1. In general, it may be stated that the difference between the yield-point load and the load for complete collapse is always less for columns than for beams.

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[7] *Bulletin,* Soc. of the Engrs. of Technology, St. Petersburg, 1913.

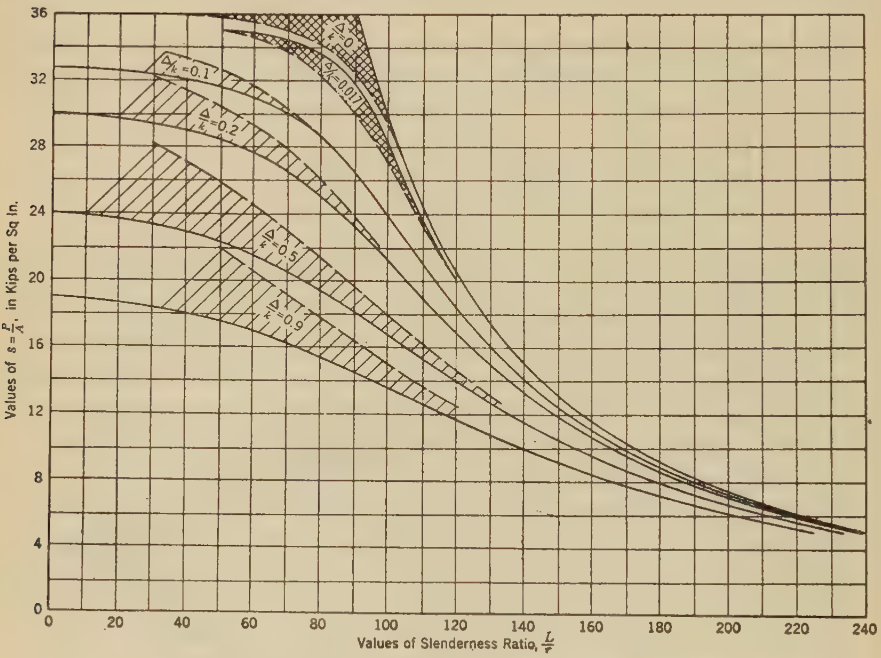

FIG. 8. — COMPARISON BETWEEN YIELD-POINT LOAD AND COMPLETE BUCKLING LOAD; INITIALLY CURVED COLUMN.

Design of a column by the trial and error method may now be made readily by the use of the curves in Fig. 7. The only factor difficult to decide upon is the proper value of the eccentricity ratio, $\frac{\Delta}{k}$, to be used. It has been shown[8] that the effect of any small accidental eccentricity of load on the column can be represented by some definite curvature of the axis. All imperfections of the column are then representable by some "equivalent curvature." Deviation from a straight line, of the axis of the column, due to accidental crookedness undoubtedly increases with the length of the column. It seems logical, therefore, that the eccentricity ratio selected to take care of such imperfections should be some function of the length. The proper function can be determined only on the basis of carefully conducted tests. Professor H. Kayser,[9] by working backward from test results to find the amount of initial curvature which must have been present, found values of Δ ranging from $\frac{L}{400}$ to $\frac{L}{1\,000}$. He recommended the use of $\Delta = \frac{L}{400}$ as a safe allowance for the effect of imperfections. E. H. Salmon[8] has made a study of this question and, on the basis of tests made by a number of experimenters, recommends the use of a value of $\Delta = \frac{L}{750}$.

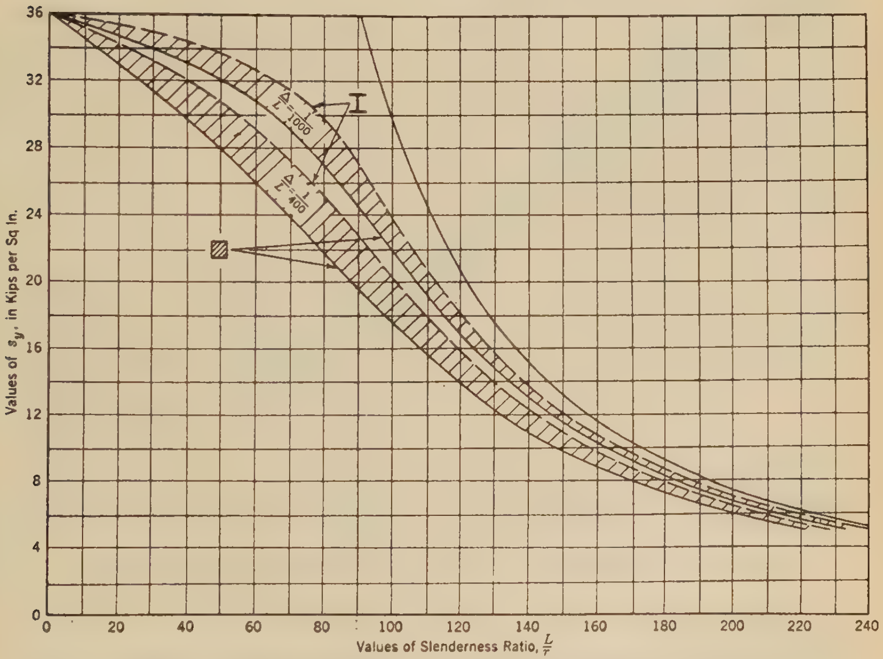

FIG. 9.—PROPOSED DESIGN CURVES FOR PIN-ENDED COLUMNS.

From the curves in Fig. 7 it is possible to derive a single curve for any given relation between Δ and L. For a steel having $f_y = 36\,000$ lb per sq in., and $\Delta = \frac{L}{400}$ and $\Delta = \frac{L}{1\,000}$, four such curves are shown in Fig. 9, for columns of rectangular cross-sections and for columns having extreme I-sections; that is, having all the material concentrated at the radius of gyration from the axis. Granting that a column of I-section is not likely to have any more initial crookedness than one of rectangular cross-section of the same length, it is seen that the I-section is more efficient in this respect.

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[8] "Columns," by E. H. Salmon, Oxford Technical Publication, p. 152.

[9] "Knickversuche mit doppelteiligen Rahmenstaben," *Die Bautechnik*, Vol. 12, Berlin, 1930.

For $\Delta = \frac{L}{400}$ the two curves in Fig. 9, plotted with a factor of safety, $n = 2.25$, are shown in Fig. 10 in comparison with standard American specifications for column design as approved by the Society,[3] the American Railway Engineering Association,[4] and the City of Boston, Mass.[5] (The value, 2.25, is selected to make the result comparable with the usual standard formulas based on a working stress of 16 000 lb per sq in.)

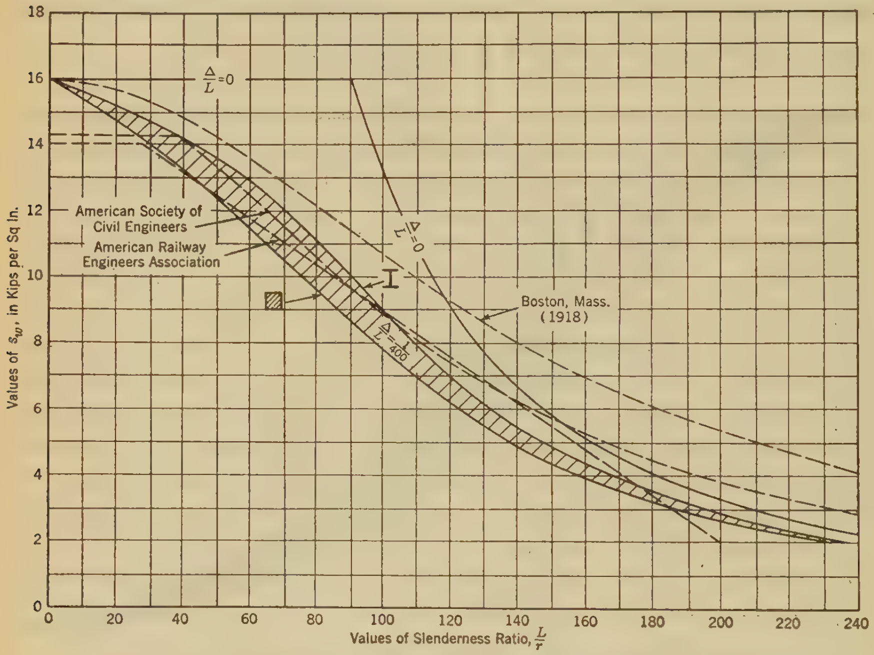

FIG. 10.—PROPOSED DESIGN CURVES COMPARED WITH STANDARD FORMULAS.

3.—IMPERFECTIONS REPRESENTED BY ECCENTRICITY OF LOAD

The effect of slight imperfections in straightness of the axis and central application of load, has been studied in Section 2 on the basis of some definite amount of "equivalent curvature." These imperfections can also be represented by some "equivalent eccentricity" of load on a straight column with a given amount of eccentricity, *e*. The most serious condition (for direct and bending stresses) will be obtained when the eccentricities are on the same side as shown in Fig. 4(*b*). For any value of the load, P, the bar assumes a definite shape and the maximum fiber stress, which occurs at the mid-cross-section, is given by the well-known secant formula which may be written in the form:

$$f_m = s\left[1 + \frac{e}{k}\sec\left(\frac{L}{2r}\sqrt{\frac{s}{E}}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots (14)$$

When the maximum fiber stress is taken equal to the yield point, Equation (14) may be written,

$$s_y = \frac{f_y}{1 + \frac{e}{k}\sec\left(\frac{L}{2r}\sqrt{\frac{s_y}{E}}\right)} \quad \ldots\ldots\ldots\ldots\ldots (15)$$

In the same manner as before, any desired factor of safety, n, can be incorporated in Equation (14) giving:

$$s_w = \frac{f_y}{n + \frac{n\,e}{k}\sec\left(\frac{L}{2r}\sqrt{\frac{n\,s_w}{E}}\right)} \quad \ldots\ldots\ldots\ldots\ldots (16)$$

Curves, almost identical with those plotted from Equation (12) (Fig. 7), may be plotted from Equation (15).

Due to the transcendental function which appears in Equation (15), it is not so readily solved as Equation (12). Approximations of Equation (15) have been suggested, which eliminate the secant function, but this simply reduces it to the same form as Equation (12), which is just another way of stating that the effects of eccentricity and initial curvature on the strength of a column are approximately the same. Since the imperfections which are to be represented by the eccentricity ratio, $\frac{e}{k}$, are of a highly indeterminate nature, there is no particular reason for representing them by "equivalent eccentricity" when they can be equally well represented by "equivalent curvature."

4.—General Case of Eccentric Loading

The General Eccentricity Formula.—The secant formula, discussed in Section 3, is in reality only a special case of eccentric loading in which the eccentricities are equal and on the same side of the axis (Fig. 4(*b*)). In general, a column may be loaded eccentrically, as represented in Fig. 11. In

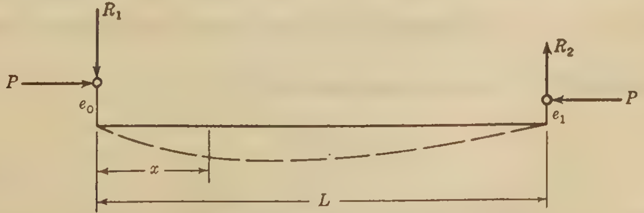

FIG. 11.—GENERAL CASE OF ECCENTRIC LOADING.

the diagram the eccentricity, e_o, at the left end is understood to be the larger of the two and is considered as positive, while e_1, the smaller eccentricity, is considered either positive or negative as it is on the same side as e_o, or on the opposite side.

An analysis of this case shows that as long as the load is less than a certain value, which will be denoted by P_q, the maximum bending moment, and consequently, the maximum fiber stress, occurs at the left end and is simply obtained from the equation (for $P < P_q$):

$$f_m = \frac{P}{A}\left(1 + \frac{e_0}{k}\right) \tag{17}$$

The value of P_q is given by the equation:

$$s_q = \frac{P_q}{A} = \frac{(\cos^{-1}\alpha)^2 E}{\left(\frac{L}{r}\right)^2} \tag{18}$$

in which, $\alpha = \dfrac{e_1}{e_0}$, and may vary from + 1 to − 1. Curves representing s_q as a function of $\dfrac{L}{r}$ for various values of α, can be drawn similar to those shown by broken lines in Fig. 12, which are for $\alpha = + 0.5$ and $\alpha = - 1.0$.

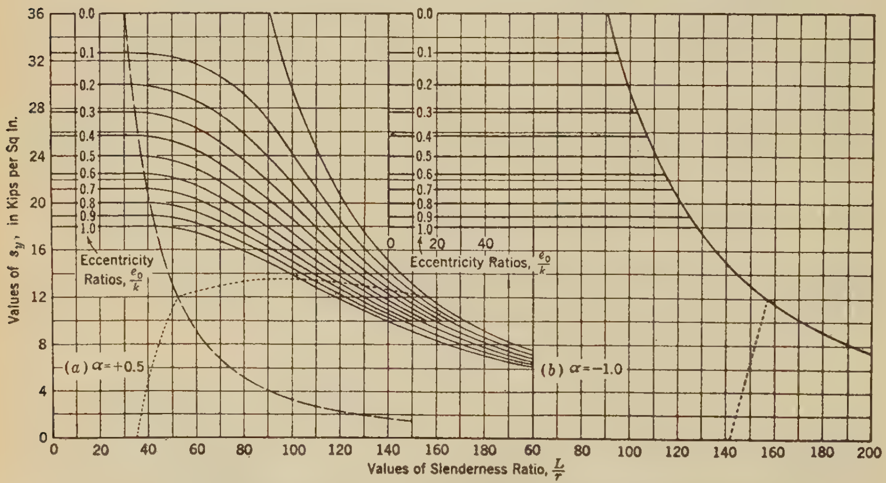

FIG. 12.—YIELD-POINT LOAD FOR VARIOUS ECCENTRICITY RATIOS: (a) ECCENTRICALLY LOADED COLUMN (a) $\alpha = + 0.5$; (b) $\alpha = - 1.0$.

In the case of $\alpha = - 1.0$ (Fig. 12(b)), this curve coincides with the Euler curve.

From Equation (17), may be written (for $P < P_q$):

$$s_y = \frac{f_y}{1 + \frac{e_0}{k}} \tag{19}$$

which serves as a basis for design loads, as long as it gives a value for the average compressive stress less than the value given by Equation (18). Looking at this limiting case in another way, by making s_q in Equation (18) equal to s_y in Equation (19) and solving for the corresponding value of $\left[\frac{L}{r}\right]_q$ gives:

$$\left[\frac{L}{r}\right]_q = (\cos^{-1}\alpha)\sqrt{\frac{E}{f_y}\left(1+\frac{e_0}{k}\right)} \quad\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Thus, as long as the slenderness ratio of the column is less than the value determined by Equation (20), Equation (19) may be used as a basis for design. It is seen that, especially for negative values of α (bending in a reverse curve), the $\frac{L}{r}$ - range within which Equation (19) may be used, is very large.

This much of the analysis, for loading as represented in Fig. 11, was presented by C. N. Ross, in 1927, who applied it to the case of secondary stresses in truss members.[10]

Referring again to Fig. 11: When the load, P, reaches a value greater than P_q, the section of maximum bending moment, and, consequently, of maximum fiber stress, occurs at some intermediate point along the column. As a result of this fact the maximum stress is now a function of the deflection of the axis, and Equation (17) is no longer valid. An analysis of this case[11] leads to the equation (for $P > P_q$):

$$f_m = s\left[1+\frac{e_0}{k}(\psi\csc\phi)\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

in which, $\phi = \frac{L}{r}\sqrt{\frac{s}{E}}$, and $\psi = \sqrt{\alpha^2 - 2\alpha\cos\phi + 1}$. As before, making $f_m = f_y$, and denoting the corresponding value of average compressive stress by s_y, gives (for $P > P_q$):

$$s_y = \frac{f_y}{1+\frac{e_0}{k}(\psi\csc\phi)} \quad\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

which serves as a basis for design loads for all cases to which Equation (19) cannot apply. It will be seen that when $\alpha = +1$ (equal eccentricities on the same side), Equation (22) will reduce to Equation (15).

For various values of $\alpha = \frac{e_1}{e_0}$ curves may be plotted from Equations (19) and (22), which will show the average compressive stress, at which yielding first begins, as a function of the slenderness ratio of the column. For a steel having $f_y = 36\,000$ lb per sq in., and values of $\alpha = +0.5$ and -1.0, characteristic curves are shown in Fig. 12.

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[10] "Equivalent Eccentricities Due to Secondary Stresses," C. N. Ross, *Transactions*, Inst. of Engrs. of Australia, Vol. VIII (1927), p. 8.

[11] See the writer's paper entitled, "Stresses in Eccentrically Loaded Steel Columns," *Publications*, International Assoc. for Bridge and Structural Eng., Vol. 1, Zurich, 1932, p. 507.

For practical purposes, a set of these curves for values of α progressing at intervals of 0.25, between the values of $+ 1.0$ and $- 1.0$, can be drawn by the successive solution of Equations (19) and (22). Results for intermediate values of α can then be interpolated from these curves.

Allowance for the usual imperfections may be made in this case by the use of an "equivalent eccentricity" rather than "equivalent curvature." These accidental factors may occur in any form, and, therefore, the "equivalent eccentricity" should be chosen in the same direction at each end, since this is the most serious type. Using a value of, say, $e = \frac{L}{400}$, this "equivalent eccentricity" can be determined for any given column. Superposing this value of e at each end on the actual eccentricities, e_0 and e_1, will simply give modified values, e'_0 and e'_1, to be used in determining α. This modified value of α will then be used in selecting the yield-point load from the curves such as those in Fig. 12. The design load is obtained by dividing by the desired factor of safety.

Columns in Rigid Frame Construction.—One of the most important cases of column action to the structural engineer occurs in rigid frame construction, examples of which are shown in Fig. 13. Due to the elastic action of other members meeting at a joint, such a column acts intermediately between pin-ended and completely fixed-ended conditions. The determination of the actual condition requires the use of a theory of the stability of a system of compressed bars. Such a theory has been developed by Mises and Ratzersdorfer.[12] Their treatment, however, is based on the assumption that the material always behaves elastically and it does not determine the stresses that may arise, but rather considers only the problem of the elastic stability of such a system. Furthermore, its application to any system except the simplest becomes extremely laborious.

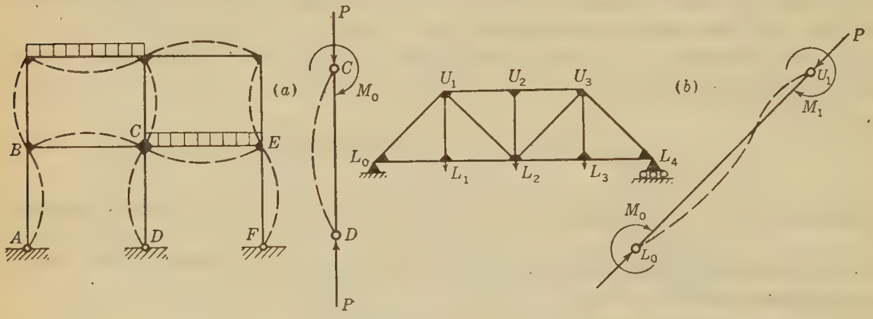

FIG. 13.—THE COLUMN IN RIGID FRAME CONSTRUCTION.

Since most compression members encountered in structural engineering are comparatively short, the maximum fiber stress is usually the governing factor in design rather than the elastic stability. In order to determine the stress that may occur in a column in a truss, consider the member, $L_0\,U_1$, shown in Fig. 13(*b*). In addition to the axial compressive force, *P*, there will be secondary end moments, M_0 and M_1, arising due to the elastic action of the other members.

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[12] "Über die Stabilitätsprobleme der Elastizitätstheorie," von R. v. Mises, *Zeitschrift für angew. Math. u. Mech.*, Vol. 3 (1923), pp. 406–422; also, "Die Knicksicherheit von Rahmentragwerken," von R. v. Mises und J. Ratzersdorfer, *Zeitschrift für angew. Math. u. Mech.*, Vol. 5 (1925), p. 218, and Vol. 6 (1926), p. 181.

An analysis of the maximum bending moment that may occur due to this loading, can be made in exactly the same manner as in the case of the type of loading represented in Fig. 11. The resulting equations for maximum bending moment will be:

$$M_m = M_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

as long as $s < s_q$, in which,

$$s_q = \frac{\left(\cos^{-1}\dfrac{M_1}{M_0}\right)^2 E}{\left(\dfrac{L}{r}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

and,

$$M_m = M_0 \ (\csc\phi) \sqrt{\left(\frac{M_1}{M_0}\right)^2 - 2\,\frac{M_1}{M_0}\cos\phi + 1} \quad \ldots\ldots\ldots(25)$$

when $s > s_q$. In Equation (25), $\phi = \dfrac{L}{r}\sqrt{\dfrac{s}{E}}$, as before.

Using Manderla's exact method for computing secondary end moments, which is well known, the values of M_0 and M_1 can be expressed in terms of the axial force, *P*, for any given truss. In this way the loading that may first produce a maximum fiber stress equal to the yield point of the material can be established. The writer has made such calculations for several simple cases, but the labor involved is considerable even for two-member and three-member frames.

It is the usual practice, in computing secondary end moments, to neglect the effect of the axial load, *P*, on the lateral deflections produced in the bar. It is generally indicated that truss members are usually of such proportion that the effect of the axial force, *P*, in modifying the secondary end moments, is seldom more than 6% and usually much less.[13] In view of this fact, the excessive labor involved in using Manderla's exact method is scarcely considered justifiable.

Neglecting the effect of the axial force, *P*, on the deflections (in computing the values of secondary moments) amounts, however, to assuming that these moments increase in direct proportion to the axial force, *P*; that is, by making this approximation, the effect of the secondary end moments in Fig. 13(*b*) may be represented by the type of loading shown in Fig. 11, in which, e_0 and e_1 will be such that $P \times e_0 = M_0$ and $P \times e_1 = M_1$. Thus, columns in rigid frame construction may be designed on the basis of such curves as those presented in Fig. 12.

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[13] "Statically Indeterminate Stresses," by John I. Parcel and George A. Maney, Members, Am. Soc. C. E., John Wiley & Sons, p. 321.

The procedure will be as follows: From the usual approximate method of calculating secondary end moments, the values of M_0 and M_1 for each member are determined. Dividing each by the axial force, P, the values of e_0 and e_1 are obtained. To make allowance for the effect of initial curvature and imperfections in general these values should be modified, as discussed previously, before computing $\alpha = \frac{e'_1}{e'_0}$ to be used in selecting the average compressive stress, s_y, from the curves (Fig. 12).

In the case of slender members this method of procedure will result in some error (on the side of safety, however). As was mentioned previously, Ross has proposed this same method of procedure for the design of compression members in trusses. He made a study[10] of a large number of existing structures of all types and found that in the great majority of cases the secondary end moments gave values of α such that the member came within the range of Equation (19) rather than Equation (22). Stated in another way, the slenderness ratio of the member was usually less than the value given by Equation (20). In all such cases, of course, the use of the curves (Fig. 12) represents no error at all. In practical cases, when the slenderness ratio is not less than the value determined by Equation (20), the error involved will usually be small and certainly on the side of safety.

PART II.—BUILT-UP STEEL COLUMNS

5.—Effect of Shear in Built-Up Columns

Stability.—Columns, as used in engineering structures, are frequently made of channels laced together by lattice-bars or batten-plates, as shown in Fig. 14. When such a column deflects laterally under load the cross-sections of the bar are no longer perpendicular to the end loads, and shearing forces are introduced. In the case of a column having a solid cross-section, the effect of such shearing forces is negligible, but in the case of built-up columns, the effect of the additional distortion due to shear must be taken into account.

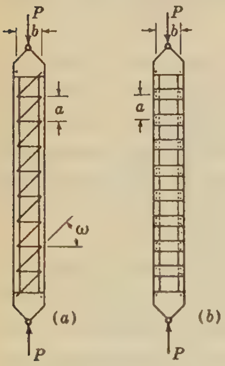

Fig. 14.—Built-Up Columns.

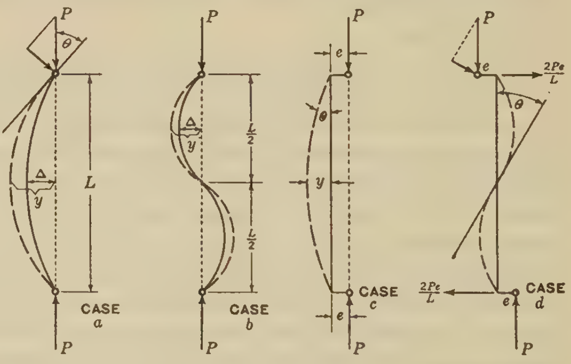

Fig. 15.—Possible Types of Initial Imperfections.

An analysis[14] of built-up columns shows that the critical load at which elastic instability occurs can be represented by the Euler equation in terms of a fictitious column having a length, L', thus:

$$P_c = \frac{\pi^2 E I}{(L')^2} \quad\text{.........................}\quad (26)$$

For the lattice-bar column (Fig. 14(*a*)), the fictitious length is given by the equation:

$$L' = L\sqrt{1 + \frac{\pi^2 E I}{L^2}\left(\frac{1}{\sin\omega\cos^2\omega\, E A_d} + \frac{b}{a E A_b}\right)} \quad\text{.......}\quad (27)$$

in which, L = the true length of the column; ω = the angle between batten-bar and diagonal; A_d = cross-sectional area of two diagonal bars; A_b = cross-sectional area of two batten-bars; a = unsupported length of channels; and, b = distance between gravity axes of channels. For the batten-plate column (Fig. 14 (*b*)), the fictitious length is given by the equation:

$$L' = L\sqrt{1 + \frac{\pi^2 E I}{L^2}\left(\frac{a\,b}{12\, E I_b} + \frac{a^2}{24\, E I_c} + \frac{K_2\, a}{0.4\, b\, A_b G}\right)} \quad\text{......}\quad (28)$$

in which, a = distance center to center of batten-plates; b = distance between gravity axes of channels; I_c = moment of inertia of one channel about its gravity axis; I_b = moment of inertia of two batten-plates, with respect to gravity axis of bending; A_b = cross-sectional area of two batten-plates; K_2 = a coefficient = 1.2 for rectangular batten-plates; and G = shearing modulus for the battens.

The derivation, upon which these fictitious lengths are based, assumes an infinitely large number of panels. Practically, however, their use will give good results for a very limited number of panels. The specifications of the American Railway Engineering Association require that the slenderness ratio of the unsupported length of channel shall not be greater than $\frac{a}{r_c} = 40$. From a series of tests made on batten-plate columns, Kayser[15] concluded that Equation (26) gave results in good agreement with the tests for spacings of batten-plates considerably greater than the foregoing requirement.

A built-up steel column may be designed on a basis of the yield-point load obtained from the upper $\Delta = \frac{L}{400}$ - curve in Fig. 9 by using the fictitious length, L', as determined from Equation (27) or Equation (28), instead of the true length. Of course, the use of this curve, in which the initial curvature to represent the effect of imperfections has been taken as a function of the length, will amount to assuming the imperfections as a function of the fictitious rather than the true length. This will only result in giving a slight additional factor of safety which is probably not out of order for built-up columns.

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[14] "Strength of Materials," S. Timoshenko, Pt. II (Van Nostrand), p. 592.

[15] "Knickversuche mit doppelteiligen Rahmenstaben," von H. Kayser, *Bautechnik*, Vol. 12, 1930.

Effect of Imperfections on Shearing Stresses.—It is present-day practice to design the lacing or batten-bars, empirically, to resist possible shearing forces. For example, the A.R.E.A. Specifications require that the lacing-bars shall be designed to resist shearing forces not less than 0.025 times the total compressive force on the column.

In discussing the design of lattice-bars and battens to resist possible shearing forces, it seems logical to proceed as before and assume, as a basis of computation, an imperfection in the form of initial curvature or eccentricity of load. When this has been selected, it will be possible to evaluate the maximum shearing force that arises for any value of the compressive load, P. This maximum shearing force will then be calculated for the value of the load, P_y, which first causes yielding in the extreme fibers. The shearing force thus calculated will be used as a basis for design. In this way the strength of the column in shear will be consistent with its strength in bending and thrust.

The imperfections should now be taken in the most serious form, as far as shearing forces are concerned. Possible types of such initial imperfections are represented in Fig. 15. Further consideration of some of these initial conditions to represent imperfection can be ruled out.

First, consider the imperfections as represented by the half wave of a sine curve shown as Case *a* in Fig. 15(a). For any value of the load, P, there will be a definite value of the angle, θ, at the end, and the maximum shearing force occurring here will be, $V = P \sin \theta$. In the second case, when the initial curvature is taken as a full sine curve (S-shaped), it will be logical to assume Δ as the same function of $\frac{L}{2}$ instead of L, and each half of the column will be identical with the previous case. However, since the column will buckle in one wave (regardless of its initial shape) at the Euler load for the length, L, this case can never give rise to as great shearing forces as Case *a*. Case *c*, likewise, will never be as serious as Case *a* because, at the load producing failure, the maximum bending moment at the center must be approximately the same for each case. In Case *a* this moment is $P \times y$, while, in Case *c*, it is $P\,(y + e)$; hence, y (and, consequently, θ, at the end) will always be less for the eccentrically loaded column. If θ is less, the shear, $P \sin \theta$, will be less.

Case *d* only, then, needs further consideration. When the eccentricities are chosen on opposite sides of the axis, definite horizontal reactions, $\frac{2\,P\,e}{L}$, arise at the ends. Remembering that θ will be a small angle, it is seen from Case *d*, Fig. 15, that the shearing force will be a maximum at the center cross-section and equal to $\frac{2\,P\,e}{L} + P \sin \theta$. It is possible that this condition may introduce greater shearing forces than will initial curvature. Cases *a* and *d*, in Fig. 15, will now be considered in more detail.

6.—SHEAR DUE TO IMPERFECTIONS REPRESENTED BY INITIAL CURVATURE

Considering Case *a*, Fig. 15, the equation of the elastic line is represented by Equation (6). Taking the first derivative of this equation with respect to x will give the slope at any cross-section as,

$$\frac{dy}{dx} = \frac{\Delta\,\pi^2}{\pi^2 - q^2 L^2}\,\frac{\pi}{L}\cos\frac{\pi\,x}{L} \quad\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

This will be a maximum at the end, where $x = 0$, and hence,

$$\tan\,\theta_m = \frac{\Delta\,\pi^2}{\pi^2 - q^2 L^2}\,\frac{\pi}{L} \quad\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Substituting the values, $q^2 = \frac{P}{EI}$ and $P_e = \frac{\pi^2\,EI}{L^2}$, Equation (30) may be written,

$$\tan\,\theta_m = \frac{\Delta\,\pi}{L}\,\frac{1}{1 - \frac{s}{s_e}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Since the maximum shearing force is $V = P\sin\theta_m = P\tan\theta_m$ (for small angles), Equation (31) may be written:

$$v = \frac{V}{A} = s\,\frac{\Delta\ \pi}{L}\,\frac{1}{1 - \frac{s}{s_e}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

For any given slenderness ratio, $\frac{L}{r}$, the value of s, at which yielding first begins, is given by Equation (12). For a fixed value of $\frac{L}{r}$, this formula may be considered as representing the relation between the average compressive stress, s_y, for yielding, and the amount of initial curvature as represented by Δ. Solving Equation (12) for Δ and substituting the value obtained into Equation (32) gives,

$$v_y = \frac{\pi\,(f_y - s_y)}{\frac{L}{k}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

Equation (33) gives the maximum value of the average shearing stress which the details are called upon to resist when the column is at its yield-point load, and, consequently, represents the desired basis for design.

When $\frac{L}{k} = 0$, $s_y = f_y$, and Equation (33) is seen to become indeterminate. To evaluate v_y for this case, $\frac{\Delta}{L}$ in Equation (32) may be made equal to the chosen ratio (for example, $\frac{\Delta}{L} = \frac{1}{400}$) for which Equation (32) becomes,

$$v = \frac{V}{A} = s\,\frac{\pi}{400}\,\frac{1}{1 - \frac{s}{s_e}} \quad \text{.....................(34)}$$

from which, the value of v_y for $\frac{L}{k} = 0$ is seen to be, $v_y = f_y\,\frac{\pi}{400}$.

It is interesting to note in this connection that while the effects of initial curvature and eccentricity of load on the bending stresses are approximately the same, this is not the case with regard to shearing stresses. Initial curvature is considerably more serious for shear, due to the fact that to begin with there is some definite slope at the end, while in the straight bar, eccentrically loaded, this slope at the end must be built up by the load.

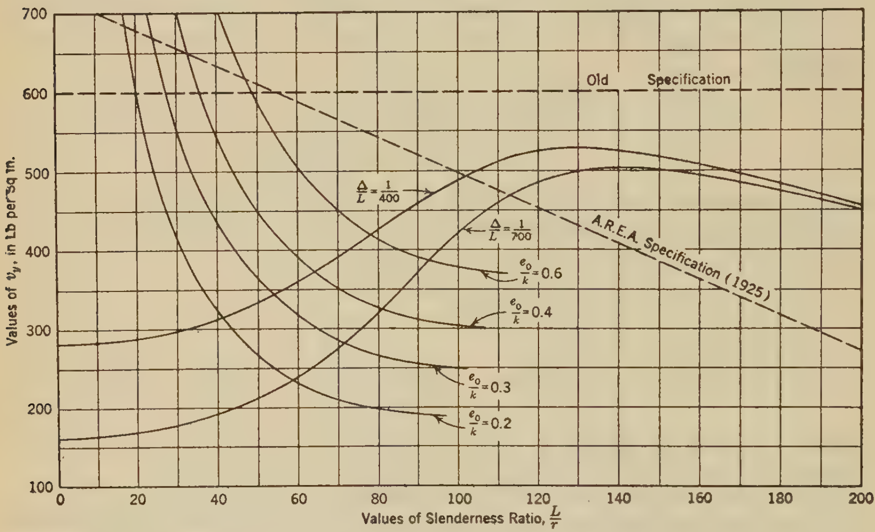

FIG. 16.—AVERAGE SHEAR STRESS DUE TO VARIOUS IMPERFECTIONS.

Taking values of s_y from the curves in Fig. 9, in which $k = r$, values of v_y may be computed from Equation (33) for various values of $\frac{L}{r}$, and any given ratio of $\frac{\Delta}{L}$. In this manner, curves may be plotted showing how the maximum average shearing stress, v_y, varies with the slenderness ratio, $\frac{L}{r}$, for any given ratio of $\frac{\Delta}{L}$. Such curves are shown in Fig. 16 for $\frac{\Delta}{L} = \frac{1}{400}$ and $\frac{\Delta}{L} = \frac{1}{700}$.

From an examination of these curves it is seen that the maximum shear occurs for columns having a slenderness ratio of about 130, which is greater than the usual allowance for slenderness made by specifications. For columns of the usual proportions the maximum average shearing stress varies from about 300 to 550 lb per sq in. in the case of the more severe allowance for imperfections. In Fig. 16, for purposes of comparison, a curve is shown which represents the A.R.E.A. Specifications of 1925 and a curve representing an older A.R.E.A. Specification. It might be judged that the new specification was unsafe for slender columns, but it must be remembered that this same specification limits the slenderness ratio of main compression members to 100 or less.

7.—SHEAR DUE TO IMPERFECTIONS REPRESENTED BY ECCENTRICITY

Considering imperfections represented in the form of eccentricity of load, as shown in Fig. 15 (*d*) and proceeding in the same manner as in the previous case (Section 6), the maximum angle, θ, at the middle cross-section is given by the equation:

$$\tan \theta_m = \frac{e}{L}\left[\frac{qL}{\sin \frac{qL}{2}} - 2\right] \quad \text{..................(35)}$$

The maximum shearing force at the middle cross-section is,

$$V = \frac{2Pe}{L} + P \sin \theta \quad \text{......................(36)}$$

Remembering that, for small angles, $\tan \theta = \sin \theta$, and substituting Equation (35), Equation (36) becomes,

$$V = \frac{Pe}{L}\left[\frac{qL}{\sin \frac{qL}{2}}\right] \quad \text{.......................(37)}$$

which gives the maximum shearing force for any value of the load.

It is interesting to note in connection with Equation (37) that $\frac{2Pe}{L}$ represents the shearing force at all cross-sections if deformation of the bar is neglected. Assuming that $q = \sqrt{\frac{P}{EI}}$ is very small, the term in the brackets in Equation (37) becomes equal to 2. However, as the load, P, is increased, this factor increases until when P reaches the Euler value it becomes equal to π. That is, for slender columns, where the load may reach the Euler value before the maximum fiber stress reaches the yield point, the shearing force may become 57% greater than the value that would be obtained by neglecting deformations.

Considering Equation (37) again, and dividing both sides by A, will give:

$$v = s\,\frac{e}{L}\left(\frac{qL}{\sin \frac{qL}{2}}\right) \quad \text{......................(38)}$$

In this case where the effect of imperfections is being represented by the particular type of eccentricity represented in Fig. 15(d), it will be more reasonable to use a constant value of e than to take it as a function of the length, since for a short column there is just as much chance for accidental eccentricity of load as for a long one.

For this case, $\alpha = -1$, and the value of s at which yielding begins, is given by Equation (19). Solving Equation (19) for the eccentricity, e, and substituting the value obtained into Equation (38) gives,

$$v_y = \frac{f_y - s_y}{\frac{L}{k}} \left(\frac{qL}{\sin \frac{qL}{2}} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Comparing Equation (39) with Equation (33), it is seen that for slender columns they give about the same result. However, for short columns, Equation (39) may give considerably higher values for the average shearing stress.

In Fig. 16 are shown four curves plotted from Equation (39), assuming an extreme I-section ($k = r$), for eccentricity ratios, $\frac{e}{k} = 0.2, 0.3, 0.4$, and 0.6. From a general study of all the curves in Fig. 16, a constant allowance for shearing stresses of 600 lb per sq in., in accordance with the old A.R.E.A. Specifications, would seem to be justifiable.

8.—SHEAR IN THE GENERAL CASE OF ECCENTRIC LOADING

Shear Stress at Yield-Point Load.—From an examination of the curves in Fig. 16 for equal eccentricities on opposite sides of the axis, it is seen that, for short columns, the shear arising due to such loading may be considerable. In the case of short stocky members in rigid frame construction, where large secondary end moments may arise, it seems advisable to know something about the possible extent of such shearing forces. The brief analysis[16] made herein will be for the equivalent case of eccentric loading, as considered in Section 4.

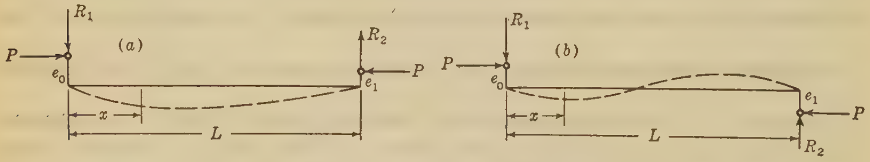

FIG. 17.—GENERAL CASE OF ECCENTRIC LOADING.

Consider the pin-ended column with eccentrically applied end loads, as shown in Fig. 17. The eccentricity, e_o, is taken as the larger so that in Fig. 17(a), $\alpha = \frac{e_1}{e_0} > 0$, and in Fig. 17($b$), $\alpha = \frac{e_1}{e_0} < 0$. By the ordinary procedure the slope of the elastic line at any point, distant x from the left end, is found to be:

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[16] For a detailed analysis, see the writer's paper, "Shearing Stresses in Steel Columns," *Publications*, International Assoc. for Bridge and Structural Eng., Vol. II. Zurich, 1934, p. 480.

$$\frac{dy}{dx} = -q\,e_0 \sin q\,x + q\,e_1 \frac{\cos q\,x}{\sin q\,L} - q\,e_0 \frac{\cos q\,x}{\tan q\,L} + \frac{e_0}{L} - \frac{e_1}{L} \quad \ldots\ldots (40)$$

For $\alpha > 0$ (Fig. 17(a)) the maximum shearing force will occur at the end of the column where $x = L$, and will be:

$$V = \frac{P\,(e_0 - e_1)}{L} + P\left[\frac{dy}{dx}\right]_{x=L} \quad \ldots\ldots (41)$$

For $\alpha < 0$ (Fig. 17(b)), the maximum shearing force will occur at the inflection point some place within the column and will be,

$$V = \frac{P\,(e_0 - e_1)}{L} + P\left[\frac{dy}{dx}\right]_{\max} \quad \ldots\ldots (42)$$

Substituting the values of $\left[\frac{dy}{dx}\right]_{x=L}$ and $\left[\frac{dy}{dx}\right]_{\max}$ into Equations (41) and (42), respectively, and dividing through by A, gives:

For $\alpha > 0$:

$$v = s\,\frac{e_0}{L}\,\phi\,\beta \csc \phi \quad \ldots\ldots (43)$$

and for $\alpha < 0$:

$$v = s\,\frac{e_0}{L}\,\phi\,\psi \csc \phi \quad \ldots\ldots (44)$$

in which, $\phi = q\,L = \frac{L}{r}\sqrt{\frac{s}{E}}$; $\beta = 1 - \alpha \cos \phi$; and $\psi = \sqrt{\alpha^2 - 2\,\alpha \cos \phi + 1}$.

For any given value of $\alpha = \frac{e_1}{e_0}$ the average shearing stress may be computed from Equation (43), or Equation (44), for any values of s and e_0.

It is desired to evaluate this average shearing stress for the particular load that first produces yielding in the extreme fibers. The corresponding values of s_y are given by Equations (19) and (22), in Section 4. Solving these formulas for e_0 and substituting the values obtained into Equations (43) and (44) gives:

$$v_y = \frac{(f_y - s_y)\,F}{\frac{L}{k}} \quad \ldots\ldots (45)$$

in which, F is a numerical factor as follows:

For $\alpha > 0$, and $P < P_q$:

$$F = \phi\,\beta \csc \phi \quad \ldots\ldots (46)$$

For $\alpha > 0$, and $P > P_q$:

$$F = \frac{\phi\,\beta}{\psi} \quad \text{.............................} \quad (47)$$

For $\alpha < 0$, and $P < P_q$:

$$F = \phi\,\psi\,\csc\phi \quad \text{........................} \quad (48)$$

and for $\alpha < 0$, and $P > P_q$:

$$F = \phi \quad \text{..............................} \quad (49)$$

Assuming an extreme I-section ($k = r$), and a steel for which $f_y = 36\,000$ lb per sq in, curves similar to Fig. 18 may be plotted from Equation (45), showing v_y as a function of the slenderness ratio for various values of α and $\frac{e_o}{k}$. Such curves may reasonably be used as a basis for the design of lacing or batten-bars in the case of short compression members in rigid frame construction, by using the secondary end moments, calculated in the usual approximate manner, to determine values of α and e_o.

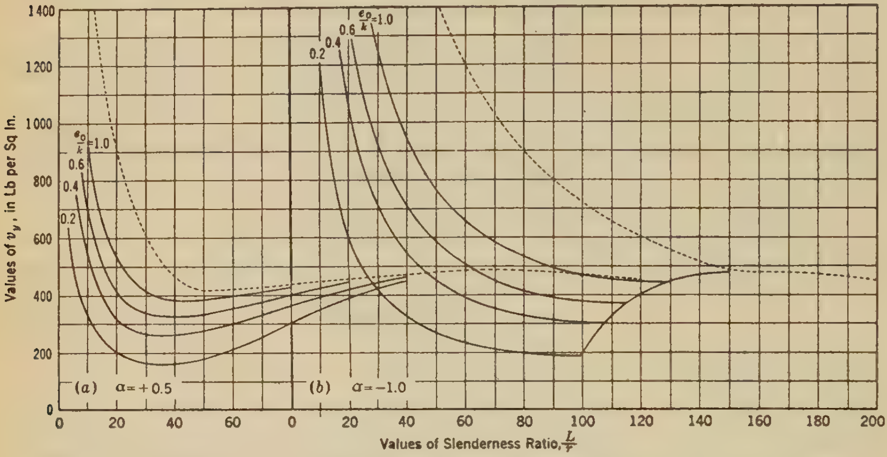

FIG. 18.—AVERAGE SHEAR STRESS DUE TO ECCENTRIC LOADING: (a) $\alpha = +0.5$; (b) $\alpha = -1.0$.

Absolute Maximum of Shear Stress.—From Equation (43) and (44), it is evident that, for a given slenderness ratio, there will be a particular combination of load and eccentricity (to produce yielding) for which the average shearing stress will be an absolute maximum. Since the eccentricity, e_o, has already been expressed in terms of the load in Equation (45), this equation may now be differentiated with respect to s and the derivative set equal to zero, to determine s as a criterion for absolute maximum shear. This procedure yields the following criteria for determining s:

For $\alpha > 0$, and $P < P_q$:

$$\frac{f_y - 3\,s}{f_y - s} = \phi\left[\cot\phi - \frac{\alpha\,\sin\phi}{\beta}\right] \quad \text{................} \quad (50)$$

For $\alpha > 0$, and $P > P_q$:

$$\frac{f_y - 3\,s}{f_y - s}\,\alpha\,\phi\,\sin\,\phi\left[\frac{1}{\psi^2} - \frac{1}{\beta}\right] \quad \text{(51)}$$

For $\alpha < 0$, and $P < P_q$:

$$\frac{f_y - 3\,s}{f_y - s} = \phi\left[\cot\,\phi - \frac{\alpha\,\sin\,\phi}{\psi^2}\right] \quad \text{(52)}$$

and for $\alpha < 0$, and $P > P_q$:

$$s = \frac{f_y}{3} \quad \text{(53)}$$

Equations (50) to (53) may be solved graphically and the resulting s - criteria for absolute maximum shear shown by dotted curves such as those in **Fig. 12.**

When the criteria call for $s = 0$ (Fig. 12) this must be interpreted as meaning an infinitely small axial force at an infinitely large eccentricity. In other words, for short columns, bending by reverse couples at the ends is the worst possible type of loading for shearing stresses. It is interesting to note that for the particular steel chosen ($f_y = 36\,000$ lb per sq in.), these curves intersect the Euler curve at $\frac{L}{r} = 157$. This means that in the case of slender columns the worst possible type of loading for shear is an axial load without any eccentricity. For the intermediate columns the criteria require some combination of thrust and bending to produce the worst conditions for shear.

Using the values of s from these curves in Equation (45), the curves of absolute maximum shear for the various values of α considered, have been plotted in Fig. 18. In each diagram, this curve of absolute maximum shear stress is shown by the dotted line.

The absolute maximum shear runs fairly high, especially for negative values of α. Therefore, in the case of short compression members in rigid frames, it would seem more advisable to use the actual values of e_0 and α obtained by secondary stress calculations in determining the shear stress, than to try to make some blanket allowance for all cases. The usual additional allowance for the effect of imperfections should be made in the same manner as outlined in Section 4. In this case, however, eccentricities on opposite sides of the axis to represent imperfections would be superposed on the actual eccentricities, e_0 and e_1, to give the modified values to be used in determining α.

ACKNOWLEDGMENT

The writer is particularly indebted to Professor S. Timoshenko, of the University of Michigan, for invaluable help and advice in the preparation of this paper.

SUMMARY

Although, in this paper, the writer does not pretend to solve, completely, the problem of column design, he offers what is believed to be a more logical basis of procedure than is represented by present-day practice. It is recommended that, as a basis of calculation, some imperfection in the form of an initial curvature or eccentricity of load be taken, and then that the working load be selected on a basis of the load first producing yielding in the extreme fibers due to the assumed form of imperfection.

For pin-ended columns the final proposal for a basis of design is represented by the curves for $\Delta = \frac{L}{400}$ in Fig. 9. For columns in rigid frame construction, in which secondary end moments arise due to the rigidity of the joints, it is proposed to use a set of curves such as those illustrated by Fig. 12 as a basis of design, in which the values of α and e_o will be determined by the usual approximate method of calculating secondary stresses.

For pin-ended columns, which are built up with lacing or battens, an average shearing stress of 600 lb per sq in. is proposed as a basis for the design of the lacing or battens.

For built-up columns in rigid frame construction, it is proposed to use curves such as as those presented in Fig. 18, as a basis for the design of lacing and battens, in which the values of α and e_0 will be found from the secondary stress calculations.

APPENDIX

NOTATION

The following symbols conform as nearly as practicable with the "Symbols for Mechanics, Structural Engineering, and Testing Materials", advanced by the American Standards Association[4]. Some of the symbols do not appear in the paper, but are included herein, as a matter of information:

a = distance, center to center, of column stiffeners (batten-plates or lattice-bars); also, a = unsupported length of channels. As a subscript, a denotes "average".

b = distance between gravity axes of channels; as a subscript, b denotes "batten-bars".

c = distance from neutral axis to extreme fiber; as a subscript, c denotes "critical".

d = depth; as a subscript, d denotes "diagonal bars".

e = initial eccentricity of load; as a subscript, e denotes "Euler".

f = unit stress; f_y = yield-point stress.

i = any integer; as a subscript, i denotes 'initial".

k = core radius of a cross-section $= \frac{S}{A}$.

m = Poisson's ratio; as a subscript, m denotes "maximum".

n = a factor of safety.

q = a substitution factor $= \sqrt{\frac{P}{EI}}$; as a subscript, q denotes "a certain value".

r = radius of gyration $= \sqrt{\frac{I}{A}}$.

s = $\frac{P}{A}$ = average compressive stress on the column; s_y = average compressive stress at which yielding in the extreme fibers first begins; s_w = allowable working stress; $s_e = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$ = stress corresponding to the Euler load; and s_q = a certain value of s.

t = thickness.

v = average shearing stress, $\frac{V}{A}$; v_y = average shearing stress when Load P is at its yield-point value.

w = load per unit distance; as a subscript, w denotes "working load".

x = variable co-ordinate of Deflection y.

y = deflection of a column, distant x from one end; y_o = initial deflection of a deflected column; as a subscript, y denotes "yield point".

A = area; A_d = cross-sectional area of two diagonal bars; A_b = cross-sectional area of two batten-bars.

C = a constant to be evaluated from the end conditions of a bar.

E = modulus of elasticity.

F = a numerical factor (Equation (45)).

G = shearing modulus for batten-plates.

I = rectangular moment of inertia with respect to gravity axis of bending; $I_c = I$ for one channel, about its gravity axis; $I_b = I$ of two batten-plates with respect to gravity axis of bending.

K = a coefficient = 1.2 for rectangular batten-plates.

L = length; L' = a fictitious column length.

M = bending moment; M_o and M_1 = secondary end moments.

P = compressive axial load on a column; P_e = the Euler load for a pin-ended column $= \frac{\pi^2 EI}{L}$; P_c = critical load for any column; P_y = load that first produces yielding, due to eccentricity; P_w = allowable working load.

S = section modulus.

V = total shear force at any cross-section.

W = total load.

α = a ratio, $\frac{e_1}{e_0}$.

$\beta = 1 - \alpha \cos \phi$.

Δ = maximum initial deflection of an initially curved column.

θ = an angle (Fig. 16).

π = 3.1416.

$\phi = q\,L = \frac{L}{r}\sqrt{\frac{s}{E}}$.

$\psi = \sqrt{\alpha^2 - 2\,\alpha \cos \phi + 1}$.

ω = angle between batten-bar and diagonal bar.

DISCUSSION

William R. Osgood,[17] M. Am. Soc. C. E. (by letter).—Since the design of compression members which would fail by buckling, if they failed, is admittedly the element in structural design least satisfactory to handle, any paper that again brings to the attention of engineers the intrinsic difficulties involved is worth while.

Mr. Young does well to call attention to the collection of column curves in the A. I. S. C. Manual,[3] for hardly a better exhibit exists showing the (unavoidable) uncertainty of engineers concerning the strength of columns used in ordinary structures. The very existence of such a large "factor of uncertainty" among qualified engineers calls for the assumption of a correspondingly large factor of safety.

It is well known that the strength of medium-long columns is affected markedly by small eccentricities or initial curvatures. Mr. Young is concerned chiefly with such accidental departures from ideal conditions, and he proposes, as has been done time and again, to assume values for these quantities, which are quite unknown in an actual structure, and then to design on the basis of maximum fiber stress. This procedure, as has also been pointed out before, is merely a method of applying a variable factor of safety. It is not a very good method because it results in excessive safety for short columns, as Fig. 8 indicates, and it makes the H-section appear stronger than the rectangular section, as Figs. 9 and 10 indicate, whereas the H-section is actually the weakest of all symmetrical cross-sections.[18] The inherently safest columns so far as the effects of accidental departures from ideal conditions are concerned are, on the one hand, very short columns in which, if failure occurs, the material itself fails, and buckling plays no part, and, on the other hand, very long columns which, if they fail, fail by pure buckling, and in which high strength of the material is of no consequence.

It cannot be emphasized too strongly that the strengths of columns, except very short ones, do not depend on any one particular stress, such as the yield point, or any other. The strength depends on the compressive stress-strain diagram as the double-modulus theory of column action shows.

Mr. Young speaks of the proportional limit in the same breath with yield point and ultimate strength, yet while the last two stresses are readily, and fairly definitely, determinable, the proportional limit may be obtained as any stress from almost zero to nearly the yield point, depending on who does it and how he does it. For practical purposes in the design of columns the highest stress for which Euler's formula applies may be considered as the proportional limit. This stress will be well above a proportional limit carefully determined from a difference curve but also well below the yield point of most structural steels. It is entirely unjustifiable, therefore, to assume the ideal stress-strain diagram of Fig. 2 in explaining the behavior of columns, except possibly at large values of the eccentricity ratio. The whole problem of column action is indeed simplified on the assumption of such a diagram; in fact, there is little left, for it is precisely in the range of stresses between the highest stress for which Euler's formula is applicable and the yield point that the action of columns is most erratic.

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[17] Materials Testing Engr., National Bureau of Standards, Washington, D. C.

[18] *Civil Engineering*, March, 1935, p. 173.

ALFRED S. NILES,[19] ASSOC. M. AM. SOC. C. E. (by letter).—The formulas developed in Section 4 to determine the maximum stress in cases of eccentric loading are not as convenient for practical use as the equivalent ones that can be derived from the same basic theory. In the case of a member with axial compression and end moments, but no transverse load, the distance, x_m, from the left end of the span to the point of maximum moment is given by the formula[20] (using the author's notation):

$$\tan q\, x_m = \frac{e_1 - e_0 \cos \phi}{e_0 \sin \phi} \qquad \text{(54)}$$

Equation (54) gives the location at which the slope of the moment curve is zero. If $e_0 > e_1$ and $\tan q\, x$ is negative, this point will be off the span, and the true maximum moment will be that at the left support. If $\tan q\, x$ is positive, the maximum moment will be on the span, and its magnitude will be:

$$M_m = \frac{P\, e_0}{\cos q\, x_m} \qquad \text{(55)}$$

In practice, it is quite as easy to apply the criterion of the sign of $\tan q\, x_m$ as to use Equation (18) or Equation (24). If it is then found that the maximum moment is in the span, it is easier to find the secant or cosine of an angle, the tangent of which is known, than to compute the quantity, ψ, which appears in Equations (21) and (25). If this line of attack is followed, Equation (25) would be superseded by,

$$M_m = M_0 \sec q\, x_m = \frac{M_0}{\cos q\, x_m} \qquad \text{(56)}$$

and Equation (22) by,

$$s_y = \frac{f_y}{1 + \dfrac{e_0}{k \cos q\, x_m}} \qquad \text{(57)}$$

The forms with the cosine are usually more convenient than those with the secant because tables of the former function are more common. If the cosine, $q\, x_m$, is very small, however, it is better to compute the secant directly from the relation, $\sec^2 x = \tan^2 x + 1$.

Just as the computation for maximum moment can be simplified by making the determination of the location of that moment a step in the computation, so can the computation of maximum shear. The expression for the total shear at any point distant x from the left end of an eccentrically loaded strut can be written,[20]

$$V = P \left[\frac{e_1 - e_0 \cos \phi}{\sin \phi}\, q \cos q\, x - e_0\, q \sin q\, x \right] \qquad \text{(58)}$$

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[19] Prof. of Aeronautic Eng., Stanford Univ., Stanford University, Calif.

[20] "Airplane Structures," by Niles and Newell, John Wiley & Sons, 1929, p. 201.

The slope of the shear curve will be zero when,

$$\cot q\,x_s = -\frac{e_1 - e_0 \cos\phi}{e_0 \sin\phi} \qquad (59)$$

When $\alpha = \dfrac{e_1}{e_0}$ is positive, Equation (59) will indicate a location that is off the span, and the maximum shear will be at the right-hand end of the span (assuming $e_1 < e_0$). Its magnitude divided by the area of the section will be,

$$v = \frac{V}{A} = s\,e_0\,q\,(1 - \alpha\cos\phi)\csc\phi = s\,e_0\,q\,\beta\csc\phi \qquad (60)$$

When α is negative, the shear stress at the point of inflection is given by,

$$v = s\,q\,e_0 \csc q\,x \qquad (61)$$

Equations (60 and (61) can be proved to be identical with Equations (43) and (44), and are much easier to use.

Continuing along the same lines, Equations (45) to (49), inclusive, can be replaced by,

$$v_y = k\,q\,(f_y - s_y)\,F \qquad (62)$$

in which, F is a numerical factor, as follows: For $\alpha > 0$ and $M_{\max}$ at $x = 0$:

$$F = (1 - \alpha\cos\phi)\csc\phi = \beta\csc\phi \qquad (63)$$

for $\alpha > 0$ and $M_{\max}$ at $x = x_m$:

$$F = (1 - \alpha\cos\phi)\csc\phi\cos q\,x_m = \beta\csc\phi\cos q\,x_m \qquad (64)$$

for $\alpha < 0$ and $M_{\max}$ at $x = 0$:

$$F = \csc q\,x_s = \sec q\,x_m \qquad (65)$$

and, for $\alpha < 0$ and $M_{\max}$ at $x = x_m$:

$$F = 1 \qquad (66)$$

It is to be noted that Equation (63) is a special case of Equation (64) for $x_m = 0$. The criteria for absolute maximum shear represented by Equations (50) and (53) remain unchanged, but Equations (51) and (52) may be replaced by,

$$\frac{f_y - 3\,s}{f_y - s} = \phi\left[\cot\phi - \frac{\alpha\sin\phi}{\beta}\right] + q\,x_m \tan q\,x_m \qquad (67)$$

and,

$$\frac{f_y - 3\,s}{f_y - s} = -\,q\,x_m \tan q\,x_m \qquad (68)$$

Equations (54) to (68) would be more convenient for plotting curves such as those of Figs. 12, 16, and 18, than the corresponding equations in the author's paper. The first step in the analysis of an eccentrically loaded strut would be the determination of the location of the point of zero slope of the moment curve from Equation (54). The distance from this point to the point of inflection, multiplied by q, will be $\frac{\pi}{2}$. It can then be seen which of the four cases is represented. Some saving of time can probably be obtained by taking advantage of the relations between Equations (63) and (64) and Equations (50), (67), and (68).

Similar formulas for locating the point of maximum moment, etc., can easily be derived for cases of axially loaded beams. Formulas for the total shear, bending moment, slope, and deflection for single span, and continuous beams of this type, with various types of transverse load, based on the work of Müller-Breslau, have been published by the writer.[21]

In the text following Equation (22), the author recommends the addition of assumed eccentricities to allow for imperfection of manufacture. When the known eccentricity, e_1, is of the same sign as e_0, it is obvious that assumed eccentricities should be added. The author might have brought out more clearly that when e_1 and e_0 are of opposite sign the assumed eccentricities should be added algebraically rather than arithmetically. Only in this way will the assumed value of the distance from the left end to the point of inflection be increased, as is necessary if the correction is to be on the conservative side.

Although some of the author's formulas could be simplified, his basic ideas and his developments from them are sound, making his paper a valuable contribution to the literature of column theory.

J. F. BAKER,[22] ASSOC. M. AM. SOC. C. E. (by letter).—Many of the methods used to-day for the design of compression members are so unsatisfactory that any paper advocating a rational approach should be carefully studied by all structural engineers. Mr. Young's attack of the problem is most sound and he may be interested to know that certain of his suggestions have already been put into practice.

His proposal to take a definite initial curvature of the axis of the member to represent all imperfections, find the load that first produces the yield stress, and then to divide this load by a constant factor to arrive at the working load was used in deriving the strut formula contained in the Steel Structures Research Committee's Recommendations for a Code of Practice for the Use of Structural Steel in Buildings, published in 1931.[23] These recommendations were adopted by the London County Council and also by the British Standards Institution, the formula being included in the London County Council's revised Code of Practice, for the design of buildings erected in London, which came into force on February 16, 1932, and in the British Standard Specification (No. 449) for the Use of Structural Steel in Buildings published in April, 1932. Thus, for the last three years almost all stanchions of steel framed buildings erected in Great Britain and many other parts of the world have been designed on this rational basis. The formula,

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[21] "Airplane Structures," by Niles and Newell, 1929, pp. 185–237.

[22] Prof. of Civ. Eng., Univ. of Bristol, Bristol; and Technical Officer, Steel Structures Research Comm., London, England.

[23] First Rept., Steel Structures Research Committee, Lond., H. M. Stationery Office, London, England, 1931.

$$n\,f = \frac{f_y + (\eta + 1)\,\epsilon_e}{2} - \sqrt{\frac{f_y + (\eta + 1)\,\epsilon_e}{2} - f_y\,\epsilon_e} \quad \ldots\ldots\ldots(69)$$

in which (following the notation of the paper), f = working stress, in tons per square inch of gross section; n = factor of safety = 2.36; f_y = yield-point stress = 18 tons per sq in.; ϵ_e = the Eulerian value, $\frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$, in tons per square inch; and $\eta = 0.003\,\frac{L}{r}$. Equation (69) is merely a convenient form of the Perry formula[24] for the maximum stress in an initially curved, pin-ended, compression member which has been derived by Mr. Young as Equation (13). The measure of the initial curvature, η, was adopted as a result of Professor Andrew Robertson's study[25] of a large number of tests on struts, whereas the factor, 2.36, was chosen from a consideration of existing strut formulas.

The Recommendations for a Code of Practice were published shortly after the Committee's inception and were only tentative. As the work involved was very lengthy, it was found impossible at the time to include a rational rule for the design of stanchions subjected to end bending moments. It would have been useless to provide the designer of a steel framed building with a strut formula that necessitated the accurate determination of the end moments since in the state of knowledge then existing it was impossible to estimate, with any accuracy, the moments transmitted from the beams through the steelwork connections. As Mr. Young states, the labor involved is considerable, but in the four years since 1931 a number of investigators working for the Steel Structures Research Committee have concentrated on the problem and their work has been under consideration for some months. While it is not possible to state exactly what the Committee's final findings will be, it is probable that design curves will be produced having the same basis as set forth by Mr. Young and applicable to stanchion lengths subjected to end moments.

Some indication of the argument which is being followed can be obtained from a paper[26] in the Committee's interim report. In that paper a method is given of dealing with an initially curved strut subjected to unequal end moments, which covers much the same ground as that of Section 4 in Mr.

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[24] "On Struts," by Ayrton and Perry, *The Engineer*, Lond., Vol. LXI, 1886.

[25] "The Strength of Struts," by A. Robertson, *Selected Engineering Paper No. 28*, Inst. C. E., Lond., 1925.

[26] "A Note on the Effective Length of a Pillar Forming Part of a Continuous Member in a Building Frame," by J. F. Baker, Assoc. M. Am. Soc. C. E., Second Rept. of the Steel Structures Research Committee, Lond., H. M. Stationery Office, 1934.

Young's paper. The method is interesting in that it uses a graphical construction due to Mr. H. B. Howard.[27]

For example, in Fig. 19, draw Lines OA and OB through the pole, O, in such a manner as to make equal angles, $\theta = \frac{L}{2}\sqrt{\frac{P}{EI}}$, with the center line, OZ. These lines are made to represent M_a and M_b to some definite scale, being measured upward from O for positive moments. A circle, $BXAO$ (Fig. 19), is constructed through Points B, O, and A. The bending moment at Point N on the beam, distant x from the center line, is then represented by the vector, ON, which is drawn so that $< NOZ = x\sqrt{\frac{P}{EI}}$. The maximum bending moment in the beam is given by the length of the diameter, OX.

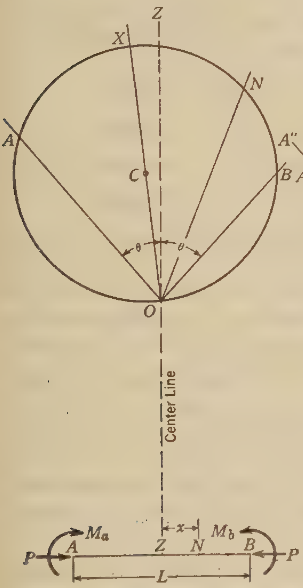

FIG. 19.

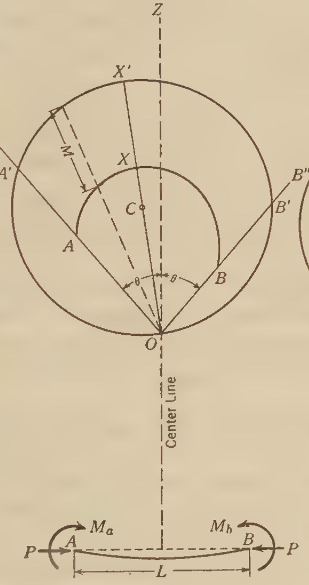

FIG. 20.

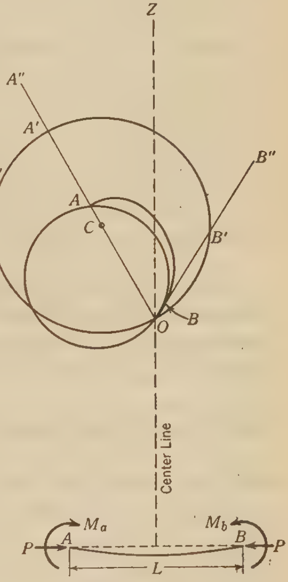

FIG. 21

Before this construction can be used for a practical column some method of allowing for the initial curvature representing the imperfections of the member must be found. It is most usual to assume the initial shape to be that of a sine curve; there is nothing axiomatic, however, about that curve, and the stresses in the column would not be seriously affected by a slight change in its initial shape. Table 1 gives a comparison between the sine curve,

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[27] "The Graphical and Analytical Determination of Stresses in Single Span and Continuous Beams Under End Compression and Lateral Load with Variations in Shear, Distributed Load, and Moment of Inertia," by H. B. Howard, Reports and *Memoranda No. 1233,* Aeronautical Research Committee, Lond., H. M. Stationery Office, 1929.

$$y = 0.003 \frac{L\,r}{a} \sin \frac{\pi\,x}{L} \quad \dots\dots\dots\dots\dots\dots\dots (70)$$

which is the initial curvature assumed in the Committee's formula (Equation (69)) and the shape assumed by an originally straight beam when subjected to a uniform moment, M, such that,

$$M = 0.024 \frac{r\,E\,I}{a\,L} \quad \dots\dots\dots\dots\dots\dots\dots\dots (71)$$

in which, a is the half width of the column and I is its moment of inertia.

TABLE 1.—Comparison of Shapes for Initial Curvature of Column

x		$y \frac{a}{L\,r}$	
		Sine curve, Equation (70)	Beam under uniform moment, Equation (71)
0	L	0	0
$\frac{L}{8}$	$\frac{7\,L}{8}$	0.00115	0.00131
$\frac{L}{4}$	$\frac{3\,L}{4}$	0.00212	0.00225
$\frac{3\,L}{8}$	$\frac{5\,L}{8}$	0.00280	0.00282
$\frac{L}{2}$	.....	0.00300	0.00300

It is clear that no serious error will arise if this latter shape is taken to represent the initial curvature of the column. Therefore, a graph may be constructed for the approximate bending moment at any point of an initially curved column subjected to end moments, M_a and M_b, as follows: Lines OA'' and OB'', Fig. 20, are drawn so that they make equal angles, θ $\left(= \frac{L}{2}\sqrt{\frac{P}{E\,I}}\right)$, with the center line, OZ. Along these two lines, OA' and OB' are measured in such a manner that $OA' = M + M_a$ and $OB' = M + M_b$, in which, $M \left(= 0.024 \frac{r\,E\,I}{a\,L}\right)$ is the uniform moment necessary to produce the initial curvature in the column. The circle, $A'\ X'\ B'\ O$, in Fig. 20 gives the bending moment in an initially straight column subjected to end moments, $(M + M_a)$ and $(M + M_b)$; the bending moment required at any point of the initially curved column can be read from the figure, AXB, obtained from the circle, $A'\ X'\ B'$, by subtracting the moment, M, from each vector. If Line OX' is a diameter of the circle, $A'\ X'\ B'\ O$, the maximum bending moment in the initially curved column will be given by the length of the intercept, OX.

It will be seen from Figs. 20 and 21 that when the center, C, of the circle, $A'OB'$, lies on Line OA'', or on Line OB'', or anywhere on the opposite sides of those lines from the position shown in Fig. 20, the maximum bending moment will be found at one end or other of the column. When this occurs the end moments will have completely masked the "strut action" and there seems to be no reason why the permissible stress in the column should be cut down below the value, 8 tons per sq in., allowed in a beam. The critical values of the end moments which make C lie on Line OA'' or on Line OB'' are, therefore, of considerable importance.

The critical values are given by,

$$\frac{M_b + M}{M_a + M} = \cos 2\,\theta \qquad (72)$$

in which, M_a and M_b are the applied end moments and,

$$M = 0.024\,\frac{r\,E\,I}{a\,L} \qquad (73)$$

The maximum bending moment in a column length, therefore, will occur at the end where the greatest end moment is applied when such moments have values defined by Equation (72); or when the greater end moment is M_a and the smaller is less than M_b; or, when the smaller end moment is M_b and the greater exceeds M_a.

In certain lengths of continuous columns a simple relation between the end moments may be assumed. For instance, in any length of an outside stanchion, except the bottom and topmost, the end moments due to dead load or to the worst combination of live loading are nearly equal in magnitude and opposite in sign, the sign convention being that of Figs. 19, 20, and 21. The critical value of the end moment is, therefore,

$$M_c = \frac{M\,(1 - \cos 2\,\theta)}{(1 + \cos 2\,\theta)} \qquad (74)$$

and it is a simple matter to calculate for any length of outside stanchion, having its axis, YY, in the plane of the outside wall and subjected to any axial load, the minimum value of the end moment that will ensure the maximum moment occurring at the end.

An examination of the measured stresses in existing steel framed buildings has shown that in many cases the maximum stress occurs at an end section of the column and, therefore, the permissible stress could be taken at the full value allowed in beams.

The foregoing analysis and that given by Mr. Young only refers to end moments which bend the member in the plane containing the initial curvature. The great majority of stanchion lengths have end moments about both principal axes, and it is essential that these should be taken into account in a practical design method. For a member bent into single curvature by end moments about both axes, all the information required is contained in Equation (75), in which the yield stress of the material is taken as 18 tons per sq in., the factor of safety is 2.36, and the allowance for initial curvature is the same as in Equation (69),

$$
\begin{aligned}
(7.63 - P)\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right]\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{ry}\right)^2\right] \\
= f_{x\text{-}x}\left[1 + \frac{0.61\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right]\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{ry}\right)^2\right] \\
+ f_{y\text{-}y}\left[1 + \frac{0.61\,P}{\pi^2 E}\left(\frac{L}{ry}\right)^2\right]\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right] \\
+ 0.003\,P\left(\frac{L}{ry}\right)\left[1 - \frac{2.36\,P}{\pi^2 E}\left(\frac{L}{rx}\right)^2\right] \quad \ldots\ldots\ldots\ldots (75)
\end{aligned}
$$

in which, P = axial load, in tons per square inch; $f_{x\text{-}x}$ = end bending stress about the major axis, XX, in tons per square inch; and $f_{y\text{-}y}$ = end bending stress about the minor axis, YY, in tons per square inch.

From this case, design curves can be produced with ease, but the procedure is somewhat more complicated than that usual in design offices to-day. Where this complication must be avoided it can be proved that safety in design is assured in all cases if a bending stress equal to the sum of the bending stresses, $f_{x\text{-}x} + f_{y\text{-}y}$, about both axes is assumed to be due to bending about the minor axis, YY, and the simpler design curves corresponding to that case are used. Considerable economy is sacrificed, however, by this simplification.

Mr. Young has mentioned the effect of the axial load in the member on the end moments. In steel framed buildings this effect of the axial load can be considered as arising from two causes: (*a*) The flexure brought about by loading the beams; and (*b*), the initial curvature. As far as the production of a safe design formula is concerned Cause (*a*) may be neglected in all cases; Cause (*b*) may be neglected when the column is bent by the beams in single curvature; but it must be taken into account when the bending is in double curvature. With the conservative allowance of initial curvature which must be made to represent the imperfections of the member, the effect may be considerably more than the 6% mentioned by Mr. Young for the modification of secondary end moments in trusses.

K. L. De Blois,[28] Assoc. M. Am. Soc. C. E. (by letter).—The subject of a rational design for steel columns, of course, is not a new one. A glance through engineering literature will show the attempt to derive a formula which is theoretical and at the same time not too difficult to apply from the designer's point of view. Mr. Young's derivation follows closely that of the well-known secant formula except that he assumes that the eccentricity is represented by an equivalent curvature, whereas the secant formula is based on an initial end eccentricity.

The author points out a serious defect in the column formulas of the straight-line, parabolic, and Rankine types in general use to-day. This defect is that the relation between the axial load and the maximum fiber stress is non-linear as shown in Fig. 6. In other words, if f_m is 30 000 lb per sq in., s_y is not 1.5 times the value of s_y when f_m is 20 000 lb per sq in. This is due to the change in eccentricity as P varies. The secant formula, Equation (14), with its transcendental function takes this into account as does the author's Equation (12). This means that if tension members of a structure were designed for a certain loading and a unit stress of 18 000 lb per sq in., they would all become stressed to 26 000 lb per sq in. under a 50% increase in load; whereas, in a compression member, if the load were increased 50% the unit stress would be greater than 26 000 lb per sq in. This is important because many bridge design specifications require that all truss members shall be uniformly stressed under a certain increase in the live load.[29]

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[28] Assoc. Highway Bridge Engr., U. S. Bureau of Public Roads, San Francisco, Calif.

The term, $\frac{e}{k}$, in Equation (14) can be written $\frac{e\,c}{r^2}$ since $k = \frac{S}{A} = \frac{I}{cA} = \frac{A\,r^2}{cA} = \frac{r^2}{c}$. The formula is then in its usual form,

$$s = \frac{f_m}{1 + \frac{e\,c}{r^2} \sec\left(\frac{L}{2r}\sqrt{\frac{s}{E}}\right)} \quad \text{..................(76)}$$

By substituting the term, $\left(1 + \frac{\left(\frac{L}{r}\right)^2}{X\left(\frac{c}{r}\right)}\right)$, for $\sec\left(\frac{L}{2r}\sqrt{\frac{s}{E}}\right)$, the resulting formula (termed the "Modified Secant Formula") follows the secant formula curve closely for values of $\frac{L}{r}$ less than 200[29]; thus:

$$s = \frac{f_m}{1 + \frac{e}{r}\left(\left(\frac{L}{r}\right)^2 \frac{1}{X} + \frac{c}{r}\right)} \quad \text{..................(77)}$$

in which, X is a constant, its value depending on f_m. In railroad work a bridge is considered ready for renewal when f_m becomes 26 000 lb per sq in. Using this value of f_m in the secant formula (Equation (76)), the modified secant formula is,

$$s = \frac{26\,000}{1 + \frac{e}{r}\left(\left(\frac{L}{r}\right)^2 \frac{1}{6\,400} + \frac{c}{r}\right)} \quad \text{..................(78)}$$

If the working stress, f_w, is 20 000 lb per sq in., Equations (76) and (78) would be divided by 1.3 to get the design unit stress. This gives slightly larger sections than by using f_m of 20 000 lb per sq in. in Equation (76), but the compression members would then be stressed consistently with the tension members when the load is increased 1.3 times. Thus, Equations (77) and (78) need not be solved by "trial and error" as in the case of the author's equations. It must be understood, however, that the non-linear relation between f_m and s is attained by varying the value of X in Equation (77) to fit the secant formula, (Equation (76), as different values are used for f_m. For design purposes where one or two values are used for f_m, Equations (77) and (78), providing a direct solution, are more desirable. However, in research and investigation work where different values of f_m are encountered, Equation (12) is certainly easier to solve than the secant formula, Equation (76).

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[29] See "A Simplified Column Formula of the Secant Type," by J. B. Hunley, *Bulletin*, A.R.E.A., Vol. 29, No. 300, October, 1927, p. 197.

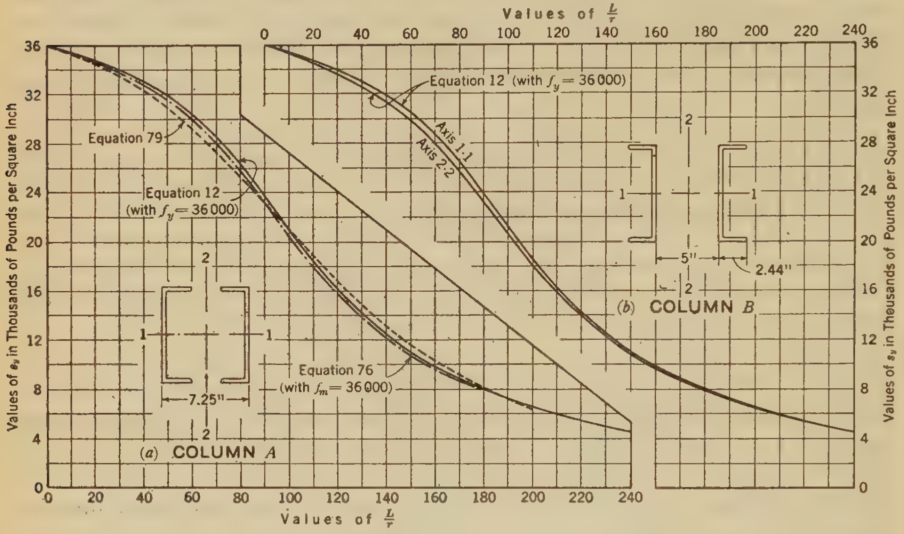

FIG. 22.—EFFECT OF $\frac{L}{r}$-RATIO ON s_y.

A comparison of the author's equations with the modified secant formulas for two different column sections, is shown in Fig. 22. The column characteristics are shown in Table 2. Let f_y = 36 000 lb per sq in.; let e and Δ both be assumed as $\frac{L}{750}$; E = 29 000 000; and X in the modified secant formula = 2 500. Then Equation (77) becomes,

TABLE 2.—CHARACTERISTICS OF COLUMNS IN FIG. 22 (TWO 8-INCH CHANNELS @ 16.25 POUNDS)

Definition	Fig. 22 (a)		Fig. 22 (b)	
	Axis 1–1	Axis 2–2	Axis 1–1	Axis 2–2
(1)	(2)	(3)	(4)	(5)
Section area, A, in square inches	9.52	9.52	9.52	9.52
Moment of inertia, I, in inches ^{4}	79.6	92.7	79.6	92.7
Radius of gyration, r, in inches	2.89	3.10	2.89	3.10
Distance, c, from neutral axis to extreme fiber, in inches	4.00	3.62	4.00	4.94
Ratio, $\frac{c}{r}$	1.39	1.17	1.39	1.59
r^2	8.35	9.61	8.35	9.61
$k = \frac{r^2}{c}$	2.09	2.65	2.09	1.95

$$s_y = \frac{36\,000}{1 + \frac{e}{r}\left(\left(\frac{L}{r}\right)^2 \frac{1}{2\,500} + \frac{c}{r}\right)} \quad \ldots\ldots\ldots\ldots\ldots (79)$$

$$\frac{\Delta}{k} = \frac{L}{750} \times \frac{c}{r^2} = 0.00133 \left(\frac{L}{r}\right)\left(\frac{c}{r}\right) \quad \ldots\ldots\ldots\ldots (80)$$

and,

$$\frac{e}{r} = \frac{L}{750} \times \frac{1}{r} = 0.00133\,\frac{L}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (81)$$

Another marked improvement in the author's formula is the term, $\frac{\Delta}{k}$, which takes into account the shape of the column. Since $k = \frac{S}{A} = \frac{r^2}{c}$, $\frac{\Delta}{k}$ may be written $\frac{\Delta\, c}{r^2}$, or $\frac{\Delta}{r} \times \frac{c}{r}$. The importance of this "eccentricity ratio," $\frac{\Delta\, c}{r^2}$ — and more particularly, $\frac{c}{r}$ — cannot be over-emphasized. It has generally been assumed that if the column section were such that the radius of gyration, r, is the same about either axis the column would have the same strength; but this is not the case unless the ratio, $\frac{c}{r}$, is the same about both axes.[29]

The columns in Fig. 22 have the same minimum radius of gyration, r, but the ratio, $\frac{c}{r}$, for the column in Fig. 22(*b*) changes the limiting axis and is weaker than the column in Fig. 22(*a*). Present-day formulas would lead to the conclusion that the columns were equal in strength since their minimum radii of gyrations are equal; but the secant formula shows that this is not true. The limiting axis of the column in Fig. 22(*b*) is 2–2 even though the radius of gyration, r, is greater than about Axis 1–1. Without regard to formulas one would expect the outstanding legs of Column *B* to be a source of weakness. Thus, the secant type of formulas enables the designer to take into account the column shape as accurately as he now computes the radius of gyration. The most economical section occurs when $\frac{c}{r}$ is a minimum about either axis. It approaches 1.00 as a minimum. A circular section, of course, would be the nearest approach to the ideal. The $\frac{c}{r}$-ratio has a marked effect on s for values of $\frac{L}{r}$ from 20 to 80, a common range in design. A variation in $\frac{c}{r}$ from 1.5 to 2.5 affects the values of s as much as a variation of $\frac{L}{r}$ from 40 to 60 or from 60 to 80.

The total effect of $\frac{c}{r} \times \frac{e}{r}$ is given by the author's use of $\frac{\Delta}{k}$, but in so doing the real significance of the $\frac{c}{r}$-ratio is lost. In applying the author's Equation (12) the ratio, $\frac{c}{r}$, appears as in Equation (80). It is best, however, to keep the $\frac{c}{r}$-ratio separate as a guide in choosing the column section.[30]

Referring to Fig. 22 the maximum variation of Equations (77) to (81) from those of the author is about 1 000 lb per sq in. With a value of n equal to 2.25 this would be only 440 lb per sq in. On Fig. 22(b) is shown the secant formula for Column A, Axis 1–1, only. The values of s are slightly less than those given by the author's formula and in close agreement with the Hunley formulas for $\frac{L}{r}$ less than 200.

The important point is that these formulas, although giving slightly different results, have the characteristics of a rational column formula, namely (1) when e or $\Delta = 0$, $s = f$; (2) they express the non-linear relation between s and f; and (3) they take into account the shape of the column as well as the slenderness ratio.

MARVIN A. GRAY, ESQ.[31] (by letter).—Long before Samson and Delilah, which is more than 3 000 years ago, columns have played an important part in tragedies. Any study of the structures of ancient times, as far back as records go, shows that an inherent subconscious fear of the combined features of lack of strength and stability in columns existed. This can be observed readily in the pictures of ancient buildings. The ancients built their columns by what they thought was a rational method of design; and yet many workmen were killed in the construction of their larger buildings and bridges. On August 29, 1907, the Quebec Bridge collapsed, one of the world's greatest structural failures; to-day, tragedies of column failure occur even in airplanes, which are more carefully and exactly designed than any other structure built. Therefore, this paper is on a most useful subject, presented at an opportune time; but it touches only lightly on the subject.

With the coming of Galileo there is record of a definite beginning in the development of the structural theory. Then came Hooke and Huygens with their conception of elasticity. In 1696 Jacques Bernoulli[32] proposed the problem[33] which led to Euler's discovery of the column formula; but this did not happen until a clever experimentalist found that column strength varied directly as the area and inversely as the square of the length. In

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[30] See "Modern Framed Structures," by Johnson, Bryan, and Turneaure, Pt. III, p. 69.

[31] Chicago, Ill.

[32] Encyclopædia Britannica, Fourteenth Edition, Vol. III, p. 457.

[33] Leonard Euler: "Methodus inveniendi Lineas Curvas maximi minimique proprietate gaudentes, sive Solutio problematis isoperimetrici lastissimo sensu accepti." Lausannae & Genevae, MDCCXLIV; Additamentum I, "De Curvis elasticis," pp. 245 to 311.

1744[33] Euler first developed his now famous column formula; in 1757[34] he developed the formula in a more general form and discussed it in great detail. Then came Thomas Young (1773–1829)[35] with his very practical conception of E. In 1855, it remained for Bessemer first to produce an economical and strong metal building material. As time has passed, many column formulas have been proposed by such men as Euler, LaGrange, Schneider, Rankine, Johnson, Engesser, etc., and special formulas by such men as Gordon, Ritter, Swain, Merriman, Considére, von Kármán, Westergaard, E. H. Salman, O. H. Basquin, Hickerson, D. H. Young, etc.

None of these men has said much more than Euler and LaGrange; and they have omitted and warped much of the original. The formula proposed in the paper is in direct contradiction with those derived by J. Prescott, for example.[36] Too much attention has been paid to producing new formulas and not enough to securing more accurate data and applying them. More tests on really high-strength rolled columns of solid cross-section or rolled structural shape are needed. This would be valuable rational data for a rational column formula.

The Engineers of the U. S. Army Air Corps have derived twelve formulas, illustrated by graphs, which are more accurate and useful than any of the other column formulas (except that by Euler).[37] These formulas involve the use of E as a constant although some steels have a proportional limit of 51 000 lb per sq in. and a yield point of 174 000 lb per sq in.[38] The author also assumes E to be constant although in Fig. 1 it is quite apparent that it may vary considerably from $f = 0$ to $f = f_y$, because then E becomes the secant modulus, and, in many tests, E has been shown to decrease from 60 to 70%, although the material is able to keep a constant load. In tests conducted at the Structural Laboratory of Northwestern University School of Engineering, the writer found that columns reach their maximum strength after they have deflected laterally. This fact was noted in tests recorded many years ago by the late George F. Swain, Past-President and Hon. M. Am. Soc. C. E.,[39] and also in the formula developed by Euler,[33] namely,

$$P = \frac{\pi^2 EI \sqrt{1 + \theta^2}}{L^2} \qquad \text{(82)}$$

Mr. Young feels that he should accept such standards as happen to exist. The type that happens to be fabricated most generally is taken as a standard; but what about the sub-average column? To the writer it seems that the standard should be a minimum fixed by engineers rather than by fabricators. He has seen structural sections 10 ft long made as accurate in cross-section as any commercially machined material with an error (or δ) of −0.01 in., or

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[34] K. Preussische akademie der wissenschaften Berlin Histoire * * * avec les memoires, 1757. (*Memoires* de l'Academie de Berlin, Tome XIII, 1757, p. 252.)

[35] Dictionary of National Biography, Vol. LXIII, printed in 1900, p. 396 (also, his lectures, ¶ 137).

[36] "Applied Elasticity," Paragraph 78; see, also, Paragraphs 76 to 93.

[37] "Airplane Design," Revised Edition, February, 1930, U. S. Army Air Corps (see Fig. *A*, p. 350).

[38] Air Corps *Information Circular No. 656*, p. 2. Table 1-A.

[39] "Structural Engineering," by George F. Swain, First Edition, 1924, Vol. I, Chapter on "Columns," p. 446, Lines 26 and 27; p. 448, Line 5 through Line 9.

$\frac{\Delta}{L} = \frac{0.01}{120} = \frac{1}{12\,000}$. This is thirty times as accurate as the author's standard. In ordinary work the steel companies can produce an accuracy of $\frac{1}{9\,220}$. From 1874 to 1898 strength of steel produced for the major bridges decreased 40% and from 1898 to 1924 it decreased (with rare exceptions in both cases) 6½ per cent.[40] Nevertheless, for the more important bridges, engineers are consistently thinking of stronger steels, which leads to more and better bridges. Unless useful study of columns is advanced so as to create less expensive bridges (which means also using material of higher specific strength as in airplanes), not many more bridges will be built—although, of necessity, a few will have to be built from time to time. Let the engineer step forward and declare what should, will, and is to be, rather than what may happen to be, taken as the standard.

Little has been concluded, definitely, concerning the strength of steel columns. Every city has its own formula; every State has one, and every group of engineers. Separate column formulas are offered for bridges, buildings, ships, airplanes, and airships. Mr. Young mentions the general fact that there are too many column formulas, but does not demonstrate how he applies his formula to the various types of columns just mentioned. They should be expressed by one formula because they are all fundamentally the same. Investigators, however, always seem to favor some new mathematical expression. In 1757 Euler favored a great number of tests on the subject in question. If one were to translate Euler's work and LaGrange's work completely and undertake a series of useful, complete, but inexpensive column tests on the basis of their findings, one would at least have a rational idea of the problem. In studying built-up columns at present, the profession seems to be complicating the column subject rather than leading the way to a stiffer and stronger column.

Although the author touches on many important points of the subject in question, it is not clear how he will apply his formula to a member of great strength, and as a guide or aid in research. His formula does not follow that of Euler in the range that Euler's has been proved valid, or for columns that are straight. The work of the Society's Special Committee on Steel Column Research[41] presents many points of importance not explained by Mr. Young's paper. The unit stresses used to-day in column design represent about 6% of the unit stresses developed in columns that have been tested. The design of rigid frames is not a complex problem of designing columns, but a combination of the analysis of column and beam action. Elasticity, crookedness, eccentricity, and end conditions of columns have been, and can be, controlled. This paper reveals the results of considerable thought and labor and this discussion is intended only to emphasize the fact that the data upon which it is based are inadequate, when E, an assumed constant, may vary from 60 to 70 per cent.

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[40] "Structural Engineering," by George W. Swain, Past-President and Hon. M. Am. Soc. C. E., First Edition, 1924, Vol. II, p. 93.

[41] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1376.

R. G. STURM,[42] ASSOC. M. AM. SOC. C. E., AND MARSHALL HOLT,[43] JUN. AM. SOC. C. E. (by letter).—Some of the fundamental principles of instability are applied, in this paper, to the design of steel columns, assuming that when the maximum stresses reach the yield strength the column becomes unstable.

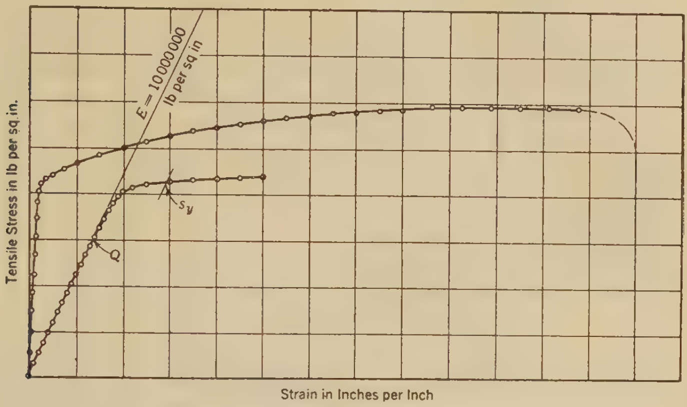

FIG. 23.—TYPICAL STRESS-STRAIN CURVES FOR A NON-FERROUS METAL.

The stress-strain curves of most non-ferrous metals differ from those shown in Fig. 1 by virtue of the fact that they depart gradually from the extended initial modulus line. A typical stress-strain curve for aluminum alloys, for example, is shown in Fig. 23. In applying these principles of instability, this stress-strain curve was idealized as one in which the value of the tangent modulus decreases as a straight line from the initial value at the upper limit of the elastic range to a value, λE, at the yield strength (0.2%), and from this decreased value along another straight line to zero at the modulus of failure. (The elastic range of the material may be determined by any standard method because reasonable variations in the actual value will not materially change the computed column strength.) Following the principles proposed by Engesser,[44] the critical load on a perfectly straight column was taken as the value given by Euler's formula, using the value of the reduced tangent modulus corresponding to the average stress in the column at buckling. A relation between the critical stress and the slenderness ratio thus obtained is shown in Fig. 24.

In the case of imperfect or eccentrically loaded columns in which the member deflects appreciably, the principles used by the author are the same as those established by T. Claxton Fidler[45] in 1887, namely, that the imperfections and eccentricities in the column can be represented by a sine curve. This leads to Fidler's formula for deflection which is the same as the author's Equation (8). This analysis was advanced still further by Ayrton and Perry[46] and formulas for the stresses in imperfect columns were developed. The formula for maximum stress at the center of an imperfect column is identical with the author's Equation (11).

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[42] Research Structural Engr., Aluminum Research Laboratories, New Kensington, Pa.

[43] Research Engr., Aluminum Research Laboratories, Arnold, Pa.

[44] "Die Knickfestigkeit gerader Stäbe," von F. Engesser, *Zentralblatt der Bauverwaltung,* Berlin, 1891.

[45] "Treatise on Bridge Construction," by T. Claxton Fidler.

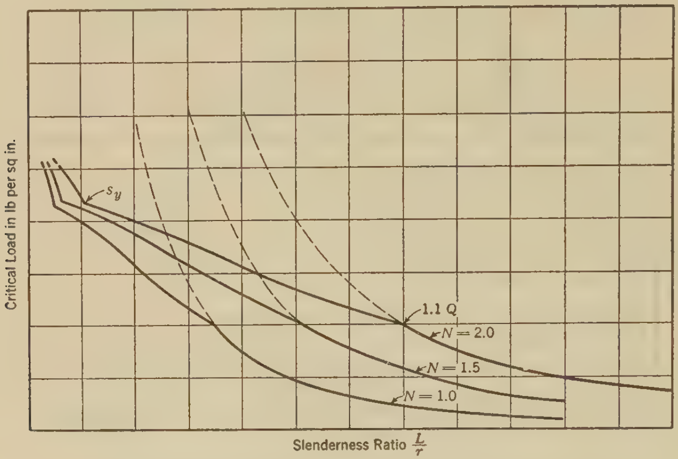

FIG. 24.—RELATION BETWEEN SLENDERNESS RATIO AND BUCKLING LOADS OF STRAIGHT COLUMNS (MADE FROM NON-FERROUS METALS).

If the initial eccentricities in the column are small, such that the maximum average stress developed in the column exceeds the elastic range of the material, the writers found, in their analysis for columns of aluminum alloys, that the value of s_e may be reduced according to the foregoing idealized stress-strain curve. The relations between the reduced value of s_e (say, s'_e) and the average stress in the column, s, are given by the following equations: When f_m exceeds 1.1 times the upper limit of the elastic range of the material, Q, but when s is still less than 1.1 Q, the value of s'_e becomes,

$$s'_e = s_e \left[1 - \frac{1}{j}\left(\frac{f_m - Q}{f_u - s}\right)^2\right] \dots\dots\dots\dots\dots\dots (83)$$

in which j is a factor depending upon the form of the member such that: $j = 4$ for solid rectangular sections; $j = 3$ for tubular sections; and $j = 2$ for I and H-sections bending parallel to the web (for bending perpendicular to the web the same as rectangles).

When s is between 1.1 Q and s_y,

$$s'_e = s_e \left[1 - (1 - \lambda)\frac{s - Q}{s_y - Q}\right] \dots\dots\dots\dots\dots\dots (84)$$

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[46] "On Struts," by Professors W. E. Ayrton, and John Perry, *The Engineer*, December 10 and 24, 1886.

in which, s_y = the stress at 0.2% permanent set; and λ is a factor that ranges from 0.05 to 0.12 depending on the particular alloy.

When s is greater than s_y,

$$s'_e = \lambda\, s_e \left[1 - \frac{s - s_y}{f_u - s_y}\right] \quad\text{.................. (85)}$$

In this way it is possible to extend Equation (11) into the plastic range for aluminum alloys. The maximum fiber stress, f_m, can thus be computed for the plastic range by a simultaneous solution of Equation (11) and Equations (83), (84), and (85).

Fixity of the ends of columns may be considered by re-defining s_e in the more general expression,

$$s_e = \frac{N^2\, \pi^2\, E}{\left(\frac{L}{r}\right)^2} \quad\text{........................ (86)}$$

in which, N is the Eulerian constant for end restraint (that is, $N = 1$ for hinged ends; and $N = 2$ for fixed ends).

It was found that columns fail by bending when the value of f_m reaches the modulus of failure for the material which for non-ferrous metals is practically equal to the tensile strength of the material, except for cases of local buckling.

The writers have used the equivalent of the author's Equation (12) to determine the ultimate load-carrying capacity of a column of aluminum alloys with the following changes: (*a*) s_y is replaced by s_u, the ultimate carrying capacity of the member; (*b*) f_y is replaced by f_u, the tensile strength of the material; and (*c*), s_e is replaced by s'_e. Thus, the ultimate strength of columns made of materials not exhibiting the definite yield point of mild steel may be computed by the following equation, similar to Equation (12):

$$s_u = \frac{f_u}{1 + \frac{\Delta}{k}\left(\frac{1}{1 - \frac{s_u}{s'_e}}\right)} \quad\text{.................... (87)}$$

It must be understood that Equation (87) applies only to columns that fail as a unit and not to those that fail by local buckling or twisting. In some cases, local buckling can be taken care of by a proper choice of the value of f_u.

These principles and equations can be applied to columns subjected to a combination of axial and transverse loads by treating the transverse load (or moment) as being applied first and considering the deflection produced by this load as an eccentricity of application of the axial load. The amount of stress that can be developed by the action of the axial load (residual stress, f_r), is equal to the modulus of failure, f_u, minus the bending stress, f_b, produced by the transverse loads. A satisfactory agreement was found between the strengths of heat-treated steel and duralumin tubes under combined axial and transverse loads computed in the foregoing manner and obtained from actual test results.[47]

For the aforementioned case of combined axial and transverse loading, Equation (87) becomes,

$$s_u = \frac{f_u - f_b}{1 + \frac{\Delta\, b}{k}\left(\frac{1}{1 - \frac{s_u}{s'_e}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (88)$$

in which, f_b = the stress resulting from the transverse bending; and $\Delta\, b$ = the deflection resulting from transverse bending, assuming the initial crookedness to be negligible compared to the deflection.

The equivalent of the author's Equation (13) becomes,

$$s_w = \frac{f_u - f_b}{n + \frac{n\ \Delta\ b}{k}\left(\frac{1}{1 - \frac{n\, s_w}{s'_e}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (89)$$

The deflection resulting from the transverse loads, Δb, may be closely approximated by,

$$\Delta\, b = \frac{f_b\, L^2}{\pi^2\, E'\, c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (90)$$

in which, E' = a reduced value of E such that,

$$\frac{\pi^2\, E'}{\left(\frac{L}{r}\right)^2} = s'_e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (91)$$

Substituting Equation (91) into Equation (89) and replacing k by its equivalent, $\frac{r^2}{c}$, Equation (89) becomes,

$$s_w = \frac{f_u - f_b}{n + \frac{n\, f_b}{s'_e}\left(\frac{1}{1 - \frac{n\, s_w}{s'_e}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (92)$$

The writers did not use the secant formula to determine the equivalent of the author's Equations (14), (15), and (16), because the expression became quite involved when a reduced effective modulus was necessary.

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[47] "Strength of Tubing Under Combined Axial and Transverse Loading," by L. B. Tuckerman, S. N. Petrenko. and C. D. Johnson, Technical Notes of National Advisory Com. for Aeronautics, No. 307, June, 1929.

Under the heading, "Columns in Rigid Frame Construction," the author refers to the elastic action of other members meeting at a joint causing columns to act intermediately between pin-ended and completely fixed-ended conditions. Although he might have intended pin-ended conditions to cover the case of columns with end moments, it should be stated that the other members framing into a joint might throw sufficient initial bending into the column so that the ultimate strength in the assembly might be less than that of a pin-ended column axially loaded (without end moments). An extension of Manderla's original work[48] taking into account the axial stresses in computing secondaries in a truss, leads to the conclusion that continuity in a compression member offers no restraint in itself to end rotation as the load on the member approaches the buckling load. It would be of value if Mr. Young would explain his conception of the end fixation of columns.

F. E. TURNEAURE,[49] M. AM. SOC. C. E. (by letter).—The analysis of columns presented by Mr. Young is a valuable contribution. In some respects it differs somewhat from that presented by the Special Committee of the Society on Steel Column Research,[50] while in other ways it is in agreement. In view of the importance of the subject and the comprehensive nature of these two studies, it will be useful to compare them briefly.

Mr. Young is in agreement with the Committee's conclusions in regard to using the yield-point strength as the critical stress to be considered, and that elastic conditions may be assumed for stresses below this point. The variation in yield point and in the form of the stress-strain curve in the vicinity of this point, both below and above, renders a more exact treatment of buckling strength of little practical value for structural columns.

As the basis of his fundamental formula Mr. Young has adopted a centrally loaded, curved column and has superimposed upon this column any eccentricity that may occur. From the first, the Special Committee on Steel Column Research used the so-called secant formula, based entirely on eccentricity, and included in the eccentricity the effect of crookedness. The Committee selected this method in order to simplify the analysis of its test results where a definite eccentricity was used, and the final analysis leading to working formulas was made by including the effect of crookedness in the assumed eccentricity. The results of these two methods of procedure are not materially different. Analyses of crookedness made by the Committee and reported in its second progress report[51] indicate a much lower value for crookedness coefficient than Mr. Young has used. From its studies the Committee adopted a crookedness value, in terms of eccentric ratio, of $\frac{ec}{r^2} = 0.001 \frac{L}{r}$. This compares with a value of about $0.003\frac{L}{r}$, which Mr. Young generally utilizes $\left(\Delta = \frac{L}{400}\right)$. This is a relatively high value and much larger than appears to be justified by the studies of the Committee.

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[48] "Die Berechnung der Sekündarspannung welche im einfachen Fachwerke infolge starrer Knotenverbindungen auftreten," *Allgemeine Bauzeitung*, 1880.

[49] Cons. Engr.; Dean, Coll. of Mechanics and Eng., Univ. of Wisconsin, Madison, Wis.

[50] *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 1485; Vol. 95 (1931), p. 1152; and Vol. 98 (1933), p. 1376.

[51] *Loc. cit.*, Vol. 95 (1931), p. 1152.

A column in a modern riveted truss is substantially a member of a rigid frame and, therefore, is subject to the secondary stresses, or end deformation moments, that occur in such frames. Mr. Young suggests that these secondary stresses be calculated and the column formula modified accordingly for each member. In its study of working formulas the Committee carefully considered this matter and suggested that for ordinary design purposes it would be reasonable to assume an eccentric ratio, $\frac{ec}{r^2} = 0.25$, for all columns and to assume a free length (between points of inflection) of 75% of the full column length in applying the theoretical formula. A column in a rigid frame having 25% secondary stress at each end, and in such a direction as to bend the member in single curvature, acts nearly like an eccentrically loaded, pivoted-end column of one-half the length and with an eccentric ratio of 0.25.

The Committee felt justified, therefore, in using a column length of three-fourths the full length for a practical working formula. The application of these ideas resulted in a secant formula representing the ultimate strength of the column, based on yield-point strength. Any desired factor of safety could then be applied to this ultimate strength formula. The working formula that was suggested was a parabolic curve coinciding substantially with the resulting secant curve within the usual working limits of $\frac{L}{r} = 40$ to 150. This working formula, adopted recently by the American Railway Engineering Association, is:

$$s = 15\,000 - \frac{1}{4}\left(\frac{L}{r}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (93)$$

Pin-ended columns are not frictionless, but the amount of secondary moment at the ends due to truss deformation is limited. If frictionless, the eccentricity of load would be limited by the effect of crookedness and other incidental column defects; if frictional resistance is sufficiently large, such columns act as riveted members. As a reasonable estimate, the Committee adopted the same eccentric ratio of 0.25 as for riveted columns, but increased the free length to 85% of the full length, leading to the parabolic working formula,

$$s = 15\,000 - \frac{1}{3}\left(\frac{L}{r}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (94)$$

For large and important structures, the writer endorses, fully, Mr. Young's suggestion that secondary stresses be computed and the column formula modified accordingly. The Committee's formula provides for eccentricity acting in the same direction at both ends. However, most columns in a truss will be bent in reverse curvature, resulting in maximum stresses at the ends; and such columns might well be designed at a fixed basic unit strength without reduction for column length. For ordinary practice as covered by general specifications, one would scarcely resort to such a separate calculation, and a single formula is preferable.

The shear analysis presented by Mr. Young appears to be quite complete, and shows clearly the high stresses possible in short members, decreasing to a smaller value with increasing length and then increasing again for long members. In this case, again, the high values assumed for crookedness result in shear values differing somewhat from those corresponding to the column formulas of the Committee. The new shear formula recently adopted by the American Railway Engineering Association was based upon the analyses by Shortridge Hardesty, M. Am. Soc. C. E., using the adopted secant formulas as a basis. This new formula is an empirical one, but is constructed so as to provide reasonably well for both short and long columns, as follows:

$$\frac{V}{P} = \frac{1}{100}\left(\frac{100}{\frac{L}{r} + 10} + \frac{\frac{L}{r}}{100}\right) \ldots\ldots\ldots\ldots\ldots(95)$$

in which, $\frac{V}{P}$ = ratio of shear to direct stress.

It appears to the writer that the paper by Mr. Young and the reports of the Special Committee on Steel Column Research, taken together, present a clear picture of the underlying factors in the rather complex column problem, and suggest rational methods of dealing with the subject.

N. J. DURANT,[52] ASSOC. M. AM. SOC. C. E. (by letter).—Simple rational formulas for the design of steel columns, which are deduced from elementary analyses, are brought to the attention of the Engineering Profession by this paper. The subject of column analysis is one of the oldest in the broad field of elasticity, but in spite of all the research conducted, the profession has used empirical formulas for years, with the result that, to-day, the engineer has a multiplicity of column formulas at his command, no two of which will give identical results.

These formulas (whether straight lines, parabolas, or of the Rankine-Gordon type), which have been in general use, are merely forms of wasted ingenuity. Usually, the only parameter in them is the slenderness ratio, $\left(\frac{L}{r}\right)$. One of the disadvantages of such formulas (whether or not they conform to test results of a particular material) is, that immediately the designer is confronted with a material with which he is unfamiliar, he feels the necessity for an analysis which will interpret for him the principles as distinct from the incidents upon which experimental results depend.

The chief disadvantage of the approximate formulas, however, lies in the fact that unless the engineer has a rational formula to serve as a standard, he is in complete ignorance regarding their factors of safety, and this alone should be a sufficient reason for their condemnation.

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[52] Senior Engr., Rendel, Palmer & Tritton, Westminster, London, England.

It follows that the engineer must be in doubt regarding the stability of the resulting structures, because it is no criterion that such structures withstand the loads for which they have been designed.

Objections to rational column formulas have been raised because they have been considered too complicated for general use. No formula for columns could be so complicated but that a graph or tabular values of the dependent variable for different values of the argument could not be readily obtained, and this is all the average designer needs. His subsequent uses of the formula, therefore, would involve no additional work beyond that entailed by the use of an approximate formula.

Furthermore, in the case of an investigation, whether of failure of a column or for other reasons, the investigator would arrive at precisely the same formulas as did Mr. Young for similarly imposed conditions.

A rational formula for columns cannot be condemned because it is complicated; neither can a straight-line formula be defended on the basis of its simplicity. Simplicity is a desideratum to be aimed at in any formula, whatever its purpose, but if all the essential variants are absent, the formula ceases to be of general value.

A formula for the design of columns should be one of general utility and if it has a theoretical basis, then by a variation of the fundamental elastic constants it would apply to any material which obeys Hooke's law and which has a compressive or tensile stress-strain curve. Moreover, if the factor of safety in the column formula is equal to that of the material in tension, the resulting structure will have, within narrow limits, a uniform strength.

The only actual uncertainty in a rational column formula is the value which should be assigned to the length of the column. If the ends were fixed in position and direction this value would be determinate, but such a condition is unattainable.

All members of a frame are subjected to varying degrees of constraint which make the effective length indeterminate, but the true effective length, of course, must be less than the geometric length. If the geometric length were inserted in the formula, the designed column would have a strength greater than that calculated, provided, of course, the material was in no way impaired and that the secondary moments were properly assessed. The value of the constraints will be less, in general, for increases in the yield stress, but no analytical law can be formulated to deal with this phenomenon.

Moreover, the more slender a member is compared with those to which it is attached, the shorter will be its effective length. This statement, although obvious, serves in some measure to fix the effective length of a column. For instance, the web members of a truss are almost invariably of more slender proportions than the chords and, in consequence, their effective lengths are less than their geometric lengths. In general, the chords receive little restraint from the web members and probably their effective lengths are their geometric lengths, approximately.

Throughout his paper the author has used the equation,

$$E\,I\,\frac{d^2y}{dx^2} = -\,M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (96)$$

which is sufficiently accurate for columns of solid cross-section.

For built-up columns, the form is of importance and it would be necessary in a rigorous investigation to use the more general equation:

$$E\,I\,\frac{d^2y}{dx^2} = -\,M + k\,\frac{E\,I}{A\,G}\,\frac{d^2M}{dx^2} \ldots\ldots\ldots\ldots\ldots (97)$$

in which k is a constant depending on the distribution and form of the area of the cross-section, and G is the modulus of rigidity.

However, considering the variation in the elastic constants and the diversity of column forms, Equation (97) would not be expedient for general use.

To avoid the anomalies in design created by the approximate formulas, it is to be hoped that a rational formula for column design will soon be incorporated in all specifications for the design of structures.

So much research work has been done on structures, particularly in America, that it appears strange that the march of progress should be marred by adherence to traditional standards, many of which, have no scientific significance.

E. C. HARTMANN,[53] ASSOC. M. AM. SOC. C. E. (by letter).—The various factors involved in the rational design of steel columns have been treated in a comprehensive manner by the author, but the writer is inclined to wonder how much of the paper is likely to be absorbed in usable form by those actively engaged in column design. What most designers want is something reasonably accurate and not too complex, which will take into account the principal variables and will fit neatly into the ordinary design specifications. In the case of a column subjected to eccentric load, or to a side load, for example, the designer would like to know what allowable stress to use on the extreme fiber.. Knowing this, he is perfectly competent to proceed with his design in the usual manner, selecting a member on which the combined stress (bending plus direct stress) does not exceed the allowable.

A simple formula for determining the allowable stress for all types of compression members, which incorporates practically all the principles in the author's paper, has been derived by the writer and his associates. For the sake of simplicity the derivation presented herein will be made for steel columns using the same basic assumption as that adopted by the author, namely, that the yield-point stress is the limit of usefulness. It will then be shown how the derivation may be extended to include materials such as the structural alloys of aluminum which have no pronounced yield point and, hence, have a range of useful stress above this point.

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Fig. 25 demonstrates a very general case of a pin-ended column having an initial crookedness, Δ, loaded eccentrically at a distance, e, and resisting a side load as well as the column load. The dotted line represents the deflected position of the member under load.

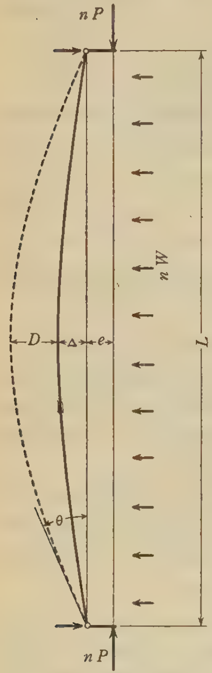

FIG. 25.

Let the column load, P, and the side load, W, be the design loads for which a factor of safety, n, is to be provided. If the column is properly designed, the fiber stress from combined bending and direct stress should just reach the yield point if P is increased to $n\,P$, and W to $n\,W$, simultaneously. Therefore, the following equation represents the condition at failure:

$$\frac{n\,P}{A} + \frac{n\,P\,e\,c}{I} + \frac{n\,P\,\Delta\,c}{I} + \frac{n\,P\,D\,c}{I} + \frac{n\,M\,c}{I} = f_y \quad (98)$$

in which, in addition to the notation of the paper, D = the total deflection which accompanies the yield-point stress, caused by all loads combined, in inches; and M = the maximum bending moment produced by transverse loads, in pounds. It will be recognized that Equation (98) is simply a summation of the five component parts of the extreme fiber stress, based on the well-known fundamental equation for combined stress: $f = \frac{P}{A} + \frac{M\,c}{I}$.

Transposing the fourth term in Equation (98) to the right-hand side and dividing through by the factor of safety, n:

$$\frac{P}{A} + \frac{P\,e\,c}{I} + \frac{P\,\Delta\,c}{I} + \frac{M\,c}{I} = \frac{f_y}{n} - \frac{P\,D\,c}{I} \quad\ldots\ldots\ldots\ldots (99)$$

The term, $\frac{f_y}{n}$, in Equation (99), may be replaced by f_b, the design stress for the member treated as a beam, which gives:

$$\frac{P}{A} + \frac{P\,e\,c}{I} + \frac{P\,\Delta\,c}{I} + \frac{M\,c}{I} = f_b - \frac{P\,D\,c}{I} \quad\ldots\ldots\ldots\ldots (100)$$

Since P and W are design loads (ultimate loads divided by the factor of safety, n), and since M is the moment produced by the design load, W, the terms on the left-hand side of Equation (100) represent the ordinary design calculations for combined stress on the extreme fiber of the member. The equation states that if this calculated stress is equal to the terms on the right, a factor of safety of n is provided against failure (yield-point stress). Therefore, the terms on the right constitute the allowable design stress, f_w; thus:

$$f_w = f_b - \frac{P\,D\,c}{I} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (101)$$

in which f_w = the allowable combined stress, in pounds per square inch, on the extreme fiber at the center of the unsupported length of any member subjected to combined bending and direct stress (includes eccentrically loaded columns); and f_b = the allowable stress, in pounds per square inch, for the member treated as a beam.

The only term in Equation (101) that is unknown is D, the total deflection accompanying the yield-point stress. This deflection may be expressed as the sum of the effects of the various loads, but a simpler and more direct solution is to express it directly in terms of the known total fiber stress, as follows:

$$D = \frac{\left(f_y - \frac{n\,P}{A}\right) L^2}{\pi^2 E\, c} \quad \text{.....................(102)}$$

Equation (102) is readily derived by the use of the well-known moment-area theorem which states that the deflection at the center of a span is the bending moment at the center produced by the moment diagram, divided by EI, considered as a load.

The total bending stress at the center is $f_y - \frac{n\,P}{A}$ and, if M' is the total moment causing this stress:

$$\frac{M'\,c}{I} = \left(f_y - \frac{n\,P}{A}\right) \quad \text{.....................(103)}$$

Therefore, in this case, the center ordinate of the moment diagram, divided by $E\,I$, is:

$$\frac{M'}{E\,I} = \frac{\left(f_y - \frac{n\,P}{A}\right)}{E\,c} \quad \text{......................(104)}$$

The constant, π^2, in Equation (102) results from assuming that the curve of moments is a sine curve and in this respect Equation (102) is an approximation. Other curves might be assumed; for example, a parabolic moment curve gives a constant, 9.6, instead of π^2.

Let s_e = the critical stress, in pounds per square inch, for the member treated as an Euler column, representing the Euler value, $\frac{\pi^2 E\, r^2}{L^2}$, as in the author's paper. Transposing terms:

$$\frac{L^2}{\pi^2 E} = \frac{r^2}{s_e} \quad \text{..........................(105)}$$

Substituting Equation (105) in Equation (102) gives the following:

$$D = \frac{\left(f_y - \frac{n\,P}{A}\right)}{s_e} \; \frac{r^2}{c} \quad \text{..................(106)}$$

The term, f_y, may be replaced by $n\, f_b$, which gives:

$$D = \frac{\left(f_b - \frac{P}{A}\right)}{\frac{1}{n}\ s_e}\ \frac{r^2}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (107)$$

Substituting Equation (107) in Equation (101) and replacing I by $A\, r^2$:

$$f_w = f_b - \frac{P}{A}\ \frac{\left(f_b - \frac{P}{A}\right)}{\frac{1}{n}\ s_e} \quad \ldots\ldots\ldots\ldots\ldots\ldots (108)$$

The derivation of Equation (108) for end conditions other than pinned is relatively simple and leads to the same formula. The method preferred by the writer is to evaluate the end condition in terms of the equivalent length of pin-end column—that is, in terms of the length between the points of contraflexure of the member. Thus, a fixed-end column acts the same as a pin-end column of one-half its length. Since the line of action of the column load, P, must follow the straight line joining the points of contraflexure, the derivation is simply a repetition of the foregoing, except that the effective length, $z\, L$, is substituted for L and the effective initial crookedness, $z\, \Delta$, is substituted for Δ. The Euler value, s_e, becomes $\frac{\pi^2 E\, r^2}{(z\, L)^2}$. The value of z for completely fixed ends is 0.5. For round or pin ends, it is 1.0, of course, whereas for the general run of framed ends many designers use 0.75 as a representative average value.

As previously pointed out, the derivation of Equation (108) is based on the line of reasoning adopted by the author in dealing with steel. For aluminum alloys and other materials which do not display the abrupt yield-point characteristic of steel, it is necessary to introduce into the derivation a suitably reduced modulus of elasticity, E_a, when the value of $\frac{n\, P}{A}$ exceeds the elastic range. A similar correction should be made even in the case of steel, because the limit of the elastic range is slightly below the yield-point stress. This matter is not as important in steel as in most other materials.

If applied to a column that is eccentrically loaded, the method described herein gives the same result as that in the author's paper. Furthermore, it has the additional advantage that it applies to columns transversely loaded; in fact, it allows any member under combined bending and direct compression to be designed on the same basis as an ordinary beam, with no departure from conventional procedure. In order to clarify the method of using Equation (108) a simple example may be cited, using current standard specifications[54] as a basis for design. Let the problem be to determine whether a 12-in., 65-lb Carnegie beam section, unbraced, on a 15-ft simple span (pin ends) is safe under the combined action of an end load of 200 000 lb and a distributed transverse load of 20 000 lb, acting in the plane of the web. The following are the calculations:

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[54] Specifications for Steel Railway Bridges (1934), Am. Ry. Eng. Assoc.

(1) $\frac{P}{A} = \frac{200\,000}{19.11} = 10\,500$ lb per sq in.; $\frac{L}{r}$ (Axis 2–2) $= \frac{15 \times 12}{3.02} = 59.6$; and, the allowable axial compression $= 15\,000 - \frac{1}{3}\,\overline{59.6}^2 = 13\,800$ lb per sq in. Consequently, the member is safe for the column load alone.

(2) The bending stress (Axis 1–1) $= \frac{20\,000 \times 15 \times 12}{8 \times 88} = 5\,100$ lb per sq in.; $\frac{L}{b} = \frac{15 \times 12}{12} = 15$; and $f_b = 18\,000 - 5 \times \overline{15}^2 = 16\,900$ lb per sq in. Therefore, the member is safe for transverse load alone.

(3) The combined fiber stress (Axis 1–1) $= 10\,500 + 5\,100 = 15\,600$ lb per sq in.; $\frac{L}{r}$ (Axis 1–1) $= \frac{15 \times 12}{5.28} = 34$; s_e (Axis 1–1) $= \frac{\pi^2 E}{\overline{34}^2} = 248\,000$ lb per sq in.; $n = \frac{33\,000}{18\,000} = 1.83$; $\frac{1}{n}\, s_e = 135\,000$ lb per sq in.; and f_w (Axis 1-1) $= 16\,900 - 10\,500 \,\frac{(16\,900 - 10\,500)}{135\,000} = 16\,400$ lb per sq in., which proves the member safe for combined load.

In the foregoing examples it should be noted that, for checking the strength of the member under column load alone, the smallest radius of gyration is used (Axis 2–2), whereas for checking combined stress conditions the radius of gyration is taken about the axis normal to the plane of the transverse bending loads (Axis 1–1).

In using Equation (108) the end conditions for the transverse loading should be consistent with those for the column loading. Thus, in the foregoing example, if the member were assumed to have riveted ends, the value of s_e would have been computed for $z = 0.75$,[55] and, therefore, in computing the bending stress from the transverse load the effective span should be reduced to 0.75 × 15.

Certain parts of the foregoing derivation can be used in arriving at a formula for estimating the maximum shear on compression members. Assuming the deflected member in Fig. 25 to take the form of a sine curve, it can readily be shown that the maximum slope of the member is:

$$\tan\theta = \frac{\pi\,(D + \Delta)}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (109)$$

The total shear on the member is:

$$n\,V = n\,B + n\,P \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (110)$$

in which B is the shear force, in pounds, produced by the transverse load, W. Neglecting the difference between $\tan\theta$ and $\sin\theta$ and dividing through by the factor of safety, n, gives the following equation:

$$V = B + \frac{\pi\,P\,(D + \Delta)}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (111)$$

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[55] Specifications for Steel Railway Bridges (1934), Am. Ry. Eng. Assoc., Appendix A.

in which V = the total shear force, in pounds, for which the column should be designed (shear at failure divided by factor of safety). Substituting the value of D from Equation (107),

$$V = B + \frac{\pi P}{L}\left[\frac{\left(f_b - \frac{P}{A}\right)}{\frac{1}{n} s_e}\frac{r^2}{c} + \Delta\right] \qquad (112)$$

Equation (112) is for pin-ended members only ($z = 1$). It may be generalized as to end conditions by using $z\,L$ instead of L, and $z\,\Delta$ instead of Δ, in accordance with the foregoing discussion of end conditions; thus:

$$V = B + \frac{\pi P}{z\,L}\left[\frac{\left(f_b - \frac{P}{A}\right)}{\frac{1}{n} s_e}\frac{r^2}{c} + z\,\Delta\right] \qquad (113)$$

in which V = the maximum total shear force, in pounds, for which compression members must be designed in order to provide a factor of safety against shear failure consistent with that against column failure. The Euler value, s_e, should be calculated for the effective length, $z\,L$. The shear, B, from transverse loads should also be that for the effective length, $z\,L$.

Equation (113) fits in nicely with Equation (108) since the two have a number of terms in common. It should be emphasized that the radius of gyration, r, should be taken about the axis normal to the plane in which the shearing force, V, is acting. The distance to the extreme fiber, c, should be taken in a direction consistent with r and, in the case of unsymmetrical members having two values of c, the smaller of the two should be taken unless transverse loads or eccentricities are present which throw the larger compression stresses on the other side.

Equation (113) like Equation (108) gives results in agreement with the author's method when applied to a column eccentrically loaded; and has the additional feature that it applies also to columns transversely loaded. In order to clarify the method of using Equation (113) an example may be given using the same set of conditions as in the foregoing example. The following are the calculations: $B = \frac{20\,000}{2} = 10\,000$ lb; $\frac{\pi}{L} = 0.0174$; $\frac{f_b - \frac{P}{A}}{\frac{1}{n} s_e} = \frac{16\,900 - 10\,500}{135\,000} = 0.0474$; $\frac{r^2}{c} = \frac{5.28^2}{6.06} = 4.6$; $\Delta = 0$; and, $V = 10\,000 + 0.0174\,P\,[0.0474 \times 4.6 + 0] = 10\,000 + 0.0038\,P = 10\,760$ lb.

The shear produced by column action in this member is less than 1% of the end load, and is a small fraction of the total shear. If the initial crookedness had been assumed equal to $\frac{L}{400}$, or 0.45 in., the resulting shear from column action would be increased to 0.0116 P, and the total shear would be 12 300 lb. In either case, of course, the column selected has ample web section.

In his handling of the shear problem, the author has apparently overlooked the factor of safety. His v_y, which is defined as "the average shearing stress when the load, P, is at its yield-point value," is used in Fig. 16 for direct comparison with values taken from the American Railway Engineering Association specifications. He should have used v_w, or $\frac{v_y}{n}$, following the same logic by which s_w in his Equation (13) is arrived at from s_y in his Equation (12). This would have had the effect of dropping all his curves in Figs. 16 and 18 considerably. The writer wonders if the selection of an average design shear of 600 lb per sq in. would not have been affected by this correction. There is little in the paper to justify a blanket value of 600 lb per sq in. under any circumstances.

The author has pointed out the effect of shearing distortions on latticed or battened members. In the writer's derivations these distortions affect only the magnitude of the deflection, D. In Equations (102 and (107) it is assumed that shearing deformations are negligible. Actually, under the effect of transverse loads, the shearing deflection in a latticed member often runs as high as the bending deflection. Some account of this might be taken in Equations (108) and (113), possibly by inserting a suitable factor by which the term, $\left(f_b - \frac{P}{A}\right)$, is multiplied when latticed or battened columns are under consideration. This is a matter worthy of further study.

The writer would like to emphasize the importance of the term, D, the total deflection of the member, in the foregoing derivations. This deflection is that which occurs just at the point of failure, or just as the extreme fiber reaches the yield-point stress. Dividing through the various equations by the factor of safety, n, does not affect the magnitude of D, so that in the formula for working stress, f_w (Equation (101)), D is still the deflection corresponding to the yield-point stress. The significance of this to the designer is simply that, in estimating the degree of fixation of the ends of a member (magnitude of effective length, $z\,L$), the conformation of the member should be visualized with respect to conditions just before failure. Many a member which is practically fixed-ended ($z = 0.5$) at low loads may be practically pin-ended ($z = 1.0$) at or near failure, and should be so considered in design.

The writer advocates the use of Equations (108) and (113) in actual design instead of such generalities as $\Delta = \frac{L}{400}$ and average shear stress = 600 lb per sq in., advocated in the author's summary. The designer should be allowed some latitude to evaluate such variables as crookedness, end conditions, and eccentricity of loading in the light of his own peculiar design conditions. Most designers will quickly simplify the application of Equations (108) and (113) by preparing charts and other "short-cuts" for the routine problems, saving the formulas themselves for the unusual cases that always arise.

Edward Godfrey,[56] M. Am. Soc. C. E. (by letter).—The idea that a column is imperfect, that it is not perfectly straight, and that imperfection (lack of straightness) is directly proportional to the length of the column, is a rational assumption. Mr. Young's attack on the problem of the column is in the right direction. A perfect column formula, except for slender columns, is not possible, because a perfect column is an impossibility. The imperfection of a column that is easiest to detect, in columns otherwise carefully made, is lack of straightness. In 1909, the writer pointed out[57] that by visual inspection one can detect deviation from a straight line of about $\frac{1}{400}$ of the length of a structural member. This is the ratio suggested by Mr. Young. The writer chose $\frac{1}{300}$ as the ratio for a rational column formula to allow for eccentric end application of the load.

Mr. Young finds an expression for the total extreme fiber stress which includes the bending stress due to the bow of the column. His design curve for a rectangular section in that column, as shown in Figs. 9 and 10, is almost a straight line for values of $\frac{L}{r}$ up to 120 and for a bow in the column equal to $\frac{1}{400}$ of the length. It is evident that a somewhat larger assumed bow in the column would give a still closer approach to a straight line.

In the writer's solution, using an average value of the radius of gyration as one-third the depth of section, and an assumed bow in the column of $\frac{1}{300}$, the curve of allowed unit stress was found to be a straight line up to $\frac{L}{r}$ equal to about 110. This agreed with the commonly used straight-line formula, $16\,000 - 70\,\frac{L}{r}$. The deviation from this straight line is very slight for slenderness ratios of 150 or less. At $\frac{L}{r} = 150$ the curve strikes the Euler curve with a factor of safety of 2.

It is significant that this is just about the upper limit of structurally permissible columns. Thus, a completely rational formula, too cumbersome to use in practice, however, based on no other supposition than a bow or imperfection that would reject a column subject to ordinary inspection, agrees almost exactly with the common, well-tried, straight-line formula that never laid claim to any except experimental, empirical backing.

The writer recommended that columns be divided into two distinct classes (structurally permissible columns and slender columns), and that design stresses in these two classes be based on entirely different considerations, not forgetting, of course, the zone where the two classes overlap. The unit stress to be used in the first class was based solely on the extreme fiber stress in a bowed column, subject to an endwise load, and that unit stress was shown to be theoretically determinate by the use of a formula of the type, $16\,000 - 70\,\frac{L}{r}$. The unit stress in slender columns was recommended to be on the basis of a factor of safety, say, about 2, based on the ultimate load as shown by the Euler formula. There is no need of a factor of safety greater than 2 in slender columns, because the steel is in no manner distressed in a slender column (even in one having a decided bow) until the ultimate load is nearly reached.

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[56] Structural Engr., Pittsburgh, Pa.

[57] *Railroad Age-Gazette*, July 2, 1909; see, also "Steel Designing," by Edward Godfrey; and *The Structural Engineer* (London), September, 1932.

There need be no confusion in the two classes of columns. Specifications for structures need give no recognition to slender columns, except to exclude their use. All structural columns can be designed safely on the same straight-line formula, because there is extra safety in the range approaching slender columns; this is of advantage, because the weight of members and other considerations make it desirable. Slender columns and formulas for them should be reserved for such work as transmission towers, airplanes, etc.

The writer is not familiar with present practice among Continental engineers, but in former years they used the Euler formula with a factor of safety of 5. This is an extravagant waste in slender columns, where the Euler load is of real significance. The Euler load has no meaning whatever in structurally permissible columns.

L. T. WYLY,[58] M. AM. SOC. C. E. (by letter).—The method of analyzing steel columns, proposed in this paper, is the most rational that has been advanced to date. The problem has been defined concisely by B. R. Leffler,[59] M. Am. Soc. C. E., as follows: "The problem of the column is one of bending moment. The very fact that a column fails with an average unit stress below the strength of the material is evidence that it is impossible to centralize the load at every section, * * *." The parabolic formulas now generally advocated and widely used omit entirely (except as covered by a constant, representing assumed average values) the important factors of eccentricity and the value, c. Any solution of a bending problem which neglects these terms can be neither logical nor complete. The writer feels, however, that before the Engineering Profession can adopt Mr. Young's procedure it must have a more general knowledge of secondary bending actions, and more definite information regarding actual eccentricities to be expected on columns, as determined by careful and comprehensive field and shop measurements. Furthermore, the rational design of steel columns must necessarily take into account a number of factors not represented in the formula but which directly affect the strength of the column. Among the latter are the following:

(1) The form of section including its strength under local buckling action;

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[58] With Bureau of Bridges, Div. of Highways, Springfield, Ill.

[59] *Transactions*, Am. Soc. C. E., Vol. LXXXVI (1923), p. 541.

(2) Adequateness of design of connection, pin-plates, bracing, and other items unfortunately too often considered as secondary details; and,

(3) The relative degree of excellence to be expected in fabrication shops and in field construction in matters such as milling, setting shoes and grillages, and maintaining the structure in proper position during construction.

In short, it is the writer's conviction that a column formula should be tied up directly with specific clauses in design and construction specifications, and the formula used with a fairly accurate idea of how these specifications will be followed. The writer desires to discuss briefly some of the evidence supporting this view.

It should be recognized at the outset that the problem is concerned with short columns primarily, or, at most, with intermediate columns. The slenderness ratio, in the plane of the truss, of main columns of heavy bridges will usually be 40, or less, except in the inclined end posts where it may be 60 or 75. In a light highway bridge, particularly if rolled wide flange beams are used as columns, the ratio may be as high as 100. The $\frac{L}{r}$ ratio normal to the truss will usually be less. In short, the Euler values do not enter into the problem.

Computed Eccentricities.—If eccentricities in the plane of the truss are to be determined by secondary stress analysis, the following factors should be taken into account: (*a*) For a large bridge with subdivided panels and heavy chords the secondary hangers or struts will be relieved of considerable primary stress by the beam action of the chords, with consequent reduction of secondary moments on the chords themselves; and (*b*) with railway loadings the local effect of heavy drivers is likely to be severe on the secondary moments in the chords, and an accurate solution may be very tedious.

The foregoing facts have been established by J. I. Parcel and G. A. Maney,[60] Members, Am. Soc. C. E. There is the further question of whether, at the present time, it would be advisable to entrust the matter of a secondary analysis of a truss, and consequent selection of a column formula for individual members, to the average designer. The writer's view is that much more work will have to be done on this problem and certain standards established by the profession before the method will be satisfactory for adoption in a specification. The same applies to eccentricities arising in a plane normal to the truss due to floor-beam or other rigid-frame action.

Measured Eccentricities.—With regard to initial crookedness of columns, the Final Report of the Board of Engineers of the Delaware River Bridge, between Philadelphia, Pa., and Camden, N. J., shows measurements taken on twenty-two columns varying in length from 29 to 41 ft. Initial crookedness of $\frac{1}{4}$ in. and more was found, and was just as great in the short members as in the long. The writer has taken careful measurements on about sixteen columns fabricated for a large highway bridge by a modern shop and has found initial crookedness of $\frac{1}{4}$ to $\frac{5}{16}$ in. These columns were as long as 100 ft, and the sectional areas up to 75 sq in. and most of them had central longitudinal diaphragms included in the section. The short columns showed as much eccentricity as the longer ones. It is the writer's belief based on the foregoing information, and on discussions with shop foremen and inspectors, (1) that initial crookedness is not necessarily a function of length or $\frac{L}{r}$; (2) that it is likely to be as much as $\frac{1}{4}$ in. for a short column; and (3) that it should not exceed $\frac{3}{8}$ in., for a member 100 ft long if shop inspection is adequate. Furthermore, the writer would point out that it seems reasonable to believe that a column with a central longitudinal diaphragm or web-plate normal to the main webs should have less initial crookedness than one in which the webs are laced together, the shop work being easier in this respect. The writer would propose a clause in construction specifications limiting allowable initial crookedness of a column to an amount equal to 0.125 in. plus 0.0002 L (in which L is length of column, in inches), and making a greater amount than this cause for rejection.

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[60] "Secondary Stresses in Kenova Bridge", *Bulletin No. 4*, Eng. Studies, Univ. of Minnesota, Minneapolis, Minn.

There are important initial end eccentricities to which columns may be subjected, which are not due primarily to rigid frame or secondary bending action but to what may be termed accidental misfits in shop or field work. Probably the most frequent are:

1.—Eccentricities due to improper column alignment due to such causes as the following: (*a*) Improper fit of lateral bracing may pull members out of line transversely; and (*b*) settlement of one or more falsework supports during riveting work.

2.—Eccentricities due to misfits at ends of columns which depend largely or wholly on direct bearing for transmission of load.

It should be noted that the methods of computing the ultimate strength of columns subjected to initial eccentricities of this type are not included by Mr. Young, since the eccentricity (that is, end moment divided by column load) is not constant, as may be assumed with eccentricities in a truss arising from secondary bending.

Any column may be subjected to eccentricities due to improper alignment, and unless careful check is maintained of the position of a structure during the construction stages such errors may easily escape detection. An example in the writer's experience is shown in Fig. 26 in which a bridge column, shop riveted to a length of 100 ft, was pulled out of line $\frac{3}{8}$ in. in the field by an improper fit of lateral bracing. This was corrected by reaming the lateral connections. Some idea of the shears that may occur in a column by this kind of accidental eccentricity is given by the curves shown in Fig. 27.

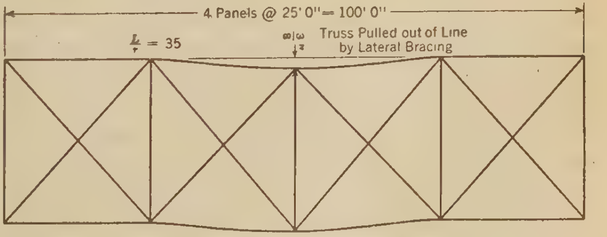
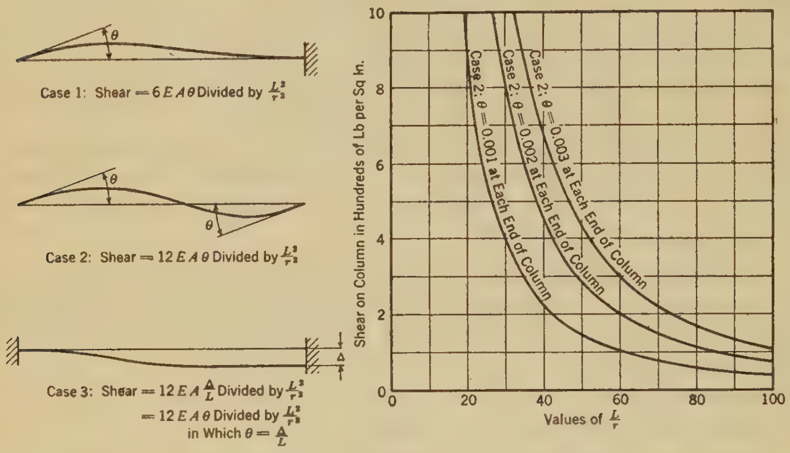

4 Panels @ 25' 0" = 100' 0"
$\frac{L}{r} = 35$
$\frac{3}{8}$
Truss Pulled out of Line by Lateral Bracing

FIG. 26.

It is the writer's belief that errors of this type occur more often than is generally realized.

Columns subject to eccentric loads due to imperfect end bearings include some of the largest and most important compression members in engineering structures. Among them are: (1) All end posts or main vertical posts at piers that bear directly on a footing, on pins, on shoes, or on grillages; (2) main truss, or building columns that depend partly or entirely on milled ends for load transfer at splices; (3) trunnion columns supporting main trunnion bearings or trunnion cross-girders in a bascule bridge, as well as columns supporting counterweights, or movable span and counterweights, in bascule or vertical lift bridges; and, (4) columns in movable bridges, that bear on main trunnions.



Case 1: Shear = $6EA\theta$ Divided by $\frac{L^2}{r^2}$

Case 2: Shear = $12EA\theta$ Divided by $\frac{L^2}{r^2}$

Case 3: Shear = $12EA\frac{\Delta}{L}$ Divided by $\frac{L^2}{r^2}$
= $12EA\theta$ Divided by $\frac{L^2}{r^2}$
in Which $\theta = \frac{\Delta}{L}$

FIG. 27.

In through bridges most portal columns are in Class (1), and these may be subject to severe bending from lateral loads in addition to direct loads. A source of serious error in fit at tops of the columns of Class (3) may be the necessity for shimming between column top and base of bearing or of cross-girder. Frequently shims of varying thickness are used in this case. As it is manifestly impossible to get a thin shim which is perfectly flat, and as usually the weight of the bearings is not sufficient to iron out small waves in the shims, except in the case of the very largest bridges, it is evident that the bearing will not take its final position on the column until considerable dead load from the structure has been added, and at that time the actual error usually can not be detected or corrected. Where a cross-girder rests on the column the problem is further complicated by the fact that the girder will be subject to flexure—often much flexure—after the fit has been made and the dead load added to the girder. In the case of the columns of Fig. 28 special adjustments were made, after careful study, to take care of this matter. If there is any error in shop or field in making the center of the trunnion normal to the plane of rotation of the truss, the members of Class (4) and their connections are subject to racking or weaving.

In order to illustrate what may be expected from present practice in this matter there are shown, in Fig. 28 and Fig. 29, the results of careful measurements taken by the writer on the milled ends of trunnion columns on two different bascule bridges. In each case the grillage top plates were finished in the shop and were set in the field with great care. The measurements were taken after the columns had been lined in both directions and had been placed in proper position to receive the cross-girders or bearings supporting the movable leaf. Measurements of fit between grillages and columns were made by inserting thickness gauges between the top of the grillage plates and the base of the column, under the main material of the column. Measurements on the tops of the columns in Fig. 29 were taken in a similar manner. All measurements were checked independently by other observers. Attention is called to the fact that in Fig. 28 the actual milled surfaces vary from the normal by an angle of 0.0012 radian, and that in Fig. 29 the surfaces vary from the normal by as much as 0.003 radian. In each case the weight of the columns themselves is sufficient to cause them to come to a firm bearing at the bearing edge or corner. It is apparent in the case of these columns that the error in milling bears no relation to the slenderness ratio of the column. The columns in these two bridges were fabricated by two different firms, each enjoying well established reputations for high grade work, and the work was inspected by professional shop inspectors. Furthermore, it should be noted that in the case of each bridge, the milling of the columns of the second leaf (not shown), was very satisfactory. In the case of the columns shown in Fig. 29, a proper bearing of the member on the grillage was obtained by tilting the columns. The Final Report of the Special Committee on Steel Column Research[61] shows even greater errors in end milling.

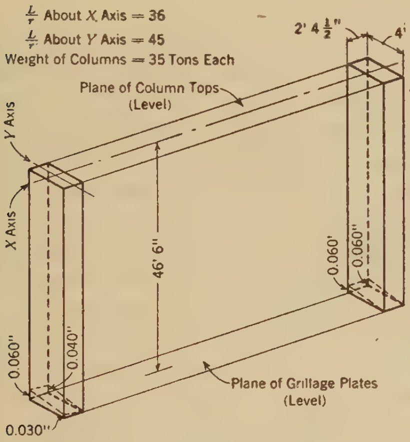

$\frac{L}{r}$ About X Axis = 36
$\frac{L}{r}$ About Y Axis = 45
Weight of Columns = 35 Tons Each
2' 4 1/2"
4' 2 1/2"
Plane of Column Tops (Level)
Y Axis
X Axis
46' 6"
0.060"
0.060"
0.060"
0.040"
0.030"
Plane of Grillage Plates (Level)

FIG. 28.

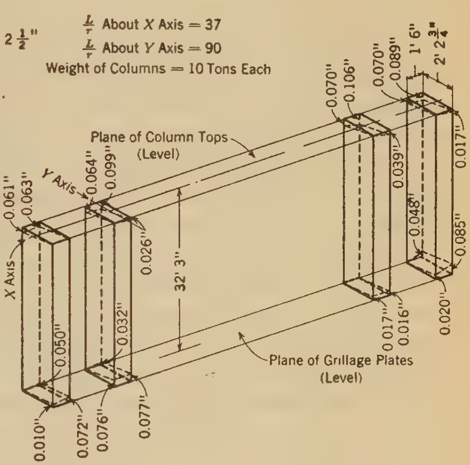

$\frac{L}{r}$ About X Axis = 37
$\frac{L}{r}$ About Y Axis = 90
Weight of Columns = 10 Tons Each
0.070"
0.106"
0.070"
0.089"
1' 6"
2' 2 3/4"
Plane of Column Tops (Level)
0.061"
0.063"
Y Axis
0.064"
0.099"
X Axis
0.026"
32' 3"
0.039"
0.017"
0.048"
0.085"
0.050"
0.032"
0.017"
0.016"
0.020"
Plane of Grillage Plates (Level)
0.010"
0.072"
0.076"
0.077"

FIG. 29.

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[61] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1412.

Other instances of measured eccentricities at ends of columns due to errors in milling or in field work are, as follows:

(1) In the case of a large movable bridge carried on a heavy cross-girder which latter was seated through castings on heavy columns the writer found the load to be off center on one column by an amount of 2½ in., and off center on another by an amount of 1½ in. These measurements were checked by independent observers also. In the case of two other similar columns on the same structure the load was found to be located centrally.

(2) Column No. 5 of the Equitable Building, in Des Moines[62], Iowa, showed stresses corresponding to a rotation of the lower end of the column through an angle of more than 0.001 radian.

(3) Column No. 4 of the American Insurance Union Building, at Columbus, Ohio[63], showed stresses corresponding to a rotation of the lower end of the column through an angle of 0.00138 radian.

Attention is directed to the implications of such eccentricities. In the case of all the columns shown on Figs. 28 and 29, the weight of the movable leaf was sufficient to force the column completely down to a bearing on the grillage plate. It should be noted that the errors due to milling at the base may well be duplicated at the top and that the eccentricities may act on the same side of the column, producing single curvature in the column and bending stresses at the middle (which may or may not be serious), or they may act on opposite sides of the column with resultant heavy shearing stresses. Fig. 27 illustrates this relation, showing the shears that will arise from this secondary bending at ends of members. The effect of direct column load, increasing bending and shears, is neglected. It is to be noted that the shears arising from this source alone are likely to be so heavy that they can not be taken by any reasonable amount of lacing.

It is the writer's belief, based on such information as the foregoing, that present methods of shop and field work and inspection may be expected to produce errors in end rotation of columns of 0.001 radian and that errors as large as 0.003 radian may occur easily. He would like to urge the necessity of a comprehensive investigation in shop and field of actual eccentricities occurring in structures.

The writer would also propose a special clause to cover milling for construction specifications and would suggest that definite limits to milling errors be set at an angle of 0.0005 radian and that larger errors than this be made cause for rejection. He would also call attention to the necessity for intelligent shop and field supervision in this work. Too often this work is entrusted to inspectors who have not adequate knowledge of the problem.

Local Failure.—The writer desires also to discuss briefly the question of local failure of the built-up short column and one or two factors producing such failure, together with its implications regarding the questions of column design raised in this paper. It is important to recognize that most of the models of large columns which have been tested and reported in available engineering literature have failed, not through integral action, but as a consequence of local distress. Consider, for example, the models of the second Quebec Bridge[64] having a stiffness ratio of $60 \frac{L}{r}$, or less; or, again, consider Models *U2M1* or *US2MS1* of the Memphis (Tenn.) Bridge.[65] For a number of years, prominent engineers even felt that it was impossible to design short columns that would not fail locally. One investigator has stated:[66]

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[62] *Bulletin* 72, Eng. Exp. Station, Iowa State Coll., Ames, Iowa, pp. 6 and 25, and Fig. 18.

[63] *Bulletin 40*, Eng. Exp. Station, Ohio State Univ., pp. 15, 16, 26, and Fig. 17.

> "Now if one point be clearer than any other in regard to built up columns, it is that they do not act as solid or homogeneous columns. Unless improperly designed they always fail locally by the flange buckling between panel points, and it is their strength in this connection which determines their strength as a whole. If, of course, the column deflect in a direction perpendicular to the plane of the web bracing (lacing) only, it will act as a solid column, and no question of built up column arises."

However, the integral failure of the ten models of the Metropolis (Ill.) Bridge columns may well be said to have established conclusively the fact that it is entirely practicable to design short columns that will act as a unit, and it seems obvious to the writer that, until knowledge and practice of designing main section, end connections, transverse bracing, and other features too often considered as secondary details, is such that local failure is avoided, it is futile to rely upon any theory of integral action in column design. To illustrate, the failure of Model *U2M1* of the Memphis Bridge, through local failure of the forked ends, at a unit stress 25% below the yield point of the material, is probably a good indication of the strength of the forked ends, but not necessarily any criterion as to the strength of the member as a whole. The writer wishes to urge the proposition that the rational design of steel columns must necessarily include the design of any construction feature the premature failure of which can precipitate the collapse of the main member. There are a number of features of this class which are not adequately treated in design specifications at the present time. Prominent among them are: (1) Lateral bracing in the plane normal to the plane of the main webs of a column, this bracing commonly being composed of lacing at present; (2) the question of the proper relation between the size and stiffness of main flange angles and the web-plates; and, (3) the form of section as influenced by Factors (1) and (2).

With regard to the matter of lacing the writer feels strongly that a complete revision of present practice is urgent. He wishes to call attention to the following propositions:

(*a*) A column is subject to bending in the plane normal to the main webs, often quite as much as in the plane of the webs. Lacing does not form a proper element to resist this bending;

(*b*) Lacing on columns at present is usually much overloaded and the design loads specified are frequently about 50% of total stress carried; and

(*c*) As used at present, lacing induces serious deformations on the webs

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[64] Final Report of Board of Engrs., Quebec Bridge, Vol. 1, pp. 197–215.

[65] *Technologic Paper No. 101*, National Bureau of Standards, U. S. Dept. of Commerce.

[66] "Columns", by E. H. Salmon, 1921, p. 185.

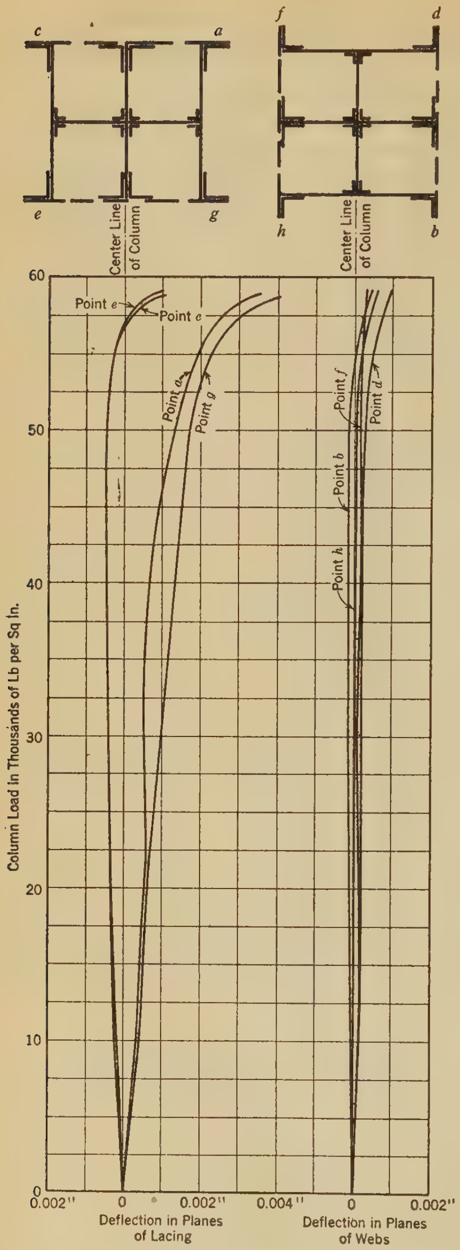

FIG. 30.

of the columns at splices, internal diaphragms, or other points of rigid support of the webs, with a direct material lowering of utilizable capacity of the main member.

The foregoing propositions will be considered in the order listed. In view of the relative flexibility of a system of lacing, as compared with a solid plate, it seems evident to the writer that the latter is essential in order to develop the maximum efficiency of a short column. To illustrate this point Fig. 30 shows the curves for deflection normal and parallel to main webs, of Model *U2U3* of the Municipal (St. Louis, Mo.) Bridge. It will be noted that at the useful limit point of the column (namely, about 50 000 lb per sq in.), the lateral deflection in the plane of lacing is about five times that in a direction parallel to the main webs. It is also to be noted that in the case of Models *HC1* and *MY1* of the Metropolis Bridge[67] that the deflection at failure in the plane of the lacing, even with the solid plate in the section, is much larger than the deflection in the plane parallel to the main webs. Furthermore, most lacing is liable to abuse in handling, shipping, and erecting; it is inherently weak as a structural detail in that it is connected at the ends with one or two rivets. The difficulties arising from the failure of lacing are too well known to require much discussion.[68] The collapse at Panama, in 1915, during the official test, and under a smaller load than that for which it was designed, of the new 280-ton floating crane built for the Federal Government, was caused by such failure of lacing. With lacing replaced by a solid central plate normal to the main webs, the effect of a loose rivet or two would not be serious.

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[67] *Technologic Paper No. 101*, National Bureau of Standards, U. S. Dept. of Commerce. Figs. 50 and 51.

[68] *Engineering News*, Vol. 73, May, 13, 1915, p. 918.

Present practice is to ignore completely the "participation" stresses induced in lacing, particularly double lacing, by deformation of the main column material under stress. Measurements on test models have shown that the lacing-bars adjacent to tie-plates, splice-plates, etc., are subjected to heavy induced stresses and, in fact, that these frequently exceed the stresses from transverse shears. In addition, the lacing on portal columns of through bridges is subjected to the shears arising from lateral loads on the structure which frequently are neglected in the design of the lacing. As a result, the lacing, at present, is often subjected to about twice the load for which it is designed. However, the main objection to present practice in lacing, from the writer's viewpoint, is that it induces distortions in the main column material at certain points and thus precipitates local failure of the member at a lower load than the material would otherwise carry. Lacing-bars acting as a pantograph, tend to force out of line the ribs of the column to which they are connected as the main material distorts axially under primary load. At points where the ribs are held rigidly in place, as at tie-plates, internal diaphragms, splice-plates, etc., this movement is prevented, but at the first connection adjacent to these points of support the lacing-bars are more or less free to move and, consequently, the distortion of the main material is greatest at these points. Careful measurement both of this movement and of the stresses induced in the webs of the column by it was made in connection with the tests on the models for the new Quebec Bridge.[69] The stresses in the main material at these points were found to be from 10 to 14% greater than elsewhere, and it is probable that the maximum stress was still greater since it was not possible to measure the stress at the sharp point of the local buckle. The number of test models of large built-up columns that fail locally as a result of this action is proportionately large. Conversely, the ten models of the Metropolis Bridge were remarkably free from local failure, and all were braced with substantial lacing supplied with transverse ties at the ends of the lacing, and all included, in the section, central longitudinal web-plate normal to the main webs. An interesting piece of evidence in this matter is afforded by the results of tests on Models *TC*4, *TC*5, and *TC*6, of the 1912 Quebec tests,[70] in which the introduction of a central diaphragm plate in the section apparently increased the capacity of the members about 18 per cent. It is to be noted that in the case of Model *U*2*U*3, of the Municipal Bridge, local failure through lacing distortion was present; and yet the model developed a high percentage of efficiency. The presence of the central longitudinal diaphragm undoubtedly contributed very materially to this result.

In view of all the foregoing considerations, the writer would propose that solid plates or diaphragms centrally located, and normal to the main web-plates, be required on all main structural steel columns having a stiffness ratio of $60 \frac{L}{r}$, or less, and that preference be given the use of a solid plate instead of lacing wherever possible. Furthermore, he would propose that all lacing on main structural columns be supplied with transverse ties at all end connections to prevent lateral displacement of the main webs, and that all such lacing be proportioned to carry the additional stress induced by this arrangement. He would further favor a more definite requirement in the specifications that all lacing and transverse webs on columns subject to lateral stresses be proportioned for such additional loads. The induced stresses in the lacing are easily and quickly computed.[71] Distribution of shears between the lacing and the solid plate when both are used should rest on a rational basis of relative rigidity of the two systems considered as a truss or a beam over the length of the member.

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[69] Final Rept. of Board of Engrs., Quebec Bridge, Vol. 1, pp. 207–208.

[70] *Loc. cit.*, pp. 201–209.

The use of the aforementioned features is well established by precedent. Columns of this type were used throughout on the Metropolis Bridge and on the Kansas City Bridge over the Missouri River. This type of lacing was also reported to have been used in 1914 on the Hoang Ho Bridge, in China.[72] The writer has also used the foregoing features on several fairly large highway bridges. He was pleased to find that the shop foremen reported that the tie-bars were an advantage in assembling the material and that no particular difficulty was encountered with either the lacing or with the central diaphragm. For smaller structures rolled or built-up I-beam sections, with solid webs, will often prove very suitable.

In the writer's judgment, present specifications regarding the relative proportions of flange angles to web are entirely inadequate. The real importance of this matter should not be overlooked. The chords of the first Quebec Bridge[73] were built of material having an elastic limit of about 40 000 lb per sq in., and they were designed originally to take a working stress of 24 000 lb per sq in. and, later, computed to take a working stress of 27 000 lb per sq in. The bridge chords failed at an estimated stress of 18 000 lb per sq in.[74] The first models[75], failing through inadequate lacing, collapsed at a unit stress estimated to be about 22 000 lb per sq in. The second model, adequate in lacing, failed by the buckling of webs and flange angles between lacing connections at a unit stress of about 30 000 lb per sq in. This type of failure is also evidenced in the test made by A. N. Talbot, Past-President and Hon. M. Am. Soc. C. E., and Professor H. F. Moore[76], in 1910. By analogy to the case of web-plates supported on the edges and partly supported by stiffeners in the center it is apparent that the factors involved in this problem are the relative stiffness of the plate and the angles, the length along the flange angles between lacing supports, and the width thickness ratio of the web-plates themselves. Inasmuch as the Bryan theory[77], on which the buckling theory of column web-plates rests, assumes complete support against lateral displacement of the edges of the web, it is apparent that the angles must furnish this support in addition to being stable under their own load. The writer would like to urge the advisability of adequate tests along this line, the work to begin where the web-plate tests made for the Delaware River Bridge ended.

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[71] *Engineering News*, October 3, 1907, Vol. 58, p. 369.

[72] *Technologic Paper 101*, National Bureau of Standards, U. S. Dept. of Commerce 1918, p. 13.

[73] *Engineering News*, Vol. 59, 1908, p. 422.

[74] *Loc. cit.*, pp. 404, 422.

[75] *Loc. cit.*, pp. 455–459.

[76] *Bulletin No. 44*, Univ. of Illinois Eng. Experiment Station, 1910, p. 34.

Finally, it seems to the writer that the foregoing considerations demonstrate conclusively that the form of section to be used must be taken into account when applying a column formula. The addition of a central longitudinal web-plate normal to the main webs will undoubtedly increase the unit stress that the member will carry, but will actually lower the allowable load if calculated on an $\frac{L}{r}$ basis alone, as is frequently the custom at present.

In conclusion, the writer desires to express his sincere appreciation and admiration for the remarkable work done by Mr. Young in developing a rational mathematical theory on what has been possibly the outstanding unsolved problem in structural engineering.

D. H. YOUNG,[78] JUN. AM. SOC. C. E. (by letter).—It is gratifying that the majority of the discussers are in agreement with the proposed basis for the design of steel columns as stated at the end of Section 1 of the paper. However, since some criticism of it has been made on the grounds that it is unjustifiable, it is, perhaps, worth while to recapitulate some of the considerations that led to its proposal.

First, in the case of a compressed column, it is an accepted fact that general imperfections, such as initial crookedness, accidental eccentricity of load, irregularities in cross-sectional area, and lack of homogeneity of the material, affect, considerably, the behavior of the column. It is also well known that the effect of such imperfections can be represented either by assuming a definite initial curvature of the column axis or by assuming a definite eccentricity of the compressive load. Thus, in discussing the behavior of actual columns under load, it is not only justifiable, but absolutely essential, to make one of the foregoing assumptions to allow for the effect of general imperfections, regardless of how indeterminate the proper degree of initial curvature or eccentricity of load may be. Such an assumption is a far more fundamental point than being "merely a method of applying a variable factor of safety" as Mr. Osgood calls it.

Accepting the logic of this assumption, the second step is to consider the general behavior of a pin-ended, initially curved, steel column under the action of a gradually increasing compressive load (see Fig. 4). Such a column begins to deflect laterally from the beginning of loading, and at a certain value of the compressive load yielding begins in the extreme fibers at the cross-section of greatest bending moment. This beginning of yielding may be termed a "localized failure" of the material and the corresponding value of the compressive load denoted by P_y. As the load continues to increase above the value of P_y it soon reaches a value at which the column buckles; that is, continues to bend laterally, without further increase of load. This condition may be termed "complete failure" of the column, and the corresponding load, P_u, is the ultimate load that the column can carry. In considering whether P_y or P_u should be taken as a basis for the selection of the working load for a steel column, the writer chose P_y for the following reasons:

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[77] "Problems Concerning Elastic Stability in Structures", by S. Timoshenko, *Transactions*, Am. Soc. C. E., Vol. 94 (1930), pp. 1010–1014.

[78] Instr., Eng. Mechanics, Univ. of Michigan, Ann Arbor, Mich.

(1) The beginning of yielding represents a definite damage to the material, if not complete failure of the column, and the load, P_y, which causes such damage may well be regarded as the limit of structurally useful strength of the column. That the Engineering Profession is well agreed on this point is evidenced by the fact that it is accepted practice to select the working loads for beams, subject to transverse loading, on the basis of that loading which first produces yielding in the extreme fibers of the beam.

(2) The determination of the ultimate load, P_u, necessitates a complicated analysis involving the properties of the material beyond the proportional limit (as ordinarily defined). The degree of accuracy with which these properties can be known is not very high and, as a result, calculations of the ultimate load, P_u, are of more academic than practical value.

(3) For most cases of actual steel columns initial imperfections must be expected of such an order of magnitude that there is no great difference between the ultimate load, P_u, and the load P_y, at which yielding first begins in extreme fibers.

When Mr. Osgood criticizes the proposed basis for design on the grounds that "it results in excessive safety for short columns, * * *, and it makes the H-section appear stronger than the rectangular section, * * *", he is using as a definition of strength the ultimate buckling load, P_u. The writer believes that, to be consistent, Mr. Osgood should object also to the present, well accepted basis for the design of steel beams, subjected to transverse loading. In this case, also, current design methods make the I-section appear stronger than the rectangular section when judged on a basis of ultimate strength. In fact, this difference is even greater in the case of beams than in the case of columns.

Finally, in calculating the load, P_y, the writer assumes the material to follow Hooke's law up to the yield-point stress. In his discussion, Mr. Osgood concludes that "it is entirely unjustifiable, therefore, to assume the ideal stress-strain diagram of Fig. 2 in explaining the behavior of columns, except possibly at large values of the eccentricity ratio." As a matter of fact, there is good justification for this assumption on two counts:

(1) The writer concludes from Fig. 8 that the "large values of the eccentricity ratio", to which Mr. Osgood refers, are probably never greater than $\frac{\Delta}{k} = 0.1$. Indeterminate as the factor, Δ, may be, there is good indication that (except, possibly, for very short columns, $\frac{L}{r} < 40$, which always fail

at $\frac{P}{A} = f_y$) general imperfections necessitating an allowance for $\frac{\Delta}{k} > 0.1$ must be expected in the case of actual columns.

(2) As pointed out by Professor Turneaure, in his discussion, "the variation in yield point and in the form of the stress-strain curve in the vicinity of this point, * * * renders a more exact treatment * * * of little practical value for structural columns."

Development of Rational Design Formulas.—Following the proposed basis, rational design formulas are developed in the paper for the case of a pin-ended column under two conditions of loading: (1) The condition of loading considered in Section 2 of the paper, which is simply that of two compressive end loads; and (2) the condition of loading, considered in the first part of Section 4 of the paper, which is that of compressive end loads applied with given (not accidental) eccentricities, e_0 and e_1. For Condition (1) it is most convenient to allow for the effects of general imperfections by assuming an initial shape of the column axis in the form of a half-sine wave having a maximum deviation from a straight line through the ends of Δ. The resulting formula for determining the loading which first produces yielding is represented by Equation (12). For Condition (2) it is most convenient to allow for the effect of general imperfections by increasing the given eccentricities, e_0 and e_1, algebraically by equal amounts, e. The resulting formulas for determining the loading which first produces yielding are represented by Equation (19) and Equation (22). As mentioned by Professor Niles, design formulas can be derived by the same procedure for other conditions of loading, for example, such as a combination of axial and transverse loads.

Considerable discussion hinged upon the writer's choice of $\Delta = \frac{L}{400}$, to allow for the effect of general imperfections in the fundamental case of a pin-ended column. One finds, among the discussions, suggested values ranging from $\frac{L}{1\,200}$, as given by Mr. Gray, to $\frac{L}{300}$ as given by Mr. Godfrey. The writer chose $\frac{L}{400}$ from a study of recommendations given by various experimental investigators. Professor Turneaure suggests that this value seems too severe and points out that, as a result of its studies, the Special Committee of the Society on Steel Column Research adopted a crookedness value which corresponds approximately to $\Delta = \frac{L}{1\,200}$. It must be kept in mind, however, that the value, $\Delta = \frac{L}{400}$, as chosen by the writer, is intended to allow, not only for crookedness, but also for accidental eccentricity of load, variations in cross-sectional area, and all imperfections in general. For the most part, the Committee used special bearing-blocks by which eccentricity of load could be controlled and, hence, no conclusions can be made from such tests, regarding the possible extent of accidental eccentricities of load in the case of columns in actual structures. The writer agrees with Mr. Wyly that a careful study conducted, not only in the laboratory, but also in the field and shop, is necessary for determining the proper allowance for the effect of general imperfections. Mr. Wyly's discussion shows clearly the variety of factors which must be considered in the adoption of a reasonable value for such a factor as Δ. Particularly does it show that initial crookedness is by no means the predominate factor among general imperfections and that errors in alignment, etc., may result in very high accidental eccentricities of load. Thus, the writer does not believe that his proposal of $\Delta = \frac{L}{400}$ is by any means ultra-conservative. Mr. Wyly, however, presents good evidence to the effect that the value taken for Δ should include a constant term plus a term that increases with the length, L.

Rational Formulas vs. Empirical Formulas.—Equation (12) will serve as a basis for a comparison between rational and empirical design formulas of the straight-line and parabolic types. The principal advantage of the rational formula is that it shows explicitly what factors affect the strength of a column and, in addition, just what effect each factor has. For example, Equation (12) shows the strength of a pin-ended steel column to depend upon the following factors: (*a*) Properties of the material, as represented by f_y and E; (*b*) the extent of general imperfections, as represented by Δ; (*c*) the slenderness of the column, $\frac{L}{r}$; and (*d*) the shape of the cross-section, as defined by the core radius, k, as compared with the radius of gyration, r.

The only argument ever advanced against the rational formula is that it is cumbersome to use in practice. Regarding empirical design formulas, it is beyond question that the only argument in their favor is their simplicity.

As to the arguments against them the writer can do no better than to refer to the discussion by Mr. Durant, with which he is in complete agreement. To Mr. Durant's arguments against the empirical formula, one more may be worthy of addition; namely, that there is a definite psychological advantage in having the designer work directly with a rational formula which, like Equation (12), for example, keeps him constantly reminded of the various factors that influence the strength of the column with which he is dealing. In this way he may be able to take advantage, whenever possible, of favorable conditions regarding one or another of the aforementioned factors and thus make some saving in material or improvement in design. On the other hand, if he works day after day with a straight-line or parabolic type of formula in which $\frac{L}{r}$ appears to be the only factor affecting the strength of a column, he may soon forget the general characteristics of column action and, particularly, the inherent uncertainties which go with it.

Columns with Other Than Pinned Ends.—Regarding the adaptation of Equation (12) to the case of a column with other than pin-ended conditions of support, the writer agrees with Messrs. Sturm and Holt and with Mr.

Hartmann that this can be done by using in Equation (12) a modified value of s_e, as given by Equation (86). This is an important point since it extends considerably the range of application of Equation (12). Furthermore, it is a point that was not stressed in the paper, and it is well that these discussions have called attention to it. However, as a general method of dealing with columns having other than pin-ended conditions of support it has its definite limitations and since these limitations were not mentioned by the discussers it is, perhaps, worth while to call attention to them.

To begin with, such a procedure evidently takes for granted that when yielding first begins in the extreme fibers, the column axis will have such inflection points that the "free length" between them will be equal to the reduced length, $L' = \frac{L}{N}$, as determined by Equation (86). In general, this circumstance will be realized only by virtue of the existence of two conditions. The first is that, to allow for the effect of general imperfections, one assumes an initial shape of the axis of the column which agrees in form with the configuration of elastic buckling consistent with the constraints. Only by virtue of such an initial shape, will the "free length" between inflection points remain constant during loading and equal to the effective length, $L' = \frac{L}{N}$. In view of the highly indeterminate nature of general imperfections, it seems justifiable in any case to assume such an initial shape, thus fulfilling the first condition in effect if not in fact. The second condition which must be realized is that, during loading, any moments arising at points of constraint must be independent of change in length of the column due to compression, because, regardless of the choice of an initial shape, if such secondary moments do arise they will cause the "free length" between inflection points to change gradually during loading. For such cases it is not justifiable to use a reduced length, $L' = \frac{L}{N}$, in Equation (12), in calculating the load that first produces yielding.

In summary, the use in Equation (12) of a modified value of s_e, as given by Equation (86), for the calculation of the load first producing yielding in the case of a column with other than pin-ended supports will be justifiable: (*a*) For all cases of columns in which the manner of support is such that the constraints are of an inelastic nature (see Fig. 31); and (*b*) for columns with elastic constraints, providing the arrangement is such that bending of the column, during loading, is independent of any change in its length due to compression (see Fig. 32). The same procedure will be unjustifiable for columns with elastic constraints where the arrangement is such that bending of the column, during loading, depends upon change in its length due to compression (see Fig. 33). Cases such as those shown in Fig. 33 represent column action in rigid-frame construction and require special consideration.

The writer does not agree with Professor Turneaure that any allowance in the form of a reduced length equal to 85% of the true length should be made for possible end constraint due to friction in the case of pin-ended columns. Such a procedure might be justifiable if all loads on a pin-connected structure were static, but in the case of bridges, subjected to moving loads, vibrations of the structure are certain to be set up and such vibrations can easily result in a gradual rotational creep of the ends of a pin-ended compression member. Thus, it seems that frictional constraint at the ends of a pin-ended column is too uncertain a factor to justify any advantage being taken of it and that under certain conditions a pin-ended column designed on the basis of Equation (94) may by no means have the intended factor of safety.

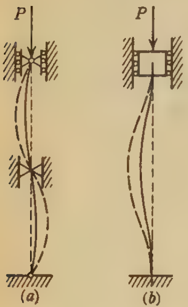

FIG. 31.

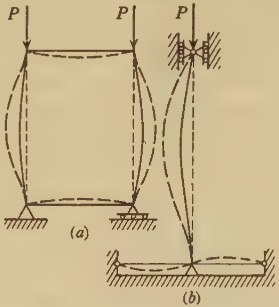

FIG. 32.

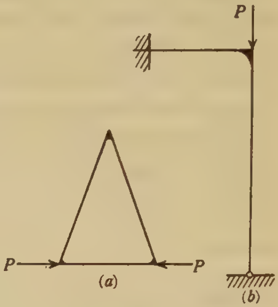

FIG. 33.

Columns in Rigid Frame Construction.—It is pointed out in the paper that an exact analysis of column action in rigid-frame construction is too laborious to be practicable. Instead, it is proposed to calculate all secondary end moments by the usual approximate method and then to use Equation (19) and Equation (22), or curves, such as those of Fig. 12, plotted from these equations, as a basis for design. Such a procedure amounts to the assumption that the effective length of the member is the same as its true length and also that the secondary end moments increase in direct proportion to the axial compressive force in the member. Both these assumptions are conservative and hence the use of Equations (19) and (22), or curves like those of Fig. 12 plotted from them must, of course, result in some extra safety. However, in many practical cases, this indeterminate degree of extra safety is by no means always as large as might be expected. For example, in the case of a truss, the chord members are usually much more rigid than the web members. As a result of this the effective length of a web member will be considerably less than its true length. On the other hand, web members are usually bent in double curvature so that the maximum fiber stress occurs at one end and the member can be designed on the basis of Equation (19) where no question of effective length is involved. Again, as emphasized by Mr. Durant, chord members receive little constraint from the web members, and their effective lengths are approximately their geometric lengths. This is particularly true when one considers the behavior of a chord member in a plane perpendicular to the plane of the truss. Thus, in such cases, there is little to prevent bending of the chord members in alternate directions in successive panels. This raises the question as to whether or not the Special Committee on Steel Column Research is justified in its selection of an effective length equal to 75% of the geometric length for all cases of compression members in a riveted truss.

Built-Up Columns.—Regarding the strength of a built-up column as a unit (that is, assuming adequate lacing, diaphragms, etc.), Equation (12) can be used as a basis for design by introducing the fictitious length, L', as given by Equation (27) or Equation (28). In this case, the fictitious length, L', is greater than the true length, L, thus compensating for the additional flexibility of the column due to shearing deformation of the lacing or battens.

Regarding the proportioning of details, such as lacing or batten bars, to make certain that the built-up column will act as an integral unit, Mr. Wyly raises considerable objection to present specifications regarding this point. The writer believes that a proper basis for the design of such details is presented in Part II of the paper. The manner of allowing for the effect of general imperfections, as regards shear, is discussed in the second half of Section 5, and corresponding design formulas (Equations (33) and (39)) are developed in Sections 6 and 7. The manner of allowing for shear due to secondary end moments which occur in the case of a column in a truss, is discussed fully in Section 8, and Equation (45) is proposed as a basis for design. It is seen that in the case of a short column under the most unfavorable conditions, both as to general imperfections and secondary end moments, the necessary allowance for shear may be so high as to prohibit the use of a column of non-solid cross-section.

Mr. Durant points out that to be perfectly rigorous in the derivation of the equations of Sections 6, 7, and 8 of the paper, the writer should have used as the basic differential equation of the elastic curve, Equation (97), instead of Equation (96). Although this is true the writer believes that the error in the average shear stress, v_y, as given by Equations (33), (39), and (45), can be somewhat compensated for by using, in these equations, or in the curves plotted from them, the fictitious length, L', as given by Equation (27) or by Equation (28). Incidentally, the error appears to be on the side of safety.

Miscellaneous.—Messrs. Sturm and Holt attempt to extend the range of application of Equation (12) to the determination of the inelastic buckling load by the use of a reduced Eulerian value, s', as defined by their Equations (83), (84), and (85). However, they give no justification for such a procedure, except that values of s_u, calculated from their Equations (87) and (88) agreed "satisfactorily" with certain test results[47]. The principal obstacle to the use, in Equation (12), of a reduced Eulerian value, s'_e, calculated by the use of a reduced tangent modulus, is that such a substitution (see Equation (87)) amounts to the assumption that the material throughout the full length of the column behaves in accordance with this reduced modulus, whereas this inelastic behavior is usually confined to a relatively short portion of the length of the column.

Mr. Hartmann proposes a column formula (Equation (108)), the use of which he advocates in actual design. Mr. Hartmann's exceedingly careful definition of the factor of safety, n, scarcely seems consistent with the manner in which, subsequently, he illustrated the application of this formula. The writer cannot agree that Equation (108), when used as Mr. Hartmann advises, bears any similarity to the basis of design proposed in the paper.

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[47] "Strength of Tubing Under Combined Axial and Transverse Loading", by L. B. Tuckerman, S. N. Petrenko, and C. D. Johnson, Technical Notes of National Advisory Comm. for Aeronautics, No. 307, June, 1929.

Professor Niles shows how the equations in Sections 4 and 8 can be simplified somewhat for practical use by retaining in them the variable, x_m, which defines the position of the section of maximum bending moment. His Equations (54) to (58) should prove useful to any one who desires to plot sets of curves like those of Figs. 12 and 18.

Mr. Hartmann questions the comparison made in Fig. 16 between the writer's equations for allowance for shear in a pin-ended column and the 1925 A. R. E. A. Specification. He assumes that the factor of safety has been overlooked and suggests that values of $\frac{v_y}{n}$ instead of v_y should have been plotted, which would have had the effect of lowering all the writer's curves considerably. As a matter of fact, what the writer did was to double all ordinates of the straight line representing the A. R. E. A. Specification on the assumption that it intended a factor of safety of approximately two. This surely is equivalent to Mr. Hartmann's suggestion.

In closing, it is worthy of notice, as brought out by Professor Baker, that a rational column formula, including the effect of all factors in Equation (12), except that of the shape of the cross-section, has been incorporated in the British Standard Specification for the Use of Structural Steel in Buildings and that a British Steel Structures Research Committee is considering the problem of steel columns subjected to secondary end moments, as in the case of compression members in rigid frame construction. It is hoped that this paper may have some influence in bringing the design of steel columns in the United States to a more rational basis than is represented by present-day practice. The writer agrees completely with Mr. Gray in his statement that the paper "touches only lightly on the subject", but he agrees also with Mr. Godfrey in that "its attack is in the right direction."

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1932

ANALYSIS OF THICK ARCH DAMS, INCLUDING ABUTMENT YIELD

BY PHILIP CRAVITZ,[1] JUN. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. I. M. NELIDOV, A. FLORIS, AND PHILIP CRAVITZ.

SYNOPSIS

A graphical solution of stresses in a circular arch under the influence of normal loads is presented herein. These curves differ from those previously published, in that they include the effect of abutment yield. Furthermore, a finite value for Poisson's number is used rather than the infinite value previously assumed for simplication.

Although no novel method of analyzing arches is introduced, the derivation of the final stress equations is traced because it combines advances made by Cain, Jakobsen, Vogt, and others. The forms of the equations are greatly altered, in order to facilitate computations and the plotting of graphs.

Some examination is made to determine the quantitative effect of variations in the necessary assumptions of the values of the modulus of elasticity of rock, and Poisson's number. An example of the use of the curves is appended.

INTRODUCTION

In the past several years, great strides toward a more rational solution of the stresses in an arch dam have been made by numerous contributors. Of necessity, the authors of these papers assumed either fixed or hinged-end conditions, and precise mathematical relations were built up on this basis. In an effort to consolidate the gains made in this particular field of indeterminates, F. H. Fowler, M. Am. Soc. C. E., prepared a series of graphs,[2] from which the stresses at the critical sections of crown and abutment may be obtained. That such a graphical procedure is necessary for general practical use may be

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NOTE.—Published in January, 1935, *Proceedings*.

[1] Design Engr., Los Angeles County Flood Control Dist., Los Angeles, Calif.

[2] "A Graphic Method of Determining Stresses in Circular Arches under Normal Loads by the Cain Formulas," by F. H. Fowler, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 92 (1928), p. 1513.

verified easily by examining the complex equations arising from the mathematical treatment.

For relatively thin arches, the assumed end conditions do not introduce an appreciable error. However, it is generally recognized that, for relatively thick arches, the assumptions give stresses that can only be considered as approximate.

That the final reliance in the design of such a structure as an arch dam must rest upon the judgment of an experienced designer can scarcely be gainsaid. However, although an exact solution of the stresses in so statically an indeterminate structure is almost never to be hoped for, the judgment of the designer can be assisted more adequately by an analysis that includes all the known factors of the same magnitude of importance than by one that neglects one or two such factors.

Quantitatively, light was first shed on the bothersome subject of abutment yield by Fredrik Vogt, Assoc. M. Am. Soc. C. E., in 1924, and again in 1927.[3] Unfortunately, the stress equations arising from the inclusion of this factor and Poission's effect, are many times more unwieldy and impractical for general design use than the equations that exclude these factors. That the complexity of these equations should be mastered is evidenced by the fact that, in many instances, there may be a possible saving of concrete when the arch is analyzed on the basis of yielding abutments, because of the more equable distribution of stress between crown and abutment.

Discussion of the division of load between cantilevers and arches is not pertinent to this paper because it was possible to construct the graphs so that the curves would be applicable regardless of the proportion of water load determined, or assumed, to be carried by the arches. Wherever possible, detailed derivation is eliminated and reference is made to the origin.

Notation.—In the notation advanced herein (see Appendix), an effort has been made to reconcile the conflicting symbols introduced by several writers, with the American Standard Symbols for Mechanics, Structural Engineering, and Testing Materials.[4]

DERIVATION OF STRESS EQUATIONS

Arch Ring Deformations.—Using the customary sign conventions (that is, linear deformations are positive when they are elongations; stresses are positive when they produce a positive deformation; and, water pressure, p_e, is taken as positive), Cain and Jakobsen derive the relation for an arch ring, shown in Fig. 1,[5] as follows:

$$\Delta_C = X \left\{ \frac{r^2_n}{E_c\, I_n} \left[r_C\,(\phi_1 - \sin\phi_1) - r_n \left(\sin\phi_1 - \frac{\phi_1}{2} - \frac{\sin 2\,\phi_1}{4} \right) \right] \right.$$
$$\left. + \frac{r_n}{E_c\, t} \left[1.94\,\phi_1 - 0.47\,\sin 2\,\phi_1 \right] \right\} - \frac{\sigma}{E_c\, t}\, p_e\, r_e\, r_n \sin\phi_1 \ldots\ldots\ldots (1)$$

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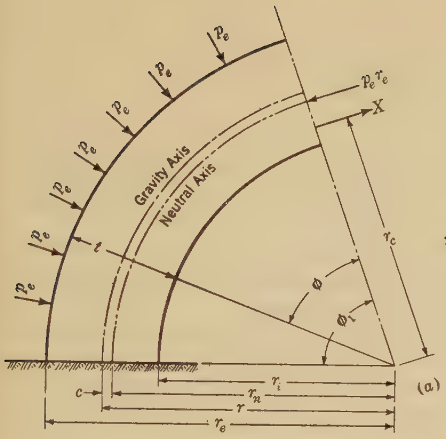
[3] "Ueber die Berechnung der Fundamentdeformation," by Fredrik Vogt, Assoc. M. Am. Soc. C. E., Math. Naturv. Klasse 1925, No. 2; also "Stresses in Thick Arches of Dams": Discussion by Professor Vogt, *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 554.

[4] A.S.A.—Z10*a*—1932.

[5] "Stresses in Thick Arches of Dams," by B. F. Jakobsen, *Transactions*, Am. Soc. C. E., Vol. 90 (1927), pp. 475 and 522.

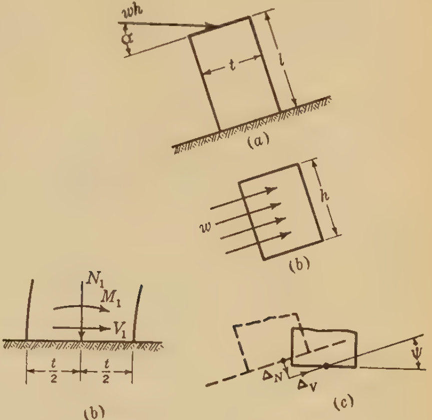
and,

$$\psi_C = (r_C\,\phi_1 - r_n \sin \phi_1)\,\frac{X\,r_n}{E_c\,I_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$



p_e
p_e
p_e
p_e
p_e
p_e
p_e
p_e
$p_e r_e$
X
Gravity Axis
Neutral Axis
t
r_c
ϕ
ϕ_1
(a)
c
r_i
r_n
r
r_e

FIG. 1.



wh
α
t
l
(a)
w
h
(b)
N_1
M_1
V_1
$\frac{t}{2}$
$\frac{t}{2}$
(b)
Δ_N
Δ_V
ψ
(c)

FIG. 2.

The following pertinent relations also are derived:

$$M = X\,(r_C - r_n \cos \phi) \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$N = p_e\,r_e - X \cos \phi \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and,

$$V = X \sin \phi \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which, M, N, and V are the moment, thrust, and shear, respectively, at any arbitrary section at an angle, ϕ, from the crown.

Foundation Deformations.—Although the ideal case, upon which Professor Vogt's foundation deformation equations are based,[3] can never be exactly attained in practice, the quantitative information derived is probably correct within a usable range of accuracy. A gratifying check on his theoretical and empirical forms has been made by the Committee of Engineering Foundation on Arch Dam Investigation.[6]

In Fig. 2, Professor Vogt presents a hypothetical cantilever of length, l, and thickness, t. In plan, the loaded area is equal to h. Under the influence of the load, $w\,h$ (assuming plane sections to remain plane), any deformation of the foundation may be resolved into the components, Δ_{n1}, Δ_v, and ψ. In terms of N_1, V_1, and M_1, at the base, their values are as follows:

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[6] Rept. by the Committee of Engineering Foundation on Arch Dam Investigation, Vol. II, May, 1934, pp. 225 to 232, inclusive.

$$\Delta_N = \frac{m^2 - 1}{m^2} \times \frac{N_1}{E_R\, t} \times \sqrt[3]{t^2 h} \ldots\ldots\ldots\ldots \tag{6}$$

$$\Delta_V = \frac{m^2 - 1}{m^2} \times \frac{V_1}{E_R\, t} \times \sqrt[3]{t^2 h} + \frac{(m-2)(m+1)}{m^2} \times \frac{M_1}{E_R\, t} \times \frac{1}{1 + 1.1\,\frac{t}{h}} \ldots \tag{7}$$

and,

$$\psi = \frac{18}{\pi} \times \frac{m^2 - 1}{m^2} \times \frac{M_1}{E_R\, t^2} \times \frac{1}{1 + 0.25\,\frac{t}{h}} + \frac{(m-2)(m+1)}{m^2} \times \frac{V_1}{E_R\, t} \times \frac{1}{1 + 1.1\,\frac{t}{h}} \ldots\ldots\ldots\ldots \tag{8}$$

in which, E_R = the modulus of elasticity of rock; $N_1 = w \sin \alpha$ = normal force per unit of length; $V_1 = w \cos \alpha$ = shear force per unit of length; and $M_1 = w\, l \cos \alpha$ = moment force per unit of length.

Where the moment is referred to the center, or gravity, axis, Equations (6), (7), and (8) are reduced to;

$$\Delta_N = \zeta \times \frac{N_1}{E_R} \ldots\ldots\ldots\ldots \tag{9}$$

$$\Delta_V = \zeta \times \frac{V_1}{E_R} + \eta\, \frac{M_1}{E_R\, t} \ldots\ldots\ldots\ldots \tag{10}$$

and,

$$\psi = \mu\, \frac{M_1}{E_R\, t^2} + \eta\, \frac{V_1}{E_R\, t} \ldots\ldots\ldots\ldots \tag{11}$$

in which the values of ζ, η, and μ are obvious by comparison.

For the arch, the moment force about the gravity line is,

$$M_1 = X\,[r_C - r \cos \phi_1] + p_e\, r_e\, c \ldots\ldots\ldots\ldots \tag{12}$$

and N_1 and V_1 are the same as N and V in Equations (4) and (5), respectively. Inserting these values of N_1, V_1, and M_1, into Equations (6), (7), and (8), and taking the components of Δ_N and Δ_V normal to the crown radius, the foundation yield would allow a crown-section movement of:

$$\Delta_F = \frac{\zeta}{E_R}\,[V_1 \sin \phi_1 - N_1 \cos \phi_1] + \eta\, \frac{M_1}{E_R\, t} \sin \phi_1 + \left[\mu\, \frac{M_1}{E_R\, t^2} + \eta\, \frac{V_1}{E_R\, t}\right] [r_n\,(1 - \cos \phi_1) - c \cos \phi_1] \ldots\ldots \tag{13}$$

or, expanded and re-arranged:

$$\Delta_F = \frac{X}{E_R} \left[\zeta - \eta\, \frac{r}{t} \cos \phi_1 \sin \phi_1 - \left(\mu\, \frac{r}{t} \cos \phi_1\right) \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right) \right.$$

$$+ \eta \sin \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \Bigg] + \frac{X\, r_C}{E_R} \left[\frac{\eta}{t} \sin \phi_1 + \frac{\mu}{t} \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \right]$$

$$+ \frac{p_e\, r_e}{E_R} \left[\eta \frac{c}{t} \sin \phi_1 - \zeta \cos \phi_1 + \mu \frac{c}{t} \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \right] \ldots\ldots (14)$$

and,

$$\psi_F = \mu \frac{M_1}{E_R\, t^2} + \eta \frac{V_1}{E_R\, t} = \frac{X}{E_R\, t} \left(\eta \sin \phi_1 - \mu \frac{r}{t} \cos \phi_1 \right)$$

$$+ \frac{X\, r_C\, \mu}{E_R\, t^2} + p_e\, r_e \frac{\mu\, c}{E_R\, t^2} \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Derivation of Variables, K and r_C, Including Yield.—Using the two known end conditions of the crown section, due to symmetry, namely, Equations (16) and (17):

$$\Sigma\, (\Delta_C + \Delta_F) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

and,

$$\Sigma\, (\psi_C + \psi_F) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

and substituting for X the product, $K\, p_e\, r_e$, two equations are derived in terms of t, r, ϕ, m, ζ, μ, and η, and the two unknowns, K and r_C. Furthermore, it is assumed that $E_c = E_R$ and that $I_n = I = \frac{1}{12} t^3$. Thus:

$$\left[\sigma \frac{r_n}{t} \sin \phi_1 + \zeta \cos \phi_1 - \eta \sin \phi_1 \frac{c}{t} - \mu \frac{c}{t} \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \right]$$

$$- K \Bigg[- 12 \left(\frac{r_n}{t} \right)^3 \left(\sin \phi_1 - \frac{\phi_1}{2} - \tfrac{1}{4} \sin 2\, \phi_1 \right) + \frac{r_n}{t} (1.94\, \phi_1 - 0.47 \sin \phi_1)$$

$$+ \zeta - \eta \sin \phi_1 \frac{r}{t} \cos \phi_1 - \mu \frac{r}{t} \cos \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right)$$

$$+ \eta \sin \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \Bigg] - K \frac{r_C}{t} \Bigg[12 \left(\frac{r_n}{t} \right)^2 (\phi_1 - \sin \phi_1)$$

$$+ \eta \sin \phi_1 + \mu \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \Bigg] = 0 \ldots\ldots\ldots\ldots (18)$$

and,

$$K \frac{r_C}{t} \left[12 \left(\frac{r_n}{t} \right) \phi_1 + \mu \right] - K \left[12 \left(\frac{r_n}{t} \right)^2 \sin \phi_1 - \eta \sin \phi_1 + \mu \frac{r}{t} \cos \phi_1 \right]$$

$$+ \left[\mu \frac{c}{t} \right] = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

The solution of Equations (18) and (19) simultaneously may be simplified greatly by a convenient grouping of variables such that:

$$Q_1 = \Bigg[-12\left(\frac{r_n}{t}\right)^3 \left(\sin \phi_1 - \frac{\phi_1}{2} - \tfrac{1}{4} \sin 2\,\phi_1\right) + \frac{r_n}{t}\,(1.94\,\phi_1 - 0.47 \sin 2\,\phi_1)$$

$$+ \zeta - \eta \sin \phi_1 \frac{r}{t} \cos \phi_1 - \mu \frac{r}{t} \cos \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)$$

$$+ \eta \sin \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)\Bigg] \dots\dots\dots\dots\dots\dots(20)$$

$$Q_2 = \left[12\left(\frac{r_n}{t}\right)^2 (\phi_1 - \sin \phi_1) + \eta \sin \phi_1 + \mu\left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)\right] \dots(21)$$

$$Q_3 = \left[\sigma \left(\frac{r_n}{t}\right) \sin \phi_1 + \zeta \cos \phi_1 - \eta \frac{c}{t} \sin \phi_1 - \mu \frac{c}{t}\left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)\right].(22)$$

$$Q_4 = \left[12 \left(\frac{r_n}{t}\right)^2 \sin \phi_1 - \eta \sin \phi_1 + \mu \left(\frac{r}{t}\right) \cos \phi_1\right] \dots\dots(23)$$

$$Q_5 = \left[\mu \left(\frac{c}{t}\right)\right] \dots\dots\dots\dots\dots\dots(24)$$

and,

$$Q_6 = \left[12 \left(\frac{r_n}{t}\right) \phi_1 + \mu\right] \dots\dots\dots\dots\dots(25)$$

Substituting Equations (20) to (25) in Equations (18) and (19), the following more manageable forms are derived:

$$Q_3 - K\,Q_1 - K \frac{r_C}{t}\,Q_2 = 0 \dots\dots\dots\dots\dots\dots(26)$$

and,

$$K \frac{r_C}{t}\,Q_6 - K\,Q_4 + Q_5 = 0 \dots\dots\dots\dots\dots\dots(27)$$

or, solving simultaneously,

$$K = \frac{(Q_3 \times Q_6) + (Q_5 \times Q_2)}{(Q_1 \times Q_6) + (Q_4 \times Q_2)} \dots\dots\dots\dots\dots(28)$$

and,

$$\frac{r_C}{t} = \frac{(Q_4 \times Q_3) - (Q_1 \times Q_5)}{(Q_2 \times Q_5) + (Q_3 \times Q_6)} \dots\dots\dots\dots\dots(29)$$

Stress Equations in Terms of K and r_C.—Professor Cain has derived the equations of stress for a thick arch in the form:

At the Crown:

$$s_e = \frac{M_c}{I} \, \frac{\dfrac{t}{2} + c}{r_e} \, r_n - \frac{P_C}{r_e \log_e \left(\dfrac{r_e}{r_i}\right)} \dots\dots\dots\dots\dots(30)$$

and,

$$s_i = -\frac{M_c}{I}\,\frac{\frac{t}{2} - c}{r_i}\,r_n - \frac{P_C}{r_i \log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots\ldots\ldots\ldots (31)$$

At the Abutment:

$$s_e = \frac{M_1}{I}\,\frac{\frac{t}{2} + c}{r_e}\,r_n - \frac{P_1}{r_e \log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots\ldots\ldots\ldots (32)$$

and,

$$s_i = -\frac{M_1}{I}\,\frac{\frac{t}{2} - c}{r_i}\,r_n - \frac{P_1}{r_i \log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots\ldots\ldots\ldots (33)$$

Equations (30) to (33) may be converted into a form convenient for computing and platting by noting that,

$$\log_e\left(\frac{r_e}{r_i}\right) = \frac{t}{r_n} \text{ (approximately)} \ldots\ldots\ldots\ldots\ldots (34)$$

$$P = p_e\,r_e - X\cos\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

$$M = X(r_C - r_n\cos\phi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

$$X = p_e\,r_e\,K \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

$$12\,\frac{r_n}{t}\left(\frac{1}{2} + \frac{c}{t}\right) = n_e \ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

and,

$$12\,\frac{r_n}{t}\left(\frac{1}{2} - \frac{c}{t}\right) = n_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Making the necessary substitutions, Equations (30) to (33) reduce to:

At the Crown:

$$\frac{s_e}{p_e} = n_e\left(K\,\frac{r_C}{t}\right) + \frac{r_n}{t}\,(1 - n_e)\,K - \frac{r_n}{t} \ldots\ldots\ldots\ldots (40)$$

and,

$$\frac{s_i}{p_e} = -\frac{r_e}{r_i} \times n_i\left(K\,\frac{r_C}{t}\right) + \frac{r_e}{r_i} \times \frac{r_n}{t}\,(1 + n_i)\,K - \frac{r_e}{r_i} \times \frac{r_n}{t} \ldots (41)$$

At the Abutment:

$$\frac{s_e}{p_e} = n_e\left(K\,\frac{r_C}{t}\right) + \frac{r_n}{t}\,(1 - n_e)\,K\cos\phi_1 - \frac{r_n}{t} \ldots\ldots\ldots (42)$$

and,

$$\frac{s_i}{p_e} = -\frac{r_e}{r_i} \times n_i\left(K\frac{r_C}{t}\right) + \frac{r_e}{r_i} \times \frac{r_n}{t}(1 + n_i)\,K\cos\phi_1 - \frac{r_n}{t} \times \frac{r_e}{r_i} \quad ..(43)$$

EFFECT OF VARIATIONS IN E_R AND m

The equations introduced in this paper correspond with comparable equations derived by Cain with the added advantages that the effect of yielding abutments and a finite value of Poisson's ratio are taken into consideration. To simplify the use of these equations a set of graphs is necessary, similar to those presented by Fowler in 1927. Figs. 3, 4, 5, and 6 are examples of such curves. These graphs are based on the following conditions, assumptions, or simplifications: (*a*) The arch is circular and of uniform thickness; (*b*) $I_n = \frac{1}{12}t^3$; (*c*) $\frac{E_v}{E_c} \times 1.2 = 2.88$ (in which, E_v = the modulus of elasticity in shear); (*d*) the trace of the abutment plane is normal to the arch rib; (*e*) the loading over the entire arch is uniform; (*f*) the stress due to vertical loads is zero; (*g*) $E_c = E_R$; (*h*) $m = 8$; and (*i*) tensile stresses are positive.

As the values of E_R and m selected for the computations are open to conjecture, the effect of a reasonable variation from the chosen values is to be examined. The result of solving a specific example, in which the effect of abutment yield is quite appreciable (since the ratio, $\frac{t}{r}$, is large), is shown in Table 1(*a*), where the stresses when $E_R = 75\%\ E_c$ and $E_R = 125\%\ E_c$ are compared with those obtained when it is assumed that $E_R = E_c$. In this example, $t = 30$ ft; $r = 60$ ft; $2\,\phi_1 = 120°$; and $\frac{h}{t} = 2$. The values in

TABLE 1.—STRESSES (IN POUNDS PER SQUARE INCH) FOR UNIT HEAD IN AN ARCH WITH YIELDING ABUTMENTS

Variant	CROWN		ABUTMENT	
	Extrados	Intrados	Extrados	Intrados
(*a*) COMPARISON OF STRESSES FOR VARIOUS VALUES OF E_R				
$E_R = E_c$	−1.35	−0.14	−0.07	−2.18
$E_R = \frac{3}{4}\,E_c$	−1.34	−0.09	−0.15	−2.08
$E_R = 1\frac{1}{4}\,E_c$	−1.28	−0.16	−0.03	−2.25
(*b*) COMPARISON OF STRESSES FOR VARIOUS VALUES OF POISSON'S RATIO				
Fixed abutments; $m = 8$	−1.18	−0.13	+0.41	−2.77
Yielding abutments; $m = 8$	−1.35	−0.14	−0.07	−2.18
Yielding abutments; $m = 5$	−1.29	−0.18	−0.10	−2.16

Table 1(*a*) are derived with the assistance of Figs. 3(*a*), 4(*a*), 5(*a*), and 6(*a*), by first entering the charts from the bottom at the given value of $\phi = \frac{120}{2} = 60$ degrees. Then continue vertically to an intersection with the proper curve of $\frac{t}{r} = \frac{30}{60} = 0.5$. The stresses for unit head on the horizontal lines through these intersections may then be read.

The greatest variation from the assumed E_R amounts to only approximately 0.10 lb per sq in. for unit head, which, for a 100-ft head, would change the stresses by only 10 lb per sq in.

In Table 1(*b*), the stresses when $m = 5$ are compared with those when $m = 8$, the value adopted for the curves. As in the previous example, $\frac{t}{r} = 0.5$, $2\,\phi_1 = 120°$, and $\frac{h}{t} = 2$, and the stresses are derived by means of Fig. 3(*a*), 4(*a*), 5(*a*), and 6(*a*). Furthermore, in order to show the comparative effect of including abutment yield, the stresses obtained with the assumption of fixed ends, as in Fowler's curves, are included.

Table 1(*b*) indicates that the stresses are quite insensitive to fairly large variations in the value of *m*. For this typical example, furthermore, the yielding abutment calculations show a more equable distribution of stress between crown and abutment by increasing the critical crown extrados stresses and reducing the critical abutment intrados stresses. In this particular example, tension at the abutment extrados found by the fixed-end assumption is substituted by a small compression upon introducing the yielding foundations.

EXAMPLE TO DEMONSTRATE THE USE OF THE CURVES

As a typical case, assume that $\frac{h}{t} = 7$; thickness of arch, $t = 40$ ft; central radius $= 100$ ft; $2\,\phi_1 = 65°$; head of water $= 150$ ft; and that the arch carries 75% of the load. This problem is solved by determining stresses separately from Figs. 3, 4, 5, and 6, for $\frac{h}{t} = 2$ and $\frac{h}{t} = 12$. The unit-head stress for $\frac{h}{t} = 7$ is then found by interpolation. The final process is illustrated by Table 2. To obtain the arch stresses from the unit-head stress, values of unit stresses in Table 2 are multiplied by 150×0.75.

TABLE 2.—STRESSES (IN POUNDS PER SQUARE INCH), IN AN ARCH WITH YIELDING ABUTMENTS

Section	Unit-Head Stress for the Following Ratios of $\frac{h}{t}$:			Arch stress
	2	12	7	
Crown extrados	−1.31	−1.37	−1.34	−151
Crown intrados	+0.72	+0.88	+0.80	+ 90.0
Abutment extrados	+0.35	+0.37	+0.36	+ 40.5
Abutment intrados	−1.77	−1.73	−1.75	−197

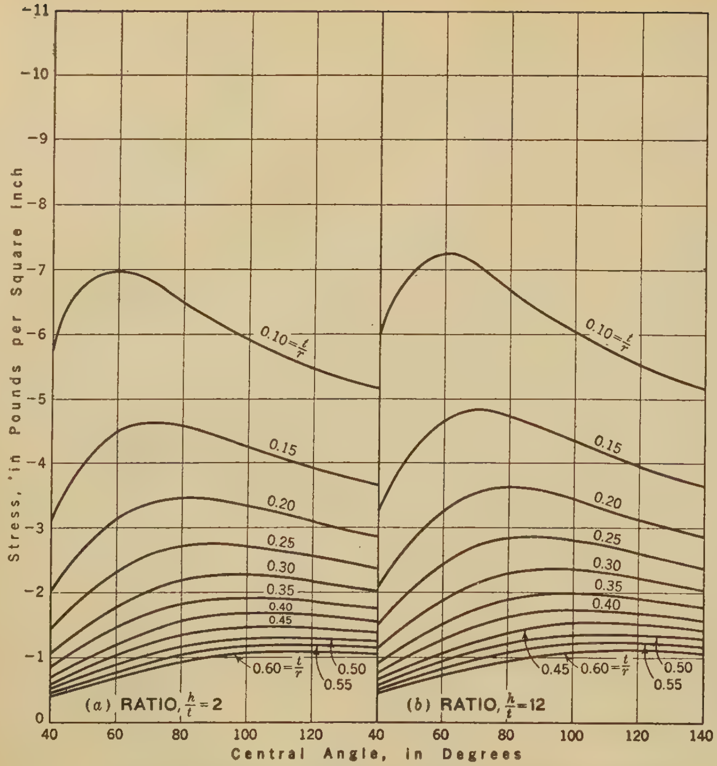

FIG. 3.—CROWN STRESSES, EXTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

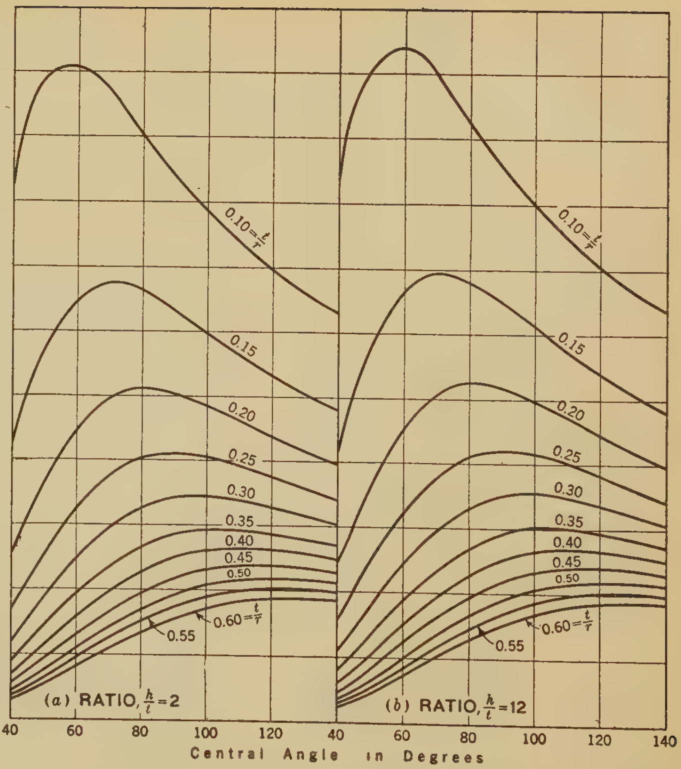

FIG. 4.—ABUTMENT STRESSES, INTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

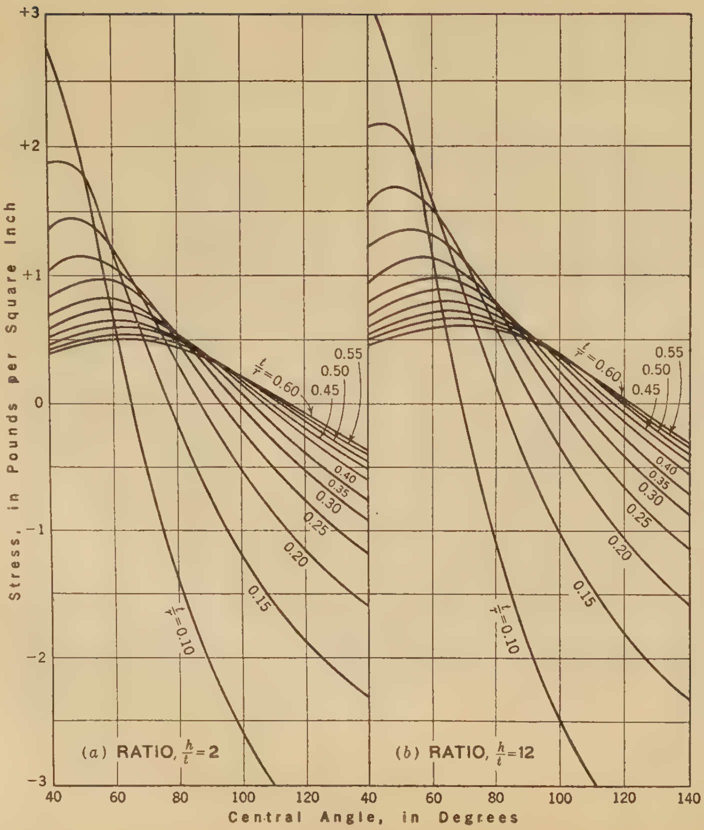

FIG. 5.—CROWN STRESSES, INTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

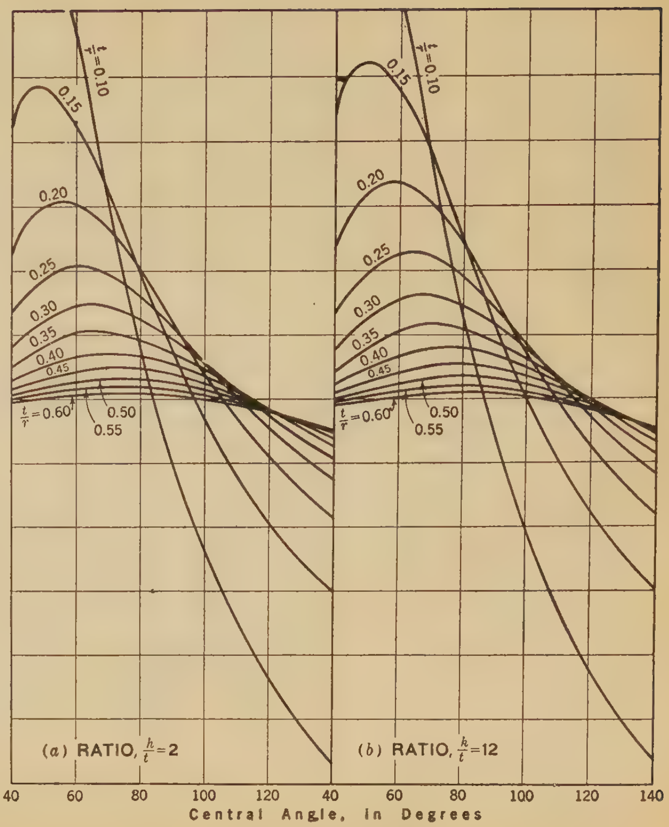

FIG. 6.—ABUTMENT STRESSES, EXTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

ACKNOWLEDGMENT

The writer wishes to express his appreciation to John L. Cox, Jun. Am. Soc. C. E., for some valuable assistance rendered in checking curve computations.

APPENDIX

NOTATION

The following notation is recommended for the guidance of discussers:

b = breadth.

c = radial distance between the neutral axis and the gravity axis of the arch barrel; as a subscript, c, denotes "concrete."

d = depth.

e = eccentricity; as a subscript, e, denotes "extrados."

f = unit fiber stress.

h = height; approximate height of loaded abutment.

i = a subscript to denote "intrados."

k = radius of gyration.

m = Poisson's number; $\frac{1}{m}$ = Poisson's ratio.

n = a substitution factor (see Equations (38) and (39)); as a subscript, n, denotes "neutral axis."

p = pressure per unit area; p_e = normal radial pressure acting on the arch along the extrados.

r = radius; radius of the gravity axis of the arch ring; r_n = radius of the neutral axis of the arch ring; r_e = radius of extrados; r_C = radial distance along the crown radius, to the point at which it is necessary to apply Force X, in order to eliminate crown rotation.

s = stress; s_e = unit stress at the extrados; s_i = unit stress at the intrados.

t = thickness; radial thickness of the arch.

v = unit shear.

y = radial deflection.

C = constant; as a subscript, C, denotes "crown."

E = modulus of elasticity; E_c = modulus of elasticity of concrete; E_R = modulus of elasticity of rock; E_v = modulus of elasticity, E, in shear.

F = a subscript denoting "yielding foundation."

H = head.

I = rectangular moment of inertia; I_n = moment of inertia about the neutral axis.

J = polar moment of inertia.

K = a constant $= \frac{X}{p_e \, r_e}$.

M = moment; moment at any arbitrary section, at an angle, ϕ, from the crown; M_1 = moment, M, at the abutment.

N = thrust at any arbitrary section, at an angle, ϕ, from the crown; N_1 = normal force, N, at the abutment.

P = a concentrated load; a thrust normal to an arch section; P_C = a thrust normal to a section at the crown; P_1 = a thrust normal to a section at the abutment.

Q = a substitution factor (see Q_1, Q_2, etc.).

R = reaction; as a subscript, R, denotes "rock."

T = temperature.

V = total shear; shear at any arbitrary section, at an angle, ϕ, from the crown; V_1 = shear force, V, at the abutment.

W = total load.

X = additional crown force applied normal to the radius passing through the crown, necessary to bring the crown section back to its original position.

α = angle with the normal, or radial.

ξ = specific gravity;

Δ = deformation; Δ_C = deformation of the crown section normal to its radius; Δ_F = deformation of the crown section due to yielding foundation.

δ = unit elongation.

ε = coefficient of thermal expansion.

ζ = a substitution factor (see Equation (9)).

η = a substitution factor (see Equation (10)).

θ = angular distance.

μ = a substitution factor (see Equation (11)).

ρ = radius of curvature.

Σ = summation of.

σ = a correction due to Poisson's effect $= \left[\frac{m-1}{2\,m} + \frac{m+1}{2\,m}\left(\frac{r_i}{r_n}\right)^2\right]\frac{r_e}{r}$.

ϕ = a central angle; angle of any radius with the radius through the crown.

ψ = rotation of a section; ψ_C = angular rotation of the crown section from a radial line; ψ_F = angular rotation of the crown section as the result of yielding in the foundation.

DISCUSSION

I. M. NELIDOV,[7] ASSOC. M. AM. SOC. C. E. (by letter).—The development of the method by which the effect of abutment deformations on stresses in a circular arch ring can be estimated, is offered in this paper. The author has plotted curves for $E_r = E_c$; $m_r = m_c = 8$; and $\frac{h}{t} = 2 - 12$, so that the stresses can be determined directly as a function of $\frac{t}{r}$ and the central angle, $2\,\phi_1$. The method considers the deformation of the neutral fiber of the arch ring decreased by the effect of Poisson's ratio and by the introduction of a finite thickness of the arch. Furthermore, it is based on the assumption that plane sections remain plane after deformation. It ignores any possible interference from adjacent arch rings. As is known, the method is conventional, being derived from the use of thin railroad arches, in which the ratios of cross-sections to the length of the axial line are small. Its advantage is in the workable form of the equations. On the other hand, any attempt to introduce the effect of the finite thickness and of Poisson's ratio in the form of an interaction between the radial and tangential stresses leads to unusually complicated formulas.[8]

When an important structure is under consideration, a method based on the general equations of the theory of elasticity should be used preferably. For instance, applying such a method as that developed by C. W. Comstock,[9] M. Am. Soc. C. E., to an arch ring with $\frac{t}{r} = 0.345$ and $2\,\phi_1 = 60°$ and with $p_e = 1.0$, one finds the results indicated in Table 3(*a*). For an arch ring with $\frac{t}{r} = 0.02$ and $2\,\phi_1 = 47°$, Table 3(*b*) will give the comparative results.

It is to be noted that when the elastic properties of the material of the arch are considered there is a notable increase of tensile stresses in arches with large ratios of $\frac{t}{r}$ and small central angles. In the development of the results in Table 3 it was assumed that the plane between the adjacent arches remains a plane and that no shearing stresses occur within this plane.

Referring to the derivations of the paper, the writer wished to arrive at the author's results by following a different line of reasoning. Consider

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[7] Senior Engr. of Hydr. Structure Design, State Dept. of Public Works, Sacramento, Calif.

[8] "Stresses in Thick Arches of Dams", by B. F. Jakobsen, *Transactions*, Am. Soc. C. E., Vol. 90 (1927), pp. 500–507.

[9] "On the Stresses in Arch Dams", by C. W. Comstock, N. Y., 1931, pub. by the author.

TABLE 3.—COMPUTATIONS BY THE THEORY OF ELASTICITY
(Tension is Positive)

Theory	UNIT STRESSES, IN POUNDS PER SQUARE INCH				PERCENTAGE OF DIVERGENCE			
	Crown		Abutment		Crown		Abutment	
	Extrados	Intrados	Extrados	Intrados	Extrados	Intrados	Extrados	Intrados
(a) $\frac{t}{r} = 0.345$; and, $2\,\phi_1 = 60$ DEGREES								
Curved beam	—2.75	1.99	3.89	—5.46	0	0	0	0
Comstock	—1.10	1.80	5.88	—5.46	—60	—10	66	48
(b) $\frac{t}{r} = 0.02$; and, $2\,\phi_1 = 47$ DEGREES								
Curved beam	—74.5	—18.1	—6.1	—89.2	0	0	0	0
Comstock	—68.4	—26.5	—14.1	—79.8	—8	46	23	—11

a circular arch ring of uniform thickness, loaded with the uniform radial load. Its main, or statically determinate, system is shown in Fig. 7(*a*). For this system the moments, thrusts, and shear are known, as follows: $M' = p_e\, r_e\, e'$; $P' = -\, p_e\, r_e$; and $V' = 0$. Furthermore, in accordance with Fig. 8:

$$e' = \frac{\displaystyle\int_{r_i}^{r_e} \left(1 + \frac{r_i^2}{r_y^2}\right) r_y\, dr_y}{\displaystyle\int_{r_i}^{r_e} \left(1 + \frac{r_i^2}{r_y^2}\right) dr_y} \qquad \text{.................(44)}$$

Equation (44)[10] indicates that the funicular frame of the loads and the neutral line of the arch are spaced at a distance, e'.

If the left abutment is fixed, as shown in Fig. 7(*b*), and the right end of the arch is supported with a rigid cantilever extending to the elastic center, *E*, three unknown redundant forces, *X*, *Y*, and *Z*, are thus introduced, which will be determined if the deformations which they should counteract are made known.

Using the general equation of elastic work and considering only the work, *W*, of the tangential and shearing stresses:

$$W = \int_{-\phi_1}^{+\phi_1} \int_{r_i}^{r_e} \tfrac{1}{2}\, s_{mt}\, dr\, \Delta_{mt}\, d\phi + \int_{-\phi_1}^{+\phi_1} \int_{r_i}^{r_e} \tfrac{1}{2}\, s_{pt}\, dr\, \Delta_{pt}\, d\phi$$
$$+ \int_{-\phi_1}^{+\phi_1} \int_{r_i}^{r_e} \tfrac{1}{2}\, s_v\, dr\, \Delta_v\, d\phi \qquad \text{.......................(45)}$$

in which, s_{mt} is the unit tangential stress due to bending moment; s_{pt} is the unit tangential stress due to thrust; s_v is the unit tangential stress due to shear; and, Δ is the corresponding total deformation.

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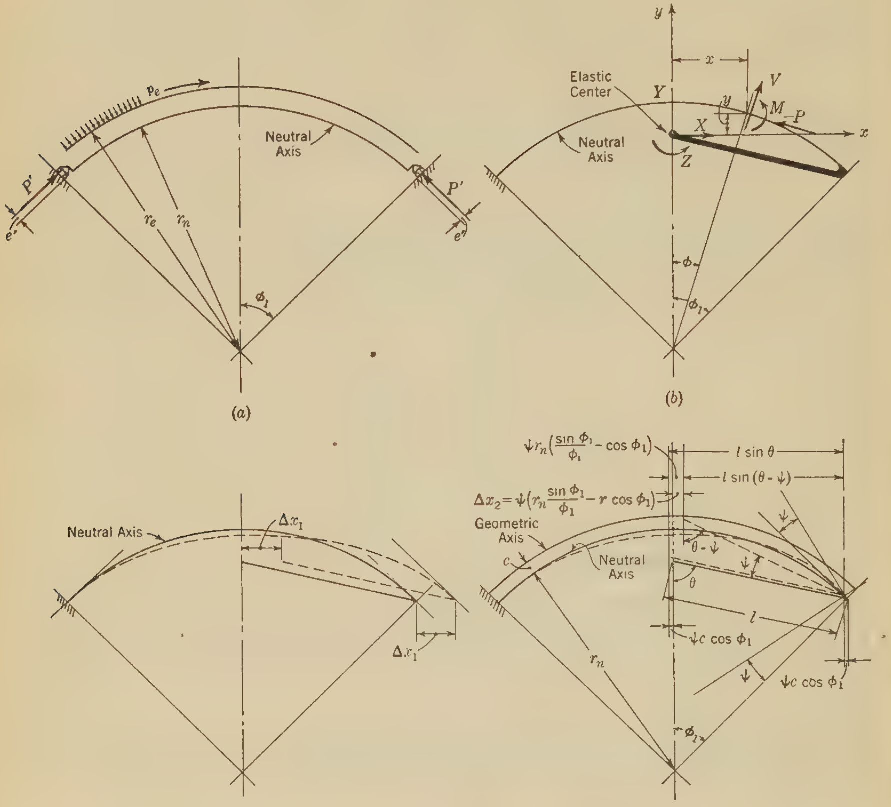
[10] "Strength of Materials", by S. Timoshenko, Pt. II, p. 532, 1931 Edition.

The unit stresses in a cross-section and the corresponding deformations are shown in Fig. 8. This stress and deformation distribution is modified near the supports by the influence of the deformation of abutments, but to an unknown extent.

The unit tangential stress due to water pressure and the force, X, will be:

$$s_{pt} = \frac{P\,r_n}{t\,r_y} = \frac{-p_e r_e + X \cos\phi\, r_n}{t r_y} \quad \ldots\ldots\ldots\ldots\ldots (46)$$



p_e
Neutral Axis
P'
P'
e'
e'
r_e
r_n
ϕ_1
y
x
Elastic Center
Y
V
M
P
X
x
y
Neutral Axis
Z
ϕ
ϕ_1
Neutral Axis
Δx_1
Δx_1
$\psi r_n\left(\frac{\sin\phi_1}{\phi_1} - \cos\phi_1\right)$
$l \sin\theta$
$l \sin(\theta - \psi)$
$\Delta x_2 = \psi\left(r_n \frac{\sin\phi_1}{\phi_1} - r\cos\phi_1\right)$
Geometric Axis
c
Neutral Axis
$\theta - \psi$
ψ
ψ
θ
l
$\psi c \cos\phi_1$
r_n
ψ
$\psi c \cos\phi_1$
ϕ_1

(a) (b)

(c) (d)

FIG. 7.

and the uniform deformation due to this stress is expressed by the equation:

$$\Delta_{pt} = -\frac{p_e r_e r_n}{E_c\, t}\,\sigma + \frac{X \cos\phi\, r_n}{E_c\, t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

in which $\sigma = \frac{r_e}{2\,r}\left[1 + \frac{r^2_i}{r^2_n} - \frac{1}{m}\left(1 - \frac{r^2_i}{r^2_n}\right)\right]$.

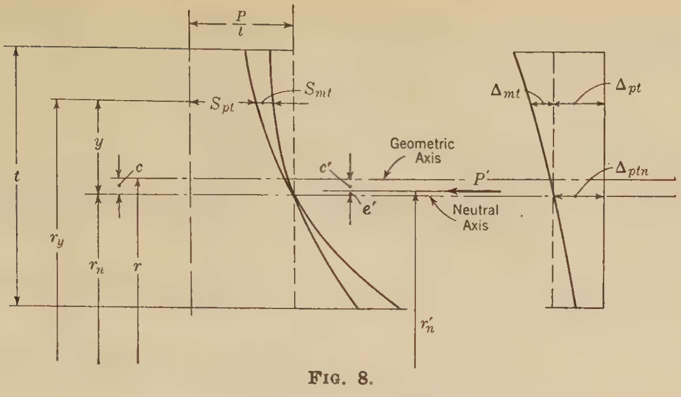


FIG. 8.

The unit tangential stress due to the bending moment produced by the water pressure and the force, X, will be:

$$s_{mt} = -\frac{p_e\, r_e}{t}\left[\frac{r_n}{r_y} - \frac{r_e}{2\, r}\left(1 + \frac{r^2_i}{r^2_y}\right)\right]$$
$$+\left[X\, r_n\left(\cos\phi - \frac{\sin\ \phi_1}{\phi_1}\right) + Z\right]\frac{r_n}{I_n}\left(\frac{r_n}{r_y} - 1\right) \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

and the total deformation due to this stress is expressed by:

$$\Delta_{mt} = \frac{-\,p_e\, r^2_e\, r_y}{2\, tr\, E_c}\left[1 + \frac{r^2_i}{r^2_y} - \frac{1}{m}\left(1 - \frac{r^2_i}{r^2_y}\right)\right] + \frac{p_e\, r_e\, r_n}{Et}\ \sigma$$
$$+\left[X\, r_n\left(\cos\phi - \frac{\sin\ \phi_1}{\phi_1}\right) + Z\right]\frac{r_n}{E_c\, I_n}\left(\frac{r_n}{r_n} - 1\right) r_y \ldots\ldots (49)$$

In accordance with Fig. 7(b), the total stresses in a section are:

$$M = M' + X\, y + Z \ldots\ldots\ldots\ldots\ldots\ldots\ldots (50a)$$

$$P = P' + X\cos\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (50b)$$

and,

$$V = 0 + X\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (50c)$$

It is noted that as with $\phi = 0$, and $V = 0$, the force, $Y = 0$, and does not enter in Equations (50).

The relative displacement of the End $E\ C$ of the rigid bracket directed along the X-axis will be in accordance with Fig. 7(c) and Fig. 7(d):

$$\Delta x = \Delta x_1 + \Delta x_2 = 2\,\Delta_N\cos\phi_1 - 2\,\Delta_V\sin\phi_1 - 2\,\psi\left(r_n\,\frac{\sin\phi_1}{\phi_1} - r\cos\phi_1\right)$$
$$= -\frac{\partial W}{\partial X} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

Along the Y-axis, Δy is unimportant because $Y = 0$. The rotation of the rigid bracket will be:

$$\Delta \theta = 2 \psi = - \frac{\partial W}{\partial Z} \qquad \text{.....................(52)}$$

The derivatives of the work for the half arch are:

$$- \frac{1}{2} \frac{\partial W}{\partial X} = - X \frac{r^3_n}{E_c I_n} \left(\frac{\phi_1}{2} + \frac{1}{4} \sin 2 \phi_1 - \frac{\sin^2 \phi_1}{\phi_1} \right) \sigma_m + \frac{p_e r_e r_n}{E_c t} \sin \phi_1 \sigma_p$$
$$- X \frac{r_n}{E_c t} \left(\frac{\phi_1}{2} + \frac{1}{4} \sin 2 \phi_1 \right) - X \frac{r}{E_c t} \left(\frac{\phi_1}{2} - \frac{1}{4} \sin 2 \phi_1 \right) \sigma_v \quad \text{...(53a)}$$

and,

$$- \frac{1}{2} \frac{\partial W}{\partial Z} = - Z \frac{\phi_1 r_n}{E_c I_n} \sigma'_m \qquad \text{..................(53b)}$$

The coefficients, σ, in Equations (52) and (53) are: $\sigma_m = \sigma_m' = 1$; $\sigma_p = \dfrac{1 + \sigma}{2}$; and $\sigma_v = 2.88$. In deriving these coefficients all the factors producing less than 1% of the divergence of the stress were omitted.

With the sign convention adopted the normal and tangential displacements and rotation of the abutment will be:

$$\Delta_N = \zeta \frac{P_1}{E_r} \qquad \text{........................(54a)}$$

$$\Delta_V = - \zeta \frac{V_1}{E_r} + \eta \frac{M_{1g}}{E_r t} \qquad \text{..................(54b)}$$

and,

$$\psi = - \eta \frac{V_1}{E_r t} + \mu \frac{M_{1g}}{E_r t^2} \qquad \text{..................(54c)}$$

in which $M_{1g} = - p_e r_e c' + X \left(r \cos \phi_1 - r_n \dfrac{\sin \phi_1}{\phi_1} \right) + Z$. Due to the small influence of e', $c' = c$ will be assumed in future operations.

After substitution of Equations (50) and (53) into Equations (51) and (52) and subsequent integration, the following expressions are obtained for evaluating the unknowns, X and Z:

$$- p_e r_e \left\{ \frac{r_n}{t} \frac{E_r}{E_c} \sin \phi_1 \sigma_p + \zeta \cos \phi_1 - \frac{c}{t} \left[\eta \sin \phi_1 + \mu \left(\frac{r_n}{t} \frac{\sin \phi_1}{\phi_1} - \frac{r}{t} \cos \phi_1 \right) \right] \right\}$$
$$+ X \left\{ \frac{r^3_n}{I_n} \frac{E_r}{E_c} \left(\frac{\phi_1}{2} + \frac{1}{4} \sin 2 \phi_1 - \frac{\sin^2 \phi_1}{\phi_1} \right) + \frac{r_n}{t} \frac{E_r}{E_c} \left(\frac{\phi_1}{2} + \frac{1}{4} \sin 2 \phi_1 \right) \right.$$
$$+ \sigma_v \frac{r}{t} \frac{E_r}{E_c} \left(\frac{\phi_1}{2} - \frac{1}{4} \sin 2 \phi_1 \right) + \zeta + 2 \eta \sin \phi_1 \left(\frac{r_n}{t} \frac{\sin \phi_1}{\phi_1} - \frac{r}{t} \cos \phi_1 \right)$$
$$\left. + \mu \frac{r_n}{t} \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right) \left(\frac{r_n}{t} \frac{\sin \phi_1}{\phi_1} - \frac{r}{t} \cos \phi_1 \right) \right\}$$
$$- \frac{Z}{t} \left[\eta \sin \phi_1 + \mu \frac{r_n}{t} \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right) \right] = 0 \quad \text{......(55a)}$$

and,

$$-p_e r_e \mu \frac{c}{t} - X \left[\eta \sin \phi_1 + \mu \frac{r_n}{t} \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right) \right]$$

$$+ \frac{Z}{t} \left[\mu + \phi_1 \frac{E_r}{E_c} \frac{r_n}{I_n} t^2 \right] = 0 \ldots\ldots\ldots\ldots\ldots\ldots (55b)$$

Equations (53) can be rewritten as follows:

$$A + B K + C K_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56a)$$

and,

$$A' + B' K + C' K_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56b)$$

in which $K = \dfrac{X}{p_e r_e}$ and $K_1 = \dfrac{Z}{p_e r_e t}$

The solution of these equations gives:

$$K = \frac{A - A' \dfrac{C}{C'}}{B' \dfrac{C}{C'} - B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57a)$$

and,

$$K_1 = - \frac{A' + KB'}{C'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57b)$$

The coefficients of Equations (57) are:

$$A = - \frac{r_n}{t} \sin \phi_1 \frac{E_r}{E_c} \sigma_v - \zeta \cos \phi_1$$

$$+ \frac{c}{t} \left[\eta \sin \phi_1 + \mu \frac{r_n}{t} \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right) \right] \ldots\ldots\ldots\ldots (58a)$$

$$B = \zeta + \eta \left(2 \frac{r_n}{t} \frac{\sin 2 \phi_1}{\phi_1} - \frac{r}{t} \sin 2 \phi_1 \right)$$

$$+ \mu \left(\frac{r_n}{t} \right)^2 \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right) \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right)$$

$$+ \frac{r^3_n}{I_n} \frac{E_r}{E_c} \left(\frac{1}{2} \phi_1 + \frac{1}{4} \sin 2 \phi_1 - \frac{\sin^2 \phi_1}{\phi_1} \right)$$

$$+ \frac{r_n}{t} \frac{E_r}{E_c} \left(\frac{1}{2} \phi_1 + \frac{1}{4} \sin 2 \phi_1 \right) + \sigma_v \frac{r}{t} \frac{E_r}{E_c} \left(\frac{1}{2} \phi_1 - \frac{1}{4} \sin 2 \phi_1 \right) \ldots (58b)$$

$$C = - \left[\eta \sin \phi_1 + \mu \frac{r_n}{t} \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right) \right] \ldots\ldots (58c)$$

$$A' = - \mu \frac{c}{t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58d)$$

$$B' = - \left[\eta \sin \phi_1 + \frac{r_n}{t} \left(\frac{\sin \phi_1}{\phi_1} - \frac{r}{r_n} \cos \phi_1 \right) \right] \ldots\ldots\ldots (58e)$$

and,

$$C' = \mu + \phi_1 \frac{E_r\, r_n}{E_c\, I_n}\, t^2 \ldots\ldots\ldots\ldots\ldots\ldots (58f)$$

Equations (58) may be checked against Equations (18) to (25) of the paper by noting that Equation (52) is $-\psi = \phi_1 \frac{r_n}{E_c\, I_n} Z$, and by comparing Equation (50a) with Equation (3) of the paper, from which it follows that $Z = X\left(r_n \frac{\sin\phi_1}{\phi_1} - r_c\right)$. After substituting these expressions into Equations (51), (52), and (53), Equations (18) to (25) of the paper are obtained.

A question of the relative importance of Poisson's ratio and of the deformation of abutments arises. Table 4 shows a comparison of stresses, in pounds per square inch, for $p_e = 1$ lb and for arch rings varying from a very thin arch, with a very large central angle, to a very thick arch with a very small central angle. Table 4 also gives the percentage of divergence of stresses from those computed by a conventional method; that is, with small $\frac{t}{r}$; with $m_c = \infty$; and with rigid abutments.

TABLE 4.—COMPARISON OF STRESSES

(Tension is Positive)

$\frac{t}{r}$	$2\phi_1$, (in degrees)	$m_c = m_r$	E_r	$\frac{h}{t}$	UNIT STRESSES					PERCENTAGE OF DIVERGENCE				
					At Crown		At Abutment			At Crown		At Abutment		
					s_e	s_i	s_e	s_i	s_s	s_i	s_e	s_e	s_i	s_s
0.01	150	∞	0	..	—44.2	—43.0	—41.9	—45.3	0.005	0	0	0	0	0
0.01	150	5	0	..	—44.2	—43.0	—41.9	—45.3	0.005	0	0	0	0	0
0.01	150	∞	E_c*	1	—44.2	—43.0	—41.9	—45.3	0.005	0	0	0	0	0
0.01	150	5	E_c*	1	—44.2	—43.0	—41.9	—45.3	0.005	0	0	0	0	0
0.01	150	∞	E_c*	30	—44.2	—43.0	—41.9	—45.3	0.005	0	0	0	0	0
0.01	150	5	E_c*	30	—44.2	—43.0	—41.9	—45.3	0.005	0	0	0	0	0
0.15	100	∞	0	..	—4.20	—1.07	+0.48	—6.78	0.28	0	0	0	0	0
0.15	100	5	0	..	—4.20	—1.07	+0.42	—6.76	0.27	0	0	—11	0	0
0.15	100	∞	E_c*	1	—4.23	—1.12	—0.11	—6.16	0.24	0	+5	†	—9	—12
0.15	100	5	E_c*	1	—4.22	—1.14	—0.14	—6.17	0.24	0	+7	†	—9	—12
0.15	100	∞	E_c*	30	—4.45	—0.78	+0.20	—6.46	0.27	+6	—27	—58	—4	0
0.15	100	5	E_c*	30	—4.43	—0.81	+0.16	—6.42	0.27	+6	—24	—66	—5	0
0.50	50	∞	0	..	—0.42	+0.17	+0.36	—1.13	0.38	0	0	0	0	0
0.50	50	5	0	..	—0.51	+0.19	+0.26	—1.08	0.37	+20	+12	—28	—4	—2
0.50	50	∞	E_c*	1	—0.78	+0.67	+0.00	—0.67	0.39	+85	+286	†	—41	+1
0.50	50	5	E_c*	1	—0.78	+0.62	0.00	—0.68	0.38	+85	+259	†	—40	0
5.50	50	∞	E_c*	30	—0.86	+0.84	—0.04	—0.54	0.40	+104	+386	†	—52	+3
0.50	50	5	E_c*	30	—0.85	+0.80	—0.05	—0.54	0.40	+101	+363	†	—52	+2

* Finite. † Reversal of sign.

The unit stresses in Table 4 were computed by the combined expressions, Equations (46) and (48). The bending stress due to water pressure expressed by the first part of Equation (48) is numerically equal to ±0.20 lb per sq in. for the arch, with $\frac{t}{r} = 0.15$, and to ±0.07 lb per sq in. for the arch, with $\frac{t}{r} = 0.50$. This stress is of minor importance at the crown, but it is of significance at the abutment, if considered.

The data in Table 4 indicate that in an arch ring of an average thickness and with an average central angle the divergence of stresses is about 10%, and only the deformation of abutments causes a greater divergence. For a thick arch ring with small central angle, the divergence due to Poisson's ratio is about 20%, whereas that due to the deformation of abutments is increased many times this amount. The divergence of shearing stresses is negligible.

The limits of the ratio, $\frac{h}{t} = 1 - 30$, were assumed, in order to visualize the effect of the extremely large deformations, although actually this ratio will never be greater than 1. It was originally applied by Fredrik Vogt, Assoc. M. Am. Soc. C. E., for the case of an abutment of constant thickness, t, and with the forces, M, P, and V, uniformly distributed along the height, h. In the case of an arch dam this condition is not fulfilled; both the thickness, t, and the acting forces vary along the height, h, producing a warped surface of an original abutment plane (which is ordinarily a warped surface from the start, due to the requirements of the excavation).

The problem becomes complex and a practical issue for its solution must be found. If the arch rings of a unit height, as well as the areas near the abutment were separate one from another, then under the forces acting they would displace in relation to each other, as the keys of a piano, each one an independent element. The relative elasticity and continuity of the foundation makes the deformation spread to the neighboring units over a certain distance. From a list of seventy-four arch dams in California the writer found that the ratio, $\frac{h}{t} = \frac{1}{t}$, at the crest varied from 0.046 to 1.000 (average 0.200) and at the base from 0.009 to 1.000 (average 0.042). If the entire height of the dam was assumed as h, the same ratios would be: At the crest between 2 and 45 (average, 16); and, at the base, between 1.4 and 19 (average, 5). As the deformation coefficients, ζ, η, and μ, increase with $\frac{h}{t}$, the selection of larger values of $\frac{h}{t}$ will be on the side of safety; but even with an increase of five to ten times the ratios referred to, h will be only about equal to 1. The ratios in which the entire height of the dam is considered do not seem to be justified.

A. Floris, Esq.[11] (by letter).—The title of this practical paper and the introductory remarks made therein, are rather misleading, because the plotting of curves is not an analysis or graphical solution of a problem. The author utilizes a known theory by arranging its results in the form of diagrams for use in practice. From this point of view, of course, the paper is a valuable addition to the literature on the subject.

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[11] Dipl. Ing., Los Angeles, Calif.

The usefulness of graphs would be greatly increased if they were made independent of the system of units. The author's diagrams, and those for arches with fixed abutments, referred to in the paper, have a common drawback. They cannot be utilized in countries in which the metric system of units is used. It is possible, however, to arrange the diagrams in such a way, as to make them independent of this restriction. N. Kelen, in his well-known book on arch and multiple-arch dams, for instance, gives graphs that can be used independently of the system of units.[12]

It would be appreciated if the author could include in his closing discussion, graphs that do not have the aforementioned restrictions. If this is not feasible, a brief outline demonstration of how such graphs are plotted would be a welcome addition.

PHILIP CRAVITZ,[13] JUN. AM. SOC. C. E. (by letter).—In an engineering paper involving such ponderous equations as those concocted by the writer, it is indeed satisfying to him to obtain a verification of his basic equations by an independent method such as that provided by Mr. Nelidov. That the more laborious general equations of elasticity should preferably be used in the design of important structures, as is stated by Mr. Nelidov, is a debatable matter of opinion. It can scarcely be denied that an exact solution for an actual case in practice can probably never be obtained, because there are always such factors as the irregularity of the dam site, dissymmetry relative to any given center line, variations in the properties of the abutments when traversed from side to side, etc.

Because of such practical considerations, the writer believes that an exact theoretical solution may prove more bothersome, and scarcely more reliable, than a method which strives to consider all stress factors of equal magnitude of importance plus the judgment of an experienced design engineer. The curves presented in the paper do enable a designer to visualize quantitatively the effect of yielding abutments on stresses in thick arch dams.

However, because of the basic fact that the stresses of a statically indeterminate structure, such as an arch dam, are fundamentally independent of the scale, regardless of the number of stress determinants that are considered, there must be some mathematical set-up, similar to that devised by the writer, in which the final equations vary with $\frac{t}{r}$ and $2\,\phi_1$. Thus, it is only a question of regimenting computations so as to evolve curves that give stresses for any number of contributing factors introducing the slightest variation in the resultant stresses.

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[12] "Die Staumauern," von N. Kelen, Berlin, 1926.

[13] Design Engr., Los Angeles County Flood Control Dist., Los Angeles, Calif.

Regarding the desire expressed by Mr. Floris to obtain the curves in a universal system of units, it may be sufficient to indicate that the stresses given by the curves may be easily converted to any other desired system by the multiplication of a constant. For example, to obtain the stresses, in kilograms per square centimeter, multiply all values obtained from the curves by 0.23 times the height of water head, in meters.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1933

ELASTIC PROPERTIES OF RIVETED CONNECTIONS

BY J. CHARLES RATHBUN,[1] M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. RALPH E. GOODWIN, HAROLD C. ROWAN, WALTER SCHOLTZ, J. F. BAKER, L. E. GRINTER, C. R. YOUNG AND K. B. JACKSON, E. MIRABELLI, R. L. MOORE, JOHN SANFORD PECK, AND J. CHARLES RATHBUN.

SYNOPSIS

In designing floor-beams for steel-frame buildings, it is customary to neglect the restraining effect of end connections and to assume that the beams are hinged at the ends. In many other problems of stress analysis in steel frames, it is customary to treat the connections as completely fixed, or rigid.

While it is recognized that the end connections do provide some degree of restraint there has been no means of allowing for it because no one has known, numerically, how much restraint there is. One could not safely take advantage of any saving resulting from the resistance due to bending, created by the connections.

Tests of eighteen end connections of the three most common types of steel beams, form the basis of this paper.[2] The writer has shown how these data may be applied in the design, not only of single beams, but also in the investigation of the stresses in frames, composed of beams and columns, joined by these typical riveted connections.

His analysis shows that economics in design can be effected safely in some cases. It also gives the engineer a more complete understanding of the action of standard beam connections, and indicates several problems for further research.

NOTE.—Published in January, 1935, *Proceedings*.

[1] Associate Prof., Civ. Eng., Coll. of the City of New York, New York, N. Y.

[2] Prepared from a dissertation presented to Columbia University in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Faculty of Pure Science.

INTRODUCTION

The purpose of this paper is to record data derived from a series of tests of standard riveted beam connections and to show the method of incorporating this information in the analysis of steel frames. These tests were conducted in the Materials Testing Laboratory of the School of Technology of the College of the City of New York. Some idea is afforded of the rigidity of several types of connections and of the effect of yielding in these connections on the stress distribution in steel structures, such as building frames.

All the so-called "exact methods" of solution of the building frame under loads are based on the assumption (among others) that the horizontal members are rigidly connected to the vertical. This may lead to serious error. Unlike the assumption that the members do not change in length under stress, the effect of yielding in the connections cannot be determined by the analyst alone. Laboratory experiments are necessary in order to establish constants to incorporate into any analysis that may be made.

Little has been published on this important subject; to date, only a few isolated tests have been made on riveted connections for the purpose of obtaining some measure of their capacity to resist moment. In 1917, a series of tests was made at the University of Illinois, Urbana, Ill.[3] C. R. Young, M. Am. Soc. C. E., has conducted tests at the University of Toronto, Toronto, Ont., Canada, primarily for the purpose of comparing riveted, with welded, connections.[4] The writer knows of no other tests, at all extensive, on the subject.

Although it is probable that many of the connections encountered in a wind-stress design of buildings will be larger than those represented in the data herein submitted, the results of this investigation should serve as a guide in evaluating the properties of the larger connections. The capacity of the testing machine limited the size of the specimens tested.

The paper consists of two parts: Part I is a report on a series of physical tests to determine the relationship between the angular change in riveted beam connections and the moment inducing these changes; and Part II shows the necessary changes that must be made in the formulas used in the several current methods of analysis in order to allow for the elasticity of the riveted connections.

PART I.—PHYSICAL TESTS ON RIVETED CONNECTIONS

TEST SPECIMENS

Ordinary shop practice was followed in fabricating the specimens used in this work, no special care being taken because of the fact that the specimens were to be subjected to tests.

The physical properties of the steel in the various specimens of these tests are given in Table 1.

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[3] *Bulletin No. 104*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

[4] The report of this set of tests was read at the Annual Convention of the American Society of Civil Engineers in Chicago, Ill., in 1933, and pub. in the *Canadian Journal of Research*, July and August, 1934.

TABLE 1.—TESTS OF STEEL

MEMBER		UNIT STRESSES, IN POUNDS PER SQUARE INCH		Percentage elongation	Percentage reduction	CHEMICAL ANALYSIS (PERCENTAGES)	
Symbol	Weight, in pounds per foot	Elastic limit	Tensile strength			Phosphorus	Silicon
H-140....	167	41 200	61 340	30.0	53.1	0.014	0.037
G- 22....	101	43 220	60 540	30.0	61.4	0.014	0.030
G- 16....	83	42 540	60 420	28.7	56.9	0.016	0.032
G- 15....	99	41 950	59 470	28.7	59.1	0.015	0.034
S- 24....	105.9	45 860	64 640	27.5	50.2	0.013	0.031
S- 18....	54.7	44 810	58 980	27.5	54.0	0.016	0.043
S- 8....	18.4	39 840	62 320	26.2	55.6	0.014	0.042
S- 6....	12.5	44 000	64 800	27.5	60.0	0.017	0.032
I- 12....	31.8	47 120	62 900	26.2	58.25	0.018	0.036

DESCRIPTION OF TEST SPECIMENS

Eighteen specimens (see Figs. 1 and 2) representing as many connections were tested for the purpose of furnishing data from which their elastic resistance to moment could be obtained. These specimens may be divided into three types: Series *A*, Specimens 1 to 7 (Fig. 1), are standard clip-angle connections; Series *B*, Specimens 8 to 12 (Fig. 1) are seat-angle connections; and Series *C*, Specimens 13 to 17 (Figs. 1 and 2) are a series of shallow wind-braced connections. Specimen 18 was designed to furnish some information on the effect of the post on the joint. In Series *A* and *B*, $\frac{7}{8}$-in. rivets were driven in $\frac{15}{16}$-in. holes. In Series *C*, except Specimens 13 and 14, 1-in. rivets were driven in $1\frac{1}{16}$-in. holes. The lighter pieces were punched and the heavier ones were drilled.

Each specimen consisted of a central plate with an I-beam abutting on either side fastened by means of the standard connection to be investigated. The specimen thus formed was tested by applying a load on the central plate, while it was supported at the outer ends of the I-beams. Thus, the specimen acted as a simple beam loaded at the center and a known moment was induced in the two connections.

Series *A* represents a type of connection that is not designed primarily to resist moment, and it is customary to consider these connections as hinged. However, they have some measure of elasticity and it was hoped that these tests would furnish the data necessary to enable a designer to estimate this elasticity in those cases where it seriously affected stress distribution. In designing Series *A*, standard connections were chosen from those published in Handbooks issued by steel manufacturers, taking the double-riveted connections and the companion single-riveted connections for the 8-in., 12-in., and 18.-in. I-beams. Thus, a study of double riveting as it affected the stiffness of the connections was made. The spread in size from the 8-in. to the 18-in. I-beams was as great as was considered practicable. As the 6-in. standard connection is probably the most common, it was added to the series. A comparison of the several connections showing the effect of adding a second row of rivets can be made by studying Fig. 3.

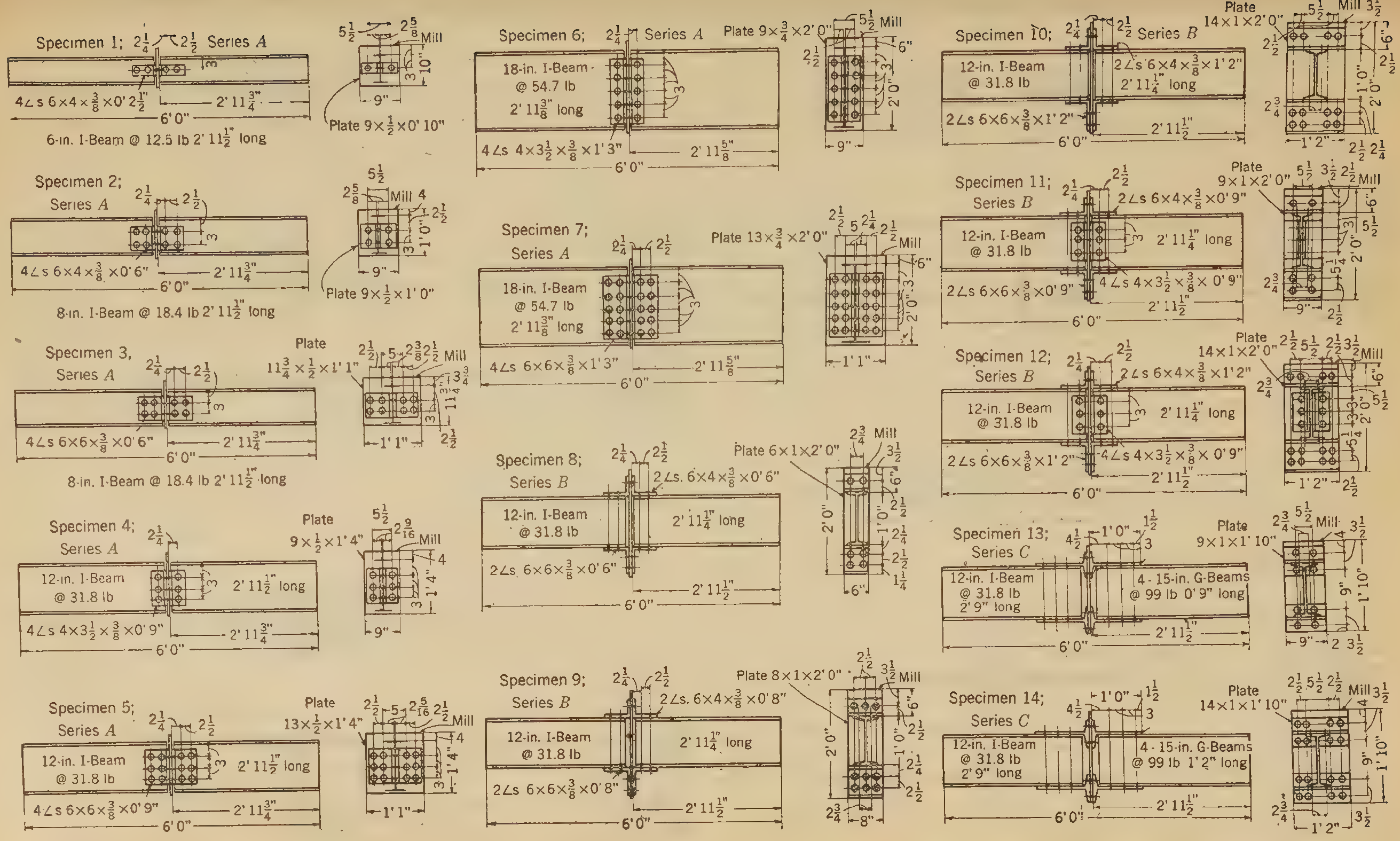

Specimen 1; Series A
4∠s 6×4×3/8×0′2½″
6′0″
2′11¾″
6-in. I-Beam @ 12.5 lb 2′11½″ long
Plate 9×½×0′10″
Mill
Specimen 2; Series A
4∠s 6×4×3/8×0′6″
8-in. I-Beam @ 18.4 lb 2′11½″ long
Plate 9×½×1′0″
Specimen 3, Series A
4∠s 6×6×3/8×0′6″
8-in. I-Beam @ 18.4 lb 2′11½″ long
Plate 11¾×½×1′1″
Specimen 4; Series A
12-in. I-Beam @ 31.8 lb
2′11½″ long
4∠s 4×3½×3/8×0′9″
Plate 9×½×1′4″
Specimen 5; Series A
12-in. I-Beam @ 31.8 lb
2′11½″ long
4∠s 6×6×3/8×0′9″
Plate 13×½×1′4″
Specimen 6; Series A
18-in. I-Beam @ 54.7 lb 2′11⅜″ long
4∠s 4×3½×3/8×1′3″
2′11⅝″
Plate 9×¾×2′0″
Specimen 7; Series A
18-in. I-Beam @ 54.7 lb 2′11⅜″ long
4∠s 6×6×3/8×1′3″
Plate 13×¾×2′0″
Specimen 8; Series B
12-in. I-Beam @ 31.8 lb
2∠s. 6×4×3/8×0′6″
2′11¼″ long
2∠s. 6×6×3/8×0′6″
Plate 6×1×2′0″
Specimen 9; Series B
12-in. I-Beam @ 31.8 lb
2∠s. 6×4×3/8×0′8″
2′11¼″ long
2∠s 6×6×3/8×0′8″
Plate 8×1×2′0″
Specimen 10; Series B
12-in. I-Beam @ 31.8 lb
2∠s 6×4×3/8×1′2″
2′11¼″ long
2∠s 6×6×3/8×1′2″
Plate 14×1×2′0″
Specimen 11; Series B
12-in. I-Beam @ 31.8 lb
2∠s 6×4×3/8×0′9″
2′11¼″ long
4∠s 4×3½×3/8×0′9″
2∠s 6×6×3/8×0′9″
Plate 9×1×2′0″
Specimen 12; Series B
12-in. I-Beam @ 31.8 lb
2∠s 6×4×3/8×1′2″
2′11¼″ long
4∠s 4×3½×3/8×0′9″
2∠s 6×6×3/8×1′2″
Plate 14×1×2′0″
Specimen 13; Series C
12-in. I-Beam @ 31.8 lb 2′9″ long
4 - 15-in. G-Beams @ 99 lb 0′9″ long
Plate 9×1×1′10″
Specimen 14; Series C
12-in. I-Beam @ 31.8 lb 2′9″ long
4 - 15-in. G-Beams @ 99 lb 1′2″ long
Plate 14×1×1′10″

FIG. 1.—TEST SPECIMENS OF RIVETED CONNECTIONS, WITH ⅞-IN. ϕ-RIVETS; SERIES *A*, SPECIMENS 1 TO 7; SERIES *B*, SPECIMENS 8 TO 12; AND SERIES *C*, SPECIMENS 13 AND 14.

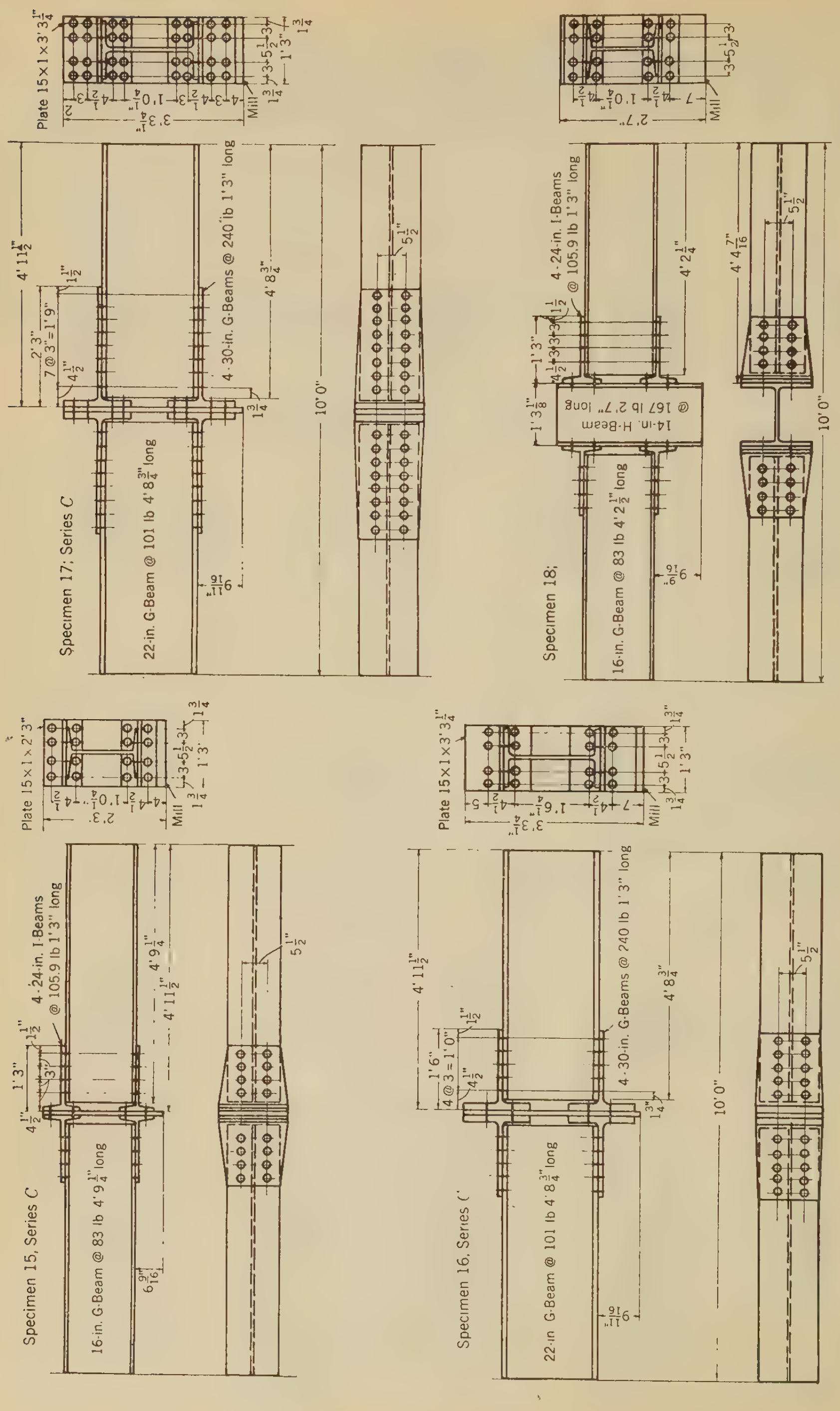


Plate 15×1×3' 3¼"
Specimen 17; Series C
22-in. G-Beam @ 101 lb 4' 8¾" long
4 · 30-in. G-Beams @ 240 lb 1' 3" long
4' 11½"
2' 3"
7 @ 3" = 1' 9"
4' 8¾"
10' 0"
Mill
Specimen 18;
16-in. G-Beam @ 83 lb 4' 2½" long
4 · 24-in. I-Beams @ 105.9 lb 1' 3" long
14-in. H-Beam @ 167 lb 2' 7" long
2' 7"
4' 2¼"
4' 4 7/16
10' 0"
Mill
Plate 15×1×2' 3"
Specimen 15, Series C
16-in. G-Beam @ 83 lb 4' 9¼" long
4 · 24-in. I-Beams @ 105.9 lb 1' 3" long
4' 9¼"
4' 11½"
2' 3"
Mill
Plate 15×1×3' 3¼"
Specimen 16, Series C
22-in. G-Beam @ 101 lb 4' 8¾" long
4 · 30-in. G-Beams @ 240 lb 1' 3" long
4' 11½"
1' 6"
4 @ 3 = 1' 0"
4' 8¾"
10' 0"
Mill

FIG. 2.—TEST SPECIMENS OF RIVETED CONNECTIONS, WITH 1-IN. φ-RIVETS, SERIES C.

Series *B* was designed and tested in order to find the stiffening effect of seat angles. Owing to the superior lever arm of these angles over those in Series *A*, these connections proved to be comparatively stiff (see Fig. 3 (*a*)). Although it is not common practice to place seat angles at both the top and bottom of the I-beam, it was thought that a better balanced connection for the purpose of these tests would be thus obtained. A variation from common practice was made in the width of the angles and in the combination of seat angles and clip angles. From a comparison of the curves thus obtained (see Fig. 3 (*a*)), it is hoped that one can form some estimate of these several factors.

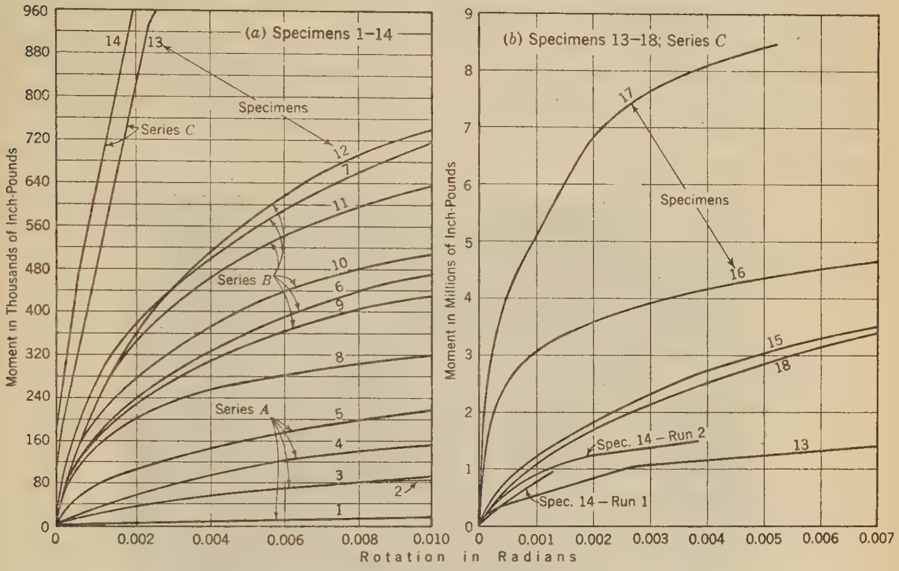


FIG. 3.—COMPARATIVE ELASTICITY OF CONNECTIONS.

Series *C* (Fig. 3 (*b*)) represents one type, and a common one, of shallow wind-braced connections. No definite standards were available, so the writer designed these units. Specimen 13 was designed so that the failure due to tension in the rivets might be anticipated and that, before such a failure would occur, the shear rivets would be badly overstressed. In Specimen 14 the tension rivets were increased in number over those in Specimen 13. Specimen 15 was a re-design of Specimen 13 for heavier construction, and Specimen 18 is a companion to Specimen 15. The comparison of these two should give some idea of the effect of inserting a post in place of the plate. Specimens 16 and 17 were designed to permit a study of the effect of a double row of tension rivets. The members in this entire series are graded in size from the small 12-in. I-beam to a specimen as large as could be handled in the laboratories.

METHOD OF TESTING

The specimens in which the anticipated load was less than 60 000 lb were tested in a screw machine of that capacity. An I-beam was laid on the bed of the machine in order to support the bearing-blocks; the load was applied at the rate of 0.05 in, per min; and the machine was kept in balance, but the load was not increased, while the readings were being taken. For specimens in which the anticipated load was greater than the capacity of the 60 000-lb machine the tests were made in a hydraulic machine of 300 000-lb capacity, and the bed was extended as before. The supporting beam was from the same stock as the specimen and deflection readings on it permitted a comparison between the beam and the connection.

All the specimens were whitewashed so as to accentuate strain lines and deformation in the rivets and plates. During each test, a careful record was kept of the deflection of the center plate relative to the end supports. This aided the operator of the machine materially in judging the increments of load and the time at which the test should be stopped. The amount of this deflection was obtained by means of two Ames dials supported by a bar that rested on rods fastened to the ends of the specimen. The stem of the dial was in contact with a bracket secured to the central plate, as shown in Fig. 4. During the test the average of the two dial readings was plotted, but the resulting curve has not been incorporated in this paper; nor has it been used in computing the accompanying curves. For such a curve to be of value, corrections would have to be made for the shear deformation of the joint and for the elastic deformation of the beam itself. Although these corrections are appreciable, the dials proved advantageous for the purpose intended. For the tests of Series *A* and *B* the side bars were brass, $\frac{1}{4}$ in. by 1 in. in cross-section; and for Series *C* and Specimen 18, a 3-in. duralumin channel was used.

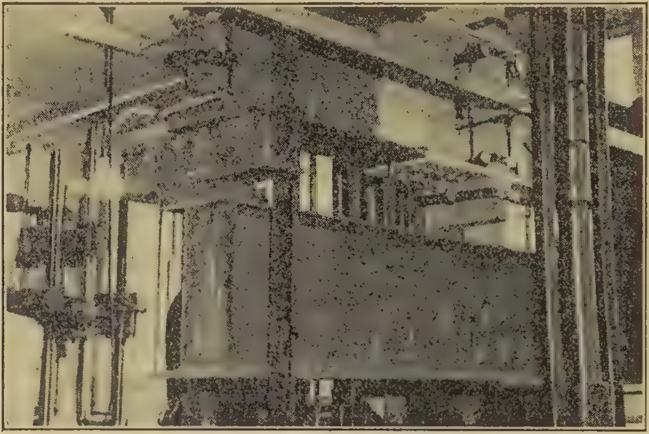
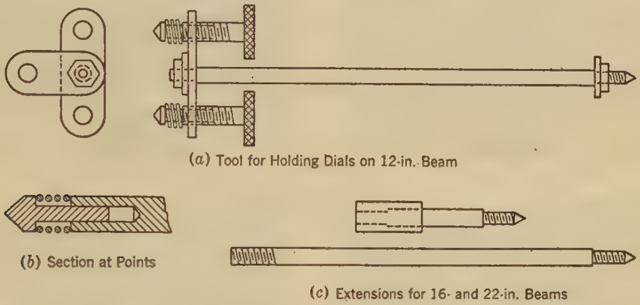


FIG. 4.—SPECIMEN 16, SHOWING POSITION OF THE SEVERAL DIALS AND METHOD OF BRACING A SPECIMEN AGAINST BUCKLING.

For a simple beam under uniform load, with a deflection of $\frac{1}{360}$ of its span, the angle of rotation of the end of the beam is about 0.009 radian. When possible, the tests were continued until the deflection of the connection exceeded this value. In a few cases this was prevented by the limited capacity of the machine.

Four dials, reading to 0.0001 in., were used in obtaining the angular rotation of each connection. In the case of Series *A* they were clamped directly to the flange of the I-beam with their stems resting against the central plate. This type of set-up could not be used in the remaining tests. In the case of Series *B*, the seat angles were in the way of the clamps, and, as some shear would occur between these angles and the I-beam, a different design was required. The flanges of the stubs of the I-beam in Series *C* interfered with the stem of the dial, and, in addition, the web of the stub interfered with the clamps as in Series *B*. The special holder shown in Fig. 5 was made to carry the dials in Series *B* and *C*. It was designed to furnish a three-point contact when clamped between the flanges of the I-beam. As first designed the pointed screws of the holder were made solid, but this proved unsatisfactory because the beam deformed under stress, causing one point to become loose. To correct this difficulty the points were placed on springs, as shown in Fig. 5(*b*).


(*a*) Tool for Holding Dials on 12-in. Beam

(*b*) Section at Points

(*c*) Extensions for 16- and 22-in. Beams

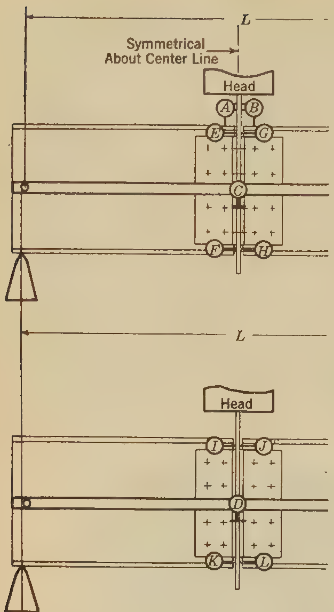
FIG. 5.—SPECIAL DEVICE FOR HOLDING DIALS THAT MEASURED ROTATION OF CONNECTION

In the tests of Series *B* the stems of the dials rested directly on the control plate. In the tests involving the stub I-beams (Series *C*, and Specimen 18), the stems of the dials could not be brought in direct contact with the central plate, so steel bands of iron (1 in. by $\frac{1}{8}$ in.), were placed around the stubs and secured to the plate by set screws at the elevations where the dial stems would come in contact with them (see Figs. 4 and 6(*c*)). Care was taken that the bands did not touch any part of the specimen except through the set screws, thus assuring the condition that they were stationary relative to the plate.

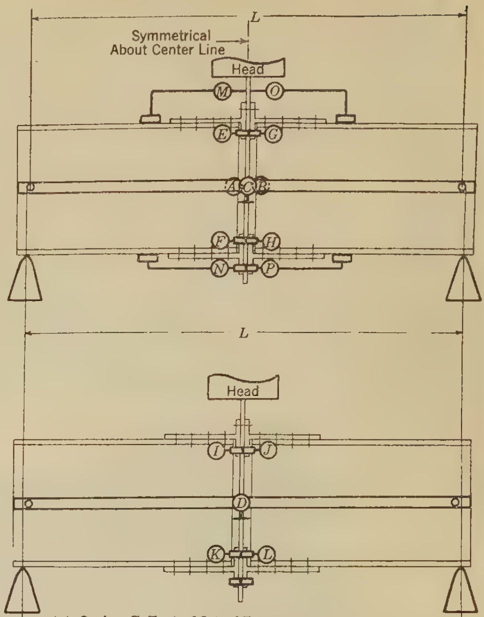
In Series *B* and *C*, another set of dials was secured to the flange of the I-beam, but back of the connection, as shown in Figs. 4, 6(*b*), and 6(*c*). Readings from these dials were for the purpose of securing data that would be of value if an accident occurred to any of the other rotation dials during the test. The data thus obtained have not been incorporated in this paper. Fig. 6 shows schematically the location of the various dials and the general set-up of the tests.

DESCRIPTION OF TESTS AND FAILURES

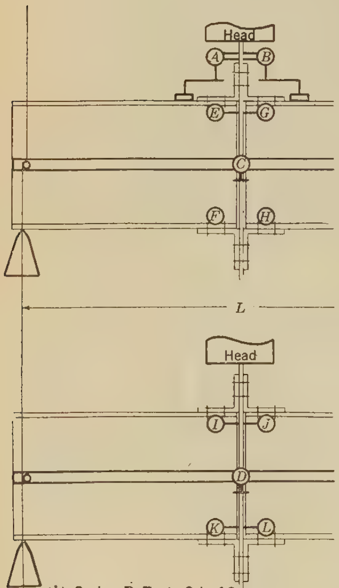
Each specimen was tested by loading it on the central plate, thus inducing a moment of one-fourth the load times the span and a shear of one-half the load in each connection. Fig. 7(*a*) illustrates the manner in which connec-


L
Symmetrical About Center Line
Head
A
B
E
G
C
F
H
L
Head
I
J
D
K
L

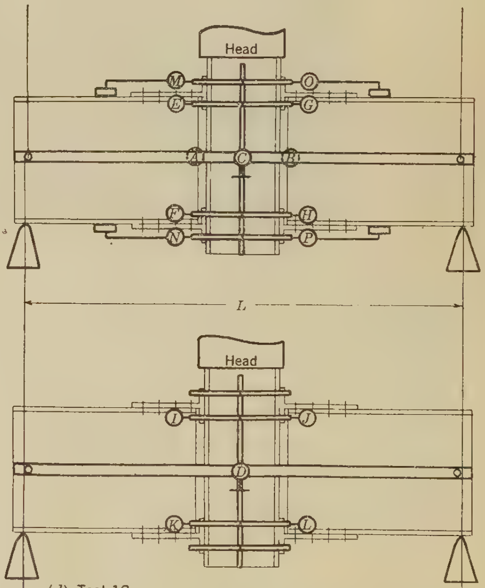
(*a*) Series *A*; Tests 1 to 7


L
Symmetrical About Center Line
Head
M
O
E
G
A
C
B
F
H
N
P
L
Head
I
J
D
K
L

(*c*) Series *C*; Tests 13 to 17


Head
A
B
E
G
C
F
H
L
Head
I
J
D
K
L

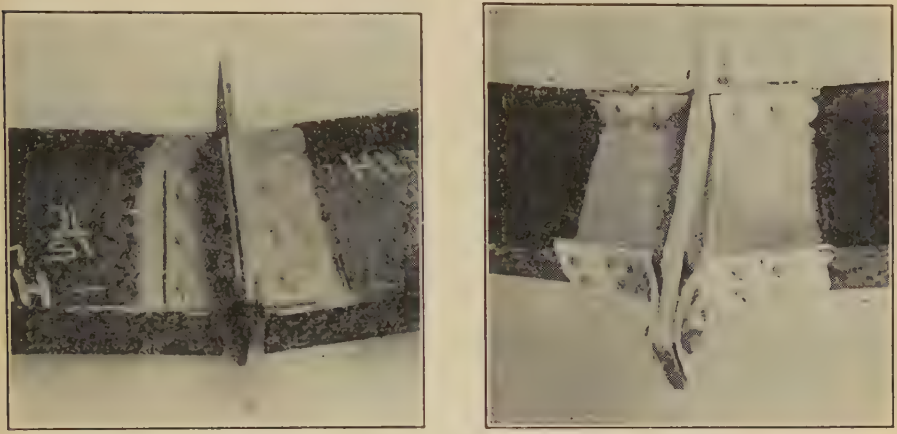
(*b*) Series *B*; Tests 8 to 12


Head
M
O
E
G
A
C
B
F
H
N
P
L
Head
I
J
D
K
L

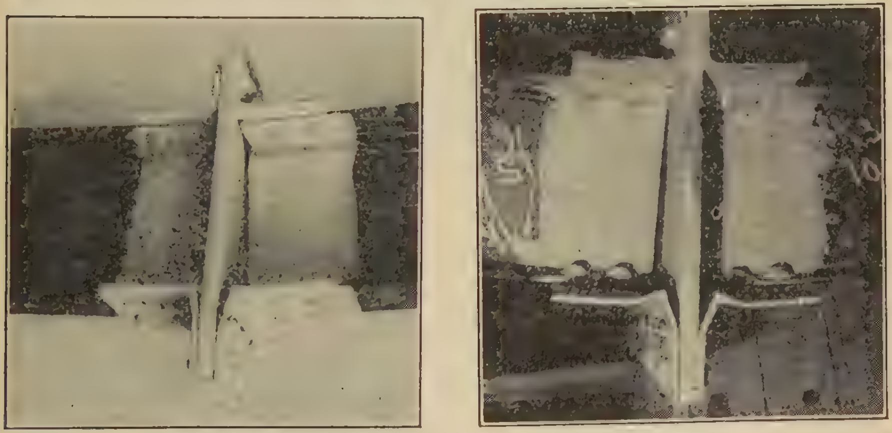
(*d*) Test 18

FIG. 6.—LOCATION OF DIALS, POSITION OF BANDS, AND METHOD OF APPLYING LOAD.

tions in Series *A* failed. The angles deformed badly without visible distress in the rivets or in the I-beams. The tests seem to confirm the general impression that this type of connection can be subjected to considerable angular deformation without seriously affecting the capacity of the connection to take shear.



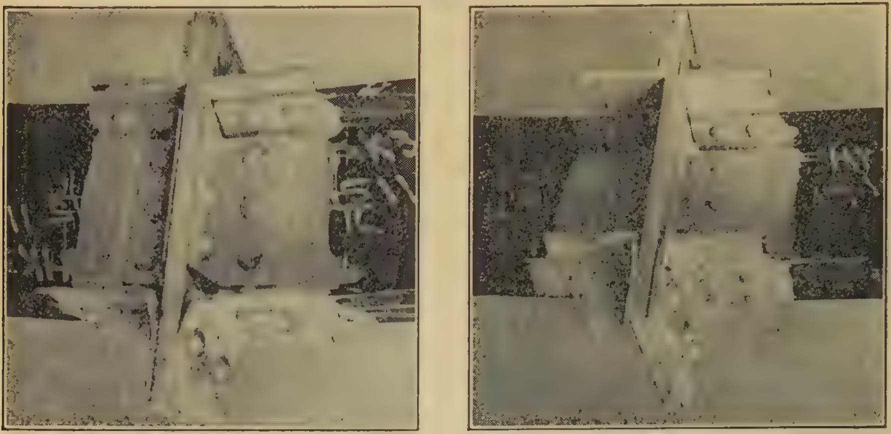
(*a*) SPECIMEN 4. (*b*) SPECIMEN 8.



(*c*) SPECIMEN 9. (*d*) SPECIMEN 10.

FIG. 7.—TYPES OF FAILURES, SERIES *A* AND *B*.

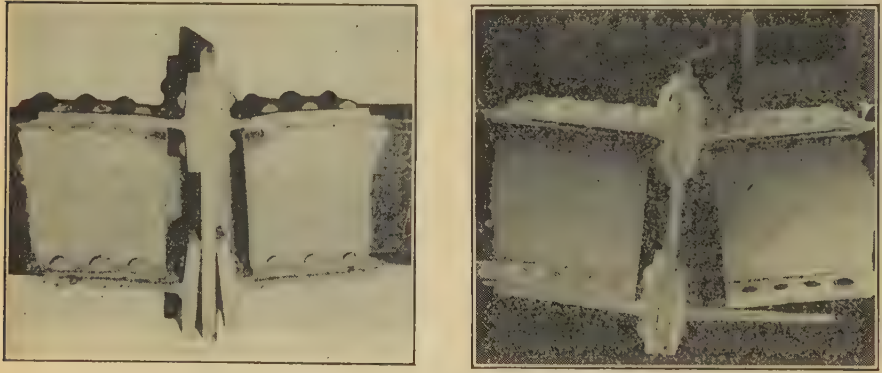
Figs. 7(*b*), 7(*c*), and 7(*d*) illustrate the manner in which the three types studied in Series *B* failed. Compared with Fig. 7(*d*), Fig. 7(*b*) shows the peculiar effect of widening the seat angles, whereas Fig. 7(*c*) shows the combination of seat and clip angles. It is to be noted that the serious stresses, apparently, are confined to the lower seat angles, and in some cases to the rivets in the lower seat angles, but not to the I-beams. From this, it might be inferred that the connection can be stiffened materially by increasing the thickness of the angles. (In the test specimens in all cases these angles are ⅜ in.) However, for a given deformation, this would lead to an increased unit stress. The effect of lengthening the seat angle may be noted by comparing the failure of Specimen 11 with that of Specimen 12, in Fig. 8.



(*a*) SPECIMEN 11. (*b*) SPECIMEN 12.

FIG. 8.—FAILURE OF SPECIMENS 11 AND 12, SERIES *B*.

Figs. 4, 9, and 10 illustrate the manner in which specimens of Series *C* failed. The rivets in Specimen 13 (Fig. 9(*a*)) failed in tension, and this connection showed weakness in the rivets early in the test. All shear rivets showed a considerable deformation and, doubtless, if the previous failure in the tension rivets had not occurred, the connection would have failed in horizontal shear, with a slight increase of load. The flanges of the I-beams indicated distress long before the failure. In Fig. 9(*a*) the deformation due to shear on the connection, as well as that due to moment, is quite evident.



(*a*) SPECIMEN 13. (*b*) SPECIMEN 15.

FIG. 9.—FAILURES OF SPECIMENS 13 AND 15, SERIES *C*.

Specimen 15 (Figs. 2 and 9(*b*)) was a case of direct shear failure. Some tensile deformation of the rivets was clearly visible and the flanges of the stub I-beam on the tension side were deformed. This connection failed suddenly. Specimen 16 (Fig. 2) failed by tension in the rivets, with some small deformation of the flanges of the I-beam, whereas in Specimen 17 (Fig. 2), there was no visible deformation under the load of 300 000 lb, the limiting capacity of the machine.

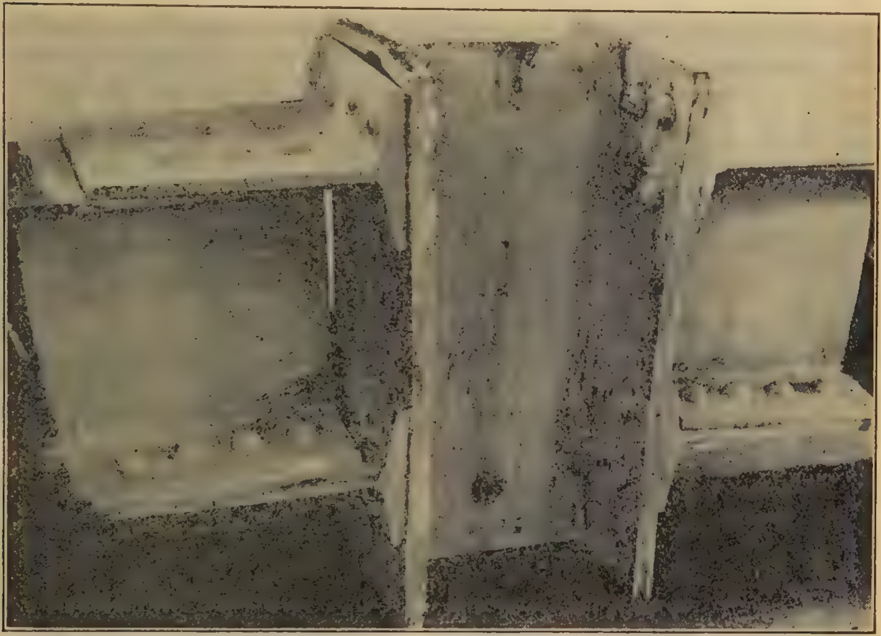


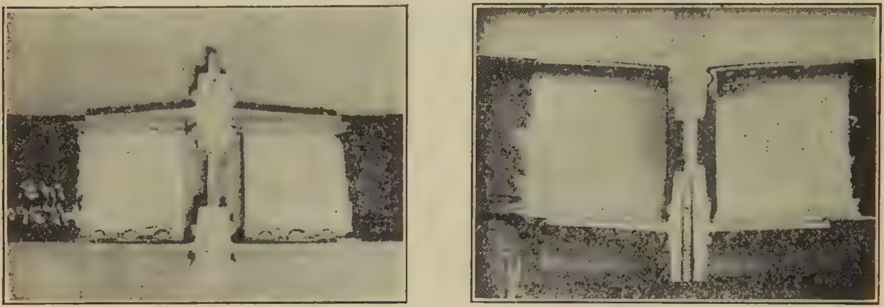
FIG. 10.—FAILURE OF SPECIMEN 18.

Sundry other failures of beams in Series C are illustrated in Fig. 11. The lines of shear in the I-beam of Fig. 11(a), are clearly visible, as is also the affect of the rivet heads due to shear. This test gave no visible evidence of tension in the rivets. The method of placing dials for the specimens of Series C, is shown in Fig. 11(b). Another view (Specimen 15) of the shallow wind-bracing connection, after the rivets had failed in shear, is shown in Fig. 11(c). In Fig. 11(d), Specimen 16 reveals distress in shear as well as rivets that have failed in tension.

Specimen 18 (Fig. 2) failed in a manner so complicated that it is difficult to compare the results of this test with those of its companion, Specimen 15. The column section showed distress before the connections had passed their yield point. From Fig. 10 one can follow the general action of this joint. The stress distribution in tension was high in the central rivets and low in the outer ones.

The column section used was so short that the curves obtained do not properly represent the effect due to a column of considerable length. The difference in the curves of Specimens 15 and 18 is probably greater than would be produced by the usual column section found in practice.

In Specimen 14 there was little distress evident due to rivet tension. Deformation due to shear on the rivets was visible and strain lines on the I-beam indicated that the connections had passed their limit of working stress. In testing this specimen, experience with Specimen 13 led the writer to anticipate a shear failure. After obtaining the stress reading for a load of 80 000 lb, the load was removed. This gave a set of curves that compare with the remainder of the series. The load was then transferred from the central plate to a point over the ends of the I-beam, thus relieving the shear on the stub beams, and the test was re-run to a load of 105 000 lb. In plotting the curve of the second run the effect of a decrease in the length of the lever arm was taken into consideration. The point of zero deflection was observed with no load on the beam; it was not read as the deflection due to the set from the previous load. Failure did not occur in the connections of this specimen but at one of the supports, where the I-beam buckled.


(*a*) Specimen 14. (*b*) Specimen 16.

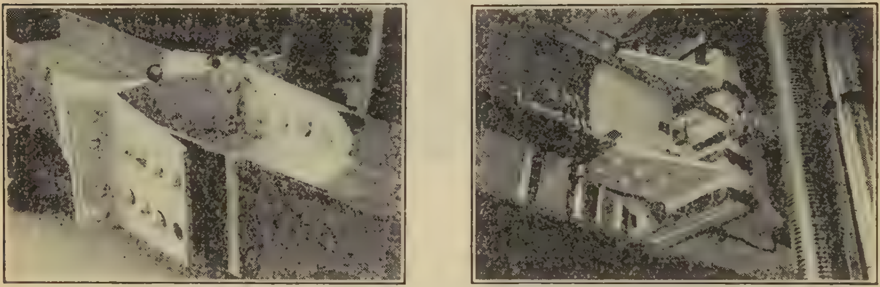

(*c*) Specimen 15. (*d*) Position of Dials, Specimen 14.

Fig. 11.—Failures of Specimens 14, 15, and 16, Series *C*.

As the specimen had not been destroyed, except for the buckling of one flange of the I-beam, it was inverted and replaced in the machine in order that a study of the reversal of stresses might be obtained. In this third run on the specimen the loads were placed over the ends of the I-beam and not on the central plate, primarily because this edge of the plate had not been milled for this type of loading. The curve obtained was of an entirely different type from that of the others in this entire series of tests. This curve suggests a subject for further investigation.

In the case of the test of Specimens 15, 16, and 17, difficulty was encountered because of the tendency of the specimen to buckle (see Fig. 4). Stiffening angles were welded at the ends of the supporting beam, and similar angles were clamped to the ends of the specimen. In addition, angles were bolted on either side of the specimen. These angles were braced to the building to prevent a failure due to the upper ends of one of these sets of angles swinging to the north, whereas the upper ends of the angles on the opposite end of the test specimen swung to the south. Care was taken to ensure that the bracing did not interfere with the rotation of the test specimen on its supports, and thus influence the test readings.

Method of Computing Rotation

The rotation of the beam relative to the plate was measured by means of Dials *E* to *L* in Fig. 6(*b*) to Fig. 6(*d*). The rotation of each connection was obtained by noting the difference in reading between the two dials on one side of the I-beam and dividing it by the distance between the two dial stems. The corresponding dials on the opposite side of the I-beam furnished a second set of observations on this same point. The readings on the second connection of the specimen were obtained in a similar manner and the two sets of observations on each connection were averaged. Thus, each test may be considered as a test of two connections, one on the right side, and one on the left side, of the plate. These two tests were practically independent of each other, except for some slight deformation of the central plate.

In order that the variation between the two tests may be seen, the curves showing the angular rotation plotted against the moment have been drawn for both connections in Figs. 12 to 23. The average is plotted in full and, from this curve, the data in Table 2 are taken. The angle of rotation is expressed in radians or, what is the same thing, as its tangent, and the moment is expressed in inch-pounds. It is to be noted that all the curves are of the same general shape and have no definite yield point. This is more clearly shown in Series *A* and *B* than in Series *C*.

TABLE 2.—Constants from Curves in Figs. 11 to 18

Series	Test No. (1)	Slope tangent at origin, in $(10)^8$ inch-pounds (2)	Slope reloading curves, in $(10)^8$ inch-pounds (3)	Ultimate moment, in inch-pounds (4)
A....	1	0.001	0.001	23 500
	2	0.29	0.095	135 000
	3	0.39	0.126	159 500
	4	0.28	....	403 000
	5	0.47	....	529 000
	6	2.9	....	1 225 000
	7	4.5	1.59	1 300 000
B.....	8	1.9	1.09	580 000
	9	2.2	1.48	665 000
B....	10	4.7	1.33	1 055 000
	11	4.2	2.09	....
	12	4.4	1.75	1 026 000*
C....	13	13.7	8.7	1 845 000
	14	8.8	8.8	1 640 000*
	15	21.3	20.0	5 700 000
	16	160.0	66.0	8 500 000*
	17	180.0	149.0	8 530 000*

* Test failed to reach the ultimate.

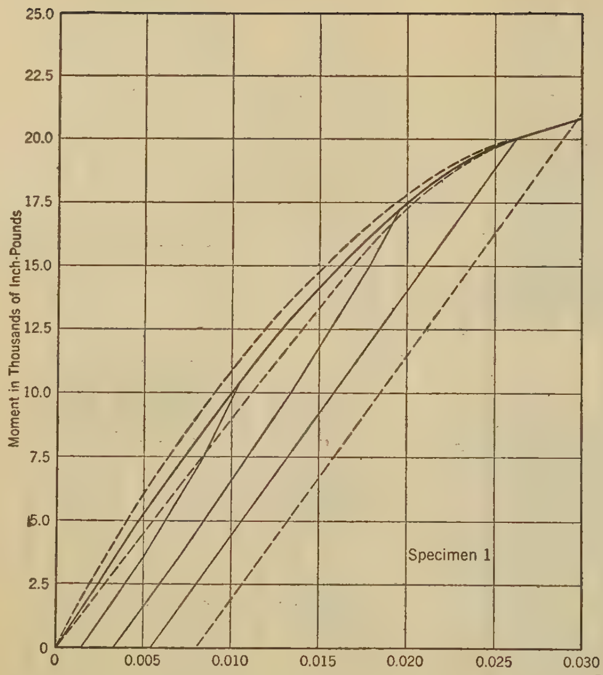


FIG. 12.—MOMENT ROTATION CURVES FOR SPECIMEN 1, SERIES *A*.

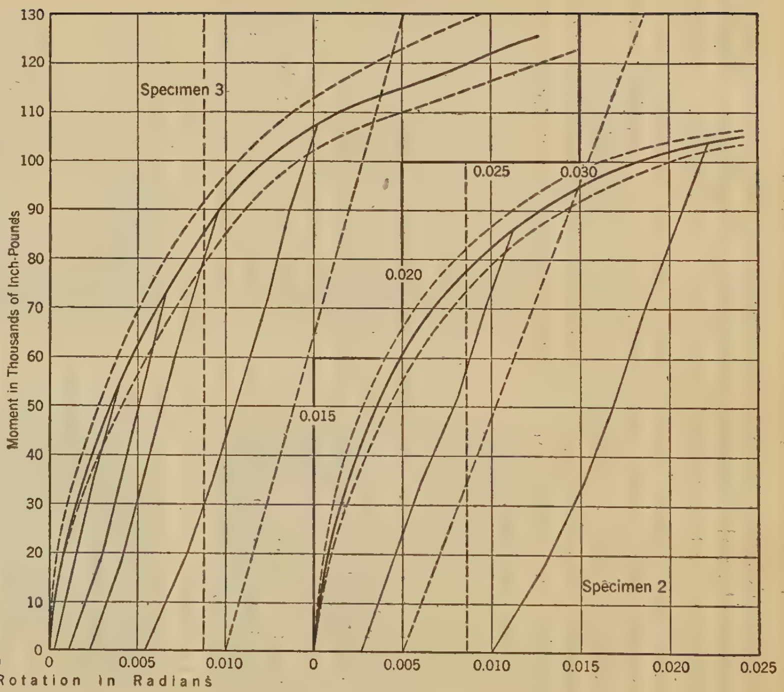


FIG. 13.—MOMENT ROTATION CURVES FOR SPECIMENS 2 AND 3, SERIES *A*.

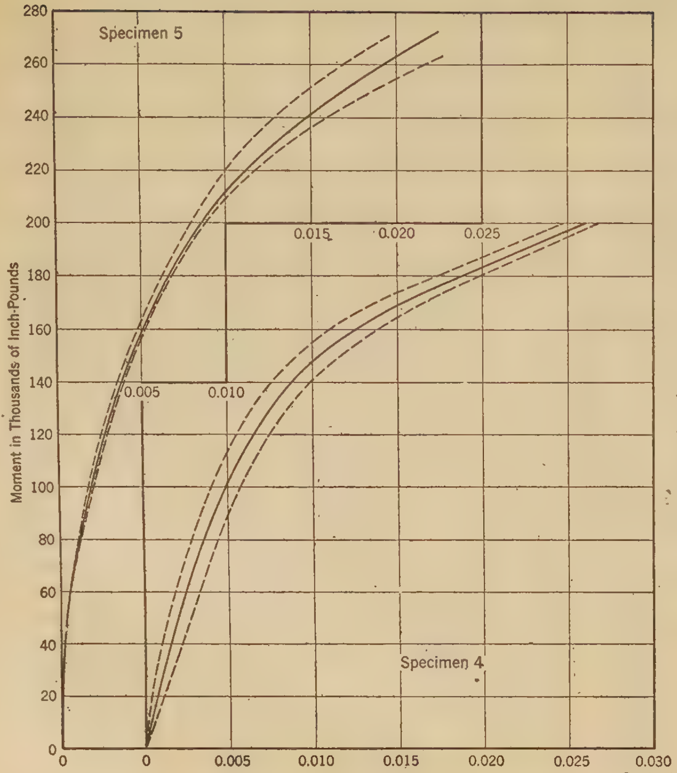


FIG. 14.—MOMENT ROTATION CURVES FOR SPECIMENS 4 AND 5, SERIES *A*.

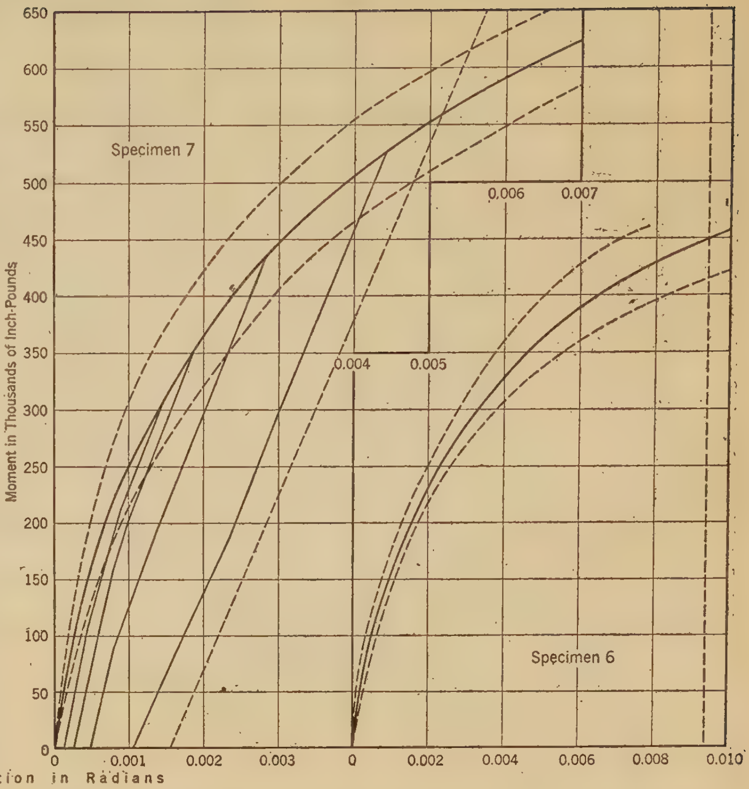


FIG. 15.—MOMENT ROTATION CURVES FOR SPECIMENS 6 AND 7, SERIES *A*.

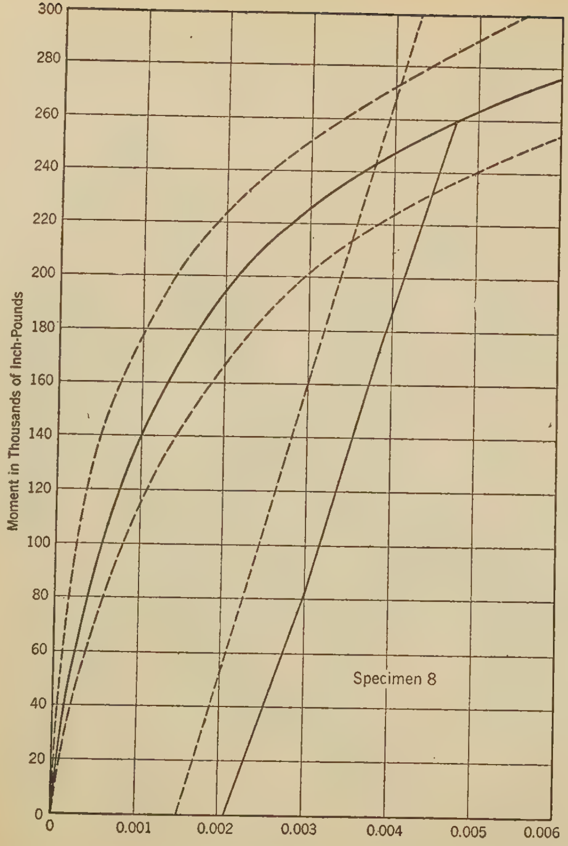


FIG. 16.—MOMENT ROTATION CURVES FOR SPECIMEN 8, SERIES *B*.

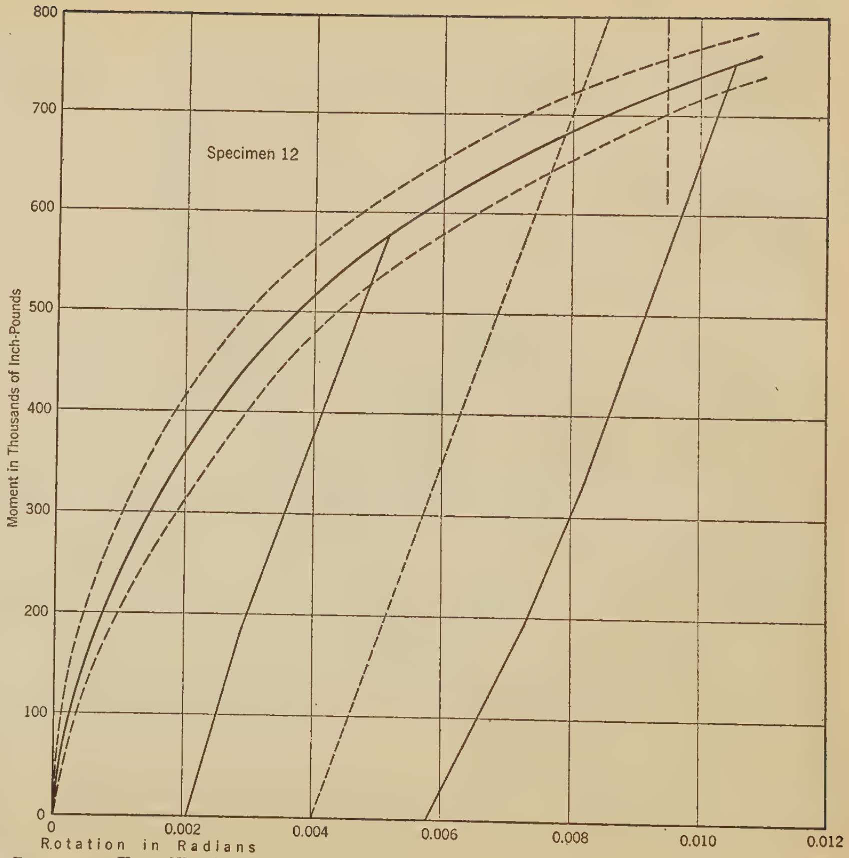


FIG. 17.—MOMENT ROTATION CURVES FOR SPECIMEN 12, SERIES *B*.

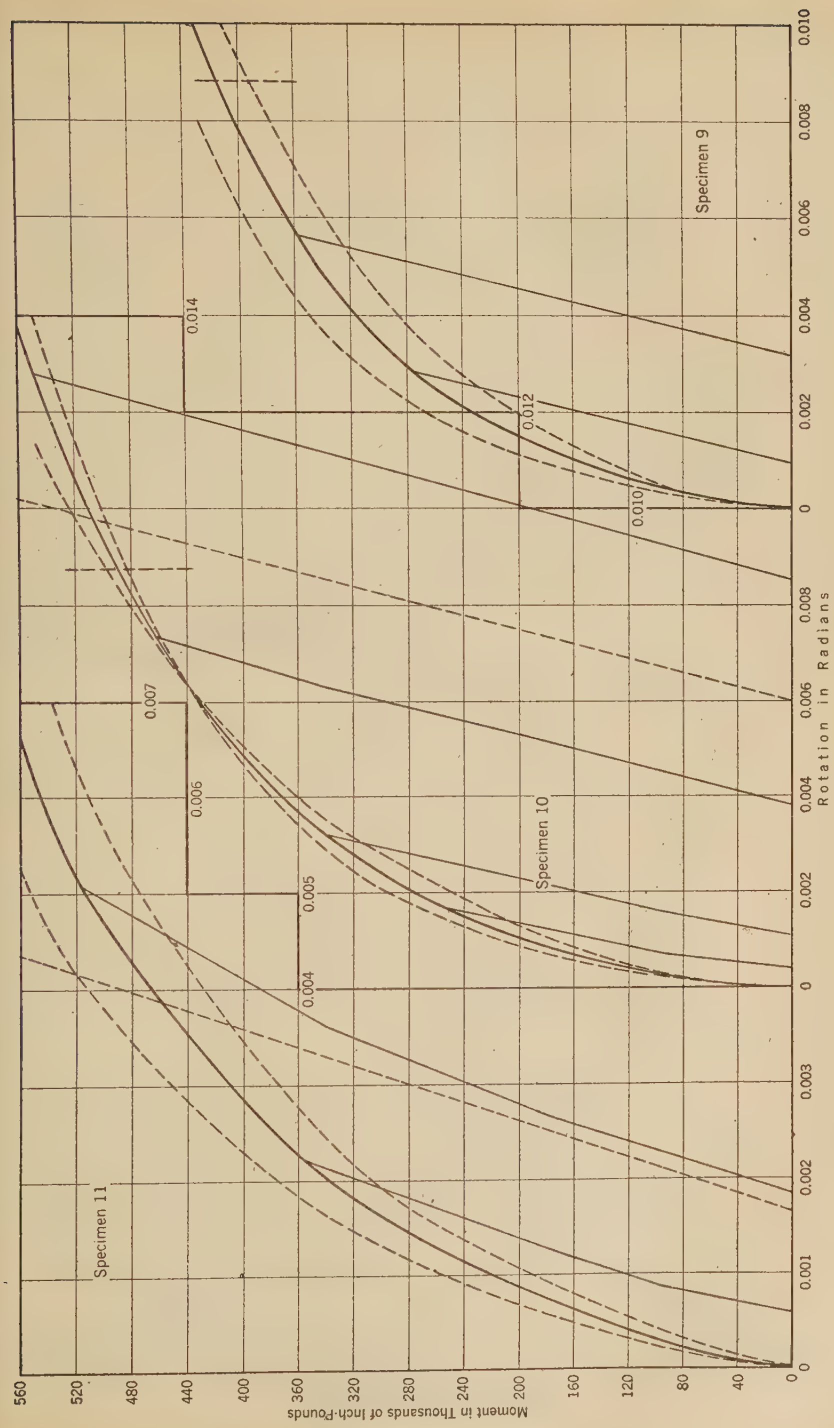


FIG. 18.—MOMENT ROTATION CURVES FOR SPECIMENS 9, 10, AND 11, SERIES *B*.

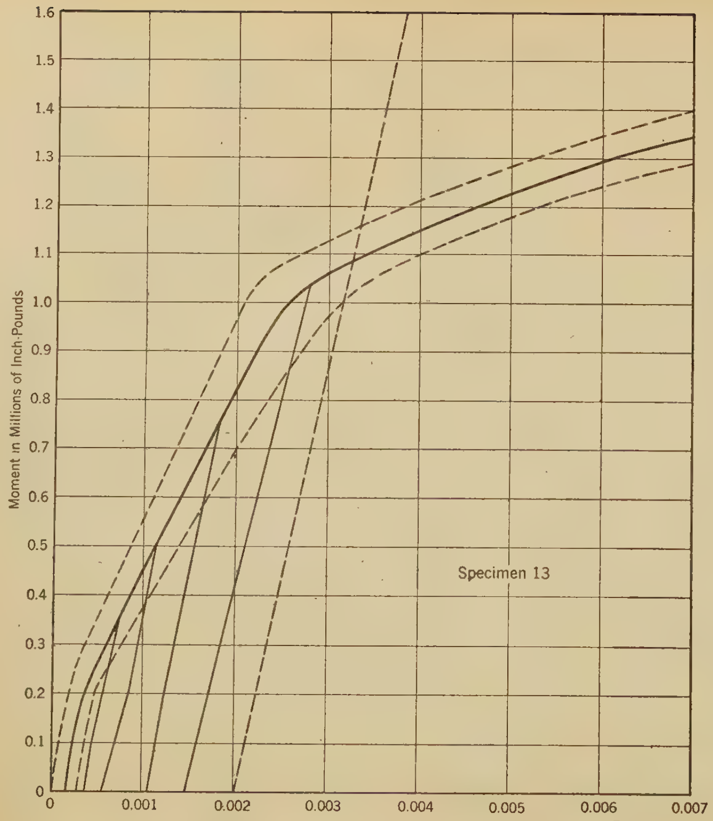


FIG. 19.—MOMENT ROTATION CURVES FOR SPECIMEN 13, SERIES *C*.

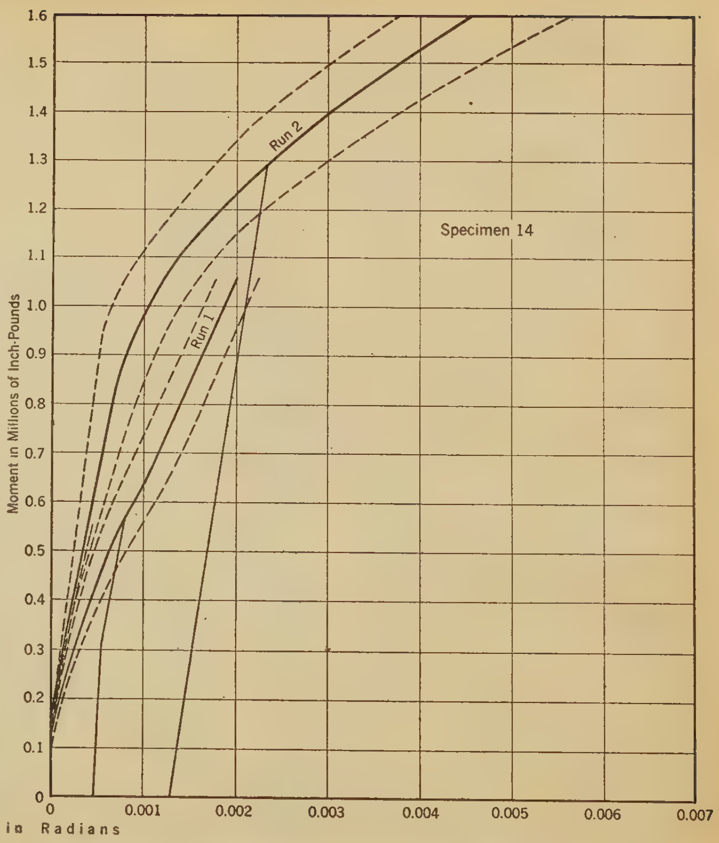


FIG. 20.—MOMENT ROTATION CURVES FOR SPECIMEN 14, SERIES *C*.

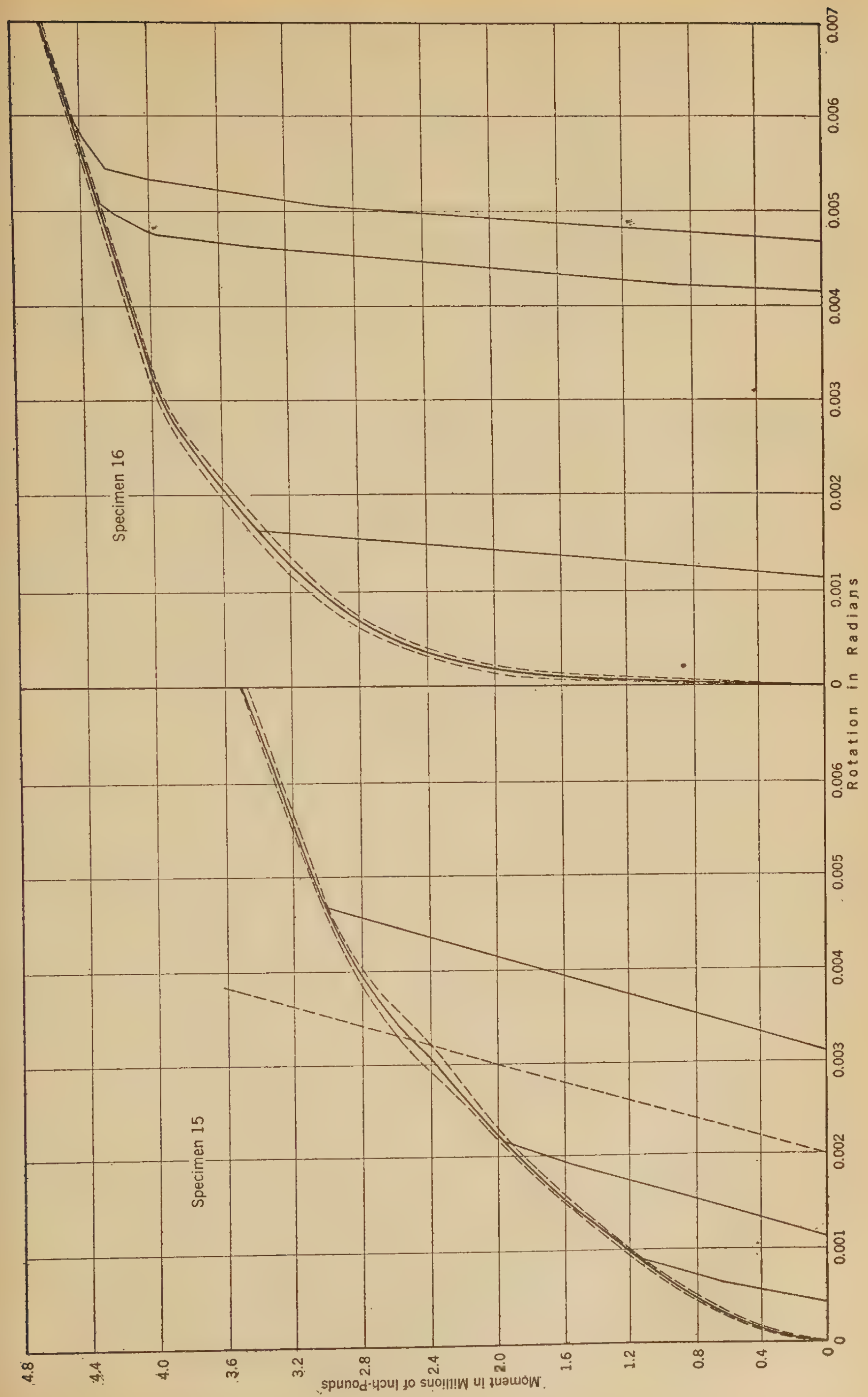


FIG. 21.—MOMENT ROTATION CURVES FOR SPECIMENS 15 AND 16, SERIES C.

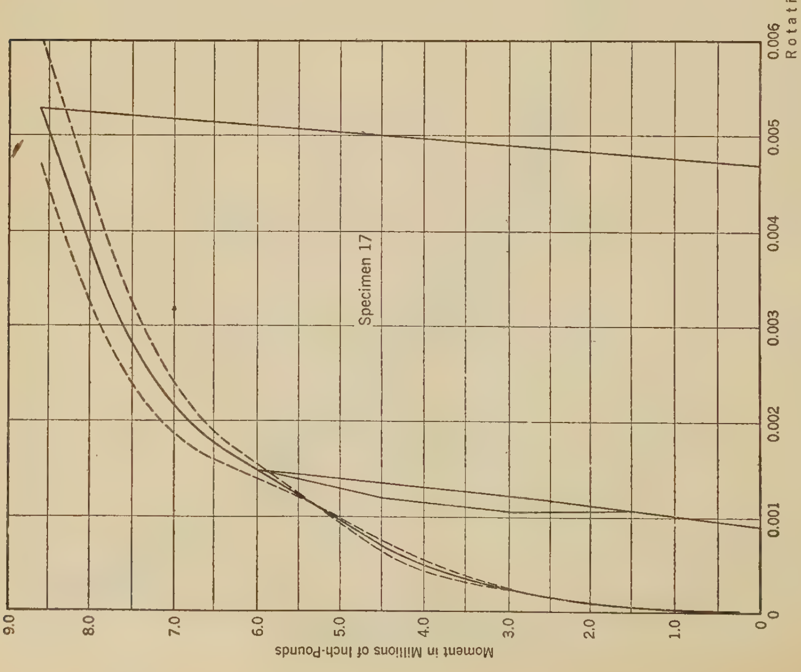


FIG. 22.—MOMENT ROTATION CURVES FOR SPECIMEN 17, SERIES *C.*

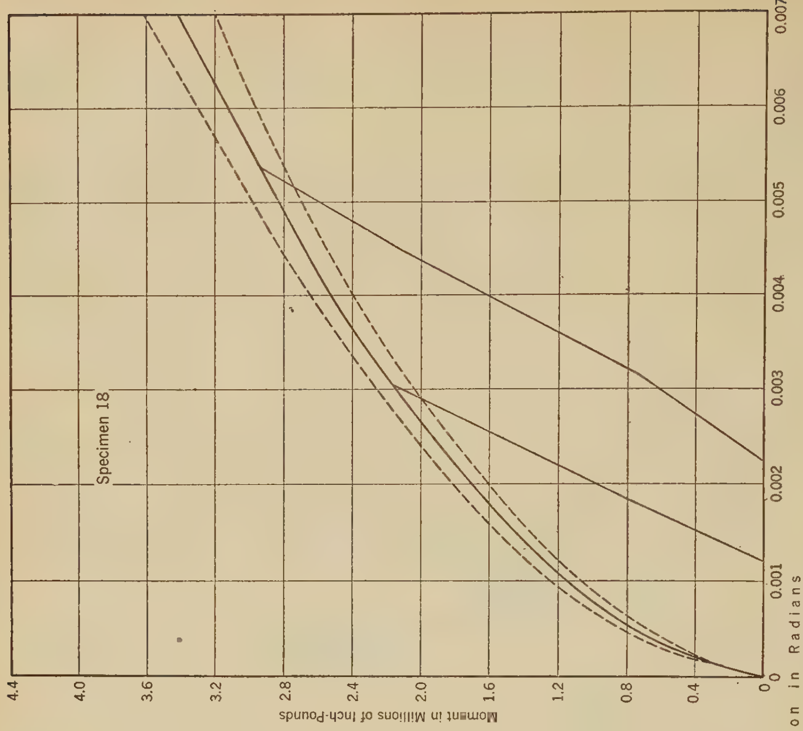


FIG. 23.—MOMENT ROTATION CURVES FOR SPECIMEN 18, SERIES *C.*

Reloading curves were taken during the tests, but in order to avoid confusion, only the curve of the average between the two connections has been plotted. These reloading curves are quite straight, and elastic constants for the connections have been obtained from their slopes.

Table 2 gives two constants for each type of connection tested, taken from the average curve. Column (2) gives the slope of the curve at its origin for a connection that has not been previously strained. Column (3) gives the slope of the reloading curves. All these slopes are in inch-pounds. The ultimate moment imposed on the connections is given in Column (4) except in the cases indicated, in which the capacity of the machine or other cause prevented carrying the test to failure.

SHEAR

In the method of testing used, the shear effect and the moment effect were not entirely separated from each other. This was not considered a serious

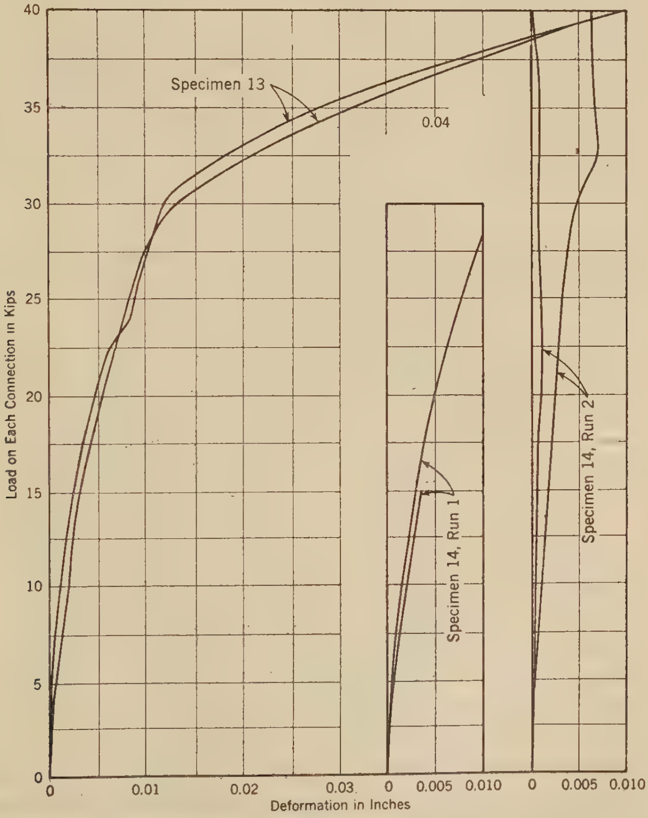


FIG. 24.—DEFORMATION DUE TO SHEAR, SPECIMENS 13 AND 14. SERIES *C*.

objection as they have little effect on each other as far as deformation is concerned, especially in the range of working stress. In actual practice both shear and moment are present in a connection when under stress. In all the tests an effort was made to obtain stress-deformation curves for the connection in shear. For Series *A* and *B* the results were not considered of much practical value and are not given in this paper. The shear stresses were low and the results were not satisfactory. The connections were designed to resist shear, but not moment. In Series *C*, however, very interesting data were obtained, which lead one to believe that this type of connection is much more satisfactory in shear than has heretofore been supposed. In no specimen of this series was special provision made for shear. Although it was seriously questioned whether these connections could take enough shear to produce ultimate failure in moment, it was thought that adding clip angles would complicate the interpretation of the results. The series of tests was designed as a study of the effect of moment and not of shear.

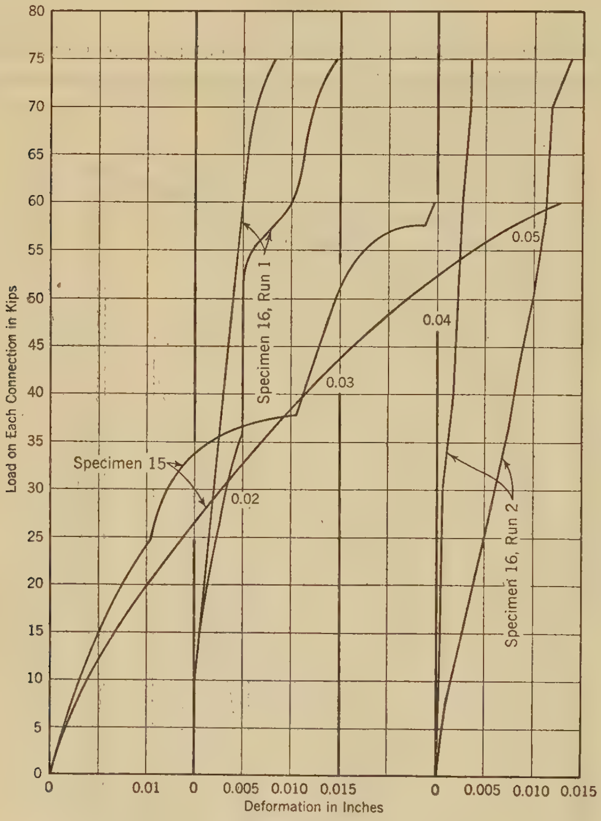


FIG. 25.—DEFORMATION DUE TO SHEAR, SPECIMENS 15 AND 16, SERIES *C*.

In Series *C* the shear dials were clamped to the I-beam as nearly midway between the flanges as possible, while the stem rested on a bracket secured to the central plate. All shear dials registered to 0.0001 in. The readings of these dials had to be corrected for the angular change of the I-beam due to rotation of the connection before the shear curves were plotted. The curves were drawn with the vertical displacement of the connection plotted against the load on the connection. This latter is one-half the load on the specimen. Figs. 24, 25, and 26 give some idea of the capacity of this type of connection to resist shear when not reinforced by clip angles, or by other means. In plotting the shear curves of the second runs of Specimens 14 and 16 (Figs. 24 and 25), deflection under no load was taken as zero and not as the set from the previous run.

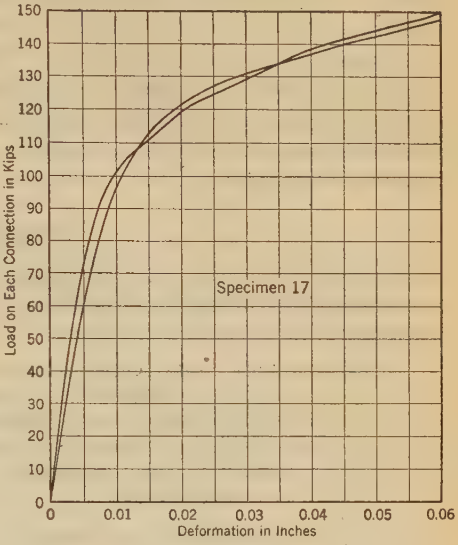


FIG. 26.—DEFORMATION DUE TO SHEAR, SPECIMEN 17.

PART II.—APPLICATION TO FRAMED STRUCTURES

INTRODUCTION

In order to facilitate the incorporation of the effect of the elasticity of the connections in the analysis of framed structures, the principal formulas of the several methods of frame analysis have been expanded to include this as a factor. The methods of Slope Deflection and Moment Distribution have been considered. The Theorem of Three Moments has been rewritten to include elastic connections and the Deformator Method of stress analysis is discussed briefly.

It is to be noted that few of the moment-rotation curves obtained experimentally in Part I approach a straight line even in the lower values. This makes it necessary to assume an arbitrary value for the moment on the connection, or for its angular rotation when substituting in the formulas involving the elasticity of connections. After the moment on this connection is computed it can be compared with the original assumption and the error corrected. Stated in another way, as the curves of Part I have no mathematical equations there seems to be no way of avoiding the trial-and-error method of computing when other than approximate results are desired.

For the purpose of yielding a first approximation and also to provide a comparison of the elastic properties of the several connections, the case of a beam uniformly loaded and connected at each end to a rigid column for each of the connections listed, has been analyzed.

In expanding the formulas of the several methods it is assumed that the reader is familiar with these methods as applied to frames with rigid connections. The formulas found in texts on the subject are first given for comparison with the expanded form. Their derivation and use are omitted as they can readily be found in numerous books on the subject. These equations are followed by the corresponding formulas, expanded to include the effect of elasticity in the connections. The derivation of the expanded formulas is then given.

NOTATION

The notation adopted for this paper conforms essentially with the American Standard Symbols for Mechanics, Structural Engineering, and Testing Materials.[5] The definitions follow:

d = depth; total depth of a member.
h = heights of supports in a continuous beam.
i = width of member at base of groove in model.
j = width of rectangular groove in model.
l = length of the member of a model, corresponding to length, L, of a prototype.
q = a coefficient for moment in Table 3 = $-\frac{M}{w\,L^2}$.
t = thickness of model sheet.
u = a coefficient for moment in Table 3 = $-\frac{M}{w\,L^2}$.
w = unit uniformly distributed load, or load per unit length on a uniformly loaded member.
x = variable distances measured parallel to X-axis; $\overline{x}$ = distance from the left support to the center of gravity of Area A; $\overline{x}_1$ = distance from right support to center of gravity of Area A.
y = deflection; also, variable distances measured parallel to the Y-axis.
A = area of the moment curve due to transverse loads, the member being considered as a simple beam.
E = modulus of elasticity; E_c = modulus of elasticity of model material.
I = rectangular moment of inertia of a cross-section of a member.
K = stiffness factor.
L = length; nominal length of a member (distance from center to center of connections); L_1 and L_2 = transformed lengths of a member as defined in Equations (18) and (19); L_a = nominal length of the member in Span a; and L_{BC} = the length of the member in Span $B\,C$, transformed by the connection at Point B.
M = moment; M_c = moment at end of a loaded beam, with rigid connections and with length, L.
S = sections modulus of a cross-section of a member.

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[5] A. S. A.—Z 10*a*—1932.

Z = a coefficient of M such that $MZ = \theta_M$ = angle of rotation of the connection due to moment, M, values of Z being taken from Figs. 12 to 23.

α = ratio of the deflection of one end with respect to the other end, to the length of the member (Fig. 27).

$\beta = \theta_A - \alpha$ = slope at left end of a beam.

$\gamma = \theta_B - \alpha$ = slope at right end of a beam.

Δ = displacement; the displacement of the central support from a line joining the two end supports in a continuous beam.

θ = an angular distance; change in slope of the end tangent to the elastic curve; the angle of rotation of a connection; $\theta_M = MZ$ = angle of rotation of the connection due to moment, M, taken from Figs. 12 to 23.

σ = scale ratio of a model.

τ = ratio of characteristics of a model and its prototype; τ_r = ratio between two expressions for rigidity in model and prototype; τ_e = ratio between two expressions for elasticity in model and prototype.

SLOPE DEFLECTION METHOD

The slope deflection method consists of the proper application of certain formulas to the analysis of rigid frames. When the connections are rigid the three fundamental formulas and their use are given in texts and treatises on the subject. The formulas for any beam, AB, are:

$$M_A = 2\,E\,\frac{I}{L}\,(2\,\theta_A + \theta_B - 3\,\alpha) - M_{cA} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

and,

$$M_B = 2\,E\,\frac{I}{L}\,(2\,\theta_B + \theta_A - 3\,\alpha) + M_{cB} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

and when the right end is hinged:

$$M_A = 3\,E\,\frac{I}{L}\,(\theta_A - \alpha) + \frac{3\,A\,\overline{x_1}}{L^2} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

When the end connections are not rigid and allowance is made for the elastic properties of the connections, Equations (1) and (2) become:

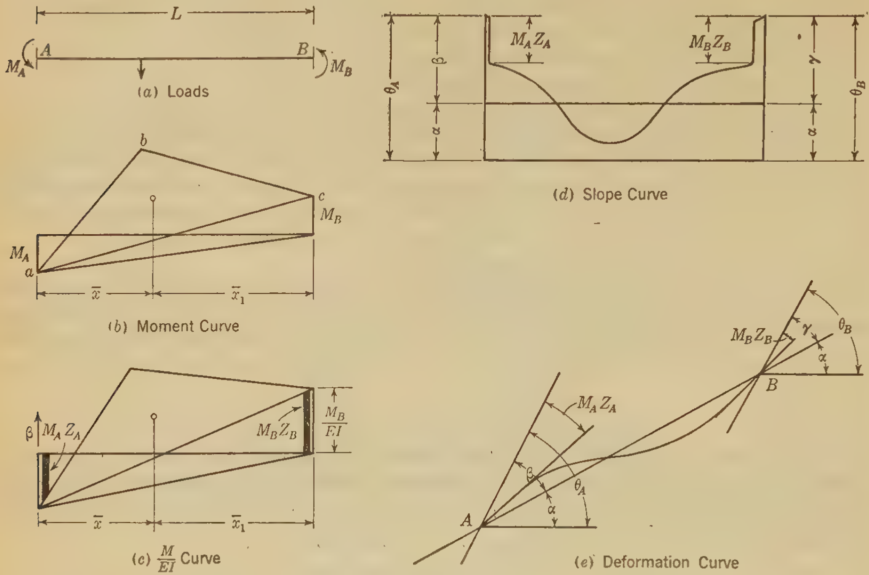
$$M_A = 6\,E\,I\,\frac{2\,L_{2B}\,(\theta_A - \alpha) + L\,(\theta_B - \alpha)}{4\,L_{2A}\,L_{2B} - L^2} - M_{cA} \quad \ldots\ldots\ldots (4)$$

and,

$$M_B = 6\,E\,I\,\frac{2\,L_{2A}\,(\theta_B - \alpha) + L\,(\theta_A - \alpha)}{4\,L_{2A}\,L_{2B} - L^2} + M_{cB} \quad \ldots\ldots\ldots (5)$$

in which, $L_2 = L + 3\,E\,I\,Z$ and Z is a coefficient of M, such that $M\,Z = \theta_M$ = angle of rotation of the connection due to the moment, M. The signs of M_A, M_B, θ_A, θ_B, and α in Equations (1) to (5) are positive as indicated in Fig. 27; that is, the moment is positive when it tends to turn the member in a counter-clockwise direction. Angular changes are positive when they move in this same direction.

M_{cA} and M_{cB} are the end moments on the beam (considered fixed at the ends by rigid connections in Equations (1), (2), and (3), and by elastic connections in Equations (4) and (5)), due to external loads on the beam. For the rigid connections these values are:



(a) Loads

(b) Moment Curve

(c) $\frac{M}{EI}$ Curve

(d) Slope Curve

(e) Deformation Curve

FIG. 27.

$$M_{cA} = -\frac{2A}{L^2}(2L - 3\bar{x}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

and,

$$M_{cB} = -\frac{2A}{L^2}(2L - 3\bar{x}_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

and, for the beam with elastic connections, they are:

$$M_{cA} = -\frac{6A}{L} \times \frac{2L_{2B}\bar{x}_1 - L\bar{x}}{4L_{2A}L_{2B} - L^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

and,

$$M_{cB} = -\frac{6A}{L} \times \frac{2L_{2A}\bar{x} - L\bar{x}_1}{4L_{2B}L_{2A} - L^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

As M_{cA} and M_{cB} are the moments at the ends of a beam, their sign follows the convention of the beam theory (positive moment produces compression on the upper side of the beam) and not the convention designated by Fig. 27. By following the signs given in Equations (1) to (5), however, no confusion should arise from this source.

By replacing L_2 by L in Equations (4), (5), (8), and (9), Equations (1), (2), (6), and (7) result.

By making L_{2B} equal to infinity (a hinged connection at B) and L_{2A} equal to zero (a rigid connection at A) in Equations (4) and (8), Equation (3) results.

In applying the slope deflection method to a rigid frame with riveted connections the only change from customary procedure is the use of Equations (4), (5), (8), and (9) instead of Equations (1), (2), (3), (6), and (7). After the values of L_2 are ascertained little more work is required in the application of the longer formulas.

Derivation of Formulas.—To derive the necessary formulas consider the member shown in Fig. 27. This beam is acted upon by loads (shown as a single load, producing a moment curve, *abc*), and the end moments, M_A and M_B. All of them acting simultaneously, produce the moment curve of Fig. 27(*b*). It is to be noted that although M_A is positive, it produces a negative moment on the moment curve. The $\frac{M}{EI}$-curve is shown in Fig. 27(*c*) and is obtained by dividing the ordinates of the curve of Fig. 27(*b*) by the corresponding EI-value of the beam. The area of this curve denotes an angular change and is dimensionless. The differential area, $\frac{M}{EI}\,dx$, is a differential angular change in the beam at the point where this area is taken. The effect of the elastic connections is to cause a change in the slope of the beam at the point where the connection is located. As stated previously, this may be taken as $M\ Z$ and, like the area of the curve, is a dimensionless expression. It is represented on the curve, Fig. 27(*c*), by the heavy lines labeled $M_A Z_A$ and $M_B Z_B$.

From Fig. 27(*e*) it can be seen that the deflection of Point B from a tangent to the curve at Point A is $(\theta_A - \alpha)\ L$. It must be remembered that such angles as θ and α are very small and not as shown in the diagram, hence the tangent of the angle may be taken as equal to the angle. This same deflection can also be computed by taking the first moment of the $\frac{M}{EI}$-curve about Point B and adding thereto the angular change at the connection, A, multiplied by the length, L. Computing the deflection of Point B from the tangent at Point A by both methods and equating them:

$$-(\theta_A - \alpha) L = -\frac{M_A L}{2EI} \times \frac{2L}{3} + \frac{M_B L}{2EI} \times \frac{L}{3} - M_A Z_A L + \frac{A\overline{x}_1}{EI}$$

which may be simplified into:

$$(\theta_A - \alpha) L - \frac{2 M_A L^2}{6EI} - M_A Z_A L + M_B \frac{L^2}{6EI} + \frac{A\overline{x}_1}{EI} = 0$$

Substituting the value of $L_2 = L + 3\ E\ I\ Z_A$:

$$(\theta_A - \alpha) L - \frac{2 M_A L L_{2A}}{6EI} + \frac{M_B L^2}{6EI} + \frac{A\overline{x}_1}{EI} = 0 \quad \ldots\ldots (10)$$

Multiplying by 6 $E\ I$ and dividing by L, Equation (10) becomes:

$$2\,L_{2A}\,M_A - L\,M_B = 6\,E\,I\,(\theta_A - \alpha) + \frac{6\,A\,\overline{x_1}}{L} \quad \text{..........(11)}$$

Taking the deflection of Point A from the tangent at Point B in a similar manner:

$$2\,L_{2B}\,M_B - L\,M_A = 6\,E\,I\,(\theta_B - \alpha) - \frac{6\,A\,\overline{x}}{L} \quad \text{..........(12)}$$

The values of M_A and M_B can be obtained by solving Equations (11) and (12) simultaneously, obtaining:

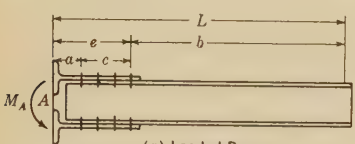
$$M_A = 6\,E\,I\,\frac{2\,L_{2B}\,(\theta_A - \alpha) + L\,(\theta_B - \alpha)}{4\,L_{2A}\,L_{2B} - L^2} + \frac{6\,A}{L} \times \frac{2\,L_{2B}\,\overline{x_1} - L\,\overline{x}}{4\,L_{2A}\,L_{2B} - L^2} \quad \text{.(13)}$$

and,

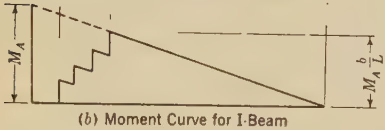
$$M_B = 6\,E\,I\,\frac{2\,L_{2A}\,(\theta_B - \alpha) + L\,(\theta_A - \alpha)}{4\,L_{2B}\,L_{2A} - L^2} - \frac{6\,A}{L} \times \frac{2\,L_{2A}\,\overline{x} - L\,\overline{x_1}}{4\,L_{2B}\,L_{2A} - L^2} \quad \text{..(14)}$$

Equations (13) and (14) are identical with Equations (4) and (5) when the Values of M_{cA} and M_{cB} are substituted from Equations (8) and (9).

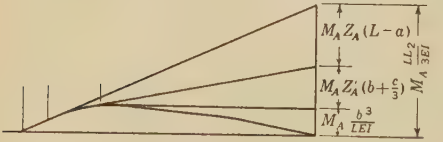
A second method of obtaining Equations (11) and (12) is by the conjugate beam method. The slope at the ends of a beam are the same as the reactions of a conjugate beam of the same length carrying the $\frac{M}{E\,I}$-curve as a load. Let these reactions be denoted by β and γ. Computing these values by taking moments first about Point B and then about Point A in Fig. 27(c) and noting from Fig. 27(d) that $\beta = \theta_A - \alpha$, and $\gamma = \theta_B - \alpha$, Equations (11) and (12) are obtained at once.



(a) Loaded Beam

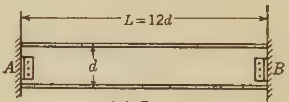


(b) Moment Curve for I-Beam

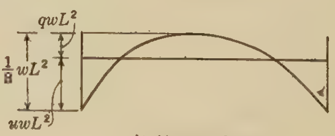


(c) Deflection Curve

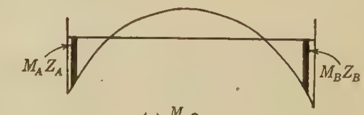
FIG. 28.



(a) Beam



(b) Moment Curve



(c) $\frac{M}{EI}$ Curve

FIG. 29.

Evaluation of L_2 and L_1.—The expression, L_2, is used instead of $L + 3\ E\ I\ Z$ for the purpose of simplifying the formulas. As $3\ E\ I\ Z$ is in units of length it may be visualized as the change in the length of the member due to the elasticity of the connections and L_2 may be visualized as the transformed length of the member.

In order to obtain a more accurate expression for L_2, that takes into account the stiffening effect of the connections of the types shown in Fig. 28(*a*), assume that the member is acted upon by a moment, M_A, at Point *A*. The moment at End *B* may be considered as zero. The moment on the I-beam is zero at the end where the instruments measuring *Z* were attached. The customary assumption is made that the shear (horizontal) is distributed equally between rivets and that the moment curve is straight in the connections instead of stepped as shown in Fig. 28(*b*). The angular change due to this stepped moment in the I-beam is, then, $M_A\,Z_{1A} = M_A \dfrac{b}{L} \times \dfrac{c}{2\,EI}$, approximately, and the total deflection of Point *B* from a tangent to the curve at Point *A* due to a moment, M_A, may be written as:

$$M_A\,Z_A\,(L - a) + M_A\,Z_{1A}\left(b + \frac{c}{3}\right) + M_A\,\frac{b}{L} \times \frac{b^2}{3\,EI} \;\ldots\ldots\; (15)$$

which was assumed as $\dfrac{2\,M_A\,L\,L_2}{6\,EI}$ in deriving Equation (10).

Placing these two expressions equal will give a corrected definition for L_2:

$$L\,L_2 = 3\,E\,I\,Z_A\,(L - a) + 3\,E\,I\left(b + \frac{c}{3}\right)Z_1 + \frac{b^3}{L} \;\ldots\ldots\; (16)$$

Dividing by *L* and substituting the value of Z_1:

$$L_2 = 3\,E\,I\,Z\left(1 - \frac{a}{L}\right) + \frac{3}{2}\left(b + \frac{c}{3}\right)\frac{b\,c}{L^2} + \frac{(L - e)^3}{L^2} = L + 3\,E\,I\,Z\left(1 - \frac{a}{L}\right)$$
$$- 3\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{2\,L^2}\,(3\,b + c) - \frac{e^3}{L^2} \;\ldots\ldots\ldots\ldots\; (17)$$

The last term is small except for short beams. When it is neglected the expression for L_2 becomes:

$$L_2 = L - 3\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{2\,L^2}\,(3\,b + c) + 3\,E\,I\,Z\left(1 - \frac{a}{L}\right) \ldots (18)$$

In the case of the beam partly restrained at the end, the expression, $2\,L_2 + L$, occurs in the equations. By substituting $3\,L_1$ for $2\,L_2 + L$, these equations can be simplified. With this definition of L_1, Equation (15) becomes:

$$L_1 = L - 2\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{L^2}\left(b + \frac{c}{3}\right) + 2\,E\,I\,Z\left(1 - \frac{a}{L}\right) \ldots (19)$$

Dropping the term, $\dfrac{a}{L}$, would, to some extent, allow for the deformation due to shear in the connection which has been neglected in deriving these formulas. Dropping this term and the others that may be negligible:

$$L_2 = L + 3\,E\,I\,Z - 3\,e \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (20)$$

and,

$$L_1 = L + 2\,E\,I\,Z - 2\,e \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (21)$$

The Beam Partly Restrained at the Ends.—Probably the case that occurs the greatest number of times in steel design is that of the beam partly restrained at the ends by the connections. It is customary in designing such a beam to consider the connections as hinged. If, however, the ends are partly restrained by these connections the moment curve is as shown in Fig. 29(*b*), and a material saving can often be made. To find the value of the negative moment at the ends of the beam, Equations (8) and (9) are used in conjunction with Equations (4) and (5).

The beam with fixed ends is only a special case of the beam of Fig. 27, in which, θ_A, θ_B, and α are all zero, and therefore the first terms of Equations (4) and (5) disappear.

The Partly Restrained Beam Under Uniform Load.—Of especial interest is the beam fixed at the ends by an elastic connection and loaded uniformly throughout its length. To find the value of the negative moment, the following values are substituted in Equation (8) or in Equation (9):

$$A = \frac{1}{8} w L^2 \times \frac{2}{3} L = \frac{1}{12} w L^3;\; L_{2A} = L_{2B} = L_2;\; \bar{x} = \bar{x}_1 = \frac{L}{2}$$

from which,

$$M_{cA} = M_{cB} = -\frac{6\, w L^3}{12\, L} \times \frac{(2\, L_{2B} - L)\frac{L}{2}}{4\, L_2^2 - L^2} = \frac{-\, w L^3}{4\,(2\, L_2 + L)} = -\frac{w L^3}{12\, L_1} \quad \text{..(22)}$$

in which, the value of L_1 is given in Equation (19).

The moment at the center of the span is:

$$\frac{1}{8}\, w L^2 + M_{cA} = \frac{1}{8}\, w L^2 - \frac{w L^3}{12\, L_1} = \frac{1}{8}\, w L^2 \left(1 - \frac{2\, L}{3\, L_1}\right) \quad \text{.....(23)}$$

If the moments at the ends and center of this span are assumed equal to $u\, w\, L^2$ and $q\, w\, L^2$, thus defining u and q:

$$u = \frac{-\, L}{12\, L_1} \quad \text{..........................(24)}$$

and,

$$q = \frac{1}{8}\left(1 - \frac{2\, L}{3\, L_1}\right) \quad \text{.....................(25)}$$

If the moment at the center of the span is not numerically greater than that at the end, the I-beam at the connection governs the limit of the load. It is recognized that the evaluation of the unit stress in the I-beam at the connection is not as simple as herein implied because, doubtless, the stresses in the connection govern, rather than those in the I-beam, near the connection.

The importance of the several terms of Equation (19) can best be demonstrated by an example. For this purpose a beam of a length equal to twelve times its nominal depth has been chosen. The load is assumed to be of sufficient value to give a fiber stress of 18 000 lb per sq in. Table 3 gives

TABLE 3.—INCREASE OF CARRYING CAPACITY DUE TO CONNECTIONS, ON A UNIFORMLY LOADED BEAM

Specimen No.	PROPERTIES OF BEAM AND CONNECTIONS					COMPUTATIONS FOR MOMENTS				
	Size of beam (1)	Moment of inertia, I, in inches (2)	Section modulus, S, in inches (3)	Distance, a, in inches (4)	Distance, e, in inches (5)	Length of beam L, in feet (6)	$1-\frac{a}{L}$ (7)	$2e\left(1-\frac{e}{L}\right)$, in inches (8)	$\frac{bc}{L^2}\left(b+\frac{c}{3}\right)$ in inches (9)	Assumed moment, M_c, at beam connection, in inch-pounds (10)
	(a) STRESS AT CENTER GOVERNS									
1	6-in. I-Beam @ 12.5 lb...	21.8	7.3	2.25	4.75	6.0	0.969	9	2	5 100
2	8-in. I-Beam @ 18.4 lb...	56.9	14.2	2.25	4.75	8.0	0.976	9	2	58 000
3	8-in. I-Beam @ 18.4 lb...	56.9	14.2	2.25	4.75	8.0	0.976	9	2	58 000
4	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	2.25	12.0	0.984	4+	....	95 000
5	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	150 000
6	18-in. I-Beam @ 54.7 lb...	795.5	88.4	2.25	2.25	18.0	0.990	4+	....	348 000
7	18-in. I-Beam @ 54.7 lb...	795.5	88.4	2.25	4.75	18.0	0.990	9	2.5	515 000
8	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	248 000
9	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	310 000
10	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	360 000
11	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	435 000
12	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	480 000
15	16-in. G-Beam @ 83.0 lb...	1 161.6	144.1	4.5	13.5	16.0	0.977	25	8	245 000
16	22-in. G-Beam @ 101.0 lb...	2 557.2	233.7	4.5	16.5	22.0	0.983	31	11	398 000
18	16-in. G-Beam @ 83.0 lb...	1 161.6	144.1	4.5	21.0	16.0	0.977	37+	13+	235 000
	(b) STRESS AT ENDS GOVERNS									
13	12-in. I-Beam @ 31.5 lb...	215.8	36.0	4.5	10.5	12.0	0.969	19+	5	648 000*
14	12-in. I-Beam @ 31.5 lb...	215.8	36.0	4.5	10.5	12.0	0.969	19+	5	648 000*
17	22-in. G-Beam @ 101.0 lb...	2 557.2	233.7	4.5	25.5	22.0	0.983	46	17	4 206 600*

TABLE 3.—(*Continued*)

Specimen No.	COMPUTATIONS FOR MOMENTS (*Continued*)						LOAD COMPUTATIONS		
	Angular change, M_cZ, in connection (taken from curve) (11)	$2EIZ$, in inches (12)	$2EIZ\left(1-\frac{a}{L}\right)$, in inches (13)	Transformed length of beam, L_1, in inches (14)	Moment at center of beam, M_d, in inch-pounds $=18\ 000S$ (15)	$M_c=\frac{2L}{3L_1-2L}$ in inches (as a check on assumption) (16)	$W=\frac{+8M_d}{L}$ in inch-pounds, load allowed on unrestrained beam (17)	Load allowed on beam restrained by connections (18)	Percentage excess load allowed (19)
	(a) STRESS AT CENTER GOVERNS								
1	0.0049	1 260	1 220	1 285	131 400	5 100	14 600	15 170	4
2	0.0045	265	258	347	255 600	57 700	21 300	26 100	22
3	0.0045	265	258	347	255 600	58 000	21 300	26 100	22
4	0.0045	613	603	743	648 000	95 500	36 000	41 300	15
5	0.0044	379	373	512	648 000	150 000	36 000	44 300	23
6	0.00435	592	586	796	1 591 200	349 000	59 000	71 860	22
7	0.00412	384	380	589	1 591 200	515 000	59 000	78 000	32
8	0.0041	224	211	348	648 000	248 000	36 000	49 700	38
9	0.00395	165	161	298	648 000	310 000	36 000	53 200	48
10	0.0037	133	131	268	648 000	360 000	36 000	56 000	55
11	0.00345	103	101	238	648 000	436 000	36 000	60 200	67
12	0.0034	91.5	90	227	648 000	475 000	36 000	62 400	73
15	0.00315	90	88	263	2 593 800	2 460 000	108 100	210 600	95
16	0.00315	121	119	363	4 206 600	3 980 000	127 600	248 100	94
18	0.0035	104	101	269	2 593 800	2 360 000	108 100	206 400	91
	(b) STRESS AT ENDS GOVERNS								
13	0.0015	30	29	159	425 000	....	36 000	59 600	66
14	0.00105	21	20	150	363 000	....	36 000	56 100	56
17	0.00055	20	20	255	1 890 000	....	127 600	184 700	45

* $M_c = 18\ 000\ S$, in inch-pounds.

the results of the computations for the several connections tested in Part I. In computing the first fifteen connections in Table 3 the values of M_u were assumed, the values of $M\ Z$ were read from the curve, after which L_1 was computed. From the formula, $M_u = M_q \left[\frac{-2L}{3L_1 - 2L}\right]$ (obtained from Equations (24) and (25)), the assumed values were checked. In these cases the values of M_q were taken as 18 000 S, in which, S is the section modulus of the beam. In the case of Specimens 13, 14, and 17, the stress at the end of the beam rather than that at the center was the determining factor. The moment, M_q, was computed from the reciprocal formula, $M_q = M_u\left(1 - \frac{3L_1}{2L}\right)$. From Fig. 29 it is seen that $-M_u + M_q = \frac{1}{8}\, w\, L^2$, the formula customarily used in solving this problem. The increase of strength obtained by considering the connection as elastic over the hinged connection is given by the expression, $\frac{-M_u + M_q}{18\,000\ S}$. This is given in Column (19), Table 3.

THEOREM OF THREE MOMENTS

The theorem of three moments may be written (see Fig. 30):

$$\frac{L_a}{I_a} M_A + 2\left(\frac{L_a}{I_a} + \frac{L_b}{I_b}\right) M_B + \frac{L_b}{I_b} M_C + \frac{6\, A_a \overline{x_a}}{L_a\, I_a} + \frac{6\, A_b \overline{x_{1b}}}{L_b\, I_b} = 6\, E\, \Delta \left(\frac{1}{L_a} + \frac{1}{L_b}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Equation (26) is found in many forms all of which can be transformed into the one here given or can be derived as special cases of it. If the elasticity of the connections is taken into consideration the theorem changes only in its second term which becomes:

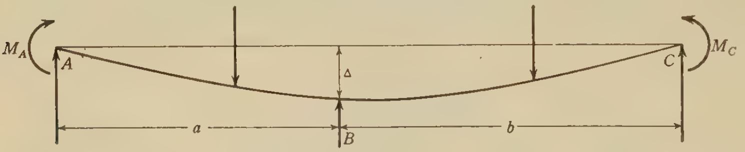


FIG. 30.

$$\frac{L_a}{I_a} M_A + 2\left(\frac{L_{BA}}{I_a} + \frac{L_{BC}}{I_b}\right) M_B + \frac{L_b}{I_b} M_C + \frac{6\, A_a\, \overline{x_a}}{L_a I_a} + \frac{6\, A_b\, \overline{x_{1b}}}{L_b\, I_b} = 6\, E\, \Delta \left(\frac{1}{L_a} + \frac{1}{L_b}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

In Equations (26) and (27) (and in Equation (33) which follows), as well as in their derivation, the signs of the moments are assumed positive when they produce compression in the top fibers of the beam rather than by the convention used in the "Slope Deflection Method." This is done to be consistent with current practice in the use of the Theorem of Three Moments. This theorem (Equation (26) and Equation (27)) may be considered as a special case readily derived from the slope deflection equations. The theorem expresses only the relationship between the loads on a continuous beam on three supports, the elastic properties of the beam, and the moment at the deflection of the central support. The end moments are considered as loads.

To derive Equation (27), it is only necessary to note that the slope of Span *AB* at Point *B*, is the same as the slope of Span *BC* at the same point.

Span *AB* may be considered as hinged at the left end and elastically connected at the right, while Span *BC* is hinged at the right and elastically connected at the left end. For a beam with a hinge at the left end, Equation (14) becomes:

$$M_B = \frac{3\,E\,I\,(\theta_B - \alpha)}{L_{2B}} - \frac{3\,A\,\overline{x}}{L\,L_{2B}} \quad \dots\dots\dots\dots\dots\dots (28)$$

when L_{2A} becomes infinite, while for a beam with a hinge at the right end, Equation (13) becomes (after changing the sign of M_A):

$$-\,M_A = \frac{3\,E\,I\,(\theta_A - \alpha)}{L_{2A}} + \frac{3\,A\,\overline{x}_1}{L\,L_{2A}} \quad \dots\dots\dots\dots\dots\dots (29)$$

For Span *A B* substitute the following in Equation (28): $M_{BA} = M_{BC} = M_B$; $A\,\overline{x} = A_a\,\overline{x}_a + \frac{M_A\,L_a^2}{6}$; and $\alpha = -\frac{\Delta}{L_a}$. Then,

$$L_{BA}\,M_B = 3\,E\,I_a\left(\theta_B + \frac{\Delta}{L_a}\right) - \frac{3\,A_a\,\overline{x}_a}{L_a} - \frac{M_A\,L_a}{2}$$

Similarly, substitute in Equation (29) for Span *BC*, $M_A = M_{BC}$; $A\overline{x}_1 = A_b\,\overline{x}_{1b} + \frac{M_C\,L_b^2}{6}$, and $\alpha = \frac{\Delta}{L_b}$, obtaining:

$$-\,L_{BC}\,M_B = 3\,E\,I_b\left(\theta_A - \frac{\Delta}{L_b}\right) + \frac{3\,A_b\,\overline{x}_{1b}}{L_b} + \frac{M_C\,L_b}{2} \quad \dots\dots (30)$$

Eliminating θ_A and θ_B between these two equations by taking advantage of the fact that they are equal:

$$3\,E\left(\theta_B + \frac{\Delta}{L_a}\right) = \frac{L_{BA}}{I_a}\,M_B + \frac{L_a}{2\,I_a}\,M_A + \frac{3\,A_a\,\overline{x}_a}{L_a\,I_a} \quad \dots\dots\dots (31)$$

and,

$$3\,E\left(\theta_A - \frac{\Delta}{L_b}\right) = -\frac{L_{BC}}{I_a}\,M_B - \frac{L_b}{2\,I_b}\,M_C - \frac{3\,A_b\,\overline{x}_{1b}}{L_b\,I_b} \quad \dots\dots\dots (32)$$

Subtracting Equations (31) and (32), Equation (27) follows.

The term, $\Delta\left(\frac{1}{L_a} + \frac{1}{L_b}\right)$, may be replaced by $\frac{h_A - h_B}{L_a} - \frac{h_B - h_C}{L_b}$. This comes from the geometry of the figure, in which, h_A, h_B, and h_C are the heights of the several supports, the assumption being made that they are in line when the beam is not strained.

For the special case in which there is no settlement of supports and the beam carries a uniform load, Equation (27) becomes:

$$\frac{L_a}{I_a} M_A + 2\left(\frac{L_{BA}}{I_a} + \frac{L_{BC}}{I_b}\right) M_B + \frac{L_b}{I_b} M_C + \frac{w\,L^3_a}{4\,I_a} + \frac{w\,L^3_b}{4\,I_b} = 0 \quad \ldots\ldots (33)$$

Moment Distribution Method

In the moment distribution method of analyzing rigid frames, three groups of formulas are required: (1) The moment induced at either end of a beam by loads when the beam is considered as fixed-ended; (2) the moment induced at one end of a beam by a moment imposed at the other; and (3) the rigidity of the beam with respect to a moment at the end.

Equations (6) and (7), and the last term of Equation (3), fulfill the requirements for the first group when no correction is made for the yielding of the connections; Equations (8) and (9) fulfill the requirements when this factor is taken into consideration.

In the second group of formulas, the carry-over factor is 0.500 in the case of rigid connections. In the case of elastic connections this factor can be obtained from Equations (13) and (14). Let the beam of Fig. 27(*a*) be the one under consideration; let $\theta_B = 0$; $A = 0$; and $\alpha = 0$; then:

$$M_A = 6\,E\,I\,\frac{2\,L_{2B}\,\theta_A}{4\,L_{2A}\,L_{2B} - L^2} \quad \ldots\ldots (34)$$

and,

$$M_B = 6\,E\,I\,\frac{L_{2A}\,\theta_A}{4\,L_{2B}\,L_{2A} - L^2} \quad \ldots\ldots (35)$$

Eliminating θ_A between Equations (34) and (35):

$$M_B = M_A\,\frac{L}{2\,L_{2B}} \quad \ldots\ldots (36)$$

The carry-over factor then is $\frac{L}{2\,L_{2B}}$ which reduces to 0.500 when L_{2B} is equal to L.

In order to find the rigidity of the beam with respect to the moment at End A or End B (which is given by the formulas, $\theta_A = \frac{M_A\,L}{4\,E\,I}$ and $\theta_B = \frac{M_B\,L}{4\,E\,I}$, for rigid connections), it is only necessary to solve Equation (34) for θ, thus,

$$\theta_A = M_A\,\frac{4\,L_{2A}\,L_{2B} - L^2}{12\,L_{2B}\,E\,I} = M_A\left(\frac{L_{2A}}{3\,E\,I} - \frac{L^2}{12\,L_{2B}\,E\,I}\right)$$

$$= \frac{M_A}{3\,E\,I}\left(L_{2A} - \frac{L^2}{4\,L_{2B}}\right) \quad \ldots\ldots (37)$$

and,

$$\theta_B = \frac{M_B}{3\ E\ I}\left(L_{2B} - \frac{L^2}{4\ L_{2A}}\right) \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

Equations (37) and (38) are the formulas from which the stiffness of the beam may be determined when computing the unbalanced moment distribution about a joint. If $L_2 = L$ in Equations (37) and (38) the usual formulas for both ends rigidly connected obtain. If End A is hinged and End B is rigid, Equation (38) reduces to $\theta_B = \frac{M_B L}{3\ E I}$, as $L_{2B} = L$ and L_{2A} = infinity.

Examples.—The changes in the moment distribution method due to elastic connections can best be illustrated by comparing the solution of a problem in which all connections are rigid with that of one in which some of the connections are taken as elastic.

Consider the beam of Fig. 31 supporting a uniform load, w, of 100 lb per ft. The connections at Points A and D are hinged and the beam, at Point C, is continuous. The beam will first be considered as continuous at Point B, and, later, the case of elastic connections at Point B will be solved. These two problems are analyzed in a parallel manner so that a comparison can readily be made between them.

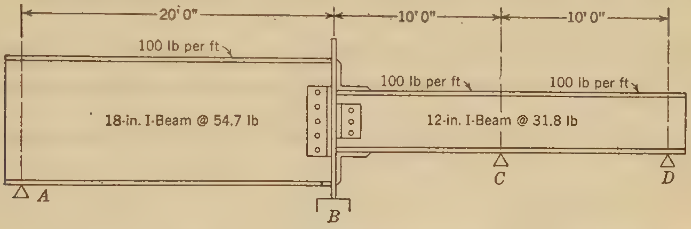


FIG. 31.—A CONTINUOUS BEAM.

Rigid Connections.—The fixed-end moments, M_c, for this case (Fig. 31) are:

$$M_{BA} = -\frac{3\ A\ \overline{x}}{L^2} = -\frac{3 \times 48 \times 10^5}{240} = -60 \times 10^3$$

$$M_{BC} = M_{CB} = -\frac{2\ A\ \overline{x}}{L^2} = -\frac{2 \times 6 \times 10^5}{120} = -10 \times 10^3$$

and,

$$M_{CD} = -\frac{3\ A\ \overline{x}}{L^2} = -\frac{3 \times 6 \times 10^5}{120} = -15 \times 10^3$$

The carry-over factor is 0.5000 for both ends of Beam BC. The stiffness factors for the several beams are evaluated as follows:

$$K_{BA}\ E\ \theta = \frac{3I}{L}\ E\ \theta = \frac{(3)\ (795.5)}{240}\ E\ \theta = 9.93\ E\ \theta$$

$$K_{CB}\ E\ \theta = K_{BC}\ E\ \theta = \frac{4\ I}{L}\ E\ \theta = \frac{(4)\ (215.8)}{120}\ E\ \theta = 7.2\ E\ \theta$$

and,

$$K_{CD}\,E\,\theta = \frac{3I}{L}\,E\,\theta = \frac{(3)\,(215.8)}{120}\,E\,\theta = 5.4\,E\,\theta$$

These values are used in the computations shown in Fig. 32.

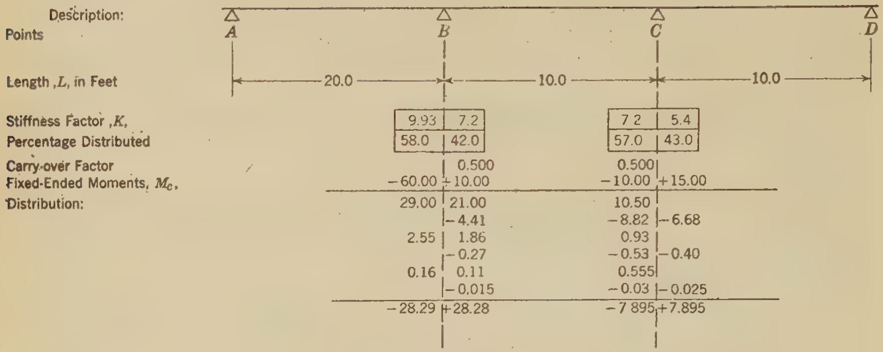

Description:
Points
A
B
C
D
Length ,L, in Feet
20.0
10.0
10.0
Stiffness Factor ,K,
9.93
7.2
7 2
5.4
Percentage Distributed
58.0
42.0
57.0
43.0
Carry-over Factor
0.500
0.500
Fixed-Ended Moments, M_c,
−60.00
+10.00
−10.00
+15.00
Distribution:
29.00
21.00
10.50
−4.41
−8.82
−6.68
2.55
1.86
0.93
−0.27
−0.53
−0.40
0.16
0.11
0.555
−0.015
−0.03
−0.025
−28.29
+28.28
−7 895
+7.895

FIG. 32.—COMPUTATIONS FOR BEAM IN FIG. 31, CONSIDERING THE CONNECTION AT POINT B AS RIGID.

Elastic Connections.—It is first necessary to compute L_2 for Beams BA and BC. In both cases the moments are low and the tangent at the origin gives $\frac{M}{MZ}$ with considerable accuracy. Therefore, the values given in Table 2 will be taken for $\frac{1}{Z}$.

From Equation (18):

$$L_{2BA} = 240 - 7.5 + \frac{(3)\,(3)\,(10)^7\,(795.5)}{(10)^8\,(2.9)} = 232.5 + 246.5 = 479$$

$$L_{2BC} = 120 - 14.5 + 3.5 + \frac{(3)\,(3)\,(10)^7\,(215.8)}{(10)^8\,(4.2)} = 109 + 46 = 155$$

$$A\,\overline{x}_{AB} = \frac{wL^3}{24} = \frac{(100)\,(20)\,(240)^3}{24} = (1\,152)\,(10)^6$$

$$A\,\overline{x}_{BC} = A\,\overline{x}_{CB} = \frac{(100)\,(10)\,(120)^3}{24} = (72)\,(10)^6$$

and the fixed moments, M_c,

$$M_{BA} = \frac{-3\,A\,\overline{x}}{L\,L_{2BA}} = \frac{-(3)\,(1\,152)\,(10)^6}{(240)\,(479)} = -\,29\,900$$

$$M_{BC} = \frac{-6\,A\,\overline{x}}{L\,(4\,L_2 - L)} = \frac{-(6)\,(72)\,(10)^6}{(120)\,(620 - 120)} = -\,7\,200$$

$$M_{CB} = \frac{-6\,A\,\overline{x}\,(2\,L_2 - L)}{L^2\,(4\,L_2 - L)} = \frac{-(6)\,(72)\,(10)^6\,(190)}{(120)^2\,(500)} = -\,11\,400$$

$$M_{CD} = \frac{-3\,A\,\overline{x}}{L^2} = -\,15\,000$$

The carry-over factor is 0.50 for the right end of Beam *BC*, but for the left end, it is $\frac{L}{2\,L_2} = \frac{120}{(2)\ (155)} = 0.387.$

The stiffness factors, *K*, for the several beams are:

$$K_{BA}\ E\ \theta_B = \frac{I}{L_1}\ (3\ E\ \theta_B) = \frac{795.5}{479}\ (3\ E\ \theta_B) = (1.66)\ (3\ E\ \theta_B)$$

$$K_{BC}\ E\ \theta_B = \frac{4\ I}{4\ L_2 - L}\ 3\ E\ \theta_B = \frac{4\ (215.8)}{620\ -\ 120} = (1.73)\ (3\ E\ \theta_B)$$

$$K_{CB}\ E\ \theta_C = \frac{4\ L_2\ I}{L\ (4\ L_2 - L)}\ 3\ E\ \theta_C = \frac{(4)\ (155)\ (215.8)}{(120)\ (500)}\ 3\ E\ \theta_B = (2.24)\ (3\ E\ \theta_C)$$

and,

$$K_{CD}\ E\ \theta_C = \frac{I}{L}\ 3\ E\ \theta_C = \frac{215.8}{120}\ 3\ E\ \theta_B = (1.80)\ (3\ E\ \theta_C)$$

Using the foregoing values and computing the moment distribution the same as in Fig. 32, the results, considering the connection at Point *B* as elastic, are shown in Fig. 33. The results may be checked by the theorem of three moments by substituting the proper values in Equation (33), with the result that, $M_B = -\ 17\,800$; and $M_c = -\ 10\,550$.

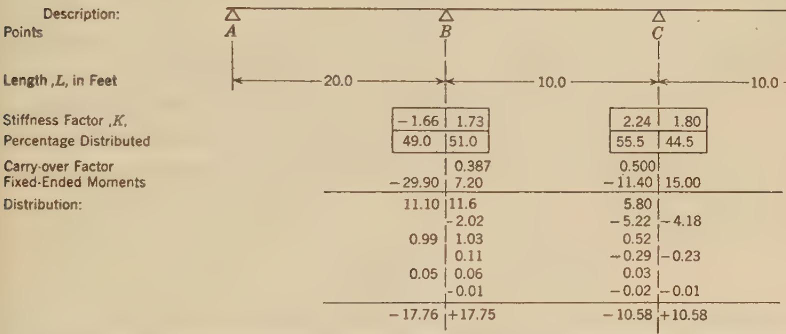

Description:
Points
A
B
C
D
Length, L, in Feet
20.0
10.0
10.0
Stiffness Factor, K,
Percentage Distributed
Carry-over Factor
Fixed-Ended Moments
Distribution:
– 1.66 | 1.73
49.0 | 51.0
0.387
– 29.90 | 7.20
11.10 | 11.6
– 2.02
0.99 | 1.03
0.11
0.05 | 0.06
– 0.01
– 17.76 | + 17.75
2.24 | 1.80
55.5 | 44.5
0.500
– 11.40 | 15.00
5.80
– 5.22 | – 4.18
0.52
– 0.29 | – 0.23
0.03
– 0.02 | – 0.01
– 10.58 | + 10.58

FIG. 33.—COMPUTATIONS FOR THE BEAM IN FIG. 31, CONSIDERING THE CONNECTION AT POINT *B* AS ELASTIC.

DEFORMETER METHOD

The analysis of a structure by means of the deformeter method requires a model cut from a sheet of celluloid or other elastic material. In designing this model, if deformation depends primarily on bending as it does in building frames, the moment of inertia of the various parts must be proportional to those of the structure. This proportion also applies to the connections. The widths of the various parts of the model, as *d*, are computed so that the cubes of these widths are proportional to the moments of inertia of the corresponding members. The rigidity of each member is proportional to $\frac{EI}{L}$ whereas that of the corresponding member in the model is $\frac{E_c t d^3}{12\, l}$, in which, E_c is the modulus of elasticity of the model material; l is the length of the member; and t is the thickness of the sheet of celluloid. If the ratio between these two expressions for rigidity is τ_r, then, $\tau_r = \frac{E_c t d^3}{12\, l} \div \frac{EI}{L}$; or,

$$\tau_r = \frac{E_c t d^3 L}{12\, l\, E I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

The same ratio should exist between the elasticity of the connections in the structure and in the model, whence: $\tau_r = \frac{E_c t i^3}{12\, j} \div \frac{1}{Z}$; or,

$$\tau_r = \frac{E_c t i^3 Z}{12\, j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

Eliminating τ_r between Equations (39) and (40),

$$\frac{j}{i^3} = \sigma \frac{E I Z}{d^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

In which, σ is the scale ratio of the model; j is the width of a rectangular groove cut in the member at the location of the connection; and i is the width of the member at the base of this groove (see Fig. 34).

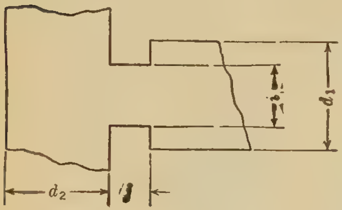


FIG. 34.

SUMMARY AND CONCLUSIONS

The tests described in Part I give the elastic properties of eighteen connections of three distinct types (see Figs. 12 and 26, and Table 2). The tests showed that the connections of Series *A* can be deformed through angles considerably beyond working conditions without affecting their capacity to resist shear. The connections of Series *B* possessed this property in a much smaller degree, but are capable of deforming within the working range. The deformations in Series *A* and *B* are due to the yielding of the angles rather than to the rivets or main members. The connections shown as Series *C* are comparatively rigid. Their method of failure can be controlled by detail of design.

Part II indicates the method of analyzing structural frames when consideration is given to the elastic properties of the connections. Expressions for the "transformed length of the member", L_2, and the "transformed length of the beam", L_1, have been developed and incorporated into the formulas for stress analysis by the slope deflection, moment distribution, and deformeter methods, as well as into the theorem of three moments. The saving resulting from utilizing the resistance of the connections in designing a simple beam is indicated in Table 3, where it is shown that the load in some cases may be increased more than 90% by utilizing the support given by the connections.

APPRECIATION

Acknowledgment is freely given to the following for material assistance in the preparation of this paper: G. E. J. Pistor, M. Am. Soc. C. E., through whose interest the steel for these tests was made available and also for many helpful suggestions in planning the tests; Frederick Skene, Dean of the School of Technology, College of the City of New York, for use of their laboratories, equipment, etc., and for his co-operation; A. H. Beyer, O. E. Hovey, and Jonathan Jones, Members, Am. Soc. C. E., for many helpful suggestions during the gathering of the data and the writing of the paper; J. Sanford Peck, Assoc. M. Am. Soc. C. E., for material assistance, co-operation, and many suggestions during the conduct of the tests; A. G. Hayden, M. Am. Soc. C. E., and Mr. C. W. Vanderbilt, of the Kalman Steel Company, for material assistance and helpful suggestions in the preliminary stages of the investigation; Mr. Leo S. Pavelle for his assistance as a photographer; Mr. Arthur Gatterdam for the suggestion of the use of L_2 and for a careful review of the derivation of the equations; the McClintic-Marshall Company for furnishing the steel which was fabricated in its Bethlehem (Pa.) shops and delivered to the laboratories; and to the American Institute of Steel Construction for its co-operation in obtaining the steel, and for many suggestions.

DISCUSSION

Ralph E. Goodwin,[6] M. Am. Soc. C. E. (by letter).—Without question, this paper is a genuine contribution to the knowledge of restrained end conditions in beams and rectangular frames and, in a timely manner, Professor Rathbun has saved students of this subject from the sin of pride and the dangers of self-complacency. Just as the profession was becoming acquainted with new means for analyzing rigid frames with some degree of facility Professor Rathbun comes forward with a bitter new "dose" in the form of further complications to problems already so complicated that few engineers other than college professors bother to solve them.

The test results are valuable and the mathematical analysis is clever and cleverly presented. The use of simple beam moments in connection with restrained end conditions in Part II of the paper simplifies the resulting expressions enormously. Further simplification results from the author's use of the symbol, L_2; but he flatters the intelligence of his readers when he assumes that the steps in his derivations and his systems of algebraic signs will be self-evident. In problems of this nature the difficulties with algebraic signs become almost unsurmountable unless the latter are explicitly defined. The tendency of experts is to grow so accustomed to their own particular methods and sign conventions that it does not occur to them that these methods and conventions may not be taken for granted by every one. Since the paper is written for specialists, some engineers who are not expert in this field will have difficulty in following the derivations and in using the resulting equations without further elucidation.

Unfortunately, the author did not make clear in his Figs. 12 to 23 the meaning of the straight diagonal dashed lines and of the straight vertical dashed lines at 0.0089 radian on some of these curves. The straight diagonal dashed lines represent the average slopes of the "set" curves, whereas the straight vertical dashed lines indicate the rotation angle which the ends of a uniformly loaded simple beam make with the horizontal when the beam is deflected one-three-hundred-sixtieth of the span. This value of the rotation angle is thought to be the maximum limit of practical significance. In taking the "set" curves the load was removed and the point taken on these curves as the load was placed back on the specimen.

The general problem of elastic connections is similar in nature to the problem of joint rotation in the slope-deflection method. The yielding of the connection has the same effect upon the end moments of the member as would be produced by joint rotation; but in the equations the two effects must be kept distinct for two reasons: (1) Joint rotation is common to all members meeting at the joint in question, whereas yielding of the connections may not be the same for all members; and (2) yielding of any connection is itself a function of the end moment. In the conventional derivation of the slope-deflection equations for rigid connections, as presented in textbooks, the principle of superposition is utilized in order to add the fixed-end moments to the moments produced by joint rotation and joint translation. This is legitimate since the two effects are assumed to be independent of each other, the angles of joint rotation (θ_A and θ_B) not being functions of the end moments (M_A and M_B); but in the author's work the elastic rotation of a connection is itself a function of the end moment which produces it and, therefore, the ordinary fixed-end moment can not correctly be superposed upon the effects of joint rotation and translation. The author's procedure in respect to this point is correct inasmuch as he derives his slope-deflection equations for elastic connections from first principles and not by superposition. As was to be expected, the restrained-end moments for a beam with elastic connections appear in the expressions for M_A and M_B (Equations (13) and (14)), not the fixed-end moments for rigid connections. It is to be regretted as increasing the chances for error that the author has used the same symbols (M_{CA} and M_{CB}) for the restrained-end moments of a beam with elastic connections as for the fixed-end moments of a beam with rigid connections.

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[6] Associate Prof., Civ. Eng., School of Technology, Coll. of the City of New York, New York, N. Y.

For moment diagrams the author wisely chooses to utilize the principle that a beam with end moments may be treated as a simple beam carrying the end moments as additional loads, so that his moment diagrams show three separate diagrams superposed, namely: (1) Moment diagram for the transverse loads on a simple beam; (2) moment diagram for end moment on the left, applied as a load on a simple beam; and (3) moment diagram for end moment on the right, applied as a load on a simple beam. This treatment appears to produce maximum simplification of the resulting equations.

In the matter of algebraic signs, always so puzzling, the author is fairly considerate. He offers much specific information in regard to this matter, but the following categorical listing of the rules used for algebraic signs in the paper may be helpful. In the first part of the paper, as far as the "Theorem of Three Moments," the author uses a combination of two sign conventions: The moments of transverse loads follow the beam sign convention; and end couples, joint rotation, joint translation, and yielding of connections follow the slope-deflection sign convention, counter-clockwise rotation being taken as positive. Although it is immaterial for the cases discussed in this paper the writer would like, on general principles, to amend the statement (following Equation (5)) that "the moment is positive when it tends to turn the member in a counter-clockwise direction." A counter-clockwise end moment should always be taken as positive whether acting on the member or on the adjacent joint, but it should be indicated whether the symbols, M_A, M_B, etc., refer to the moment couples acting on the member or to the ones acting on the joint. In the paper the symbols, M_A, M_B, etc., refer to the moment couples acting on the member, so that a positive value of M_A indicates a counter-clockwise couple acting on the member with a corresponding clockwise couple ($-M_A$) acting on the adjacent joint. In plotting moment diagrams when using the direction-of-rotation sign convention the several component diagrams must all be plotted consistently; that is, all must refer to the same face of a section through the member. As regards algebraic signs the author's moment diagrams show the moment couples acting on the left-hand face of a transverse cut through the beam. On this face of the section, the algebraic signs of the moment couples will be the same when determined by the direction-of-rotation sign convention as when determined by the beam sign convention. The foregoing remarks explain the author's statement (under the heading, "Derivation of Formulas") that: "It is to be noted that although M_A is positive, it produces a negative moment on the moment curve." Some may consider it objectionable to combine two different systems of algebraic signs as has been done in the paper, but no difficulty will be experienced in altering the equations by those who desire to make a change in this particular.

In the derivation of the theorem of three moments with elastic connections the beam sign convention is used as more appropriate to this case. In his work the author simplifies his mathematics by treating Joints A and C as hinged and acted upon by the couples, M_A and M_C, considered as loads. He allows for the effect of these couples by treating them as loads on a simple beam and including their effect with that of the transverse loads. In the case of a hinged connection at A, L_{2A} becomes infinite and Equation (14) becomes indeterminate in the form in which it is printed. Equation (28) is obtained by first dividing both numerator and denominator of Equation (14) by L_{2A} and then setting L_{2A} equal to infinity. The equation for L_{BA} M_B may be obtained in a different manner by eliminating θ_A from Equations (13) and (14). Similar procedure is followed in obtaining Equations (29) and (30). The assumption is made in the derivation that the differences in elevation between Supports B and A and Supports B and C are the same and that each is equal to Δ. This is equivalent to rotating the pair of spans until Supports A and C are at the same elevation; or, which is the same thing, assuming as a reference axis the line joining Supports A and C. The result is valid when all three supports are at different elevations, and it may be shown that,

$$\Delta\left(\frac{1}{L_a}+\frac{1}{L_b}\right)=\left(\frac{\Delta_a}{L_a}+\frac{\Delta_b}{L_b}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

Perhaps the mathematical part of the paper would appeal to a wider public if the author had been more liberal with explanations, but the subject-matter is of the highest interest, and the purpose of the present discussion is mainly to endorse the paper and to clarify it for the benefit of those who are not expert in this field. Any suggestions made herein are of trivial weight in comparison with the importance of the original work.

HAROLD C. ROWAN,[7] ESQ. (by letter).—The necessity for information as to the rigidity of steel-frame building connections has been widely recognized by structural engineers. In reporting the results of these tests on standard connections the author has provided much useful information.

In the "Introduction" it is stated that no extensive tests have been published on this type of investigation. The writer would like to call attention to the work of the Department of Scientific and Industrial Research of Great Britain.[8] A report published in May, 1934, contains, among other material, the results of a comparatively extensive series of tests conducted at the University of Birmingham, under Professor Cyril Batho. The writer participated in the work.

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[7] Commonwealth Fund Fellow, Univ. of Illinois, Urbana, Ill.

There appears to be no serious discrepancy between the results obtained by the author for the moment-deformation curves and those obtained at Birmingham. The manner in which flexible connections failed is meaningless because it is impossible in practice for even a small fraction of the deformation involved in failure to occur. The only possibility of failure occurring in such connections is by fatigue. Although it may be true that the strength of the specimens in Series *A* and *B* in shear is not impaired by considerable flexure under moment, their manner of failure alone does not justify that conclusion. Furthermore, the effect of shear on connections of Series *B* is to increase the rigidity. This point may be quite important when considering the rigidity of a dual connection, such as Specimen 11.[9]

The author remarks that the most serious stresses occur in the lower seat angles of Series *B*. It would be noted that the connections were tested upside down as compared with their position in a building, and it is really the top angle which is highly stressed and results in most of the deflection. The same applies to the rivets in the vertical legs of the angles. Specimens 8, 9, 10, 11, and 12 should possibly have been tested the other way up.

The correctness of the author's inference that a seat-angle connection may be stiffened materially by increasing the thickness of the angles was clearly demonstrated, within reasonable limits, in the Birmingham tests.

The type of specimen in which a single plate is used instead of a column section is not representative of the actual conditions obtained in a building. On account of the fact that the column rivets are common to both sides of the specimen, the connections behave as if a rigid plane existed at the mid-section of the plate. When the connection is to a column flange, this is not true, and flexure of the flange will cause lower rigidity. This effect will be largest in the wind-bracing connections, especially where four rows of rivets are used.

The method given by the author for the solution of semi-rigid frames using constants to represent the degree of rigidity of the connections is interesting. It may be mentioned that the British report[8] gives three methods of solution using the same assumptions. A method of interpreting results which does not involve the straight-line assumption, and in which the moment-deformation curve is used to yield immediate results without resorting to trial and error, is also given in the report.[10] By its use the end moments resulting from a uniform load on a beam having the connections under investigation are thus found by drawing a single straight line on the experimental curve and reading the fixing moments directly from the graph. For purposes of interpretation it is best to assume that the beam is attached to rigid columns, but the method is also extended to include the effect of flexure in surrounding members, assuming that their rigidity may be estimated approximately. The effect of a change of load, change of stiffness of a beam due to altering its length or moment of inertia, change of stiffness of surrounding members, or the substitution of some other connection, can be seen immediately. The method has since been extended to cover the cases of unsymmetrical loading and unequal end conditions, or both.

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[8] Second Rept. of the Steel Structures Research Comm. of the Dept. of Scientific and Industrial Research, pub. by His Majesty's Stationery Office, London, England, 1934.

[9] *Loc. cit.*, p. 93.

[10] Second Rept. of the Steel Structures Research Comm. of the Dept. of Scientific and Industrial Research, pub. by His Majesty's Stationery Office, London, England, 1934, p. 92.

The rapid design of steel-frame buildings, which is so often necessary, renders it highly improbable that any method involving complete analysis can ever be used in design, and certainly not any complete analysis using trial-and-error methods. Although it is possible in any single case to choose a constant for the moment-deflection relation which will give the same resulting moments as a curve, it may be shown that any such method involves a change in that constant for different lengths of beam, different loads, and different surrounding members. In any method, of course, the accuracy cannot be greater than the reliability of the connections to act according to the assumed or experimental graph, and tests have shown that the curves extend over a fairly wide range.

There are other considerations which make any analysis approximate. The most important is probably the fact that the members and connections of a steel-frame building are generally encased in concrete, and this will probably affect the distribution of the moments and the stresses. Another uncertainty is the capacity of a building to sway sidewise under unsymmetrical loads. Walls, partitions, and floors tend to stiffen the frame, and it is possible that an analysis which neglects side-sway more nearly represents the actual conditions than one which takes account of side-sway.

It may be mentioned that a large number of tests have been made at Birmingham since the writing of the second report mentioned previously.

WALTER SCHOLTZ,[11] JUN. AM. SOC. C. E. (by letter).—In the solution by moment distribution presented in the Part II of this paper, modified slope deflection equations were used to obtain expressions for fixed-end moments, distributed moments, and carry-over factors. The labor of solving these somewhat complicated expressions could be greatly simplified. For example, referring to Fig. 31, consider the elastic connection as an additional, or phantom, member in the frame. For the connection in this example the stiffnesses given for each side are: Left side, $\frac{1}{Z_L} = 2.9 \times 10^8$; and, right side, $\frac{1}{Z_R} = 4.2 \times 10^8$. For both acting together,

$$\frac{1}{Z} = \frac{1}{Z_L + Z_R} = \frac{1}{\frac{1}{2.9 \times 10^8} + \frac{1}{4.2 \times 10^8}} = 1.69 \times 10^8,$$

in which $\frac{1}{Z}$ is the absolute stiffness, $\frac{M}{\theta}$, for the connection as a member. Its carry-over factor obviously is -1. The analysis now can be performed by ordinary moment distribution, the only precaution to be observed being that of utilizing the absolute stiffness for all members so that the correct relationship to $\frac{M}{\theta}$ for the connection is established. The ordinary fixed-end moments and the carry-over factor of $\frac{1}{2}$ are used.

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[11] Designing Engr., Austin & Ashley, Los Angeles, Calif.

(1) Points	A		B_1		B_2		C	D
(2) Length, L	20' 0"		Phantom Member for Connection		10' 0"		10' 0"	
(3) Stiffness, $\frac{M}{\theta} \times 10^{-8}$		2.98	1.69	1.69	2.16	2.16	1.62	
(4) Percentage Distributed		63.8	36.2	43.9	56.1	57.1	42.9	
(5) Carry-over Factor			−1	−1	0.500	0.500		
(6) Fixed-End Moments		−60.00			+10.00	−10.00	+15.00	
(7)	Standard Cross Distribution (Six Cicles)							
(8)		−16.89					+10.78	

FIG. 35.

In the solution shown in Fig. 35 the writer has neglected the reduction of the moment at the ends of members due to transfer of shear to the connection at each row of rivets; but the effect of this factor is clearly of small magnitude. The results obtained are identical with those that would be obtained by using $L_2 = L + 3\ E\ I\ Z$ in the equations given by Professor Rathbun. Instead of the stiffness factor, K, in Fig. 32, the writer suggests the factor of absolute stiffness, KE, or $\frac{M}{\theta}$. For Span B_1A (Fig. 35, Line 3) $\frac{M}{\theta} = 9.93 \times 3 \times 10^7 = 2.98 \times 10^8$; for Span B_2C, $\frac{M}{\theta} = 7.2 \times 3 \times 10^7 = 2.16 \times 10^8$; and, for Span CD, $\frac{M}{\theta} = 5.4 \times 3 \times 10^7 = 1.62 \times 10^8$. (Note that, for the opposite end fixed, $KE = \frac{4\ I}{L}$ and for the opposite end free, $KE = \frac{3\ I}{L}$). The physical picture on which Fig. 35 is based will be visualized easily by any one familiar with the Cross method, and it permits a simple application in design calculations of the valuable data reported in this paper.

It should be noted that the loading used in this example is only a fraction of the allowable load for these beams. Inasmuch as the slopes of the rotation-moment curves for all the connections decrease materially as the load increases, it follows that for design loadings the connection stiffnesses would be materially less than those used in Fig. 35, which were found by taking the slope of the tangent to the rotation-moment curve at the origin.

J. F. BAKER,[12] ASSOC. M. AM. SOC. C. E. (by letter).—A better understanding of the behavior of steel work connections is essential if the design of steel frames is to be put on a more rational basis and full economy of material obtained. This understanding will follow only from a careful study of such tests as those conducted by Professor Rathbun. It is unfortunate, however, since the behavior of connections is so complex and the field to be covered is so wide, that Professor Rathbun was unaware of the work that has been done in Great Britain since 1930 for the Steel Structures Research Committee of the Department of Scientific and Industrial Research, which was appointed in 1929 on representations from the steel industry backed by an offer of support from the British Steelworks Association. As mentioned in the Committee's First Report, published in 1931, a series of laboratory experiments on beam-to-stanchion connections was undertaken. A detailed report of certain of these tests (which were conducted by Professor Cyril Batho, of Birmingham University, on a series of connections, including all the types dealt with by Professor Rathbun) is given in the Second Report of the Committee, published in 1934.[13] It would be out of place to give a summary of that work here, but it should be studied by all interested in steel design. Characteristics of the connections were given by curves similar to those in the paper under discussion.

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[12] Prof. of Civ. Eng., Univ. of Bristol, Bristol, England; Technical Officer, Steel Structures Research Committee, London, England.

Professor Rathbun's analysis of framed structures is clearly presented, although certain of the assumptions made in introducing Equations (15) to (21), to obtain a more accurate expression for L_2, are of doubtful value. In the First Report of the Steel Structures Research Committee already mentioned, a general equation was given applicable to frames with semi-rigid connections, to which Professor Rathbun's Equations (4), (5), (8), and (9) will be found to lead. The method of attack in stress analysis is largely a matter of taste, but it is interesting to compare this work. The derivation of this general equation is given in the Second Report, together with expressions for the factors necessary for the application of the moment distribution method. These expressions will be found to agree with Professor Rathbun's factors in the particular case of the continuous beam (see the heading, "Moment Distribution Method"). It must be emphasized that in applying this method to a frame with semi-rigid connections it is essential to take into account the widths of the stanchions if serious errors are not to be made.[13] These methods of analysis are based on the assumption that the relation between moment transmitted by a connection and the relative rotation of the members joined, is linear. This is not true of actual connections and, therefore, in applying the methods an estimate, which may need subsequent correction, must first be made of the moment transmitted, as has been done in Table 3 of the paper. A direct determination of the restraining moments for such cases as those of Table 3, when a beam carries a uniformly distributed load, can be made by a simple graphical construction given by Professor Batho in the Second Report of the Steel Structures Research Committee, previously mentioned. If OPQ, Fig. 36, is the relation between angular rotation and moment for the connections, as determined by experiment, and if OA is measured off equal to M_F, the moment at the end of the beam which would cause complete fixity (that is, $\frac{WL}{12}$, in which W is the total load on the beam and L is its length), and OB is measured off equal to $\frac{M_F L}{2\,EI}$, and

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[13] Second Rept., Steel Structures Research Committee, Dept. of Scientific and Industrial Research, pub. by H. M. Stationery Office, Lond., 1934.

AB is joined, the point of intersection, *P*, of this line with the curve, *OPQ*, gives the end-restraining moment directly.

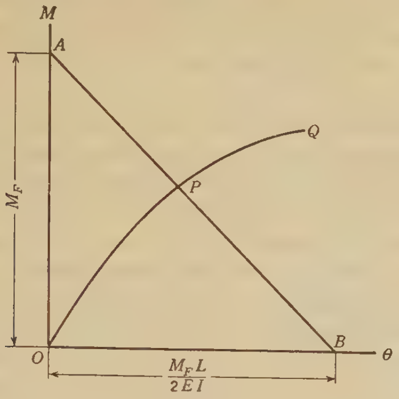

FIG. 36.

Mention has been made in Professor Rathbun's paper of the use of mechanical methods of stress analysis. Considerable attention has been given to this by the Steel Structures Research Committee. An attempt was made to use the deformeter on semi-rigid frames, the connection being obtained by reducing the rigidity of the beam as suggested by Professor Rathbun. This was found to be unsatisfactory due probably to the difficulty of making the necessary reduction with sufficient accuracy and also to the creep of the celluloid at the reduced section. A perfectly satisfactory direct loading method,[13] was evolved, however, in which the connections were made up in much the same manner as they are in practice.

L. E. GRINTER,[14] ASSOC. M. AM. SOC. C. E. (by letter).—It is encouraging that another writer has thrown the weight of his opinion, backed by the evidence of his tests, upon the side of those engineers who have continually insisted that riveted steel structures should be designed for partial continuity based upon the actual conditions of restraint. It is further encouraging that the author's tests present data that make it possible for one to estimate the probable restraint for a considerable range of beam sizes and types of riveted connections. Many more data of this kind are badly needed by the profession. In a country in which the annual construction program is several billion dollars, one has difficulty in understanding why the engineer must guess at such factors as end restraint when the cost of a comprehensive series of tests would be insignificant by comparison.

The tests presented are mainly on specimens of the connected beam type. Only Specimen 18 contained a column section between the beams. Hence, one must keep this fact in mind when using the author's data. It is evident that a girder-to-column connection will permit a greater total rotation than a girder-to-girder connection because of the duplication of tension rivets and the deformation of the column section. This effect can be evaluated for a specific case from the author's data. Specimens 18 and 15 have identical connections except for the insertion of the column section into the former. The moment-rotation curves, Figs. 21 and 23, show that Specimen 18 permitted an average rotation of about 20% more than Specimen 15. Of course, one could not expect this same relationship to hold for joints of considerably different types.

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[14] Prof. of Structural Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

Of particular interest is the author's extension of the theory of continuous structures to take account of the elasticity of the connections. Much of the apparent complication of the procedure disappears when one has mastered the terminology involved. However, it seems to the writer that further simplification would be permissible without in any way destroying the true value of the analysis. For instance, consider the introduction of the terms, (a), (c), and (e), which greatly complicate Equations (17), (18), and (19). For all practical purposes the factor, (a), which is the distance from the face of the column connection to the first rivet through the beam flange, and the length, (e), which is the distance to the farthest rivet, might be taken as zero. Then, the factor, (c) = (e) − (a), would also be zero, and the effect upon Equation (18), for instance, would be as follows:

$$L_2 = L - 3\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{2\,L^2}\,(3\,b + c) + 3\,E\,I\,Z\left(1 - \frac{a}{L}\right)$$

which becomes,

$$L_2 = L + 3\,E\,I\,Z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

when (e) = (c) = (a) = 0.

The physical significance of the suggested change is simply that the elastic action of the connection is assumed to be concentrated at the end of the beam instead of being distributed over the length of the end connection. Since there is still much to be learned about the true action of the connection itself, it seems quite as reasonable to the writer to assume that the elastic properties of the connection are thus concentrated. Undoubtedly, one could design special connections for which the assumption of zero length for the connection would be about as accurate as the author's procedure, which is based upon the assumption of uniform distribution of shear between rivets and a straight-line variation of beam moment within the length of the connection.

In order to demonstrate the insignificance of any error involved in the assumption of zero length for the connection, the writer has prepared Fig. 37, which is an analysis of the author's continuous beam example shown in Fig. 31 and analyzed in Fig. 33. The revised constants are determined, as follows:

$$L_{2BA} = 240 + \frac{3 \times 3 \times 10^7 \times 795.5}{10^8 \times 2.9} = 486$$

$$L_{2BC} = 120 + \frac{3 \times 3 \times 10^7 \times 215.8}{10^8 \times 4.2} = 166$$

$$A\,\overline{x}_{AB} = \frac{w\,L^3}{24}\;\frac{100 \times 20 \times 240^3}{24} = 1\,152 \times 10^6$$

$$A\,\overline{x}_{BC} = A\,\overline{x}_{CB} = \frac{100 \times 10 \times 120^3}{24} = 72 \times 10^6$$

$$M_{BA} = -\frac{3\,A\,\overline{x}}{L\,L_{2BA}} = -\frac{3 \times 1\,152 + 10^6}{240 \times 486} = -29\,500$$

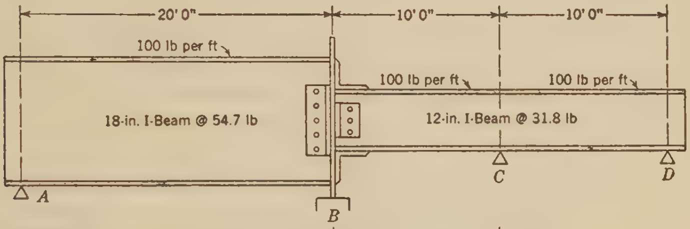
$$M_{BC} = +\frac{6\,A\,\overline{x}}{L\,(4\,L_2 - L)} = +\frac{6 \times 72 \times 10^6}{120\,(664 - 120)} = +6\,600$$

$$M_{CB} = -\frac{6\,A\,\overline{x}\,(2\,L_2 - L)}{L^2\,(4\,L_2 - L)} = -\frac{6 \times 72 \times 10^6 \times 212}{120^2 \times 544} = -11\,700$$

and,

$$M_{CD} = +\frac{3\,A\,\overline{x}}{L^2} = +15\,000$$

Signs agree with the writer's convention[15] that an end moment tending to rotate an adjacent joint clockwise is positive; the reversed moment is negative.



Stiffness Factors	1.63	1.59	2.19	1.80	
Distribution Factors	50.7%	49.3%	54.8%	45.2%	
Carry-over Factors	← 0	+0.5 →	← +0.361	0 →	
Fixed-End Moments	−29.5	+6.6	−11.7	+15.0	in.-lb ÷ 1 000
Balancing Moments	+11.6	+11.3	+5.6		
		−1.8	−4.9	−4.0	
	+0.9	+0.9	+0.4		
			−0.2	−0.2	
Final Balanced Moments	−17.0	+17.0	−10.8	+10.8	

FIG. 37.—CONTINUOUS BEAM ANALYSIS FOR ELASTIC CONNECTION OF ZERO LENGTH AT JOINT *B*

The carry-over factor is different from + 0.5 for the left end of Beam *BC* where it is $\frac{L}{2\,L_2} = \frac{120}{2 \times 166} = 0.361$.

The stiffness factor for each beam is, as follows:

$$K_{BA} = \frac{I}{L^2} = \frac{795.5}{486} = 1.63$$

$$K_{BC} = \frac{4\,I}{4\,L_2 - L} = \frac{4 \times 215.8}{664 - 120} = 1.59$$

$$K_{CB} = \frac{4\,L_2\,I}{L\,(4\,L_2 - L)} = \frac{4 \times 166 \times 215.8}{120 \times 544} = 2.19$$

and,

$$K_{CD} = \frac{I}{L} = \frac{215.8}{120} = 1.8$$

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[15] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 15.

A comparison of the results obtained by three analyses of this continuous beam is given in Table 4. It becomes evident from this study that the probable error involved in treating the elastic end connection as of zero length would be of no great importance in the design of an ordinary structure. The important fact (which is usually neglected) is that the elasticity of the end connection produces changes in moments of between 30 and 40% at both Point *B* and Point *C*. As compared with this large discrepancy, the small effect of the length of the end connection becomes of little importance.

It is interesting to speculate upon the difficulties involved in extending the author's treatment of the simple three-span beam involving only a single elastic connection to a continuous frame with a dozen or more semi-elastic joints. Evidently, the labor involved would be a rather serious matter by any of the theoretical methods suggested by the author. In fact, a practical study would seem to be limited to the assumption that the moment-rotation curve remained a straight line or that the connection remained elastic within the limit of working stresses. The same limitation would necessarily apply to a deformeter study because it would clearly be impossible to represent the curved diagrams of the published tests by a stress-strain curve of celluloid. However, despite these apparent limitations, the author has performed a great service in calling attention to the serious effect upon moments and stresses that may be involved in the neglect of the elasticity and plasticity of the connections.

TABLE 4.—COMPARISON OF MOMENTS FOR THE CONTINUOUS BEAM ANALYZED IN FIG. 37.

Item No.	Assumptions	MOMENTS	
		M_B	M_C
	Moments, in Inch-Pounds, for:		
1..........	Inelastic connections..............................	28 300	7 900
2..........	Elastic connections of zero length..................	17 000	10 800
3..........	Elastic connections by author's assumptions........	17 800	10 550
	Percentage Change from:		
4..........	Items Nos. 1 to 2..............................	−39.9	+36.7
	Items Nos. 1 to 3..............................	−37.1	+33.5
	Items Nos. 2 to 3..............................	+4.7	−2.3

C. R. YOUNG,[16] M. AM. SOC. C. E., AND K. B. JACKSON,[17] ESQ. (by letter). —Within the scope of his tests the author has done a remarkably useful and convincing piece of work, and the rotations found for the connections tested can be used, no doubt, for estimating the probable behavior of similar details under certain conditions A more complete diagnosis of the cause of the rotations obtained, however, would have served the useful purpose of indicating the most effective remedies.

The design of the specimens raises certain questions in the minds of the writers who, some time ago, conducted a somewhat similar series of tests. The variety of types and sizes adopted by the author is most comprehensive, but it would have been desirable also to vary the length of the specimens in order to obtain the moment-shear relationship that exists in the simulated beam of length, $L = 12\ d$. Moreover, the column used in Specimen 18 was too short to develop its normal restraint on the connection.

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[17] Asst. Prof. of Applied Physics, Univ. of Toronto, Toronto, Ont., Canada.

Some difference of opinion may also arise as to the effect on the angular rotation of the omission of shear connections. The form assumed by the stems of the T's in flexure will not be the same for the top and bottom attachments. For the former the stem pulls away from the beam flange between the end of the beam and the first pair of rivets in it, whereas for the latter the contact is firm from the end of the beam to the end of the connection. The curves of the two stems being different, additional rotation will naturally arise.

The method of measurement is also open to criticism. Stress distribution must be particularly irregular in all these specimens, and, therefore, the method of attaching the Ames dials and the location of their stem contacts are matters of first importance. In Series *B* and *C* (Fig. 6), the special device for holding Dials *E* to *L* required sleeve-jointed spring points; it was attached to the beam flange near the inner end of the connection, and the stem of the dial was kept in contact with a surface that moved transversely due to vertical shear. Can such mounting conditions warrant measurements to 0.0001 in.? Dials *M* to *P* were clamped to that section of the flange, just outside the connection, where buckling would first occur and the transverse movements of the dial stems on the plate due to flexure and shear might well excuse the inclusion of the results.

The writers are of the opinion that more reliable results are to be obtained by measuring the relative rotation of cross-sections of the beams beyond the local effects of the connections by means of flange extensometers that encircle the plate or column, and comparing them with similar measurements made on an uncut beam specimen similarly loaded. Intermediate extensometer measurements may be made to locate the sources of the rotation obtained, for example, elongation of the column flange tension rivets, flexure in the T, or deformation of the shear rivets in the beam flange. An independent check may be obtained by deflectometer measurements.

As the author has suggested, much additional information might have been obtained regarding the behavior of these connections from the systematic application of normal and reversed loads. Only by this means is it possible to dissociate the effects of flange attachment and stress on deformation and to detect the extent of elastic and non-elastic slip.

The phenomenon of greatly increased rotation on the first reversal of load for Specimen 14 is characteristic of riveted specimens. The reason for it is, of course, non-elastic slip. Had the reversal of loading been continued until the specimen reached a steady deformational state the rotation would have been the same for reversed load as for normal load.

It would be interesting to know to what relative extent the tension and compression attachments contributed to the rotation. The results of the writers' investigations indicate that the tension attachment is responsible for from 20% to 50% more deformation than the compression attachment.

In seeking ways and means of improving end restraint, especial attention should be given, therefore, to the capacity of an attachment to resist tension with a minimum of yield.

University of Toronto Tests.—Incidental to an investigation of the degree of restraint developed at the ends of steel beams and girders with welded connections, a study was made of the rigidity of four riveted T-connections of two types at the University of Toronto, at Toronto, Ont., Canada, in 1930–32.[18]

The connections were similar to those of the author's Series *C*, Specimens 14 and 15, except that shallow welded shear connections were added. They consisted of T's cut from 30-in., 149-lb, Bethlehem I-beams, connecting two 18-in., 47-lb. Carnegie beams to a central 1-in. plate for two duplicate specimens, and to the flanges of a 4-ft length of 12 by 12-in., 110-lb, Carnegie H-beam for two other duplicate specimens. For convenience, connections of the first type may be termed "plate specimens" and those of the second type, "column specimens." For the former, $\frac{7}{8}$-in. rivets were used throughout, but since the grip of the rivets in the flange of the T's was considerably less for the column specimens than for the plate specimens, they were made $\frac{3}{4}$ in. in diameter, the better to balance the design. This was in conformity with the expected strength of tension rivets of different grips, as indicated by tests made some years ago at the University of Toronto.[19]

A nominal factor of safety of between 2.25 and 2.50 was used in the design of rivets, selection of the working stresses being made with a view to rendering failure of the rivets more likely than failure elsewhere. Although the factor of safety adopted by the author for his Specimens 14 and 15 is not mentioned, it appears to have been greater than the foregoing.

Unlike the author, the writers distrusted the vertical shear rigidity of T's and, consequently, in order that the angular rotation of the connections at the face of the support might not be unduly influenced by vertical shear deformation, or that shear failure might not precede moment failure, two shear plates were welded on the web of each beam for all specimens, with a factor of safety of 3.4 in the welds.

In order to study the characteristics of the connections plain beam specimens corresponding to the cut-and-connected specimens were tested in exactly the same manner, and the effect of the connections was determined by differences.

Instruments.—Two types of instruments were used to measure the deformation of the specimens: (*a*) Yoke extensometers, with accessories, to measure horizontal deformation between points on or near the flanges of the beam; and (*b*) deflectometers, with accessories, to measure the vertical movement of points on the neutral axis of the beam.

Whereas the author measured his deflections with respect to a bar resting on rods fastened to the ends of the specimen, the writers measured the vertical movement of points on the neutral axis of the beams as related to a bridge hung from, and aligned at right angles to, the central vertical element of the specimen. Yokes carrying Ames dials were fixed to the specimens at the neutral axis and were of such length that the plungers of the dials bore on the bridge. The particular advantage of this method of measurement was that it gave the deformation of each half of the specimen with respect to the central vertical section separately, and, in the case of the connected specimens, indicated the relative rigidity of the two connected cantilevers.

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[18] "The Relative Rigidity of Welded and Riveted Connections," by C. R. Young, M. Am. Soc. C. E., and K. B. Jackson, *Canadian Journal of Research*, July and August, 1934.

[19] "Permissible Stresses on Rivets in Tension," by C. R. Young, M. Am. Soc. C. E., and W. B. Dunbar, *Bulletin No. 8*, Section No. 16, School of Eng. Research, Univ. of Toronto, 1928.

Test Procedure.—Although a loading in only one sense is required to determine the degree of restraint or continuity developed by a connection, a reversed loading is necessary for a wind-bracing connection, which may be subjected to complete reversal of stress, possibly involving both elastic and non-elastic slip. Furthermore, initial set occurring during the first load alters the behavior of the specimen during subsequent loads. Consequently, to determine the final rigidity of a wind connection, an adequate series of loads must be applied to eliminate initial set, and to reveal the full value of slip due to reversal of load, as well as the normal elastic deformation of the specimen. Inasmuch as the author's tests do not involve reversal of load (except in one incidental case) his rotation angles cannot apply strictly to wind connections after the first change in wind direction.

To fulfill the requirements mentioned each specimen was subjected to five loads applied and released in 5 000-lb. or in 10 000-lb. stages, from 0 to 75 000 lb, and a sixth load to failure. The first two loads, termed "normal loads," were applied so that the connection was stressed as by a gravity load in the simulated beam. The third and fourth, termed "reversed loads," were applied in the opposite direction, and the last two loads were again normal loads. Readings on the four extensometers and six deflectometers were made after each stage in the application and release of each load.

Results.—In the writers' investigation an effort was made to discover the sources of the deformation, an inquiry that the author did not attempt. For purposes of analysis the connections were divided longitudinally into three regions, I, II, and III, and transversely into two halves termed the top and bottom flanges. Region I comprises the half width of the column and, consequently, existed only for column specimens. Region II extended from the face of the plate or column $2\frac{1}{2}$ in. along the specimen, and Region III extended from the section at $2\frac{1}{2}$ in. to one at 12 in. from the face of the plate or column. For the plain beam specimens the origin was at mid-span. In referring to the specimens it was sometimes found convenient to group all three regions into one region—VI.

The flexural slope developed in a given flange and region is determind by: (1) The type of connection; (2) the flange included, top or bottom; (3) the stress involved, tension or compression; and (4) the loading sequence, including or excluding non-elastic slip.

In some types of connections the effect of a change in Factors (2), (3), or (4) is small, while in others it is considerable. A comprehensive test of any type of connection, therefore, must involve both flanges in both kinds of stress and the loading sequence must be such as to include any non-elastic slip that may occur. Consequently, the application of one-direction loads, as in the author's tests, can not yield the final answer to the problem of wind-bracing rigidity. It was for this reason that the writers adopted a load varying from capacity loading in one direction to capacity loading in the other direction.

Analysis of the flexural deformation occurring in each flange and region of the specimens under the reversing design load, combined with the evidence obtained during failure loads, forms the basis of certain observations which attempt to explain the deformations indicated in terms of the structural details involved.

In Region I the most obvious deformational characteristic was that the tension half of the connection was the critical element. Elastic slip was slightly greater in compression than in tension, but was about twice as great in the top flange as in the bottom. Proportional deformation was 2.6 times as great in tension as in compression, but only slightly greater in the top flange than in the bottom. This excessive deformation in tension is due to the outward bending of the column flanges and the elongation of the connecting rivets.

The deformations in Region II included longitudinal deformation of the attachment within $2\frac{1}{2}$ in. of the column face, bending of the T-flanges, and deformation of the connecting rivets. The deformations of the top and bottom flanges were very similar in the riveted specimens, for which the top and bottom attachments were the same.

The deformations indicated for Region III included the longitudinal deformation of the attachment and beam, and the shear deformation of the connecting rivets or welds with any slip that might have occurred. Non-elastic slip produced more than one-half the deformation and constituted nearly one-third of the total deformation of this type of connection, elastic slip was negligible, and proportional deformation was similar to that of the plain beam.

An analysis of the deformation in Region VI disclosed the particularly interesting fact that for plate and column specimens the deformation arising in Region III is 75 and 50%, respectively, of the deformation in the entire connection. This substantiates Professor Rathbun's observation of marked deformation in the rivets through the beam flanges in the case of certain of his specimens.

Analysis of the total flexural deformation that occurred in each type during the reversing load and the relative effect of tension and compression, top and bottom flanges, and normal and reversed loads in producing it, justifies the following observations: (1) About 30 to 45% of the deformation consisted of non-elastic slip; (2) the aggregate elastic deformation in tension was from 56 to 70% greater than that in compression; (3) opposite flanges deformed equally; and (4) the deformations under normal and reversed loads were identical.

Coefficients of Restraint.—The coefficients of restraint (that is, the relation between the end moment developed in a uniformly loaded beam and the fixture moment), was first found for the initial load assumed as carried to failure. Account was thus taken of initial set, or slip, or both. Obviously, this is the correct loading for total moment calculations, as unrecoverable deformation in the connections permanently lessens their participation in the total moment developed in the span.

Although coefficients of restraint greater than unity are found for welded specimens, the values for riveted T-specimens were less than this. For the plate specimens, it ranged from 0.99 for 0.2 of the design load to 0.93 at the design load. For the column specimens, it ran from 0.92 at 0.2 of the design load to 0.84 at the design load. Above the design load the values dropped off rapidly. Calculations show that for Professor Rathbun's Specimens 14 and 15 the coefficients of restraint at design loads were 0.96 and 0.73, respectively; for uniformly distributed gravity loads the ideal value, 0.75, permits an increase in load of 100 per cent.

A more typical situation with respect to end restraint arises when any one of the innumerable loadings following the initial load is applied. The connections have then suffered permanent deformation and permanent impairment of their restraining value. Calculations of the coefficient of restraint, therefore, should include a constant amount for the permanent deformation resulting between the beginning of the intial load and the beginning of the repeat load.

As was to be expected, the development of a large permanent deformation during the early loads reduced the coefficients of restraint under the repeat load in its early stages to values much less than those for the initial load. For the plate specimens, it varied from 0.63 at 0.2 of the design load to 0.9 at the design load; for the column specimens, it was much lower, varying from 0.3 at 0.2 of the design load to 0.75 at the design load.

Factors of Safety.—The plate specimens failed by flange buckling, with an average ratio of test factor of safety to design factor of safety of 0.97. The column specimens failed by rupture of the tension rivets, with an average ratio of test factor of safety to design factor of safety of 0.89. The rivets failed at an average tension of 26 900 lb each, or 61 000 lb per sq in., based on the size before driving, as compared with an assumed ultimate strength in pure tension of 30 000 lb. The fact that the column specimens gave smaller relative factors of safety than the plate specimens was due, no doubt, to a lesser uniformity of stress distribution among the tension rivets connecting the T to the column flange. Furthermore, the shorter grip of the smaller rivets in the column flange failed to compensate for their use in the column specimens and for the leverage effect of the pressure at the toes of the T. As the moments on the author's comparable specimens did not produce rivet tension in excess of 55 100 lb per sq in. (Specimen 15), the influence of flange bending and leverage on ultimate tensile strength was not disclosed.

Effective Slope Angles Due to Connections Only.—So far as the effect on positive moment in the beam or on story drift is concerned the slope angle due to the connection may be taken as the difference in flexural slope, at any section beyond the connection, between a beam integral with the column and a similar cut-and-connected beam under the same load.

For a load producing a moment varying from capacity positive moment to capacity negative moment, the slope angles attributable to the connections alone, were found by the writers to be 0.00222 radian for the plate specimens and 0.00492 radian for the column specimens. It is to be noted that in each case the shaft of the column is assumed to be straight, and that although local bending in the simulated column web is not included, bending in the column flange is included.

For a capacity moment in one sense only the values of the slope angles may be taken as one-half those given for the reversing load. The mean value for plate specimens under a load in one direction only (that is, 0.0011 radian) is approximately equal to that obtained by Professor Rathbun on reloading Specimen 15 at a moment of 40% of the ultimate moment, which was the ratio of design to probable ultimate load observed in the Toronto tests. It is not possible to make a similar comparison with the author's single-column specimen, No. 18, as the ultimate moment is not given.

E. MIRABELLI,[20] M. AM. SOC. C. E. (by letter).—In Table 3 the author intimates that the carrying capacity of beams under gravity loading may be increased by as much as 95% by making use of the end restraint provided by the connections. This percentage approaches, very closely, the theoretical maximum of 100% which would occur with a perfectly balanced design when, under full working load, the connections yield just enough to make the bending moment at mid-span equal to the restraining moment at the connections. Such percentages may lead to unwarranted optimistic conclusions regarding the extent to which economy is possible in an actual design. The possibility of extremely large savings from utilizing resistance of connections is doubtful for an ordinary building frame, for the following reasons:

(1) The elasticity of the connections differs from that of the remainder of the beam and, as a result, the stresses developed in the beam are not proportional to the applied load. The bending moment at mid-span increases more rapidly than the loading, whereas the bending moment at the end increases less rapidly than the loading. The variation for Specimen 10 (see Table 3) is shown by the calculated curves in Fig. 38. Because of this characteristic, the factor of safety based on unit stresses is larger than that based on loading. For example, if the factor of safety is 2 measured to the yield point for the beam of Fig. 38, the resisting moment at the yield-point stress is twice that at the working stress. As indicated by the broken lines, the applied load at the yield point is less than twice the working load. The factor of safety based on loading for the case shown is only 1.78, whereas the stress intensities indicate a factor of 2.00. The importance of dealing with applied loads rather than with unit stresses may be appreciated

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[20] Asst. Prof., Structural Eng., Mass. Inst. Tech., Cambridge, Mass.

if an extreme case is considered in which the increase in moment is so rapid that an increase of, say, 5% in loading beyond the working load causes the bending stress to be doubled.

The percentages of permissible excess loading shown in Column (19) of Table 3 were computed with the factor of safety based on unit stresses. Table 5 indicates how some of these percentages are reduced if the factor of safety is based on applied loads. The allowable excess load is reduced by an average of about 30 per cent.

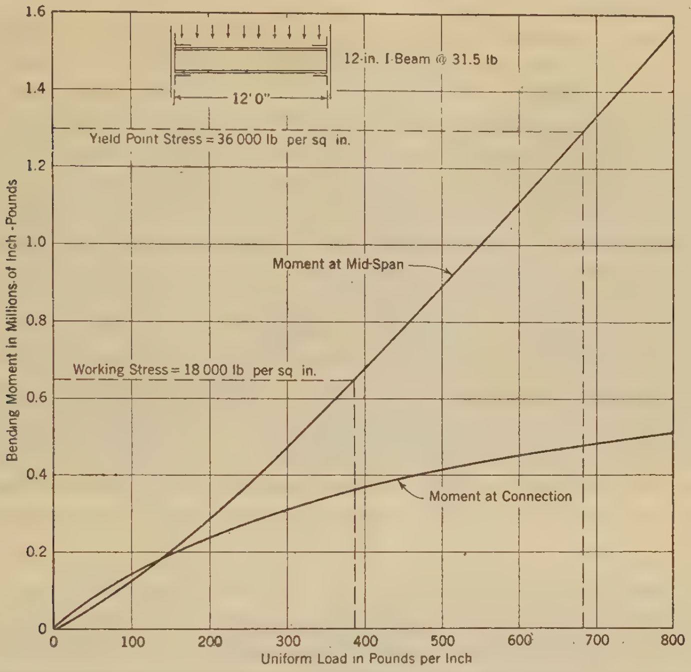

FIG. 38.—RELATION BETWEEN APPLIED LOAD AND BENDING MOMENT

TABLE 5.—COMPARISON OF EXCESS (UNIFORM) LOADS DUE TO CONNECTIONS (1 Kip = 1 000 Pounds)

Specimen (see Table 3) (1)	Span length, *L*, in feet (2)	Load allowed on unrestrained beam, in kips (3)	LOAD, IN KIPS, ALLOWED ON RESTRAINED BEAM WHEN THE FACTOR OF SAFETY IS BASED ON:		PERCENTAGE ALLOWABLE EXCESS WHEN FACTOR OF SAFETY IS BASED ON:	
			Stress (4)	Load (5)	Stress (6)	Load (7)
2	8	21.3	26.1	24.5	22	15
4	12	36.0	41.3	39.9	15	11
6	18	59.0	71.8	67.4	22	14
8	12	36.0	49.7	44.5	38	24
10	12	36.0	56.0	49.2	55	37
12	12	36.0	62.4	54.7	73	52
15	16	108.1	210.6	184.3	95	71
16	22	127.6	248.0	202.0	94	58
18	16	108.1	206.0	188.2	91	74
..	..	.....	.....	.....	....	....

(2) The author's percentages are based on the use of perfectly rigid supporting columns. Flexibility of the columns has the effect of reducing the allowable excess loading. The greatest effect from this source is likely to be in the upper stories of a building where column sections are relatively small, and the least effect is in the lower stories of a tall building where column sections are large.

(3) Bending moment in the beams of a building frame is reduced at the expense of the columns. Unless the arrangement is such that the end moments in adjacent beams balance each other, the restraining moments are transferred to the columns, and much of the saving in material of beams may be offset by an increase in required material for columns.

(4) Table 3 was compiled for uniform loading which leads to more favorable results than concentrated loading. As an illustration, with a concentrated load at mid-span the percentages for Specimens 10 and 15 in Table 5 are changed from 55 and 95 to 43 and 68, respectively, and Percentages 37 and 71 are changed to 31 and 55, respectively.

(5) From the possible economy in material must be deducted the added cost of the more complex analysis involved in the use of end restraint. The method of analysis undoubtedly would be simplified through experience, but would always be more costly than the analysis with unrestrained connections.

Regardless of whether or not very large savings can be accomplished, information of the type given in this paper is of great value in providing an essential "missing link" in the precise determination of stresses in building frames.

R. L. Moore,[21] Esq. (by letter).—In view of the emphasis that has been placed on methods of analyzing continuous frames, it seems that the question of joint rigidity in riveted structures has received too little attention. The author of this paper has made a valuable contribution to the knowledge in this field.

From the standpoint of design, the evaluation of the rigidity of the joints in terms of total fixity has certain advantages over the procedure outlined by the author, particularly where the method of moment distribution is used. It may be shown that the distribution factors for any joint are proportional to the product of the $\frac{I}{L}$ -values and the degrees of fixity for the members at that joint. It also follows that the carry-over factors at one end of a member are equivalent to one-half the degree of fixity at the other end. After computing fixed-end moments and making adjustments for the estimated degrees of fixity at the joints, the procedure becomes one of simple moment distribution. The author's illustrative problem, shown in Fig. 33, may be solved readily in this manner. On the basis of the results presented in this discussion, however, the writer would alter somewhat the values for the fixed-end moment used. The author has assumed M_{BA} and M_{BC} in Fig. 33, to be the equivalent of about 50% and 70% fixed, respectively. Although the latter percentage is in accord with the writer's tests, the fixity for the clip-angle connection on the 18-in. I-beam, on the same basis, would not be taken at more than about 20 per cent.

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[21] Research Engr., Aluminum Research Laboratories, Aluminum Co. of America New Kensington, Pa.

In the tests upon which these conclusions are based, two-span continuous beams, having a joint or splice at the center support, were used. By comparing the behavior of such specimens with that of an unspliced beam, having 100% continuity, a direct measure of the efficiency of the joints in transmitting bending moments was obtained.

Figs. 39 and 40 show the eight specimens on which tests were made. Specimens 1 and 2 were representative of cases of 100% and 0% continuity, respectively, whereas Specimens 3 to 8 included several common types of beam connections. All specimens were 10 ft 6 in. in over-all length and were made of 8-in. aluminum alloy I-beams and aluminum alloy angles and plates. The section elements of the I-beams were as follows:

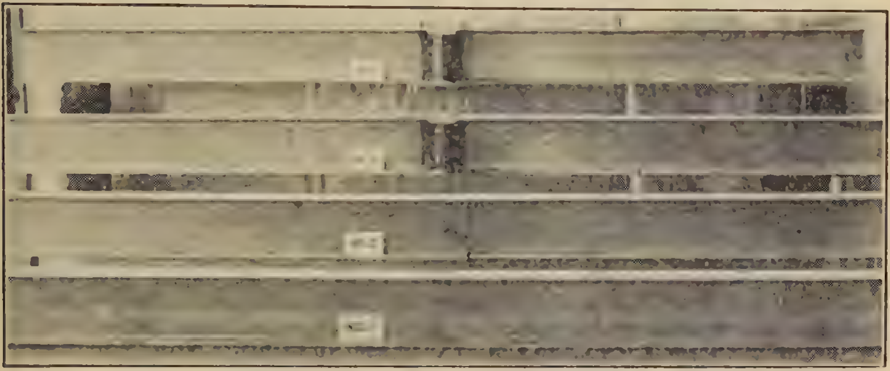


FIG. 39.—SPECIMENS 1, 2, 3, AND 4.

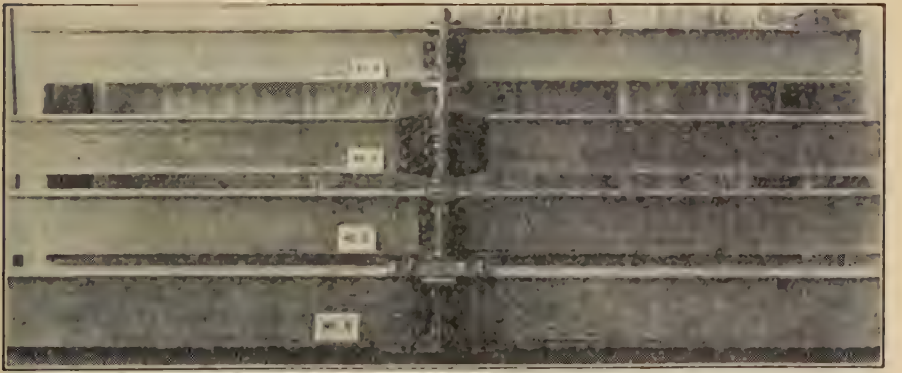


FIG. 40.—SPECIMENS 5, 6, 7, AND 8.

Web thickness, in inches	0.28
Weight, in pounds per foot	5.90
Area, in square inches	4.88
Moment of inertia, I, in inches^4	5.06
Section modulus, S, in inches^3	12.6

TABLE 6.—TENSILE PROPERTIES OF 8-INCH ALUMINUM ALLOY* I-BEAM

Specimen	Yield strength (set = 0.2%), in pounds per square inch	Tensile strength, in pounds per square inch	Percentage elongation, in 2 in.
Web (longitudinal)	43 200	62 900	19.0
Web (transverse)	44 100	65 800	17.0
Flange (longitudinal)	45 400	65 000	20.0

* Duralumin.

The tensile properties are listed in Table 6. The joints were fabricated by means of hot-driven steel rivets $\frac{5}{8}$ in. in diameter. The make-up of each may be described briefly, as follows:

Specimen	Type of Joint
3	Four clip angles, 3 by 3 in. by $\frac{3}{8}$ in. by 6 in. long.
4	Four clip angles, 3 by 3 in. by $\frac{3}{8}$ in. by 6 in. long. Two splice-plates, $\frac{7}{16}$ in. by 4 in. by 8 in.
5	Four clip angles, 3 by 3 in. by $\frac{3}{8}$ in. by 6 in. long. Two splice-plates, $\frac{7}{16}$ in. by 4 in. by 1 ft long.
6	Four clip angles, 3 by 3 in. by $\frac{1}{2}$ in. by 6 in. long. One connection of the joint riveted; the other pin-connected (diameter of pin, $1\frac{1}{2}$ in.).
7	Four clip angles, 6 by 3 in. by $\frac{1}{2}$ in. by 10 in. long.
8	Four clip angles, 3 by 3 in. by $\frac{3}{8}$ in. by 4 in. long. Four seat angles, 3 by 3 in. by $\frac{1}{2}$ in. by 6 in. long.

Fig. 41 shows the nature of the two-span beam tests. The reactions on the auxiliary loading beam of the testing machine used (capacity, 300 000 lb) were 5 ft, center to center, whereas the loads on the specimen were applied at the center of each span. Deflections were measured by a dial-gauge graduated to 0.001 in., with respect to a reference beam under the specimen, as shown in Fig. 41. Stresses were determined by a 2-in. strain-gauge, readings

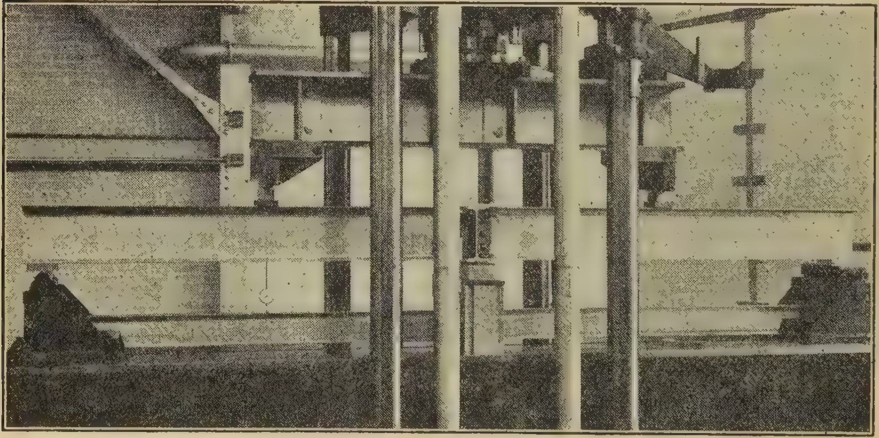


FIG. 41.—TWO-SPAN BEAM TEST OF SPECIMEN 4.

being taken on the top and bottom flanges at a sufficient number of sections to determine the distribution of bending moments.

As a preliminary step in the investigation, deflections and permanent sets were determined for each specimen for loads up to 30 000 lb. The beams were then turned over and the procedure was repeated. The behavior was similar in both cases, and permanent sets were on the order of only 0.002 to 0.004 in.

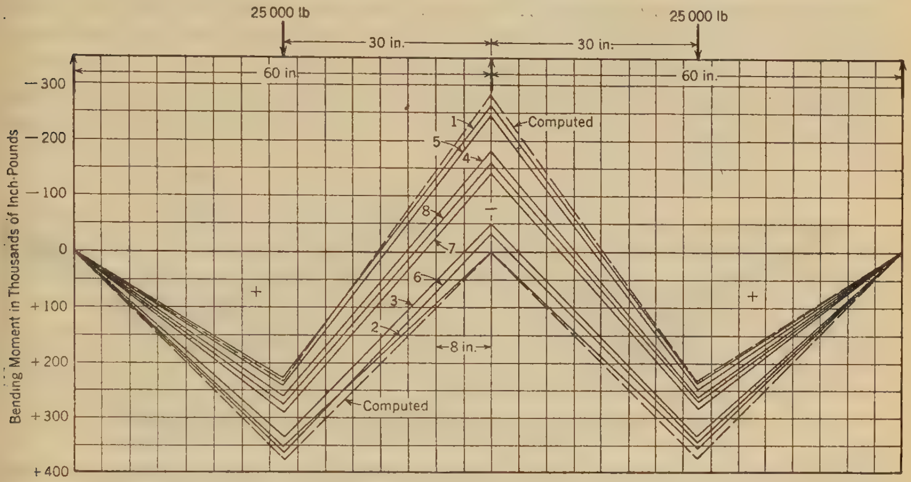


FIG. 42.—BENDING MOMENT CURVES FOR A 50 000-POUND LOAD.

Fig. 42 shows a set of bending moment curves constructed from the measured stresses for a 50 000-lb load. The ratios of the moments developed

TABLE 7.—TESTS OF 8-INCH, ALUMINUM ALLOY I-BEAMS

Specimen	Two-Span Tests						(d) Strength of Joints as Determined by Simple Beam Tests on a 48-Inch Span			
	(a) Continuity Factors Based on Measured Bending Moments at Center Support for a 50 000-Pound Load		(b) Comparison of Measured and Computed Center-Span Deflections, in Inches, for a 50 000-Pound Load		(c) Ultimate Loads and Corresponding Maximum Bending Stresses at Center of Spans				Maximum Computed Stresses,† in Pounds per Square Inch	
	Bending moment, in inch-pounds	Continuity factors (percentage)	Measured	Computed	Ultimate loads, in pounds	Maximum stresses,* in pounds per square inch	Beam proportional limit, in pounds	Ultimate load, in pounds	At beam proportional limit	At failure
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	258 000	100	0.140	0.145	80 100	29 400	40 000	56 800	37 900	53 800
2	0	0	0.240	0.260	55 500	31 000	......	......	......	......
3	50 000	19	0.231	0.216	62 300	33 100	4 000	8 800‡	63 000	139 000
4	185 000	72	0.170	0.149	69 000	28 600	20 000	37 500	42 000	72 000
5	245 000	95	0.154	0.146	75 500	29 300	30 000	58 800	33 000	67 000
6	30 000	12	0.238	0.234	53 900	30 300	......	......	......	......
7	145 000	56	0.181	0.165	66 000	30 000	15 000	21 800	47 000	68 400
8	160 000	62	0.180	0.157	72 500	30 800	15 000	34 400	36 700	84 000

* Estimated on the basis of average measured stresses for a 50 000-lb load.
† Stresses computed at point where ultimate failure occurred.
‡ Load at which test was stopped; beam continued to yield without fracture.

at the joints over the center supports to those observed in Specimen 1 gave a direct measure of the continuity factors or joint efficiency for each type of construction. From the results given in Table 7 (*a*) it will be noted that none of the joints was the equivalent of an unspliced beam. The highest degree of fixity — 95% — was found for Specimen 5, which had long splice-plates in addition to web clip angles. Continuity factors for the other spliced specimens ranged from 72 to 12%, the latter being found for Specimen 6 which had one span pin-connected near the joint. The standard clip-angle connection in Specimen 3 developed about 19% fixity.

Fig. 43 shows the influence of joint fixity upon the deflections of the different specimens. The relative positions of the curves will be recognized as about the same as those shown in Fig. 42 for the bending moments. The average measured center-span deflection of Specimen 1, being 100% continuous, was only 58% of that observed for the two simple spans of Specimen 2, whereas all the deflections of the other specimens were between these limits. Although the deflection measurements obtained were not sensitive enough to the degree of fixity at the center support to be used in determining continuity factors, as was done in the case of the bending moments, the agreement between the measured and the computed values given in Table 7 (*b*) was reasonably good.

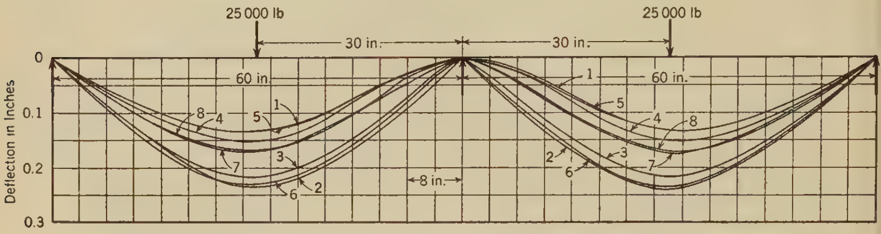


FIG. 43.—COMPARISON OF DEFLECTIONS FOR A 50 000-POUND LOAD.

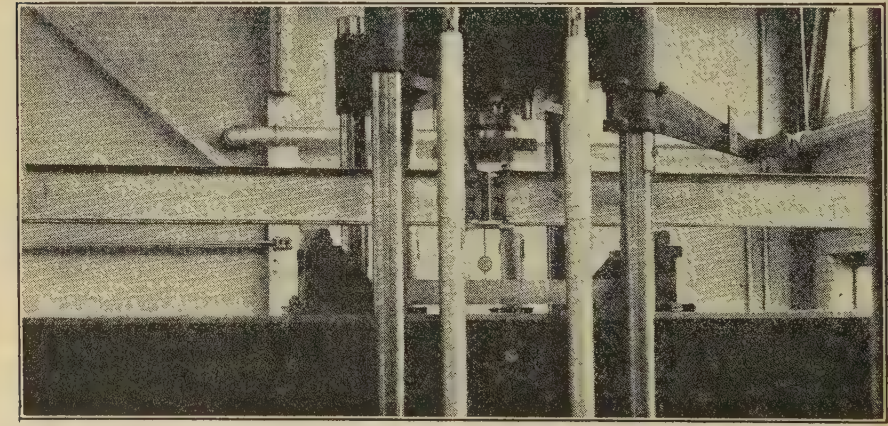


FIG. 44.—BEAM TEST OF SPECIMEN 4 ON 48-INCH SPAN.

In comparing bending moments and deflections for a single 50 000-lb load, attention should be called to the fact that both were proportional to the loads applied up to approximately the point of failure. The latter occurred in every case by a lateral buckling of the compression flanges of the beams without any apparent injury to the joints. Table 7 (*c*) gives the ultimate loads carried and the corresponding maximum computed flange stresses at the centers of the spans.

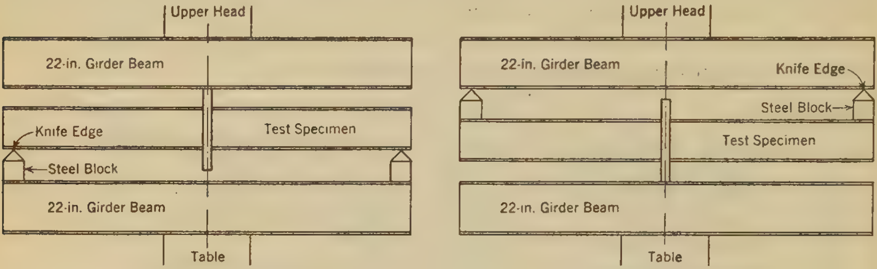
Since none of the specimens was damaged in the two-span tests all but Specimens 2 and 6, which did not have rigid joints, were subjected to a simple beam test on a 48-in. span as shown in Fig. 44. The purpose of this test was to determine more nearly the ultimate bending resistance of the joints. Table 7 (*d*) gives a summary of the maximum loads carried and the corresponding computed stresses at the points where failure occurred. The manner of failure in each case was, as follows:

Specimen	Type of Failure
1	Lateral buckling of the flange accompanied by the buckling of the web at the center load point.
3	Yielding of the web in bearing, around the clip-angle rivets.
4	Shearing failure of the rivets in the lower splice-plate.
5	Tension failure of the lower splice-plate.
7	Tension failure of the web at the outside line of rivets.
8	Shearing failure of the lower seat-angle rivets.

It is significant that with one exception (Specimen 7) the moments resisted in the simple span tests were from about 70 to 90% higher than those developed in the two-span tests. Furthermore, it appears that this bending resistance may be safely estimated on the basis of the ultimate strength of the material at the point subjected to the maximum stress.

Although it is appreciated that these tests have been limited to comparatively few specimens, the results obtained do indicate some useful limits for design purposes. For standard clip-angle connections it appears that a degree of fixity of from not more than 20 to 40% may be expected, depending upon whether single or double lines of rivets are used in the web. The use of seat angles in addition to clip angles on the web may increase continuity factors up to 50% or 60%, whereas the use of splice-plates on the flanges may result in continuity factors of from 75 to 100 per cent. In using these percentages, it is obviously essential that the joints be made sufficiently strong to carry the moments attributed to them. It should be pointed out that these percentages are in substantial agreement with the results indicated in Table 3 of the paper. The 15% increase in carrying capacity of Specimen 4, for instance, corresponds to a joint fixity of about 20%, as given herein for standard clip-angle connections. The increase in carrying capacity of Specimens 11 and 12, having seat angles, corresponds to a joint fixity of about 60 per cent. Although none of the writer's tests included specimens similar to those in Series *C*, the joint efficiencies obtained for this type of construction ranged from about 65 to 100 per cent.

John Sanford Peck,[22] M. Am. Soc. C. E. (by letter).—Under "Description of Tests and Failures" (see following Fig. 11) the author states: "The curve obtained was of an entirely different type from that of the others in this entire series of tests. This curve suggests a subject for further investigation." He was referring to Specimen 14, Run 3, in which a specimen that had been tested beyond the elastic limit, but not to destruction, was loaded in the reverse position, thus simulating the condition arising in a wind-bracing connection when the direction of the wind changed 180° (see Fig. 3(*b*), Specimen 14, Run 3.). A further inquiry into this condition, therefore, is justifiable. Two types of connections were fabricated for testing in the laboratory of the College of the City of New York, identical with Specimens 13 and 14 (see Fig. 1), except that the over-all length was 10 ft instead of 6 ft. Two specimens of each type were made up, the entire investigation thus comprising tests of two pairs of specimens.



(*a*) POSITIVE MOMENT (*b*) NEGATIVE MOMENT

Fig. 45.—Method of Loading.

The experimental technique was identical with that followed by the author. The dials used in measuring the angular rotation were placed as shown in Fig. 6, except that Dials *M*, *N*, *O*, and *P*, were omitted. The method used to reverse the direction of the application of the load is shown in Fig. 45, the knife-edges, *AA*, being changed from the bottom to the top position when it was desired to change the direction of loading.

TABLE 8.—Sequence of Loading

Loading cycle	Load, in Pounds From:	Load, in Pounds To:	Successive increments, in thousands of pounds
1	1 000	8 000	2
2	8 000	1 000	4
3	1 000	8 000	4
4	8 000	1 000	8
	DIRECTION REVERSED		
5	−1 000	−8 000	−2
6	−8 000	−1 000	−4
7	−1 000	−1 000	−4
8	−8 000	−1 000	−8
9	− 1 000	− 8 000	− 8
10	− 8 000	−16 000	− 2
11	−16 000	− 1 000	− 4
12	− 1 000	−16 000	− 4
13	−16 000	− 1 000	−16
	DIRECTION REVERSED		
14	1 000	16 000	2
15	16 000	1 000	4
16	1 000	16 000	4
17	16 000	1 000	16
18	1 000	16 000	16
19	16 000	24 000	2
20	24 000	1 000	4
21	1 000	24 000	4
22	20 000	1 000	20
	DIRECTION REVERSED		
23	− 1 000	−24 000	− 2
24	−24 000	− 1 000	− 4
25	− 1 000	−24 000	− 4
26	−24 000	− 1 000	−24
27	− 1 000	−24 000	−24
28	−24 000	−40 000	− 2
29	−40 000	− 1 000	− 5
30	− 1 000	−40 000	− 5
31	−40 000	−1 000	−40
	DIRECTION REVERSED		
32	1 000	40 000	5
33	40 000	1 000	5
34	1 000	40 000	10
35	40 000	1 000	40

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The specimens were loaded according to the schedule given in Table 8, the loads being the total applied to the central plate of the specimens. In Figs. 46(*a*) and 46(*b*), the angular rotation, expressed in radians, of the connection was computed by the same method as that used by the author, and the results of each side were averaged. The results from the tests of the two identical specimens were then averaged so that the data shown, represent the average of tests of four connections, respectively.

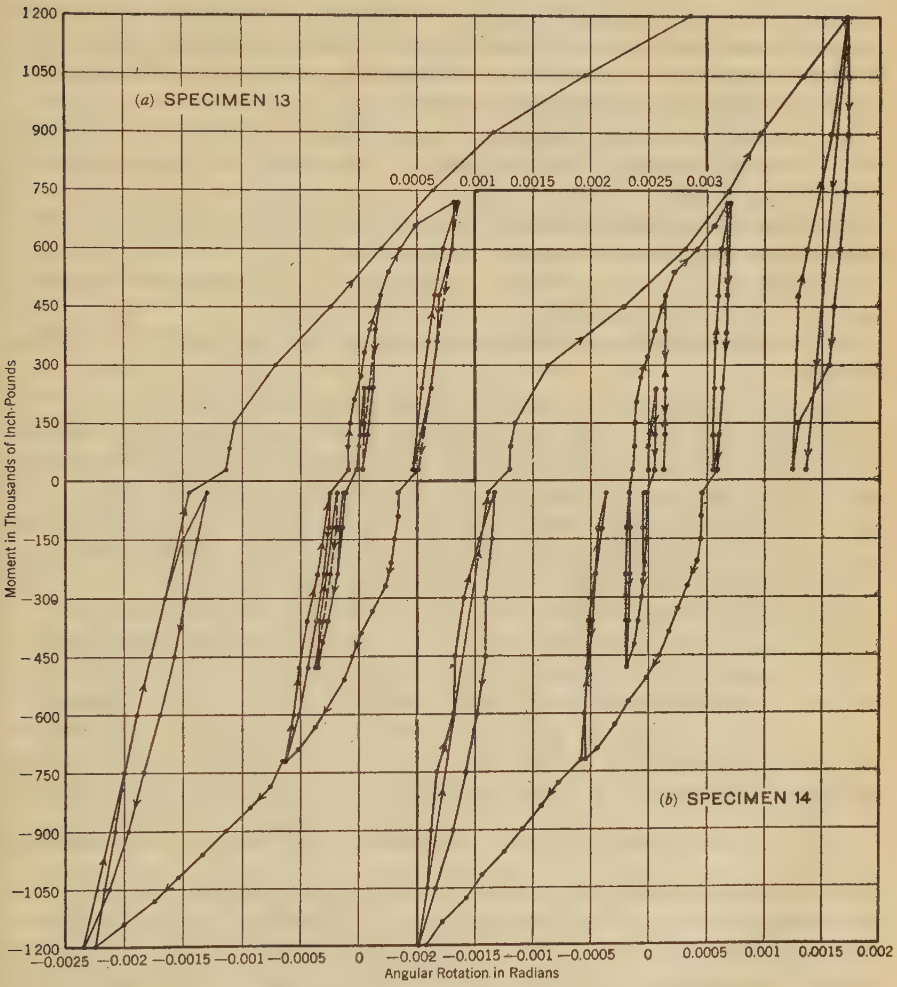

FIG. 46.—RESULTS OF TESTS ON SPECIMENS.

The writer presents these curves with the idea that the reader can draw his own conclusions from the data. The small number of tests involved makes it difficult to generalize too extensively on the action of wind-bracing connections under a reversal in wind direction. The tests do indicate trends which may be summarized, as follows:

(1) Under repeated loads in the same direction within the elastic limit, the connections behaved elastically because the re-loading curves may be considered to be straight, allowing for hysteresis effect.

(2) Under a reversal of the direction of loading, a slip occurred in the connection, due to some factor which was not determined in this investigation. (It might possibly have been the slackness of fit of the rivets in the holes.) After this slip had occurred the joint continued to behave elastically under repeated loads in the same direction and within the elastic range, until another reversal in the direction of loading, when a slip again occurred followed by a second phase of elastic behavior.

(3) Beyond the elastic limit there was a large increase in the magnitude of the slip on a reversal in the direction of loading and the connection deformed plastically.

(4) The effect of the additional row of rivets was to increase the resistance of the connections to angular deformation.

(5) The slope of the re-loading curves for equal positive and negative moments appeared to be equal.

(6) The slopes of the re-loading curves appeared to decrease with increasing moments.

Direct comparison of the data presented herein with those presented by the author is difficult for the reason that the conditions under which the two sets of tests were made were not identical. The use of a longer span decreased the effective shear on the connections as tested by the writer and, although it may be argued from theoretical consideration that the shear on the joint has no effect on the angular rotation, there is not sufficient experimental evidence supporting that point of view to warrant the comparison of two sets of data having different conditions of shear acting on the connections.

The writer's tests, however, do serve to accentuate a point which the author has failed to emphasize particularly. Figs. 46(*a*) and 46(*b*) show clearly that the slope of the reloading curve is not a constant quantity, but is dependent on the degree to which the joint has previously been strained. The author has chosen his coefficient, Z, such that $M Z = \theta$, in which θ is the angular rotation of the joint for a given moment, M. From the foregoing equation, $Z = \frac{\theta}{M}$, in other words, Z is the reciprocal of the slope of the re-loading curve. From an inspection of Figs. 46(*a*) and 46(*b*) it is readily seen that for a given moment there can be several values of Z depending upon the previous history of the connection. This is not clearly brought out by the author either in his curves or in his test. For one who is familiar with the interpretation of experimental data and the general technique of re-loading curves, the information is available with which to make that deduction, but it is doubtful whether the average reader is familiar enough with laboratory technique to be able to read that most important fact for himself from the data as presented, and it is to be regretted that the author did not bring the point out more explicitly. In view of the fact that Z is a variable coefficient that differs with the size and type of the joint and with the degree to which the joint has previously been strained, it would seem that the use of a numerical value for Z in actual steel design must be based upon the completion of a long series of laboratory investigations undertaken to determine its exact numerical values.

The author has undoubtedly made a major contribution to steel design in developing his theory of the elastic behavior of riveted connections, but the writer foresees a long period of necessary laboratory work running into several years before this theory can be used in practical designing. At present, those connections which developed relatively small resistance to moment (namely, Series A, Specimens 1 to 7), are taken as being free end beams whereas those connections which developed large resistance to moment (Series B and C, Specimens 8 to 14) are designed as completed fixed-end connections. The use of the author's method of design, therefore, would not affect the design of the clip-angle connections and would increase the amount of steel necessary in the wind-bracing connections; hence, the result would be less economy of steel than at present.

J. CHARLES RATHBUN,[23] M. AM. SOC. C. E. (by letter).—The discussions of this paper emphasize the fact that the problem presented was being studied in at least three other laboratories at the same time that the work was being conducted by the writer, indicating that the subject of stress analysis in steel frames has reached a stage where the properties of the connections should be considered. It is fortunate that the synopsis and reference of each of the works of the other investigators have thus been published in order that those who are interested further in this subject may read the originals if they choose. The writer undertook the experiments described in his paper because he was unable to find any data on the subject. Certain phases of the experiments touched upon by the discussers (such as the reversal of moment, the distribution of deformation among the parts of the connection, and the effect of shear) were omitted because they were not needed for the solution of the problem of which this investigation was a part.

Professor Goodwin has taken advantage of his familiarity with the writer's work and methods to clarify the paper in some respects, and his comments are appreciated. The question of signs is one that arises quite often in studies of this nature, partly because textbooks are often confusing on the subject. The system used by the writer for beams is to accept the conventional sign for moment on a beam as positive when the upper fibers are in compression. As E and I are always positive, the signs for shear, load, slope, and deflection follow at once by direct integration or differentiation of the mathematical expression for moment. Some confusion has arisen because the foregoing statement leads to negative loads when their direction of action is negative (downward) and to negative deflection when this deflection is downward. The desire to avoid these two situations is probably the reason for using other sign systems in computations. The convention for the signs of the moment tending to rotate a joint is used as positive when counter-clockwise for the reason that counter-clockwise angles, when small, have positive tangents. Thus, stress is proportional to strain and the signs are consistent in this relation. Those who wish to consider clockwise motion as positive will find no error in their work.

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[23] Associate Prof., Civ. Eng., Coll. of the City of New York, New York, N. Y.

The beam and joint sign conventions cannot be consistent with each other at all times. In the case of a beam a counter-clockwise moment is positive at one end and negative at the other. The writer knows of no better method of minimizing this difficulty than the introduction of two self-consistent sign systems. As Professor Goodwin states if the reader is familiar with another sign system than that used by the writer the formulas are developed in the paper and it is a very simple matter to go over them changing the signs to the more familiar system. The writer feels that Professor Goodwin has done a service in clarifying this point.

Mr. Rowan and Professor Baker infer that the writer should have mentioned the work of Professor Batho[10] in his "Introduction". This omission was unintentional and was due to the fact that the paper was submitted before Professor Batho's report was available. The writer is in accord with Professor Baker that those who are working on any particular problem should be in closer touch with each other. Discussion reveals that the British Steel Works Association has been working on this subject for several years without the knowledge of those committees of the American Institute of Steel Construction and of this Society as well as of Columbia University, which were consulted during the investigation, or examined the writer's paper before it was published. At the same time, these organizations have been familiar with the plans of the investigation since 1931, and the tests have been advertised locally as they were conducted, so that those who were interested might view them and suggest ideas. It is unfortunate that practical difficulties prevent a closer co-operation in the advancement of the field in which several investigators are interested.

Professor Batho's work is similar in many ways to that of the writer. It contains many ideas not brought out in the paper or discussions. The duplication of the several tests is not an economic loss as a study of the additional number of tests lends weight to the value of the constants used in computations.

The remarks concerning the only possibility of connection failure being due to fatigue are not correct. A frame that consists of columns and beams under horizontal loads relies upon the connections for its strength unless they are stronger than the beam itself. It is true that failures of this type do not occur in buildings, probably because horizontal loads are not applied to buildings with as much force as would be necessary to cause the trouble suggested. The problem is of more than academic interest, however. One method of bracing for horizontal loads is to use wind-brace connections of the type of Series C, because such connections have a high resistance to moment.

Mr. Rowan's suggestion that column sections introduced between the connections would produce results of a different character is undoubtedly true. As there are so many combinations of column section and connections, the writer is of the opinion that the properties of columns as affecting the rigidity of joints can be made a separate study. The combination of connection and column can then be developed, as can the combination of several connections. The writer tested only one column, because his paper deals with connections. As Professor Grinter has indicated, some idea of the effect of columns can be obtained from this test. The field is full of subjects for investigation but, to be of value, a paper must confine itself to the subject under discussion. The subject of shear, therefore, was omitted from the paper, perhaps unwisely. The question of partitions and concrete in buildings is also entirely out of the province of the subject.

One reason shear has not been discussed is that the data of the tests have a wide spread. To conduct a series of tests varying the shear would involve considerable labor and might lead to a minor correction in the results which by their nature are erratic. If one is interested, this is a subject for separate investigation, but the writer feels that he has made his paper of sufficient length within its well-defined scope.

Shear angles were omitted on the connections of Series *C* in order to observe the effects of such omission. A study of the shear curves will give designing engineers the data from which they can decide whether or not such angles are necessary.

Mr. Rowan's remarks about "complete analysis" and "trial-and-error method" are not consistent with modern progress. Twenty-five years ago some engineers might have agreed, if they were strong opponents of the progress being made in stress analysis. As to the "trial-and-error" philosophy, the writer respectfully refers Mr. Rowan to the published opinions of Hardy Cross, M. Am. Soc. C. E.

Mr. Scholtz is to be congratulated in indicating how the labor of allowing for the connections can be reduced in an analysis. The writer feels that the improvements in this investigation that are to be made are in theory rather than in laboratory tests. Although more tests are always of value the digesting of these tests and the incorporating of their results in engineering design is another and more important matter.

Professor Batho's graphical method of analyzing connections is one that deserves a more complete treatment than the outline given in the discussion. It should help materially in popularizing the idea of including connection data in a frame analysis.

Both Professors Baker and Grinter suggest that some of the terms in Equations (15) to (21) could be omitted to advantage. This is the intent of the writer. Having given the several terms that theoretically affect the results, Table 3 is constructed so as to include these terms in the computations. If one does not wish to use the corrections, the consequent error can be approximated after a study of the table.

The writer is inclined to agree with Professor Grinter that in the practical case the moment-rotation curves may be considered as straight lines. In fact, the reloading curves are not far from straight. The experimental data indicate that after a connection has received some stress the reloading curves govern. This should tend to simplify the computations. The slope of the curve must be based on judgment after the probable history of the connection has been considered.

As the tests described by Professors Young and Jackson were conducted with objectives differing from those of the writer, a thorough study of their work should reveal many ideas not treated by the writer. The suggestion that the writer neglected the study of shear, for example, is mentioned by several of the discussers. To have made an investigation along these lines would have multiplied the number of tests, the number of specimens, the time, the labor, and the expense very much. The effect of shear, the writer feels, is small compared with the variation in the properties of connections of the same design. Consequently, he has left it for others to study.

Although the laboratory technique used by the writer differed from that of the several discussers, he feels that it was as satisfactory as the others. It was adopted after considerable study of the idea of fastening the instruments beyond the influence of the connection. No trouble was experienced with the dials slipping on the bands, due to shear deformation. The data were not recorded, nor the curves plotted to a degree of accuracy comparable to readings of 0.0001 in., and the writer does not wish to create the impression that the work was done with that refinement. Dials of this degree of accuracy were used because they were easy to read and were readily available.

The question of reversal of moments was considered while planning the series of tests made by the writer. This aspect of the subject was not studied for several reasons: One was that the paper had already assumed an excessive length; another was that an investigation of the deflection of one of the tall buildings in the City of New York (which deflected less than 6 in. in eighty stories during a 90-mile wind) led to the impression that the wind would not reverse the dead moment on a building connection under any but the most exceptional circumstances. The writer felt that he may have been in error in neglecting this phase of the problem and, therefore, a study of reversals was undertaken at the College of the City of New York as soon as the eighteen tests made by the writer were completed. These tests were made by Professor Peck who has presented his findings in a separate discussion.

The question of whether the top or bottom attachment is responsible for the largest percentage of deformation was not studied and was not treated in the paper although the necessary data were taken during the tests. This phase of the problem did not have a bearing on the work that the writer had in mind. As illustrated in the curves, the right half and the left half of the specimen each constituted a separate test of a connection. These tests were both plotted and published. This was also done in the case of reversed loading, but in the latter case, only the average curve is published, for the sake of clearness.

Professor Mirabelli demonstrates, by means of curves, the effect of the change in the properties of the connections as the load is increased, with the conclusion that the effect is less as the load increases. This is a material contribution to the paper. His statement that the writer's percentages are based on rigid columns is true; the necessary changes can be made when the design is known. This makes the data different from those of the problem investigated by the writer, with the result that different answers will be obtained.

The third point indicates that the transfer of the moment from the beam to the column will increase the size of the column. The fact is that the moment is in the column whether it is computed or not. It follows, therefore, that it is in the interest of economy and safety to evaluate this moment as nearly as possible when designing columns. This is an argument against the present practice of designing beams with connections considered as having a rigidity of either infinity or zero.

It is true, as stated, that the more accurate analysis is the more expensive and that the approximate methods will be used in most commercial work. The question that must be solved in each individual case is whether the more correct solution is worth the expense involved.

Mr. Moore has given an entirely different approach to the problem in which he shows that similar investigations are being conducted in aluminum. He introduces the degree of fixity, or the ratio of the bending moment developed in the spliced beam, over the support, to that obtained in a continuous beam. To obtain this ratio he observed the stresses at about eight sections of the beam and computed back to the moment at the connection. This avoided the errors that might have arisen from measuring directly at the connection itself. It is unfortunate that the mathematical equations resulting from this method of attack were not given so that they might be compared with those in the paper, the best in each system being retained.

The tests made by Professor Peck were designed to obtain curves that can be compared directly with those of the paper, furnishing information on the effect of reversal of moment on the connections. The only difference between the tests of Specimens 13 and 14 and those offered by Professor Peck is the ratio between shear and moment. The reason for the change in length of the specimens from those previously used was that the early tests indicated serious deformation in shear before the moment curves had been developed as much as was desired. Although the shear may have some effect on the action of the connection, the writer believes that this question can be approached from a theoretical angle as was done by Professor Batho, but an experimental approach would be very expensive and difficult.

The decrease in the slope of the loading curve, when the connection receives high stresses, is of interest. This is only one of the properties of connections that must be considered when designing a structure. The history of the connections is of necessity a part of the problem of stress analysis. This history is not difficult to anticipate when the frame with its anticipated loads is given.

Tests of the type discussed in the paper are, by their nature, expensive and, therefore, the number cannot be great for financial reasons, although tests are always of value. The writer does not agree with Professor Peck that a long series of laboratory investigations are necessary before the theory can be used in practical design. The question is entirely an economic one.

Sufficient tests have been made so that a general idea of the constants of the connections can be anticipated over the range covered by these tests and even beyond.

It is hoped that the properties of riveted connections will receive further study from the profession because, undoubtedly, they affect the distribution of stress. It is the writer's opinion that the greatest progress can be made by perfecting the method of analysis rather than in increasing the data by laboratory tests. When a method of analysis that is simple and easily applied is devised, it will be accepted by designers. This does not mean that further tests are not of great value but rather that the value of further research in the theory is greater.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1934

HYDRAULIC LABORATORY RESULTS AND THEIR VERIFICATION IN NATURE

BY HERBERT D. VOGEL,[1] ASSOC. M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. W. F. HEAVEY, CHILTON A. WRIGHT, PAUL S. REINECKE, MORROUGH P. O'BRIEN, JOHN A. JAMESON, JR., J. C. STEVENS, PAUL W. THOMPSON, SAMUEL SHULITS, AND HERBERT D. VOGEL.

SYNOPSIS

Data contained in this paper were assembled in 1933, added to in January, 1934, and again augmented in the summer of the latter year. Nevertheless, it is quite certain that the intervening months have produced many new facts corroborating the information set forth herein, and while it was not originally intended to present anything like a complete inventory of cases, still it would appear desirable to bring out all new facts in the order of their determination. From this standpoint it would seem that the proper purpose of the paper is to discuss in general the principles of verification of hydraulic model results, to present a number of concrete examples, and to open the way for subsequent discussions. Since the subject can never become closed, it would be presumptious to claim more than a "scratching of the surface" in the present instance. In spite of this, the hope is held that the following pages will serve to assure the profession that efforts are being made to verify the reliability of results obtained by hydraulic model experimentation.

INTRODUCTION

What reliance can be placed by field engineers on the solutions obtained from model studies and what proof is there of the statement that a river will re-act to treatment the same as its miniature in a laboratory? These are the questions most frequently asked of those engaged in the task of wresting river secrets from small-scale models, and—strangely enough—they are the

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NOTE.—Presented at the meeting of the Waterways Division, New York, N. Y., January 18, 1934, and published in January, 1935, *Proceedings*.

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most difficult to answer convincingly. Not that insufficient work has been done in laboratories to afford proof of one kind or another, but positive data and positive facts are so few in the general case as to make positive proof next to impossible. It is noteworthy that of the topics assigned for discussion at the XVth International Congress of Navigation (Venice, 1931), one was:

> "The study of hydrotechnical questions by means of laboratory researches on reduced scale models. Comparison of the results of such researches with those of direct observations of the natural phenomena, with a view to ascertaining how far the law of similitude is true", * * *

and that of all the papers submitted only one or two authors showed the hardihood to come down to definite cases of comparison. Furthermore, that German paper which most boldly attacked the subject offered as one of its strongest items of evidence the experiments for determining the spillway discharge of Keokuk Dam, in Iowa, reported in 1929 by Albion Davis and the late Floyd A. Nagler, Members, Am. Soc. C. E.[2] It was shown, as a result of tests on a 1:11 scale model and the actual structure, that there was close agreement between discharge coefficients and other hydraulic functions in model and Nature.

There are many reasons for the difficulties in citing actual instances of complete verification, in Nature, of model findings. Were this not true it would follow either that model tests are entirely unreliable, or that no attempts have been made to present evidence of verifications. The latter assumption having been shown to be in error, and the best authorities having agreed that beneficial results may be obtained from a well conducted model study, it remains to cite a few of the more common reasons for the difficulties that have been encountered.

To begin with, only the more difficult problems are submitted to the laboratory for analysis by experimental means. This is quite natural, of course, since there is no reason for making model studies of problems which present no doubtful elements, or for which there are already satisfactorily established practices. In the second place, most of the problems submitted to the laboratory are accompanied by schedules of detailed proposed plans, each of which is to be tested separately in the model. As a result of the experiment it may be necessary to modify some of the plans in order to find one or more that will produce the desired results. By the time any of these plans has been accepted and instituted in the field it has almost invariably been somewhat further modified to suit the character of changes in local conditions that have taken place in the mean time. Naturally, the difficulty of comparing effects of the work finally installed with those indicated as being most suitable by the model increases with the degree of the modification.

Thirdly, the time required in Nature for structures to become effective differs greatly with the type of the works involved and the character of local conditions. Works, for instance, which are designed to produce changes in the river bed by inducing deposition of silt would be more quickly effective on the heavily silt-laden streams of the West and Southwest than on such comparatively clear rivers as the Ohio, or the Upper Mississippi. If properly located, spur-dikes installed to increase depth of channel at low water by their scouring action, become effective relatively soon in rivers having mud or sand bottoms, whereas in localities in which there are deposits of gravel such dikes may achieve the desired results only at a much later date.

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[2] *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 777.

To obtain the results, in Nature, indicated by a model study, the improvement works should be installed as soon as possible. This is particularly important when the river, or the part involved, is changing actively. In a large alluvial river, such as the Lower Mississippi, delay of one or more working seasons in instituting a plan may be attended by such extensive changes in the river as to affect the success of the proposed plan materially.

In the case of the United States Waterways Experiment Station, experimental work was begun in January, 1931, and the first experiments were not completed until from six months to a year later. A similar period necessarily elapsed before any works, tested during these experiments, could be installed. Consequently, sufficient time has not elapsed to demonstrate conclusive results in every case. However, proofs are not entirely lacking, and a number of cases in which field checks on model results have been obtained, either partly or completely, will be discussed in this paper.

VERIFICATION OF MODEL RESULTS

General.—The present Mississippi River navigation program (adopted January 21, 1927) provides for the procurement of a navigable channel 9 ft deep and 300 ft wide, for all stages of the river, between St. Louis, Mo., and Baton Rouge, La. In this program the principal reliance has been placed on regulatory works, supplemented, where necessary, by channel dredging. Standard practice has been generally followed in the design and location of these works, but in more and more instances, and especially when unusual or baffling conditions are presented, the problems are referred to the U. S. Waterways Experiment Station for study and recommendations.

A technique was quickly evolved for the study of this particular type of problem, because new methods, involving movable-bed models, were required, and special questions, such as those of limiting slopes and of character of sand necessary for the sand bed, required almost immediate answers. Where exigencies of the situation demanded, it was found possible to design and construct a model within less than a month after the receipt of survey data from the field, and to begin the release of experimental data immediately thereafter. This is seldom desirable, however, and with less haste necessary, experiments proceed at a more leisurely pace, some of the studies having been extended for as long as $1\frac{1}{2}$ years.

In certain instances, models have been operated concurrently with the installation of improvement works in the prototype. In such cases, the various construction details, together with any natural changes, have been reproduced as they progressed in Nature, and findings have been forwarded to the field engineers either by mail or by radio.

Field representatives of the various districts for which models have been built are frequent visitors to the Laboratory for the purpose of inspecting the progress of the experiments. Only rarely do these visiting engineers fail to point to some phenomena in the model which are verified by existing conditions in Nature. It may, perhaps, be the formation of a sand bar or a gravel bar, an eddy, a caving bank, or scour in a certain vicinity; or, it may be the development of the general configuration of the bed of the model which bears such a close resemblance to that of the natural reach that it becomes difficult to distinguish between maps of the two. It is noticeable that these engineers nearly always find in the model a simulation of those characteristics of the river with which they are most familiar. This fact lends an added value to such testimonials. Numerous instances are on record in which developments in the natural stream have been predicted from the model studies many months in advance of their actual occurrence.

Hydraulic models have two general functions: First, by their aid a more exhaustive and detailed estimate and evaluation may be made of existing conditions in the prototype, and, as a result of the information thus secured, plans for improvement may be conceived with greater likelihood of effectiveness and economy than would otherwise be possible; and, second, the merit of these plans may then be determined in the model itself. Before a model can be relied upon, confidently, to give such important information, its capacity to simulate known past and present conditions must first be demonstrated. In laboratory parlance, the ascertainment of such facts is termed the "verification of the model".

Theory of Verification.—The first step in the verification of a model is a manual one, that of checking it dimensionally to insure that geometric similarity between the model and the corresponding region in Nature has been obtained. Following this preliminary check, to which every model is subjected, it remains to prove that hydraulic similarity exists, or that results of tests conducted on the model may be used as the basis of designing regulatory works for the full-sized river, harbor, etc. In general, this hydraulic similarity can be verified in any one or all of several manners.

Verification (1).—In the case of a movable-bed model, checks can be made by subjecting it to "moulded-bed" or to "flat-bed" verification runs. If two or more surveys of the vicinity are available, the first method is used. The movable-bed part of the model is moulded to conform to known conditions of the earlier date; the model is subjected to several cycles of flow at varying stages, representing average hydrographs in Nature; it is considered trustworthy only if its capability to reproduce the known conditions of the later date can be demonstrated.

In instances in which field information taken at different dates is not available, the second method is used. The movable-bed part of the model is moulded flat; water is passed through the model at varying stages in simulation of average conditions in Nature; and the test is completed when the model has moulded its bed so that it conforms to the hydrographic conditions in Nature at the time of the survey from which the model was constructed. The reasoning behind such a test lies in the fact that a stream adjusts the configurations of its bed in accordance with its hydraulic regimen and the alignment of its banks. If the dimensional check of the model proves its geometric similarity to the prototype and, if hydraulic similarity exists, the model should mould its own bed in simulation of Nature. All the movable-bed models studied at the U. S. Waterways Experiment Station are subjected to one or the other of these verification tests, and many of them are given both checks.

Verification (2).—The more practical application of this type of check is obtained when a comparison can be made between conditions as they develop in a stream after the completion of a model study and the conditions which the model indicated would develop. This type of field verification can be made in the case of: (*a*) Movable-bed models by comparing bed configurations in streams with the configurations predicted from model studies; (*b*) fixed-bed river models (when it is desired to determine the changes in gauge heights, velocity distributions, etc., resulting from cut-offs or other regulatory work) by a comparison of model predictions with actual developments in Nature; and (*c*) models of spillways, turbines, etc., by similar comparisons.

The work of the U. S. Waterways Experiment Station is too new to have yielded many conclusive verifications of this kind. However, several such comparisons are available, and some of them will be discussed in detail. It should be pointed out that these two types of verifications are fundamentally the same. In each case, a comparison is made between the development of the model and the corresponding development of the prototype. The only difference lies in the time at which the model study is made—whether at the end of the period of development, or at the beginning.

Verification (3).—A third check may be obtained if two or more models to different scales have been used for the study of the same problem. A transference of the results of the small-scale model (by means of the laws of similitude) to the scale of the larger model provides a check on the accuracy of the results of both models. If the two are in fair agreement, it is reasonable to assume the next step in the extrapolation of the model data to the full scale of the prototype. Several examples of each type of verification are described herein.

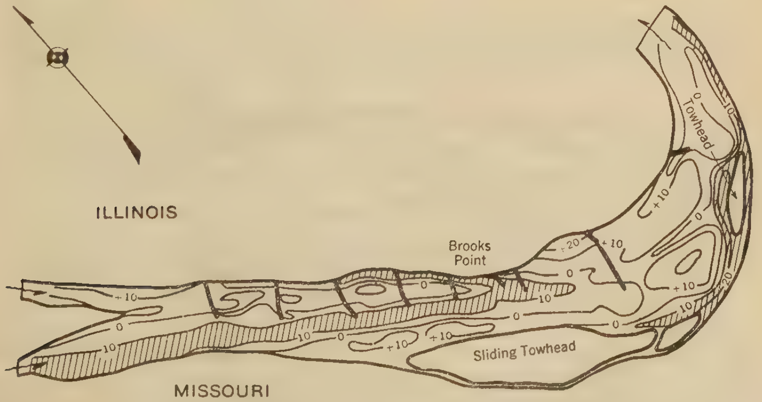

Fig. 1.—Condition Existing in Nature, September, 1932, in the Mississippi River at Brooks Point, Ill. (Contour Elevations Are Referred to Local Low-Water Plane. Hatched Areas Represent Depths Greater Than 10 Feet).

The Brooks Point Model.—The model of Brooks Point, Ill. (Fig. 1), was built in simulation of a 10-mile stretch of the Mississippi River lying between Mile 20 and Mile 30 above Cairo, Ill. The purpose of the model study was to devise a means for improving the depth of the navigation channel over the crossing at that point during low-water stages. Only one general survey of the region was available to the Laboratory. This was made in September, 1932, and the model was constructed from the data thereof. Fig. 1 shows the conditions in the prototype as revealed by this survey. Lacking any other survey data, the model was subjected to the flat-bed verification test. At the end of 100 hr of operation, it was found that the model was in close similarity with Nature. The map made of its bed at this time is shown in Fig. 2. Attention is invited to the faithfulness with which details in the full scale were simulated by the model.

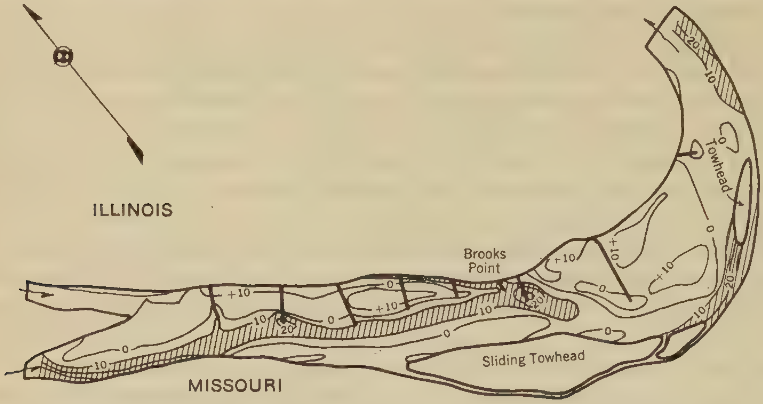

FIG. 2.—FLAT BED VERIFICATION RUN, MODEL OF MISSISSIPPI RIVER AT BROOKS POINT, ILL. (CONTOUR ELEVATIONS ARE REFERRED TO LOCAL LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 10 FEET).

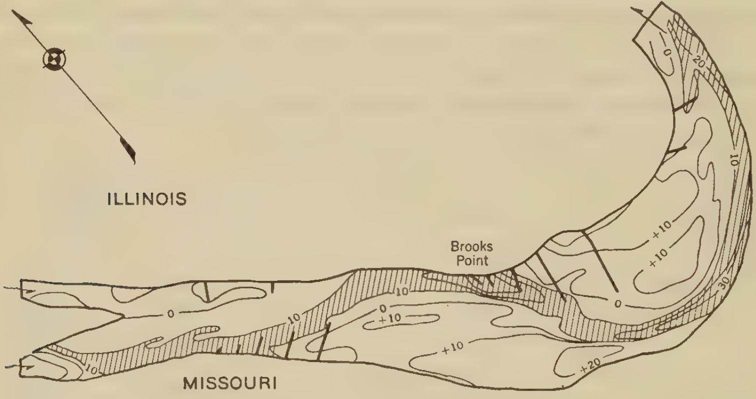

FIG. 3.—INDICATED SOLUTION, BY LABORATORY TEST, OF CHANNEL IMPROVEMENTS AT BROOKS POINT, ILL. (CONTOUR ELEVATIONS ARE REFERRED TO LOCAL LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 10 FEET).

While the experiments on the model of Brooks Point were in progress the Station was visited several times by the engineers in charge of the channel improvement works in the vicinity represented by the model. They were unanimous in voicing their opinion that the model simulated to a marked degree the conditions of Nature.

This model was successful in demonstrating several practicable plans for the improvement of the channel at Brooks Point. One of these, which provided for the construction of five spur-dikes on the right bank just above the shoal crossing, and the removal of three existing dikes on the left bank at a point just opposite, was selected. The details of this plan are shown in Fig. 3. In order to preserve the best conditions for navigation, the removal of the three dikes was conducted concurrently with the construction of the five new dikes. The new dikes were completed in August, 1933, but only sections of the old dikes had been removed at this time. A survey of this vicinity, made about two months later (October, 1933), is shown in Fig. 4. It can be seen that in this short interval of time the river has shown unmistakable signs of adopting the new channel alignment. Fig. 5 is a view of this model with new spur-dikes in place and being tested.

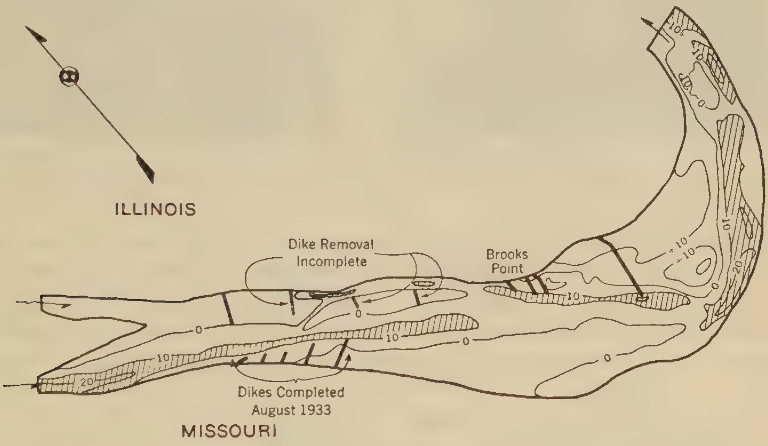

Fig. 4.—Conditions Existing at Brooks Point, Ill., October, 1933 (Contour Elevations Are Referred to Local Low-Water Plane. Hatched Areas Represent Depths Greater Than 10 Feet).

The Fort Chartres Model.—A problem similar to that at Brooks Point was presented in the case of the study for channel improvements at Fort Chartres, Ill. (see Fig. 6). This model represented that section of the Mississippi River between Miles 120 and 137 above Cairo, Ill. The purpose of the experiment was to determine methods for improving channel depths over the crossing at Fort Chartres, East, situated at about Mile 131. The model was built in accordance with a survey of this region (see Fig. 7) made during October, 1932. The experiment showed that a navigable channel could be procured at low stages if a system of spur-dikes was built on the

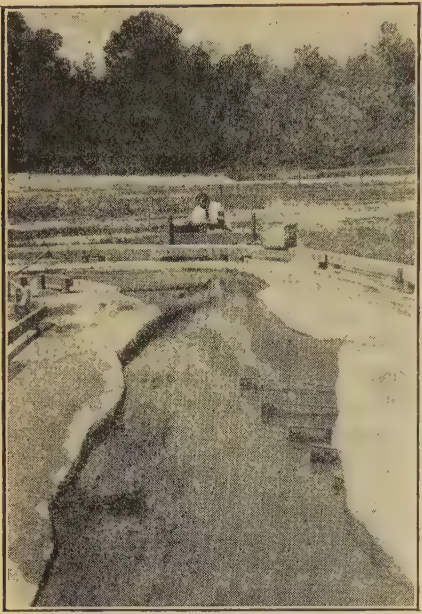

FIG. 5.—LOOKING DOWN STREAM AT BROOKS POINT MODEL. PROJECTED SPUR-DIKES IN PLACE AND UNDER TEST.

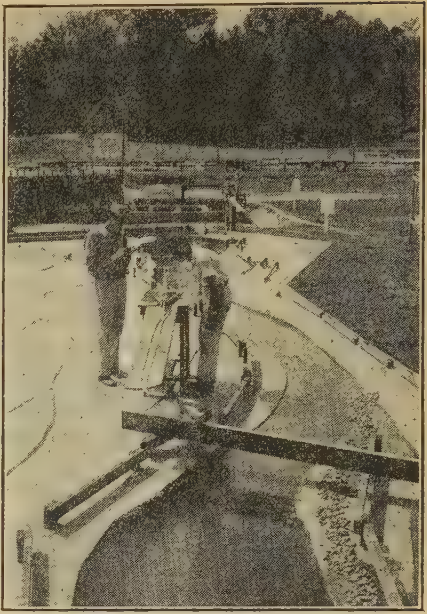

FIG. 6—MEASURING VELOCITIES IN THE FORT CHARTRES MODEL WITH PITOT TUBE AND INCLINED MANOMETER.

right bank between Miles 130 and 133. The details of the proposed system of dikes and the channel which they would effect, as indicated by the model, are shown in Fig. 8. The plan was adopted for construction with slight modification, and work was begun early in the working season of 1933. A survey showing the status of construction and existing channel conditions was made of this region in March, 1934, and is shown in Fig. 9. Although sufficient time has not elapsed for the dikes to become completely effective, it will be noted that they have already caused a relocation of the channel along the desired alignment and that project depths are being obtained along this trace.

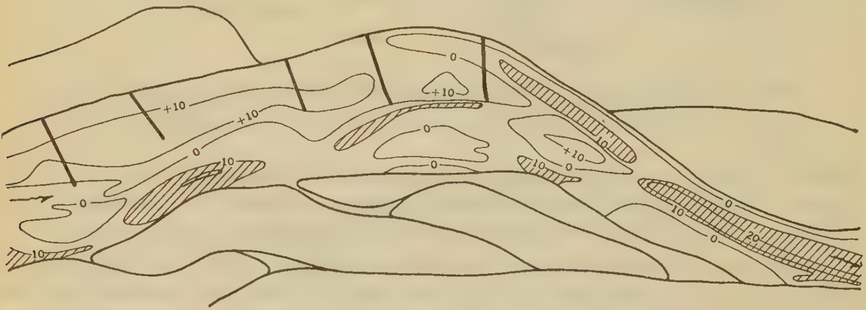

FIG 7.—CONDITION OF RIVER BED AT FORT CHARTRES, ILL., AS SHOWN BY SURVEY OF OCTOBER, 1932.

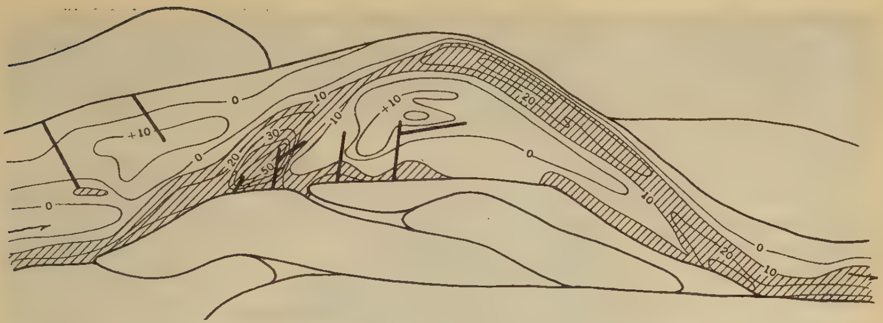

FIG. 8.—PLAN FOR IMPROVEMENT OF CHANNEL AT FORT CHARTRES, ILL., INDICATED BY MODEL TEST.

The Point Pleasant Model.—The Point Pleasant (Mo.), reach of the Mississippi River lies about 80 miles below Cairo, Ill. It is one of several localities in which the low-water channel is characterized by excessive width and consequent deficient depth. A system containing three permeable pile-dikes with a combined length of 8 232 lin ft was completed during 1932 for the improvement of depths over the crossing-bar. Surveys made during August, 1932, showed no improvement in the channel depth opposite the dikes and their ability to improve the channel was questioned. Some authorities thought that the dikes should have been placed on the opposite side of the stream, and advocated their removal. At the direction of the President of the Mississippi River Commission, a model was constructed which simulated an 8-mile stretch of the river, extending from Mile 75 to Mile 83, with the Point Pleasant dikes near the center (see Fig. 10). After the model had been properly verified, it was moulded to conform to conditions shown by the latest field survey, which was made during August, 1932 (see Fig. 11). The findings of the model not only substantiated conditions in the field, but indicated that the Point Pleasant dikes were properly designed and located, and eventually would cause the channel to scour to navigable depths (see Fig. 12).

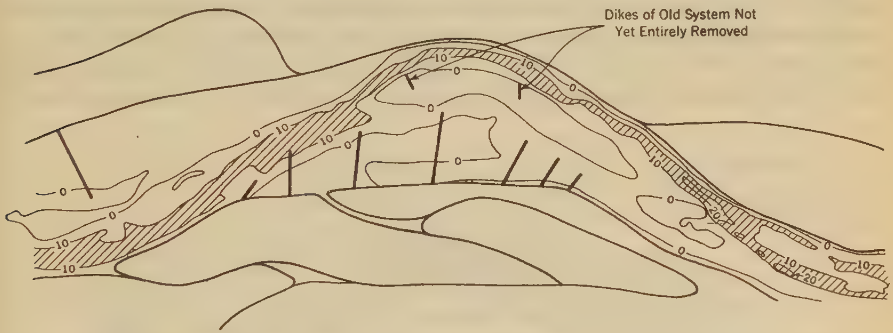

FIG. 9.—PLAN UNDER EXECUTION IN NATURE, CHANNEL AT FORT CHARTRES, ILL., SHOWING CONDITION IN MARCH, 1934.

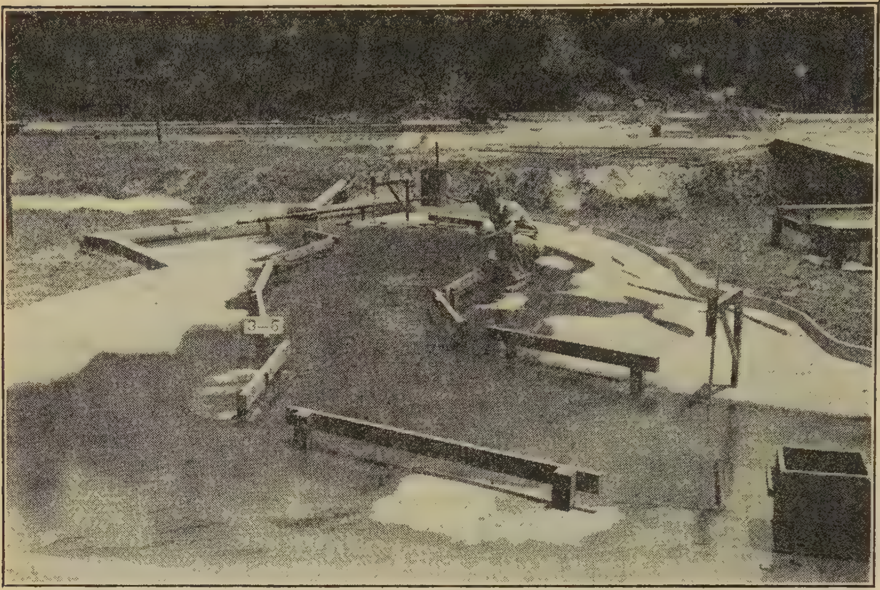

FIG. 10.—VIEW OF POINT PLEASANT MODEL.

Construction of the Point Pleasant model was begun during the first months of 1933, and the experiment was completed before the recession of the annual high water. The model indicated that with the advent of periods of low water the dikes in the river would develop a navigable channel with controlling depths of about 10 ft below mean low water. A survey made by the Area Engineer during July, 1933 (see Fig. 13), showed the crossing-bars to be shaped similarly to those indicated by the model, and revealed depths of 10 ft almost entirely across the critical section. During the next month a continuous channel, 10 ft deep, would probably have developed,

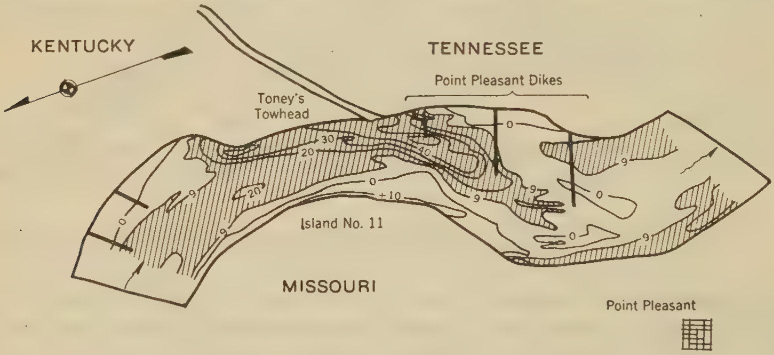
KENTUCKY
TENNESSEE
Point Pleasant Dikes
Toney's Towhead
30
20
40
0
+10
9
20
Island No. 11
MISSOURI
Point Pleasant

FIG. 11.—CONDITION AT THE POINT PLEASANT, ILL., REACH, IN AUGUST, 1932 (CONTOUR ELEVATIONS ARE REFERRED TO MEAN LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 9 FEET BELOW MEAN LOW-WATER).

as scouring occurs on the crossings during low water. This would have given a complete verification of the model predictions. In order that there should be no jeopardy to navigation, however, some dredging was done at that time (July). Thereafter, the Area Engineer reported under date of October 27, 1933, that the channel gave no trouble whatever. In fact, as of that date, it had scoured approximately 3 ft. deeper. This was the first time in years, that the crossing was not dredged several times.

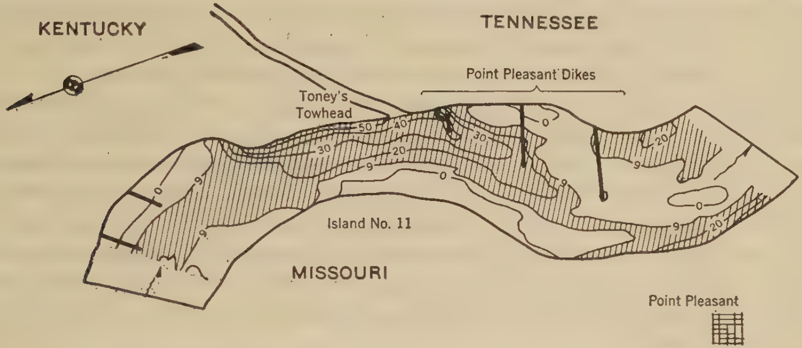
KENTUCKY
TENNESSEE
Point Pleasant Dikes
Toney's Towhead
50
40
30
20
9
0
Island No. 11
MISSOURI
Point Pleasant

FIG. 12.—LABORATORY TEST OF POINT PLEASANT, ILL., MODEL INDICATED EVENTUAL EFFECTIVENESS OF STRUCTURES IN ABOUT ONE YEAR (CONTOUR ELEVATIONS ARE REFERRED TO MEAN LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 9 FEET BELOW MEAN LOW-WATER).

Thus, there is—in the foregoing case—an instance in which, through the instrumentality of a model, the eventual effectiveness of existing structures was shown. Such results may be termed "ratifying". The aggregate cost of the Point Pleasant dikes, at an average cost per linear foot of between $30 and $35, was approximately $265 000. Since it is frequently as expensive to remove dikes from a stream as to build them, it would seem that results which lead to such economies are of as great value as those which indicate new plans of attack.

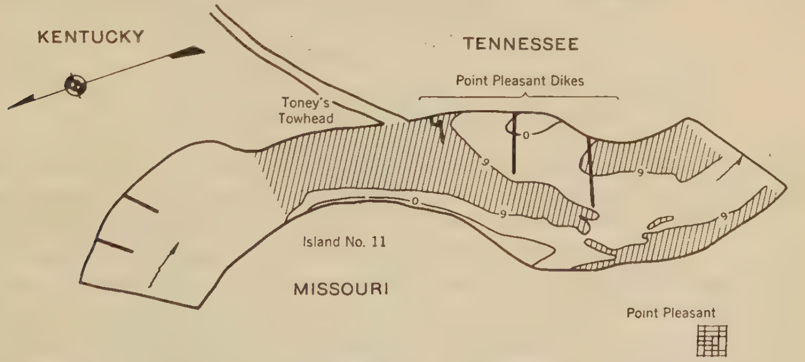
KENTUCKY
TENNESSEE
Point Pleasant Dikes
Toney's Towhead
9
0
Island No. 11
MISSOURI
Point Pleasant

FIG. 13.—CONDITION OF CHANNEL IN POINT PLEASANT REACH, IN JULY, 1933, ELEVEN MONTHS AFTER ORIGINAL SURVEY (CONTOUR INTERVALS ARE REFERRED TO MEAN LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 9 FEET BELOW MEAN LOW-WATER).

The Morrison Towhead Model.—Another instance in which a model gave so-called "ratifying" results, is that of the model of the Morrison Towhead. The contraction works program on the Lower Mississippi River provides for the closure, where indicated, of certain secondary or back channels, in order to increase the volume of flow in the main channel during low stages. At Morrison Towhead, 70 miles below Cairo, Ill., it was proposed to block the flow back of the island (Morrison Towhead) by the construction of a single, permeable, deflecting dike at the entrance to the channel.

Before arriving at a decision, however, a model study was undertaken which had for its objective the determination of the proper location for the dike. The model indicated that no bed-load was passing into the back channel and that such deterioration as this channel was experiencing was due entirely to sedimentation which would be retarded by the dike. On the strength of these findings the construction of the dike was omitted. Subsequent surveys were made of the vicinity during May, 1932, and June, 1933. Comparison of these surveys with previously existing conditions affirmed the prior findings of the model.

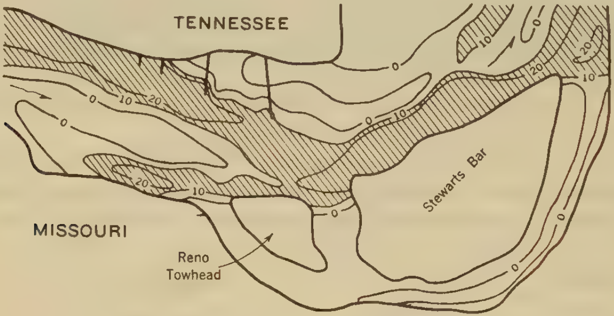

FIG. 14.—PLAN FOR IMPROVEMENT, STEWARTS BAR, AS INDICATED BY LABORATORY TEST MADE IN 1932.

The Stewarts Bar Model.—Stewarts Bar is an island about 2 miles long, situated in the Mississippi River approximately 93 channel miles below Cairo, Ill. (see Fig. 14). At about Mile 91 the stream enters a bend and the channel swings over from the left to the right bank. The crossing-bar, which is situated near the head of Stewarts Bar, has been a hindrance to navigation for many years and periodic dredging has been necessary during low stages. Several alternate arrangements of pile-dikes were proposed for the improvement of this reach. The situation was complicated by the fact that over-contraction might re-open the secondary channel behind Reno Towhead and Stewarts Bar and so reduce the flow in the main channel as to cause a deterioration rather than an improvement of navigation conditions. The problem was referred to the Laboratory for study and report.

A movable-bed model, having a horizontal scale of 1:1 000 and a vertical scale of 1:100, was constructed and tests were made during 1932 of each of the proposed plans. In these tests the engineers of the Laboratory considered not only various locations and lengths for the dikes, but dikes of different height and inclination with respect to the flow of the current. A report was rendered which indicated that any one of several of the proposed plans would be effective in producing the desired results.

One of the plans which indicated satisfactory results provided for the construction of four spur-dikes on the left bank between Miles 90.8 and 92.0 (see Fig. 14). With some slight modifications this plan was adopted, approved for construction, and the dikes were completed in December, 1932.

To ascertain what effect the dikes were having, a re-survey was made of the Stewarts Bar Crossing during November, 1933 (see Fig. 15). An examination of the results of this survey shows that the dikes had already secured a channel of the desired depth. Comparing Figs. 14 and 15 a remarkable agreement will be found. In this connection, it should be borne in mind that the full effects of a system of dikes are seldom realized during the first season.

Under date of October 27, 1933, the District U. S. Engineer's Office, at Memphis, Tenn., reported that no dredging was done in this reach during the low-water season, 1933, because, for the first time in years, the channel was in such excellent condition as to cause no concern. The dikes were noted to have been very effective in inducing a heavy fill, and had maintained that crossing at a depth of 10 ft or more all that season, despite extreme low-water conditions.

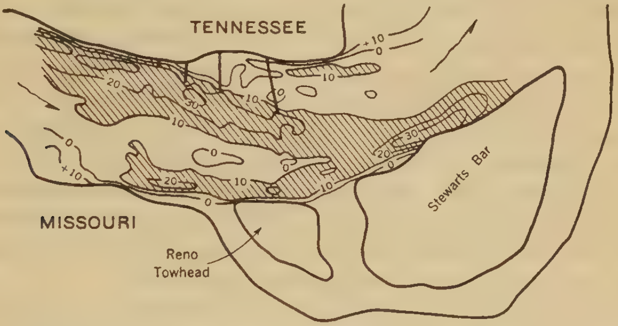

FIG. 15.—CONDITION OF STEWARTS BAR AS SHOWN BY SURVEY OF NOVEMBER, 1933.

The Walkers Bar Model.—The project for the canalization of the Ohio River, from Pittsburgh, Pa., to Cairo, Ill., was completed in 1929. This project was not designed to provide the required depth of 9 ft at all bars, because it was judged more economical to secure the needed depth at the worst crossings by dredging rather than by increasing the height or number of dams. At one location, Walkers Bar, where excessive dredging had been required annually, it was thought that impermeable dikes might provide the required depths more economically.

At the request of the District Engineer of the Louisville, Ky., District, a model study was undertaken to ascertain the feasibility of the plan, and the most effective system of dikes for the improvement of this vicinity. An engineer from the Louisville District familiar with local conditions was detailed to assist in the experiment. The experiment extended over a 17-

month period and was completed in September, 1932. During this time twenty-two separate plans were studied exhaustively and at least two were found which, in the model, gave the desired results.

On May 31, 1934, the U. S. District Engineer Office, at Louisville, reported on an up-to-date survey of Walkers Bar to note the effect of erecting dikes, and the extent of the scour and fill that had occurred since the dikes were put in operation. Benefits to the channel were unquestioned although the survey was made before the end of the high-water season.

The Island No. 9 Model.—For many years deficient depths during the low-water seasons at the Donaldson Point Crossing (Mile 56 below Cairo, Ill.) have proved a hindrance to navigation. In the past it has been the custom to provide relief during low stages by channel dredging. Since the low-water season of 1927, a total of 1 750 000 cu yd of material has been dredged at this crossing. The crossing is situated near the lower end of a reach which, for about 10 miles, lies practically in a straight line. Throughout the greater part of the reach the average width between banks at low stage is approximately 3 000 ft. For a short distance above Donaldson Point Crossing the channel width is about 5 000 ft at low water. The program for the improvement of this reach proposed to eliminate the necessity for annual low-water dredging by increasing the depth over the crossing by constructing four permeable pile-dikes in the part of the reach having excessive widths.

From an engineering viewpoint, the improvement of the reach presented no unusual difficulties. An outstanding factor in connection with the plans was the fact that although only slight increase in depth was desired, a total of 8 500 lin ft of dike seemed necessary to produce the needed results. Hence, the problem was largely one of engineering economics.

After the problem was submitted to the Laboratory, the experiment was conducted with the objective of finding a more economical method of producing the same effect as would be obtained by the proposed 8 500 ft of dike. Several modified plans were found, all of which indicated that the required deepening could be attained at a considerably lower cost than was originally contemplated. The most promising of the several plans indicated that four short pile-dikes on the opposite side of the stream, having combined lengths of only 1 600 ft, built in conjunction with five sand and gravel dikes, would give the required deepening and, at the same time, would furnish protection to the bank in a certain locality where it was being actively caved.

This latter plan, modified somewhat as to physical appearance, but similar as to hydraulic effect, was adopted for field construction, and work was begun in the early part of 1933. A re-survey of the area was made during June, 1933. Although it was too early at this time, of course, to expect that the effects of the works had been accomplished, the survey showed that the structures were tending to produce the desired result although several seasons may be required for its consummation.

Cut-Off Studies.—One of the most excellent opportunities for comparing effects produced by identical changes on similar models of differing scales is that afforded by the experiments designed to determine the results of artificially made cut-offs on the Mississippi River. Among the first models built at the U. S. Waterways Experiment Station was a small replica of the Greenville Bends, its horizontal scale being 1:4 800 and its vertical scale 1:360. With this model, tests were made to determine the effects of all possible cut-offs in the Greenville Bends region.

Some months later a model study was ordered to determine effects of ten dredged cut-offs in the Mississippi River between Rosedale, Miss., and Point Breeze, La. These ten newly proposed cut-offs included two that had been tested by the first model. The second model was built outdoors to scales of 1:2 400 and 1:120, and constitutes, even at this time, the largest model of its kind in the world.

During 1933, two new models were built to a considerably larger scale than had been attempted previously for a study of this kind. These last models, designed to accommodate movable beds of sand, were constructed to a horizontal scale of 1:1 000, and a vertical scale of 1:100. The purpose of their operation was to determine the amount of bed lowering that might be expected up stream from the several cut-offs. Incidentally, water-surface lowerings were also observed in order to check the results previously obtained from the other models. Tables 1 and 2 show comparisons between results obtained from the several models.

TABLE 1.—COMPARISON OF EFFECTS ON GAUGE HEIGHTS OF CUT-OFFS IN THE GREENVILLE BENDS FOR A 1929 FLOOD DISCHARGE

River gauge	Cut-Off at Leland Neck		Cut-Offs at Leland and Tarpley Necks	
	1:4 800 model	1:2 400 model	1:4 800 model	1:2 400 model
96	0.0	0.0	0.0	+0.2
99	—2.5	—2.1	—3.2	—3.0
101	—2.2	—2.3	—3.6	—3.0
103	—2.2	—1.9	—2.9	—2.2
104	—1.8	—1.7	—2.1	—2.3

TABLE 2.—COMPARISON OF EFFECTS OF CUT-OFFS

River gauge	River mile	Stage Changes, in Feet	
		1:2 400 model	1:1 000 model
(*a*) Effect of a Flood Discharge Equal to That of 1929, at Diamond and Yucatan Points			
80	574.9	—1.9	—1.85
73	617.9	—3.5	—3.2
(*b*) Effect of Cut-Offs at Giles Bend, Under the Conditions of a 40-Foot Stage, at Natchez, Miss.			
67	657.2	—2.5	—2.5
St. Joseph	662.4	—2.6	—2.5
64	681.4	—3.9	—4.0
62	687.8	—4.0	—4.0

Following tests on the 1:2 400 model, certain cut-offs were actually made on the Mississippi River. Subsequent observations have been in line with what would be expected for the current state of development.

Hydraulic Flume Tests Corroborated by Observations on Rhine River

Not all cases of similarity between model and prototype are as apparent as the foregoing, but, on the other hand, numerous instances are on record in which experimental data have been checked daily by observed phenomena in Nature. An interesting case of this kind occurred in connection with studies of Island No. 35, up stream from Memphis, Tenn. (see Fig. 16). The U. S. Waterways Experiment Station has a tilting hydraulic flume as a part of its fixed equipment. This flume was used in an extensive series of tests which had as its objective the study of the tractive force of flowing water and the transportation of bed-load material in natural streams. In one phase of the experiments the procedure was as follows: The bottom of the flume was covered to a given depth with sand which had been subjected previously to a mechanical analysis. The inclination of the flume was set to a designated slope, which remained fixed until the completion of the run. Water was then passed through the flume and regulated by means of the weir and the tail-gate so as to have the exact slope of the sand bed. The depth of the water was next increased by successive increments, observations being made and recorded of all essential hydraulic data for each observed depth. Fig. 17 shows in graphic form the results of a typical run. In this diagram depths are plotted as abscissas, and mean velocities, rates of bed-load movement, and experimentally derived values of Manning's n, as ordinates.

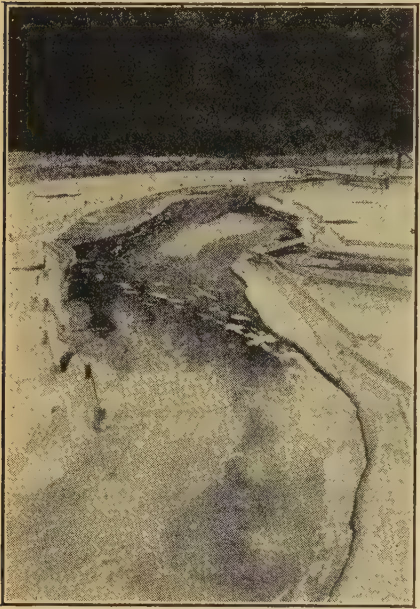

Fig. 16.—Model Study of Navigation Difficulties Encountered at Island No. 35, Mississippi River.

Several interesting facts were brought to light by this experiment. It was noted that the mean velocity curve broke first at the point where the type of flow changed from laminar to turbulent; the next break in this line occurred simultaneously with the beginning of movement of the bed material. Most authorities are in agreement that when movement of the bed material in a stream begins there is an accompanying reduction in the rate of increase of the mean velocity of the water, with a corresponding change in the roughness coefficient of the stream. This was plainly shown to have occurred in all the foregoing experiments. Although the graphs shown in Fig. 17 are the results of only a single test, several like experiments, in which different sands and different slopes were used, were also performed, and the results were similar in each instance. Fig. 18 is a graph presented by Ph. Forchheimer,[8] which shows data taken on the Rhine River at Basle, Switzerland. The data have been plotted in such a manner that the slope of the curve is a measure of the roughness. The break occurs at the time movement of the river bed begins. In commenting on this graph, Forchheimer states[8] in effect that Kutter knew that the coefficient, C, in the Chezy formula decreased when the "geschiebe" (bed-load material), began moving; du Boys, also, was familiar with this fact and mentioned it in his writings; and it is common knowledge among hydraulic engineers that the roughness of a river bed changes as the stage increases.

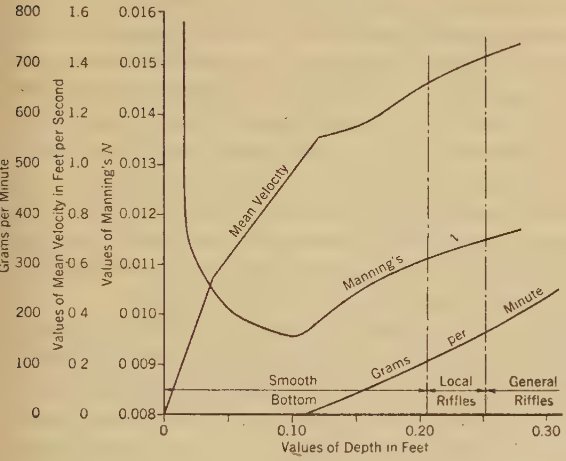

FIG. 17.—HYDRAULIC FLUME TEST OF BED MATERIAL, MISSISSIPPI RIVER, ISLAND 35.

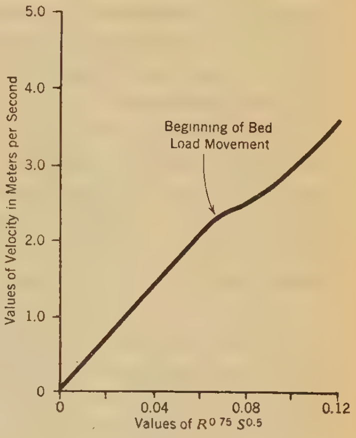

FIG. 18.—RHINE RIVER AT BASLE, SWITZERLAND.

Fig. 18, the result of actual observations, was introduced to show that at the point where bed movement begins there is a change in the variation of the roughness coefficient.

CONCLUSIONS

In the foregoing no attempt has been made to present an inventory of field verifications pertinent to model investigations conducted at the U. S. Waterways Experiment Station. That the list is incomplete is obvious and, in many cases not cited, sufficient similarity has been observed in Nature to compel faith in results of the tests. In some instances verifications have been more difficult—if not altogether too difficult—to obtain. The main point to note is that the science, which was a veritable "infant" only a few years ago is now (1935) a lusty "'youngster", rapidly coming of age, and demanding that its importance be recognized by every one.

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[8] "Hydraulik", von Ph. Forchheimer, Leipzig und Berlin, 1930.

DISCUSSION

W. F. Heavey,[4] M. Am. Soc. C. E. (by letter).—In 1933, an undistorted model of more than two miles of the St. Clair River, in Michigan, was constructed at the United States Waterways Experiment Station, which is the underlying subject of Lieut. Vogel's paper. The scale of the model was 1:100 and its purpose was to predict the behavior of submerged sills in the St. Clair River, which were proposed to raise the levels of Lakes Michigan and Huron in order to compensate for authorized diversions and for the enlargement of connecting channels. The series of model tests with different types of sills developed the conclusion that the desired compensation of 0.55 ft can be effected by as few as eight submerged sills, instead of the sixteen or seventeen originally estimated in 1926. To obtain maximum effectiveness from each sill it was found, from the model studies, that the up-stream face of each sill should be vertical, or nearly so, and that 60% of its effectiveness would be lost should material be deposited on the up-stream side of the sill to reach its top, which is 30 ft below low-water datum. Fortunately, the section of the St. Clair River at the outlet of Lake Huron carries very little material in suspension, and it is not expected that much fill will be so deposited.

Due to the peculiar difficulty of a mathematical solution to this problem and to the accurate and extensive data available for comparative purposes, the actual construction of these sills will afford an excellent opportunity to compare laboratory results with the action of Nature. The writer looks forward to successful vindication of the laboratory results, and hazards the prediction that not more than six sills will be required. Present plans are to construct only two sills as the first step and to observe the results obtained therefrom before constructing additional sills.

In 1929 a new concrete caisson breakwater at Milwaukee, Wis., was damaged by a severe storm. At that time, there were several theories as to the reason for the failure. Apparently, it had failed because of scour on the harbor side, which had been thoroughly riprapped only a few weeks before the storm, but with smaller stones than had been used on the lake side.

A model of a small section of the breakwater on a scale of 1:24 was constructed in an indoor tank, 4 ft wide by 10 ft long by $3\frac{1}{2}$ ft deep. Small flux stone was crushed to scale and a device for creating waves corresponding to natural waves 20 ft high was installed. A cross-sectional view of the breakwater with rip-rap on both sides was available through a glass window in the side of the tank. As the size of the wave increased, it was observed that the drag of the receding wave and the waterfall action of the wave spilling over the breakwater were working stones loose on the harbor side, moving them back and forth until they were rolled some distance from the breakwater. After 60 hours of continuous wearing down of the rip-rap on the harbor side, the model breakwater suddenly failed. The larger sized rip-rap on the lake side was undamaged. This and subsequent experiments with larger rip-rap proved conclusively that, contrary to popular belief, heavy well-placed rip-rap is as essential on the harbor side as on the so-called "exposed" side of a breakwater in all cases where waves spill over the breakwater.

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[4] Major, Corps of Engrs., U. S. Army; Chf., Personnel Section, U. S. War Dept., Washington, D. C. Major Heavey resigned his membership on December 31, 1935.

These examples are cited in support of the author's contention that model experiments may be profitably used in "verification" of the action of Nature, as well as in foretelling what the action of Nature will be on work not yet undertaken.

CHILTON A. WRIGHT,[5] ASSOC. M. AM. SOC. C. E. (by letter).—Field verification of the results obtained from model tests of hydraulic structures and rivers in several instances in which mobile sand beds were used in the models, is the basis of this excellent paper. Observations and surveys in the field yielded a good check as to the main features of the bed configuration and of the water stages developed in the model.

The present interest in the possibility of predicting from model tests the effects of control works on actual rivers impels the writer to cite an instance observed in Sweden in the summer of 1933. The regulating gates of the dam at the new low-head hydro-electric plant at Vargön are to be used to regulate the weekly flow of the Göta River at Lake Vänner, a natural lake about 2 140 sq miles in area.[6] The discharge of the river varies up to 30 000 cu ft per sec, the up-stream water level varies by 13 ft, and the down-stream level by 4.3 ft. These facts preclude the use of the usual discharge formulas.

The problem was taken to the Hydraulic Laboratory of the Royal Technical University, at Stockholm, where models of various types of sector gates were constructed in the hydraulic flume at a scale of 1:25, in order to determine the discharge as a function of the up-stream and down-stream depths and the gate-opening. Another model study of the entire project was made later at a scale of 1:36, in order to determine the interrelation of the discharges through the various gates and turbines.

The discharge diagrams obtained from the laboratory tests were compared with an official discharge curve developed previously for the Göta River from current meter measurements, and the diagrams showed a deviation of only −2% from the model results. A new set of discharge measurements was made at the weir in 1932 by means of a current meter, during a period of comparatively low water. No variation from the discharge curves based on the model tests could be observed over the range of these measurements.

This example corroborates the evidence given by Lieut. Vogel that in many cases the actual performance of hydraulic structures built according to designs determined by means of model studies in a laboratory can be predicted from these tests with a high degree of accuracy. Although the bed of the model was fixed in the case cited, rather than built of sand, as in the tests which Lieut. Vogel reports, the results obtained will be of equal interest.

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[5] Associate Engr., Hydr. Laboratory, National Bureau of Standards, Washington, D. C.

[6] "Untersuchungen betreffend die Abflussverhältnisse an Regulierwehr bei Vargön für die Wochenregulierung des Gota Älv," by Wolmar Fellenis and Erik G. W. Lindquist, M. Am. Soc. C. E., Meddelande från Vattenbyggnadsinstitutionen vid Kungl. Tekniska Högskolan, Stockholm, Sweden, No. 7, June 1933.

Paul S. Reinecke,[7] M. Am. Soc. C. E. (by letter).—The tests applied at the U. S. Waterways Experiment Station, at Vicksburg, Miss., to "prove" the correctness of a solution are described, clearly and concisely, in the paper. The "tests" are simple, understandable, and reasonable even to those who are not laboratory experts.

One reason why the solutions have been so uniformly satisfactory is that not only were visitors from the field invited to witness the tests, as explained by the author, but in the early three years of the Station's history (and the writer believes the custom is still being continued), it was obligatory for a field engineer, familiar with conditions at the site of the proposed improvements, to be present at the laboratory at least during the crucial periods of the test and preferably throughout the testing period. For this reason "theoretical" results were carefully "tied in" at all times with existing conditions and with practical results. Such a requirement prevented much waste motion in following a line of research which could not prove practical on account of field conditions that were not consonant with the theoretical assumptions made.

During the first few years after the establishment of the Waterways Experiment Station its personnel necessarily worked under a severe strain. Congress had adopted the Flood Control Plan which had to be set in motion promptly in order to avoid the danger of damage from another great flood. (So great was the early progress made that, in 1929, after less than one year's work, the repaired and improved Mississippi River levees withstood, without a single break, a flood equal to, or greater than, any previous flood in the river's leveed history, excepting only the Great Flood of 1927—a hitherto unprecedented record.) Consequently, it was necessary to have reasonable answers for the details of "location" problems long before the laboratory could build up a well-trained staff or develop a thoroughly tested method of procedure. The only alternative was for the U. S. District Engineers in the Mississippi Valley to proceed along lines previously laid down and found reasonable, without any benefit of model study. In fact, it was necessary to proceed with considerable work before laboratory solutions were available and, naturally, in some few cases errors of judgment were made. For instance, training dikes were placed in several locations where satisfactory results were not obtained. Here, again, the Experiment Station came to the rescue of the country's pocketbook. Instead of scrapping these expensive completed works—costing more than $100 000, and building a new set elsewhere at similar expense—the laboratory solution indicated how a relatively slight modification of the existing system could be made to produce satisfactory river conditions. In this regard, it must be borne in mind that, in the present state of knowledge, the hydraulics—especially of stream flow—the construction, and the location of improvement works still involve more art than science, and, consequently, there may be more than one "satisfactory" manner of improving any certain reach of river.

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[7] Major, Corps of Engrs., U. S. A.; U. S. Dist. Engr., St. Louis, Mo. (Formerly Asst. to President, Mississippi River Comm., in Chg. of Operations and Planning, Vicksburg, Miss.).

With the assistance that model study was able to give, it was possible to avoid repeating the failures of approximately 50 yr ago in locating contraction or training works in the Lower Mississippi, where the great force and changing direction of the attack of the current during the varied changes of river conditions between high and low waters, exert such terrific strains on river structures. Due partly to their location and partly to their method of construction, the dikes and dams built in the Plum Point (Tenn.) Reach in 1881–85 were uniformly disappointing in permanent results obtained. The laboratory solution of the problems in this same reach bids fair to assist in improving permanently a troublesome stretch of river.

Model study, verified by the tests indicated by Lieut. Vogel, affords a reasonable answer, not only of the question as to where to build, but also of the question whether to build at all—as witness the problem at Morrison Towhead, below Cairo, Ill. Before the results of model study were available for this site, there was considerable argument whether the closing dike should be built at the head of the back channel or chute, at its foot, or near the middle. Many "pet ideas" were put forth, with the usual chance that the longest winded and loudest protagonist among all concerned would finally win. However, the results of trying out the different ideas in a model indicated quite apparently that the best answer was to let Nature alone at this place and save the Government's money.

In his remarks concerning studies on the Island No. 9 Model, Lieut. Vogel touches on the question of costs. The quoted figures indicate an average of about 30 000 cu yd dredged annually during the preceding 6 or 7 yr. At a cost of 5 cts per yd, the annual charges amount to $1 500 per yr (disregarding expense of delays to navigation). The original plan of contraction works (8 500 lin ft at $35 per ft) would cost about $297 500, and the annual carrying charges at 5% for maintenance until the dikes have been silted in permanently, and for interest, would be nearly $15 000 per yr—or ten times the cost of dredging. The approved solution of the laboratory, requiring only 1 600 lin ft of dikes, would cost only $1 600 to $1 700 annually. Consequently, in comparing the annual cost of the dikes with the cost of dredging, plus the expense to shipping on account of the delays, it is obvious that the improvement of the river by contraction and training works is warranted at this locality.

The foregoing analysis should help answer the question as to whether the building of contraction works in the Lower Mississippi is a waste of public funds, as some "dredging only" enthusiasts maintain. These latter do not in the writer's opinion give sufficient consideration to the extra expense caused by delays in transit while waiting for dredges to open up channels through numerous sand-bars developed after every high water. If it is possible, at reasonable cost, to contract and train the river to scour out these bars as the waters subside, and thus do away with the necessity of so many dredges, then by all means such training works should be constructed and the delays in transit due to shoal waters avoided. The laboratory affords a cheap and satisfactory method of estimating the efficiency and necessary length of training structures, and, consequently, their cost, to compare with the average cost of dredging in the same reach of river in its unimproved condition.

Very properly, Lieut. Vogel insists that any problem assigned for laboratory study must be assigned "without strings." A laboratory is a good and inexpensive place to try out "hobbies," provided it is not limited in experimentation only to those hobbies. Every possible and reasonable solution must be investigated, and the best ones recommended. In the early years of the Station (and also since as far as the writer knows personally) the Station Director was never limited to only one or two preconceived answers, but was urged and directed to get the best answer possible. The high type of engineering-scientific personnel on duty at the U. S. Waterways Experiment Station fully entitles them to the confidence placed in them.

MORROUGH P. O'BRIEN,[8] ASSOC. M. AM. SOC. C. E. (by letter).—The importance of verifying the results of river model experiments by comparisons with Nature is emphasized in this paper. One criticism that may be offered is that the comparisons presented are restricted to river models, whereas the author implies in the "Introduction" that hydraulic models generally are being examined.

While he was a student in Germany, the writer was struck by the fact that, although great reliance was placed on model experiments of all kinds, few quantitative comparisons of models with Nature had been published. In "Hydraulic Laboratory Practice,"[9] which was a review of model theory and results, almost no such verifications are to be found. In France, England, and the United States, the question of the reliability of models has been given more attention. An elaborate comparison of models of different sizes has been made in a paper by Carmichel and Escande.[10] This study covered the following subjects: Weirs, orifices, and tubes; uniform flow in open channels; transportation of steel spheres; surfaces of discontinuity behind obstructions; vortices, whirlpools, and gyrating motions; spillways and intakes; and flow in pipes at high velocities. In general, these experiments showed a scale effect that was not great. Carmichel and Escande also presented a complete and concise survey of model laws.

Concerning models of movable river beds and estuaries, Osborne Reynolds and Vernon-Harcourt felt the need for careful studies of the reliability of their results, but later experimenters have not always been as cautious. As to results obtained, Reynolds stated that, after a long period of operation, his model resembled the estuary of the Mersey River as well as charts, made at different times, resembled each other.

In the model of the Severn Estuary,[11] Professor A. H. Gibson investigated the discrepancy between results obtained with different vertical scales and different sands, and also compared his model with Nature. The scales finally adopted were 1 : 8 500 horizontally and 1 : 200 vertically, with a model sand size of 0.007 in. as compared with an average size of 0.009 in. Professor Gibson stated that these models reproduced, with a high degree of accuracy, the behavior of the tides at all points in the estuary, both as regards their height, range, and rate of rise and fall. Furthermore, the phenomenon of the bore was also reproduced with a very close agreement between the height and speed as measured in the model and as observed in the estuary. Observations of the drift of floats, and of current velocities at various points above and below the English Stones, he reported, agreed well with those at corresponding points in the estuary. Generally, all the phenomena connected with the movement of the water in the estuary were reproduced with a remarkable degree of accuracy in the model.

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[8] Associate Prof., Mech. Eng., Univ. of California, Berkeley, Calif.

[9] "Hydraulic Laboratory Practice," Edited by the late John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., A. S. M. E., New York, N. Y., 1929.

[10] XVth International Congress of Applied Mechanics, Venice, 1931.

[11] "Construction and Operation of a Tidal Model of the Severn Estuary," H. M. Stationery Office, London, 1933.

A comparison of the charts and cross-sections of the sand banks, silt deposits, and the main channels that were prepared from surveys of the model at times corresponding to 1886, 1901, and 1924, with corresponding charts and cross-sections of the estuary itself, showed a good general agreement over at least 90% of the area of the estuary. According to Professor Gibson, this agreement was as close as could be expected,

> "In view of the fact that the model was not exposed to the effect of storms, which are liable to produce relatively large changes in any natural estuary. While there are differences in detail, these are not so great as the corresponding differences in detail in the estuary itself at the times of the different surveys. The only points of marked difference occur where, owing to some rapid deepening of the bed of the estuary, the slope of its sand banks is so great that the angle of repose of the sand in the model would have to be exceeded in order to give accurate reproduction."

The changes in the mean level of this estuary were reproduced with considerable accuracy in the model, and, generally, the agreement was bound to be such as to justify the assumption that "changes similar to those produced in the model by the introduction of a barrage would also be reproduced under similar conditions in the estuary."

For some time, the staff of the Hydraulic Laboratory at the University of California has been investigating the correspondence between models and prototypes. A few of the experimental results obtained by students have been published by the writer.[12] Additional work has been done by A. W. Kidder, Assoc. M. Am. Soc. C. E., the late John A. Jameson, Jr., Jun. Am. Soc. C. E., and others, and this work is being continued.

The discussion by Mr Jameson, published herewith, was prepared before his death to serve as his contribution to the data which Lieut. Vogel has offered in various papers. Its application to the present paper by Lieut. Vogel will be apparent from the content.

As would be expected, agreement between model and prototype is best when the reduction in size is not great and when the absolute depths and velocities considerably exceed certain critical values depending upon viscosity and surface tension. Furthermore, phenomena, such as flow over spillways, in which changes in pressure, momentum, and elevation are large as compared with the forces of friction, can be represented in models more faithfully than those in which friction is a dominant factor, as in models of rivers. In addition to the errors resulting from incorrect reproduction of the friction forces in fixed-bed models, river models having movable beds suffer from a lack of information on bed movement sufficient for the formulation of a proper model law. The author's comparisons of models and Nature show that in Nature there is a tendency toward the conditions found in the model, but on the basis of the data presented, one should not conclude that quantitative correspondence may be expected.

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[12] "Checks on the Model Law for Hydraulic Structures," *Transactions*, Am. Geophysical Union, 1932.

JOHN A. JAMESON, JR.,[13] JUN. AM. SOC. C. E. (by letter).—For the past few months (1934) the writer has been engaged, at the Hydraulic Laboratory of the University of California, in a systematic study of the limits of hydraulic model correspondence, in the course of which certain observations have been made, which may be of interest.

True Models.—A model (scale 1:20) has been built of a concrete-lined tunnel conduit with its inlet and outlet transition connections to a concrete flume. In operating this model the elevation of the water surface in the flume below the outlet transition was controlled so as to correspond to that observed in tests[14] made on the prototype and the discharge for the model, Q_m, was adjusted according to the relation, $q = l^{\frac{5}{2}}$, or,

$$Q_m = \frac{Q_n}{1\,790} \quad \text{.........................} (1)$$

[in which q = the scale ratio of discharge, $\frac{Q_m}{Q_n}$; l = the scale ratio of length; Q_m = discharge through the model; and, Q_n = discharge through the prototype.]

Photographs of the prototype show a fish-tail wave in the outlet transition, which closely resembles that observed in the model. In Nature, fish-tail waves were observed at the outlet from the tunnel and a similar phenomenon occurred in the model.

Flow in the tunnel section of the model was characterized by standing waves. The first of these waves which occurred just below the tunnel entrance, had a height of about 15% of the water depth, while down stream the wave height decreased progressively. According to the observers who conducted the test on the prototype, no waves were observed when sighting through the tunnel, the flow being smooth. The possible reasons for their appearance in the model are: (1) The proximity of the depth of flow to the critical point; (2) the impossibility of obtaining correspondence of friction factors between the model and the prototype; and (3) the shortness of the approach channel.

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[13] Research Asst. in Mech. Eng., Univ. of California, Berkeley, Calif. Mr. Jameson died September 26, 1934.

[14] Data presented to the Special Committee on Irrigation Hydraulics by Fred C. Scobey, M. Am. Soc. C. E. (Unpublished).

Using the observed field data for the points at the ends of the tunnel, the value of Manning's n was found to be 0.0105. In a previous paper,[15] the author derives from the Manning formula the relation, $q = \frac{a^{\frac{5}{3}} s^{\frac{1}{2}}}{n\, p^{\frac{2}{3}}}$, or,

$$n = \frac{a^{\frac{5}{3}} s^{\frac{1}{2}}}{p^{\frac{2}{3}} q} \qquad \text{.........................(2)}$$

[in which a = a scale ratio for areas, $\frac{A_m}{A_n}$; s = a scale ratio for slope, $\frac{S_m}{S_n}$; n = a scale ratio for roughness coefficient; and, p = a scale ratio for wetted perimeters, $\frac{P_m}{P_n}$.]

Since $a = l^2$, $s = 1$, $p = l$, and $q = l^{\frac{5}{2}}$, Equation (2) yields,

$$n = l^{\frac{1}{6}} \qquad \text{.............................(3)}$$

For the model in question, $l = 20$ and $n = (20)^{\frac{1}{6}} = 1.65$, so that N_m for the model should be $\frac{0.0105}{1.65} = 0.00635$, in order to satisfy the foregoing relations. Actually, N_m was found to average about 0.0081 when computed from measured data for the same two points considered in the prototype computation.

The shortness of the approach section between the forebay and the model proper and the rather sudden transition from the forebay supply tank to the channel were necessary because of limitation of space. It was impossible to make the approach longer than about 18 in. (or about six times the depth of flow), and this condition resulted in undesirable turbulence in the flume approaching the tunnel. Several different types of baffles were investigated, but none seemed to have an appreciable effect on the standing waves in the tunnel, which may indicate that the latter were not caused by the shortness of the approach section.

When the results of the model tests are transferred to the prototype, the forebay level is about 10% higher in the model than in the prototype for the same discharge and down-stream water-surface elevation.

Distorted Models.—The distorted models included three open channels of rectangular section lined with painted galvanized iron, each consisting of two tangents connected by a 90° elliptical bend. These channels were the same depth (6 in.), but their horizontal dimensions were in the ratio of 1:2:4 (that is, considering the 12-in. channel as the prototype, the 6-in, and the 3-in. channels had horizontal scale ratios of 2 and 4, respectively). The bottoms of all three channels were horizontal and all opened into a large forebay tank at the upper end and discharged into a return channel at the lower end. In these tests each channel entrance was fitted with a galvanized-iron transition section to smooth out entrance eddies, and each had a galvanized-iron sharp-crested weir at both entrance and exit to isolate the section of the channel being studied from disturbance up stream or down stream. The entrance weirs operated in a submerged condition and their crests were fixed at the same elevation (about 2.5 in. above the channel bottoms). When the channels were calibrated individually it was found that the discharges were nearly in the ratio 1:2:4 (the quantity per foot of weir crest being: 12-in., 0.230 cu ft per sec; 6-in., 0.234 cu ft per sec; and, 3-in., 0.232 cu ft per sec. for a head on the weirs of 0.166 ft).

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[15] "River Laboratory Hydraulics", *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 118.

With water flowing through the channels sand was introduced into each immediately below the upper weir, the weights of sand being proportioned to the squares of the horizontal scales (that is, 1:4:16). When the water was shut off, and the channels were drained, it was observed that there was a definite line in the two larger channels above which there was no deposition whatever, due to the scouring action of the water below the entrance weirs. This line was parallel to that of the entrance weirs and not at proportional distances down stream as would be expected in scale models. In other words, the horizontal dimensions are controlled by the vertical, and distortion is not permissible. This condition also existed in model tests for the Hastings Dam, where a distorted model was used to study scour below gates.[16] The same line of scour would undoubtedly have appeared in the 3-in. channel if there had been longer tangent sections before the bends. As it was, the scour line for the two larger channels intersected the small channel in the curved part of its length so that no definite line appeared, although no sand remained in the up-stream tangent section.

Another factor that must be considered in using distorted models for experiments with bed load is the effect upon friction coefficients of the different proportions of bottom and sides making up the wetted perimeter in the model and prototype. In another series of tests, the entrance weirs were removed, and the lower weirs adjusted to compensate for the aforementioned friction differences. This was accomplished by raising the weir crests in the two smaller channels by computed amounts so that the slope of the water surface and the mean velocity would be approximately the same for all three channels. Wooden spur-dikes, 1 in. high, were placed radially at corresponding locations in the three channels, and the entire bottom of each channel was covered with about 0.4 in. of sand. The water passed through the channels at a depth of about 0.34 ft and at a velocity just sufficient for sand movement. The times of flow were in the ratio of 1:2:4. Although there was qualitative correspondence, it was observed that in spite of the fact that the weirs were adjusted carefully, so as to give equal mean velocities, the scour in the 3-in channel was more rapid than in the other channels.

J. C. Stevens,[17] M. Am. Soc. C. E. (by letter).—"Man marks when he hits but fails to mark when he misses." The author has given some excellent examples confirming the behavior of open river models by prototype performances. It is truly remarkable that the small-scale models and distortions between horizontal and vertical dimensions which it was necessary to adopt, should show such excellent similitude of behavior, and this is all the more remarkable when one realizes that the model sands could not have differed greatly from the prototype sands. To reduce sizes of model sands to simulate prototype sands is impossible, of course. If it were done the model material would be colloidal.

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[16] "Laboratory Tests on Hydraulic Models of the Hastings Dam," *Bulletin No. 2*, Univ. of Iowa Studies in Engineering.

[17] **Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.**

It would have been most instructive if the author could have drawn from his magic bag an example or two in which prototype performance failed to conform to model behavior, and explained the "why" of the failure. It is vitally important to be able to recognize the limitations of models and to stay within those limits. Failures are the most potent means of defining those limitations. The writer, therefore, offers the following negative "verification" with the belief that it will be just as instructive as the author's positive ones.

Field tests of three of the seven Leaburg siphons were made by the writer, and the results have been fully described.[18] Three models of Siphon No. 7, to scales of 1:20, 1:15, and 2:15, had been made and tested independently, and a study of the published[19] results will reveal the following points of dissimilarity between model and prototype performances:

1.—It was impossible to simulate the cavitation effects found in the prototype. Velocities practically corresponding to a head of 1 atmosphere were found in the Leaburg siphons near the summits. The live stream passing around the summit curves would swing from one side of the barrel to the other, leaving the intermediate spaces filled with pronounced eddies at high negative pressures.

In the model the live stream completely filled the barrel of the three models, and no effect of cavitation was detected.

2.—The coefficients of discharge for the models were higher than for the prototype. Based on the outlet area of the siphon the coefficients of flow,

$C = \dfrac{Q}{A\sqrt{2gH}}$, were:

Models	Coefficient of Flow	Percentage Error of Model
Prototype, No. 7 Siphon	0.63	....
1:20 model	0.68	+ 8.0
1:15 model	0.69	+ 9.5
2:15 model	0.71	+12.7

The models showed consistently higher discharges than the prototype, and the larger the model the greater the error. This is believed to be due to the eddy losses in the prototype as a result of cavitation that did not exist in the model.

3.—The depth of flow on the crest of the siphon summit necessary for priming was much less in the prototype than in the model. At first, this was attributed to air leakage around the priming gates of the prototype. In the models, as the forebay rose air was compressed in the summit which precluded the possibility of obtaining sufficient depth of flow over the crest to prime quickly.

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[18] "On the Behavior of Siphons," by J. C. Stevens, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 986.

[19] *Transactions*, Am. Soc. C. E. Vol. 99 (1934), p. 1008; also, *Engineering News-Record*, August 18, 1932, p. 187.

A small air outlet at the summit of the 2:15 model was provided to simulate the air leakage at the summit of the prototype which improved somewhat the sensitiveness to priming of the model. The question arose: If the leakage around the primer of the prototype were stopped would it require a greater depth of flow to prime? In order to test this, the priming gates of all three siphons were later sealed tightly and tested for priming head. The siphons became more sensitive and primed with less depth of flow than before sealing the gates, as the following will show:

Head Required to Prime, in Feet	Siphon No. 7	Siphon No. 6
Primer unsealed......................	0.41	0.34
After sealing primer..................	0.19	0.20

The explanation of why the siphon models failed in these important particulars to predict prototype performances probably lies in the fact that the tests were based on the Froudian similitude for gravity forces, whereas friction forces, including eddies and impact, have effects of nearly equal magnitude in producing the phenomenon of flow through them. These friction forces become of great importance in the prototype on account of the high velocities, but are of lesser moment in model performance because of the relatively low velocities that obtain in them.

PAUL W. THOMPSON,[20] JUN. AM. SOC. C. E. (by letter).—The author "hits a nail on the head" when he refers to the infrequency of the cases in which actual recorded data from a prototype have checked data previously secured from a small-scale model of that prototype. His paper presents the results of what is probably the most extensive, and successful, attempt yet made to confirm model results by actual data from the prototype.

It appears that the illustrations of the paper offer almost conclusive proof to the proposition that small-scale models are capable of yielding dependable qualitative data, even in the cases of problems involving erosion and transportation of débris by rivers. It might even appear from the illustrations that quantitative results have been obtained from the models; however, such appearance is probably coincidental rather than real. There are a few stubborn facts, each of which seems to preclude the possibility of any precise quantitative interpretation of data derived from small-scale models, at least in so far as problems in erosion and transportation of débris are concerned.

To begin with, he who would study problems involving the transportation of débris in natural rivers is confronted by a phenomenon about which little is known. It would not appear to be possible to simulate exactly a phenomenon when the phenomenon itself is imperfectly understood.

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[20] Lieut., Corps of Engrs., U. S. Army; Care, Military Atttaché American Embassy, Berlin, Germany. (Formerly Asst. to Director, U. S. Waterways Experiment Station, Vicksburg, Miss.)

The troubles of the experimenter are not confined to the difficulty of attempting to simulate characteristics about which he knows little. Frequently, he finds himself unable to simulate those factors about which he does know something. Thus, most of the models described by the author were built to a horizontal scale of 1 to 1 000 and to a vertical scale of 1 to 100. Such distortion of linear dimensions was necessary and, as the results of the experiments show, was justified. However, there is no question but that distortion affects the behavior in a model. Just what the effects may be seems to be beyond accurate determination at the present time; but it does appear unreasonable to believe that a model which is geometrically similar to a river ten times as deep as the one being investigated, can yield precise quantitative results in regard to the latter.

The fact that data from small-scale models are often not quantitatively dependable should cause no one to conclude that model studies are a waste of time and money. That carefully designed models are capable of yielding data which are qualitatively and comparatively dependable is enough to justify the use of such models in almost any problem in hydraulics involving large expenditures of money. Perhaps a time will come when models which produce quantitative results can be built; however, by that time science will probably know all the answers that the models can give. Meanwhile, the Engineering Profession is indebteded to Lieut. Vogel for another convincing presentation of the possibilities of model research.

SAMUEL SHULITS,[21] JUN. AM. SOC. C. E. (by letter).—In a convincing manner, Captain Vogel presents the practical value and qualitative reliability of river models. However, it is the problem of the quantitative evaluation of fluvial models that confronts laboratories everywhere to-day and further systematic research is necessary to clarify this phase of river-model usefulness. At present, the writer is inclined to agree with Koerner[22] that the results of properly planned experiments with movable-bed models are always qualitatively transferable to the prototype in Nature. In other words, it is always determinable which of the investigated construction measures promises the greatest efficacy in the desired direction. According to Koerner, it is still necessary, however, to exercise great care in applying quantitative conclusions derived from river-model studies to predict the behavior of prototypes.

Although the path may be difficult, there is gratifying evidence that movable-bed models can yield quantitative results. For example, the Schoklitsch bed-load formula[23] was developed rationally, its constants were determined in a laboratory flume, and the results have been found to hold for several European rivers. It correlates bed load, size of material moved, energy gradient, and discharge.

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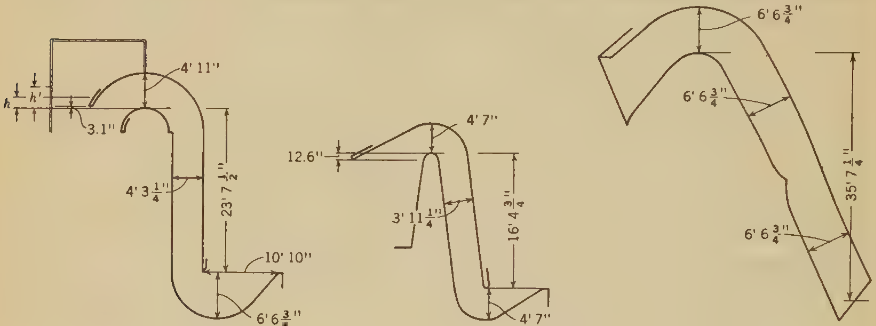
[21] Asst. Engr., U. S. Bureau of Reclamation, Denver, Colo.

[22] "Modellversuche für einen Fluss mit starker Geschiebebewegung ohne erkennbare Bankwanderung", by H. D. Krey, prepared and edited by Burghard Koerner, Berlin, 1935, p. 38.

[23] "Der Geschiebetrieb und die Geschiebefracht", by A. Schoklitsch, *Wasserkraft und Wasserwirtschaft*, No. 4, 1934, p. 37; also "The Schoklitsch Bed-Load Formula" by Samuel Shulits, *Engineering*, June 21 and 28, 1935, pp. 644 and 687.

Furthermore, it has actually been possible to find an accurate quantitative relation between the discharge, the head, the size of the bed material, and the pool depth below a jet falling freely over a sharp-crested weir[24] in a small model. In this case, eight different sands of non-uniform composition were used. To be sure, some may say that the relation thus obtained is valid only for the small model, but it must be realized that the search for even such a relationship was considered futile not long ago.

Lack of agreement between siphons and their models has been reported by A. Veronese.[25] Three models, of different scales, were made of siphons (Fig. 19) in Italy; the purpose of the tests was to determine the relationship between the minimum head (h, in Fig. 19), and the corresponding time, T, required to prime, in both model and prototype. If the subscripts, m and n, denote, respectively, model and prototype, and if the scale ratio is expressed as $1:a$, then, with Froude's law as a basis, the relations between priming head and priming time should be:



(a) CAMUZZONI (b) CARRON (c) S. CATERINA

FIG. 19.—SIPHONS INVESTIGATED BY ALESSANDRO VERONESE.

$$h_n = a\, h_m \qquad (4)$$

and,

$$T_n = T_m \sqrt{a} \qquad (5)$$

but Veronese's experiments led him to conclude that Equations (4) and (5) are invalid, and he derived the following from the data:

$$h_n = h_m\, a^b \qquad (6)$$

and,

$$T_n = \frac{T_m}{\sqrt{a}} \qquad (7)$$

in which $b = \frac{24 \sqrt{a}}{T_m}$. The agreement (which can be termed sufficient, per-

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[24] "Kolkbildung unter Ueberfallstrahlen", by A. Schoklitsch, *Die Wasserwirtschaft*, 1932, No. 24.

[25] "Ricerche sulla relazione che intercede tra l'altezza di adescamento dei sifoni autolivellatori sperimentati in modello e quella dell'originale", by Alessandro Veronese, *L'Energia Elettrica*, July, 1934, p. 517.

TABLE 3.—Comparison of Observations and Computations by Veronese's Equations

Siphon	Model scale, *a*	Minimum Priming Head, *h*, in Inches			Priming Time, *T*, in Seconds		
		Observed	Prototype Computations		Observed	Prototype Computations	
			By Froude's law	By Equation (6)		By Froude's law	By Equation (7)
Camuzzoni (Fig. 19(*a*))	20	1.04	20.80	6.97	170	760	38.03
	10	1.11	11.10	4.72	120	379	37.97
	5	1.33	6.65	3.90	80	179	35.71
	1	5.12	5.12	5.12	....	....	....
Carron (Fig. 19(*b*))	20	2.90	58.00	9.61	270	1 208	60.40
	12	2.88	34.55	8.03	210	727	60.69
	8	2.93	23.42	6.73	170	481	60.07
	1	7.20	7.20	7.20	60	60	60
S. Caterina (Fig. 19(*c*))	25	1.90	47.50	19.87	165	825	33.00
	15	1.95	29.23	15.70	120	465	31.00
	8	2.72	21.76	12.91	90	255	31.80
	1	15.75	15.75	15.75	....	....	....

haps, although not close) between the observations and values computed with Veronese's equations is shown in Table 3. The usual equations based on Froude's law do not give as close an agreement. The prototype priming time was measured only on the Carron siphon.

Herbert D. Vogel,[26] Assoc. M. Am. Soc. C. E. (by letter).—In 1929 there was not a single hydraulic structures laboratory in the United States, in a strict sense, although plans were being considered for eight to be established at various places.[27]

A recent *Bulletin*[28] of the National Bureau of Standards contains descriptions of fifty-three American laboratories, twenty-two of which are shown to be operated largely for the solution of open-channel problems. This phenomenal development may be due partly to the great increase in public work activities during the past few years, but inference is strong that some credence should be given the explanation that model studies return ample dividends. It should follow as a corollary, then, that small-scale models may be depended upon for the procurement of trustworthy information.

In an effort to find proof—or disproof—of the reliability of open-channel experimentation, this paper was tossed out as a challenge. The results, while disappointing, were nevertheless illuminating in their paucity, and the question could now be asked, "Are current studies being undertaken: (1) Because of the proved validity of the method; (2) because models are a cheap means of approximation; or (3) simply because it is fashionable to carry on 'scientific' research?" To run down the answer to this query circular letters were sent out from the U. S. Waterways Experiment Station during August, 1934, to more than fifty laboratories or individuals engaged in hydraulic research.

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[26] Capt. Corps of Engrs., U. S. A., 3d Engrs., Schofield Barracks, Honolulu, Hawaii.

[27] "Hydraulic Laboratory Practice". Edited by the late John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., p. 731, Am. Soc. Mech. Engrs., N. Y., 1929.

[28] "Hydraulic Laboratories of the United States", *Hydraulic Laboratory Bulletin, Series B*, National Bureau of Standards, First Revision, October 1, 1935.

It was requested in each case that information be furnished regarding the performance of any model tests, the results of which had been substantiated subsequently by results obtained in the field. Twenty of these letters were directed to addresses within the United States and the remainder were sent to foreign countries.

Replies were received from twenty-eight of those addressed, twelve of these being from laboratories within the United States. With a few notable exceptions, the replies contributed scarcely any information directly pertinent to the subject. From Germany came the citation of one instance of a model (scale, 1: 45) of a dam that had produced results which, when compared with recorded data of the completed structure, were found to be accurate within 10 per cent. One correspondent in Italy reported that tests on siphons had failed to confirm the laws of similitude, whereas another stated that models of siphon spillways had been tested and found to reveal true information as to what could be expected in Nature.

A comparative wealth of data was received from Holland, outstanding among which were the following:

(*a*) Model studies to determine the efficiencies of two centrifugal pumps gave values varying from 0.75 to 0.87. In the prototype the efficiencies were found to be 0.80 in each case.

(*b*) After improving the Harbor of Breskens the intensity of wave action was reduced somewhat, as had been indicated by a model study.

(*c*) The results of six experiments with models (scale, 1: 20) to determine the movement of stone on fascine mattresses were found to be in close agreement with the results obtained later in the Maas River.

(*d*) A 1:50 model of a discharging sluice gave depths of scour that were almost exactly verified in the field.

(*e*) A model (scale, 1: 25) was operated to determine the discharge capacity of a sluice. Exact comparison with the data derived from full-scale performance has not been possible, but there appears to be a close agreement.

(*f*) A 1:5 scale model was built to study the flow of viscous fluids in suction pipes. Prototype and model flows were found to be exactly similar.

(*g*) Six models of navigation locks were experimented with in an effort to determine times of filling and forces acting upon the ships during lockage. When these locks had been built to full scale it was found that for two of them the times of filling were within 10% of those figures indicated by the model tests. Exact comparisons were not possible in the other cases, but it appeared that the general results were as had been predicted. Some indications of the produced forces were obtained from the full-scale structures, from which it appeared that differences of 15 to 25% were about the least that could be expected.

Sweden supplied information of two model tests which were subsequently verified by performances in full scale. One of these was to determine the degree of erosion that might be expected below the sector gates of a power plant; the other pertained to the procurement of data on a regulating weir.

Verification was received from England on the performance of tidal models; and data of surge tank experiments were also presented, together with those obtained later in the field. The results indicated an almost exact agreement between surge heights in model and in Nature, and revealed close agreement in periods.

Three of the twelve American laboratories that replied gave verifications of data obtained from model spillways, locks, and dams. Little or no information was received in justification of movable-bed models.

There can be no doubt that field verifications of small-scale experiments are difficult to obtain, but if models are to continue in their present high place as auxiliary aids for the hydraulic engineer, it will be necessary for their sponsors to procure positive proof of their reliability. Furthermore, it is just as important that the failures be revealed and recorded because, unless this is done, undue reliance may be placed on the results of certain studies from time to time. The scientist and the engineer must face facts as they are, prepared to modify their convictions as the need for so doing is evidenced. The several discussions of the paper carry this thought as an underlying consideration. The time for proselyting is past; careful scrutiny and study are now in order as means of determining what types of models can be relied on for quantitative data; what kinds will give reasonably accurate qualitative effects; and which are capable of nothing but deceit.

An extremely common sense viewpoint of the entire question has been presented[29] editorially. One paragraph appearing particularly worthy of quotation is as follows:

"The publication of results, with particular reference to any new form of technique evolved, is greatly to be encouraged, in all hydraulic model work, no less, of course, than in any other branch of applied mechanics; for the guidance so provided may serve not only to help others to avoid the many pitfalls, but may effectively reduce the cost and labour involved. Certainly the cost of such investigations cannot be lightly undertaken, the labour entailed both in the laboratory and often also in the very important work of collecting essential data for the experimental work from the field, is considerable. Speaking generally, however, in cases where the need for model investigations is felt, the cost will be small in comparison with the scheme in hand."

This comment was written with the writer's paper in mind.

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[29] *Engineering* (London), May 31, 1935.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1935

THE HYDRAULIC JUMP IN TERMS OF DYNAMIC SIMILARITY

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WITH DISCUSSION BY MESSRS. HUNTER ROUSE, SHERMAN M. WOODWARD, ROBERT E. KENNEDY, L. STANDISH HALL, MORROUGH P. O'BRIEN, F. V. A. E. ENGEL, BALDWIN M. WOODS, J. C. STEVENS, NOLAN PAGE, ANDREI I. IVANCHENKO, F. T. MAVIS AND ANDREAS LUKSCH, I. M. NELIDOV, AND BORIS A. BAKHMETEFF AND ARTHUR E. MATZKE.

SYNOPSIS

Since Bidone's classical "Memoire"[3], the first to describe the hydraulic jump, this fascinating and puzzling phenomenon has been the subject of repeated experimental investigation. Most of the work has dealt with what may be termed the "vertical" elements of the jump, such as the relation between the lower and upper stages, the height of the standing wave, etc. Scarcely any data are available, at least in systematic form, with regard to what may be termed the "longitudinal elements", such as the length of the jump, the profile of the surface of the roller, etc. The importance of such data is obvious. In designing stilling-basins at the toes of spillways, in laying out devices to prevent erosion below sluices, and in other similar cases, knowledge of the longitudinal elements is indispensable.

During 1932–33 the longitudinal elements of the jump was the subject of systematic research in the Fluid Mechanics Laboratory of Columbia University, in New York, N. Y. A particular feature of the work was, that in interpreting and systematizing the results obtained, recourse was taken to the principle of dynamic similarity and the final data were presented in generalized dimensionless form. Dimensionless presentation in terms of

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NOTE.—Published in February, 1935, *Proceedings*.

[1] New York, N. Y.

[2] New York, N. Y.

[3] *Memoires,* Acad. de Turin, 1820.

dynamic similarity is known to have yielded splendid results and has become a matter of course when dealing with flow in closed conduits. On the other hand, its application to open flow, with the exception of models of rivers, has not been as widespread as could be expected. It also appears that the basic premises which should govern the approach to open-flow problems, are not always clearly understood and that at times investigators are prone to select parameters non-judiciously. An example is the frequent use of the so-called "Boussinesq number."

The experiments described herein were referred to a general dynamic characteristic, "the kinetic flow factor"[4]. The results obtained seem to have confirmed the usefulness of the general methods applied and are claimed to have given the first comprehensive picture of the longitudinal features of the hydraulic jump in general.

THEORETICAL PREMISES

To facilitate a comprehensive approach, a brief recapitulation will be first given of the theoretical premises, which underlie the subsequent treatment.

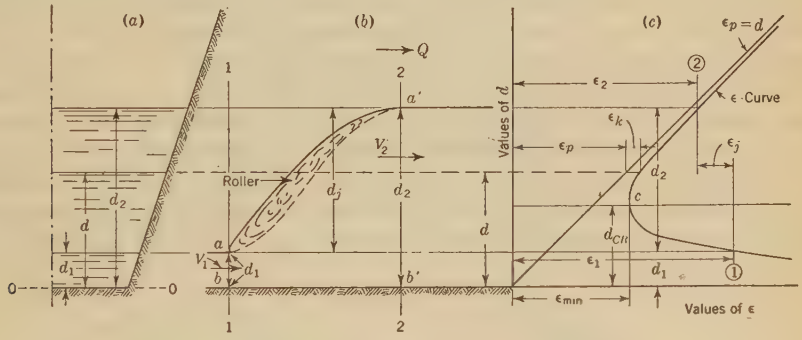

FIG. 1

Specific Energy Curve.—In treating problems of varied flow in open channels, it has become customary to refer the motion to a "specific energy curve"[5] represented in Fig. 1(*c*). The constant discharge in a canal of given cross-sectional area, *A*, with a varying depth, *d*, is expressed by *Q*. For each depth one may estimate the energy head (the energy in a unit weight of liquid), referred to a datum line, 0-0, drawn through the bottom of the canal. The potential energy head is $\epsilon_p = d$; the kinetic energy head is,

$$\epsilon_k = \frac{V^2}{2g} = \frac{Q^2}{2\,g\,A^2_d} \quad \text{........................(1)}$$

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[4] "Hydraulics of Open Channels", by Boris A. Bakhmeteff, M. Am. Soc. C. E., Engineering Society Monographs, McGraw-Hill Co., 1932, p. 64.

[5] *Loc. cit.*, p. 32.

in which, A_d is the cross-sectional area at the depth, d. Their sum is equal to:

$$\epsilon = \epsilon_p + \epsilon_k = d + \frac{V^2}{2g} = d + \frac{Q^2}{2\,g\,A^2_d} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

which is the total specific energy of flow, being a function of the depth, d, only. The graph of the ϵ-curve (Fig. 1(*c*)) indicates, that the energy contents vary with the depth, increasing indefinitely for very large and very small values of d. For each discharge there is a definite point, c, which marks the lowest possible energy, ϵ_{min}, compatible with the given discharge and the given canal cross-section. The flow at the point, c, is said to be "critical", the corresponding depth, d_C, being the "critical depth".

For the important case of a rectangular channel, with a width, B, and with $q = \frac{Q}{B}$, the discharge per unit width, the specific energy is:

$$\epsilon = d + \frac{q^2}{2\,g\,d^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and the critical depth at which the energy content is minimum is expressed by,

$$d_C = \sqrt[3]{\frac{q^2}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The Hydraulic Jump.—Referred to the specific energy curve, the hydraulic jump is a local phenomenon by means of which flow passes in a rather abrupt manner from a lower stage, d_1, corresponding to Point (1) on the lower branch of the energy curve, to the upper stage, d_2, marked by Point (2) on the upper branch (see Fig. 1(*b*)). Obviously, the essence of the phenomenon is the change in the form of energy. At the lower stage, energy in kinetic form prevails; in the jump, kinetic energy is transformed into potential energy, which obviously predominates on the upper branch of the curve. On the energy diagram (Fig. 1(*c*)), Points (2) and (1) which correspond to the so-called conjugated depths before and after the jump, do not lie on the same vertical. Their horizontal distance, $\epsilon_j = \epsilon_1 - \epsilon_2$, represents the energy head lost in the jump.

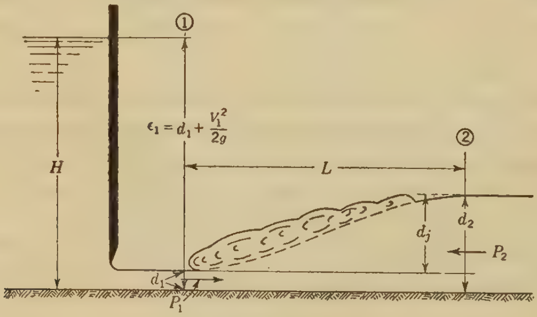

FIG. 2

Similar to many other cases of fluid dynamics in which one deals with abrupt variations in the forms of flow, a comprehensive solution is reached by applying (as first suggested by Bélanger) the "momentum principle".

It is customary to make certain fundamental assumptions when applying the momentum principle. These assumptions underlie the familiar equations relating to the hydraulic jump and should be kept clearly in mind.

In Fig. 1(*b*), or in Fig. 2, Sections (1) and (2) refer to the "beginning" and to the "end" of the jump. Thus, they delimit the jump, schematically, separating it from the adjoining reaches of gradually varied flow. Motion at Section (1), preceding the toe of the roller, is free and still is not influenced by the roller. The stream lines are parallel. Therefore, the pressure distribution is assumed to follow the hydrostatic law. This means that the potential energy referred to 0-0 (Fig. 1), throughout the stream is the same and is equal to the depth, d, which permits expressing the average energy by the simple expression, Equation (2). Section (2) represents the "end" of the jump. It is supposed to be selected so that the expansion of the live jet under the roller has ceased. In other words, the curvature of the stream lines, which features the flow in the jump proper, is no longer present and the stream filaments have again become parallel, with the hydrostatic distribution of pressure again restored. For this reason the energy may again be computed by the simple expression, Equation (2).

The assumption of parallel filaments of flow in Sections (1) and (2) which leads to hydrostatic distribution of pressure, permits one to estimate the pressure components across these sections, which effectuate the change of momentum in the liquid body as the respective resultants of the hydrostatic pressure,

$$P = \gamma A_d z_0 \dots\dots\dots\dots\dots\dots (5)$$

in which, γ is the specific weight of the fluid, and z_0, the depth of the center of gravity of the respective section below the free surface.

For a jump in a horizontal flume, the only other force component acting in the direction of the flow (and thus contributing to the change of momentum) is the resultant of "external" friction forces, acting between the liquid body, $a\ a'\ b'\ b$ (Fig. 1(*b*)), and the solid boundaries (the bottom and the walls) of the canal. Compared to the action of the highly disturbed and turbulent motion, engendered inside the liquid body in the roller and in the expanding vein, and which naturally results in pronounced losses of energy head, the shearing stresses acting in the boundary layer between the jump and the walls are taken to be insignificant in their ultimate effect. It is customary, therefore, simply to neglect these outward forces and to eliminate them from "the picture" when establishing the equation embodying the momentum principle. In other words, the familiar equations, giving the ratios between the stages before and after the jump, are obtained by neglecting the friction forces, acting on the comparatively short and insignificant reach, L (Fig. 2), over which the jump extends.

For a rectangular channel, considering a unit width with a discharge, q, the difference of the hydrostatic pressure is:

$$P_1 - P_2 = \gamma \left(\frac{d_1^2}{2} - \frac{d_2^2}{2} \right) \dots\dots\dots\dots\dots\dots (6)$$

The change of momentum per second, with the mass flow, $\frac{q\,\gamma}{g}$, is $\frac{q\,\gamma}{g}\,(V_2 - V_1)$. Equating and transforming, one obtains:

$$\frac{2\,q^2}{g} = d_1\,d_2\,(d_1 + d_2) \qquad \text{.......................(7)}$$

from which are developed the well-known equations, giving the relations between the conjugated depths:

$$d_2 = \frac{d_1}{2}\left[-1 + \sqrt{1 + \frac{8\,q^2}{g\,d^3_1}}\right] \qquad \text{..................(8)}$$

and,

$$d_1 = \frac{d_2}{2}\left[-1 + \sqrt{1 + \frac{8\,q^2}{g\,d^3_2}}\right] \qquad \text{..................(9)}$$

Another form is obtained by substituting d_C from Equation (4), thus:

$$d_2 = \frac{d_1}{2}\left[-1 + \sqrt{1 + \frac{8\,d^3_C}{d^3_1}}\right] \qquad \text{.................(10)}$$

and,

$$d_1 = \frac{d_2}{2}\left[-1 + \sqrt{1 + \frac{8\,d^3_C}{d^3_2}}\right] \qquad \text{.................(11)}$$

Dynamic Similarity.—The Kinetic Flow Factor.—In Equations (10) and (11) the relation between the stages is made to depend on the ratio of the critical depth to the respective depths, d_2 and d_1. With reference to Fig. 1(*c*), the ratio, $\frac{d_C}{d}$, marks the position of the flow on the energy curve. The larger the ratio, $\frac{d_C}{d}$, the greater will be the velocity of flow and, consequently, the relative part of the kinetic energy component in the total energy head. In fact, each point on the energy curve characterizes a "state of flow", which may be numerically qualified by what has been termed,[4] the "kinetic flow factor"; thus:

$$\lambda = 2\,\frac{\epsilon_k}{\epsilon_p} = \frac{V^2}{g\,d} = \frac{q^2}{g\,d^3} \qquad \text{....................(12)}$$

which gives a measure of the "kineticity of flow", expressed by twice the ratio of the kinetic energy head to the potential energy head contained in each pound of liquid, flowing at the depth, *d*. The reason for the coefficient, 2, in Equation (12) is that, in the critical state (that is, when the flow is at the critical depth, d_C), the value of λ becomes:

$$\lambda_C = \frac{\frac{q^2}{g}}{d^3_C} = \frac{\frac{q^2}{g}}{\frac{q^2}{g}} = 1 \qquad \text{....................(13)}$$

In other words, the kinetic flow factor at the critical depth is $\lambda = 1$. Flow on the upper branch of the ϵ-curve, with $d > d_C$, is governed by the condition, $\lambda < 1$; flow on the lower branch, with $d < d_C$, is governed by the condition, $\lambda > 1$. Obviously, the factor, λ, is a general dimensionless characteristic of the dynamic conditions of flow. In fact, in the terms ordinarily used in studies of dynamic similarity the kinetic flow factor, λ, is equivalent to the so-called Froude number.

It is important, in this instance, to make as clear as possible the premises that underlie the treatment of open-flow problems in terms of dynamic similarity. Two basic cases must be distinguished: First, that in which the most important agencies are resistances of the frictional type, analogous to those occurring in a pipe; and, second, that in which the features of the phenomenon are, for the most part, determined by the action of gravity. The phenomena in the first case were first studied comprehensively by Osborne Reynolds.[6] Flow in these circumstances will be dynamically similar, when the same ratio persists between the inertia effects (mass times acceleration, in the Newtonian equation of motion) and the frictional resistances, the source of the origin of which is the general property of matter known as viscosity. In this case the factor expressing, numerically, the ratio of the inertia forces to the viscous forces, has the form of the so-called Reynolds number:

$$\mathsf{R} = \frac{V\,l}{\nu} \qquad (14)$$

in which, $\nu = \frac{\mu}{\rho}$, the kinematic viscosity, and l, an appropriate longitudinal parameter, such as the diameter or the radius of a closed conduit. A number similar in structure to Equation (14) would be in order, for example, when studying the frictional resistances in uniform canal flow. In that case, l could be the depth, or any other characteristic geometric dimension, provided the cross-sections compared were geometrically similar.

In the second case, the phenomena are principally determined by action of gravity. This was the problem faced by Froude, when seeking to compare the resistances offered by surface waves created by the motion of a vessel. Dynamic similarity, in this case, representing a constant ratio between inertia forces and gravity action, is expressed by a factor known as the Froude number, and usually given in the form:

$$\mathsf{F} = \frac{V}{\sqrt{g\,l}} \qquad (15)$$

in which, l is an appropriate longitudinal factor, such as the length of the ship.

In analyzing open-flow cases it must be first remembered, that the principle in general may be applied to geometrically similar cases only. Many mistakes are known to have arisen from a failure to observe this fundamental rule.

Furthermore, one must determine which of the two types of forces, gravity or friction, are preponderant in each specific case and which of the two, therefore, should be accepted as the principal agent, affecting the flow. As is well known, one cannot "cater" to both types of forces simultaneously and, therefore, a definite choice must be made.

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[6] *Philosophical Transactions*, Royal Soc., 1883.

Finally, it must be borne in mind, that the longitudinal parameter, l, in Equation (14) or Equation (15), cannot be selected at random. In fact, the parameter must correspond to the essential features of the particular case; it must be connected inherently with the physical agencies that influence one or the other of the two cases. For example, in the case of the hydraulic jump, the friction forces are neglected deliberately. The phenomenon of the "jump" is considered to depend entirely on the action of gravity, which expresses itself in the hydrostatic pressures and causes the change of momentum.

Thus, a dynamic characteristic of the Froude type is in order. A longitudinal parameter, germane to the physical essence of the phenomenon, which embodies transformation of the form of energy, is the depth, d, representing the potential energy of the flow. Hence, the Froude number, $\mathsf{F} = \frac{V}{\sqrt{g\,d}}$, which is equivalent to the kinetic flow factor, $\lambda = \frac{V^2}{g\,d}$.

If $\lambda = \frac{V^2}{g\,d}$ is an appropriate dynamic characteristic of the Froude type, it would naturally follow that flow in the two cases, possessing similar values of λ, would assume similar forms. Thus, for example, the ratio of the conjugated depths, $\frac{d_2}{d_1}$, or, respectively, $\frac{d_1}{d_2}$, in a hydraulic jump (see Equations (10) and (11)) may be expressed in terms of the kinetic flow factor, by substituting Equation (12) in Equations (8) and (9) which gives,

$$\frac{d_2}{d_1} = \frac{1}{2}\left[-1 + \sqrt{1 + 8\,\lambda_1}\right] \ldots\ldots\ldots\ldots (16)$$

and,

$$\frac{d_1}{d_2} = \frac{1}{2}\left[-1 + \sqrt{1 + 8\,\lambda_2}\right] \ldots\ldots\ldots\ldots (17)$$

in which, λ_1 and λ_2 are the respective kinetic flow factors in the sections before and after the jump.

Referring to the so-called Boussinesq number which, as indicated previously, is being used improperly to an ever-increasing extent:

$$\mathsf{B} = \frac{V}{\sqrt{g\,r'}} \ldots\ldots\ldots\ldots (18)$$

is an expression of the "Froude type" (Equation (15)), in which, $r' = \frac{2\,A}{\text{wetted perimeter}}$, is twice the value of Chezy's hydraulic radius. The hydraulic radius is related inherently with friction effect. It would be quite appropriate as a factor in a characteristic of the Reynolds type (Equation (14)) as, for example, in a study of friction forces in uniform flow; but it is entirely out of place in a case (such as the "jump", or flow through a structure) in which the motion is primarily dependent on gravity action. Gravity has no relation whatever to the hydraulic radius, the latter being thus a parameter that bears no physical relation to the essence of the case.

Dimensionless Representation.—Equations (16) and (17) are given in terms of ratios; that is, in a dimensionless form, which is becoming common practice in engineering science. Dimensionless representation results in equations of the most general type. Equations (16) and (17), for example, do not refer to any particular jump. They apply equally to a jump at the foot of the Boulder Dam and a small-scale model in a laboratory flume. The ratio of the depths will be the same when, and if, the kinetic flow factor is identical.

Assume, for example, that the depth of flow in any structure is d_p, with a unit-width discharge, q_p. Assume, furthermore, that the model is to be reproduced on a geometrical scale of 1:10 so that the depth of flow is $d_m = \frac{d_p}{10}$. The discharge in the model, reproducing dynamically similar conditions, requires identity of the Froude number, or an identical value of λ. Thus,

$$\frac{q^2_m}{d^3_m} = \frac{q^2_p}{d^3_p} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

or, with a scale of 1:10:

$$q_m = q_p \sqrt{\frac{1}{1\,000}} = 0.0316\, q_p$$

General Dimensionless Characteristics of the Jump in Terms of λ.—The different features of the jump may be expressed in a useful form as ratios, $d'_1 = \frac{d_1}{\epsilon_1}$; $d'_2 = \frac{d_2}{\epsilon_1}$; etc. The physical meaning is made clear by referring to Fig. 2, in which, ϵ_1 is equal to the head, H, provided the outflow losses are omitted.

Referring to Fig. 2: $d'_1 = \frac{d_1}{\epsilon_1}$; $d'_2 = \frac{d_2}{\epsilon_1}$; $d'_j = d'_2 - d'_1$; $V^2_1 = 2g\,(\epsilon_1 - d_1)$; and, therefore,

$$\frac{(V'_1)^2}{2g} = 1 - d'_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Furthermore, $\lambda_1 = \frac{2\,(1 - d'_1)}{d'_1}$; and,

$$d'_1 = \frac{2}{2 + \lambda_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

By means of Equation (21) and bearing in mind that $(u'_2) = u'_1 \left(\frac{d'_1}{d'_2}\right)$ one can also determine the energy in Section (2) after the jump,

$$\epsilon'_2 = d'_2 + \frac{(V'_2)^2}{2g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

and the loss in the jump is,

$$\epsilon_j' = 1 - \epsilon_2' \quad \text{.........................(23)}$$

The different elements of the jump may be traced[7] in terms of d'_1, or they may be presented directly in terms of the kinetic flow factor resulting in the curves of Fig. 3.

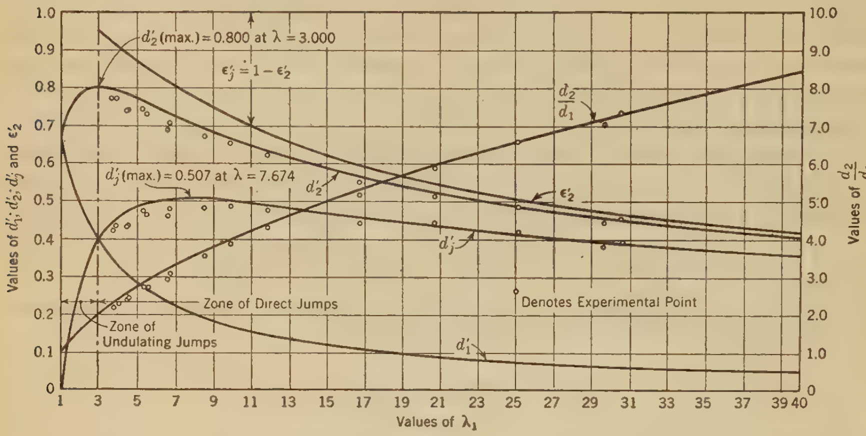

FIG. 3

These curves represent the most general dimensionless characteristics of the jump in terms of dynamic similarity. The following special features of the curves as drawn are to be emphasized as they bear directly on the experiments to follow: (1) The highest possible theoretical stage, $d'_2 = 0.800$, is reached at $\lambda_1 = 3$, the corresponding value of the initial depth being $d'_1 = 0.4$; and, (2) the d'_j-curve shows that the jump attains its maximum height at $\lambda_1 = 7.674$, with $d'_j = 0.507$.

It should be emphasized that the characteristics as given in Fig. 3 result from theoretical relations obtained by using the momentum principle. They apply to a jump in a horizontal flume, the slight effect of friction forces on the outward boundaries being neglected. It remains for experiments to show, in general, whether the foregoing premises, which permit a simple theoretical approach, are justified. The curves in Fig. 3 are supported by results obtained from experiments described subsequently. On the whole, theory and observation agree most satisfactorily. As one should expect the actual observed values of d_2 and d_j are somewhat lower than the theoretical values. This is due to the neglected effect of external friction. In general, experiments in flumes of larger size and on a larger scale, in all probability, would make the deviations still smaller.

EXPERIMENTAL PROCEDURE

The Flume.—The experiments supporting this paper were run in the varied-flow flume of the Fluid Mechanics Laboratory, at Columbia University. The tilting device, a general outline of which is given in Fig. 4, was 20 ft long, 6 in. wide, and 22 in. high; it was made of welded metal. Water was fed to the supply basin, by a 5-in. vertical pump, the basin being provided with suitable baffles and a regulated overflow.

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[7] "Hydraulics of Open Channels", Eng. Societies Monograph, 1932, p. 248.

Due to limited space and to peculiarities of the foundations the sump was of rather restricted volume, built into the lower part of the supply basin with a return channel substantially at the same level, placed under the flume. The discharge was measured by a special device designed on the principle of the "critical depth meter". The device was calibrated by volumetric measurement and was also compared with a sharp-crested weir, using the Rehbock and the Swiss Engineering Society's coefficients. The two methods gave practically identical results. It is assumed that the discharge is known accurately to within 1 per cent.

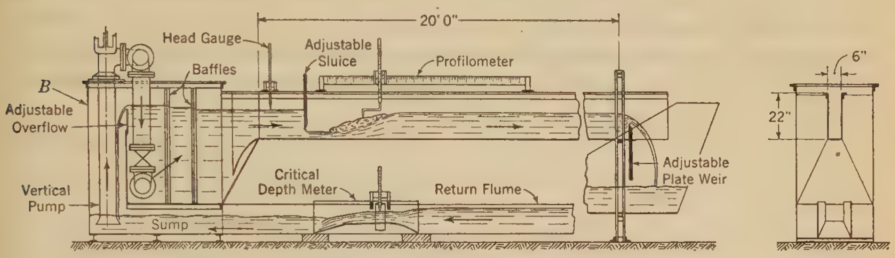

FIG. 4

In the particular experiments referred to in this paper, the bottom of the flume was kept horizontal, and the "rapid" flow necessary for the formation of a jump, was produced by an adjustable sluice. The tail-water was regulated, and thus the location of the jump was controlled by means of an adjustable plate weir at the end of the flume. A profilometer permitted readings that were accurate to within 0.001 ft. A view of the jump "in action" has been presented elsewhere[8].

A few words of explanation may be useful to clarify some of the problems connected with measuring the hydraulic jump. The jump appears to the eye as an erratic phenomenon subject to violent and apparently inordinate oscillations. Thus, the position of the toe of the jump varies continuously, up stream and down stream. Even greater agitation is noticeable in the roller to which, at first, one may be led to doubt whether any uniform or stable laws apply. However, it is known that irrespective of the apparent convulsions, the basic dynamic relations applying to the jump (such as the ratio between the depth, d_1 and d_2) are unexpectedly uniform and constant when taken as average values over a given period. Moreover, no matter how violently the position of the jump appears to vary, a stable and definite average reading of a vertical element may be obtained over a period when, for example, the position of the pointer is adjusted so that the flow oscillates to one side and then the other, around the selected position, at more or less equal time intervals. It is in this sense that an average profile of the roller covering the expanding "live vein" of the jump may be obtained by successive vertical readings at chosen points.

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[8] *Civil Engineering*, November, 1934, p. 564.

Further explanation is necessary with regard to the term, "length of the jump". The beginning of the jump, at the toe of the roller, is well defined; but there has been some confusion as to what constitutes the "end of the jump". Some writers[9] assume the end to be substantially at a point where adverse flow is no longer observable. In the writers' experience, this definition would determine the end of the roller, and not the end of the jump.

The roller is usually shorter than the jump as a whole. Furthermore (and this concept is essential), the roller is an accompanying feature only. The essential function of the jump is to transform the energy from kinetic to potential. This is accomplished in the expanding vein under the roller. The jump is thus characterized pre-eminently by an increasing depth of flow. On the contrary, beyond the jump, after expansion has ceased and the stream filaments have become parallel (in the sense defined herein under the heading, "The Hydraulic Jump"), the depth begins to diminish—at least in a horizontal flume. In other words, in this region, flow is characterized by the lowering of the surface level. It is logical under such circumstances to conceive of the end of the jump as a section at which the surface level reaches its maximum height and at which the rising curve of the expanding vein under the roller passes into the "drop" surface of the subsequent reach of gradually varied flow. In other words, as shown schematically in Fig. 2, the end of the jump is at Section (2) where the depth, d_2, has reached its maximum value. In practical experimental work the difficulty lies in the fact that the entire surface curve in this region is very flat and subject to continuous oscillations. In most cases, therefore, it is rather impossible to determine the position of the end section by direct observation. The method used was to trace profiles based on average observations, and then to select the end point from the drawing thus obtained. It was necessary to trace such profiles for each run (see Fig. 5). Even then, due to the flatness of the curves, all one can do

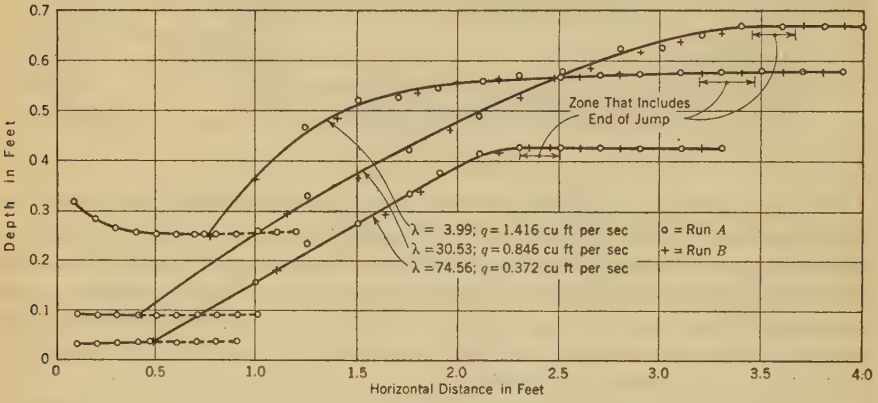

FIG. 5

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[9] See, for example, *Civil Engineering*, May, 1934.

is to establish a zone within which the position of the end section is more or less arbitrary. Obviously, in this instance, as in so many other cases of experimental engineering, one is dealing with a "transition zone".

Another delicate operation is the measurement of the initial depth at the lower stage in Section (1), (Fig. 2). The higher the kineticity, the smaller will be the value of d_1. In Equation (12), the depth, d_1, appears as a third power. Any experimental error, therefore, affects the final result seriously. These facts define a natural limit to experimentation. One cannot increase λ_1 by decreasing d_1 beyond a reasonable amount. Then (see Fig. 2), the initial depth must be measured at the toe of the roll, which is in an oscillating state and which obviously interferes with the pointer of the profilometer. The procedure used was again to trace an auxiliary profile of the free vein over a certain stretch by moving the jump somewhat downward. After the profile was thus determined, the jump was brought back into position, and the respective depth, d_1, was taken from the auxiliary tracing. The procedure is clearly demonstrated in Fig. 5.

The actual experiments were run during a period of about ten months, beginning December, 1932. At least one-half the time was spent on calibration and preliminary work, in which methods of procedure were gradually developed. Most of these preliminary results were discarded and, with the experience and skill gained, a second and final set was run. For each run a certain discharge, q, and a certain depth, d_1, were maintained over the entire period of the experiment. Then, a double series of profile observations were made, one going down stream and then returning up stream. The discharge was measured at the beginning and at the end of each run. Only such runs were considered satisfactory for further use and interpretation, which showed permanent discharge and which, in addition, registered coinciding profile readings. On the whole, experimenting with the jump is tedious work, demanding patience and perseverance. At best, in a day of 8 or 9 hr, only two successful runs were possible.

RESULTS AND CONCLUSIONS

The principal results of the experiments are summarized in Table 1. The observed values are q, d_1, d_2, and the length of the jump, L. For the latter (Table 1, Columns (4) and (5)) the experimental data are given in the form of a range as shown by Fig. 5. The other tabular values were computed. The basic factor is, $\lambda_1 = \frac{q^2}{g\, d^3_1}$; and the initial energy, $\epsilon_1 = d_1 + \frac{q^2}{2g\, d^2_1}$. The ratio, $\frac{d_2}{d_1}$, in Column (12), Table 1, is the quotient, and d_j in Column (7) is the difference of the values in Columns (3) and (2). The dimensionless factors in Columns (9), (10), and (11), Table 1, are obtained by dividing Columns (2), (3), and (7) by Column (8).

Length of the Jump.—The first and immediate aim of the research undertaken was to obtain comprehensive data concerning the length of the jump. Columns (13) to (20), Table 1, give the length in terms of dimensionless

TABLE 1.—OBSERVATIONS ON THE HYDRAULIC JUMP; EXPERIMENTS AT COLUMBIA UNIVERSITY

Run No.	Unit discharge, q, in cubic feet per second	DEPTH OF FLOW d, IN FEET (SEE FIG. 2):		LENGTH OF JUMP L, IN FEET, RANGING:		Kinetic flow factor, λ_1	Height of jump, d_j, in feet	Energy head, ϵ_1, in feet	RATIOS d' AT SECTIONS:	
		Section (1)	Section (2)	From	To				$d'_1\left(=\frac{d_1}{\epsilon_1}\right)$	$d'_2\left(=\frac{d_2}{\epsilon_1}\right)$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
S 27	1.400	0.253	0.561	2.30	2.70	3.76	0.308	0.729	0.347	0.769
S 30	1.416	0.251	0.574	2.44	2.84	3.94	0.323	0.746	0.337	0.769
S 40	1.286	0.225	0.541	2.23	2.63	4.51	0.316	0.733	0.307	0.739
S 43	1.312	0.228	0.550	2.47	2.87	4.52	0.322	0.743	0.307	0.740
S 25	1.680	0.254	0.693	2.86	3.36	5.35	0.439	0.934	0.272	0.742
S 45	1.648	0.249	0.681	2.86	3.26	5.47	0.432	0.930	0.268	0.732
S 41	1.590	0.228	0.681	3.23	3.53	6.63	0.453	0.984	0.232	0.692
S 24	1.816	0.248	0.765	3.65	4.05	6.72	0.517	1.081	0.229	0.708
S 28	2.060	0.249	0.881	4.19	4.49	8.55	0.632	1.313	0.190	0.671
S 26	2.284	0.254	0.989	4.87	5.17	9.90	0.735	1.511	0.168	0.654
S 29	2.030	0.221	0.957	4.95	5.25	11.87	0.736	1.533	0.144	0.624
S 36	1.600	0.168	0.869	4.35	4.65	16.79	0.701	1.578	0.107	0.551
S 18	1.148	0.125	0.742	3.60	3.90	20.75	0.617	1.397	0.091	0.515
S 6	0.988	0.107	0.705	3.23	3.53	25.10	0.598	1.443	0.074	0.488
S 17	0.820	0.089	0.630	2.94	3.24	29.66	0.541	1.409	0.063	0.447
S 39	0.846	0.090	0.663	3.05	3.25	30.53	0.573	1.464	0.062	0.453
S 35	0.586	0.062	0.536	2.42	2.62	44.81	0.474	1.451	0.043	0.369
S 37	0.508	0.053	0.503	2.24	2.44	53.89	0.450	1.481	0.036	0.340
S 32	0.456	0.047	0.477	2.01	2.21	62.25	0.430	1.510	0.031	0.316
S 34	0.376	0.040	0.435	1.90	2.00	68.70	0.395	1.414	0.028	0.308
S 33	0.372	0.039	0.428	1.81	2.01	74.56	0.389	1.454	0.027	0.294
S 38	0.288	0.032	0.379	1.47	1.77	78.69	0.347	1.291	0.025	0.294

Run No.	Ratio, $d'_j\left(=\frac{d_j}{\epsilon_1}\right)$	Ratio, $\frac{d_2}{d_1}$	RATIOS OF COMPARISON, RANGING:							
			$\frac{L}{d_1}$		$\frac{L}{d_2}$		$\frac{L}{d_j}$		$\frac{L}{\epsilon_1}$	
			From	To	From	To	From	To	From	To
	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
S 27	0.423	2.217	9.09	10.67	4.10	4.81	7.47	8.77	3.15	3.70
S 30	0.433	2.287	9.72	11.31	4.25	4.95	7.55	8.79	3.27	3.81
S 40	0.432	2.406	9.91	11.69	4.12	4.86	7.06	8.32	3.04	3.59
S 43	0.433	2.412	10.83	12.59	4.49	5.22	7.67	8.91	3.32	3.86
S 25	0.471	2.730	11.26	13.23	4.13	4.85	6.51	7.65	3.06	3.60
S 45	0.463	2.733	11.49	13.09	4.20	4.79	6.62	7.55	3.07	3.51
S 41	0.460	2.987	14.17	15.48	4.74	5.18	7.13	7.79	3.28	3.59
S 24	0.479	3.087	14.72	16.33	4.77	5.29	7.06	7.83	3.38	3.75
S 28	0.481	3.538	16.83	18.03	4.75	5.10	6.63	7.10	3.19	3.42
S 26	0.486	3.892	19.17	20.35	4.92	5.23	6.63	7.03	3.22	3.42
S 29	0.480	4.330	22.40	23.75	5.17	5.49	6.73	7.13	3.23	3.42
S 36	0.444	5.173	25.89	27.68	5.01	5.35	6.21	6.63	2.76	2.95
S 18	0.441	5.913	28.80	31.20	4.85	5.26	5.83	6.32	2.58	2.79
S 6	0.415	6.615	30.19	32.99	4.58	5.01	5.40	5.90	2.24	2.45
S 17	0.384	7.079	33.03	36.40	4.67	5.14	5.43	5.99	2.09	2.30
S 39	0.391	7.372	33.89	36.11	4.60	4.90	5.32	5.67	2.08	2.22
S 35	0.327	8.645	39.03	42.26	4.51	4.89	5.11	5.53	1.67	1.81
S 37	0.304	9.490	42.26	46.04	4.45	4.85	4.98	5.42	1.51	1.68
S 32	0.285	10.159	42.77	47.02	4.21	4.63	4.67	5.14	1.33	1.46
S 34	0.279	10.888	47.50	50.00	4.37	4.60	4.81	5.06	1.34	1.41
S 33	0.267	10.974	46.41	51.54	4.23	4.70	4.65	5.17	1.24	1.38
S 38	0.269	11.860	45.94	55.31	3.88	4.67	4.24	5.10	1.14	1.37

ratios, the observed L-values, Columns (4) and (5), being divided by Columns (2), (3), (7), and (8). The results are represented in Fig. 6. In tracing the curves, it is obvious that they must pass within the plotted regions, indicating the possible range of the L-values. Then, the investigator should be guided by the interrelation between d_2, d_1, d_j, etc., as illustrated by Fig. 3. In other words, when a certain region happens to be marked by particularly scattered data, the curve in question can be traced, nevertheless, by making use of some other curve, which in that particular region will be experimentally defined with better precision. The upper limit of λ_1 practicable in the case, could not be extended beyond 70 to 80 (see Fig. 6). The lower limit (slightly more than 3.5) is determined, on the other hand, by certain physical features inherent in the phenomenon. As indicated previously in another connection,[10] the maximum d_2-point in Fig. 3, corresponding theoretically to $\lambda_1 = 3$, delimits two possible forms of the jump: (1) The direct jump illustrated by Fig. 5, and the undulating jump in Fig. 7. Obviously, the region near $\lambda_1 = 3$ is a zone of transition, since it is reasonable to anticipate that the change to the undulating form is gradual and that the undulations increase as λ_1 is diminished. Fig. 7 gives a striking confirmation of the foregoing statements. The discharge being kept constant, the sluice was gradually opened and the kineticity decreased. The undulations become more and more pronounced. No numerical values for λ are given in the diagram, because the vein at the lower stage ceased to be parallel. Due to curvature the distribution of pressure is no longer hydrostatic and the simple expression (Equation (2)) for ε is no longer applicable. One may gather from Fig. 3 that the effect of boundary friction (disregarded in the Bélanger theory) would be to shift the actual point, d'_2 (maximum), somewhat to the right. For this reason, the zone dividing direct and undulating jumps would be actually at a somewhat greater value than the theoretical, $\lambda = 3$; possibly $\lambda = 3.5$, or 4.0.

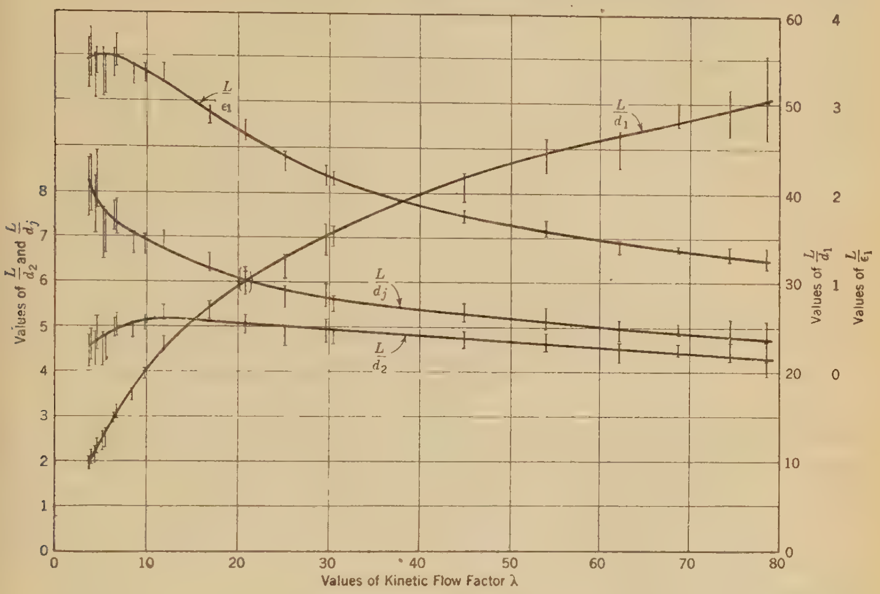

FIG. 6

Generalized Profiles of the Jump.—In plotting the profiles, in Fig. 5, the writers became impressed by the fact that the outlines of apparently such an irregular and capricious phenomenon as the jump finally proved to be unexpectedly regular. The idea naturally arose to reduce all the observed profiles to some unified dimensionless form. After certain trials, it appeared expedient to use the height of the jump, $d_j = d_2 - d_1$, as the parameter for reference. The different profiles were referred to a system of co-ordinates, shown in Fig. 8(*a*). The zero point coincides with the beginning of the jump. The co-ordinates of the surfaces are made dimensionless by introducing the ratios, $\frac{x}{d_j}$ and $\frac{y}{d_j}$. A series of profiles was traced in this manner, a selection of which is given in Fig. 8. Intermediate profiles are omitted in order to avoid obscuring the curves.

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[10] "Hydraulics of Open Channels", Eng. Societies Monograph, 1932, p. 249

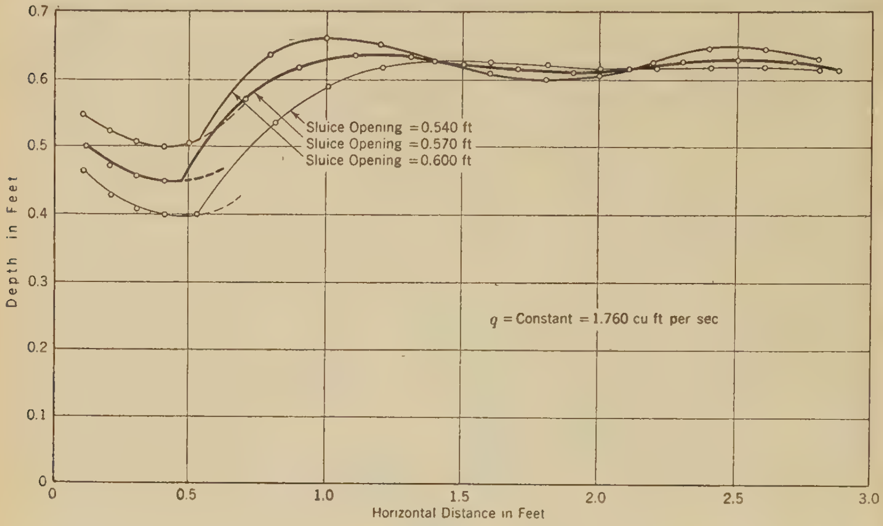

Fig. 7

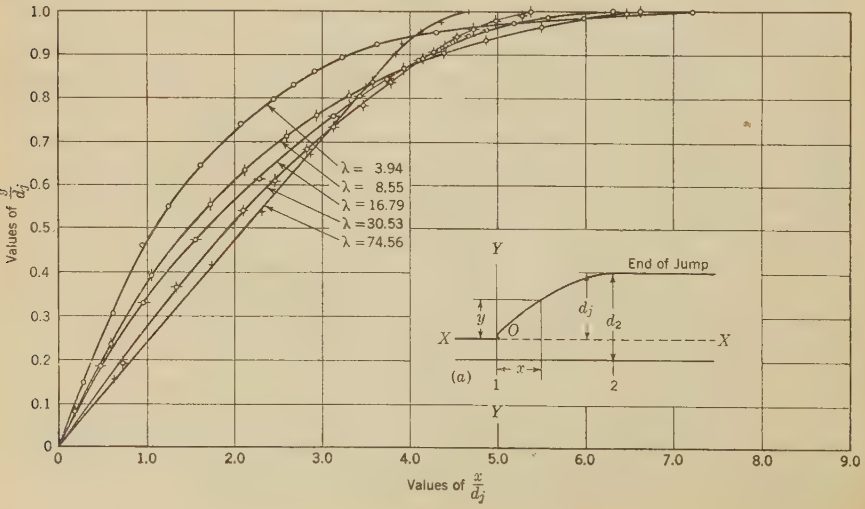

Fig. 8

The jumps embodied in these profiles are typical of a large variety. The initial depth varies from 0.03 to 0.25 ft; the length, L, from 1.5 to more than 5 ft; and the ratio, $\frac{d_2}{d_1}$, changes from about 2 to nearly 12. Notwithstanding this variety, the profiles, when presented in terms of dimensionless coordinates, range within a rather restricted area. A further step is reached by using the tri-dimensional presentation given in Fig. 9.

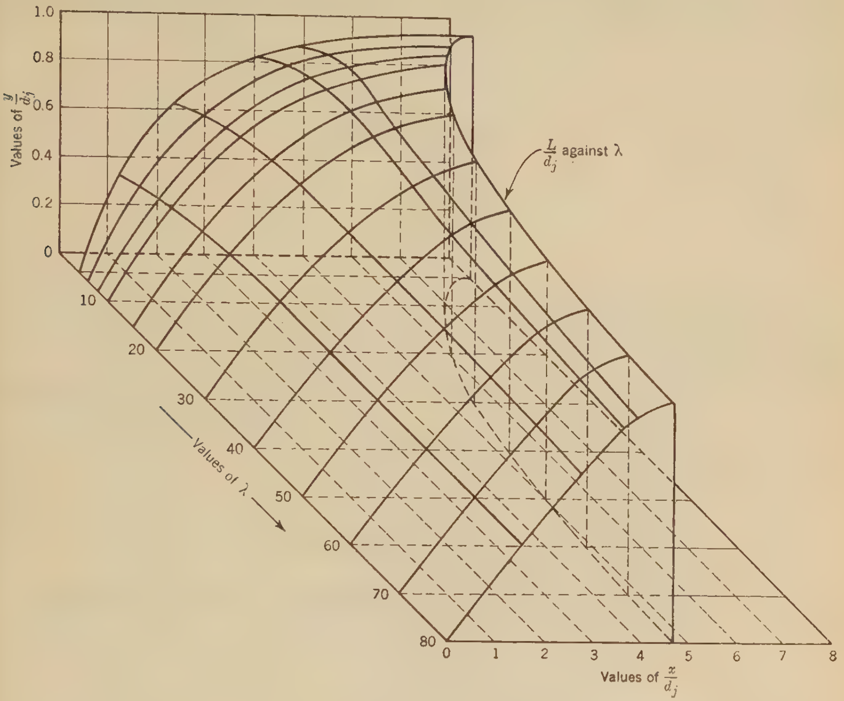

FIG. 9

Conclusions.—The results obtained by the writers may be summarized by stating that, by plotting dimensionless ratios against the kinetic flow factor, a consistent unified set of curves has been obtained. The most consistent regularity is that of the $\frac{L}{d_2}$-curve, which is in accord with certain practical indications scattered throughout recent periodical literature. The form of the $\frac{L}{d_2}$-curve, which shows a maximum point, should be considered in connection with the outline of the curves in Fig. 3.

The curves are offered to the Engineering Profession as seemingly the broadest summarized experimental evidence available at the moment. It is felt, that they may be useful in connection with practical engineering design.

The writers are fully conscious of the fact that the size of the flume and certain imperfections of its construction militate against a greater precision in the results. It is believed, however, that the general aspects of the phenomenon of the hydraulic jump have been made clearer.

Acknowledgment

The experiments described in this paper were conducted in the Fluid Mechanics Laboratory, Department of Civil Engineering, Columbia University. Special acknowledgments are due Dr. Hunter Rouse for the preparation of Fig. 9 and for other valuable assistance.

APPENDIX

Notation

The following notation, introduced in this paper, is consistent with the American Tentative Standard Symbols for Hydraulics[11]:

d = depth of flow = potential energy head, ε_p; d_p = depth of flow in the prototype; d_m = depth of flow in a model.
g = acceleration due to gravity.
j = a subscript denoting "hydraulic jump".
k = a subscript denoting "kinetic"
l = an appropriate longitudinal parameter.
m = a subscript denoting "model"
p = intensity of pressure; as a subscript, p denotes "prototype".
q = discharge per unit width.
r = radius; $r' = \dfrac{2\,A}{\text{wetted perimeter}}$.
x = variable distances from the Y-axis
y = variable distances from the X-axis.
z = elevation head; z_0 = depth of the center of gravity of a section below the free surface.
A = area of section; A_d = area at depth, d.
B = Boussinesq number.
C = Chezy's coefficient; as a subscript, C, denotes "critical".
F = force; total force.
F = Froude's number.
H = total head at any point.
L = length of hydraulic jump
P = total pressure.
Q = rate of discharge.

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[11] A. S. A.—*Z10b*—1929.

R = hydraulic radius.
R = Reynolds number.
V = average velocity at a section.
γ = specific weight of a fluid.
ϵ = energy head, or total specific energy of flow; ϵ_p = potential energy; ϵ_k = kinetic energy, ϵ_j = energy loss in the hydraulic jump.
λ = kinetic flow factor.
μ = absolute viscosity = $\nu \rho$.
ρ = density.
ν = kinematic viscosity $= \dfrac{\mu}{\rho}$.

DISCUSSION

Hunter Rouse,[12] Esq. (by letter).—In full accord with modern trends in the science of fluid mechanics, the authors have recognized in this familiar hydraulic phenomenon a physical problem which may be fully understood only through physically correct methods of approach. By means of dimensionless analysis of the phenomenon according to the Froude criterion of dynamic similarity, the hydraulic jump has finally been given definite form and character in terms of a universal parameter, with the determination of general relationships involving jumps of every size and nature.

Too much credit cannot be given for the devotion of many hours to tedious measurement, computation, and discussion, during which the writer was often a privileged witness. Only those who have experienced the high degree of turbulence and accompanying fluctuation in the region of the jump can appreciate the care such measurements entail; yet far from the expected scattering of points, the authors' results show remarkable coordination. Aside from presenting new data covering systematically a great range of conditions, and a correct physical interpretation of these data, the authors' paper serves the excellent purpose of clarifying a number of basic points often confused in previous investigations. The writer adds the following discussion in the hope that it may throw further light upon so important a type of fluid motion.

During the century following Bidone's first description of the jump, hydraulicians were slow to conclude that the popular energy principle had to be abandoned for the principle of momentum before the height of the jump could be computed successfully. Although the application of the momentum principle had been suggested shortly before the middle of the Nineteenth Century by Bélanger, emphasized later by Bresse, and finally discussed at length by Unwin in the Encyclopedia Britannica in 1880, as late as 1916 many publications, both in the United States and abroad, still held persistently to the assumption of constant energy according to the Bernoulli theorem.

Various writers, however, developed empirical equations for the relation of the two depths, believing that energy loss must exist; typical of these is that of Merriman, included in the early editions of his text on hydraulics.[13] Whatever the relationship used, abundant experimental results were quoted by the different writers, the best-known data being those of Bidone (1818), Darcy-Bazin (1856), Ferriday-Merriman (1894), Gibson (1913), and Riegel-Beebe, of the Miami Conservancy District (1917). With the perfection of the indoor hydraulic laboratory in Germany considerable study was given the jump, of which the experiments carried out under Koch at Darmstadt are good examples.[14] Rehbock made constant use of the basic momentum principle in the design of spillway aprons,[15] and the much-quoted thesis of Böss on open-channel flow[16] discussed the position of the jump with regard to channel slope and discharge. It is noteworthy, however, that little mention was made by any of these investigators of the longitudinal profile; nor had successful attempts been made to treat the problem as one of dynamic similarity.

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[12] With Soil Conservation Service, California Inst. of Technology, Pasadena, Calif. Elected an Assoc. M. Am. Soc. C. E., on January 13, 1936.

[13] "A Treatise on Hydraulics," by Mansfield Merriman, M. Am. Soc. C. E., Fourth Edition, John Wiley & Sons, New York, 1894.

[14] "Bewegung des Wassers," von Koch-Carstenjen, Julius Springer, Berlin, 1926.

[15] "Die Verhütung schädlicher Kolke bei Sturzbetten," von Th. Rehbock, *Der Bauingenieur*, 1928, Heft 4, 5.

Apparently, Groat and Kennison were the first to make use in this case of a term similar in form to the Froude number,[17] but it remained to Safranez,[18] in a series of experiments conducted in the laboratory of the Technische Hochschule, at Berlin, to undertake a two-dimensional physical study of the problem, in which constant use was made of the foregoing dimensionless term—without realization that it was actually the Froude number. Safranez gave an empirical relationship for the length of the jump, stating, in addition, that it was approximately 4.5 times the down-stream depth and that the slope of the jump was practically constant. (Refer to Figs. 8 and 9 of the authors' paper these two diagrams deserve careful study, as they show in comprehensive form the actual variation of the length and the shape of the jump with the Froude number, and the transition to the undular form as this number approaches a minimum.)

Following closely upon the publication of the foregoing papers, a controversy arose abroad over the part played by the surface roller in the apparent energy loss accompanying the jump. The roller was held by several (in particular by Kozeny[19]) to be of negligible influence, and various experimental and theoretical methods are still being used and misused to prove both sides of the case. Einwachter, whose recent paper on the length of the jump[20] has been supplemented by as yet unpublished experiments, has attempted to show the importance of the roller by replacing it with a fixed upper boundary and noting the resulting change in depth; Schoklitsch,[21] on the other hand, has analyzed the action of the roller from the viewpoint of regions of discontinuity (sources of turbulence) between the roller and the actual flow resulting in turbulent interchange of momentum.

The most recent departure from traditional research has been the endeavor of Engel[22] to study the jump as a three-dimensional phenomenon in the diverging section of a Venturi flume. It was by Engel that the Boussinesq number was devised, thus incorporating in a universal parameter characteristic of gravitational action a width term having no bearing whatever upon the force of gravity.

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[16] "Berechnung der Wasserspiegellage," von P. Böss, V. D. I. Verlag, Berlin, 1927.

[17] "The Hydraulic Jump in Open Channel Flow at High Velocity," by Karl R. Kennison, M. Am. Soc. C. E., *Transactions* Am. Soc. C. E., Vol. LXXX (1916), p. 338.

[18] "Wechselsprung und die Energievernichtung des Wassers," von K. Safranez, *Der Bauingenieur,* 1927, Heft 49; "Untersuchungen über den Wechselsprung," *Der Bauingenieur,* 1929, Heft 37, 38; and "Energieverzehrung der Deckwalze," *Der Bauingenieur,* 1930, Heft 20.

[19] "Wassersprung und Energieumwandlung," von J. Kozeny, *Wasserkraft und Wasserwirtschaft,* Heft 27, 1932.

[20] "Berechnung der in der Wehrbreite gemessenen Langenausdehnung von Deckwalzen," von J. Einwachter, *Wasserkraft und Wasserwirtschaft,* Heft 14, 21, 1932; Heft 17, 1933.

[21] "Ueber die Energievernichtung durch Walzen," von A. Schoklitsch, *Die Wasserwirtschaft,* Heft 16, 17, 1932.

[22] "Non-Uniform Flow of Water," by F. Engel, *The Engineer,* April 21 and 28, and May 5, 1933; and "Wassermengenmessung mit offenen seitlich eingeschnürten Kanälen," von F. Engel, *Zeitschrift* des Verein Deutscher Ingenieur, December 2, 1933.

The authors have distinguished carefully between the significance of the Froude number and of the Reynolds number, stating that the Froude number is the correct criterion in cases of gravitational action only, and the Reynolds number in the case of viscous action only, and that it is practically impossible to obtain complete dynamic similarity if both viscous and gravitational forces are present to an appreciable degree.

It is quite possible that, in the past, the loss of energy accompanying the jump has been the indirect cause of the failure to consider the jump in terms of the Froude number, for the hydraulic engineer has become prone to think of energy loss as synonymous with conversion into heat, the latter being possible in fluids only through viscous action. Thus, it would seem at first thought that viscous forces are by no means negligible in the hydraulic jump. In fact, discrepancies between computed and measured values of depths are generally attributed to friction losses along the bottom and sides of the channel. Actually, however, such losses are probably extremely small over the short length of flow involved; the discrepancy arises rather from a failure to consider the velocity defect along bottom and sides in computing the true mean velocity head; it is a known fact that such failure may cause considerable error in measuring the actual energy of flow, in particular since this defect is exaggerated by such abrupt deceleration of flow as occurs in the jump.

Clarity in the matter of energy loss may be attained only through a revision of the impressions received in elementary courses in hydraulics. "Loss" of hydraulic energy is seldom synonymous with immediate conversion into heat energy; in fact, it is only in the case of pure laminar flow that such immediate conversion takes place. In turbulent flow through smooth pipes, for instance, similar direct loss occurs in the laminar boundary film adjacent to the pipe walls, but within the turbulent region the situation is quite different.

In the familiar Bernoulli theorem, the specific energy of flow is expressed as the sum of velocity head, pressure head, and elevation. What the theorem fails to include is a fourth term expressing the energy of turbulence, often of great magnitude, which the hydraulician has had as yet no means of measuring or of utilizing; this missing term should denote the kinetic energy of rotation, since the velocity head includes only the kinetic energy of translation. It is the conversion to this turbulent head which corresponds in large measure to the apparent energy "loss" caused by valves, "stilling" racks, abrupt changes in section, flow through rough pipes, and by the hydraulic jump. In each of these cases the hydraulic "loss" or formation of excessive turbulence is largely or entirely independent of viscous action. Although turbulence itself may be reduced in intensity only through viscous shear and accompanying conversion into heat, this latter process is by no means immediate, but on the contrary is extremely gradual. Hence it is that the resistance coefficient for rough pipes at high Reynolds numbers (where turbulence is caused by the action of the roughness projections) is a constant term, independent of changes in viscosity; for the same reason the energy "loss" or conversion from energy of translation into energy of turbulence in the hydraulic jump is practically independent of viscosity and hence is not a function of the Reynolds number.

Were it possible to express energy of turbulence as a fourth term in the Bernoulli equation, it would follow at once that the jump could be treated correctly by means of the energy principle, since the actual energy of the individual particles is practically unaffected by thermal loss during the short time of passage through the jump. It is to be remarked that the general problem of turbulent energy is of outstanding importance in modern fluid research.

The question of the part played by the surface roller should be quite clear once this conception of turbulence is understood. The roller is as indispensable a part of the direct jump as the turbulent wake caused by flow past a valve is an inseparable part of the function of the valve. Those who have observed carefully the behavoir of the roller through the glass walls of an experimental flume will recall vividly enough the fact that the roller is not the idealized, distinctly outlined region of flow shown schematically in illustrations, but an active participant in the diverging flow. In other words, no portion of fluid involved at any instant in the reverse motion of the roller remains in the roller for more than a brief space of time; instead, violently rotating masses of this roller are torn away constantly into the live stream below, this continuous process resulting in the high degree of turbulence—and hence of apparent loss—which is characteristic of the phenomenon.

It should be obvious that such recent expedients as omitting the roller from the physical analysis of the phenomenon or replacing it by a sloping upper boundary held in place above the flow are quite beside the point. On the other hand, although the roller is inseparably associated with the jump, its length is not a true measure of the jump itself; nevertheless, European experimenters have often located the apparent end of the jump by determining visually that point at which the reversal of surface movement terminates. It is to the credit of the authors that their analysis is particularly definite and correct in all these troublesome points.

SHERMAN M. WOODWARD,[23] M. AM. SOC. C. E. (by letter).—After reading such an admirable exposition of a multiplicity of involved mathematical relationships, one feels a necessity for trying to reduce the complicated series of consecutive steps to some logical system simple enough to be pictured in one's mind as an entity. To do this it is necessary to review all the steps of the discussion and to attempt to isolate the fundamental connecting bases.

The entire mathematical theory for the hydraulic jump in a horizontal frictionless rectangular channel is based on two fundamental equations which, using the author's notation, are:

$$V_1\, d_1 = V_2\, d_2 \dots\dots\dots\dots\dots\dots\dots\dots (24)$$

and,

$$\frac{\nu\, V_1\, d_1}{g}\,(V_1 - V_2) = \frac{\nu}{2}\,(d_2^2 - d_1^2) \dots\dots\dots\dots\dots (25)$$

Equation (24) is commonly designated in hydraulics as the "law of continuity"; Equation (25) is a change-of-momentum equation, a form of Newton's second law of motion. In Equations (24) and (25): ν = density of the liquid; g = acceleration of gravitation; V_1 = velocity before entering the jump; V_2 = velocity after leaving the jump; d_1 = depth before entering the jump; and, d_2 = depth after leaving the jump. Considering ν and g as known constants, the other four quantities remain as variables. From Equations (24) and (25) any one of the four variables, as may be desired, may be eliminated. If V_2 is eliminated, for example, the commonly used formula for the jump is obtained:

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[23] Cons. Engr., T.V.A., Knoxville, Tenn.

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2\,V_1^2\,d_1}{g}} \quad \ldots\ldots\ldots\ldots (26)$$

In addition to the foregoing variables, the author introduces, among others, the following, all defined in terms of the original four:

$$q = V_1\,d_1 \quad \ldots\ldots\ldots\ldots (27)$$

$$d_j = d_2 - d_1 \quad \ldots\ldots\ldots\ldots (28)$$

$$d_c = \sqrt[3]{\frac{q^2}{g}} \quad \ldots\ldots\ldots\ldots (29)$$

$$\epsilon_{k1} = \frac{V_1^2}{2\,g} \quad \ldots\ldots\ldots\ldots (30)$$

$$\epsilon_{k2} = \frac{V_2^2}{2g} \quad \ldots\ldots\ldots\ldots (31)$$

$$\epsilon_1 = d_1 + \frac{V_1^2}{2\,g} \quad \ldots\ldots\ldots\ldots (32)$$

$$\epsilon_2 = d_2 + \frac{V_2^2}{2\,g} \quad \ldots\ldots\ldots\ldots (33)$$

and,

$$\epsilon_j = \epsilon_1 - \epsilon_2 \quad \ldots\ldots\ldots\ldots (34)$$

With Equations (24), (25), and (27) to (34), inclusive, it is possible, theoretically, to eliminate any nine of the twelve variables, giving 222 different equations, each containing three variables. Each of these equations can be solved for each of the variables and, therefore, can be written in three different ways, making a total of 666 possible equations, any of which may be used if one should so desire. (Actually, the number would be slightly less, due to the fact that some of the equations would contain only two variables.)

Unfortunately, an equation containing three variables cannot usually be represented graphically by a single line, or in any other equally simple manner. It can be represented by a surface, however, or in a plane by a family of lines. To handle an equation of three variables, then, it is a common device to divide by one of the variables, or reduce the variables to ratios, in such a way as, in effect, to reduce one of them to unity, thus obtaining an equation which can be platted as a single line. With twelve variables, ignoring reciprocals, sixty-six different ratios can be obtained, all of which can be used to swell to still greater numbers the list of possible equations. It would even be possible to extend the combinations further by taking ratios of ratios; but it is to be remembered, in contemplating such an endless series of mathematical gymnastics, that the entire mass contains no more knowledge than the two original equations and the definitions of the variables.

It is not extremely laborious to plot a family of curves; several families can be put on the same diagram; therefore, a few diagrams would suffice to introduce all the variables desired. However advantageous it may be to develop a large number of equations for purposes of research, of theoretical study, and of gaining an insight into the nature of the phenomenon, for the purposes of the practicing engineer, it seems desirable to adhere as closely as possible to the basic equations together with tables and diagrams to represent them.

Robert E. Kennedy,[24] M. Am. Soc. C. E. (by letter).—It may be of interest to apply the data given in Table 1 to the formulas for length of the hydraulic jump suggested[25] by Donald P. Barnes, Jun. Am. Soc. C. E., who reports certain investigations made in Germany by Dr.-Ing. Kurt Safranez and quotes conclusions and formulas by Professor A. Ludin. The authors take issue with Dr. Safranez and contend that the jump is longer than that distance subtended by the visible roller as arbitrarily defined by him. Professor Ludin suggests an $\frac{L}{d_2}$-value of 4.5. The average of values in Column (4), Table 1, for the observed minimum length of jump divided by values in Column (3), for d_2, the down-stream water depth, happens to be exactly 4.5. The average for values in Column (5), the maximum length, divided by d_2 is 5.0 and ranges from 4.6 to 5.5 which is from 92 to 110% of the average. This is a span of 18 points. An $\frac{L}{d_2}$-curve is shown in Fig. 6.

Professor Ludin proposes a more accurate formula which may be derived from the following:

$$R = \frac{V_1}{V_c} \quad \dots\dots\dots\dots\dots\dots\dots\dots (35)$$

and,

$$\frac{d_2}{L} = \frac{1}{4.5} - \frac{1}{6\,R} \quad \dots\dots\dots\dots\dots\dots\dots\dots (36)$$

in which, in addition to the notation of the paper, R = a "flow index."[16] By combining Equations (35) and (36) and solving for L, the length of jump, a formula is obtained which, in the notation of Fig. 2, is:

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[24] With Dept. of Investigations and Project Studies, U. S. Bureau of Reclamation, Denver, Colo.

[25] *Civil Engineering*, May, 1934, p. 262.

$$L = \frac{9\, d_2\, V_1}{2\, V_1 - 1.5\, V_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

in which V_c is the critical velocity obtained from Equation (4).

The results of applying the data in Table 1 to Equation (37) are shown in Table 2. The observed maximum length varies from 55% of the computed for the lowest ratio of $\frac{d_2}{d_1}$, in Column (12), Table 1, to 86% for the highest ratio. If the percentages in Column (7), Table 2, are plotted against the corresponding values of $\frac{d_2}{d_1}$, two curves may be drawn to indicate the approximate outside limits of the range of coefficients applicable to the Ludin formula and the discrepancy of this formula as based upon these data. Judging from these curves it appears to vary from 57 to 65%, or 8 points, for a $\frac{d_2}{d_1}$-ratio of 2.5 at the lower end of the curves and from 83 to 87%, or 4 points, for a $\frac{d_2}{d_1}$-ratio of 12 at the upper end.

TABLE 2.—COMPUTED LENGTH OF JUMP COMPARED TO MAXIMUM OBSERVED LENGTH

Run No. (see Table 1)	Critical depth, d_c, in feet	VELOCITY, IN FEET, PER SECOND*		LENGTH OF JUMP, L, IN FEET		Ratio: Column (6) / Column (5) (percentages)
		V_c	V_1	Computed	Observed†	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
S 27	0.394	3.55	5.54	4.87	2.70	55.4
S 30	0.397	3.57	5.64	4.91	2.84	57.8
S 40	0.372	3.46	5.72	4.46	2.63	59.0
S 43	0.377	3.48	5.76	4.52	2.87	63.5
S 25	0.444	3.78	6.62	5.46	3.36	61.6
S 45	0.439	3.75	6.62	5.33	3.26	61.2
S 41	0.428	3.72	6.98	5.10	3.53	69.2
S 24	0.468	3.88	7.32	5.72	4.05	70.8
S 28	0.509	4.05	8.28	6.26	4.49	71.8
S 26	0.546	4.19	9.00	6.84	5.17	75.5
S 29	0.504	4.03	9.18	6.42	5.25	81.8
S 36	0.430	3.72	9.52	5.52	4.65	84.2
S 18	0.345	3.33	9.18	4.59	3.90	85.0
S 6	0.312	3.17	9.24	4.27	3.53	82.7
S 17	0.276	2.97	9.22	3.73	3.24	86.9
S 39	0.281	3.01	9.40	3.93	3.25	82.7
S 35	0.220	2.66	9.45	3.05	2.62	85.9
S 37	0.200	2.54	9.58	2 82	2.44	86.5
S 32	0.186	2.45	9.70	2.65	2.21	83.4
S 34	0.164	2.29	9.40	2.39	2.00	83.7
S 33	0.163	2 28	9.54	2.35	2.01	85.5
S 39	0.137	2.09	9.00	2.06	1.77	85.9

* See Fig. 2. † From Column (5), Table 1.

For an illustration assume $d_1 = 0.36$ ft, $d_2 = 2.40$ ft, $V_1 = 17.2$ ft per sec, and $V_c = 5.7$ ft per sec. By the rough average approximation, $L = 5\, d_2$, the length is $5 \times 2.40 = 12.0$ ft. By Equation (27) $L = 14.4$ ft; or, applying the correction corresponding to a $\frac{d_2}{d_1}$-ratio of 6.7 which ranges from 82.5 to 86.5% and averages 84.5%, the length is equal to $14.4 \times 84.5\% = 12.2$ ft. In this case the "rough" approximation is not so rough!

L. STANDISH HALL,[26] ASSOC. M. AM. SOC. C. E. (by letter).—Designation of the component parts of the hydraulic jump in terms of a common parameter has reduced the characteristics of this phenomenon to a more readily understandable basis. The curves presented summarize the results of the experiments performed by the authors in a very concise manner. The differentiation between the use of the Froude and the Reynolds numbers is also notable.

With regard to the use of the results in engineering design, there are certain factors to be borne in mind which were not disclosed in the authors' experiments, which were made by producing a flow of high velocity by means of discharging water under a sluice. Flow of this type occurs from sluices in canals or from sluices at the toes of dams. A notable example of the latter is to be found in the flood-control dams of the Miami Conservancy District, in Ohio. Hydraulic jumps also occur at the bottoms of chutes, either in canals or on spillways from reservoirs. They also form at the toes of overflow dams.

In all cases where the flow of water is allowed to accelerate over a sufficient distance, air is mixed with the stream, and the phenomenon of "white water" occurs. The same admixture of air with water can be observed in waterfalls, or in natural stream channels, when the velocity is sufficiently high to draw air into the water by its agitation. Very few data appear to be available as to the volume of air which will become mixed with the water under these conditions. Such data as are available indicate that it is in proportion to the velocity; in other words, a velocity of 10 ft per sec would entrain 10% of air by volume and a velocity of 40 ft per sec would entrain 40% of air by volume.

The entrained air increases the bulk of the water near the bottom of a chute, or the apparent value of d_1 is increased. When this mixture strikes the pool at the foot of a chute, the agitation in the pool is greatly increased. In an open pool, it is not probable that the depth, d_2, is changed from the value normally expected, as the air is rapidly expelled. However, the length of the jump is materially increased, and it is usually found in practice that the length of pool as determined by the ordinary hydraulic formulas is inadequate. Additional free-board is also necessary at the bottom of the chute and in the pool, due to the great agitation of the water before the air is expelled.

More data are required for the design of such structures, both with regard to the volume of air entrained at high velocities and also with regard to the effect of the air on the hydraulic jump. Data on the first phase of the problem are rather difficult to secure by laboratory methods. It would appear that the velocity, the distance traveled, and the roughness of the surface of the conduit had some bearing.

However, data on the effect of entrained air on the hydraulic jump could be easily determined in the laboratory. In the experimental flume used by the authors, air could have been introduced in the bottom down stream from the sluice and the effect on the jump noted. It is hoped that the authors will be able to expand their experiments to cover this phase of the hydraulic jump. In the meantime, designers should increase the ratio of $\frac{L}{d_2}$ to values greater than 5 when conditions are such that it is to be expected that air will be entrained in the water above the jump.

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[26] Chf. Hydrographer, East Bay Municipal Utility Dist., Oakland, Calif.

MORROUGH P. O'BRIEN,[27] ASSOC. M. AM. SOC. C. E. (by letter).—The preoccupation of engineers with the vertical elements of the hydraulic jump to the neglect of the horizontal dimensions is attributable to general amazement at the almost perfect agreement between theory and experiment as regards the depths and velocities involved. That so turbulent a phenomenon needed no "coefficient of ignorance" was both surprising and encouraging, and interest in it has helped greatly in advancing the application of the momentum principle to the solution of hydraulic problems.

One feature of the hydraulic jump which deserves attention is the fact that the equations apply to a frictionless liquid and yet a loss of energy is indicated. The theory provides no means of dissipating energy and one is left to assume that the energy at Point (2) in Fig. 2 is the same as at Point (1), but that a portion of it has become unavailable to cause further flow. The explanation appears to lie in the fact that although the sums of the pressure forces and dynamic reactions are equal and of opposite sign at the two points, the lines of application do not coincide and the fluid moving through the region of the jump is subjected to a continuously applied moment which superimposes a rotational velocity on the average velocity of forward motion. The energy involved in this rotational motion is the "loss" obtained from the combination of the momentum and energy equations. It should be emphasized that the mere existence of a surface "roller" could not cause this energy loss because little energy could be dissipated in the roller itself. However, if parts of this roller are continually torn away from it and pass down stream as vortices, the total energy lost is the summation of the energies of these vortices. In any real liquid, viscosity soon brings about the conversion of this macroscopic rotational energy into thermal energy. That the jump is not a reversible phenomenon in which the depth can decrease in the direction of flow from d_2 to d_1, which is possible from momentum considerations alone, is indicated by the impossibility of recovering this random thermal energy as energy of directed motion.

Another feature of the hydraulic jump which deserves more attention is the distribution of velocity in vertical sections, and the rapidity with which it approaches that resulting from friction alone. It does not appear to be safe to assume that, at the depth, d_2, the velocity distribution is normal for the reason that the highest velocities exist initially very near the bottom and there is no general criterion indicating the distance necessary for them to be reduced to a certain fraction of their initial intensity. It may be that a phenomenon occurs which is similar to "separation" along the blades of propeller pumps and turbines. Considering the layers immediately adjacent to the bottom, momentum and velocity must be lost at a rate sufficient to overcome both friction and the adverse pressure gradient and a point is reached at which the forward velocity is reduced to zero. The main stream may then separate from the bottom and pass over an eddy of sign opposite that of the surface roller.

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[27] Associate Prof., Mech. Eng., Univ. of California, Berkeley, Calif.

In recent years, some difficulties have been encountered in the operation of hydraulic jump drops on irrigation canals. In these structures, the jump occurs in a pool at the bottom of an incline, and it may be that the usual pool length is not sufficient for the bottom velocities to be reduced to a safe value.

In most problems of open-channel flow, the energy and momentum curves can be computed from the average velocity (quantity divided by area), but where local conditions cause a considerable deviation from the normal velocity distribution, a correction factor must be applied to take into account the fact that the average of the momenta and energies transported across elementary areas exceeds the momentum and energy based on average velocity.[28] The authors assume that this correction factor is unity.

F. V. A. E. Engel,[29] Esq. (by letter).—The investigations reported in this paper are of particular interest to the writer and they are a valuable contribution to the application of a dimensionless presentation of hydraulic test results in open-channel flow. In these days the application of the Reynolds number is a common occurrence in analyzing the frictional coefficient in closed conduits, the discharge coefficient of orifices and Venturi tubes, etc. Nevertheless, few papers deal with the specific problems of open-channel flow on that basis.

Several pages of the paper refer to "Theoretical Premises," which involve the Reynolds, Froude, and Boussinesq numbers. The remarks concerning the Reynolds number are quite consistent with the usual conception of that expression in hydraulic engineering, but the writer would like to suggest some modification of the sections on the Froude and Boussinesq numbers, especially as he feels himself responsible for introducing a term which has been called the "Boussinesq" number.

The kinetic flow factor, known as the Froude number, was applied for the first time in a similar form by Th. Rehbock in 1919 for the calculation of the head losses for water flow through bridge piers. This factor (see Equation (2)) is given as twice the ratio of the kinetic energy head to the potential energy head contained in each pound of liquid flowing at the depth, d. The velocity, V, is the mean velocity obtained by dividing the rate of flow through the cross-section of the stream. This expression certainly is very convenient and may also be sufficient in some cases; but Coriolis, de Saint Venant, and others have shown that it is often necessary to use a value which takes into account the velocity distribution over the cross-section. For such conditions the kinetic energy is greater than the kinetic energy based on the mean velocity,[30] and, therefore, Equation (2) becomes:

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[28] "Velocity Head Correction for Hydraulic Flow," by M. P. O'Brien and J. W. Johnson, *Engineering News-Record*, August 16, 1934.

[29] Director of Research, Electroflo Meters Co., Ltd., London, England.

$$\epsilon = \epsilon_p + \epsilon_k = d + \frac{\alpha\, V^2}{2\, g} \text{ . . . . } (38)$$

in which α is greater than unity.

Equation (12), therefore, would become:

$$\lambda = 2\, \frac{\epsilon_k}{\epsilon_p} = \frac{\alpha\, V^2}{g\, d} \text{ . . . . } (39)$$

This expression, including α, would certainly be rather tedious to evaluate, and the reason for presenting it is mainly to show that the relations given by the authors are not quite as simple as they would indicate. They point out that "the hydraulic radius is related inherently with friction effect." In turn, the friction effect influences the velocity distribution and also the new term, α, in Equations (38) and (39). It seems quite reasonable to combine α and d in Equation (39) in such a manner that it becomes:

$$\lambda = \frac{V^2}{g\, \frac{d}{\alpha}}$$

and to replace the ratio, $\frac{d}{\alpha}$ (which has the dimensions of a length, but also takes into account the velocity distribution), by r', the hydraulic mean radius, obtaining Equation (18), the dimensionless quantity which has been called the Boussinesq number.

The writer has made extensive use of both the Froude and Boussinesq numbers in his present investigations on flow through side-contracted open channels (Venturi flumes).[31] The Froude number characterized a type of flow which is known as "free discharge." For a given rate of flow "free discharge" is defined as that condition under which the up-stream depth is unaffected, whatever the down-stream conditions may be.

The occurrence of a hydraulic jump could not always be determined by a "critical" Froude number. Rehbock[32] and other investigators also found that the "critical depth" had not yet been reached by the water level in the narrowest section of the contraction in spite of the appearance of surface rollers, which are commonly attributed to the presence of a hydraulic jump. Rehbock regarded this "partial change to rapid flow" ("Teilweiser Fliesswechsel") as being due to an uneven velocity distribution. In some section of the water stream, there is a sufficiently high velocity to produce a state of rapid flow.

The writer has analyzed[33] the expression which Professor Rehbock used to determine the first occurrence of a hydraulic jump. Rehbock's relation is a function of the Froude number and for that reason is not satisfactory to determine the change from streaming flow to rapid flow. The "critical" Boussinesq number was found to be more reliable over a wide range of flow to characterize the change from one type to another.[33]

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[30] "Elemente der Technischen Hydromechanik," by R. v. Mises, Leipzig and Berlin, 1914, pp. 153–155.

[31] "Non-Uniform Flow of Water: Problems and Phenomena in Open Channels with Side Contractions," by F. V. A. E. Engel, *The Engineer*, Vol. 155 (1933), pp. 392 to 394; 429 to 430; and 456 and 457; and "The Venturi Flume," *The Engineer*, Vol. 158 (1934), pp. 104 to 107; and 131 to 133.

[32] *Zentralblatt der Bauverwaltung*, Vol. 39 (1919) pp. 197 to 200.

[33] "Venturi-Kanalmesser: Die messtechnischen Eigenschaften in Abhängigkeit von den Strömungsarten," by F. V. A. E. Engel, *Archiv für technisches Messen*, No. 45, March, 1935.

The attacks on the Boussinesq number seem to the writer to be relatively unfounded. Three different applications have proved the usefulness of this number. The practical engineer must judge the merits of a new expression, especially with regard to the extent to which it simplifies the presentation of hydraulic test results, even if the definite theoretical proof is not yet established.

In three cases the application of the Boussinesq number has been found to yield some simplification in presenting the following hydraulic features:

(1) The discharge coefficient of a Venturi flume in the range of rapid flow is a linear function of the Boussinesq number;[31]

(2) The dissipation of energy in the case of a Venturi flume for the "limit of free discharge" can be presented in a simplified way by plotting the ratio of the down-stream depth to the up-stream depth against the Boussinesq number for a section in the throat;[31] and

(3) The change from turbulent or streaming flow to rapid flow is given by a critical Boussinesq number which determines the first occurrence of a hydraulic jump.[33]

It would have been very interesting if the authors had published the different features of the hydraulic jump in cases where the up-stream depth, or the depth at the foot of the hydraulic jump, exceeds the hydraulic mean radius. From Table 1, the writer perceives that the investigators cease their investigations at a point where the depth at the foot of the jump exceeds the value of the hydraulic mean radius. Furthermore, in Fig. 3, some test results in the range of the undulating jump (that is, for values of the kinetic flow factor smaller than 3) would be of general interest.

BALDWIN M. WOODS,[34] ESQ. (by letter).—As one interested in fluid mechanics, although more especially in aerodynamics, the writer desires to endorse heartily the plan of attack followed in this paper and the presentation of results in generalized, dimensionless form through the aid of the principle of dynamic similarity. From practical considerations alone, the growing interchange of material between countries having differing systems of units, such as the English and the metric, makes it highly important to have results equally useful regardless of the basic system of units. Furthermore, as has been so often noted, the method of dynamic similarity causes one to give careful scrutiny to the variables involved and to the most significant simple combinations of these variables into coefficients or parameters.

There is also much to be said in favor of the tri-dimensional representation of the profile, as given in Fig. 9. Engineers generally need to cultivate familiarity with these three-dimensional presentations and facility in reading them. They are labor-saving devices, much simpler in actual use than the combination of several two-dimensional charts, which are otherwise necessary. The entire Engineering Profession is indebted to the authors for the elegance of treatment resulting from the discrimination exercised in selecting this form of presentation.

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[34] Prof. of Mech. Eng., Univ. of California, Berkeley, Calif.

J. C. STEVENS,[35] M. AM. SOC. C. E. (by letter).—In presenting the results of their research on the length of the jump, the authors have made a most valuable contribution to hydraulics. They might have made a more happy choice of symbolism, however. The writer has found the notation, which follows Professor Bakhmeteff's book entitled "Hydraulics of Open Channels," [36] quite a "hazard" in mastering this paper.

There are only two independent characteristics of the hydraulic jump, the initial depth and the initial velocity. From them are derived the initial energy head, the final depth, final velocity, the final energy head, the lost energy head, and the empirical characteristic—length. If these factors are expressed in terms of any other, a series of ratios is obtained which expresses the characteristics of the jump in dimensionless terms of dynamic similarity.

The authors have chosen the initial energy head as the denominator of their dimensionless terms. Mr. E. S. Crump[37] has expressed the jump characteristics in terms of Belanger's critical depth. Mr. G. A. M. Brown[37] has expressed them in terms of the lost energy head. The writer following Woodward[38] and others has chosen to express them in terms of the initial depth. Thus, the dimensionless terms, in reality, become jump dimensions for a unit initial depth. Viewed in this light, such a jump may for brevity be termed a "unit jump," and a far simpler conception of it is had by thinking of it as such than through any other set of dimensionless terms. There appears to be no advantage in the use of the "kinetic flow factor," λ, over the ratio of initial kinetic to potential energy. This ratio, $k = \frac{V^2_1}{2\,g\,d_1}$ (which equals $\frac{\lambda_1}{2}$), has the limiting value of 0.5 for critical velocity, which expresses the simple criterion for critical flow that, in any shape of channel whatever, the velocity head is one-half the mean depth.

The authors' height of jump, $d_j = d_2 - d_1$, in practical applications is quite inferior to the ratio heretofore used by Woodward and others, namely, $J = \frac{d_2}{d_1}$, particularly when dealing with jumps in channels other than rectangular.

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[35] Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.

[36] Eng. Societies Monographs, McGraw-Hill Co., 1932.

[37] "The Standing Wave or Hydraulic Jump," *Publication No.* 7, Central Board of Irrig., Simla, India.

[38] Technical Repts., Pt. III, Miami Conservancy Dist.

The writer suggests the following simplified notation: Let d_1 = initial depth; d_2 = final depth; h_1 = initial velocity head; h_2 = final velocity head; E_1 = initial energy head = $d_1 + h_1$; E_2 = final energy head = $d_2 + h_2$; I = energy lost by impact and eddies = $E_1 - E_2$; and L = length of jump.

Then the following dimensionless terms apply in a "unit jump": $k = \frac{h_1}{d_1}$; $J = \frac{d_2}{d_1}$; $e_1 = \frac{E_1}{d_1}$; $e_2 = \frac{E_2}{d_1}$; $i = \frac{I}{d_1}$; and $l = \frac{L}{d_1}$.

In a rectangular channel, of unit width, the fundamental relationship applies:

$$J^2 + J = 4\,k \qquad \text{(40)}$$

which may be verified from Equation (16) by substituting $2k = \lambda_1$. The jump characteristics may now be expressed in terms of either k, or J, or both. Thus:

$$e_1 = k + 1 \qquad \text{(41)}$$

$$e_2 = J + \frac{k}{J^2} \qquad \text{(42)}$$

and,

$$i = e_1 - e_2 = \frac{(J - 1)^3}{4\,J} \qquad \text{(43)}$$

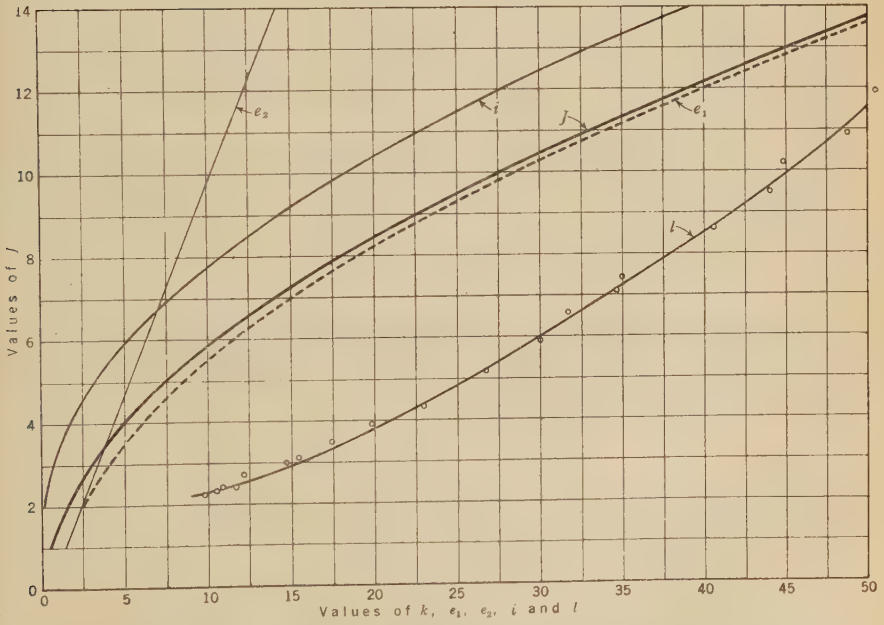

FIG. 10.

Fig. 10 shows the dimensionless characteristics of the unit jump in terms of either k or J. Curve J, as a function of k, is fundamental, and through it values of e_1, e_2, i, and l are found in terms of either k or J. The length is an empirical ratio obtained by averaging the values in Columns (13) and (14), Table 1.

It is significant to note in Fig. 10 that the characteristic curves are without points of maxima or minima. There is no maximum height of jump. The authors' maxima for d'_2 and d'_j, in Fig. 3, are not physical characteristics of the jump at all, but merely consequences of the terminology in which its characteristics are set forth. As a matter of fact these so-called maxima are for very low jumps: $d'_2 = 0.800$ for $\lambda_1 = 3$ corresponds to $k = \frac{3}{2}$ and $J = 2$, also $d'_j = 0.507$ at $\lambda_1 = 7.674$ corresponds to $k = 3.84$ and $J = 3.45$.

The advantage of the writer's terminology becomes apparent when expressions for the characteristics of the jump are desired for channels other than rectangular. Fig. 10, like Fig. 3, is applicable only to rectangular conduits and to unit widths. The fundamental relationships between J and k, however, may be found for a wide variety of conduits.[39] Once this relationship is found the other characteristics follow logically.

The general dimensionless formula for height of jump in a trapezoidal channel is:

$$J^4 + \left(\frac{5}{2}\,t + 1\right) J^3 + \left(\frac{3}{2}\,t^2 + \frac{5}{2}\,t + 1\right) J^2$$
$$+ \left[\frac{3}{2}\,t^3 + t - 6\,k\,(t + 1)\right] J - 6\,k\,(t + 1)^2 = 0 \quad \ldots\ldots.(44)$$

in which $t = \dfrac{b}{s\,d_1}$; b = bottom width; and s = the side slopes (that is, s on 1); if the two side slopes are different, s is the average of the two. For a rectangular channel $s = 0$, and it can be shown that Equation (44) reduces to Equation (40). For a triangular conduit $b = 0$, and Equation (44) reduces to:

$$J^4 + J^3 + J^2 - 6\,k(J + 1) = 0 \quad \ldots\ldots\ldots\ldots\ldots(45)$$

Considering a conduit with a cross-sectional area that may be expressed by the exponential function, $a = Kd^n$, in which K is a determinable constant, the height of the jump is given by,

$$\frac{J^n\,(J^{n+1} - 1)}{J^n - 1} = 2\,k\,(n + 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(46)$$

For a parabolic section, $n = \frac{3}{2}$, and Equation (46) becomes,

$$J^4 - (5\,k + 1)\,J^{\frac{3}{2}} + 5\,k = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(47)$$

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[39] "The Hydraulic Jump in Standard Conduits," by J. C. Stevens, M. Am. Soc. C. E., *Civil Engineering*, October, 1933, p. 364; also, discussion by G. H. Hickox, Jun. Am. Soc. C. E., *Loc. cit.*, May, 1934, p. 270. In these papers the ratio, *r*, is used instead of *k*. The latter is preferable and avoids the possibility of confusing it with the hydraulic radius.

In channels, the cross-sectional areas of which are segments of circles, a table of hydrostatic pressures is essential. Such a table and an explanation of its use in finding the height of the jump in circular conduits will be found in the writer's paper previously cited herein.[39]

In irregular channels the characteristics of the jump may be found graphically from the energy and momentum-plus-pressure curves as outlined[40] by Julian Hinds, M. Am. Soc. C. E.

All the foregoing expressions are dimensionless and for any given channel, involve only the two simple ratios, k and J, which are sufficient for the complete solution of the hydraulic jump. While the expressions are of many degrees, they are in effect single valued, since for every value of k there is only one value of J that will fulfill the physical conditions, and *vice versa*. Once k is known J is readily found by trial.

The writer questions the significance given to the "zone of undulating jumps" (Fig. 3). Fig. 7 shows a gradual transition from stable to undulating jumps as the ratio of kinetic to potential energy is diminished. Quite properly, the authors refrain from putting these results in terms of the initial energy because of the accelerations involved and the consequent departure from hydrostatic pressures, in the water prism.

In reality is not this "undulating jump" merely a "hang-over" from the oscillations that always accompany critical flow? At critical flow depths may vary considerably without appreciably changing either the energy or momentum-plus-pressure values. As a consequence the surface is in an unstable state and pronounced waves occur. For such conditions there is no jump, and in all the foregoing equations $J = 1$ and $k = \frac{1}{2}$ (the authors' $\lambda = 1$). If, now, the kinetic energy is increased and the potential energy is reduced ever so slightly (as, for example, by closing the sluice-gate) both k and J increase in value and a low jump occurs; but the flow is still barely removed from the critical, and the surface waves persist. These waves gradually fade away, as the ratio of kinetic to potential energy increases, and sensibly disappear for $k = 2$, say ($\lambda_1 = 4$). In reality, are there any valid grounds for stating that "the region near $\lambda_1 = 3$ is a zone of transition?" The range of undulations is for all values of λ_1 less than 4, and the less the value of λ_1 the greater the undulations. The point the writer wishes to bring out is that there appears to be no more reasons for setting up two kinds of jumps[41]—the "undular form" and the "direct form"—than for drawing a distinction between big jumps and little jumps.

The data on length of jumps are particularly interesting. The length expressed in terms of $d_2 - d_1$, varied from about 8 to 4.5, the latter value probably approaching a lower limit for which one may state that the kinetic energy expended can not cause the water to stand on a much steeper slope than 1 on 4. The upper limit, of course, is infinity for critical flow—at which there is no jump. The writer believes that for the sake of consistency the authors' ratio, $\frac{L}{d_1}$ (Table 1), is to be preferred, and this is the ratio given in Fig. 10.

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[40] "The Hydraulic Jump and Critical Depth in Design of Hydraulic Structures," *Engineering News-Record*, November 25, 1920, p. 1034; see, also, the *M*-function. "Hydraulics of Open Channels," by Boris A. Bakhmeteff, M. Am. Soc. C. E., p. 232.

[41] "Hydraulics in Open Channels," by Boris A. Bakhmeteff, M. Am. Soc. C. E., Eng. Societies Monograph, 1932, p. 228.

The length of the jump must be the resultant of two motions: (1) The translatory motion of the prism down stream; and (2) the vertical motion due to the rate of conversion of kinetic to potential energy. The physical phenomena involved in this conversion are but little known. It appears that the greater the initial kinetic energy the more rapidly is this conversion effected; hence, the shorter lengths accompany the expenditure of the greatest amount of kinetic energy.

One practical value of these data is that now something tangible is available by which the length of the apron at the foot of overflow dams, and the spacing of baffles, can be determined rationally. Knowing the fall from forebay to apron, and the discharge, the initial depth on the apron is readily found. The height, and now the length, of the jump may be expressed in terms of this depth.

Nolan Page.[42] Jun. Am. Soc. C. E. (by letter).—A research problem with which the writer recently had contact, involved a study similar in general to that presented by the authors, but resulted in values for length of jump sufficiently different to be of interest. This problem is one of several which have been and are under study by the Corps of Engineers, War Department, at the Hydraulic Laboratory, University of Iowa, Iowa City, Iowa, in connection with construction activities in the Upper Mississippi River and Ohio River Divisions.

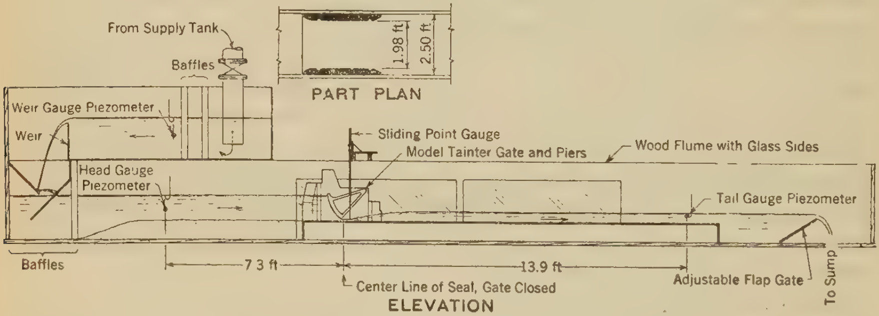
From Supply Tank
Baffles
1.98 ft
2.50 ft
PART PLAN
Weir Gauge Piezometer
Weir
Sliding Point Gauge
Model Tainter Gate and Piers
Wood Flume with Glass Sides
Head Gauge Piezometer
Tail Gauge Piezometer
Baffles
7.3 ft
13.9 ft
Center Line of Seal, Gate Closed
Adjustable Flap Gate
To Sump
ELEVATION

Fig. 11.—Model of Stilling-Basin Design.

As a step in the development of criteria for design for stilling-basins to protect navigation dams being built in the Upper Mississippi River, the simplified model shown in Fig. 11 was constructed and tested through a range of conditions indicated in Table 3. The model consisted of one Tainter gate and one-half of each supporting pier, built one-twentieth the size of the gates for Mississippi River Dam No. 20, and set up in a glass-sided flume. Instead of the designed sill and stilling-basin, a smooth, level, wooden floor was installed, as shown in Fig. 11. Water was supplied to the model through a 10-in. line from a constant-head tank two floors above the flume, passed through the apparatus as indicated, and discharged into a sump where it was re-pumped to the constant-head tank. Model quantities were determined by means of a rectangular, suppressed weir. Depths up stream from the model gate and down stream from the hydraulic jump were measured by vernier hook-gauges in stilling cans connected to the piezometers indicated in Fig. 11. Depths below the Tainter gate up stream from the hydraulic jump, were measured with a vernier point gauge.

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[42] Asst. Engr., U. S. War Dept., Hydraulic Laboratory, Iowa City, Iowa.

TABLE 3.—Test Data

Test No.	Measured flow, Q, in cubic feet per second	Energy head, $\epsilon_1 = d_0$ in feet	Measured Depths of Flow, in Feet		Measured length of jump, L, in feet	Ratio, $\frac{L}{d_2}$	Kinetic flow factor, $\lambda'_1 = \frac{\epsilon_1 - d_1}{d_1}$	Ratios:		
			d_1	d_2				$d'_1 = \frac{d_1}{\epsilon_1}$	$d'_2 = \frac{d_2}{\epsilon_1}$	$\frac{q_1}{\epsilon_1^{\frac{3}{2}}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
11	0.404	1.004	0.031	0.295	1.6	5.42	62.8	0.031	0.294	0.203
13	0.800	1.004	0.058	0.396	2.1	5.30	32.6	0.058	0.394	0.398
14	0.800	1.004	0.072	0.406	2.1	5.18	25.9	0.072	0.404	0.398
16	0.800	0.764	0.066	0.364	2.05	5.63	21.2	0.087	0.476	0.606
17	0.800	0.763	0.072	0.376	2.1	5.58	19.2	0.094	0.493	0.607
19	1 205	1.002	0.084	0.472	2.95	6.25	21.9	0.084	0.471	0.606
20	1.207	1.001	0.090	0.484	2.5	5.17	20.2	0.090	0.483	0.606
22	1.980	1.000	0.140	0.584	3.1	5.31	12.3	0.140	0.584	1.000
23	1.988	1.002	0.153	0.599	3.0	5.01	11.1	0.153	0.597	1.000
25	2.000	0.765	0.180	0.529	2.1	3.97	6.50	0.235	0.691	1.510
26	1.992	0.768	0.213	0.565	2.5	4.42	5.21	0.278	0.736	1.498
28	2.990	1.004	0.227	0.682	2.65	3.89	6.85	0.226	0.680	1.502
29	2.996	1.006	0.256	0.719	3.0	4.17	5.86	0.254	0.715	1.498
31	2.010	0.763	0.187	0.549	1.8	3.28	6.17	0.245	0.720	1.523
32	2.985	0.827	0.273	0.625	1.7	2.72	4.06	0.330	0.755	2.005
33	2.985	0.826	0.274	0.632	1.8	2.85	4.03	0.331	0.765	2.006
34	2.990	0.827	0.274	0.642	1.85	2.88	4.03	0.331	0.776	2.008
36	4.00	1.002	0 341	0.778	2.5	3.21	3.88	0.340	0.775	2.010
38	5.00	1.162	0.446	0.931	2.4	2.58	3.21	0.384	0.800	2.015
40	5.01	1.001	0.463	0.836	1.75	2.09	2.32	0.462	0.835	2.520
41	5.00	0.999	0.453	0.800	......	......	2.41	0.454	0.801	2.525
46	4.98	1.163	0.389	0.906	3.1	3.42	3.98	0.334	0.778	2.010
49	6.00	1.312	0.422	1.000	3.2	3.20	4.21	0.322	0.761	2.020

Table 3 contains only those data taken in tests on this model which are pertinent to this discussion. The lengths of jump tabulated were measured directly in the model, taking the distance from the beginning of the jump, which was always sharply defined, to the point of highest rise down stream from the jump as observed through the glass sides of the flume. The accuracy of measurements has been indicated in the tabulation. It will be noted that the values, ϵ_1, in Table 3 are not computed from d_1, but were taken as equal to d_0, the height of pond-water surface above the temporary floor of the flume. This was considered more accurate than computing ϵ_1 because of the magnified effect of slight inaccuracies in measuring d_1. Theoretically, d_0 should have been corrected for velocity of approach and the resulting ϵ_0 for friction loss at the Tainter gate, but neither correction was made, the one tending to offset the other and each being small.

In an apparatus such as this model, in which the hydraulic jump expanded laterally as well as vertically, measured values of d'_2 could not be expected to follow the curve for a straight-sided channel given in Fig. 3. However, with the widths at Section (1) and Section (2) known, the lateral expansion could be taken into account in theoretical equations for height of jump. This was done for the model and resulted in the curve marked "d'_2, computed for model" in Fig. 12. Except in the range near the maximum height of jump, measured values fall very close to the computed curve. As was the case in the authors' tests, the range in which the departure of measured from computed values occurs was one of unstable flow. In some tests a standing wave condition alternated with a hydraulic jump, the alternations following an approximately equal period for a given model setting.

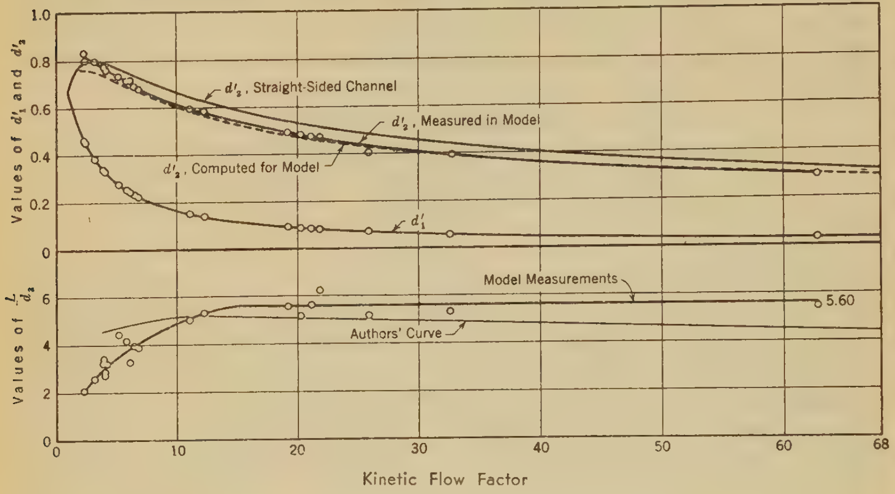

FIG. 12.

Lengths of jump measured in the model have been expressed as ratios of the depths down stream from the jump, and are plotted in Fig. 12. The authors' curve expressing the relation of this ratio to λ_1 has been transferred from Fig. 6 to Fig. 12 in order to compare directly the results obtained in the model with those obtained by the authors. It will be noted that, although the two curves assume a uniform slope at about $\lambda_1 = 14$, the authors' curve is below the curve drawn through model test points and slopes downward as λ_1 increases, whereas the model curve in this range was drawn with zero slope at a value of $\frac{L}{d_2} = 5.60$. It is thought that the jump may have been shortened in the authors' apparatus by friction on the sides of the narrow channel.

From the viewpoint of practical design, the abscissa, λ_1, involving the value, d_1, makes the curves of Fig. 12 awkward to use. Ordinarily, the known values from which dimensions for toe protection must be derived, are the quantity passing a dam site, the upper pool water-surface elevation, and the tail-water elevation corresponding to the quantity. When using the curves of Fig. 12, it is necessary first, to assume a value for d_1; from that, compute an elevation for the stilling-basin floor; and, then, compute by means of d_2, an elevation of tail-water surface, repeating the process until the computed and the given tail-water elevations agree. In this procedure, successive values of d_1 are not directly indicated by computed values of tail-water elevations.

A curve that is more directly useful as a means of computing stilling-basin floor elevations is that shown in Fig. 13. This curve was first called to the writer's attention by the late Floyd A. Nagler, M. Am. Soc. C. E., and was used in the design of Mississippi River Dam No. 2.[43] The abscissa in this case involves only the quantity per unit length of stilling-basin, which is readily determined from known river quantity and length of spillway, and the energy head up stream from the hydraulic jump, which may be taken as equal to the height of the pond-water surface above the stilling-basin floor. In using this curve, a stilling-basin elevation is assumed to provide data from which a basin elevation is computed, this process being continued until both elevations agree. Although trial and error methods must be used with this curve as with that of Fig. 12, computed and assumed values are directly related and converge rapidly. The basis for the abscissa of Fig. 13 is the equation for energy head at Section 1,

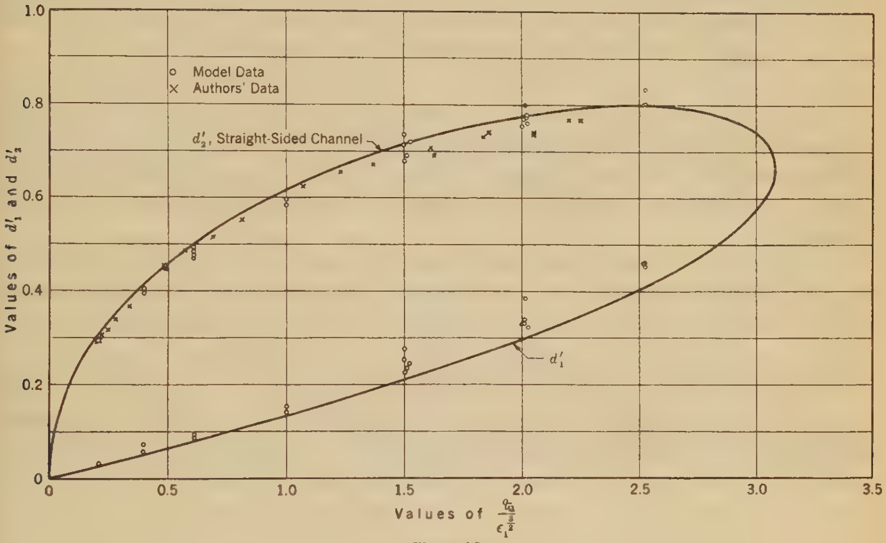

FIG. 13.

$$\epsilon_1 = d_1 + \frac{V_1^2}{2g} \quad \text{.........................} \quad (48)$$

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[43] "Laboratory Tests on Hydraulic Models of the Hastings Dam," by Martin E. Nelson, *Bulletin 2*, Univ. of Iowa Studies in Eng.

transformed to:

$$\frac{q_1}{\epsilon_1^{\frac{3}{2}}} = \sqrt{2\,g}\,\frac{d_1}{\epsilon_1}\sqrt{1 - \frac{d_1}{\epsilon_1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (49)$$

The curve presented in Fig. 13 is open to criticism in two respects; First, the test results are not entirely consistent with it; and, second, the value, $\frac{q_1}{\epsilon_1^{\frac{3}{2}}}$, is not dimensionless. Nevertheless, Fig. 13 should not be rejected as inadequate because of this. If theoretical curves are used, and they are usually sufficiently accurate for design purposes, either Fig. 12 or Fig. 13 will give identical results.

Acknowledgment is made to the District Engineer, U. S. Engineer Office, St. Paul, Minn., for permission to use the data given in this discussion. The tests were run by Frederick S. Witzigman, Jun. Am. Soc. C. E.

Andrei I. Ivanchenko,[44] M. Am. Soc. C. E. (by letter).—The results reported in this paper fully confirm the usefulness of the general methods of analysis applied. The authors have presented the first comprehensive explanation of the hydraulic jump, a phenomenon which was once considered quite puzzling.

Little can be added to the clear graphical representation of the relationships between various elements of the jump, and particularly the longitudinal characteristics illustrated in Figs. 6 and 9. The writer is particularly interested in the curve of $\frac{L}{d_j}$ plotted against λ in Fig. 9. This curve was considered by the authors as the basis of their solution of the longitudinal elements of the jump. The writer proposes the following expressions for the equation of this curve, obtained by the customary method:

$$\frac{L}{d_j} = 10.6\,\lambda^{-0.185} \quad \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

or, solving for L,

$$L = 10.6\,d_j\,\lambda^{-0.185} \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

in which $d_j = d_2 - d_1$, or,

$$L = 10.6\,(d_2 - d_1)\,\lambda^{-0.185} \ldots\ldots\ldots\ldots (52)$$

Equation (52) is quite simple, being derived from experiment; and, assuming that it is based on methods of approach that are physically correct, it is more justified than any other known formula for the length of the jump. In fact, the length of the jump is doubtless the function of the height of the jump $(d_2 - d_1)$, and the state of flow, which is represented by λ, the kinetic flow factor.

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[44] Civ. Engr.; Assoc. Prof., Hydraulics and Water Power Eng., Industrial Inst., Novocherkassk, Union of Socialistic Soviet Republics.

The writer has checked the values of the ratio, $\frac{L}{d_j}$, by means of Equation (50) for all values of λ taken from Table 1. The results are given in Table 4.

TABLE 4.—CHECK COMPUTATIONS OF THE RATIO, $\frac{L}{d_j}$

Run No.	Kinetic flow factor, λ	Values $\frac{L}{d_j}$				Run No.	Kinetic flow factor, λ	Values $\frac{L}{d_j}$			
		By Experiment:		By Equation (50)	From the curve, $\frac{L}{d_j} = f(\lambda)$, in Fig. 6			By Experiment:		By Equation (50)	From the curve, $\frac{L}{d_j} = f(\lambda)$, in Fig. 6
		From	To					From	To		
(1)	(2)	(3)	(4)	(5)	(6)	(1)	(2)	(3)	(4)	(5)	(6)
27	3.76	7.47	8.77	8.297	8.23	36	16.79	6.21	6.63	6.290	6.30
30	3.94	7.55	8.79	8.225	8.12	18	20.80	5.83	6.32	6.046	6.05
40	4.51	7.06	8.32	8.022	7.87	6	24.80	5.40	5.90	5.852	5.85
43	4.52	7.67	8.91	8.019	7.80	17	29.66	5.43	5.99	5.662	5.66
25	5.35	6.51	7.65	7.773	7.65	39	30.53	5.32	5.67	5.630	5.63
45	5.47	6.62	7.55	7.740	7.55	35	44.81	5.11	5.53	5.246	5.25
41	6.63	7.13	7.79	7.470	7.35	37	53.89	4.98	5.42	5.070	5.10
24	6.72	7.06	7.83	7.452	7.30	32	62.25	4.67	5.14	4.940	4.95
28	8.55	6.63	7.10	7.127	7.13	34	68.70	4.81	5.06	4.847	4.85
26	9.90	6.63	7.03	6.936	6.94	33	74.56	4.65	5.17	4.774	4.80
29	11.87	6.73	7.13	6.710	6.70	38	78.69	4.24	5.10	4.727	4.72

The writer is fully aware that the data in Column (6), Table 4, indicate certain discrepancies. However, obviously, as the authors state, the tracing of the curves was carried out so "that they must pass within the plotted ranges, indicating the possible range of the *L*-values." For all the curves of Fig. 6 the range of *L*-values from Runs Nos. 25 and 45, as well as for Run No. 28, lies below the curves. This may indicate some degree of unreliability in the experimental data for these runs, because the proposed formula, Equation (50), agrees very well with experimental data in all other cases.

F. T. MAVIS,[45] ASSOC. M. AM. SOC. C. E., and ANDREAS LUKSCH,[46] ESQ. (by letter).—The authors' observations of the length of the hydraulic jump in a narrow channel are a welcome addition to the literature of hydraulic research. The transfer of data from the model tests to the prototype should be made with caution, however, in the light of the authors' statement (following Equation (15)) that "in analyzing open-flow cases * * * the principle [of similitude] in general may be applied to geometrically similar cases only."

Referring to Equations (16) and (17) presenting the theory of the hydraulic jump the authors state that these equations "apply equally to a jump at the foot of the Boulder Dam and a small-scale model in a laboratory flume." It should not be overlooked, however, that in the derivation of these equations the effects of channel friction are wholly neglected. It is not unreasonable that the height of the jump should be little affected by boundary friction, and that a well-defined relation should exist between depths of flow at alternate stages. The location at which the jump occurs in a level channel of rectangular cross-section can be changed, however, by seemingly insignificant changes in resistance to flow. For a given depth of flow and a given kinetic flow factor it seems reasonable that the length of jump in a narrow channel would be less than in a channel relatively much wider, even if the side walls of both channels were made of glass. Due to the effects of the side walls the length of the hydraulic jump in a narrow channel may be relatively less than in a wide channel. Further tests are needed to show what effect the relative width of a channel may have upon the length of the hydraulic jump under otherwise similar conditions.

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On the basis of tests conducted for the Miami Conservancy District, Riegel and Beebe[47] stated that,

> "The length of the jump is approximately five times its height. While the beginning of the jump is fairly definite, its lower end is indefinite, and this figure represents merely a general estimate. The lower end was taken as the place where the water surface became and remained sensibly level, a place which was variable in position and difficult to locate."

In 1927, Safranez[48] studied the hydraulic jump experimentally at the Technische Hochschule in Berlin, Germany. The tests were conducted in a glass channel 19.6 in. wide. Lengths of jump were reported[49] for eighteen tests in which the kinetic flow factor, $\lambda_1 = \frac{v^2_1}{g\,d_1}$ varied from 3 to 365, lengths of jump from 1.0 to 3.3 ft, and heights of jump from 0.18 to 0.76 ft. Safranez proposed a tentative formula for the length of the jump which, using the authors' symbols, reduces to:

$$L_j = 6\,d_1\sqrt{\lambda_1} \qquad \text{.......................(53)}$$

An analysis of these data[49] for kinetic flow factors less than 80 (the range reported by the authors in Table 1), showed that the simple formula:

$$L_j = 5.2\,d_j \qquad \text{.......................(54)}$$

was in better agreement with the observations than Equation (53). The observed ratios, $\frac{L_j}{d_j}$, for the nine tests ranged from 4.6 to 5.5 and the mean absolute error in calculated lengths of jump, using Equation (54), was 6 per cent. For kinetic flow factors greater than 80 the length of the jump was 4.5 times its height, ranging from 4.0 to 5.1 for nine tests, with a mean absolute error of 5 per cent. The question naturally arises whether this general reduction in the ratio, $\frac{L_j}{d_j}$, for higher values of λ_1 may not be due to the resistance offered by the side walls and whether the ratio might not be substantially constant for a wide channel?

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[47] "The Hydraulic Jump as a Means of Dissipating Energy," by Ross M. Riegel and John C. Beebe, Technical Rept., Pt. III, Section XVI, p. 85, Miami Conservancy District (1917).

[48] "Untersuchungen ueber den Wechselsprung," von Kurt Safranez, *Der Bauingenieur*, v. 10, Heft 37, p. 38 (1929).

[49] *Loc. cit.*, Table 4.

As a part of a general study of stilling pools below spillways, tests were made at the Iowa Institute of Hydraulic Research to determine the geometrical properties of the hydraulic jump. The tests were conducted in a rectangular flume, 26 in. wide, lined with galvanized sheet metal. Water flowed under a sluice-gate with a rounded lower lip upon a level apron of steel plate and smooth cement mortar. The rates of flow varied from 0.48 to 1.44 cu ft per sec per ft width of channel, the depths of flow above the jump, from 0.05 to 0.15 ft, and the heights of jump, from 0.22 to 0.73 ft. In these tests the length of jump varied from 1.4 to 4.5 ft. Fig. 14(*a*) summarizing the data, shows as ordinates the ratio of length to height of jump and as abscissas, the kinetic flow factor. Fig. 14(*b*) shows as ordinates the ratio of length of roller to height of jump. An analysis of the tests conducted in a flume, 26 in. wide, indicated that the length of jump was approximately equal to 6.2 times its height and that the length of roller was approximately equal to 5.2 times its height.

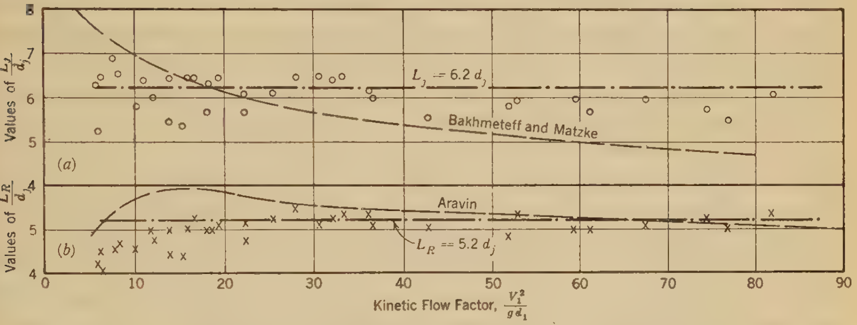

FIG. 14.—SUMMARY OF OBSERVATIONS, LENGTH OF HYDRAULIC JUMP.

V. J. Aravin[50] has presented a semi-rational analysis for determining the length of the hydraulic jump in which he followed the suggestion of Professor M. A. Velikanov "to divide the stream, when computing the losses, into two regions: dead zones, where only rotatory motion of the liquid is taking place, and zones where progressive motion is observed." Empirical coefficients were obtained from tests by Bakhmeteff, Pietrkowski, Safranez, Einwachter, and Aravin. The results obtained by the simplified formula suggested by Aravin, are:

$$\frac{L}{d_c} = 0.33\,A\,\frac{(\xi_2 - \xi_1)^3\,\xi^2_1}{\xi_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(55)$$

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[50] "The Determination of the Length of the Hydraulic Jump", by V. J. Aravin, *Transactions*, Scientific Research Inst. of Hydrotechnics, Vol. 15, pp. 48–57 (1935). (Translated from the Russian by Andreas Luksch.)

in which L = length of the hydraulic jump (roller); d_c = critical depth; $A = 0.54\,\xi_2^{4.35} + 75$; $\xi_2 = \frac{d_2}{d_c}$; $\xi_1 = \frac{d_1}{d_c}$; and d_1 and d_2 = depths of flow above and below the jump, respectively.

These data have been re-arranged for purposes of comparison and are shown in Fig. 14(*b*). Within the range of kinetic flow factors included in the authors' tests the ratio, $\frac{L_j}{d_j}$, which follows from Aravin's analysis may be replaced by a constant value, 5.4, with a mean absolute error of approximately 5 per cent.

I. M. Nelidov,[51] Assoc. M. Am. Soc. C. E. (by letter).—A graphical interpretation of the experimental results pertaining to the determination of the length of a hydraulic jump is presented in this paper. The actual depths of flow varied from 0.032 ft to 0.253 ft and the corresponding lengths of the jump, from 1.62 ft (average of the values in Table 1) to 2.50 ft.

Two questions arose in the writer's mind relative to these experiments. The first was, whether the experiments with small depths and volumes apply to the depths measured in feet and to quantities amounting to hundredths of cubic feet per second per second. The situation with respect to weirs shows that, notwithstanding the laws of similarity, the discharges of full-sized weirs cannot be estimated with a reasonable degree of accuracy. It appears that some other factors have a preponderant influence when the scale of a phenomenon is increased. These factors may be the air entrained in water flowing with high velocity, different roughness conditions, pulsation of flow, assymmetry in conditions of approach, and, perhaps, some others.

Another question was whether it is entirely correct to designate as the length of a jump—the length of its top roller, which ordinarily is longer than the length of the transition from one stage to another. For the most part, the surface curve of this transition is concave upward and only near the conjugate depth, d_2, does it acquire a sharp curvature concaved downward.[52] This indicates that the velocities of a high order may occur farther down stream than is ordinarily anticipated.

As a matter of reference the writer wishes to mention a study made by V. J. Aravin[53] based on the theorem of Bernouilli and using the computation of losses occurring in the top roller as a means for determining its length. The results were plotted on a curve for $\frac{d_2}{d_c}$ against $\frac{L}{d_c}$, L being the length of the roller, for the limits, 1.5 to 3.5 and 0.4 to 1.6, correspondingly. This curve agrees well with the experiments by Bakhmeteff, Pietrowsky, Safranez, Einwachter, and Aravin, plotted on the aforementioned curve.

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[51] Senior Engr. of Hydr. Structure Design, State Dept. of Public Works, Sacramento, Calif.

[52] "Ueber den Heutigen Stand des Wasserbaulichen Versuchswesen", von Prof. E. Meyer-Peter, *Schweizeriche Bauzeitung*, February 11, 1922, Fig. 3.

[53] "The Determination of the Length of a Hydraulic Jump", by V. J. Aravin, *Transactions*, Scientific Research Inst. of Hydrotechnics, Vol. XV, Leningrad, 1935.

BORIS A. BAKHMETEFF,[54] M. AM. SOC. C. E., AND ARTHUR E. MATZKE,[55] JUN. AM. SOC. C. E. (by letter).—The many sided and helpful discussions prompted by the paper are gratifying. The endorsement of the manner of treatment by Professor Woods and the appreciative remarks with regard to its practical usefulness by Mr. Stevens are ample reward for the efforts of the writers to clarify a complex and, at times, puzzling field of study. Many new questions have been raised which cannot be answered adequately in the light of the knowledge available at present and which, therefore, usefully indicate subject matter for further research.

The writers are fully conscious of the limitations imposed on the results by the size of the flume. They agree entirely with the remarks by Messrs. Mavis and Luksch, Nelidov, and others regarding the possible effect of the channel dimensions on the friction and the longitudinal dimensions of the jump. In fact, final clarity on this subject can be obtained only through systematic and well-planned observations on a scale exceeding any of the experimental work heretofore cited. Furthermore, in comparing observations, it will be imperative in the future to adopt some unified basis in determining "the end of the jump." In fact, some of the discrepancies suggested by Fig. 13 and Fig. 14 may be partly due to differences in estimating where the jump terminates.

The uncertainty of all these features would seem to militate, at least for the time being, against any premature attempts to crystallize the available knowledge in that, or in any other empirical, formula for computing the length of the jump (such as those suggested by Professor Ludin, and referred to in Mr. Kennedy's discussion; the formula proposed by Professor Ivanchenko, Aravin, etc.).

The writers further appreciate the possible effect of "white water", as indicated by Messrs. Hall and Nelidov. This is also a question that can be answered only by special experiments. However, one should bear in mind that the basic formulas, Equations (8) to (17) do not carry, explicitly, the value of the specific weight, γ. Therefore, it would seem that presence of air in itself should not substantially affect the numerical relations in the jump, provided the average contents of air throughout the jump is not changed.

Any possible effect of the size of the streaming, to be considered eventually in practical engineering computations, will be limited to longitudinal elements only. As justly emphasized by Professor O'Brien, the vertical elements observed, even in the smallest flumes, are in such marvelous agreement with the theoretical curve, that scarcely any further "correction" is necessary. This fact would seem to justify the practical usefulness of Equations (8) and (16) and of dimensionless diagrams of the type of Figs. 3, 10, and 12, in their simplest form, without introducing any possible corrections, as mentioned by Professor O'Brien and Dr. Engel, to take into account the unevenness of the velocities and their actual distribution in the cross-sections. This correction, as established by Coriolis[56], requires multiplying the average velocity group in the momentum equation, $\frac{\gamma}{g}\ Q\ V = \frac{\gamma}{g}\ A\ V^2$, by a correction factor, $\alpha = \frac{\int v^2\,d\,A}{V^2 A}$; and in the energy equation the group, $\frac{\gamma}{g}\ Q\ \frac{V^2}{2} = \frac{\gamma}{g}\ A\ \frac{V^3}{2}$, by a factor, $\alpha' = \frac{\int v^3\,d\,A}{V^3\,A}$, in which the respective integrals represent a summation over the entire area of the cross-section. The French hydraulicians of the Nineteenth Century gave much consideration to these points. Boussinesq in particular sanctioned the use of an average factor of about $\alpha' = 1.1$ and $\alpha = 1.03$. These numerical values were abstracted from observations on fully established, uniform flow patterns in canals and circular conduits. Now, flow in the section preceding the jump, where the kinetic energy is at its highest, usually follows a zone of intense acceleration (such as below a sluice or at the bottom of a spillway), the effect of which is to straighten out the velocities of the different filaments and to bring the velocity pattern in the lower-stage section before the jump close to practical uniformity. In the upper-stage section the influence of the kinetic energy is comparatively slight. However, to obtain some idea of the velocity picture in this region, special experiments were made in the Fluid Mechanics Laboratory of Columbia University, in New York City, the results of which are shown in Fig. 18. The velocity variance does not seem to be excessive.

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[54] New York, N. Y.

[55] New York, N. Y.

From the discussions by Messrs. Stevens and Page it would appear that dimensionless unified diagrams of the type of Figs. 12, 10, and 3, have become the order of the day. The value of such generalized diagrams lies exactly in eliminating the needless "series of mathematical gymnastics", so strongly objected to by Professor Woodward. There is no essential difference between the diagrams of Fig. 3 and those given by Mr. Stevens (Fig. 10) and Mr. Page (Fig. 12). Obviously, a relationship that exists between certain interdependent factors in a jump can be depicted equally well by selecting any of the factors that enter the problem as the independent variable against which all the other elements are plotted.

The choice of that or some other independent variable as a basis is largely a matter of taste and individual preference. The writers prefer λ, as in Fig. 3, or $d'_1 = \frac{d_1}{\epsilon_1}$[57], because the presentation in this case happens to be in direct and organic relation to the practical means of obtaining a jump as illustrated in Fig. 2. It is precisely for this reason that the curves in Fig. 3 indicate certain important physical features, which are entirely eclipsed in Fig. 10, or which can be derived from Fig. 12 only by indirect inference. The first is the course of d_j and its maximum value. Mr. Stevens takes exception to the notion itself, by stating: "There is no maximum height of jump * * *. The authors' maxima for d'_j and d'_2 in Fig. 3, are not physical characteristics * * * but merely consequences of the terminology * * *". This is a point of substantial importance. Obviously, the writers failed to make the matter sufficiently lucid. There is no better means of clarifying a point than by direct observation. Therefore, a special experiment was made using a layout similar to Fig. 2. The height, H, was kept constant and equal to 1.402 ft. The sluice-opening was changed, increasing the depth, d_1, and the discharge. By regulating the tail-water the jump was maintained in position near the sluice. With varying discharges, the lower and the upper stage depths (and thus the height of the jump, d_j) were measured. The data are assembled in Table 5.[58] As noted in Fig. 15 the height of the jump first increases until it reaches a maximum value at $d_1 = 0.27$, after which d_j decreases. The value of

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[56] *Annales des Ponts et Chaussées*, 1836; see also, *Transactions*, A. S. M. E., Vol. 54, 1932, p. 57.

[57] "Hydraulics of Open Channels", by B. A. Bakhmeteff, M. Am. Soc. C. E., Eng. Soc. Monograph, p. 248.

$$\frac{d_1}{H} = \frac{0.27}{1.4} = 0.193$$ accords very closely with the theoretical value of $d'_1 = 0.206$ corresponding to $\lambda = 7.67$, at which the maximum is supposed to occur.

TABLE 5

No.	Q	d_1	d_2	d_j	V_1	λ	$\frac{d_2}{d_1}$
1.......	1.720	0.440	1.027	0.587	7.83	4.34	2.33
2.......	1.515	0.374	0.984	0.610	8.12	5.49	2.63
3.......	1.296	0.312	0.930	0.618	8.33	6.94	2.98
4.......	1.070	0.252	0.874	0.622	8.50	8.99	3.47
5.......	0.822	0.191	0.792	0.601	8.62	12.07	4.15
6.......	0.578	0.126	0.687	0.561	9.18	20.78	5.47

Mr. Stevens further questions the significance of the zone of undulating jumps which, as the writers claim, corresponds in Fig. 3 to the zone of the mounting d'_2-curve between $\lambda = 1$ and the maximum at $\lambda = 3$. The writers fully appreciate the potency of Mr. Stevens' remarks about the undulatory form of motion in the vicinity of the critical depth in general. However, they cannot escape the feeling that the undulatory jump is a separate and well determined phenomenon in itself. Strangely, it was the undular jump

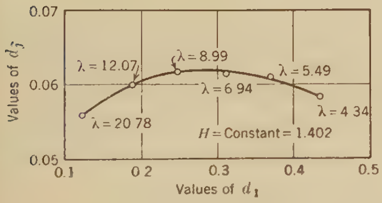

FIG. 15.

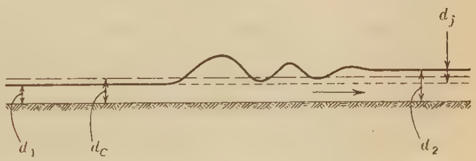

FIG. 16.

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[58] In Table 5, H = constant = 1 402 ft.

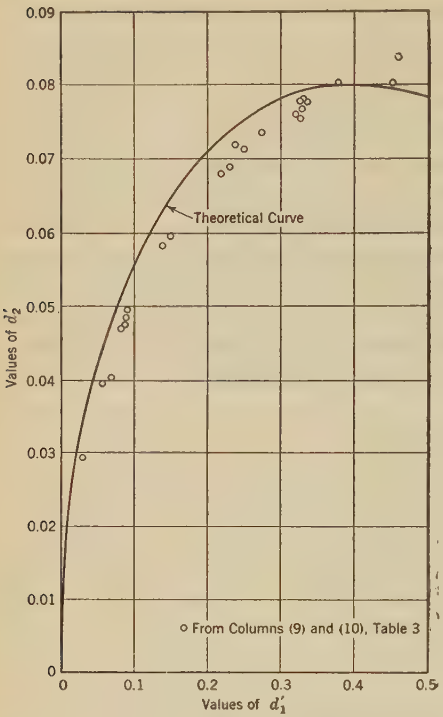

FIG. 17.

that was observed and described by most of the earlier investigators, for the simple reason that they worked with jumps that occurred in channels of steep slope, where flow in uniform motion was in a sub-critical (rapid) state. Compared to conditions at the foot of a dam or below a sluice the kineticity in uniform rapid flow is comparatively small, usually being far from sufficient to bring the jump into the direct roller form characterized by $\lambda \geq 3$ and $d'_1 \leq 0.4$. A schematic representation of an undulatory jump is given in Fig. 16. What differentiates these jumps from undulatory motions in the vicinity of the critical depth is the very substantial difference between the upper and lower stage, the depth ratio at times approaching the theoretical limit of $\frac{d_2}{d_1} = 2$.[59]

A distinct feature of the undular jump is the fact that, due to the curvature of the streaming, the momentum equation ceases to apply, and that observed values of d_2 are in excess of values computed from Equations (8) and (16).[60] This fact is confirmed once more by Mr. Page's observations. In fact, plotting the d'_2-values from Table 3 against d'_1, one arrives at Fig. 17. The points in the zone of direct jumps ($d'_1 < 0.4$) follow the theoretical curve rather satisfactorily, whereas the points for $d'_1 > 0.4$ fall outside the curve.

A point of particular interest is emphasized by Professor O'Brien, namely, the physical nature of the energy losses in a jump. The thought is that at least a part of the apparent loss may be due to a fallacious manner of estimating the kinetic energy in the initial and final section. As matters stand, these energies are computed on the basis of the prevailing average forward velocities $\left(\frac{V^2_1}{2g} \text{ and } \frac{V^2_2}{2g}\right)$, whereas it does stand to reason that at the upper stage, in addition to the average forward velocity, there might possibly be a substantial rotational effect. The senior writer is frank to admit that for a time he was disposed to attribute a considerable influence to this fact. He remembers having stated such views in informal discussions with members of the profession. Basically, the question reverts to the form of the velocity curve in Section d_2. To make the matter clear, at least qualitatively, special experiments were made at Columbia University which are illustrated by Fig. 18. By means of a Pitot tube, velocity diagrams were obtained in the respective cross-sections corresponding to the "end of the jump"; then, a foot up stream nearer the roller, and, finally, down stream at a distance of $3\frac{1}{2}$ ft, where tranquil flow was unquestionably re-established. In the curves as given, the abscissas represent the combined pressure and velocity heads registered

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[59] For detailed data see "Recherches Hydrauliques", par Darcy Bazin, Paris, 1865; also, "Berechnung der Wasserspiegellage", von Böss, V. D. I. *Forschungsarbeiten No. 284.*

[60] "Hydraulics of Open Channels", p. 251.

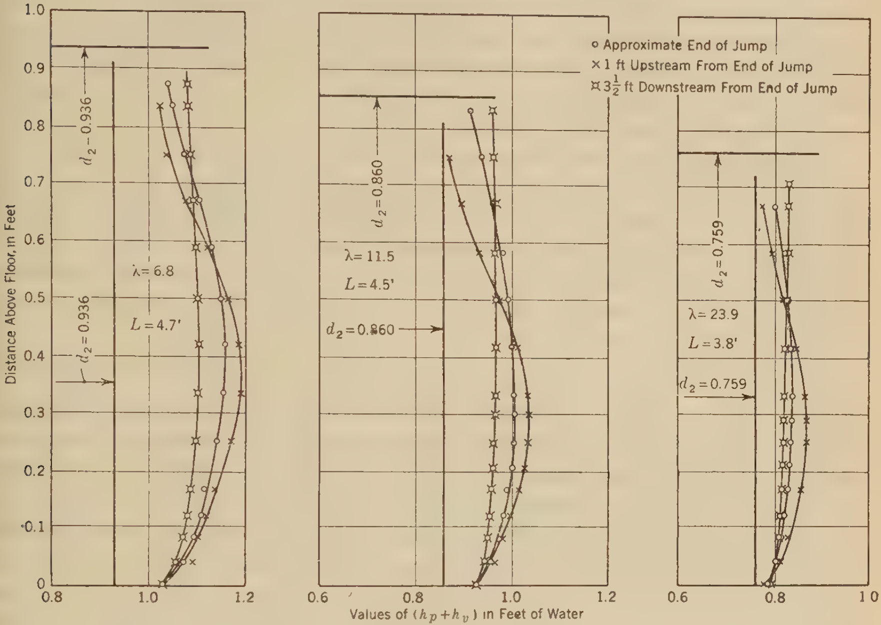

FIG. 18.

by the Pitot tube and referred to the bottom of the flume. The pressure head, corresponding to hydrostatic distribution at the end section, with h_p assumed equal to d_2, is given by the d_2-vertical. Invariably, a certain relative rotation in an anti-clockwise direction is present in the "end-of-the-jump" section. This rotation is entirely eliminated $3\frac{1}{2}$ ft down stream. The rotation is relatively larger at small values of λ, and much less noticeable at the highest values of λ, where the loss of energy should be the greatest. Although these experiments are of a preliminary nature and must be substantiated by more detailed work, they would seem to testify against attributing any too great influence to relative rotation. It would seem that the correct explanation of the loss is that given in the discussion by Mr. Rouse. When expressing the kinetic energy in terms of the average forward velocity, one neglects the fact that a considerable portion of initial energy is absorbed by the kinetic energy of the additional turbulent agitation, which can be symbolized mathematically thus:

$$\Sigma \rho \frac{1}{2} [(u')^2 + (v')^2 + (w')^2] \, dx \, dy \, dz \ldots\ldots\ldots\ldots (56)$$

in which u', v', and w' are the local turbulent velocity components, variable in time and space, whereas the summation must be taken over the entire volume of the jump. As Mr. Rouse points out, the final molecular absorption of this energy occurs in the form of heat created by the additional viscous friction caused by excessive turbulent agitation. Obviously, such viscous friction is not limited to the average forward motion and is greatly intensified at every point by the presence of turbulent components complementary to the average motion. There is every reason to believe that the dissipation of this superfluous energy into heat is not consummated within the confines of the jump itself. In other words, the fluid leaves the end section of the jump still carrying a high content of excessive turbulence which is finally dissipated and brought down to the normal state of turbulent loss in established parallel flow only over a considerable distance.

A few more words of elucidation are required in connection with the remarks of Dr. Engel. There was no desire whatever on the part of the writers to "attack" the so-called "Boussinesq number" introduced by Dr. Engel in connection with his very valuable investigations of flow in Venturi flumes. The history of hydraulics shows that in the past many empirical parameters proved to be most useful. Consider, for example, the parametric basis, expressed in terms of $\left(a + \frac{b}{\sqrt{R}}\right)$, which Kutter and, later, Bazin used when building their empirical formulas for estimating open-channel roughness. Dealing with a complex problem of tri-dimensional flow, Dr. Engel may have struck a very fortunate chord in using a parameter of reference, to which the name "Boussinesq number" was given. What the writers believe, however, is that an empirical factor, no matter how practical and useful, should not be interpreted for a condition it cannot represent—namely, a mark of dynamic similarity. If the writers understand Dr. Engel correctly he believes that, in view of the friction effect (which is disregarded in setting up the customary momentum equation for the jump, but which, obviously, is nevertheless present), jump data can be systematized more adequately by referring them to the Boussinesq number instead of using as a base the kinetic flow factor (or Froude number). In Dr. Engel's opinion this should become particularly evident in jumps where the depth at the foot of the jump exceeds the hydraulic mean radius.

To clarify this interesting question a special narrow channel, 2 in. wide, was built into the regular flume and a series of jumps was observed. The data are presented in Table 6.

TABLE 6.

Run No.	d_1	d_2	$\frac{d_2}{d_1}$	λ	R	$\frac{d_1}{R}$
N–1	0.222	0.764	3.44	9.31	0.0619	3.58
N–2	0.133	0.671	5.05	17.01	0.0522	2.55
N–3	0.078	0.551	7.06	29.38	0.0409	1.91
N–4	0.048	0.442	9.22	52.30	0.0308	1.56

As seen, the ratio of the initial depth to the hydraulic radius reaches $\frac{d_1}{R} = 3.58$; nevertheless, the experimental points in Fig. 19 do not deviate much from the theoretical curve. To clarify the matter further a still more direct experimental proof was sought. By appropriately varying the discharge and the initial depth, d_1, it was attempted to obtain a series of jumps at practically identical values of λ, but with substantially varying Boussinesq numbers. If the contention of Dr. Engel were correct, the observed data referred to λ would scatter. On the other hand, when plotted against B (see Fig. 20), the points should manifest better regularity. The runs marked in Table 7 by the letter, C, were observed in the 2-in. flume with a value of λ actually varying between 7.93 and 7.56, or a variation of 5%, whereas the Boussinesq

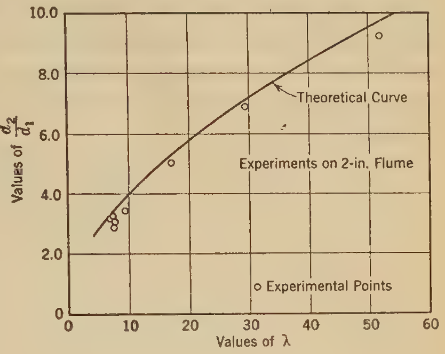

Fig. 19.

TABLE 7.

Run No.	d_1	d_2	V	r'	λ	B	$\frac{d_2}{d_1}$
C–1A......	0.300	0.896	8.76	0.135	7.93	4.19	2.99
C–1B......	0.242	0.744	7.84	0.128	7.89	3.88	3.07
C–1C.	0.162	0.521	6.32	0.113	7.65	3.27	3.22
C–1D......	0.107	0.344	5.10	0.096	7.56	2.90	3.21
N–1........	0.222	0.764	8.16	0.124	9.31	4.08	3.44
S–45.......	0.249	0.681	6.62	0.227	5.47	2.44	2.73
S–41.......	0.228	0.681	6.97	0.238	6.63	2.57	2.99
S–24.......	0.248	0.765	7.33	0.249	6.72	2.59	3.09
S–28.......	0.249	0.881	8.28	0.249	8.55	2.92	3.54
S–26.......	0.254	0.989	8.99	0.252	9.90	3.14	3.89
S–29.......	0.221	0.957	9.18	0.235	11.87	3.34	4.33

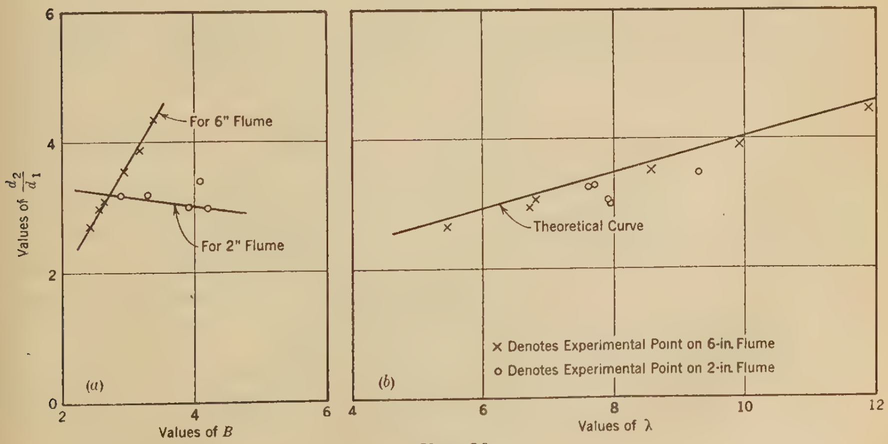

Fig. 20.

number varied about 36 per cent. Obviously, friction makes itself felt, but the observed ratios, $\frac{d_2}{d_1}$, form a compact group, with a variation of 7 per cent. For further clarification Table 7 contains certain data, obtained at neighboring values of λ on the 6-in. flume, as given in Table 1. All the observed data when referred to λ, range quite consistently close to the theoretical curve in Fig. 20. By contrast, when referred to the Boussinesq number, the points for the 2-in. flume and the 6-in. flume form two separate and incompatible curves.

AMERICAN SOCIETY OF CIVIL ENGINEERS
TRANSACTIONS
Founded November 5, 1852

Paper No. 1936

FRICTIONAL RESISTANCE IN ARTIFICIALLY ROUGHENED PIPES

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WITH DISCUSSION BY MESSRS. WARREN E. WILSON, RICHARD G. FOLSOM, RALPH W. POWELL, AND VICTOR L. STREETER.

SYNOPSIS

The flow of fluids in smooth pipes is so well understood that the losses due to friction may be predicted to within 5 per cent. The problem of estimating friction losses in old pipes, or in those roughened by use, however, has been inadequately solved. In order to clarify the problem completely much study and experimentation are necessary. This paper, which is an effort in that direction, presents the results of an experimental investigation of frictional resistance in artificially roughened pipes. It has been undertaken in an effort to show, qualitatively, the effect on the friction factor of certain artificial irregularities—varying in shape and size—that were introduced in pipes used for the tests. With water as the fluid, loss of head was investigated for roughened, 2-in., brass pipe, the Reynolds numbers ranging from 20 000 to 1 250 000. The roughness elements consisted of grooves cut spirally into the pipe. The degree of roughness was varied by changing the depth, shape, and number of grooves per inch.

The loss of head due to friction was found to increase with an increase in the depth of the grooves, provided the same general shape was maintained. The shape of the grooves, however, seems to have almost as much affect on the loss of head as the depth. The experiments also indicate that the roughness elements may be placed so close together that the loss of head is reduced. The work of previous investigators is summarized briefly; the apparatus, preliminary experiments, experiments for obtaining data, and computations are described; and the results of this investigation are discussed.

NOTE.—Published in February, 1935, *Proceedings*.

[1] Freeman Scholar, A. S. M. E., Karlsrühe, Germany.

NOTATION

The symbols in this paper are introduced in the text as they occur and are summarized for reference in Appendix I. An effort has been made to conform essentially with "Symbols for Hydraulics,"[2] compiled by a committee of the American Standards Association, with Society representation, and approved by the Association in 1929.

PREVIOUS INVESTIGATIONS

Investigations of fluid flow in pipes have been concerned with the development of general formulas, and, more recently, with studies into the effect of artificial roughness on the friction factor.

General Considerations of Fluid Flow.—In the evolution of formulas dealing with the flow of fluid in pipes, the findings of Hagen, Poiseuille, Darcy, Reynolds, and Blasius are especially significant. In 1839, Hagen[3] observed that there were two different types of fluid flow through pipes, now commonly designated as stream-line and turbulent flow. Thirty years later he stated that the point at which transfer from one form of flow to the other occurred depended on the radius, the velocity, and the temperature of the water.

In 1845, Poiseuille[4] experimented with the flow of water through capillary tubes. He deduced the following law for stream-line flow, known as Poiseuille's law, which has since been derived theoretically:

$$F_l = \frac{64\,\mu\,L\,V}{2\,g\,D^2} \qquad (1)$$

in which, F_l = loss of force, or pressure; μ = absolute viscosity; L = length of experimental pipe, in feet; V = average velocity, in feet per second; g = acceleration due to gravity; and D = mean diameter of pipe, in feet. Equation (1) may be expressed in terms of the Reynolds number and the Darcy friction factor, as follows:

$$f = \frac{64}{\mathsf{R}} \qquad (2)$$

in which, f = Darcy's friction factor; and R = Reynolds number.

In 1857, Darcy[5] conducted a series of experiments on the flow of water in cast-iron pipes with diameters ranging from 0.5 in. to 20 in., obtaining the formula,

$$H_f = \frac{V^2\,L}{c^2\,D} \qquad (3)$$

in which, H_f = loss of head due to friction; and c^2 = a coefficient which Darcy considered to be constant for each type of surface. The Darcy formula is now usually written in the form,

$$H_f = \frac{f\,L\,V^2}{2\,g\,D} \qquad (4)$$

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[2] A. S. A.—*Z10b*—1929.

[3] "Uber die Bewegung des Wassers in engen zylindrischen Röhren," von G. Hagen, *Poggenborff's Annalen*, Bd. 46, S. 423, 1839.

[4] "Recherches expérimentales sur le mouvement des liquides dans tubes de très petits diamètres," par Poiseuille, *Comptes Rendus*, Vol. 11, 1840, pp. 961, 1041; Vol. 12, 1841, p. 112; also, *Mémoires* des Savants Etrangers, Vol. 9, 1846.

[5] "Recherches expérimentales relatives au mouvement de l'eau dans les tuyaux." par H. Darcy, *Mémoires* des Savants Etrangers, Vol. 15, 1858, p. 141.

in which, $f = \frac{2g}{c^2}$. The friction factor, f, will be used throughout this paper as the measure of frictional resistance.

By a study of Stokes' equations of motion, Reynolds[6] conceived the idea that a criterion to determine whether flow would be stream-line or turbulent might be expressed in the form,

$$\mathsf{R} = \frac{D\,V\,\rho}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which, ρ = density, in pounds per cubic foot. In 1883 he confirmed this theory experimentally by observing dye introduced into flowing water in glass tubes. The value of Reynolds number, $\frac{D\,V\,\rho}{\mu}$, at which change from one flow form to the other takes place, is modified more or less by conditions at the inlet of the pipe, by roughness, the amount of curvature, etc. A reasonable mean value to accept as the transition point between stream-line and turbulent flow is $\mathsf{R} = 2\,100$.

Working with the experimental data of Saph and Schoder[7] on flow of water in smooth brass pipe, Blasius[8] obtained the empirical formula,

$$f = 0.316\ \mathsf{R}^{-0.25} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Reynolds numbers for these experiments were less than 100 000. Later experiments on smooth pipe show that the Blasius formula does not fit the data for Reynolds numbers greater than 100 000. Nikuradse[9] obtained the relation,

$$f = 0.0032 + 0.221\ \mathsf{R}^{-0.237} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

from his experiments on smooth pipe, with R as high as 3 200 000. Drew, Koo, and McAdams[10] obtained the formula,

$$f = 0.0056 + 0.500\ \mathsf{R}^{-0.32} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

from a study of all available experimental data on smooth pipes.

Experiments with Artificially Roughened Pipes.—During the fifteen years since about 1920, Schiller, Hopf, Fromm, and Nikuradse have experimented with artificially roughened pipes to determine whether roughness follows the law of similitude. In order to have similitude, corresponding surface measurements or dimensions of roughness elements in pipes must be in the same ratio as the radii, such that, if k is the measure of absolute roughness and r is the radius of the pipe, the ratio, $\frac{k}{r}$, is a constant. Factor k, expressed in inches, is a complete measure of the size, shape, and distribution of the individual roughness elements. As there is at present no method for measuring or defining roughness, the symbol, k, is hypothetical.

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[6] "An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall Be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," by Osborne-Reynolds, *Philosophical Transactions*, Royal Soc., Lond, 1883; or *Scientific Papers*, Vol. II, p. 51.

[7] *Transactions*, Am. Soc. C. E., Vol. LI (1903), p. 253.

[8] "Das Ahnlichkeitsgesetz bei Reigungsvorgängen," von H. Blasius, *Physicalische Zeitschrift*, Vol. 12, p. 1175, 1911; or, *Forschungsarbeiten Ingenieur*, Heft 131.

[9] "Gesetzsmässigkeiten der turbulenten Strömung in glatten Röhren," von J. Nikuradse, Verein Deutscher Ingenieur, *Forschungsheft*, No. 356, 1932.

[10] "The Friction Factor for Clean Round Pipes," by Drew, Koo, and McAdams, *Journal*, Am. Inst. of Chemical Engrs., Vol. 28, p. 56, 1932.

Schiller[11] experimented with artificially roughened pipe of three sizes: 8 mm, 16 mm, and 21 mm, in inside diameter (1 in. = 25.3998 mm). The roughness consisted of spiral threads cut into the inner surfaces. Two arrangements of threads were utilized. In one, the pitch was 0.8 mm, and the depth, 0.6 mm; and in the other, the pitch was 0.4 mm, and the depth, 0.3 mm. The results of Schiller's tests are shown in Fig. 1. (Curve (3) indicates results for the 8-mm pipe with the finer roughness, and Curve (2), the 16-mm pipe with the coarser roughness. If the experiments on the two pipes conformed to the principles of similitude, Curves (3) and (2) should coincide since the roughness elements were proportional to the radii. The discrepancy may be attributed to the short calming lengths which appear to have been inadequate, and also to the fact that the roughness in the calming section of each tube differed from that in the test portion.

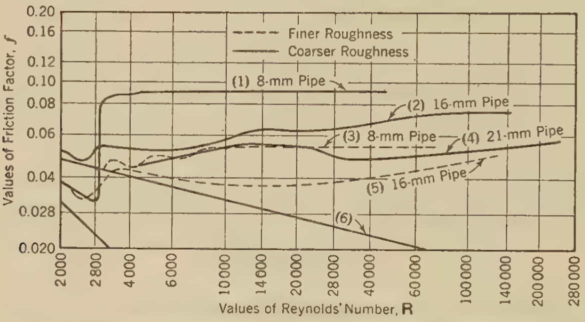

FIG. 1.—RESULTS OF EXPERIMENTS BY SCHILLER.

In investigating roughness in rectangular closed conduits, Ludwig Hopf[12] came to the conclusion that the friction factor depended upon three quantities: The cross-section of the conduit; Reynolds number; and a number which characterizes the roughness. He stated that the number for specifying the roughness appeared to require several dimensionless quantities to specify completely its size, shape, and distribution. Hopf also decided that there were two kinds of roughness, which he called "wall-waviness" and "wall-roughness." An example of the first is wood pipe or asphalted iron pipe, and an example of the second type is provided by the bare surface of cast-iron pipe.

Fromm's[13] investigations of relative roughness were carried on shortly after those of Hopf and with the same experimental equipment. In each of the roughnesses tested, several values of the hydraulic radius were used. For those cases in which f becomes constant for larger values of R, the effect of the change of hydraulic radius may be expressed by the formula,

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[11] "Uber den Strömungswiderstand von Röhren verschiedenen Querschnitts und Rauhigkeitsgrades," von L. Schiller, *Zeitschrift angewandete Mathematik und Mechanik*, Vol. 3, pp. 2–13, 1923.

[12] "Die Messung der hydraulischen Rauhigkeit," von Ludwig Hopf, *Zeitschrift angewandete Mathematik und Mechanik*, Vol. 3, pp. 329–339, 1923.

[13] "Strömungswiderstand in rauhen Röhren," von K. Fromm, *Zeitschrift angewandete Mathematik und Mechanik*, Vol. 3, pp. 339–358, 1923.

$$f = (\text{constant})\ d^{-0.314} \qquad (9)$$

in which, d is the depth of section, and the constant is dependent upon the absolute roughness, k', of the surface.

Whereas Hopf, Fromm, and Schiller investigated relative roughness over a comparatively limited range of Reynolds numbers, Nikuradse[14] obtained values of f within the range of turbulent flow up to a Reynolds number of 1 000 000. He used roughnesses varying from fairly smooth surfaces to projections equal to one-fifteenth of the radius. These roughnesses were produced by glueing sand to the inside of three sizes of pipes (diameters, 2.5 cm, 5 cm, and 10 cm). The diameter of the sand grain, k', was taken as the measure of absolute roughness, k, the relative roughness being $\frac{k'}{r}$. Sand was selected of such diameter that $\frac{k'}{r}$ was the same for the three different sizes of pipes. Six different values of $\frac{k'}{r}$ were used. The results of the experiments are shown in Fig. 2. The tests show that the principle of similitude applies accurately to pipes roughened in this manner. For the range of Reynolds numbers in which f is constant, Nikuradse gives the formula,

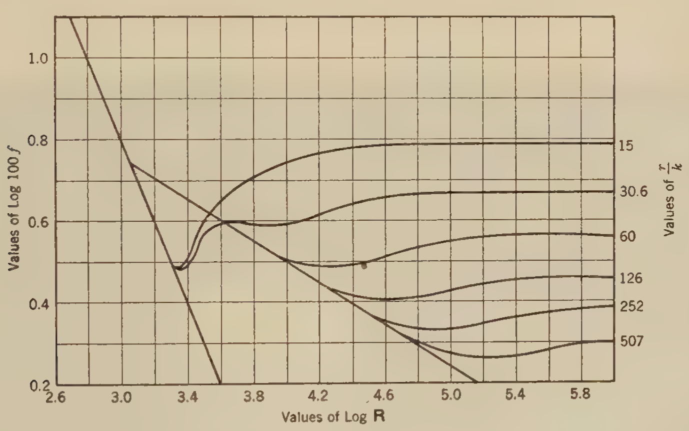

FIG. 2.—RESULTS OF EXPERIMENTS BY NIKURADSE.

$$f = \frac{1}{\left(1.74 + 2\log_{10}\frac{r}{k'}\right)^2} \qquad (10)$$

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[14] "Strömungsgesetze in rauhen Röhren," von J. Nikuradse, Verein Deutscher Ingenieur, *Forschungsheft*, No. 361, 1933.

APPARATUS

The experiments undertaken in this investigation were conducted at the University of Michigan, Ann Arbor, Mich. The apparatus is illustrated in Figs. 3 and 4. The quantity of water used in the experiments varied from 10 to 400 gal per min. Two orifices were used for measuring, one for small, the other for large, flows. Fig. 5 shows the orifice layout. Two differential manometers were used to measure the pressure drop across the orifice. One of the gauges contained mercury and the other acetylene tetra-bromide (specific gravity, 2.97). Water filled all the spaces between the gauge liquid and the pipe.

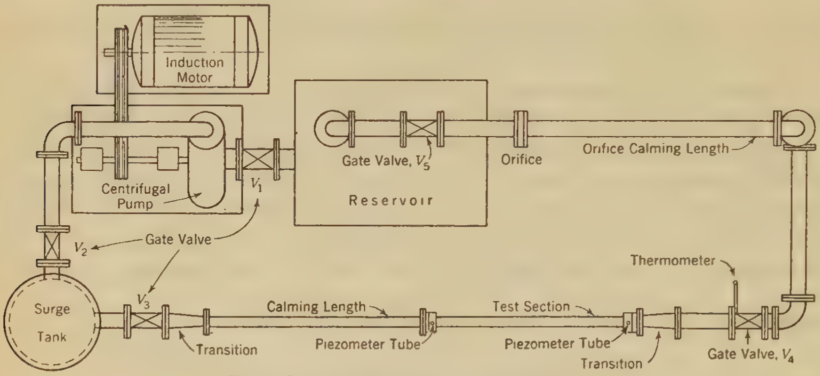
Induction Motor
Centrifugal Pump
V_1
Gate Valve, V_5
Orifice
Orifice Calming Length
Reservoir
V_2
Gate Valve
Thermometer
Surge Tank
V_3
Calming Length
Test Section
Transition
Piezometer Tube
Piezometer Tube
Transition
Gate Valve, V_4

FIG. 3.—EXPERIMENTAL EQUIPMENT.

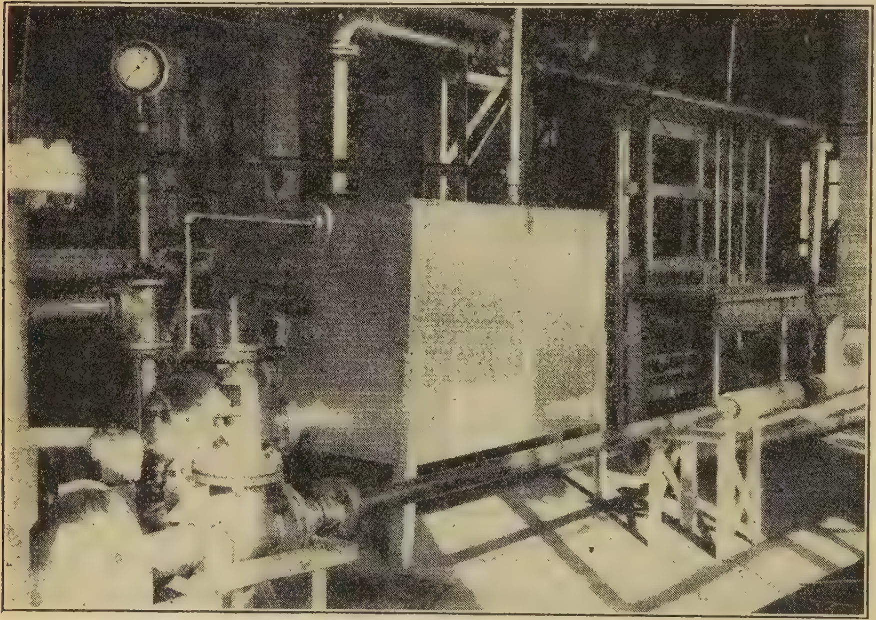

FIG. 4.—VIEW OF EXPERIMENTAL EQUIPMENT.

As the accuracy of piezometer rings is questionable in rough pipes, piezometer tubes of special design (see Fig. 6) were used to measure the pressure drop due to friction. These tubes were closed at the end, with a transverse opening $\frac{1}{4}$ in. from the end of the tube. A diagram of the test-section manometers is shown in Fig. 7. Details of connections of copper tubing to manometer glass are illustrated in Fig. 8. Manometer *A*, Fig. 7, contained mercury; Manometer *B* contained acetylene tetra-bromide; and Manometers *C* and *D* contained solutions of acetylene tetra-bromide and xylene, with specific gravities, respectively, of 1.3 and 1.05. All of them were differential manometers with water acting as one liquid.

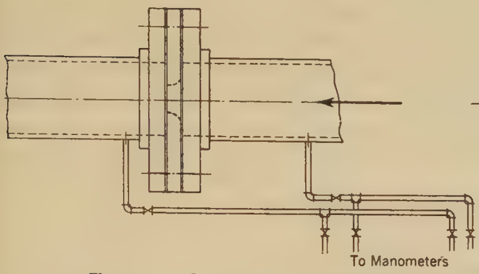
To Manometers

FIG. 5.—ORIFICE LAYOUT.

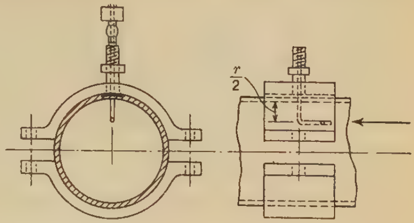

FIG. 6.—PIEZOMETER TUBE.

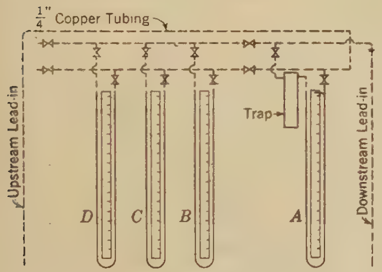
¼" Copper Tubing
Trap
Upstream Lead-in
Downstream Lead-in
D
C
B
A

FIG. 7.—TEST SECTION, MANOMETER SYSTEM.

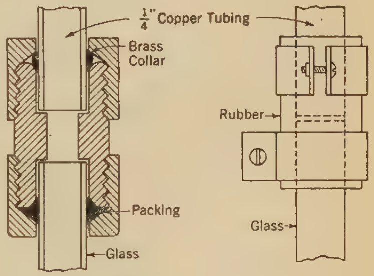
¼" Copper Tubing
Brass Collar
Packing
Glass
Rubber
Glass

FIG. 8.—MANOMETER CONNECTIONS.

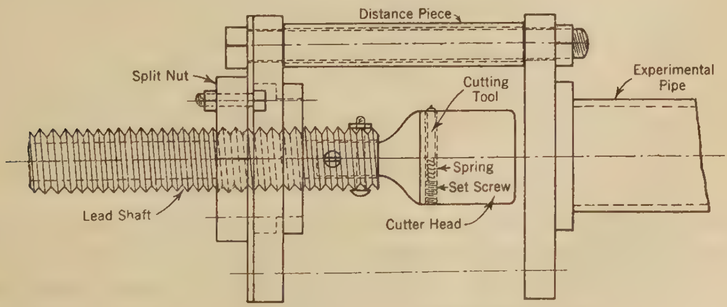
Distance Piece
Split Nut
Cutting Tool
Experimental Pipe
Spring
Set Screw
Lead Shaft
Cutter Head

FIG. 9.—ROUGHENING APPARATUS.

A steam coil in the reservoir was used to heat the water to the desired temperature. A thermometer, reading to 110° C, was inserted in the valve at the down-stream end of the test section to obtain the temperature of water in the pipe (see Fig. 3). To insure isothermal flow through the test section, the pipe was insulated with felt covering.

In order to obtain a definitely known roughness on the inside surfaces of a 2-in. brass pipe, 10 ft. long, which would permit the size, shape, and distribution of the roughness to be varied, a device was designed that would cut grooves with various known dimensions and with different spacings (see Fig. 9). The pitch of the thread on the shaft was 11.5 to the inch. As the cutting tool could contain from one to four cutting elements within $\frac{1}{11.5}$ in., the distribution of the grooves could be varied.

Preliminary Experiments

The preliminary experimental work included orifice calibrations, piezometer tests, and smooth pipe tests. The orifices were calibrated over the range of flows, for temperature of 72° and 158° F. Partial calibrations were made at temperatures of 55° F and 122° F. The experimental data for temperatures of 72° and 158° are shown on the calibration curves in Fig. 10 (in which, Δ H represents the pressure drop across the orifice, in centimeters of water). Curves for 55°, 95°, and 122° were obtained from the curves for 72° and 158°, by assuming the change of discharge for a given pressure drop

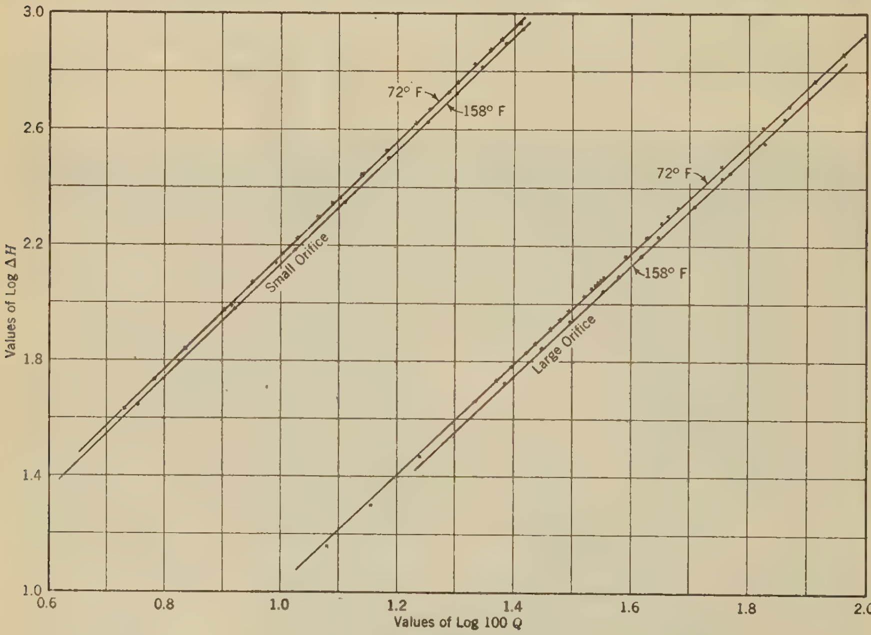

Fig. 10.—Orifice Calibration.

across the orifice to be directly proportional to the change in viscosity of the water. A change of temperature from 55° to 158° increased the discharge by as much as 6.5% for a given pressure drop.

In order to be sure that the two piezometer tubes would give the correct difference in pressure in the test section, they were inserted into the same cross-section of a 2-in. pipe, and both were connected to a sensitive manometer. If the piezometer tubes had exactly the same characteristics and if the distribution of velocities in the cross-section were symmetrical, the manometer reading would be zero for any flow. These test readings were taken for several velocities, and they all showed the pressure drop across the

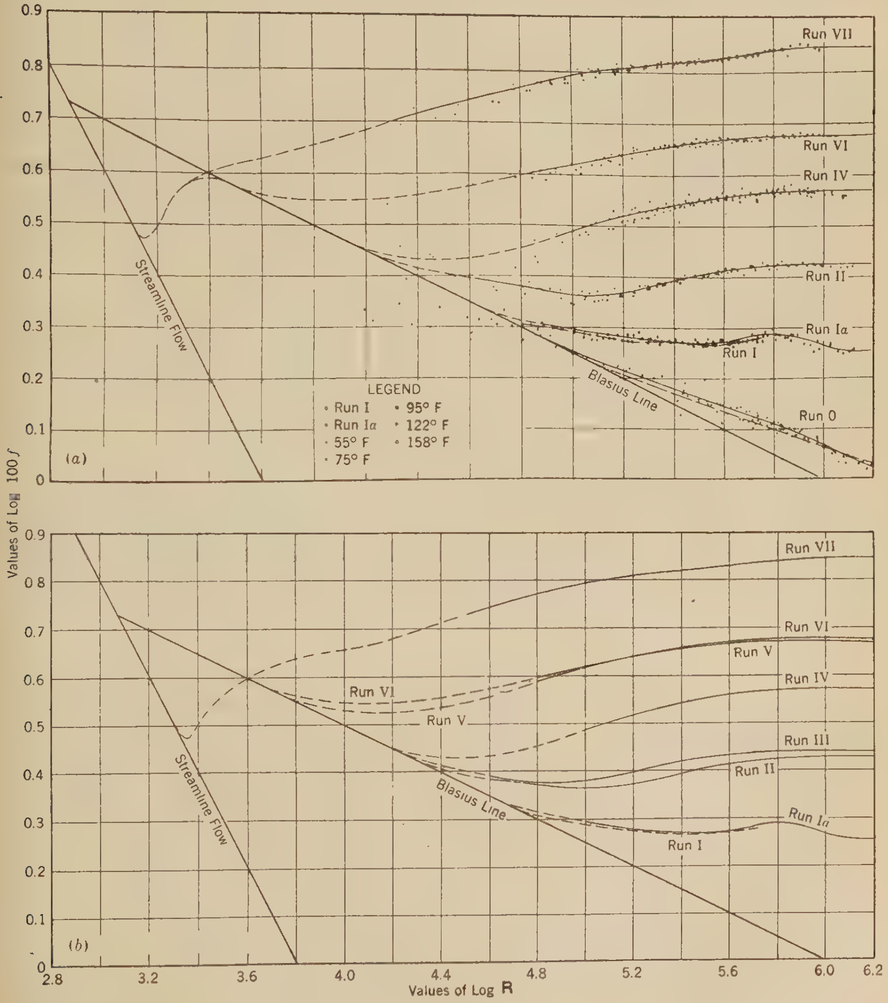

FIG. 11.—EXPERIMENTAL DATA: (*a*) FOR RUNS 0, I, I*a*, II, IV, VI, AND VII: (*b*) CURVES OF *f* PLOTTED AGAINST R OBTAINED FROM TESTS ON ROUGHENED PIPES.

manometer to be small compared with the pressure drop at the same velocity that would occur between the two ends of the test section. Accordingly, it was not necessary to add a correction to the pressure-drop measurements.

Tests were made on smooth pipes in order to obtain a basis for comparison with other investigators, and also to enable operators to become familiar with the equipment. Two sets of experiments were made with drawn brass pipe to obtain curves showing the relation of f to the Reynolds number for different temperatures (see Run 0, Fig. 11(a)). These runs were conducted in the same manner as those on rough pipe.

Experiments for Obtaining Data

The experiments on rough pipes consisted of roughening the pipes, determining the diameter, controlling the temperature, and taking readings. The test section was first roughened by running the cutter head (Fig. 9) through the pipe with the desired tool inserted in it. In order to obtain a sample of the roughness, the cutter head was run through a short piece of 2-in. brass pipe similar to that of the test section. After the sample was obtained, the 10-ft calming length was roughened in the same manner as the test section. Even with the greatest care, it was not possible to keep brass particles from collecting on the tool, so that the roughness became gradually less sharp as the cutter head advanced.

As the friction factor for a pipe varies as the fifth power of the diameter, it was important to determine the diameter accurately. This was done indirectly by determining the weight of water necessary to fill the 10-ft test section. With the weight of water, length of pipe, and density of water known, the mean diameter could be determined accurately.

The series of experiments performed on each test section was divided into eight runs, of which four were conducted with each orifice. Each run was made at a practically constant temperature. Runs were made with each orifice at temperatures of 55°, 95°, 122°, and 158° F. Each run consisted of readings of the orifice manometer, test-section manometer, and temperature for twelve to twenty different velocities.

Computations

In computing the friction factor, f, and the Reynolds number, **R**, the orifice manometer readings were converted into centimeters of water, and the test-section manometer readings into feet of water. From the proper orifice calibration curve the flow of water was read in cubic feet per second. The mean velocity of water in the test section was obtained by dividing the discharge by the mean area, in square feet. With this velocity (see first column, Table 2, Appendix II) the velocity head, $\frac{V^2}{2g}$, was obtained. Since $\frac{L}{D}$ is a constant for each roughness tested, when H_f (Appendix II) and V have been determined, the corresponding value of f can be computed from Equation (4) for each set of observations. The fraction, $\frac{\rho}{\mu}$, contained in the

Reynolds number is a function of the temperature, and, therefore, values of T can be traced through corresponding values in the third column of Table 2 (Appendix II), A table of values of $\frac{\rho}{\mu}$, in seconds per feet^2, was made up[14a] for each 0.1° C from 10 to 75 degrees. With V in feet per second, D, in feet, and $\frac{\rho}{\mu}$, in seconds per feet^2, the Reynolds number is seen to be dimensionless, and in consistent units. Finally, values of R and f, can be computed from the values listed in the fourth and fifth columns of Table 2 (see Appendix II).

Discussion of Results

Seven artificial roughnesses were investigated, three of which were cut into one set (the test section and the calming length, Fig. 3) of brass pipes and the remaining four into another set. A roughness typified by very fine grooves was first used in each set of pipes and this was followed by roughnesses with successively coarser grooves. In each case an effort was made to choose roughness designs that would conform as closely as practicable to the design that was to be used next. In spite of every precaution, however, each roughness made the one that followed more irregular than it would have been if cut directly into smooth pipe.

Relation of Friction Factor to Reynolds Number.—The experimental values of log (100 f) and log R for Runs 0, I, I*a*, II, IV, VI, and VII, are plotted in Fig. 11(*a*). As Run III had approximately the same friction factors as Run II, and Run V approximately the same as Run VI, their results are not given. The curves obtained from plotting log (100 f) against log R for all the runs on rough pipes are shown in Fig. 11(*b*). Run 0, Fig. 11(*a*), was made on smooth pipe. The dotted line through the smooth-pipe data is a graph of Equation (7). This line is seen to be in good agreement with the data. The left-hand portions of the curves of Fig. 11(*a*) and Fig. 11(*b*) are broken to indicate that these sections were not obtained experimentally, but constructed according to Nikuradse's curves (Fig. 2).

The runs were made at different temperatures with the twofold purpose of increasing the range of Reynolds numbers and of obtaining information concerning the value of f, for rough pipes, as a function of the Reynolds number. As the ratio, $\frac{\mu\,(55°)}{\mu\,(158°)} = 2.98$, and as the density decreases very slightly with increases in temperature, changing the temperature of the water from 55° to 158° F increases the value of $\frac{\rho}{\mu}$ almost three times. With this temperature change, therefore, it was possible to obtain a variation in Reynolds numbers almost three times greater than would be possible, for the same velocities, with water at one temperature. From the standpoint of results obtained, experimenting with water at these temperatures was practically equivalent to running tests with other liquids which had corresponding values of $\frac{\rho}{\mu}$. To obtain the values indicated by any of the points on the curves shown in Fig. 11(*a*) would require velocities about three times as great if tests were run at a temperature of 55° F as would be required at a temperature of 158° F. In spite of the change in impact brought about by this velocity change, the friction factor remains the same. As comparatively few experiments have been conducted on artificially roughened pipes with viscosities other than those of water at temperatures between 32° and 75° F, these tests over a greater viscosity range furnish additional proof that, for a given rough pipe, f is a function of the Reynolds number.

[14a] The values of ρ and μ were taken from "Elements of Chemical Engineering," by Badger and McCabe.

Photo-Micrographic Study of the Roughnesses.—As a means of determining as closely as possible the shape of grooves in each test section, photo-micrographs, magnified to four times the actual size, of the pipe sample and cutting tool were taken for each roughness. In Figs. 13 and 14, photo-micrographs are magnified to 2.94 times their actual size. In some cases a groove intermediate between that of the cutting tool and the sample seemed to represent best the form of the roughness. In other cases (depending upon the number of previous cuts in the pipe, the wear on the cutting tool, and the clearness of the photo-micrographs), the roughness appeared to be better represented either by the sample or by the tool. The number used herein to designate a type of roughness, corresponds to that used in Figs. 11(*a*) and

TABLE 1.—COMPARISON OF VALUES OF k' WITH ROUGHNESS DIMENSIONS

Roughness (see Fig. 12) (1)	f (= a constant) (2)	$\log \frac{r}{k'}$ (3)	Radius, r, in inches (4)	Radius of sand grains, k', in inches (5)	Average depth of grooves, in inches (6)	Number of grooves per inch (7)	Roughness (see Fig. 12) (1)	f (= a constant) (2)	$\log \frac{r}{k'}$ (3)	Radius, r, in inches (4)	Radius of sand grains, k', in inches (5)	Average depth of grooves, in inches (6)	Number of grooves per inch (7)
I.........	0.0188	2.775	1.036	0.0017	0.005	46	IV.......	0.0370	1.727	1.043	0.0196	0.012	23
Ia.........	0.0188	2.775	1.036	0.0017	0.005	46	V........	0.0465	1.45	1.043	0.0370	0.012	23
II.........	0.0266	2.195	1.034	0.0066	0.005	23	VI.......	0.0473	1.43	1.047	0.0390	0.016	11.5
III.......	0.0272	2.155	1.035	0.0072	0.005	23	VII......	0.0700	1.02	1.052	0.1005	0.022	11.5

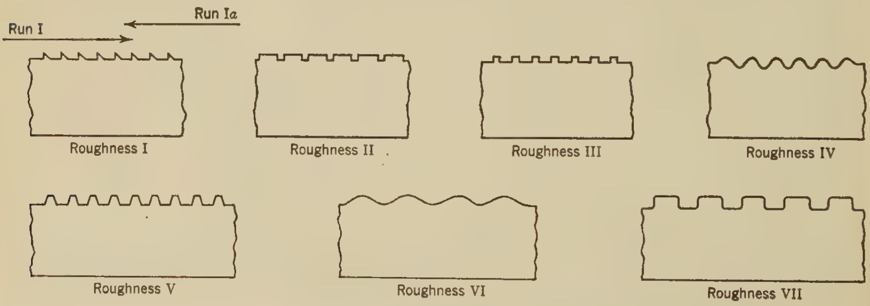

FIG. 12.—LONGITUDINAL SECTIONS OF ROUGHENED PIPE.

11(*b*) to indicate the run in which this roughness was used. Essential dimensions are classified in Table 1 and the roughness is shown diagrammatically for each case, in Fig. 12.

Roughness I. — The photo-micrographs of the sample and the tool for Roughness I are shown in Figs. 13(*a*) and 13(*b*), respectively. This was the first cut through the first set of pipes, and, also, the finest roughness. Unfortunately, the sample photo-micrograph (Fig. 13(*a*)) does not show the profile of the roughness in such a way that measurements can be taken from it. The cutting tool (Fig. 13(*b*)), also, was badly damaged in going through the calming length, which had been slightly deformed as a result of some mishandling in transit. The arrows in Fig. 12 show the direction of flow. Run I*a* was made with the same roughness, but with the direction of flow reversed.

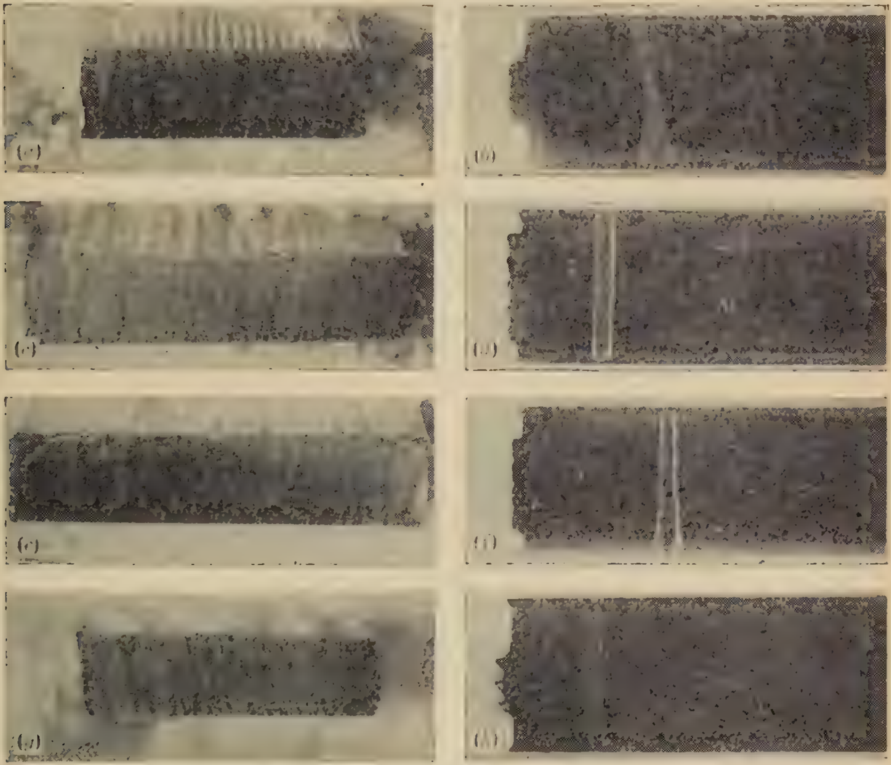

Fig. 13.—Photo-Micrographs of Longitudinal Sections of Pipe and Cutting Tools: *a*, *c*, *e*, and *g*, Respectively, Denote Roughnesses I, II, III and IV, with *b*, *d*, *f*, and *g*, the Corresponding Cutting Dies.

Roughness II.—The photo-micrographs for Roughness II are shown in Fig. 13(*c*) and the cutting die in Fig. 13(*d*). It was the first cut into the second set of brass pipes, and, therefore, the shape and dimensions of the roughness can be taken from the tool. The photo-micrograph of the sample does not aid much in obtaining the dimensions of the elements.

Roughness III.—The roughness indicated by Fig. 13(*e*) and the corresponding die, Fig. 13(*f*), was obtained by cutting over Roughness II with a tool designed to leave the roughness shown in Fig. 12. This was the only case of a roughness being cut to the same depth as that which preceded it. The cutting was probably 75% efficient (that is, 75% of pipe surface had the roughness shown by the tool, and the remainder was characterized by Roughness II). Roughness III as shown in Fig. 11(*b*) had a higher friction factor than Roughness II. This indicates that a projection causes greater turbulence than a groove.

Roughness IV.—Fig. 13(*g*) and the corresponding die, Fig. 13(*h*), show that this roughness varied in height from 0.018 in. at the end where the cutting began, to 0.006 in. at the other end of the pipe. This was the second roughness in the first set of pipes. The shape in Fig. 12 was a mean between tool and sample.

Roughness V.—The roughness, indicated by Fig. 14(*a*) and Fig. 14(*b*), had about the same average height as Roughness IV, and the same distribution, but was much more rough and abrupt in shape. This was the third roughness in the second set of pipes. The effect of shape is strikingly shown by the difference in friction factors of Roughnesses IV and V. In Fig. 14(*a*), the threads on the lower side are the standard for 2-in. pipes.

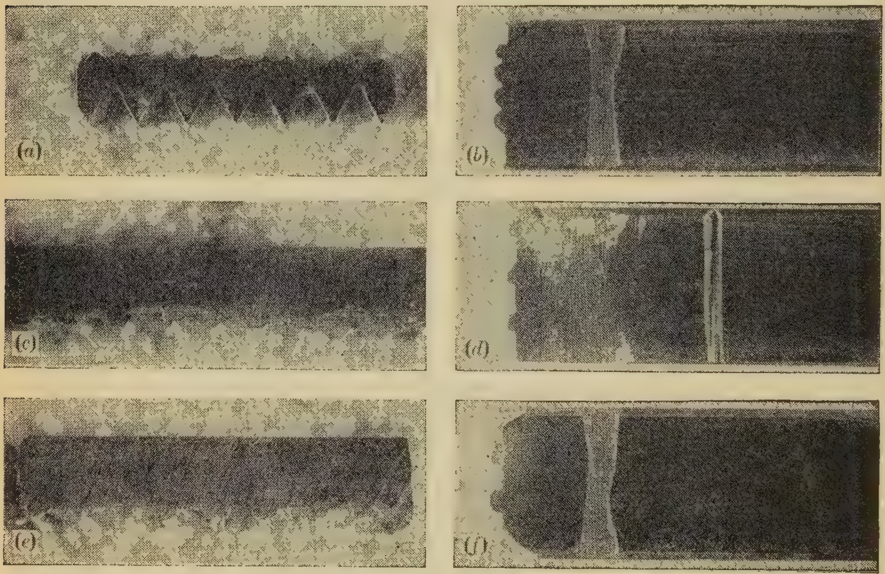

Fig. 14.—Photo-Micrographs of Longitudinal Sections of Pipe and Cutting Tools: the Top Side of *a* and the Bottom Sides of *c* and *e* Denote Roughnesses V, VI, and VII, Respectively; *b*, *d*, and *e*, the Corresponding Cutting Dies.

Roughness VI.—The third and last roughness of the first set of pipes (Roughness VI) is shown in Fig. 14(*c*) and Fig. 14(*d*). The shape was wavy in character, without sharp projections, as shown in Fig. 12. This roughness had a slightly greater friction factor than Roughness V, which had smaller elements and with twice as many peaks per inch. The decidedly abrupt shape of Roughness V seems to account for its friction factor being almost as large as that of Roughness VI.

Roughness VII.—Fig. 14(*e*) and Fig. 14(*f*) show Roughness VII, the fourth cut through the second set of pipes. As would be expected with its large grooves and sharp projections, this roughness gave the greatest friction factor of any shape investigated.

Relative Roughness.—While the description of the roughnesses and their comparisons have been made on the basis of actual size, it should be remembered that this can be done only because the radius of the pipe has changed little in comparison with the changes in roughness. This investigation neither proves nor disproves the similitude theory as applied to relative roughness. Nikuradse has made a satisfactory experimental demonstration of the validity of this theory.

The friction factor is a function of the Reynolds number and the relative roughness, or,

$$f = \phi\left(\mathsf{R}, \frac{k}{r} \right) \qquad (11)$$

The relative roughness factor, $\frac{k}{r}$, must have the same value for any two pipes if they are to have the same values of f for all values of R. At present, there is no known method of determining k. It is undoubtedly some function of the size, shape, and distribution of individual roughness elements. The problem of determining numerical values of k that will define certain standard surfaces is an important one that will doubtless receive the attention of future investigators.

Comparison of Results with Those of Other Investigators.—A comparison with Nikuradse's experiments was made by investigating conditions in the region of large Reynolds numbers where the curve representing the mean of his results is approximately horizontal. Values of k' were computed from Nikuradse's formula (Equation (10)), the friction factor, f, being taken from the right side of the curves in Fig. 11(*b*). These values of k' are the diameters of sand grains required to give the same roughness effect as the corresponding grooves. Table 1 gives the value of k' obtained in this manner. In all cases, except Roughness I and Roughness I*a*, the diameters of sand grains are larger than the depths of the grooves. The shape of the rough surfaces seems to have almost as much effect as the depth of the grooves. In Runs I and I*a* the closeness of the roughness particles appears to account for the small friction factor. Run III has the same depth of groove as Run I, and the shape is not radically different; and yet the sand grains for Run III have about four times the diameter of those for Run I. Apparently, this increase in the value of f is caused entirely by the distribution of the roughness elements.

The ratio of the depth of groove to the radius of the pipe in Schiller's experiments was larger in all cases than those used in this investigation. The distributions used by Schiller were about 32 and 64 grooves per in. Because of the relatively larger grooves and the smaller range of Reynolds numbers covered by his experiments, it is difficult to establish relationships between the results of the two investigations.

ACKNOWLEDGMENTS

The writer wishes to express his grateful appreciation to H. W. King, M. Am. Soc. C. E., and to Professor W. L. Badger, of the University of Michigan, for their interest and suggestions during this investigation; and to Mr. Weyburn M. Dodge for his aid in constructing the equipment.

CONCLUSION

The experiments described herein indicate that the shape of the roughness elements is quite as important as the size. This is evident from an examination of Columns (5) and (6) of Table 1, as well as Fig. 11. Runs I, II, and III, while having the same depth of groove, have a wide variation in friction factor. As the shape of Runs I and III are somewhat similar, the difference in friction factor may be attributed to the difference in distribution of the roughness elements.

APPENDIX I

NOTATION

The following symbols, adopted for use in this paper, conform in all essential respects with "Symbols for Hydraulics"[2] compiled by a committee of the American Standards Association, with Society representation, and approved by the Association in 1929:

c = coefficient; a constant for each type of roughened surface (not to be confused with Chezy's coefficient, C).

d = depth; depth of flow; depth of section.

f = friction factor used in expressing loss of head in pipes; Darcy's friction factor; as a subscript, f denotes "friction."

g = gravity constant; acceleration due to gravity.

h = head; as a subscript, h denotes "hydraulic radius."

k = a measure of absolute roughness; k' = diameter of sand grains, taken as a measure of absolute roughness.

r = radius of pipe

A = area; cross-section area of pipe

C = Chezy's coefficient (not to be confused with c, as introduced in the paper).

D = mean interior diameter of a pipe.

F = force; total pressure; F_l = loss of pressure.

H = total head; H_f = total loss of head due to friction.

L = length of experimental pipe sections.
R = hydraulic radius (not to be confused with R (see Equation (5)).
R = Reynolds number; R_h = Reynolds number based on the hydraulic radius
T = temperature.
V = velocity; average velocity in a section.
μ = viscosity, absolute.
γ = viscosity, kinetic $= \frac{\mu}{\rho}$.
ρ = density
ϕ = function of.

APPENDIX II

SUMMARY OF COMPUTATIONS

TABLE 2.—COMPUTATIONS TO DETERMINE FRICTION FACTOR, f, AND REYNOLDS NUMBER, R

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:			Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:		
		$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log R	log (100 f)			$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log R	log (100 f)
(*a*) RUN 0, PART *a*; $A = 0.0233$ SQUARE FEET; $D = 0.1722$ FEET; AND $\frac{L}{D} = 55.152$.									
5.71	0.555	74 800	4.867	0.2975	18.19	3.99	126 440	5.598	0.148
1.85	0.059	74 800	4.377	0.3035	19.44	4.52	126 960	5.629	0.144
1.39	0.0514	74 800	4.253	0.492	20.35	4.94	127 740	5.651	0.143
3.14	0.159	74 000	4.602	0.274	21.2	5.34	127 740	5.669	0.145
3.84	0.26	73 200	4.685	0.314	22.15	5.74	128 260	5.69	0.135
4.243	0.333	78 800	4.760	0.334	22.8	6.06	128 580	5.704	0.136
5.45	0.488	80 060	4.876	0.282	24.36	6.85	129 300	5.734	0.132
5.895	0.570	80 480	4.9120	0.282	26.4	7.95	129 820	5.772	0.124
9.69	1.275	125 400	5.322	0.200	28.2	8.89	130 340	5.802	0.114
11.71	1.775	125 400	5.44	0.178	29.4	9.64	130 600	5.820	0.1125
13.13	2.175	125 400	5.454	0.166	30.45	10.2	130 600	5.836	0.108
14.59	2.635	125 400	5.499	0.159	32.38	11.4	130 860	5.862	0.104
15.75	3.05	125 660	5.532	0.156	34.1	12.57	130 860	5.886	0.101
16.3	3.21	125 660	5.548	0.1495	35.6	13.5	131 120	5.906	0.095
17.2	3.61	125 920	5.572	0.153	38.6	15.37	131 380	5.941	0.08

TABLE 2.—(*Continued*)

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:			Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:		
		$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)			$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)
(*b*) RUN 0, PART *b*; $A = 0.0234$ SQUARE FEET; $D = 0.1720$ FEET; AND $\frac{L}{D} = 55.23$									
2.38	0.0945	99 550	4.610	0.288	7.19	0.5535	178 340	5.344	0.096
3.717	0.2224	100 700	4.808	0.273	8.72	0.878	178 340	5.428	0.128
4.96	0.3855	100 700	4.9345	0.2595	12.85	1.831	177 800	5.594	0.112
5.395	0.454	100 700	4.970	0.2595	17.08	3.186	177 800	5.718	0.1045
5.99	0.547	100 470	5.0144	0.249	19.86	4.273	178 340	5.784	0.102
6.64	0.6575	100 470	5.0593	0.240	23.56	5.909	177 800	5.858	0.094
7.67	0.850	100 700	5.1233	0.226	27.4	7.75	177 800	5.924	0.081
8.71	1.090	100 940	5.180	0.224	30.8	9.86	177 800	5.974	0.084
9.68	1.328	100 940	5.225	0.218	34.7	11.95	177 800	6.026	0.062
10.87	1.628	100 940	5.2755	0.2055	36.6	13.09	177 800	6.050	0.056
9.51	1.27	99 550	5.212	0.2135	39.6	14.49	177 800	6.090	0.031
13.20	2.31	99 320	5.353	0.205	8.68	0.911	229 000	5.534	0.150
15.93	3.24	99 550	5.436	0.172	13.5	2.02	229 810	5.728	0.111
18.42	4.16	99 780	5.50	0.155	17.34	3.255	229 540	5.836	0.100
21.52	5.69	99 780	5.3875	0.148	19.65	4.07	229 270	5.890	0.088
23.95	6.65	99 320	5.612	0.130	22.23	5.122	229 000	5.922	0.0795
27.36	8.45	99 090	5.668	0.120	24.0	5.85	229 270	5.976	0.073
30.20	10.30	99 550	5.7138	0.1195	27.44	7.295	229 540	6.0355	0.052
32.7	11.97	99 780	5.749	0.115	30.43	8.81	229 540	6.080	0.440
34 86	13.43	99 320	5.7754	0.1096	33.43	10.60	229 270	6.124	0.042
36.7	14.57	99 320	5.797	0.160	36.27	12.18	229 270	6.159	0.034
37.88	15.20	99 550	5.812	0.0918	38.62	13.60	229 810	6.194	0.027
8.90	0.975	170 240	5.416	0.156	6.84	0.625	232 700	5.437	0.192
12.78	1.918	169 700	5.572	0.136	8.28	0.907	233 300	5.522	0.198
16.36	3.028	169 700	5.679	0.120	13.71	2.163	232 000	5.738	0.127
18.8	3.97	169 700	5.740	0.1175	16.98	3.181	231 180	5.830	0.1105
21.2	5.00	169 700	5.792	0.112	20.47	4.506	231 730	5.912	0.098
23.33	5.885	169 970	5.834	0.100	23.6	5.790	232 700	5.976	0.082
26.58	7.40	170 240	5.892	0.088	27.6	7.41	232 700	6.044	0.055
29.8	9.045	169 700	5.940	0.074	30.92	9.10	232 700	6.093	0.0455
32.5	10.8	169 430	5.977	0.076	34.5	10.97	232 700	6.144	0.030
34.8	12.11	169 700	6.008	0.067	37.1	12.30	232 700	6.176	0.017
37.2	13.44	169 700	6.042	0.055	39.6	13.71	232 000	6.203	0.026
39.55	14.50	169 700	6.062	0.034	.....	......	....	......	......
(*c*) RUN I: $A = 0.2333$ SQUARE FEET; $D = 0.1727$ FEET; AND $\frac{L}{D} = 54.99$.									
1.20	0.0265	75 600	4.195	0.332	9.88	1.567	91 600	5.193	0.2735
1.72	0.0625	76 000	4.354	0.393	10.43	1.747	91 600	5.218	0.272
2.35	0.1025	74 800	4.482	0.337	10.9	1.89	91 600	5.236	0.270
2.785	0.138	80 900	4.590	0.318	9.68	1.554	100 010	5.2214	0.2880
3.01	0.1595	83 210	4.636	0.314	10.80	1.90	99 780	5.2669	0.2788
3.43	0.203	85 100	4.703	0.3055	11.59	2.185	100 470	5.3032	0.2788
3.93	0.263	85 730	4.765	0.300	12.94	2.70	100 700	5.3522	0.2742
4.31	0.321	85 940	4.806	0.305	14.11	3.195	100 700	5.3838	0.2748
4.61	0.366	86 300	4.838	0.303	15.48	3.815	101 180	5.4322	0.2718
4.91	0.409	86 780	4.867	0.296	17.11	4.63	102 140	5.4800	0.2648
5.09	0.442	87 200	4.884	0.300	18.58	5.43	102 860	5.517	0.2648
5.51	0.5165	87 640	4.922	0.299	19.26	5.84	103 100	5.5353	0.2648
5.99	0.605	88 300	4.960	0.296	20.36	6.55	104 300	5.5697	0.2672
6.63	0.7305	89 180	5.009	0.290	21.56	7.44	105 020	5.5916	0.2768
7.20	0.858	89 620	5.046	0.288	23.07	8.45	105 740	5.6243	0.2695
7.54	0.934	90 060	5.068	0.284	25.18	10.2	106 220	5.6646	0.2742
7.96	1.02	90 500	5.095	0.2755	27.05	11.82	106 940	5.6981	0.2765
8.43	1.15	90 720	5.120	0.278	29.12	13.6	107 660	5.7332	0.2742
8.80	1.254	91 160	5.142	0.278	30.73	15.18	107 900	5.7574	0.2742
9.27	1.39	91 380	5.164	0.2765	.....	......	....	......	......
(*d*) RUN I*a*: $A \doteq 0.2333$ SQUARE FEET; $D = 0.1727$ FEET; AND $\frac{L}{D} = 54.99$									
9.65	1.556	110 400	5.265	0.290	19.4	6.08	115 150	5.587	0.2765
11.28	2.073	110 900	5.334	0.2815	20.63	6.77	115 400	5.615	0.2695
13.21	2.818	111 900	5.407	0.276	21.76	7.55	115 650	5.638	0.272
14.78	3.46	112 150	5.457	0.268	22.6	8.14	116 900	5.658	0.272
16.15	4.16	112 900	5.498	0.2705	23.4	8.80	117 400	5.676	0.2735
17.31	4.79	113 400	5.531	0.2705	24.2	9.41	117 900	5.692	0.274
18.35	5.39	114 150	5.558	0.272	25.1	10.45	118 900	5.712	0.288

TABLE 2.—(*Continued*)

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:			Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:		
		$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)			$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)
(*d*) RUN I*a*: $A = 0.2333$ SQUARE FEET; $D = 0.1727$ FEET; AND $\frac{L}{D} = 54.99$. (*Continued*)									
26.4	11.45	119 650	5.737	0.2835	9.1	1.336	119 150	5.272	0.276
27.78	12.69	120 150	5.760	0.284	8.28	1.130	121 400	5.240	0.2855
28.75	13.62	120 400	5.776	0.286	1.653	0.0515	74 200	4.326	0.343
29.65	14.48	121 650	5.794	0.298	7.60	0.922	144 380	5.278	0.272
30.55	15.45	122 400	5.810	0.288	7.555	0.908	152 700	5.299	0.270
31.5	16.40	123 150	5.826	0.286	10.72	1.801	160 500	5.473	0.263
32.4	17.15	124 900	5.844	0.281	10.67	1.785	180 500	5.522	0.265
32.93	17.91	130 340	5.870	0.287	10.57	1.760	196 080	5.552	0.266
32.93	17.88	132 940	5.877	0.285	10.53	1.750	201 680	5.564	0.266
14.43	3.25	130 600	5.513	0.261	9.34	1.437	85 310	5.138	0.285
17.66	4.89	142 820	5.638	0.263	11.07	2.023	86 780	5.219	0.287
21.28	7.245	147 500	5.734	0.272	13.17	2.835	87 200	5.297	0.282
24.3	9.54	154 000	5.810	0.276	15.7	3.964	94 260	5.408	0.276
26.27	11.29	157 900	5.855	0.281	17.82	5.144	100 010	5.489	0.277
28.21	13.20	161 800	5.896	0.287	18.97	5.815	104 300	5.534	0.278
33.04	16.92	167 540	5.982	0.258	20.3	6.645	128 000	5.652	0.275
30.55	14.26	174 020	5.962	0.252	22.0	7.812	137 880	5.718	0.276
33.12	17.17	185 160	6.026	0.2625	23.5	9.25	143 860	5.766	0.293
2.763	0.141	112 650	4.731	0.3335	33.64	17.84	151 660	5.946	0.265
3.87	0.2605	113 400	4.88	0.308	27.15	11.82	157 380	5.868	0.274
4.705	0.3741	112 650	4.961	0.296	29.8	14.16	162 320	5.922	0.2695
5.15	0.448	114 900	5.009	0.296	31.62	16.04	172 670	5.976	0.2735
5.58	0.506	114 650	5.042	0.280	33.6	17.91	182 660	6.027	0.2695
6.025	0.584	114 650	5.076	0.2755	33.77	17.40	194 960	6.058	0.252
7.25	0.866	115 150	5.098	0.286	25.62	10.30	199 440	5.946	0.262
7.506	0.931	115 400	5.175	0.286	33.78	17.29	205 040	6.08	0.2495
11.0	1.911	115 650	5.342	0.266	33.78	17.30	215 800	6.102	0.250
10.5	1.778	115 650	5.322	0.275	33.77	17.41	221 800	6.114	0.2525
9.96	1.611	117 150	5.305	0.278	33.78	17.50	224 700	6.120	0.255
(*e*) RUN II: $A = 0.02331$ SQUARE FEET; $D = 0.1723$ FEET; AND $\frac{L}{D} = 55.14$.									
10.69	2.25	74 800	5.140	0.3615	12.49	3.32	128 000	5.440	0.396
9.80	1.90	75 000	5.103	0.3625	10.2	2.11	128 520	5.355	0.384
9.0	1.61	75 400	5.068	0.3655	8.34	1.442	128 000	5.265	0.3825
8.145	1.319	75 600	5.026	0.3645	10.42	2.286	128 000	5.362	0.3895
6.93	0.97	76 000	4.952	0.372	9.77	2.002	129 040	5.337	0.389
5.825	0.6845	76 600	4.886	0.372	8.98	1.662	129 040	5.300	0.3815
5.107	0.545	76 800	4.830	0.388	7.85	1.242	128 780	5.242	0.372
4.197	0.372	77 000	4.745	0.392	6.96	0.955	128 260	5.188	0.362
3.22	0.2192	77 400	4.634	0.392	5.97	0.684	127 480	5.119	0.3505
2.143	0.107	78 000	4.460	0.435	5.08	0.496	127 740	5.048	0.3505
1.582	0.0613	78 400	4.330	0.456	4.073	0.3185	128 000	4.954	0.3505
2.575	0.1421	79 010	4.545	0.398	2.96	0.174	128 000	4.815	0.364
3.615	0.2665	79 220	4.692	0.3765	2.283	0.1065	128 000	4.703	0.378
4.70	0.4504	78 800	4.806	0.376	3.482	0.232	128 000	4.886	9.348
5.505	0.615	78 400	4.872	0.374	4.565	0.3945	128 000	5.004	0.344
33.4	24.95	76 000	5.642	0.416	5.55	0.5905	128 000	5.088	0.350
31.6	22.1	76 200	5.618	0.411	10.6	2.44	167 540	5.486	0.4035
29.76	19.5	76 400	5.593	0.410	9.85	2.092	166 480	5.452	0.400
28.0	17.2	76 600	5.568	0.4075	9.04	1.73	166 220	5.414	0.392
25.95	14.61	76 800	5.536	0.404	8.025	1.34	166 480	5.363	0.386
24.18	12.54	76 800	5.505	0.3985	7.04	1.011	166 740	5.306	0.3765
22.23	10.63	77 000	5.470	0.400	6.09	0.7445	167 000	5.244	0.369
20.36	8.82	77 400	5.434	0.396	3.91	0.300	166 220	5.051	0.360
18.4	7.125	77 600	5.391	0.391	2.86	0.1665	165 180	4.911	0.375
16.52	5.72	77 800	5.346	0.388	2.122	0.100	164 920	4.781	0.414
14.47	4.33	77 800	5.288	0.383	3.462	0.228	164 920	4.994	0.346
12.58	3.276	78 000	5.228	0.3825	4.445	0.381	165 700	5.104	0.352
10.86	2.403	78 000	5.165	0.376	5.505	0.599	165 960	5.197	0.362
9.35	1.753	78 200	5.10	0.370	33.9	26.06	167 000	5.989	0.422
33.43	25.62	128 520	5.870	0.426	32.1	23.4	167 000	5.966	0.424
32.0	23.42	128 520	5.851	0.425	29.86	20.2	167 000	5.934	0.4215
29.7	19.7	128 000	5.816	0.416	27.36	17.02	167 000	5.896	0.424
27.2	16.54	128 520	5.781	0.416	25.2	14.37	167 000	5.860	0.422
24.15	12.94	128 000	5.726	0.414	22.75	11.79	167 000	5.816	0.4255
21.6	10.4	128 520	5.6805	0.415	19.9	9.07	167 000	5.758	0.428
19.36	8.30	128 520	5.632	0.413	18.11	7.43	167 000	5.717	0.422
17.1	6.39	128 000	5.577	0.408	15.75	5.535	167 000	5.656	0.4145
14.7	4.64	128 000	5.511	0.398	13.21	3.82	167 000	5.580	0.4065

TABLE 2.—(*Continued*)

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of:			Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of:		
		$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)			$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)
(*e*) RUN II: $A = 0.02331$ SQUARE FEET; $D = 0.1723$ FEET; AND $\frac{L}{D} = 55.14$. (*Continued*)									
11.53	2.886	167 000	5.521	0.402	10.6	2.53	221 700	5.608	0.4185
9.64	2.027	167 000	5.443	0.405	9.84	2.157	222 900	5.578	0.414
34.0	26.02	222 900	6.116	0.420	9.00	1.778	222 900	5.540	0.408
32.2	23.24	222 900	6.093	0.418	7.975	1.372	222 900	5.487	0.402
30.4	20.62	222 900	6.068	0.4165	6.94	1.010	222 900	5.426	0.3895
27.7	17.35	222 900	6.028	0.420	6.02	0.757	222 900	5.365	0.388
25.6	14.87	222 900	5.994	0.422	5.11	0.532	222 900	5.294	0.376
23.55	12.68	222 900	5.957	0.427	4.005	0.318	222 900	5.188	0.364
21.44	10.6	222 900	5.917	0.429	2.82	0.160	222 900	5.036	0.3705
19.44	8.75	222 900	5.874	0.432	1.902	0.0825	222 900	4.865	0.4255
17.6	7.16	222 900	5.831	0.431	3.55	0.244	223 200	5.136	0.354
15.3	5.345	222 900	5.776	0.424	4.51	0.4045	223 200	5.240	0.364
13.17	3.94	222 900	5.704	0.424	5.595	0.6455	222 900	5.332	0.381
11.11	2.777	222 900	5.631	0.418	.....	....	....	....	....
(*f*) RUN III: $A = 0.02336$ SQUARE FEET; $D = 0.1725$ FEET; AND $\frac{L}{D} = 55.09$.									
31.96	23.77	74 600	5.614	0.434	5.58	0.631	128 000	5.091	0.374
30.2	21.19	74 800	5.590	0.434	6.02	0.736	128 000	5.124	0.376
28.55	18.7	75 200	5.580	0.428	32.02	24.11	166 220	5.964	0.4365
26.58	16.1	75 600	5.538	0.426	30.35	21.38	166 480	5.940	0.433
24.46	13.58	76 000	5.507	0.422	28.4	18.69	166 480	5.912	0.432
22.45	11.45	76 200	5.470	0.424	26.46	16.16	167 000	5.882	0.430
20.55	9.505	76 400	5.433	0.420	24.02	13.40	166 480	5.839	0.433
18.53	7.69	76 600	5.388	0.418	22.0	11.37	166 480	5.801	0.437
16.4	5.91	77 000	5.339	0.410	20.0	9.40	166 740	5.760	0.438
14.21	4.48	77 400	5.277	0.413	18.09	7.61	166 740	5.716	0.435
11.36	2.819	77 600	5.183	0.407	16.0	5.89	166 740	5.663	0.430
9.07	1.74	77 800	5.086	0.393	13.5	4.192	167 000	5.590	0.428
10.77	2.442	77 800	5.160	0.392	11.36	2.957	167 000	5.515	0.428
10.06	2.13	77 800	5.130	0.390	9.8	2.186	166 740	5.450	0.4245
9.245	1.771	77 800	5.094	0.384	9.2	1.928	166 480	5.423	0.424
8.43	1.473	77 800	5.054	0.384	10.74	2.655	169 700	5.498	0.428
7.55	1.17	78 000	5.006	0.3795	10.07	2.281	169 160	5.468	0.420
6.55	0.875	78 200	4.946	0.378	9.19	1.874	168 350	5.426	0.414
5.495	0.62	78 600	4.870	0.381	8.243	1.48	168 350	5.378	0.406
4.56	0.4345	78 600	4.790	0.387	7.225	1.149	168 350	5.321	0.410
3.74	0.2922	78 800	4.706	0.388	6.195	0.796	167 270	5.252	0.3845
2.87	0.174	79 010	4.592	0.392	5.20	0.571	167 270	5.176	0.392
2.078	0.087	79 220	4.454	0.372	4.126	0.359	167 270	5.076	0.3905
1.35	0.0467	79 430	4.267	0.476	3.083	0.1988	167 270	4.95	0.3875
1.628	0.0639	80 060	4.352	0.450	1.964	0.0890	167 540	4.754	0.431
2.483	0.1251	80 900	4.540	0.375	1.61	0.0624	167 540	4.668	0.449
3.295	0.216	80 900	4.662	0.366	4.72	0.4468	167 000	5.133	0.369
4.16	0.356	80 480	4.672	0.380	5.725	0.672	165 700	5.214	0.380
5.10	0.5395	80 060	4.848	0.3845	32.4	24.22	222 900	6.096	0.431
32.0	24.2	126 180	5.844	0.4415	30.73	21.61	221 400	6.070	0.428
30.08	21.33	127 480	5.821	0.439	28.82	19.29	221 400	6.042	0.432
28.22	18.51	128 000	5.795	0.4325	27.12	17.03	222 900	6.018	0.432
25.83	15.41	128 260	5.758	0.430	24.83	14.37	222 900	5.980	0.434
22.72	12.18	128 260	5.701	0.440	23.0	12.38	222 900	5.947	0.436
20.81	10.31	128 000	5.663	0.444	20.86	10.29	222 900	5.905	0.440
18.75	8.345	128 000	5.617	0.443	18.58	8.14	222 900	5.854	0.440
16.81	6.64	128 000	5.570	0.438	16.54	6.48	222 900	5.804	0.442
14.73	5.055	128 000	5.513	0.434	14.0	4.577	222 900	5.732	0.435
12.28	3.48	128 000	5.433	0.4315	11.7	3.218	222 900	5.654	0.438
10.43	2.492	128 000	5.353	0.426	10.89	2.727	224 280	5.626	0.4295
9.325	0.978	128 000	5.314	0.4235	10.07	2.323	223 420	5.589	0.428
10.67	2.522	127 480	5.370	0.414	9.30	1.978	221 400	5.550	0.426
9.71	2.10	129 820	5.338	0.415	8.29	1.549	221 400	5.501	0.420
8.94	1.733	129 820	5.301	0.4035	7.20	1.141	221 400	5.439	0.412
8.095	1.416	129 300	5.257	0.402	6.17	0.846	221 400	5.373	0.4145
7.05	1.052	129 300	5.197	0.393	5.20	0.585	221 400	5.298	0.403
6.02	0.7355	128 520	5.126	0.376	4.163	0.3625	222 000	5.203	0.388
4.992	0.505	128 000	5.042	0.374	3.303	0.2325	222 900	5.104	0.3965
3.92	0.302	128 000	4.937	0.361	2.571	0.1372	222 900	4.995	0.385
2.857	0.1549	128 000	4.800	0.346	1.893	0.081	222 900	4.862	0.422
2.32	0.108	128 520	4.712	0.370	3.67	0.2656	222 900	5.150	0.362
3.437	0.224	128 520	4.882	0.346	4.69	0.469	223 420	5.257	0.3965
4.425	0.396	128 780	4.992	0.373	5.745	0.720	222 900	5.344	0.4065

TABLE 2.—(*Continued*)

(*g*) RUN IV: $A = 0.02375$ SQUARE FEET; $D = 0.1739$ FEET; AND $\frac{L}{D} = 54.63$.

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:			Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	VALUES OF:		
		$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log R	log (100 f)			$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log R	log (100 f)
8.87	2.275	94 030	5.162	0.530	15.68	7.51	170 240	5.666	0.5565
10.79	3.395	94 490	5.248	0.536	12.8	4.97	170 510	5.580	0.551
12.19	4.424	95 180	5.304	0.548	9.82	2.89	170 780	5.464	0.5475
14.19	6.025	95 870	5.372	0.546	5.77	0.969	170 780	5.234	0.534
16.8	8.51	96 560	5.450	0.550	30.2	28.45	170 780	5.952	0.566
18.8	10.81	97 710	5.504	0.554	30.22	28.6	170 780	5.953	0.566
20.75	13.13	98 400	5.550	0.556	29.4	27.1	171 320	5.942	0.5665
22.86	16.18	99 320	5.596	0.564	7.66	1.825	219 000	5.465	0.564
29.61	27.83	100 240	5.821	0.572	30.56	28.72	225 000	6.077	0.5585
26.44	22.28	102 620	5.674	0.574	30.66	28.95	229 000	6.087	0.558
27.9	24.36	105 500	5.709	0.565	1.861	0.0805	222 900	4.858	0.4365
30.4	29.28	112 900	5.775	0.571	2.702	0.188	228 100	5.030	0.481
30.35	28.68	128 000	5.829	0.564	3.639	0.352	229 270	5.162	0.4945
29.17	26.66	128 260	5.813	0.567	4.355	0.52	230 100	5.242	0.508
27.96	24.67	129 560	5.800	0.570	4.95	0.685	229 000	5.295	0.517
26.63	22.20	130 340	5.782	0.565	5.50	0.8825	228 550	5.340	0.536
25.0	19.40	128 520	5.747	0.563	6.46	1.22	228 100	5.409	0.5365
23.14	16.59	127 480	5.709	0.562	7.003	1.442	228 100	5.444	0.5395
21.4	14.19	125 400	5.669	0.562	7.70	1.756	227 650	5.483	0.542
30.4	29.01	120 400	5.803	0.566	8.48	2.158	227 650	5.525	0.5475
3.168	0.2515	129 560	4.854	0.469	9.008	2.445	228 100	5.553	0.5495
3.84	0.380	130 340	4.940	0.482	9.47	2.735	229 270	5.578	0.554
4.50	0.5295	129 560	5.005	0.487	10.1	3.11	229 270	5.605	0.555
5.005	0.670	127 740	5.046	0.498	10.87	3.618	229 270	5.637	0.556
5.36	0.774	127 480	5.076	0.5005	11.63	4.06	229 270	5.666	0.548
5.645	0.870	127 480	5.097	0.506	13.6	5.54	230 370	5.736	0.548
6.305	1.10	128 260	5.148	0.513	16.0	7.90	229 540	5.806	0.559
6.84	1.31	129 040	5.186	0.518	18.47	10.6	229 000	5.866	0.5635
7.54	1.61	129 040	5.228	0.523	20.9	13.77	228 550	5.920	0.569
8.18	1.92	129 040	5.264	0.528	23.2	16.80	228 550	5.965	0.5645
8.70	2.167	129 040	5.291	0.528	25.5	20.22	228 550	6.006	0.562
9.21	2.443	129 040	5.316	0.531	27.33	23.1	228 100	6.035	0.5605
9.71	2.745	128 780	5.337	0.535	30.2	27.82	228 100	6.078	0.5545
10.29	3.096	128 520	5.362	0.536	6.7	1.330	227 650	5.423	0.543
10.81	3.450	128 000	5.382	0.5395	10.28	3.207	228 550	5.611	0.552
3.147	0.2464	167 000	4.962	0.466	13.52	5.51	229 540	5.733	0.5505
4.005	0.4146	166 740	5.065	0.482	17.53	9.70	229 270	5.845	0.570
4.815	0.615	166 740	5.145	0.494	20.03	12.76	229 000	5.902	0.572
5.445	0.811	167 000	5.199	0.5075	21.8	15.17	229 270	5.939	0.574
5.71	0.8975	167 000	5.220	0.510	23.7	17.68	229 540	5.976	0.5685
6.04	1.011	168 080	5.246	0.514	27.0	22.82	229 000	6.031	0.566
6.485	1.71	167 270	5.276	0.516	30.4	28.32	229 000	6.082	0.557
7.365	1.541	166 740	5.328	0.5235	2.86	0.2143	227 380	5.053	0.4885
7.825	1.760	166 740	5.355	0.529	4.075	0.451	229 000	5.210	0.5035
8.445	2.063	167 000	5.389	0.532	5.00	0.699	230 100	5.302	0.5165
9.19	2.482	167 000	5.426	0.5385	5.435	0.855	229 540	5.336	0.5335
9.55	2.71	166 740	5.442	0.544	6.105	1.092	228 550	5.386	0.538
10.12	3.05	167 000	5.469	0.544	6.78	1.353	228 100	5.430	0.5405
10.72	3.46	167 000	5.494	0.598	7.56	1.714	228 100	5.478	0.548
14.98	6.79	167 270	5.640	0.551	8.70	2.296	229 000	5.540	0.552
16.85	8.72	167 270	5.690	0.558	9.46	2.760	229 820	5.577	0.559
18.6	10.7	167 270	5.734	0.561	10.23	3.228	229 540	5.612	0.560
20.5	13.28	167 270	5.776	0.5685	11.03	3.772	229 540	5.644	0.561
22.25	15.68	167 540	5.812	0.570	30.45	28.42	236 200	6.098	0.556
23.9	18.39	166 740	5.841	0.578	28.64	25.56	237 640	6.073	0.562
26.0	21.22	166 480	5.877	0.566	27.3	23.1	238 020	6.053	0.5615
27.95	24.38	166 480	5.908	0.564	25.82	20.8	237 640	6.028	0.564
29.6	27.05	166 740	5.934	0.5595	24.1	18.0	237 640	5.998	0.562
30.75	28.99	167 000	5.951	0.557	22.0	15.23	236 880	5.956	0.569
29.21	26.67	170 510	5.938	0.564	19.85	12.46	237 640	5.913	0.5765
28.36	25.41	170 780	5.925	0.570	17.83	10.08	237 640	5.867	0.572
27.17	23.2	170 780	5.906	0.568	15.27	7.225	237 640	5.799	0.562
25.15	20.42	171 050	5.872	0.580	11.68	4.245	237 640	5.683	0.563
23.5	17.39	170 510	5.843	0.568	7.99	2.00	236 880	5.517	0.5665
21.13	14.21	170 240	5.799	0.5665	5.51	0.921	236 880	5.356	0.552
19.03	11.31	170 240	5.750	0.565	......	......	....	.....	.. ..

TABLE 2.—(*Continued*)

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of:			Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of:		
		$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)			$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)
(*h*) Run V: $A = 0.02376$ Square Feet; $D = 0.1739$ Feet; and $\frac{L}{D} = 54.62$.									
10.07	3.761	75 600	5.122	0.640	28.37	31.75	167 000	5.916	0.6675
11.62	5.05	75 600	5.184	0.644	27.10	28.75	167 000	5.896	0.663
12.52	5.91	75 600	5.218	0.6465	25.62	25.71	166 740	5.871	0.663
14.88	8.35	75 800	5.283	0.648	23.8	22.18	166 740	5.839	0.664
17.15	11.20	76 000	5.355	0.652	22.0	19.18	167 000	5.806	0.670
19.10	14.03	76 000	5.401	0.656	19.96	15.83	167 000	5.763	0.670
20.96	17.0	76 000	5.442	0.660	17.84	12.63	167 000	5.714	0.670
23.07	20.83	76 400	5.486	0.664	16.03	10.0	167 000	5.669	0.661
28.14	30.83	76 600	5.572	0.662	14.10	7.66	166 480	5.611	0.657
26.79	27.87	76 800	5.554	0.6605	10.20	4.005	166 220	5.470	0.657
25.20	24.78	77 200	5.530	0.662	11.10	4.708	166 220	5.507	0.654
23.96	22.25	77 400	5.508	0.660	12.21	5.75	166 480	5.548	0.6565
1.241	0.0372	78 000	4.226	0.454	1.241	0.0362	166 480	4.556	0.4425
1.509	0.0583	78 000	4.310	0.4695	2.182	0.1432	167 000	4.802	0.550
2.238	0.1419	79 010	4.488	0.524	3.097	0.3123	169 430	4.960	0.584
3.031	0.276	79 430	4.622	0.549	4.10	0.586	169 430	5.082	0.614
3.95	0.509	79 010	4.735	0.584	5.105	0.929	167 000	5.172	0.623
4.94	0.8315	78 600	4.830	0.604	4.60	0.745	166 220	5.124	0.6175
4.465	0.6805	78 400	4.786	0.604	3.565	0.4325	166 480	5.014	0.602
3.56	0.415	78 600	4.687	0.586	2.762	0.248	166 740	4.904	0.583
2.675	0.2162	78 800	4.564	0.552	1.691	0.0885	166 740	4.690	0.563
1.98	0.1186	79 220	4.436	0.553	5.675	1.172	167 000	5.217	0.632
1.443	0.0643	79 430	4.299	0.560	10.80	4.565	166 740	5.496	0.6635
5.54	1.062	79 430	4.883	0.6105	9.80	3.73	166 740	5.453	0.660
10.48	4.05	78 800	5.157	0.638	8.955	3.08	167 000	5.416	0.656
9.55	3.335	79 010	5.118	0.634	8.125	2.504	167 000	5.373	0.650
8.70	2.73	79 220	5.078	0.628	7.125	1.907	167 000	5.316	0.646
7.755	2.166	79 220	5.028	0.628	6.30	1.453	167 000	5.270	0.6355
6.88	1.69	79 220	4.976	0.624	28.3	31.64	222 900	6.040	0.668
6.08	1.295	79 430	4.924	0.616	27.22	29.0	222 900	6.024	0.6625
28.2	31.59	128 000	5.798	0.669	25.8	26.23	222 900	6.000	0.666
26.85	28.3	128 000	5.776	0.664	24.05	23.73	222 900	5.969	0.658
25.4	25.19	128 000	5.752	0.662	22.1	19.4	222 900	5.933	0.671
23.3	21.1	128 000	5.715	0.660	20.2	16.28	222 900	5.894	0.6715
21.48	18.09	128 000	5.680	0.664	18.23	13.2	222 900	5.850	0.670
19.52	14.91	128 000	5.638	0.664	16.08	10.27	222 900	5.795	0.670
17.6	11.97	128 000	5.594	0.658	13.85	7.445	222 900	5.730	0.660
15.72	9.46	128 000	5.544	0.654	10.28	4.23	222 900	5.601	0.6725
13.47	6.97	128 000	5.477	0.656	11.37	5.07	222 900	5.644	0.664
11.92	5.407	128 000	5.422	0.6515	12.2	5.845	222 900	5.675	0.6645
11.09	4.65	128 000	5.392	0.649	10.42	4.29	222 900	5.607	0.666
10.28	4.00	128 000	5.359	0.649	9.655	3.673	223 560	5.574	0.667
1.158	0.0306	130 080	4.418	0.431	9.04	3.182	223 920	5.546	0.661
2.38	0.171	131 120	4.735	0.5505	8.250	2.636	223 920	5.506	0.6595
3.10	0.3115	133 200	4.856	0.581	7.325	2.05	223 560	5.454	0.654
3.82	0.4905	132 680	4.944	0.598	6.535	1.612	223 200	5.405	0.6495
4.95	0.8625	130 080	5.048	0.618	5.92	1.327	222 900	5.362	0.650
4.44	0.683	129 560	5.000	0.610	5.405	1.10	222 900	5.322	0.648
3.60	0.4375	129 560	4.910	0.599	4.395	0.706	222 900	5.232	0.6325
2.70	0.2322	129 300	4.784	0.574	3.421	0.4195	222 900	5.123	0.626
2.024	0.1291	129 300	4.659	0.570	2.703	0.2475	222 900	5.021	0.600
5.50	1.065	129 040	5.092	0.6175	1.831	0.1101	222 900	4.852	0.588
10.81	4.49	127 740	5.380	0.655	1.211	0.0458	222 900	4.672	0.5655
9.92	3.76	127 740	5.342	0.652	1.372	0.0618	222 900	4.726	0.586
9.10	3.124	128 000	5.306	0.648	2.424	0.191	222 900	4.974	0.584
8.07	2.428	128 000	5.254	0.642	3.235	0.364	223 560	5.100	0.612
7.045	1.819	128 260	5.196	0.635	2.981	0.5655	223 560	5.190	0.6235
6.39	1.431	128 260	5.154	0.616	4.945	0.8995	222 900	5.283	0.6365
(*i*) Run VI: $A = 0.02392$ Square Feet; $D = 0.1745$ Feet; and $\frac{L}{D} = 54.44$.									
28.4	31.4	78 600	5.591	0.663	17.56	11.82	83 630	5.408	0.656
26.76	28.1	79 220	5.568	0.667	15.85	9.67	84 260	5.368	0.659
25.8	26.08	80 270	5.558	0.667	13.65	7.15	84 680	5.304	0.657
23.76	21.98	80 690	5.524	0.664	11.84	5.34	84 890	5.244	0.653
21.6	18.02	81 540	5.488	0.661	10.11	3.84	85 310	5.178	0.6475
20.38	15 88	82 160	5.466	0.656	8.51	2.66	85 730	5.105	0.638
18.8	13.6	82 790	5.434	0.658	5.72	1.11	86 150	4.934	0.604

TABLE 2.—(*Continued*)

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of: $\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	Values of: log R	Values of: log (100 f)	Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of: $\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	Values of: log R	Values of: log (100 f)
(*i*) RUN VI: $A = 0.02392$ SQUARE FEET; $D = 0.1745$ FEET; AND $\frac{L}{D} = 54.44$. (*Continued*)									
1.348	0.0532	81 320	4.282	0.540	14.91	8.66	166 220	5.637	0.6635
1.943	0.0880	83 000	4.450	0.440	13.09	6.61	166 740	5.580	0.660
10.67	4.198	82 370	5.186	0.640	11.68	5.28	167 540	5.533	0.6615
10.0	3.69	82 580	5.158	0.640	10.57	4.255	167 270	5.490	0.654
9.30	3.158	82 580	5.126	0.6345	12.26	5.82	167 000	5.553	0.6615
8.70	2.73	82 370	5.096	0.630	11.0	4.74	168 [illegible]80	5.509	0.666
7.89	2.233	82 370	5.054	0.628	10.34	4.17	165 700	5.476	0.664
6,945	1.71	82 370	4.999	0.622	9.60	3.59	165 700	5.444	0.663
5.995	1.238	82 580	4.936	0.610	8.80	2.983	166 220	5.407	0.6585
5.243	0.937	82 370	4.876	0.606	7.90	2.378	166 480	5.382	0.654
4.876	0.808	82 370	4.846	0.6045	6.98	1.836	166 740	5.307	0.6495
4.357	0.6445	82 370	4.797	0.604	5.855	1.269	166 740	5.232	0.6415
4.005	0.539	82 370	4.760	0.599	5.03	0.892	166 480	5.186	0.621
3.66	0.445	82 580	4.722	0.594	4.495	0.710	165 960	5.114	0.6195
3.268	0.3465	82 580	4.673	0.584	3.76	0.492	165 700	5.037	0.6145
2.855	0.2603	82 580	4.614	0.577	3.11	0.325	165 960	4.955	0.599
2.402	0.1835	82 790	4.541	0.575	2.595	0.2203	166 480	4.898	0.588
1.86	0.1151	82 790	4.430	0.5955	2.077	0.1428	166 740	4.780	0.592
4.06	0.5235	128 260	4.959	0.574	28.6	32.6	223 920	6.048	0.674
28.35	32.18	127 740	5.800	0.6755	27.3	29.83	223 920	6.028	0.675
26.46	27.78	128 000	5.772	0.670	26.24	27.62	223 920	6.011	0.6755
24.6	23.74	128 520	5.742	0.667	25.36	25.77	223 920	5.996	0.676
23.01	20.8	128 000	5.712	0.6665	24.36	23.93	224 280	5.980	0.6785
21.6	18.35	128 000	5.684	0.668	23.15	21.62	224 280	5.958	0.6795
20.13	15.92	128 260	5.654	0.668	21.95	19.42	224 280	5.934	0.6785
18.23	13.11	128 780	5.612	0.669	20.42	16.92	224 280	5.904	0.6815
16.03	9.94	128 260	5.556	0.660	19.1	14.71	224 100	5.874	0.6785
14.4	7.95	128 000	5.508	0.656	17.6	12.5	224 100	5.838	0.6785
12.23	5.795	128 000	5.437	0.660	15.77	9.90	224 280	5.791	9.673
3.139	0.325	128 260	4.846	0.591	13.84	7.615	224 100	5.734	0.6725
10.89	4.56	126 960	5.383	0.659	12.33	5.995	224 100	5.684	0.668
10.59	4.32	126 700	5.368	0.6595	11.05	4.84	223 920	5.636	0.6705
10.02	3.85	127 480	5.348	0.656	10.6	4.45	224 100	5.618	0.6725
9.395	3.363	128 000	5.322	0.654	9.90	3.886	224 460	5.589	0.6715
8.66	2.838	128 000	5.288	0.6505	9.28	3.395	224 460	5.560	0.6685
7.85	2.318	127 480	5.242	0.648	8.56	2.862	224 460	5.526	0.665
7.085	1.842	127 480	5.198	0.638	7.62	2.252	224 460	5.476	0.662
6.20	1.381	127 480	5.140	0.629	6.66	1.692	224 460	5.417	0.655
5.406	1.028	127 740	5.080	0.6195	5.85	1.30	224 280	5.361	0.652
4.66	0.746	128 000	5.018	0.610	5.05	0.946	224 280	5.297	0.642
2.36	0.1827	128 520	4.724	0.5885	4.37	0.695	224 280	5.234	0.6335
28.53	32.45	167 270	5.921	0.674	3.598	0.464	224 280	5.150	0.627
27.3	29.8	127 270	5.902	0.6745	2.74	0.2535	224 280	5.032	0.602
26.18	27.24	167 000	5.883	0.673	2.103	0.1499	224 280	4.916	0.6025
25.0	24.77	167 000	5.862	0.6705	14.1	7.805	225 360	5.744	0.6665
23.8	22.5	167 000	5.841	0.672	12.75	6.37	222 900	5.696	0.666
21.85	19.05	167 000	5.804	0.6735	11.08	4.85	221 400	5.632	0.670
20.37	16.60	167 270	5.774	0.676	9.94	3.94	222 300	5.5860	0.6735
18.4	13.45	167 000	5.730	0.6725	8.80	3.075	222 900	5.534	0.671
16.85	11.20	166 740	5.690	0.6695	7.98	2.53	223 200	5.493	0.672
14.96	8.795	166 480	5.660	0.668	......	......	....	......	......
(*j*) RUN VII: $A = 0.02418$ SQUARE FEET; $D = 0.1754$ FEET; AND $\frac{L}{D} = 54.145$.									
10.54	5.956	75 400	5.184	0.803	2.12	0.192	79 850	4.473	0.706
9.63	5.00	75 600	5.107	0.806	2.983	0.4203	80 270	4.624	0.748
8.825	4.22	76 000	5.070	0.806	3.917	0.7375	79 640	4.738	0.756
7.97	3.423	76 200	5.027	0.806	4.31	0.908	79 220	4.778	0.763
7.11	2.597	76 200	4.978	0.7855	24.6	34.2	74 400	5.506	0.827
6.158	1.928	76 600	4.918	0.781	23.46	30.73	74 800	5.470	0.8205
5.476	1.519	76 800	4.868	0.779	22.0	27.04	75 200	5.463	0.822
5.05	1.289	77 000	4.834	0.778	20.43	23.37	75 400	5.432	0.821
4.38	0.9595	77 400	4.774	0.7725	18.57	19.26	75 400	5.390	0.822
3.46	0.5705	77 800	4.674	0.7525	16.71	15.41	75 800	5.347	0.8165
2.648	0.3197	78 000	4.560	0.733	14.8	12.11	76 200	5.297	0.8175
1.783	0.1421	78 200	4.388	0.726	12.85	9.095	76 200	5.235	0.8145
1.347	0.07995	78 400	4.268	0.718	11.26	6.98	76 600	5.180	0.815
1.57	0.1026	79 220	4.339	0.694	9.69	5.095	76 800	5.116	0.810

TABLE 2.—(*Continued*)

(j) Run VII: $A = 0.02418$ Square Feet; $D = 0.1754$ Feet, and $\frac{L}{D} = 54.145$. (*Continued*)

Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of:			Mean velocity, V, in feet per second	Loss of head, H_f, due to friction, in feet	Values of:		
		$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)			$\frac{\rho}{\mu}$, in $\frac{\text{seconds}}{\text{feet}^2}$	log **R**	log (100 f)
24.59	34.61	127 480	5.740	0.833	8.895	4.36	163 880	5.408	0.816
23.24	30.82	127 480	5.720	0.830	7.875	3.38	164 660	5.357	0.8115
21.82	26.82	128 000	5.691	0.8295	6.783	2.505	165 700	5.295	0.812
20.23	23.36	128 000	5.658	0.830	5.76	1.779	166 480	5.227	0.8035
18.69	19.69	127 480	5.622	0.826	5.105	1.371	167 000	5.175	0.796
16.97	16.16	127 740	5.580	0.824	4.295	0.959	166 480	5.099	0.790
15.09	12.64	128 000	5.530	0.820	3.33	0.5605	166 480	4.988	0.778
13.03	9.45	128 000	5.468	0.820	2.369	0.2735	166 480	4.840	0.762
11.7	7.58	128 000	5.420	0.8175	1.697	0.1331	167 000	4.696	0.740
9.93	5.50	128 000	5.354	0.821	2.876	0.4155	167 000	4.925	0.776
10.46	6.05	131 900	5.384	0.818	3.815	0.7465	167 000	5.048	0.784
9.6	5.088	130 600	5.343	0.816	10.1	5.755	222 600	5.596	0.826
8.75	4.195	129 560	5.298	0.812	9.26	4.82	223 200	5.560	0.8245
7.81	3.33	129 040	5.246	0.812	8.41	3.942	223 200	5.518	0.821
6.76	2.441	129 040	5.184	0.802	7.325	2.95	223 200	5.458	0.815
5.76	1.753	128 780	5.115	0.798	6.41	2.238	222 900	5.400	0.8115
4.895	1.257	128 520	5.043	0.7945	5.55	1.675	222 900	5.337	0.8105
4.22	0.8975	128 000	4.977	0.7785	4.96	1.33	222 900	5.288	0.808
2.966	0.429	128 000	4.824	0.763	4.335	1.01	223 200	5.230	0.8045
1.571	0.1076	128 000	4.548	0.714	3.08	0.505	223 560	5.082	0.801
2.321	0.2486	128 000	4.718	0.739	1.633	0.1348	223 560	4.807	0.7795
3.683	0.685	128 780	4.920	0.7775	2.42	0.3003	223 560	4.978	0.7855
24.44	34.62	167 000	5.856	0.830	3.75	0.748	222 600	5.166	0.800
23.22	31.59	167 000	5.833	0.842	24.5	34.97	222 900	5.982	0.840
21.59	27.2	167 000	5.801	0.841	23.62	32.87	222 900	5.966	0.8445
20.02	23.39	167 000	5.768	0.840	22.0	28.73	222 900	5.935	0.849
18.53	19.82	167 000	5.735	0.836	20.4	24.91	222 900	5.902	0.852
17.05	16.60	167 000	5.698	0.8315	18.88	21.13	222 900	5.868	0.848
15.44	13.61	167 000	5.656	0.8315	17.3	17.60	222 900	5.831	0.844
13.52	10.23	167 000	5.598	0.822	15.51	14.02	222 900	5.784	0.841
11.81	7.82	167 000	5.540	0.822	13.83	10.98	222 900	5.734	0.834
10.71	6.357	167 000	5.497	0.818	12.21	8.50	222 900	5.680	0.830
10.36	6.00	168 890	5.488	0.822	10.68	6.46	222 900	5.621	0.827
9.62	5.138	164 400	5.444	0.8185	......	....	....	.....	.....

DISCUSSION

WARREN E. WILSON,[15] JUN. AM. SOC. C. E. (by letter).—A well-planned program of research is indicated by this paper; the author has obtained results which, without doubt, are a valuable addition to the present knowledge of the characteristics of flow in pipes. The writer will make no attempt to evaluate the results, but will endeavor only to comment on the methods used to measure the loss of head in the test section of pipe.

Differential manometers were used with the evident purpose of obtaining large readings when the loss of head was small. If one assumes that the difference in elevation of the menisci in a differential manometer gives a true measure of the actual pressure difference which it is desired to observe, and if one bases his computations on the usual formula:

$$z_m = \frac{w}{w - w_k} H \qquad \text{(12)}$$

(in which z_m is the manometer reading; w, the specific gravity of water; w_k, the specific gravity of the gauge liquid; and H, the pressure difference under observation), the value of H which is calculated will generally be in error, as has been demonstrated on a previous occasion.[16]

The magnitude of the error in any manometer reading depends on the liquids and the form of the menisci. The height of rise of a liquid interface in a capillary tube may be computed from the expression,

$$h = \frac{4\,\sigma \cos\theta}{g\,D_t\,(w - w_k)} \qquad \text{(13)}$$

in which σ is the surface tension expressed in dynes per centimeter; h, the height of rise, in centimeters; θ, the angle of contact between meniscus and wall of the tube; g, the gravitational constant; and D_t, the diameter of the tube, in centimeters.

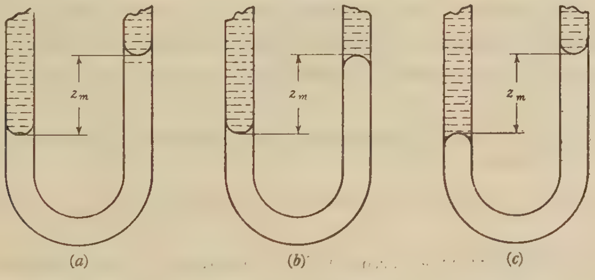

FIG. 15.

If the menisci form as in Fig. 15(a) the error of the reading is zero, providing the angle of contact is the same in the two tubes. If the formation is as shown in Fig. 15(b) the reading, z_m, is smaller than Equation (12) would indicate by an amount not greater than 2 h, in which h is given by Equation (13), since each meniscus has an effect which probably does not exceed that of a single meniscus in a capillary tube. If the formation is as shown in Fig. 15(c), the reading, z_m, is larger than Equation (13) would indicate by an amount not greater than 2 h.

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[15] Madison, N. J.

[16] "Differential Manometers Investigated," *Civil Engineering*, January, 1934, p. 30.

As a specific case, consider the manometers used and described by Mr. Streeter. The inside diameter of the tubes was about 6 mm. The liquids were acetylene tetra-bromide, and xylene, with surface tension values of approximately 36 to 38 dynes per cm. The specific gravities of the solutions were about 1.3, 1.05, and 2.97. Assuming the specific gravity of water to be unity (that is, neglecting the effect of temperature), the values of h may be computed by means of Equation (13).

The foregoing comments have shown that the error in a reading could vary from zero to 2 h. This error will correspond to an error in the computed head, H, which is H_f and will depend on the multiplication factor of the manometer. Table 3 which is based on an assumption of the value of σ as 37 dynes per cm, and of the value of θ as zero, indicates the possible limitations of the manometer.

TABLE 3.—POSSIBLE ERROR IN H_f WHEN COMPUTED FROM z_m BY EQUATION (13)

Specific gravity (1)	Multiplication factor (2)	Height of rise, h, in centimeters (3)	Possible error in z_m, in centimeters (4)	POSSIBLE ERROR IN H_f WHEN COMPUTED FROM z_m, IN EQUATION (13)	
				In centimeters (5)	In feet (6)
1.05	20.0	5.03	10.06	0.503	0.0165
1.30	3.33	0.838	1.68	0.503	0.0165
2.97	0.508	0.128	0.256	0.503	0.0165

It is obvious that the possible error in the measurement of the loss in head does not depend on the multiplication factor of the gauge, but on the diameter of the tubes and the surface tension at the interfaces. The possible error may be expressed by Equation (14) when H_e denotes the error in the measurement of H_f which is possible for any given manometer:

$$H_e = \frac{8\,\sigma}{g\,D_t}\text{ (in centimeters) } = 0.000268\,\frac{\sigma}{D_t}\text{ (in feet)}\ldots\ldots\ldots(14)$$

in which the specific gravity of water is assumed to be 1.0, and θ is assumed to be zero.

It should not be inferred from these statements that Equation (14) will give an exact value for a correction to be applied to the computed loss of head. There are too many important factors which have been neglected to permit such a simple method of correction. For instance, the cleanliness of the tubes controls the magnitude of θ to a great extent. In practice, it is found that the form of the meniscus varies widely and hence the angle is by no means constant.

The writer wishes to emphasize the magnitude of the error that may occur in any one reading of a differential manometer. It is his belief that the water-air manometer is best suited to the type of work described by the author, and will give quite as satisfactory results as a differential manometer using any liquid known to him. It must not be concluded that any serious errors exist in the research reported in this paper. It seems very probable that the wide scattering of the points in Fig. 11 for low values of the Reynolds number, may be at least partly explained by attributing an error to the manometers. For the lowest head observed, 0.0265 ft in Run I, the possible error is 62%, but for all heads of 1.65 ft, or more, the error is less than 1 per cent. However, the writer has found that the variation of the readings from the value predicted by Equation (12) seldom reaches the value given in Equation (14) and probably averages about one-third this value. However, it seems reasonable to believe that, with the ordinary water-air manometer, one can obtain an accuracy of 0.003 ft of water, which is better than what might be expected with the author's manometers even when the values given by Equation (14) are reduced by two-thirds.

RICHARD G. FOLSOM,[17] ESQ. (by letter).—Valuable experimental data regarding characteristics of rough surfaces are recorded in this paper. The method of expressing roughness in terms of the equivalent sand size is an interesting and practical one, particularly as Nikuradse[14] has obtained excellent results in the application of Equation (10). A further understanding of the frictional resistance requires a study of the mechanism of flow past the roughened surface.

In 1931, the writer conducted a series of tests[18] to investigate the characteristics of ultra-rough surfaces formed on the concentric core of an annular section. Flow pictures of aluminum-bronze particles and friction-factor determinations were made. Three views for different degrees of roughness are presented in Fig. 16 which demonstrates clearly the totally different types of flow existing for various groove dimensions.

Fig. 16(*a*) illustrates conditions when the projections causing roughness are placed relatively close together. A whirl is formed in the grooves which prevents the flow of fluid into these spaces; hence, this type of roughness produces only small divergencies from flow over a smooth surface. Fig. 16(*b*) and Fig. 16(*c*) illustrate the flow when the grooves are large enough to allow the inflow of fluid behind the whirl. For Fig. 16(*b*), the spacing of projections is a minimum for this type, and hence gives a maximum frictional resistance. Fig. 16(*c*) shows that the same flow conditions occur at wider separations of projections. Motion pictures of these flows show successive formation and sweeping out of the whirls or vortices. Measurements indicated that maximum friction occurred when the depth of the groove was about one-fourth the spacing, this relation holding throughout the investigated range of relative roughnesses (ratio of projection length to length between walls).

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[17] Instr. in Mech. Eng., Univ. of California, Berkeley, Calif.

[18] "An Experimental Investigation of the Phenomena Produced by the Highly Turbulent Flow of Water Past a Series of Sharp Obstacles," by R. G. Folsom, presented to California Inst. of Technology in 1932, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

These and other photographs indicate that a rough surface produces disturbances which spread throughout a large part of the flowing fluid. Contrary to the author's statement (under heading "Discussion of Results: Roughness III") that, "this indicates that a projection causes greater turbulence than a groove," the disturbance depends on the shape of the groove due to its control of the action of the vortices. The whirl or vortex formed has a diameter of the same order of magnitude as the depth of the groove, but the size and shape of the groove determines when the vortex will pass out into the main flow and dissipate its energy. As the roughness becomes greater, the disturbances increase with a resulting increase in energy dissipation, the latter being measured as a larger frictional drag.

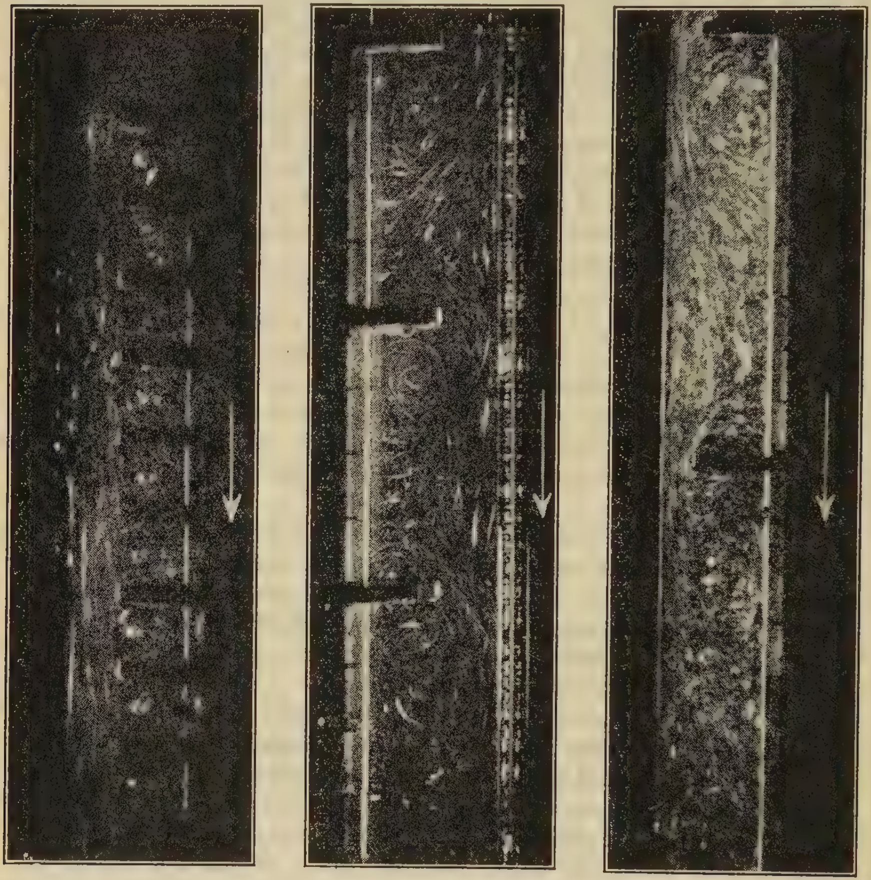

FIG. 16.—FLOW PICTURES OF ALUMINUM BRONZE PARTICLES. (ARROWS INDICATE DIRECTION OF FLOW).

Experiments at the University of California with unsymmetrical projections in an ultra-rough surface indicate that in some cases the frictional drag is very different, depending upon the direction of flow. Thus, the characteristics shown in Runs I and I*a* as being almost independent of the direction of flow cannot be considered as general.

Table 4 contains the author's results in comparison with those given in a British report[19] for wavy or corrugated surfaces. Houghton's experiments were conducted in a 3-ft square wind tunnel with air as the fluid. The value of f in Table 4 corresponds approximately to Reynolds' number of 10^6. The relative roughness is the ratio of groove depth to pipe radius, or equivalent dimension. Even if the corrugation shapes and relative roughnesses differ, the author's friction factors appear to be high. In some measure at least, this result is caused by the spiral motion of the water, which is the result of the method of making the artificial roughness.

TABLE 4.—COMPARISON OF RESULTS

Experimenter	Wave length, in inches	Ratio of groove depth to wave length	Relative roughness	log 100 f
Streeter, IV	0.043	0.276	0.011	0.58
Streeter, VI	0.087	0.184	0.015	0.68
Houghton	3.0	0.4	0.067	0.60
Houghton	1.0	0.4	0.022	0.48

Although the author presents an excellent brief account of principal investigators in the field, the writer believes the work of Fage[20] should have been noted. This material includes pressure and friction measurements for a surface composed of pyramids. The ultra-microscopic method is used to study velocity distributions and turbulence. The magnitude of the fluctuating velocities was about 2.5 times those for smooth pipes. The data show an approximate check with von Kármán's generalized velocity representations although the disturbances due to the wall extend further into the flowing fluid with the rough surface.

RALPH W. POWELL,[21] M. AM. SOC. C. E. (by letter).—Not only is this paper based on an extensive series of experiments, clearly and completely reported, but it is an excellent summary of past work in the same field. Perhaps the most noticeable omissions in the author's references are to the work of Stanton and Pannell[22] and to the extensive collection of data published by Kemler.[23]

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[19] "Note on the Velocity Distribution in the Neighborhood of a Corrugated Sheet," by R. Houghton, Tech. Rept., Aero. Research Comm., 1932–33, R and M No. 1466.

[20] "Fluid Flow in Rough Pipes," by A. Fage, Tech. Rept., Aero. Research Comm., 1934, R and M. No. 1585.

[21] Assoc. Prof. of Mechanics, Ohio State Univ., Columbus, Ohio.

[22] "Similarity of Motion in Relation to the Surface Friction of Fluids," by T. E Stanton and J. R. Pannell, *Transactions*, Royal Soc., Vol. 214 (1914), pp. 199–224.

[23] "A Study of the Data on the Flow of Fluids in Pipes," by Emory Kemler *Transactions*, A. S. M. E., Vol. 55, HYD, pp. 7–32 (August 31, 1933).

In an unpublished study made in 1914[24] the writer offered some suggestions for future experimenters, among which was the following:

> "The most fruitful field for experiment seems then to be on rough pipes from ¼ in. to 6 in. in diameter. It has been suggested[25] that threading pipes on the inside throughout their length would give a very rough pipe whose roughness could be duplicated at different diameters. The suggestion seems good, although it opens the question, not considered in this thesis, as to what size threads on a ¼-in. pipe would correspond to a given size thread on a 6-in. pipe."

As this study applies directly to the subject of this paper, and as the results confirm many of the author's statements, it is, perhaps, not out of place to offer a brief summary.

The thesis consisted of a study, on the basis of Reynolds' number, of the previously published data on pipe flow that recorded the temperature. The principal part of the data was that published by Saph and Schoder,[7] which it was afterward found had already been studied from the same standpoint by Blasius.[8]

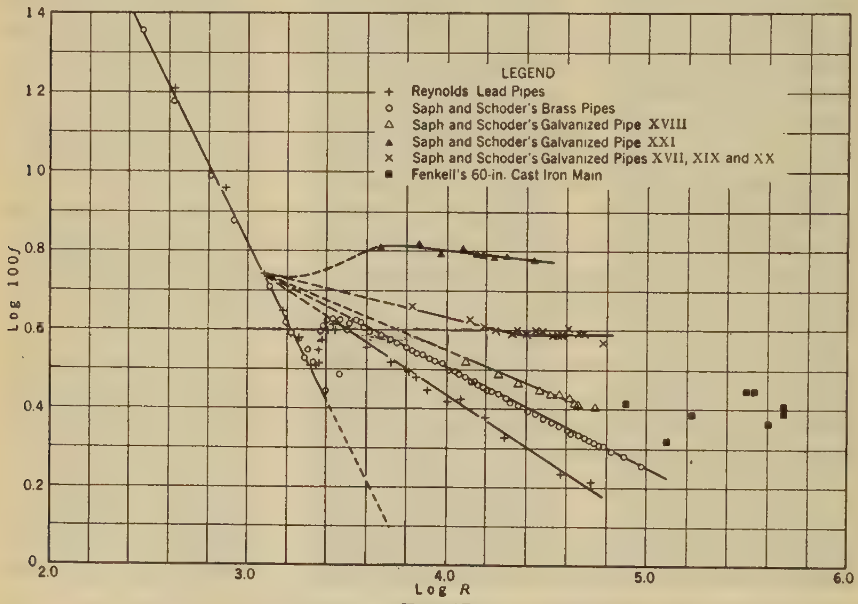

FIG. 17.

The results of the entire study are condensed into Fig. 17, which gives a plotting of f against R and corresponds to the author's Figs. 2 and 11. The points for the brass pipes studied by Saph and Schoder, represent 831 separate observations on 15 pipes ranging from 1.5 in. to 0.1 in. in diameter and at temperatures from 35.5° F to 84.0° F. These observations were re-arranged in the order of the Reynolds' numbers irrespective of what pipe they were taken on, and averaged in consecutive groups. (The only exception was in the critical region where cases in which stream-line flow seemed to be persisting were kept separate from those in which the flow was clearly turbulent.) Each point up to R = 2 500 represents the average of four or five determinations, generally on two or three pipes. For higher values the number in the group increased to six and then to ten; and between R = 4 000 and R = 40 000, it is twenty. One of these groups of twenty taken at random was found to include runs on eight different pipes, ranging in diameter from 1.25 in. to 0.10 in. The largest departure of any of these twenty values from the line that best represented all the runs was 3.6 per cent. The equation of this line was found to be:

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[24] "The Flow of Water in Straight Pipes," by Ralph W. Powell, presented to Cornell Univ. in 1914 in partial fulfillment of the requirements for the degree of Civil Engineer.

[25] *Transactions*, Am. Soc. C. E., Vol. LXXVII (1914), p. 890.

$$f = 0.329 \ R^{-0.253} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

which agrees very well with Equation (6). As noted by the author, Saph and Schoder did not extend their tests to a value of R quite high enough to show the upward curve in the line, which was discovered later by Stanton and Pannell and confirmed by Nikuradse and others.

The plotting of Reynolds' original experiments on two lead pipes ($\frac{1}{4}$ in. and $\frac{1}{2}$ in. in diameter) was made in the same way, except that each point represents the average of only four runs. These experiments are interesting for two reasons: (*a*) They show that, compared with lead pipe even "smooth" brass pipe must be considered as having a definite roughness; and (*b*) they show noticeably larger discrepancies from a smooth line above the critical velocity and from Poiseuille's formula below it, than the experiments of Saph and Schoder. In this connection the writer stated,[24] in 1914, that "it is worthy of note that Saph and Schoder's experiments prove Reynolds' formula better than those he himself performed and used to derive it." This is all the more noteworthy because Saph and Schoder seem to have taken little "stock" in Reynolds' work, and their results had been in print for a number of years before the agreement was ever investigated.

The five galvanized pipes studied by Saph and Schoder seem to fall into three definite groups as far as roughness is concerned. Pipe XVIII (diameter, 0.85 in.) was not so much rougher than the brass pipes and each point in Fig. 17 is the mean of only two runs. On the other hand, Pipe XXI (diameter, 0.35 in.) was extremely rough and each point in Fig. 17 represents the mean of three runs. The other three pipes (diameters, 1.042, 0.626, and 0.486 in., respectively) fall between the other two and differ very little among themselves. They have been grouped together, each point representing the average of four runs.

In order to check the methods for larger pipes the values for a 60-in. cast-iron main given by George H. Fenkell, M. Am. Soc. C. E.,[26] were reduced by the same method and each point in Fig. 17 represents a single run. Only the mean temperature for all the runs was given, but temperature has little effect in this case, as the curve is nearly horizontal. Perhaps it should be noted that the slope of the curve measures the influence of temperature on the friction head. Below the critical velocity where the slope is one, the resistance varies as the first power of the kinematic viscosity. Above the critical velocity where the slope is, say, one-fourth, the exponent of R in Equation (6) is −0.25, and the resistance varies as the fourth root of the kinematic viscosity; but where the curve is horizontal, f is independent of R and change of temperature has no effect on friction head.

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[26] *Transactions*, Am. Soc. C. E., Vol. LI (1903), p. 323.

One other point may be noted: All the lines as drawn in Fig. 17 intersect at the point, $R = 1\,162$ and $f = 0.0551$ ($\log R = 3.0652$ and $\log 100\,f = 0.7408$). This agrees very well with the intersection of the Blasius line with the stream-line flow line on Fig. 11. Therefore, the lower critical velocity will always be greater than $R = 1\,162$; as the author states, it is generally about 2 100.

If it were true that all these lines were straight and radiated from a single point, the formula would be,

$$f = 0.0551 \left(\frac{1\,162}{R}\right)^{2-m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which m is the exponent of the velocity in the ordinary exponential formula for friction head in pipes, and varies from about 1.67 for lead pipe to 1.80 for wrought-iron or steel pipes in good condition. In fact, this formula holds fairly well for all these pipes except Pipe XXI for the range from $R = 4\,000$ to $R = 100\,000$ (and also, of course, below the critical range where $m = 1$); but it does not hold for higher values of R, and, as the author shows, no formula can tell the entire story in the case of very rough pipes. The relationship between f and R is so complicated that only a plotting, such as Fig. 11, will suffice.

VICTOR L. STREETER,[27] JUN. AM. SOC. C. E. (by letter).—In his analytical discussion of manometers, Mr. Wilson presents a method of finding the maximum error for one reading. The scattering of experimental results (Fig. 11(*a*)) for values of R less than 10^5 can be attributed, unquestionably, to inaccuracies of the manometers. Results and conclusions drawn from the paper were based on friction factors for Reynolds numbers of 10^6. For values of R equal to 10^6 and greater, losses of head in all cases, using rough pipes, were greater than 17.0 ft. As stated by Mr. Wilson, the maximum error for one reading of the manometer is less than 1% for pressure differences greater than 1.65 ft of water. The manometer with a solution of specific gravity equal to 1.05 was used for only a few readings. Its action was so sluggish that it would frequently require 30 min to reach equilibrium.

Recently, the writer has been using the micro-manometers described by Nikuradse[9], which are constructed in such a way that air-water or water-mercury may be used as fluids. This arrangement provides for a large range of pressure differences and a high degree of accuracy.

Dr. Folsom's investigation clearly shows the effect of spacing of roughness elements on turbulence formation. A new work by H. Schlichting[28], based on experiments made at the Kaiser Wilhelm Institute for Fluid Research, at

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[27] Freeman Scholar, A. S. M. E., Karlsruhe, Germany.

[28] *Ingenieur Archiv*, 1936.

Göttingen, presents a simplified method of obtaining friction factors for practically any type of reproducible roughness. The results show that the maximum friction factor is obtained with comparatively small density of roughness elements. In his paper, Schlichting also uses Equation (10) as a basis for comparing various roughnesses.

In the roughnesses utilized by the writer a pitch of 11.5 to the inch was used. In a 2-in. pipe the angle between the groove and a plane perpendicular to the axis of the pipe is 0° 48′. This slight divergence from circular grooves perpendicular to the axis of the pipe undoubtedly has some tendency to increase the friction factor. The writer is of the opinion, however, that the differences in friction factors in Table 4 result from differences in the shape of roughness elements, which appears to have as much effect as their size.

In Fig. 17, Professor Powell presents an interesting study of friction factor-Reynolds number relationships. Equation (15) agrees very well with the Blasius Equation (6). The friction factors for Reynolds lead pipes, however, are low, resulting in values as much as 25% less than for the brass pipes used by Saph and Schoder. This variation for "smooth" pipes was not confirmed by Drew, Koo, and McAdams[10], who determined that the friction factor for smooth pipes would not vary more than ± 5% from Equation (8). This conclusion was based on a study of 1 400 observations made by several investigators on various kinds and sizes of smooth pipes. Determinations of diameters 3% to 5% smaller than actually existed in the pipes would account for these differences in friction factors.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1937

STABILIZING CONSTRUCTED MASONRY DAMS BY MEANS OF CEMENT INJECTIONS

By D. W. Cole[1], M. Am. Soc. C. E.

With Discussion by Messrs. Oren Reed, F. F. Fergusson, Joseph Wright, Charles W. Comstock, V. L. Minear, James B. Hays, and D. W. Cole.

Synopsis

The title of this paper carries the implication of many difficulties, which to a gratifying degree were surmounted in the process described.

In scope the work embraced three principal gravity dams of rubble masonry, faced with ashlar, aggregating 14 000 ft in length and ranging from 30 ft to 190 ft in height. These structures, situated in the Western Ghauts of India, about sixty miles inland from Bombay, at 2 200 ft above sea level, have been in use since about 1917 for the storage of water for hydro-electric development under the available head of 1 700 ft. In recent years the increase of seepage through and under the dams gave rise to some apprehension as to their continued stability, and the remedy of cement injections was prescribed by a committee of consulting engineers.

Accordingly, the paper describes the methods and results of the borings and the injection of 64 000 bbl of Portland cement into 380 000 lin ft of drill holes in the three dams, working under full reservoir conditions. Incidentally, observation of the process during 2½ yr seems pertinent to the currently moot questions of gravity dam design, particularly as to the uplift and sliding influences, and as to reliance upon the usual assumptions of a monolithic structure.

General Conditions

These reservoirs, all within a narrow zone of 20 miles along the crest of the Ghauts, are situated near the head of drainage areas which develop streams flowing eastward across India into the Bay of Bengal. The dams are the

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Note.—Published in February, 1935, *Proceedings*.

[1] Civ. Engr., Phœnix Eng. Corporation, New York, N. Y.

means of intercepting this drainage and storing the catchment from the annual rainfall of from 150 to 250 in. precipitated during the monsoon of June to October each year. The reservoirs have a carry-over capacity for equalizing any ordinary "failure of the monsoon." The water is diverted westward through tunnels and penstocks to two power stations at the foot of the escarpment where the operating head of about 1 700 ft is utilized through impulse turbines for electric generation of power supplying Bombay and vicinity.

The climate is tropical and frostless, with four months of wet weather and eight months of dry weather, each in superlative degree. The country rock is basaltic, commonly a hard, heavy, dense, blue-gray lava formation, weathering to black in the cliffs, with relatively few fissures or partings, but frequently having a top crust of porous and unsound formation. The top-soil of the district is of volcanic origin, red in color, and of clayey texture; it burns to inferior brick, occurring in irregular pockets of the hills, as alluvium along the streams, or in thin deposits over the extensively bare outcrop of the country lava.

All the dams are constructed of this native rock, quarried and shaped at the several sites. The foundation rock is of the same character, with any unsound top crust presumably removed, although it is suspected that this lacked thoroughness in some instances. In general, the bed-rock and the rock component of all the masonry may be considered good, ranging from reasonably good in the lowest dam, Walwhan, to very good in the longest, Shirawta, and exceptionally good in the highest dam, Thokerwadi.

The other elements—mortar and workmanship—were less satisfactory. The mortar, which is peculiar to Indian construction (known as "surkhi mortar"), was prepared according to custom, wherein: (*a*) The local top-soil of the best clay characteristics is moulded and burned into a crude brick: (*b*) the brick is pulverized and mixed with a due proportion of a grade of lime which is presumed to have certain hydraulic properties; (*c*) to Mixture (*b*) is added the desired proportion of sand (which in these dams consists of the pulverized rock of the sites); and (*d*) the complete mortar ingredients are then ground together, with water, for at least 20 min. The grinding (Step (*d*)) is done in a kind of pug-mill which may be a stone wheel drawn through a circular trough by bullocks, or, on large-scale operations, may be a motor-driven heavy wheel revolving in a steel trough containing the mortar mixture. In any case the resulting mortar is handled subsequently in the usual manner; it sets slowly, ultimately attaining strength and hardness about equal to good lime mortar in brick masonry. Furthermore, the surkhi mortar has, in some degree, the qualities of hydraulic cement which promote its setting and hardening under water.

The Hindu masons are adept with their crude tools at cutting and shaping the small ashlar facing stones, and the stone setting to the neat lines of the structure is also well performed; but no derricks were used for stone setting on these dams, and both the facing and the rubble in the "hearting" were mostly of "one man size," lifted or rolled into position. Therefore, it may be inferred that the stones were imperfectly bedded in the mortar, the joints incompletely filled, and the bonding was insecure. In brief, it was not a high-grade job of hydraulic masonry, although fairly well founded on excellent bed-rock.

The resulting seepage through the myriads of porous joints tended to increase, with some evidences of leaching or erosion of mortar. Hence, the problem arose of sealing the joints, fissures, and voids in a manner best calculated to solidify the structure and stop the leaks. The objectives set up were: (1) Solidification, with assurance of continued stability of the dams; and (2) the reduction of leakage to the greatest extent practicable. It was required that all work be done under full reservoir conditions, with uninterrupted operation of the two power stations (which have a total capacity of 100 000 kw).

Plan of Operation

From the beginning it was recognized that a controlling factor would be the progress of boring. Therefore, ample equipment of several types of drills was provided and a program of use was outlined for the Shirawta Dam. This structure was most urgently in need of attention and the proposed work was most extensive—probably exceeding the work on the other two dams combined.

Electric power was the first requisite and was provided from Company lines through a few miles of branch line terminating in a small sub-station at one end of the dam. A three-phase transmission on **T**-rail poles at 50 to 100-ft intervals fixed in the top of the dam and clamped to the parapet for the entire length of the dam (7 600 ft), served for the operation of the water pumps, the electrically operated drills, and other small motors.

Air compressors, of the electric-drive, stationary type, in units of about 300 cu ft free-air capacity, were installed conveniently below the dam and served for the operation of from four to eight pneumatic hammer drills and a variable number of the air-driven cementation pumps. Occasionally, air was used for cleaning, testing, and pumping water. A 5-in, water main on the up-stream parapet extended one-half the length of the dam, continuing as a 3-in. line to the far end.

The 5-in. air main was laid along the ground at the toe of the dam for a mile, continuing as a 3-in. pipe the remainder of the way, with 3-in. or 2-in. branches to the top of the dam at intervals of 200 ft or 400 ft. Nine "percussion" (pneumatic hammer) drills were used intermittently. Twelve electrically driven rotary drills, cutting with steel shot, were used for all core drilling.

All the cement pumping equipment was actuated by compressed air, and was built for high pressure, but was used for pressures less than 100 lb per sq in. Four of these horizontal pumps were double-acting with 3-in. plungers; eight others had 2-in. plungers; and all of them had steel ball valves arranged for convenient detachability to facilitate cleaning away encrusted cement at frequent intervals. The valves of all cement pumps, and of the safety valves, were $1\frac{1}{2}$-in. steel balls, steel seated; and they were very satisfactory in operation, with only moderate wear and replacement, or dressing of the removable seat rings.

Electrically driven centrifugal pumps were used for the camp water supply and for maintaining a pressure of 100 lb per sq in. in the water main on the dam to serve all drilling and cementation operations. Each of the cementation pumps required two mechanical mixing tanks (diameter, 30 in.; and capacity, 10 cu ft) for the various grout mixtures, with platforms for cement handling and with storm shelters during the monsoon season. These units of from two to four cement pumps, with accessories, were installed at intervals of 600 ft along the top of dam, with air and water connections. They were supplied with cement over a narrow-gauge track (2-ft) on steel ties running the length of the dam and on the incline, with power hoist, from cement storage and repair shops below the dam.

A machine shop with the usual equipment, lathe, drill press, shaper, and other motorized tools; a blacksmith shop; drill sharpening outfit, with oil furnace and temperature control for the accurate tempering of steel; electric repair shop; carpenter shop; pipe fitters' shop; a small stone crushing plant for making fine sand; a store house for the large supply of imported fittings and special repair parts (for the upkeep of machines coming from Europe or America), all were needed and provided for making apparatus and for maintaining it in good working order.

A camp for 500—officers, mechanics, and coolies—was constructed, Indian style, of corrugated galvanized-steel sheets on teakwood pole frames. The floors were of concrete and the roofs were covered with grass for heat insulation.

The contract for this work was awarded to the Francois Cementation Company, Limited, of Doncaster, England, which firm detailed a force of twenty-one engineers and foremen to direct operations and to train the local labor. The work progressed in three daily shifts, first at Shirawta Dam, October, 1931, to May, 1933; then at Walwhan and Thokerwadi Dams, November, 1932, to April, 1934.

DRILLING OPERATIONS

Spacing of Borings.—A cross-section of Shirawta Dam (Fig. 1) shows the position of the several series of borings. It was of first importance to determine by experiment the spacing of drill holes which would best serve for the diffusion of cement grout throughout the mass of masonry. Preliminary assumptions were that the dam was so porous that its mass could be "impregnated" with the grout from borings 50 ft apart; or, at least, that a water-tight cement curtain or cut-off might be placed through such borings at 50-ft intervals lengthwise of the dam.

The first 500-ft length of dam was drilled under this assumption, the holes being bored vertically from the top of the dam 5 ft from the water face and penetrating 10 ft into bed-rock foundation. Simultaneously with this series of top holes, a series of toe holes (termed the *X* and *Y*-series) was drilled from the ground at the toe of the dam, with the same spacing interval, but staggered between the positions of the top holes. These toe holes were designed to act first as "relief" holes while injecting the opposite top holes under pressure, and, afterward, to be themselves injected if they failed to be filled under pressure from the upper holes.

The plan was attractive, but it was not fully successful, as will be discussed under "The Cementation Process." Consequently, the spacing of primary borings was made $12\frac{1}{2}$ ft, longitudinally, beginning about $5\frac{1}{2}$ ft from the water face at the top of the dam and drilling either vertically or on a 20 on 1 incline down stream to bed-rock. The drilling was then continued from 10 to 40 ft into the bed-rock, according to indications of hardness of rock, fissures, loss of drill water, and suspicions of "ground leaks" that appeared at the toe of the dam. Even with this spacing it was found impossible to inject grout, or water, or to trace the color from fluorescein dye throughout the intervening mass of masonry which would connect with all the leaks or seepage showing in the down-stream face.

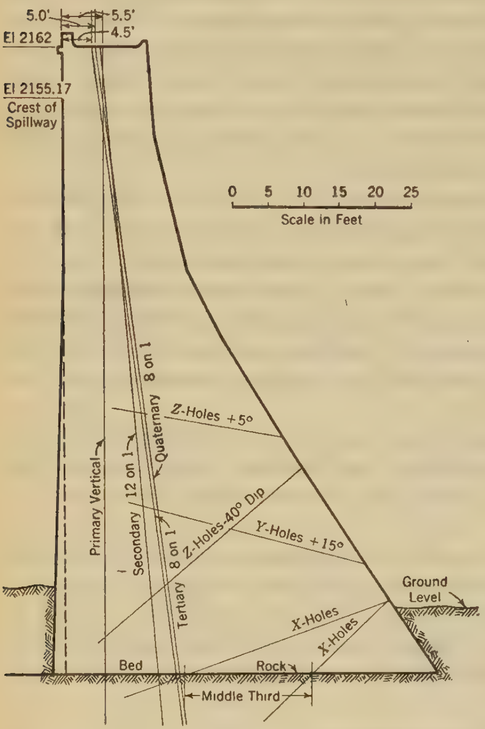


FIG. 1.—TYPICAL SECTION OF SHIRAWTA DAM SHOWING POSITION OF THE SEVERAL SERIES OF BORINGS.

Close Drilling.—As the cementation progressed it was eventually proved that close drilling was indispensable for both the top and the toe series of holes; hence, the spacing for secondary holes was made 6 ft; and, with the following tertiary, quaternary, and special intermediates, the average interval between all holes of all series became 2.1 ft lengthwise of the dam, as shown in Table 1 (Item No. 9) and Table 2. This experience applied only to the systematic location of borings throughout the length of the dam. The location of holes for intercepting and sealing the several large isolated and persistent leaks was a separate problem for each case, requiring special position, direction, depth, and methods of treatment, as hereinafter described.

This drilling experience, with the accessory cementation process, tended to confirm an original opinion that the degree of success in solidifying the dam by cement injections through borings, would be proportioned directly to the number of borings symmetrically placed within the ground plan of the structure.

Definitive Boring.—Innumerable efforts were made to intercept the path of obvious leakage by borings begun directly over the assumed path, or inclined at various angles. These borings were aimed so as to cut the path, or in any manner estimated to provide a conduit for forcing cement grout into the fissure, crack, or void that seemed to be the source of the observed leak.

Most of such efforts failed, due to the obscure relation between the outlet, the inlet, and the course of these sluggish types of seepage or small leaks. Effective boring connection with such leaks seemed to be purely fortuitous.

TABLE 1.—STATISTICS PERTAINING TO THE THREE DAMS

Item No.	Description	Shirawta Dam	Walwhan Dam	Thokerwadi Dam
1	Length of section*, in feet	....	....	800
2	Height of dam, in feet, above river bed	83	71	190
3	Maximum height of dam, in feet, above bed-rock	132	81	195*
4	Maximum depth of boring, in feet	149	107	230
5	Maximum depth, in feet, bored into bed-rock	45	64	44
6	Usual depth, in feet, bored in bed-rock	10	10	10
7	Number of holes that penetrated bed-rock for 20 ft., or more.	29	35	30
8	Area, in thousands of square feet, of a longitudinal section from the crest of the spillway to rock	555.1	275.0	116.0
9	Average spacing, in feet, of all holes, measured along the longitudinal axis of the dam	2.1	1.7	4.0
10	Maximum quantity of cement retained in any boring, in tons †	77.4	16.7	115.45
11	Average consumption of Portland cement, in tons † per boring; all series	1.50	0.77	16.10
12	Average loss of cement vented through open joints, in tons † per boring; all series	0.34	0.05	0.10
13	Average cement retained in the dam, in tons † per boring	1.16	0.72	16.00
14	Average cement retained in the dam, in tons † per linear foot of dam	0.50	0.42	3.94*
15	Average cement retained in the dam, in tons per cubic yard of masonry	0.007	0.008	0.013
16	Average ratio of cement to masonry, by weight	0.004	0.0045	0.0073
	Total Leakage Through the Dam, in Cubic Feet per Second:			
17	Before treatment	22	10	4
18	After treatment	2	1	0.25
19	Percentage improvement by cementation	91	90	94
20	Total length of dam, in feet, including spillway lengths in Item No. 22	7 600	4 472	2 300
21	Total length of dam requiring treatment, in feet	7 600	4 472	1 700
22	Length of spillway, in feet	1 000	1 100	500
23	Estimated solids in solutions of soda silicate and alumina sulfate injected under 80-lb. pressure (in twenty-five borings only), in tons	9.0	....	....
24	Estimated solids per boring	0.36	....	....
25	Estimated solids per linear foot of dam (300 ft.)	0.03	....	....
26	Working time, in thousands of hours of drilling	54.4	29.7	25.2
	Over-All Costs per Linear Foot of Boring:			
27	Plant and equipment, allowing 20% salvage	$0.32	$0.29	$0.91
28	Sundry stores and supplies	0.31	0.21	0.77
29	Labor, including foremen, office expenses, etc	0.83	0.48	1.32
30	Materials, mainly Portland cement	0.33	0.17	0.96
31	Supervision (contractors and engineers)	0.54	0.35	1.20
32	Total cost per linear foot of boring	$2 33	$1 50	$5.16
33	Total cost per linear foot of dams	$63 32	$43 30	$128.50
34	Total cost, each dam	$481 300	$194 800	$219 100

* Middle section, from Station 650 to Station 1 450.
† Long ton = 2 240 lb.

Considering that the first proposed 50-ft spacing of holes involved driving the grout through masonry interstices at least 25 ft lengthwise of the dam, when the cross-wise distance to a free outlet was much less—and that the probable shrinkage cracks would lie cross-wise rather than lengthwise of the structure—there was no logical basis for the hope of constructing a continuous curtain of cement through borings 50 ft apart lengthwise.

It will be understood that this drilling and the accessory cementation were performed for the most part while the reservoir was full, or nearly full, of water. The resulting pressure against the dam maintained the observed seepage and leaks, and tended to drift the drilling water, as well as the cement grout, toward the land face of the dam. The escape to the water face, through the shorter distance, however, interfered most with the process of cement injection.

TABLE 2.—COMPARABLE CHARACTERISTICS OF OPERATIONS ON THREE DAMS

Description	Shirawta Dam	Walwhan Dam	THOKERWADI DAM		
			North section	South section	Middle section
Specific gravity of trap-rock in foundation and in masonry	2.68	2.60	2.90	2.90	2.90
Weight, in pounds per cubic foot, of composite masonry cores tested	146.0	143.0	160.0	160.0	160.0
Over-all rate of drilling, in feet per hour of drilling: Shot	1.67	2.75	....	....	1.47
Over-all rate of drilling, in feet per hour of drilling: Percussion	5.06	9.14	3.41	3.41	....
Average consumption of steel shot, in pounds per foot bored	1.33	0.76	....	....	1.34
Average spacing of all borings, in feet, lengthwise of the dam	2.1	1.7	5 0	6.0	4.0
Portland cement: Average consumed in tons* per boring	1.50	0.77	0.65	0.35	16.10
Portland cement: Average lost, in tons* per boring	0.34	0.05	0.04	0.02	0.10
Portland cement: Average retained, in tons* per linear foot of dam	0.56	0.42	0.102	0.067	3.94
Portland cement: Average retained, in tons* per cubic yard of dam	0.007	0.008	0.003	0.0017	0.013
Ratio of cement to masonry, by weight	0.0040	0.0045	0.0017	0.0010	0.0070

* The long ton = 2 240 lb.

Character of Drilling.—For training the Hindu drill-runners, work was begun with the air-hammer drills on tripod mounting at the toe of the dam for the *X*-series of holes. As shown in Fig. 1 these holes were projected to intersect bed-rock at the upper and lower limits of the middle third of the base of the dam and then to penetrate the rock a minimum of 10 ft. The *Y*-holes were drilled consecutively with the *X*-holes and aimed upward to penetrate within 6 to 10 ft of the water face of the dam. A year later (spring of 1933), as the finishing touch, the *Z*-holes were drilled, from a traveling stage, at 5-ft spacing, penetrating in the directions shown, to within 6 ft of the water face (see Fig. 1).

Soon after beginning the *X*-holes the hammer-drill runners acquired sufficient skill to undertake the deep vertical holes from the top of the dam. The tripod mounting, set on a timber triangle with steel shoes, to prevent puncturing the pavement, was used most of the time, although a stanchion mounting was made in the local shop and used with good effect in speeding up the changes of drill rods. This device might easily be improved and standardized for use in lieu of the tripod in similar situations.

At first, this top drilling was done in one set-up to a finish at 10 ft in rock; then the "stand-pipe," or grouting connection of 2-in. wrought-iron pipe, 4 to 10 ft long, would be caulked and cemented in the collar of the hole, with a standard pipe coupling at the top. Later, it was found that a more secure setting of the pipe could be made by first drilling the hole only to the depth needed for the stand-pipe, then cementing the pipe firmly in place, coming back 24 hr, or more, later, and drilling through the pipe to the bottom of the hole.

With accurate bit sharpening and gauging it was frequently feasible to drill all the way from the top to the bottom of an 80-ft hole with a 1$\frac{7}{8}$-in. bit through the 2-in. pipe collar.

It became general practice to drill through the 2-in. pipe to any depth less than about 130 ft, using the 1$\frac{7}{8}$-in. bit with changes at 10-ft rod lengths until bed-rock was reached. Then the bit changes might be needed every few inches of depth and the gauge of the bits would be stepped down to a minimum of 1$\frac{1}{2}$ in. This practice was feasible first, because of the relatively easy cutting ground of rubble masonry in surkhi mortar, and, secondly, because of the good drilling equipment and skilled management of the labor. Particular mention should be made of the special type of drill rod provided by the contractor for use with the drifter machines. These rods were of round hollow steel, upset to a diameter of 1$\frac{5}{16}$ in. at the ends for connection with square threaded and shouldered nipples of the same outside diameter, which were tightened by rotation opposite the drill rotation.

The rods were supplied in even 10-ft lengths, with a sufficient number of 2-ft and 5-ft lengths, to facilitate changes in depth. Bits of cruciform pattern were found best for the local ground.

With this outfit of rods it was feasible to drill easily to depths of 150 ft with few delays and little breakage or wear of the nipple couplings, which were a controlling feature. In speed, this air-hammer drilling was from two to four times faster than core driling with shot bits in the same ground (see Table 2).

The cutting of both types of drill was faster and easier in this surkhi rubble than would be the case in well made Portland cement concrete of the same stone aggregates. Nevertheless, many of the individual rubble stones, as shown by tested cores, were as heavy, hard, and tough as the bed-rock itself. The average toughness, however, was less and the combination with the rather soft and brittle mortar made it possible for both types of drill to maintain good clearance and rapid cutting, with only occasional raveling or caving of the hole or jamming of the bit in cracks or fissures.

Core Drilling.—The core drills were excellent, heavy, electrically-driven, machines using rotary shot bits. At first, it was assumed that these drills would be used for all top-hole, deep drilling; but the hammer type of drill "walked away" from the rotary in speed of cutting so that there was a tendency to favor the use of the hammer type. This tendency was disapproved when it came to drilling the thin section of Walwhan Dam, and in all drilling of closely spaced finishing holes where the vibration of the hammer drilling was suspected of shattering the weak masonry, or of interfering with the setting, hardening, and preservation of the films of cement previously injected through adjacent borings.

The progress of the core machines was advanced first by improvements in the rod-handling derricks and in the skill acquired by the operators. Greater improvement was made by substituting smaller bits on these core machines. At first, the borings were made with 4$\frac{1}{2}$-in. bits, requiring 5-in. flanged stand-pipes in the collar for grout connection.

This waste of effort and materials was corrected by use of 2¾-in. (outside diameter) bits, recovering a 2-in. core and requiring only a standard 3-in. stand-pipe with coupling. No advantage was realized from the borings of larger diameter for cementation purposes. Hollow rods in 10-ft sections, and with excellent couplings, were used. Sludge tubes were used only with the larger bits and core barrels.

To quite an extent the core machines, without derrick, were utilized for starting holes in advance for both types of drilling. The swiveling head on the core drills facilitated starting, or drilling to completion, the inclined holes which became an important feature of the work at Walwhan Dam.

In general, there was good recovery of core for determining the composition and weight of the masonry and the bed-rock. The proportions of rock to surkhi mortar proved to average about 60% rock to 40% mortar. The mortar was firm enough to permit good cores to be recovered although not in as large a proportion as in rock. Diamond drilling was not considered suitable for this ground.

Test Borings.—In the routine of work practically all borings became test holes of a sort: First, in revealing the nature of the local top crust and by indicating the length of stand-pipe needed for reaching solid ground which would not be lifted by the grouting pressure; next, by showing the depth at which partial or total loss of the drilling water would indicate a crack or fissure which must have special attention in the cementation to follow; and, on entering bed-rock, the occasional loss of water (and always the change in cutting speed, the character and color of the sludge, and the general "feel" of the ground) would confirm the height of the dam at that point and develop a profile of the foundation, of which there was no authentic record.

The X, Y, and Z holes served to test the outside shell of masonry and were useful in showing the degree to which the dam was saturated. With full reservoir, the structure was mostly saturated from the bottom to within 2 or 3 ft of the lake level, but at the toe the depth at which the drill would tap water sufficient to fill the boring or to maintain a spouting flow from the "stand-pipe," varied greatly. Such penetration to water from the down-stream face varied from 2 to 20 ft, but commonly it was about 12 ft; the flow seldom increased from beyond that depth, except in the vicinity of bold leaks.

Penetrating the structure from the top, the depth at which drill water would be lost varied from about 30 to 70 ft. Usually, if it occurred at all, it was about 5 ft or 10 ft above bed-rock rather than immediately at the rock contact. This latter experience, applying to all three of the dams, confirmed the judgment that, in general, the dams were well seated in water-tight contact with their foundation rock. The few exceptions to this general rule were in the vicinity of the several serious "ground leaks" appearing at the toe of the dam, as will be further described. The borings from the top were utilized for testing with compressed air, suggesting by bubbles rising in the lake the approximate position, depth, and size of the connecting joints or cracks. In the more open ground the air would show at the joints in the down-stream face.

Next to air the most searching test was a fluorescein dye solution in the water pumped into the boring under the approved grouting pressure—50 to 80 lb per sq in. This vivid green color, derived from a teaspoonful, or less, of the red powder dissolved in a pail of water, would show in face joints or in the boring pipes in from 3 min. to about an hour, and at a distance of from 10 to 100 ft away, according to the size and direction of the connecting passages.

Either or all of these tests were useful in some degree for determining: (*a*) What mixture, pressure, and quantity of cement grout would be required for a given boring; (*b*) whether wastage under water and into the lake was beyond control; (*c*) whether wastage through the down-stream joints or pipes could be stopped by caulking; or (*d*) if the use of a sand mixture might be anticipated.

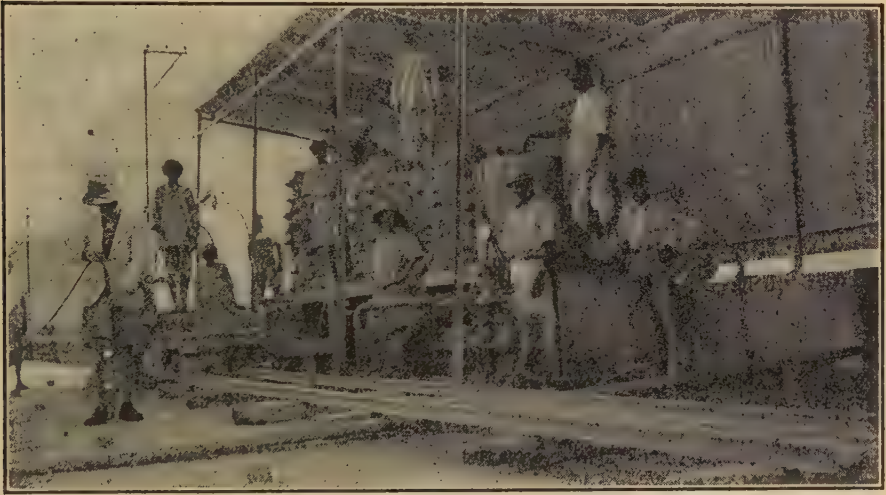


FIG. 2.—VIEW OF CEMENTATION EQUIPMENT, THOKERWADI DAM; OBSERVER IS READY TO LOWER PERISCOPE INTO A BORING.

An ocular inspection was made of a few borings in Thokerwadi Dam by means of a "periscope" devised for the purpose. This consisted of two mirrors set at 45° opposite an opening in the side of a 3½-in. galvanized tube, with a 75-watt lamp suspended between the mirrors (see Fig. 2). With this tube lowered by the light cord into the boring the illuminated side of the boring was reflected to the top by the upper mirror, giving a good picture (the obverse of the core), with a view of cracks and void spaces, down to a depth of about 40 ft for the naked eye and about 70 ft with the aid of binoculars. The device would be more valuable with better equipment and in 6-in., or larger, borings.

In the primary work the drilling got well ahead of the cementation process which, as the work progressed, tended to modify the drilling program and introduced, among other questions, the arguments for and against cementation in "stages" of depth of hole.

Valuable data pertaining to the boring operations at Shirawta Dam, from October, 1931, to May, 1933, may be summarized, as follows:

Total linear feet of percussion boring	172 578
Total linear feet of shot boring	33 951
Total linear feet, all boring	206 529
Shot Used:	
Total, in tons	20.1
Average, in pounds per linear foot of boring	1.33
Average Over-All Penetration Speed, in Feet per Hour:	
Percussion drills	5.06
Shot drills	1.67
Total Consumption of Equipment Materials, in Linear Feet:	
Drill steel	540
Percussion rods	1 700
Shot rods	280
Core barrels	430

The percussion rods used on the work were $1\frac{5}{16}$ by $\frac{7}{8}$ in. in diameter, weighing 3 lb per lin ft, with nipples. The percussion (X) bits were $1\frac{7}{16}$ in. by $\frac{3}{8}$ in. in diameter, weighing 5 lb per lin ft. The shot rods were $1\frac{3}{4}$ in. in outside diameter, weighing 6 lb per lin ft, with couplings. All threads were square, with a pitch of $\frac{1}{3}$ in. Fig. 3, depicting the borings between Sections 35 and 40 (500 ft) of Shirawta Dam, demonstrates graphically the extent of drilling required. Only the top holes are shown, the X, Y, and Z series of toe holes being omitted.

The Cementation Process

The numerous questions arising in the beginning of operations were resolved successively by experiment as the work progressed. Questions as to the character of cement to be used, its strength and periods of setting, its proper mixture with water, the feasible mixtures with sand, the manner of application, the pressures for safe application under the varying conditions, the sequence of boring and grouting, the control of waste due to cement escaping out of sight under the lake level, or out of the down-stream face joints which might be, or perhaps should not be, caulked, the effective handling of the machinery, and the setting up of a system and routine which could be entrusted to the available operatives — all such questions, depending for answer on trial, observation, and discriminating judgment, came up for decision.

Incidentally, critical observation was directed to questions of foundation security, the ascertainable influences of the grouting pressures toward the uplifting effect, with its locality and extent, and to the general question of gravity dam design and some of the assumptions of that science.

Character of Cement.—Since an Indian brand of Portland cement was available, which possessed modern standard qualities, there was little thought of using anything else. There was some debate as to desirable fineness and setting time of the cement for such use, but experience demonstrated that a straight Portland grade of cement of average setting time has the requisite qualities, particularly as to hardening in place in the presence of an excess of water.

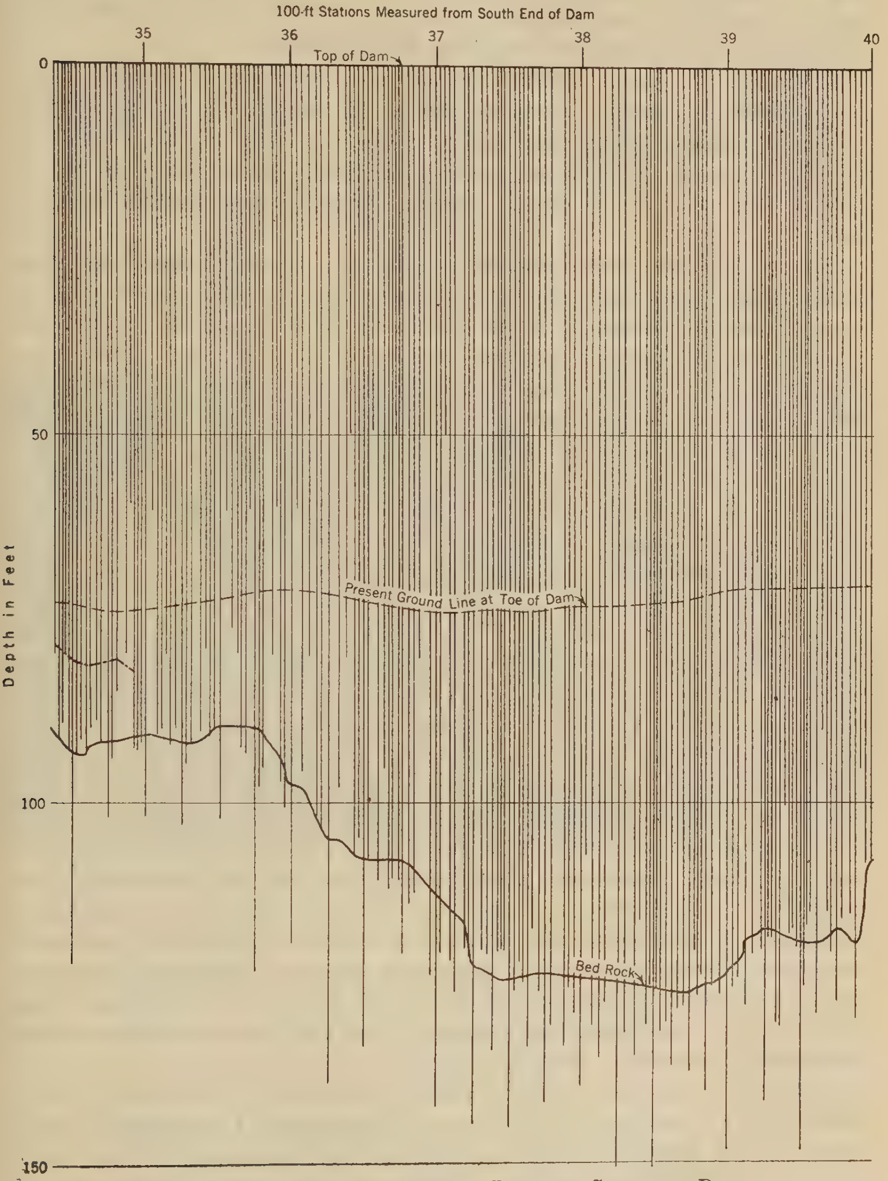


FIG. 3.—TYPICAL PART OF BORING DIAGRAM, SHIRAWTA DAM.

Throughout the job this cement never failed to set with sufficient promptness and to develop a hardness in place in the dam, under water, on the lake bottom, on the ground below the dam, from slop-over or waste through the joints, and in all crevices where it was injected. The degree of hardness and strength attained was proportional in general to the pressure of placement with accompanying extrusion of water. This was shown clearly by the superior density of cored specimens from the drills at the lower levels where the primary injection pressure had been at a maximum.

There was no conflict with the cement-water ratio law of strength, and it was understood that the excessive use of the water vehicle for grouting involved a sacrifice of strength; but strength of cement in this work is not the prime requisite. The object is to stop the cracks and fill the voids with the heaviest durable material that is feasible for delivery and solidification in place.

Toward the last, in attempting to stop the residual seepage where the regular process had not been fully effective, the standard cement was put through a bolting screen of 150-mesh wire cloth in a trial run. There was a barely perceptible benefit from this removal of all trash, lumps, and coarser bits of clinker, and it seems probable that for cement injections into tight ground some better efficiency may be attained by the use of extra finely ground and resifted cement—with due precautions against its flash setting.

Grouting Mixtures.—The question of widest discussion and experiment concerned the suitable mixtures of water and cement. Proportions were specified by percentages of weight; but for convenience of handling both the water and the cement were measured in bulk when combining them in the mixing tanks.

Cement was purchased by the long ton (2 240 lb) and delivered in jute bags of 1-cwt (112-lb) content. Approved mixtures ranged from 1% (1 part cement to 100 parts water, by weight), in tight ground, to as much as 150% (1½ parts cement to 1 part water) in open or leaky ground of masonry. The practice was to start all holes with a thin mix and progress to the heavier mixtures as the widely varying conditions demanded.

Ordinary practice in primary holes in Shirawta Dam required a starting mixture of 4 per cent. At Thokerwadi, in the more open masonry of the middle section, the holes were started with an 8% mix and progressed to one of 100% (1 part cement to 1 part water) after a few hours, according to indications of pressure, or the showing of cement color in face-joints or in weep-holes. In the final series of holes in all dams the thinnest mixture (1%), would frequently be rejected at first, but, in general, these final tight holes would run for several hours taking 1% or 2% mixtures under maximum permissible pressures of 70 to 80 lb per sq in.

In none of the dams was the mass of the masonry sufficiently open to permit mixture even of the finest sand in the grout. The ordinary porosity or intersticial spaces of the masonry were not interconnected or large enough to accept any solids coarser than the cement flour in suspension, and, frequently, even clean water dyed with fluorescein would not penetrate to a showing of color at a distance of more than a few feet from the hole under the pumping pressure.

The large use of cement at Shirawta Dam was in the numerous definite cracks, joints, leaks, and relief holes through and under the dam, and it was in this type of structure that the heavy mixtures of cement were required in combination with the prepared sand, graded through a 20-mesh sieve. The further complication of leakage and wastage of grout into the lake, through the masonry joints connecting with the holes under the injection pressure, at times made the sand mixtures imperative.

Methods of Application.—It was originally proposed to install the cement pumps in two batteries, each of six pumps at each end of Shirawta Dam, 1½ miles apart, for delivery of grout through 1½-in. or 1-in. pipes to all borings in the top and toe of the dam.

The actual beginning was made with a group of four pumps set at the cement shed near the toe of the dam 80 ft below the top, with grout delivery to the top and toe holes, through 1-in. pipes under pressure of 70 lb at the top of the dam (see Fig. 4).

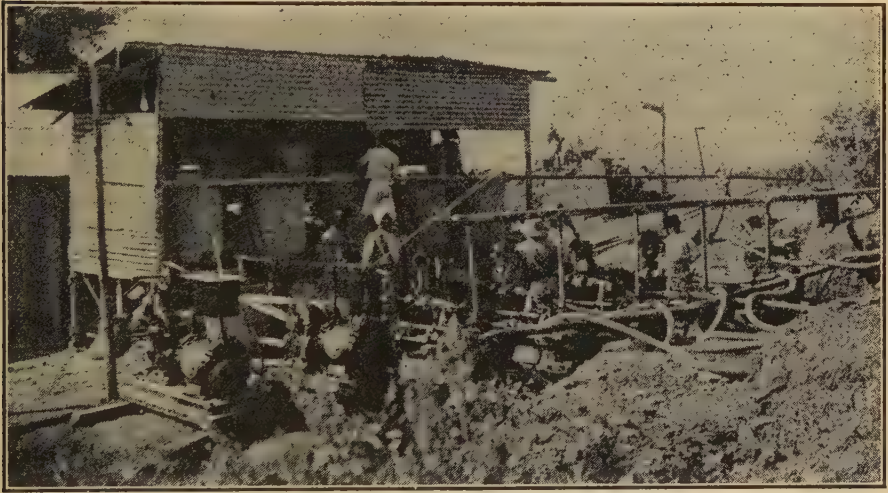


Fig. 4.—Shirawta Dam; the First Pumping Plant, Near Toe of Dam.

It was soon realized, however, that difficulties would be encountered in regulating the pressure in the borings, and that incrustation of cement in pipes and pumps was a factor that militated against distant and indirect delivery. Hence, the plan was adopted for setting the pumping plants at 600-ft intervals in tandem position against the parapet, occupying about one-half the 12-ft top width of the dam, leaving room for drilling and transport operations. This plan was applied similarly to the other dams, giving close regulation of all pressure grouting, whether in the top or in the toe borings. Portability of these cementation plants was required for the successive series of borings in the same ground, and this was readily achieved by shifting the several units of pumps, tanks, stages, and accessories on the small flat-cars used for cement transport on the narrow-gauge track—the water, air, and electric connections being ready for repeated use at each 600-ft station.

Each such plant consisted of two 3-in. plunger pumps and two 2-in. plunger pumps set at opposite ends of a double stage, 4 ft wide, 4 ft high, and 16 ft long, on which the cement in bags was stocked for mixing first through a 90-gal tank on the stage and then through a similar tank on the floor used for the pump suction chamber.

The cement stage and tanks were enclosed during the monsoon. Each tank, 30 in. in diameter, was equipped with a stirring paddle at two heights on a vertical shaft and geared to a countershaft with sprocket connections for operating the upper and lower tanks together from an electric engine under the stage. The prescribed portions of cement were measured in boxes and dumped through a coarse screen into the upper tank. The paddles being continuously rotated, served for good preliminary mixing in this tank, which was discharged into the lower tank through a screen of $\frac{1}{8}$-in. perforations made tight around the edges to prevent any lumpy or fibrous material from entering the pump suction.

The grout from the pumps was delivered to the borings through $1\frac{1}{2}$-in. wrought-iron screw pipe for the 3-in. pumps and 1-in. pipe for the 2-in. pumps, each pump having individual suction and delivery connections to avoid the clogging and stoppage incident to interconnection with other pumps. Flexible connection with suitable hose and couplings was provided both at the pump and the boring ends of the grout ranges. At each boring connection with the stand-pipe in the hole there was placed a lever type of safety valve with adjustable weight for blow-off at 70 lb, or other prescribed, pressure. This safety-valve connection was also applied at the top of the dam to the toe holes, which likewise were treated from the pumps on top of the dam.

Other fittings of the boring intake included a stop-cock on the stand-pipe above which a 4-way fitting provided the grout intake on one side. A blow-off or relief valve was placed on the opposite side, with the safety valve above or on a "tee" of the grout intake, while provision for feeding sand into the grout stream was made in the top of the 4-way cross.

A sanded mixture passing through the pumps was inadvisable for obvious reasons, so other methods were devised for introducing sand to the limited extent that was found to be beneficial. When sand was used it was practically all in connection with stopping the leakage of grout into the lake from the top holes, or wastage into ground leaks from the toe holes. In either case the sand could be run in a small continuous stream, or in handfuls, through the open-top 4-way fitting into the grout stream from the pump—falling freely into the top boring or into the down-pipe delivering to a toe boring. This process might continue for several hours until the gravity head would build up to the overflow level at the fitting, when the sand dosing would be stopped and the pump pressure applied to the neat grout stream only.

In some of the worst leaks to the lake a method was devised for delivering sanded mixtures by gravity head from two of the regular mixing tanks set on a trestle 10 ft above the top of dam. This proved quite satisfactory within short ranges of delivery pipe, but was not effective for general use (because of the clogging of the pipes and the fittings) and was abandoned in favor of the method first described. In some of the larger leaks, furthermore, there was resort to a single-drum pneumatic grouting machine, which happened to be available, but under all the local conditions this method was not as flexible, controllable, and effective as the straight pumping for the neat grout and the combination of the pumping and gravity delivery for the sanded mixtures.

Suitable Injection Pressures.—The degree of pressure safely applicable to this peculiar masonry was a matter for trial and judgment.

It was undesirable forcibly to replace any of the weak surkhi mortar with new and better cement, even had this been practicable in the section of dam as shown, with full lake static pressure against one side, atmospheric pressure on the other, and the injection pump between.

It was not desirable to apply a disrupting pressure in the process of cementation, but it was desirable to take advantage of the most effective pressure for driving the cement-laden water into the smallest passages connected with the borings and for expulsion of the excess water while compacting the cement in all accessible interstices.

With the perfectly controlled pumps, delivering through adjustable safety valves, as mentioned, it was possible (guided by the drilling log of the hole and the previously described tests) to fill the hole with the starting mixture and gradually to build up pressure to about 50 lb, at which there was little danger of any harmful effect.

In a number of early trials it was found that full air-line pressure of 100 lb per sq in. seemed to be harmless, yet, apparently, no more effective for injecting greater weight of cement. Other trials of pressure greater than 75 lb per sq in. produced some signs of distress, such as a slight upheaval of the top pavement, a slight displacement of the facing ashlar masonry, excessive leaks to the lake, or excessive discharge into relief borings. From these trials, and the general sense of caution, the limiting pressure of 70 lb per sq in. was adopted for all three dams with a few special exemptions under personal observation where 80 lb per sq in. could do no harm and might be of some help—but it was never proved that results were any better with pressures greater than 70 lb per sq in.

This regulation of pressure was applied to the normal conditions where borings could be filled with the approved starting mixture under pressure. In the very leaky ground there were many borings that could not be so filled and, therefore, required to be "coaxed" along with varying mixtures, including sand, until it became possible to staunch the waste into the lake or toward the down-stream face or the toe of the dam.

It was in this latter class of borings that the principal difficulties were encountered, particularly in the numerous cases where it was impossible for a time to tell where the grout was going or how to keep it in the wall; and, hence, a tendency was developed to persist for hours, or days on end, in pouring thin grout down a hole, siphoning through the pump and falling freely into the boring.

Sequence of Operations.—From the first it was a question whether to grout the primary holes consecutively or to skip one or more holes so as to isolate the areas under pumping pressure. Furthermore, it was a question whether to grout the top holes first with the toe holes open for "relief" (if relief were needed), or *vice versa*; and whether to grout the top (vertical) holes in separate stages of depth, from the top down, so as to "seal" the restricted horizontal zones of the dam all the way down to bed-rock before sealing the central and toe areas. It was argued by some that there was danger of inducing uplift at the heel such as to threaten the stability of the structure if the grouting was begun at the toe.

The first section of 600 ft of dam became the laboratory for testing these questions. The conclusions reached were that: (1) Isolated grouting effected no benefits; (2) toe holes appeared to be unnecessary and ineffectual for relief; and (3) the only advantage of "stage" injections was realized when the leakage from the boring was so great that the pump could no longer deliver the grout to fill the hole and build up the pressure.

Conclusion (1).—Isolated grouting proved to be of no benefit; rather, it interfered with economy, and showed some tendency to hinder the flow of grout from subsequent holes. The method of maintaining the forward side of the borings always unobstructed by previous grouting, was deemed most satisfactory.

Conclusion (2).—Toe holes seemed to be more useful for the initial grouting of bed-rock and the bed joint. This plane (the bed joint) was intersected at the third points of the base, as shown in the cross-section, Fig. 1. Incidentally, and without appearing to induce uplift at the base, or other objectionable stresses within the structure, the earlier grouting through these toe holes tended to produce a container for the upper grouting by sealing the base and the lower zone of face-joints. Caulking was indispensable in this matter, making for the retention of cement in the wall, permitting the escape of excess water in the grout, and preventing the excessive waste of cement to the ground below the dam.

Conclusion (3).—The only advantage of "stage" injections was in cases where the leakage from the boring was so excessive that the pump lacked the capacity to deliver the grout fast enough to fill the hole and build up pressure. The great disadvantage was in the delay, expense, and interference with orderly progress. Consequently, the stage system was not employed when pumping capacity was available for initial pressure on the hole. It was adopted systematically in two instances, however; First, at Walwhan Dam where the thin top section (20 ft high) had to be grouted under lower pressure than was required in the base section; and, second, at Thokerwadi Dam (190 ft high) where the upper measures were of concrete and the lower measures of loosely built rubble masonry, requiring different and separate treatment.

Conclusions (1) and (3) were confirmed by all subsequent experience on the three dams. Conclusion (2) was modified by further experience at Shirawta in places where the lower face-joints were relatively tight. It was found expedient, in such places, first, to grout the top holes and thereby in some degree to form a curtain or cut-off in the up-stream portion of the dam from top to bottom. This was the preferred method when the cement could be retained in the dam.

In localities where the leaky fissures and joints permitted excessive waste of cement through passages that could not be caulked, however, it was necessary first to seal such vents by grouting the base and lower measures of the dam through the toe holes. Consequently, the "top-toe" precedence became a question of expediency, with change of sequence to meet local conditions, in which the apprehension of uplift appeared not to be justified by any detectable symptoms.

Induced Pressures.—Observation throughout the work, with occasional testing by pressure gauges, stand-pipes, and stop-cocks, revealed a limited transference or transmission of the pump-injection pressure through the body of the masonry between adjacent holes. There was occasional, or perhaps frequent, interconnection of borings, at from 5 ft to as much as 50 ft apart, through more or less definite cracks or jointing planes. Furthermore, there were many instances in which the thin grout traveled some distance and appeared in face-joints 20 to 75 ft from the borings when under injection pressure.

In many cases with toe holes being injected, the adjacent—or, frequently, the alternate—hole would become filled with grout under nearly undiminished pressure, so that with a stop-cock on the companion hole the diffusion of grout would be completed within the area influenced by the several holes.

In the larger leaks and in the case of ground leaks, on the other hand, interconnections that occurred transferred the injection pressure, or even the low gravity pressure, through a considerable length of duct, if not over an extensive area. In general, however, there was no such spread of pressure along jointing planes from borings being injected as would result in a menace to the stability of the masonry.

Exceptions to this general rule were noted in the special situations where "ground leaks" occurred—at Stations 6.5, 17.0, 30.0, and 34.0, of Shirawta Dam. In these special cases the routine system was modified to include many additional borings and devices for throttling the gushing leaks gradually while aiming to avoid dangerous intensity and areas of applied pressure.

Routine of Cementation.—Usually, holes were drilled well in advance of the grouting so that the latter process, in a given series of borings, could be advanced in the consecutive manner, with the forward breast of the dam as free as possible for the infiltration of grout. The first move would be to wash out the one or more holes to be injected, making sure of the removal of all drill sludge or subsequent ravelings, down to the bottom of every hole, with a tolerance of 1 ft, or 2 ft, where complete clearance was difficult.

There was a tendency for these borings to accumulate sediment after standing some time, even when they had been plugged at the top after the drilling was completed. The drilling log would be studied for indications of loss, or inflow, of water, which study would be supplemented by the air, water, or dye tests as required, all having the effect of cleaning the walls of the boring and flushing out any cracks or fissures that would serve as conduits of grout into the masonry. Usually, a leakage test would be applied by measuring the rate of inflow under gravity head from the top of the dam. This rate varied from 3 to 30 gal per min, or more.

The top fittings, stop-cock, safety valve, blow-off, and the intake for sand, would then be placed, and the grout connection made. One tank of clear water would go through the pump, to expel the air, fill the boring, and, in tight ground, initiate the pressure, although in most ground of primary to tertiary borings the total delivery of the pump up to 30 gal per min would fall down the hole without producing pressure.

The addition of from 1% to 8% of cement to the water at first would sometimes initiate, say, 20 lb of pressure at the top of the dam, and this would be considered a satisfactory working pressure for half an hour, or half a day, according to the "feel" of the ground. There was always the danger of choking the hole and sacrificing the penetration of the finer connecting fissures by too much cement, too much pressure, or too much speed of delivery. Therefore, the vigilant regulation of mixture, pressure, and speed of pump was required continuously.

For safety, the rule was: "When in doubt, continue the thin mix and low pressure." Consequently, the operating foreman tended always to produce a tenuous mix and low pressure, or no pressure. The high record for this cautionary state of mind was the "injection" of one of the primary holes by pumping the 2% to 4% mix to the intake fitting and letting it fall down the hole nearly continuously for three weeks before the grout clogged and became subjected to pressure. Afterward, it was discovered that much of the cement had accumulated and set hard in a flat cone at the bottom of the lake against the dam. A somewhat less flagrant case of caution occurred on the opposite side of the dam in a boring that consumed nearly 100 tons of cement during a week of pumping in a mixture of from 2 to 100 per cent. Most of this cement was accounted for later by trenching in the rock spoils below the dam where a fairly good grade of concrete had been secretly produced by the infiltration method. (See Table 3 for estimated losses of cement.)

Much experimentation was devoted to the determination of the optimum mix, with ideal pressure and rate of delivery. The optimum was never found. The behavior of each hole was a law unto itself. Gradually, however, the judgment was developed that the extremely thin mixture, less than 2%, was not desirable or effective and that initial pressure on each hole was necessary to secure the best effect, the rate of delivery to be adjusted accordingly.

After washing and testing a hole with clear water, therefore, the approved routine was as follows: (1) A more careful measurement and adjustment of the mixture; (2) constant inspection of pumps, pipes, fittings, strainers, etc., for assurance that there was no stoppage along the line and that the indicated pressure was on the boring rather than on some plugged connection; (3) beginning with a 4% mixture, delivered fast enough to fill the boring, and to develop 10-lb pressure at the top within half an hour; (4) holding that pressure as long as any good effect appeared in the form of grout spread in the down-stream face-joints, or otherwise as might be judged; (5) thickening the mix to 8%, 16%, and as much as 100% if no waste was detected in lake soundings or in uncaulked down-stream joints or vents; (6) continuing to adjust the mixture and the pressure to the best balance for maintaining the injection during a maximum period and for a maximum weight of cement retention in the dam; and, (7) continuing the injection to "refusal" at a pressure of 70 lb per sq in.

TABLE 3.—SUMMARY OF CEMENTATION OPERATIONS

Series	BORINGS					CEMENT CONSUMED, IN TONS (2 240 POUNDS)		
	Number		Depths in Linear Feet					
				Average				
	Total	Total drilled into bed-rock	Total	Per hole	Depth per hole drilled into rock	Total	Lost	Retained
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(*a*) THOKERWADI DAM; FEBRUARY, 1933, TO MARCH, 1934; MIDDLE SECTION (STATION 650 TO STATION 1 450)								
Primary	70	69	11 136	159	13	2 162	22	2 140
Secondary	65	65	10 596	163	13	794	8	786
Tertiary	30	30	5 194	173	20	122	2	120
Quaternary	18	18	3 297	183	11	33	1	32
Quinary	15	12	2 525	168	6	78	1	77
Total	198	194	32 748	165	14	3 189	34	3 155
(*b*) SHIRAWTA DAM; OCTOBER, 1931, TO MAY, 1933.								
Primary	453	363	38 202	84	11	3 040	776	2 264
Secondary	213	64	15 556	73	9	407	47	360
Tertiary	391	190	32 714	83	2	196	29	167
Quaternary	744	451	67 244	90	2	239	34	205
X	689	536	21 717	31	6	1 113	301	812
Y	480	....	12 098	25	....	362	44	318
Z	616	....	16 818	27	....	111	6	105
Chemical	25	25	2 180	87	3	1	....	1
Total	3 611	1 629	206 529	57.2	6	5 469	1 237	4 232
(*c*) WALWHAN DAM; OCTOBER, 1932, TO FEBRUARY, 1934								
Primary	611	605	46 799	77	10	1 220	83	1 137
Secondary	273	272	21 423	78	9	134	7	117
Tertiary	398	166	24 564	62	6	195	10	185
Quaternary	57	42	4 199	74	3	25	4	21
X	431	426	16 415	38	5	174	9	165
Y	423	....	6 884	16	....	44	2	42
Z	353	1	8 361	24	5	67	4	63
Sluice roof	4	3	177	44	7	9	....	9
Gate shafts	.....	....	....	.....	....	16	3	13
Duct line	2	2	44	22	10	23	2	21
Roadway	15	15	1 291	86	76	68	1	67
Total	2 567	1 532	130 157	51	8	1 965	125	1 840

Thoroughness, rather than speed, was always the rule of action; hence, it was not the practice to force the pressure at once to the maximum, but rather to give every opportunity for the most fluid mixtures of cement to penetrate into the smallest and most remote fissures or voids which were accessible from the boring under treatment. As the work advanced through successive series of borings, the masonry became tighter, absorbed less grout per boring, and the flow became more sluggish.

Accordingly, because the pumps and pipes became encrusted with cement, the practice was developed of connecting two, three, or four adjacent borings to a single pump through its 1-in. delivery pipe, thereby speeding the main flow while retaining the desired pressure and obviating the clogging of apertures and pipes. These multiple connections could be made only with holes of similar absorption; otherwise, a loose hole in the group might take an undue proportion of grout and prevent the other holes from securing their effective pressure and share of the fluid.

Caulking Vents.—An important part of the cementation process was to caulk the face-joints of the masonry in order to retain the cement in the wall. At innumerable joints between the small-sized ashlar facing stones, in both faces of the dam, the pointing mortar had long since disappeared. On the up-stream face, in the narrow zone exposed above lake level, these joints were readily caulked against any escaping cement. Below lake level this was not expedient without the aid of a diver, and it seemed inadvisable to retain a diver for full time at the site to repair the occasional bad leaks under water. Many of the under-water leaks could be stopped by adjusting the cement mixtures and pressures or, in obstinate cases, by dosing the mixture with sand. The introduction of sawdust or rice chaff was suggested for "choking" these troublesome leaks to the lake, but this was discouraged, and only the selected sand was approved as being heavy and imperishable.

The most serious difficulty up stream was to indentify a leak at a great depth under water. The volume and time interval of rising air bubbles would show roughly the extent and position of some of the larger leaks, whereas soundings with a cutting-edge and a 2-in. pipe trap-valve, would detect deposits of cement on the lake bottom below the leak. On the down-stream face the joints were caulked systematically by a gang of about twenty men during the primary grouting, diminishing gradually toward the last, when only occasional joints required such attention.

At first, an effort was made to clear the joints of loose mortar and to insert more or less continuous strands of old rope packing; but it was soon found best to leave all joints open for the free escape of water and to resort to caulking only when the cement that escaped was of the same density, approximately, as the mixture that was being injected.

Sometimes a hole, or a group of holes, would be under pressure for hours before the cement color would show in the extruding water. At other times, the full mix would show almost immediately after starting an injection, with water and cement being expelled over large areas of face-joints opposite and at considerable distances lengthwise of the dam from the positions of the borings.

Equipped with ladders, slings, kits of tools, a supply of hemp rope strands, old bag strips, and wooden wedges, the caulking force was in constant readiness to staunch the waste of cement as soon as the inspector would decide that sufficient water had escaped and it was desirable to hold all cement in the wall and force the water and grout to further penetration. In primary work large quantities of the old rope strand were caulked in tight or held in place against the pressure by wooden wedges driven into the joints. Toward the end (and for the most part, at Walwhan Dam), strips of old cement sacking, caulked lightly in place, were found most convenient and effective in the tighter joints having less residual pressure from the injection.

A notable phase of these caulking operations was the occasional uplifting, or displacement, of the down-stream ashlar, which was poorly bonded into the rubble masonry. In fact, the facing ashlar was a veneer rather than an integral part of the dam. In numerous cases a single ashlar block would be lifted out like a loose tooth, by the grout pressure combined with the surrounding wedging for holding the cement in the wall. In a few cases the veneer peeled until a considerable area was loosened into a "blister" and all operations ceased until the condition could be remedied. The caulking operations were an essential and successful feature of the proceedings, both at Shirawta Dam and at Walwhan Dam; at Thokerwadi Dam only a little caulking was required.

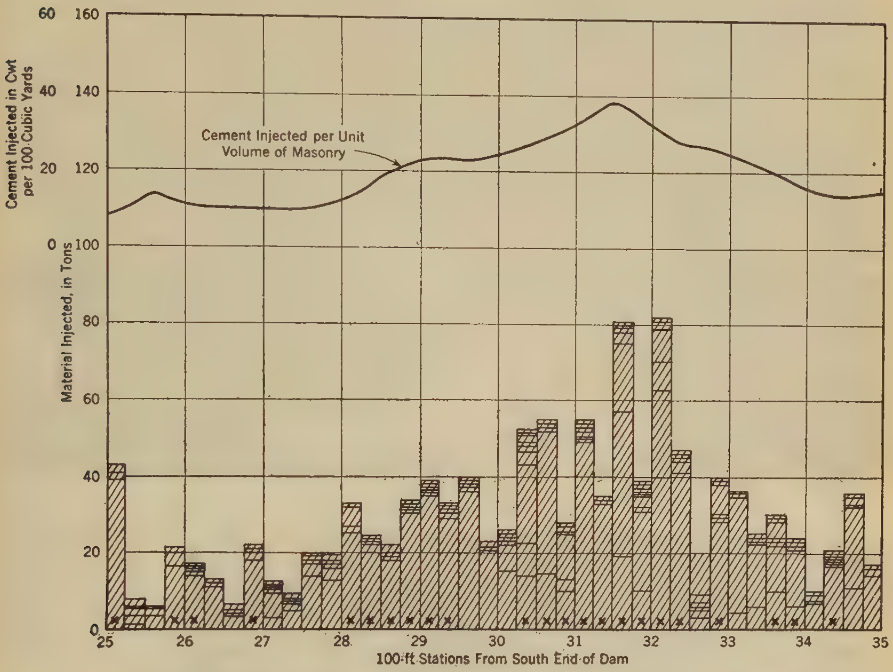


FIG. 5.—TYPICAL CURVE OF CEMENT CONSUMED AT SHIRAWTA DAM, IN EACH 25-FOOT BLOCK, LENGTHWISE OF DAM (1 CWT=112 LB).

The cement consumed at Shirawta Dam is illustrated by the typical curves in Fig. 5 (Stations 25 to 35). Sections marked x are those in which reservoir leaks, or ground leaks, caused excessive loss of cement. Figs. 6 and 7 are views of a leak at the toe of Shirawta Dam before and after cementation. Other views are shown in Figs. 8 to 12. Fig. 8 shows some typical bad leaks (including the "big leak") and small spurts before remedies were attempted. Fig. 9 demonstrates a gradual approach to the stopping of the "big leak" as it appeared in May, 1932. In Fig. 10 the ashlar masonry has been removed after a typical blow-out. No disturbance was noted in the interior masonry (see Fig. 11). In this case the pressure on the 6-in. pipes was released and borings were driven in an effort to intercept the flow along the stratum at the floor of the old inspection tunnel and below it. A view above the site of the big leak, one year after its treatment, is shown in Fig. 12. Note the 6-in. pipes.

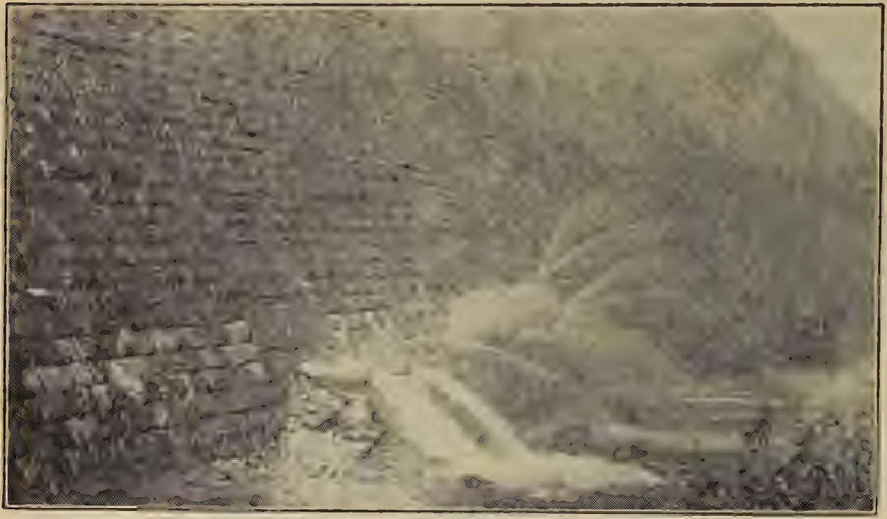


Fig. 6.—Shirawta Dam; Leaks at Toe Before Cementation.

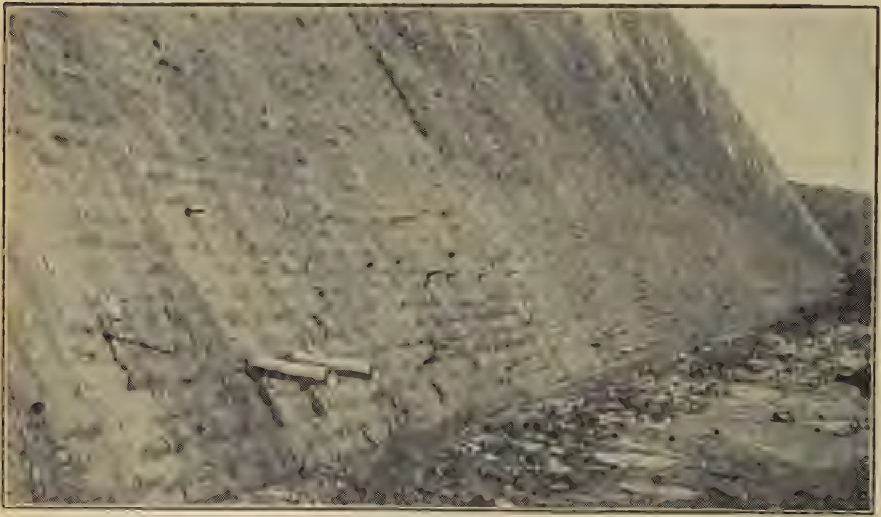


Fig. 7.—Toe of Shirawta Dam After Cementation.

Stopping the Largest Single Leak ("Big Leak")

The total leakage through Shirawta Dam was 22 cu ft per sec. Of this about 20% appeared in joints and pipes near Station 6.5 from the south end of the dam. At this point an inspection tunnel had been drifted into the dam

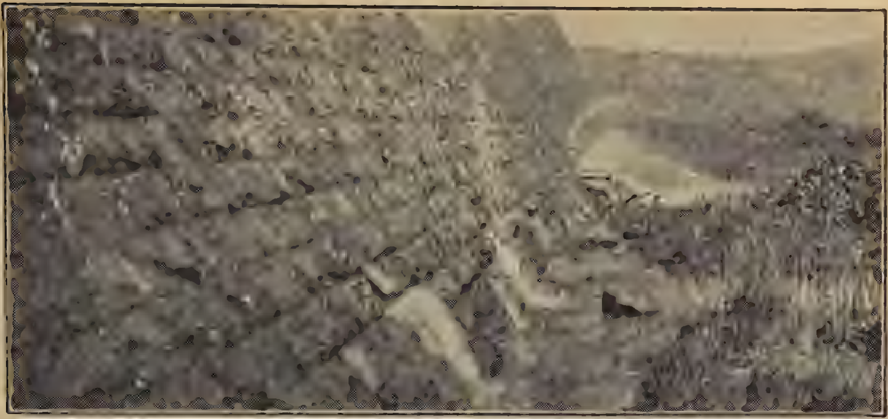


FIG. 8.—SHIRAWTA DAM LOOKING NORTH FROM STATION 6, BEFORE BEGINNING WORK.

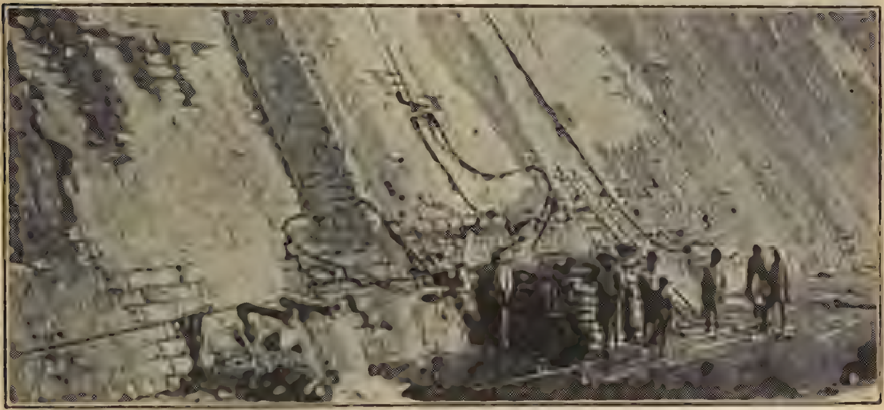


FIG. 9.—GROUND LEAK BLOCKED BY SAND BAGS AS A GRADUAL APPROACH TO STOPPING "BIG LEAK."

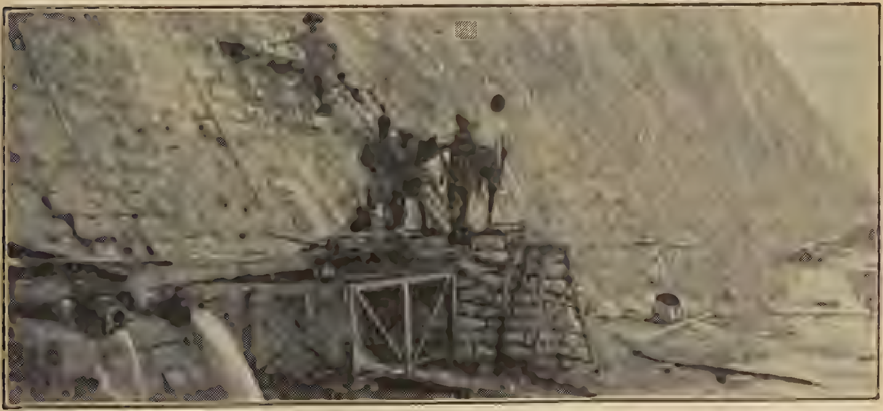


FIG. 10.—ASHLAR MASONRY REMOVED AT BLOW-OUT, SHIRAWTA DAM.

to within about 12 ft from the water-face. This tunnel had been back-filled with cement rubble and drained by two 6-in. pipes and one 4-in. pipe, perforated with 1-in. holes and bedded in the floor.

Fig. 6 shows the leakage (about 5 cu ft per sec) in the pipes and joints of this vicinity before beginning cementation. Fig. 7 shows the condition after completing the work. The final result of work on the entire dam, 1½ miles long, was the reduction of the total leakage from 22 to 2 cu ft per sec (see Tables 1, 2, and 3). The exploratory tunnel had entered a stratum of masonry 1 ft to 3 ft in height near the base of the dam in which it appeared that the mortar had been particularly inferior and seriously eroded, thereby permitting the heavy concentration of leakage at the site of this drift, or tunnel.

FIG. 11.—SHIRAWTA DAM, SHOWING FACING STONES REMOVED FROM BLOW-OUT LEAK (NO HARM DONE).

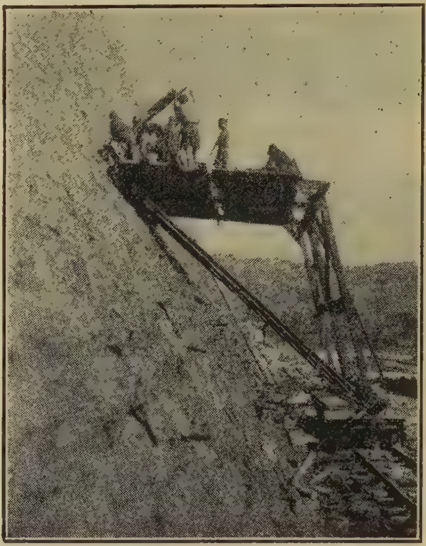


FIG. 12.—SHIRAWTA DAM, SHOWING Z-HOLES BEING DRILLED ABOVE SITE OF "BIG LEAK" ONE YEAR AFTER SUCCESSFUL CEMENTATION.

The cementation program throughout the length of the dam included the control and sealing of this most conspicuous leak, as well as several other serious concentrations later discovered at the base of the dam. No sealing medium other than Portland cement grout (pumped under pressure) was used, and the work was done under full reservoir conditions.

Sequence of Attack.—The first move was to proceed with the routine grouting of primary, secondary, and toe holes in the flanking sections of the dam, aiming to solidify the masonry on both sides and beneath the area of gushing leakage. In doing this there was such interconnection through the masonry, between the several borings and the 6-in. pipe drains, that much grout was wasted.

It was not considered safe to close the drains and grout the entire area under the reservoir head as magnified, or influenced indeterminately, by the pumping pressure. The policy, therefore, was to progress from one hole at a time with such adjustment of mixture and pressure as would gradually seal the face-joints, after due caulking, on both flanks of the big leak through the pipes. Narrowing the outlet in this manner resulted in troublesome loss of cement under water in the lake on that side while at the down-stream toe a bad "ground leak" was developed which rose to the surface from the base of the dam or from inaccessible joints a little above the base.

The leaks to the lake were stopped by the sand mixtures. The ground leak was first weighted with sand bags (see Fig. 9), and then intercepted by a number of drill holes suitably grouted under balanced pressure until the boils ceased and the area was solidified for retaining the subsequent injections within the dam. Further attempts to check the main gushers seemed to set up pressures in the relief borings which might become a menace to the stability of the dam, in the somewhat crippled condition of the section which at best was none too strong (see *X*-section Fig. 1).

At one stage of the proceedings a warning of trouble was given by the outburst of a leak through the down-stream face about 20 ft from the point where injection pressure was assumed to be at a maximum. This leak soon raised a blister under the ashlar veneer (afterward stripped as shown in Fig. 10), whereupon work was stopped pending a new plan of attack. During the pause in operations it was found, by air test and soundings in the lake, that leakage of grout on that side had occurred at a depth of about 70 ft, or a little above the bed of the lake. This suggested the possibility that there might be such a concentration of open jointing in that vicinity as would induce a definite intake and flow toward the drain pipes in the bed of the old drift.

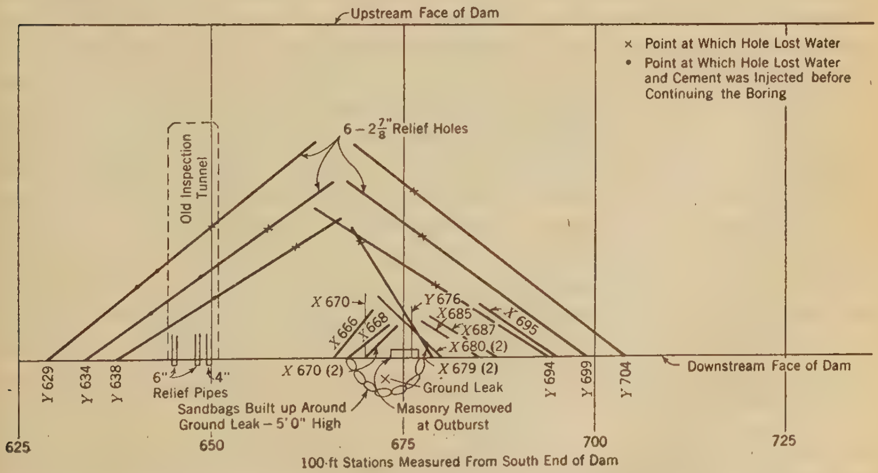

Upstream Face of Dam
× Point at Which Hole Lost Water
• Point at Which Hole Lost Water and Cement was Injected before Continuing the Boring
Old Inspection Tunnel
6 – 2⅞" Relief Holes
X 670
X 666
X 668
Y 676
X 685
X 687
X 695
X 680 (2)
X 670 (2)
X 679 (2)
Ground Leak
Downstream Face of Dam
Y 629
Y 634
Y 638
6"
4"
Relief Pipes
Sandbags Built up Around Ground Leak – 5' 0" High
Masonry Removed at Outburst
Y 694
Y 699
Y 704
625
650
675
700
725
100-ft Stations Measured From South End of Dam

FIG. 13.—SHIRAWTA DAM, SHOWING METHOD OF STOPPING "BIG LEAK" AT STATION 6.5.

A supply of coal ashes mixed with top-soil and a small "sweetening" of cement was assembled on the parapet and sifted down into the water against the dam directly above the suspected area of intake. A dose of the fluorescein dye was also included on the chance of tracing the color in the gushing leaks. On first trial the dye showed color within a few minutes. This was followed after a time by a trace of color in the soil, and continuing the process for some hours produced a marked reduction of flow and pressure in the 6-in. drain pipes.

This important checking of the leakage along the path of the old tunnel-drift greatly facilitated the further progress which consisted of drilling a series of six borings, 3 in. in diameter, from the down-stream toe at about 3 ft below the tunnel floor and 25 ft each side of its axis. Entering the dam at 45°, these holes converged in two sets of three holes, one in each flank, to an intersection under the tunnel containing the 6-in. pipes at a distance of 25 to 30 ft from the down-stream face of dam (see Fig. 13).

The effect of these borings was first to take away most of the remaining flow from the 6-in. pipes. Then they were grouted (with a 1:1 mix) one at a time, with controlled interconnections while using the opposite set of borings for pressure relief as needed. To make the job complete a similar group of borings was made a little higher up for sealing the entire tunnel area. These converging holes, supplementing the numerous previous holes in the regular and special series, had the effect of stopping all sub-surface leakage and solidifying the entire base of the dam up to the floor of the tunnel.

Thereafter, it became a relatively simple, and personally comfortable, matter to seal the ground above the tunnel by pressure grouting through the top holes, with a regulated vent through the 6-in. pipes. Finally, all leakage through the drain pipes and the masonry joints of that vicinity was stopped in this manner (see Fig. 7).

Other Bold Leaks.—In attempting to grout primary top holes between Station 30.0 to 34.0 (see Figs. 14 and 15), it was found impossible to hold the cement in the dam merely by caulking down-stream face-joints above the ground surface, and the usual methods failed to stop the boils rising through the back-filling at the toe of the dam. The toe of the dam, at 15 ft below the back-filled surface, was explored by digging trenches as required. A number of bad leaks were exposed, partly in the lower face-joints; the worst was a leak that rose from a bed-rock fissure 3 ft beyond the toe (see Figs. 14 and 15). Another connected series of leaks was discovered directly on the surface of the bed-rock, under the toe of the dam which, for some distance, was found to rest on the flat smooth surface of rock without any indentation or key seating. These leaks in and on the rock could not be caulked, even against the residual of static reservoir pressure. They were controlled successfully by trapping the flow in a series of 2-in. and 3-in. pipes in the bottom of the trench under concrete placed to serve also as a toe retainer for the dam in the rock-bound trench.

The pipe traps, rising to the surface, became at first the pressure relief vents for the water, and then for grout injected through top holes in the dam. By throttling these relief pipes as the grout discharge thickened and finally closing them, or extending a succession of stand-pipes until the pressure became balanced at 30 ft above ground, this section of the dam was made tight, and all toe leaks disappeared.

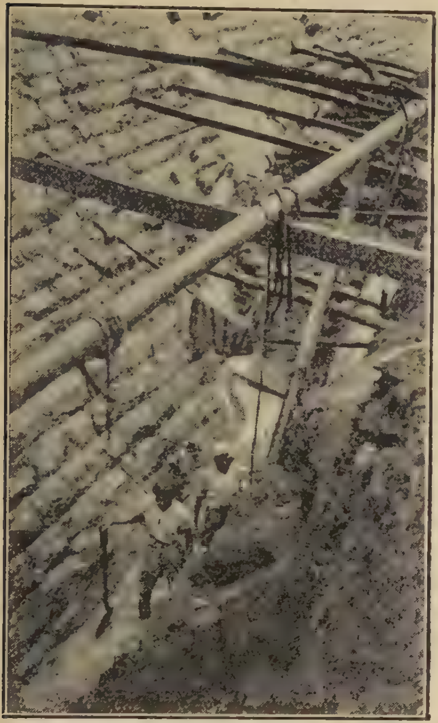


FIG. 14.—SHIRAWTA DAM SHOWING SECTION (STATIONS 30 TO 35) WHERE WORST GROUND AND TOE LEAKS OCCURRED.

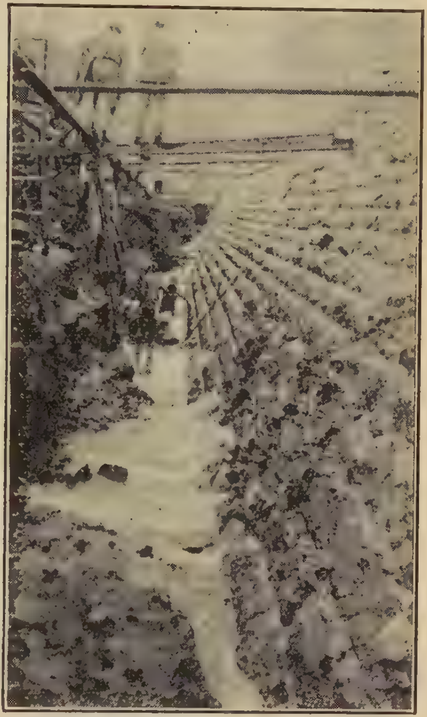


FIG. 15.—SHIRAWTA DAM; ANOTHER VIEW OF STATIONS 30 TO 35.

EXPERIMENTAL INJECTIONS

The Interior Tube Method.—Several trials were made at Shirawta and at Walwhan with a small injection pipe extending nearly to the bottom of the boring and fitted with a gland or loose stuffing-box at the top, the object being to consolidate or seal the ground from the base up. This method has succeeded in foundations where raveling ground, soft rock, or sand has required a degree of solidification in relation to other features. In the dams treated in this paper, however, the trials demonstrated that the method was less effective than the adopted standard procedure with full boring under initial pressure—variable and adjustable but continuous, from the top.

Lubrication Methods.—After the primary and secondary injections of cement grout had left something to be desired in the way of more complete sealing of residual seepage, an effort was made to induce penetration of the finer voids by the addition of some form of lubricant to the cement grout. It was suggested that compressed air might have some such effect, but when tried, produced no perceptible benefit. Calcium chloride solution mixed in the grout was no better—in fact, it made the worst record, presumably on account of its acceleration of the set. A soda-silicate, water-glass, solution appeared to help a little, but the average record of the eighty-seven borings under trial indicated that the straight 4% cement-water grout under the usual direct pumping pressure gave slightly better results than were secured with any attempted lubrication.

Chemical Treatment

Following the trials of "lubrication," as a last resort, and for the information to be derived, an attempt was made to substitute for the cement a combination of commercial salts which were reputed to have been highly beneficial in high-pressure injections for sealing wet ground in mines and shafts. In this case a section of Shirawta Dam was selected, 300 ft long, which had first been fully treated in five series of borings with the regular cement injections, and yet continued to show slight seepage over much of the down-stream face.

New borings were drilled from the top to bed-rock, at 12½-ft intervals, in this section of the dam; there were twenty-five holes in all, as exhibited in Table 1 (Items Nos. 23, 24, and 25). These borings were injected first with a soda-silicate solution immediately followed by an alumina sulfate solution. The silicate was supposed to smooth the path for the sulfate, which crystallizes and solidifies in place. Injections were made first in stages of 15-ft depth from the top down; then in 50-ft stages without any perceptible difference in effect. The maximum pressure was 80 lb per sq in. The result was a visible improvement in sealing the residual seepage, but not practically measurable in the small flows involved, which were complicated by the time interval and reservoir levels.

The conclusions regarding the chemical treatment were that:

(1) It is only applicable after thorough cementation in the usual manner for stopping all large leaks; hence, a complete re-boring of the dam is necessary, with spacing not greater than 12½ ft;

(2) The improvement in density and strength of the masonry is inappreciable;

(3) The reduction of seepage below that already accomplished by cementation is not appreciable, its durability is not proved, and the effect of the injected salts on the surkhi mortar and cement grout is a matter of doubt; and,

(4) The small immediate or visible improvement will not justify the expense.

It is probable that the reputed success of "chemical treatment" has been in rigidly confined ground where pressures of several hundred pounds per square inch were permissible for driving the fluid, with no granular solids in suspension, into the smallest passages and interstices. Manifestly, such high pumping pressure could not be permitted in a rather loosely jointed structure of relatively small dimensions resisting unbalanced static pressures, such as these dams.

The suggested limitation of benefits by the cementation process alone, opens a further field of usefulness for cement chemists and testing laboratories. A safe and reliable means of sealing the voids in otherwise completed masonry structures is needed, in order to make them virtually impermeable by water. This problem involves the production, at moderate cost, of a substance of demonstrable durability, having the property of setting or hardening in place without loss of volume or injurious reaction to any masonry constituent, and which can be carried in solution in water under pressure while maintaining a fluidity equal to, or greater than, water itself.

It may be claimed by some that the current patented or proprietary waterproofing compounds would serve for this purpose; or, it might be imagined that the old Sylvester wash of soap and alum could be injected under pressure where any solids in a water vehicle would not go; but the field is open for modern scientific experimenting.

Cementation of the Other Dams

Walwhan Dam.—The foregoing has been largely in relation to Shirawta Dam where most of the problems were first encountered. At Walwhan Dam (see Figs. 16 and 17) the same equipment and methods were used, with about the same results, as shown in Tables 1 and 2. The thin section, arched design, low specific gravity of rock and masonry, and the more general spread rather than concentration of leakage at Walwhan, introduced special difficulties requiring modification of methods.

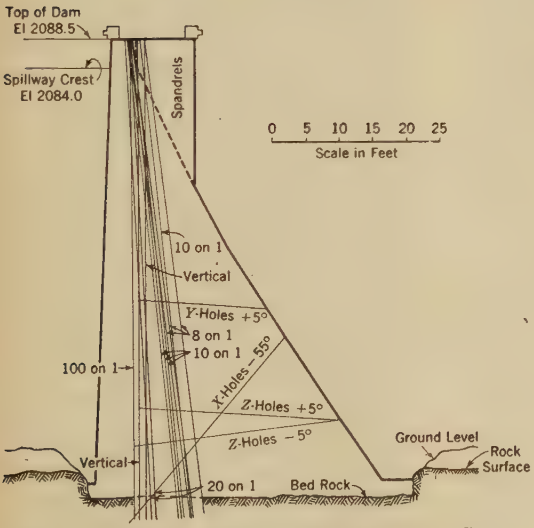

Top of Dam
El 2088.5
Spillway Crest
El 2084.0
Spandrels
0 5 10 15 20 25
Scale in Feet
10 on 1
Vertical
Y-Holes +5°
8 on 1
10 on 1
X-Holes −55°
100 on 1
Z-Holes +5°
Z-Holes −5°
Ground Level
Rock Surface
Vertical
20 on 1
Bed Rock

Fig. 16.—Typical Section, Walwhan Dam, Showing Position of the Several Series of Borings.

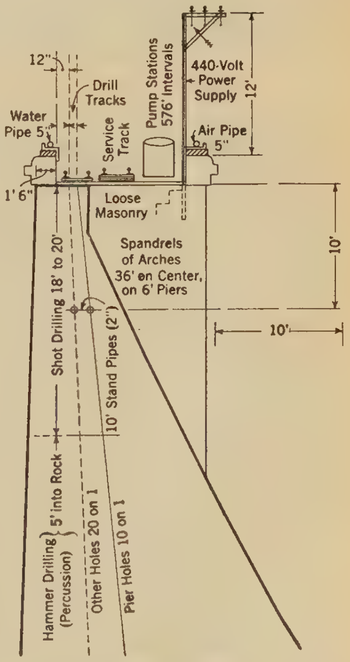

12"
Drill Tracks
Water Pipe 5"
Service Track
Pump Stations 576' Intervals
440-Volt Power Supply
12'
Air Pipe 5"
1'6"
Loose Masonry
Spandrels of Arches 36' en Center, on 6' Piers
10'
10'
Shot Drilling 18' to 20'
10' Stand Pipes (2")
5' into Rock
Hammer Drilling (Percussion)
Other Holes 20 on 1
Pier Holes 10 on 1

Fig. 17.—Typical Section, Walwhan Dam, Showing Arrangement of Plant.

The principal difficulty was to place the borings in positions where they would be sufficiently covered by masonry to hold the grout under any effective pressure (see Fig. 18). The top 20 ft of dam had to be treated as a first stage under low injection pressure, and even then this upper stage gave insufficient support under the crown of the arches; hence, the boring was required to be done only with the shot drills, in order to avoid vibrations, and at such slopes as would pass through the piers and spandrels to reach the lower measures of the masonry where higher injection pressure was needed. Views of this work are presented in Figs. 19, 20, 21, and 22. Cement and grout are plainly visible in Figs. 20 and 21.

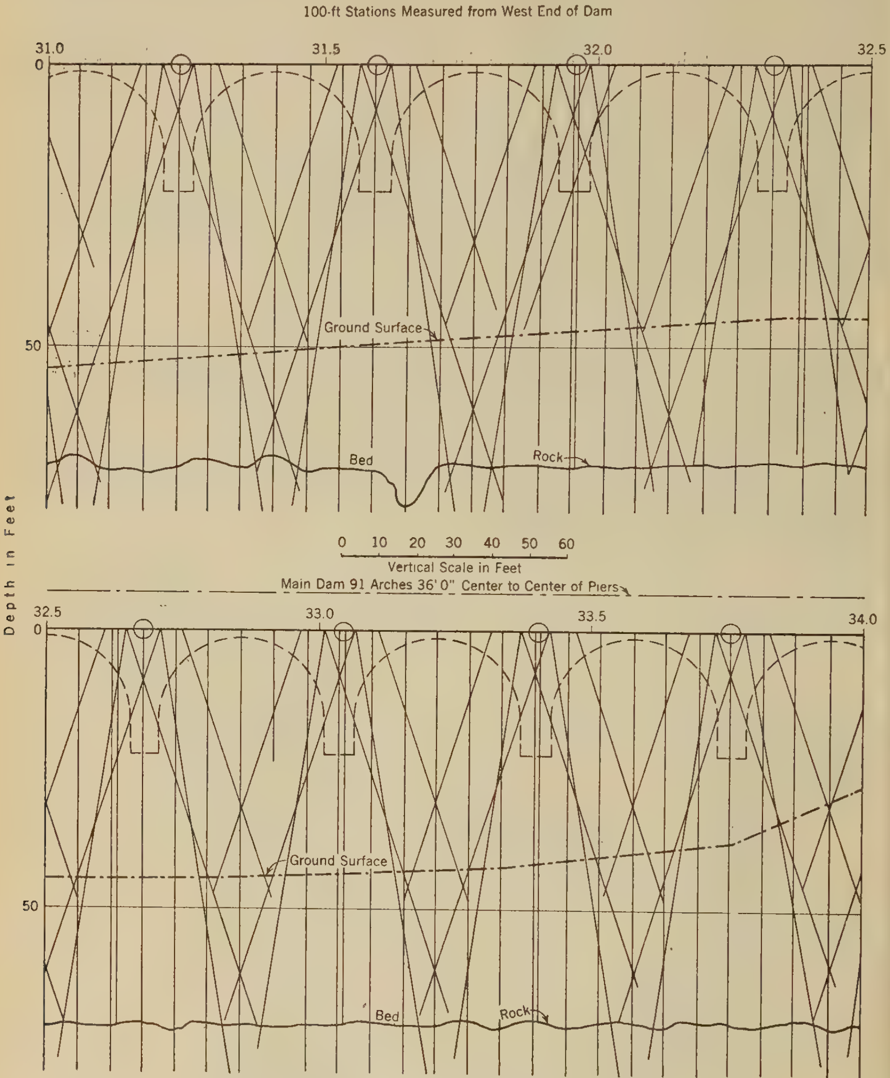

100-ft Stations Measured from West End of Dam
31.0
31.5
32.0
32.5
0
50
Ground Surface
Bed
Rock
0 10 20 30 40 50 60
Vertical Scale in Feet
Main Dam 91 Arches 36' 0" Center to Center of Piers
Depth in Feet
32.5
33.0
33.5
34.0
0
50
Ground Surface
Bed
Rock

FIG. 18.—TYPICAL SPACING OF TOP BORINGS, WALWHAN DAM.

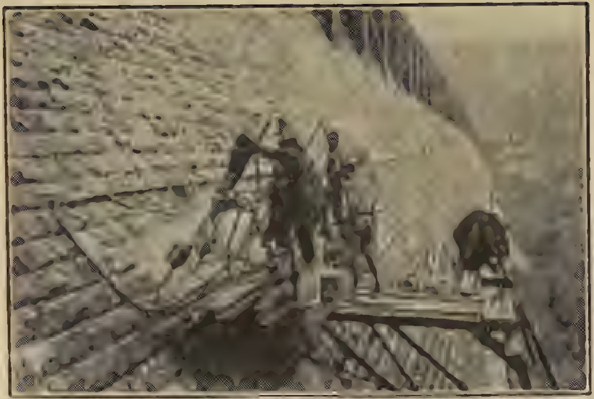


FIG. 19.—DRILLING Z-HOLES, WALWHAN DAM; PERCUSSION DRILLS ON JUMBO STAGE.

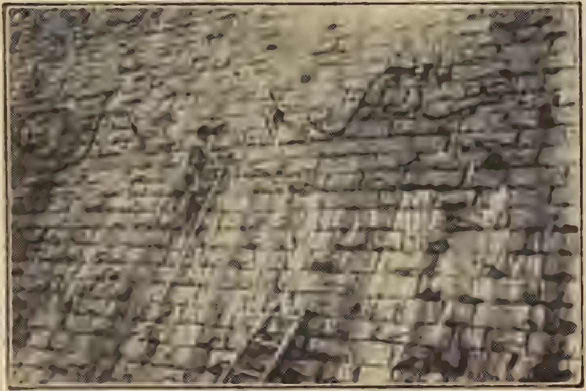


FIG. 20.—GROUTING LOWER Z-HOLES IN PAIRS, WALWHAN DAM.

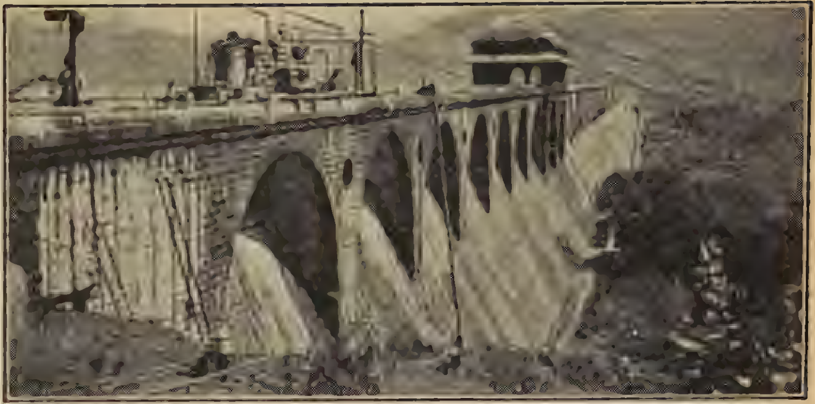


FIG. 21.—VIEW OF CEMENTATION PUMPS AND GATE-HOUSE, WALWHAN DAM.

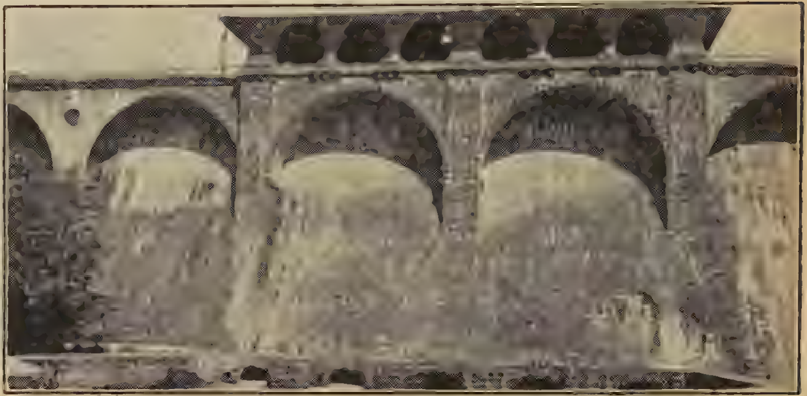


FIG. 22.—VIEW OF GATE-HOUSE AND DUCT HEADWAY.

Considerable joint caulking was required in down-stream face-joints, but this was done more readily with strips of sacking and less forcible wedging than was required at Shirawta. Grout was prevented, fairly well, from escaping to the lake by the reservoir head against a thorough pointing of joints which had been done several years previous—an advantage which was lacking at Shirawta.

The travel of grout through the wall, with interconnection of borings and venting of water and grout in the face-joints, 50 to 75 ft distant from borings under pressure, was more general than in the other dams. The progressive tightening of the masonry, with diminishing consumption of cement, in the successive series of borings and injections was equally significant; and, finally, the leakage index, 90% reduction, was substantially the same as at Shirawta.

Thokerwadi Dam. — The situation at Thokerwadi (see Fig. 23) was modified by the different design of dam, different types of masonry (all surkhi mortar, however), harder and heavier rock in the aggregates and in the foundation, together with the great difference between the high middle section and the low end sections, as separated by the buttresses which had been built over the shrinkage cracks in the dam as originally constructed. The record of cement consumed is shown in Fig. 24.

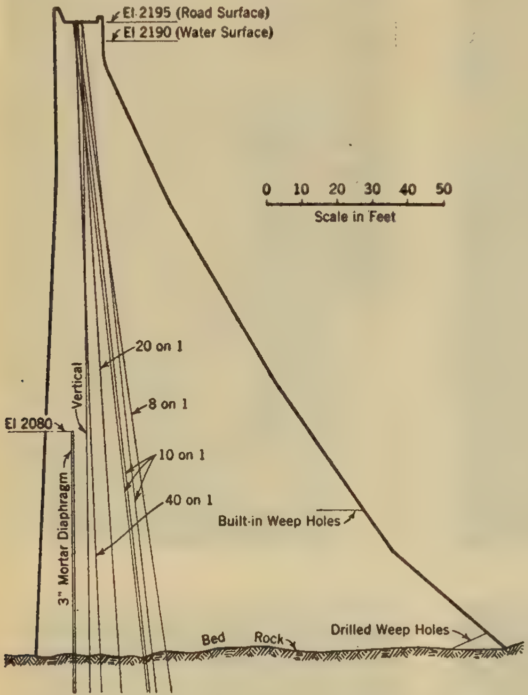


FIG. 23.—TYPICAL SECTION (MIDDLE) OF THOKERWADI DAM, SHOWING POSITION OF THE SEVERAL SERIES OF BORINGS.

Peculiarities of the design (gravity type) included a 3-in. mortar diaphragm plastered against the hearting masonry aproximately 10 ft in from the up-stream face and only from bed-rock up to a level 115 ft below the top of the dam. The design also contemplated draining the masonry through built-in ducts emerging in 4 by 8-in. rectangular weep-holes through the down-stream face. Fig. 25 shows one line of these weep-holes which carried a large part of the total leakage through the dam (4.0 cu ft per sec, which was reduced to 0.25 cu ft per sec by the cementation process). Leakage through these weep-holes was entirely stopped by the interior injections of cement (see Fig. 26).

Another large part of leakage emerged in the re-entrant angles of the buttresses where similar openings had been made for venting any water reaching the buttress through the shrinkage crack. The upper part of the dam was built of surkhi concrete with the heavy basaltic aggregates, including crushed sand. This part was first bored and grouted, with consumption of but little cement.

The plan of work was adapted to these conditions, which indicated drilling and grouting only from the top of the dam. The objective was to form an impervious curtain for extending and supplementing the "diaphragm," which had been attempted but discontinued in the original construction. The further purpose was to penetrate bed-rock and seal the bedding joint throughout the base of the dam, at the same time filling all interstices in the rubble masonry of the lower measures of the dam that could be reached by such borings in order to consolidate the structure to the utmost. (See Figs. 23 and 26.)

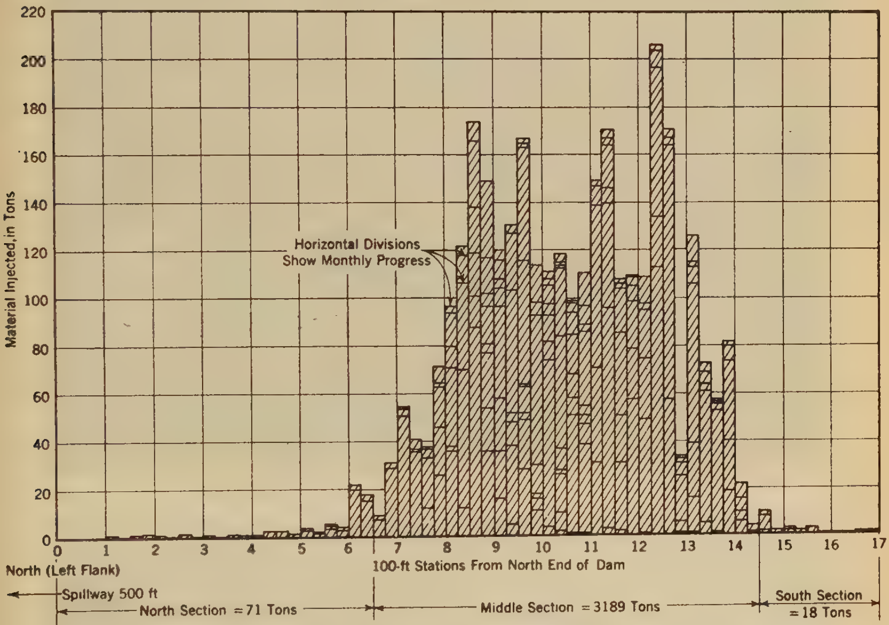


FIG. 24.—CEMENT CONSUMED, THOKERWADI DAM, IN LONG TONS (2 240 LB) PER 25-FOOT LENGTH OF DAM.

More than one-half the length of the dam (both ends) was of concrete, well seated in bed-rock, and the borings at 5-ft spacing consumed relatively little cement. The shrinkage cracks through the dam at the two buttresses, 700 ft apart, were grouted and well sealed by injection through borings intersecting each crack at 50 ft, 100 ft, and full depth. The section between the buttresses, 150 to 190 ft high, was of the same concrete in the top 100 ft; below that level the masonry was of loose, jointed rubble that raveled in the borings, permitted loss of drilling water, and required injection in several stages of depth.

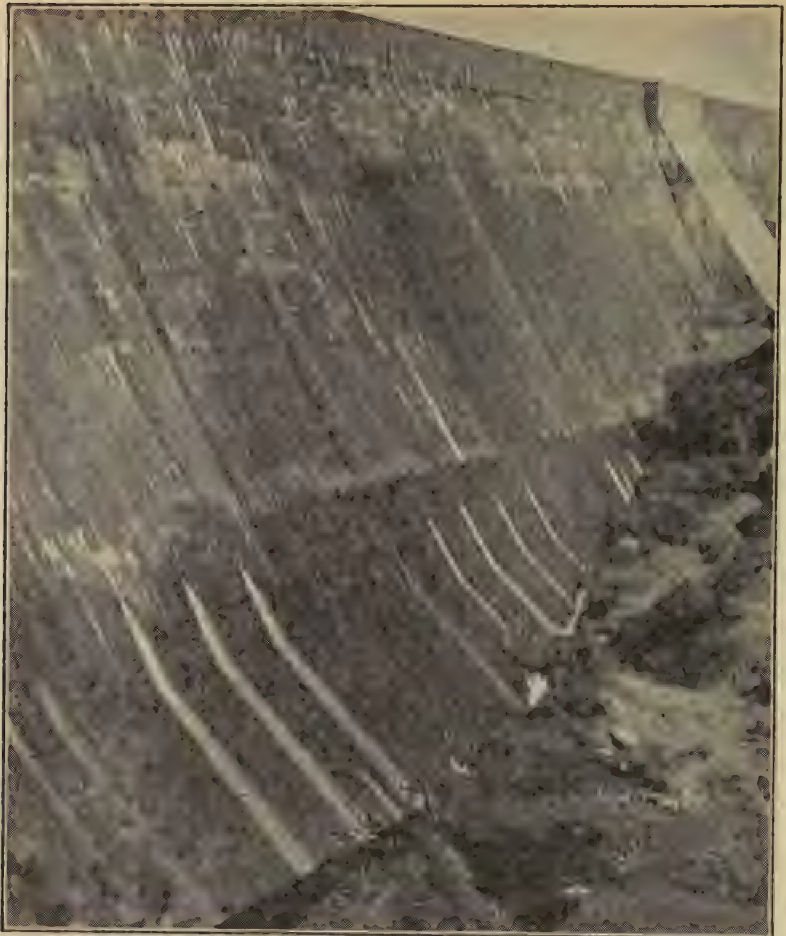


FIG. 25.—THOKERWADI DAM. BEFORE CEMENTATION.

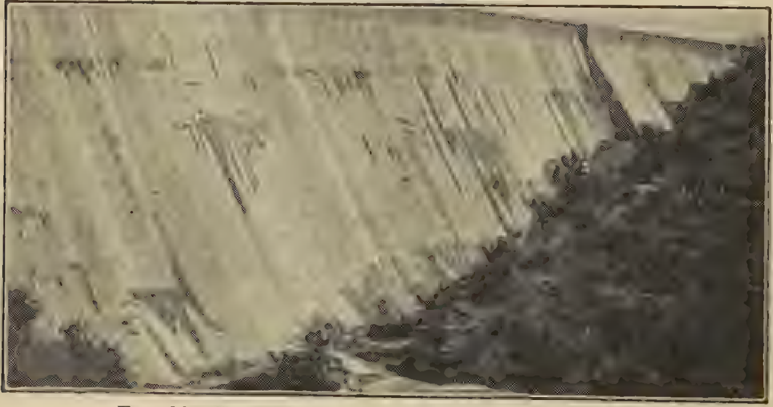


FIG. 26.—THOKERWADI DAM, AFTER CEMENTATION.

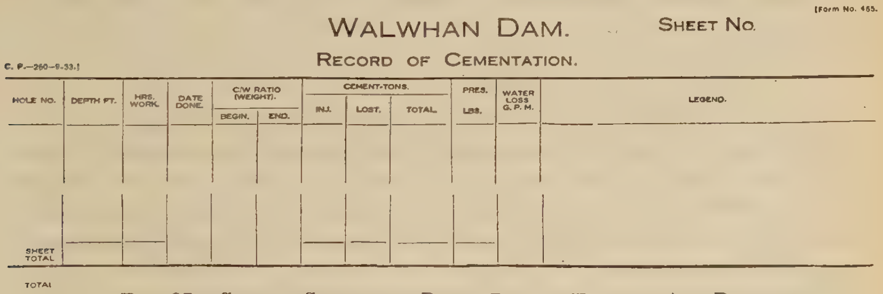
Each of the numerous borings in this section of the dam required 200 to 300 bbl of cement, in 100% mixture (1 part cement to 1 part water by weight), and the record hole took 700 bbl. No cement was lost into the lake at this dam.

There was abundant evidence that the cement became diffused widely within the masonry. For example, there was a progressive decrease in leakage of water through all the weep-holes and through the joints in the down-stream face, accompanied by a showing of cement in these vents, many of which required caulking.

No toe holes were drilled for injections from the down-stream face, as the diffusion of grout from the several series of top holes appeared to seal the bed joint and to consolidate the masonry satisfactorily. A remarkable evidence of the latter result was the progressive tightening of the drilled masonry, so that the holes ceased to ravel and cave, and the loss of drill water became infrequent. The final series of borings produced good cores of cement and refused thin grout under maximum pressure. On the whole, the results at Thokerwadi were considered rational and satisfactory, with the best showing of the three dams in $\frac{\text{cement}}{\text{masonry}}$ ratio and reduction of leakage (Tables 1, 2, and 3).

Record and Costs

A detailed record and report of all the work included a drilling log, and a report of the behavior of all borings during the injection process, in the form of a "cementation record," as shown in Fig. 27. The cost of the work is listed briefly in Table 1 (Items Nos. 27 to 34).



[Form No. 465.

WALWHAN DAM. SHEET No.

RECORD OF CEMENTATION.

C. P.—260—9-33.]

HOLE NO.	DEPTH FT.	HRS. WORK.	DATE DONE.	C/W RATIO (WEIGHT).		CEMENT-TONS.			PRES.	WATER LOSS G. P. M.	LEGEND.
				BEGIN.	END.	INJ.	LOST.	TOTAL.	LBS.		
SHEET TOTAL											

TOTAL

FIG. 27.—SAMPLE SHEET FOR DAILY RECORD USED AT ALL DAMS.

These data of over-all costs cover the total expense of the work under the five items of Plant, Supplies, Materials, Labor, and Supervision. These cost figures would be applicable for estimation purposes in other parts of the world, by the use of a suitable factor of performance in drilling, under any local conditions. Knowing the cutting speed of the types of drill to be used in the local rock and masonry, either by previous experience or by a trial run, consideration of Indian labor cost would be eliminated.

The cost of plant, equipment, stores, and repairs, as shown in Table 2, should represent a maximum expense for these items elsewhere. The cost of cement in India is about the same as the average in the United States. The cost of supervision, as shown in Table 2, is a reasonable value for use anywhere to cover adequate direction and inspection of this special type of operations.

CONCLUSIONS

The main objective of this work, on all three dams, was to make the structure as solid and stable as practicable. Incidentally, the reduction of leakage, or seepage, was an object, index, and measure of success. Evidences of success in the main objective may be reviewed, as follows:

(*a*) The marked change in the character of drilling after the primary grouting, as demonstrated by increased firmness of masonry with much less raveling and choking of the borings;

(*b*) The much less frequent loss of drilling water in the secondary and subsequent series of borings as cementation progressed, until in the final series most holes were water-tight, or consumed a minimum of cement under maximum pressure;

(*c*) The greatly decreased number of cases of interconnection of borings and of leaks to the lake, or otherwise, beyond easy control;

(*d*) The recovery of numerous cores showing good penetration of cement into fissures and interstices. (The cement in these cores was found to be well hardened in place, promising durability and permanence);

(*e*) The progressive decrease in cement consumed by the successive series of borings, particularly from primary to secondary holes, shown in Table 3, until at last many holes refused grout immediately; and, finally,

(*f*) The reduction of leakage by more than 90% is good evidence of success in solidification.

The weight of evidence seems to warrant the conclusion that these dams have been stabilized and secured (largely by isolating and restricting the continuity of uplift areas), a gratifying conservation of water has been effected, and the total cost of the work was reasonable and well justified.

REFLECTION ON THE STABILITY OF GRAVITY DAMS

It will be observed that these dams of the "gravity" type, straight in plan, are of more slender profile than could be approved under orthodox analysis. (Figs. 1, 16, and 23). Probably it could be shown that there is tension in the up-stream masonry, which is certainly not of a character to withstand appreciable tensile stress.

Under the best conditions of workmanship with the given materials it could scarcely be assumed that any section of these dams would be of the monolithic character contemplated in the ordinary computations for overturning moment; and, yet, all these dams have been in successful service about fifteen years, with such deterioration from their original conditions as is evidenced by the increasing percolation during that period. Against this background, the observations incidental to the cementation process have tended to overturn, undermine, or invalidate the observer's conceptions of why a dam stands up. His net reaction is in accord with the declaration of "the necessity for abandoning the middle-third theory and the sliding factor as useful elements in masonry dam design."[3]

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[3] "Stability of Straight Concrete Gravity Dams," by D. C. Henny, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1041.

In view of all the phenomena presented by the cementation of these dams, however, the writer hesitates to follow Mr. Henny all the way regarding the critical question of "uplift." Rather, it might be concluded from the evidence of these thousands of borings, under their manifold pressure testing, that the occurrence, persistence, and practical effect of uplift have each been unduly magnified in the fundamental assumptions of dam design. On the other hand there would appear, from these experiences, to be good reasons for the adoption of a rational shear factor in substitution or modification of the usual sliding factor. Moreover, the common assumption of a monolithic cantilever section comes under suspicion.

While the writer was stationed for some weeks at the toe of Shirawta Dam, below a full reservoir, with the monsoon pouring water on his head and the "big leaks" sloshing water on his feet, watching the effect of pumping water and cement into the middle of the structure, he could perceive no good reason why that dam should remain stable; but it did remain stable, although sections of it 25 to 50 ft long appeared to be merely floating in water and grout. It seemed as if it had uplift enough to move it.

For the most part the bedding on rock was good and the vertical bonding of the rubble masonry was fair; but there must have been planes where uplift operated to reduce the sliding resistance. The saving factors must have been shear resistance and horizontal beam, or voussoir resistance arching between abutments of contiguous sections of the dam which had a little surplus of stability—possibly by taking the resultant thrust longitudinally.

At long last, the one conspicuous result of the cementation program was to localize, delimit, or destroy the continuity of all considerable areas of hydrostatic uplift; but, of course, this result should have been accomplished in the original construction. The lesson to be derived from this experience is, that builders of gravity dams may take some chances on their design of profiles and the governing theories of stability, but they should not take chances on the soundness of their materials or the competence of their workmanship and field direction.

Designers should "keep their feet on the ground" and beware of using values of uplift from bed-rock, based on over-inflated formulas. They should also keep their eyes critically on field practices, to the end that the essentials of durable construction shall become incorporated with approved mechanical principles.

DISCUSSION

Oren Reed,[3] Assoc. M. Am. Soc. C. E. (by letter).—A helpful service to those members of the profession who are interested in the design and maintenance of masonry dams, has been rendered by Mr. Cole. Leaky masonry dams are difficult to repair successfully.

An attempt was made to stabilize the Ringedalsvand Dam, in Western Norway, by cement injections but this method was soon abandoned for another. The writer had occasion to inspect this structure in 1930. The first section, built in 1912–1913, was constructed of granite blocks pointed up with 1:2 mortar. This section has a maximum height of 54 ft. From 1914 to 1918, the dam was raised by constructing a concrete structure back of the initial section and incorporated with it to give a maximum height of 112 ft, The body of the dam is constructed of 1:5:6 concrete with about 30% "plums." The up-stream and down-stream faces are of granite. Back of the stone facing on the water side there is a tightening layer of $1:2\frac{1}{2}:3$ concrete. The thickness of this rich concrete, measured from the water face, varies from 3 ft at the crown to 10 ft at the top of the original masonry dam. Immediately behind the rich tightening concrete is a drainage system consisting of a double row of vertical drain pipes on 24-in. centers. These vertical pipes are connected to a horizontal header, which empties into the inspection gallery, whence the water is conducted to the down-stream face of the dam.

An effort was made to inject cement into the tightening layer during the summer of 1927, in two adjacent blocks of the structure, which constituted a length along the crown of 180 ft. In all, thirty-seven grout holes and six observation holes were bored for testing and grouting. They were located about 3 ft from the up-stream face and were bored vertically to depths varying from 50.4 ft to 72.8 ft. Near the bottom, the holes were 5.9 ft from the up-stream face and 4.6 ft from the vertical drain pipes. A total of 2 300 ft of ordinary injection holes were bored, together with 390 ft of observation holes.

The first holes were spaced 9.8 ft apart, but the distance was found to be too great and intermediate holes were bored. Pressure tests and observations indicated that the 4.9-ft spacing was still too great, but it was not decreased. The holes were not bored to their full depth at one time, but in stages, depending on the soundness of the concrete. The tightness of each length of hole was tested by water pressure. Cement was injected at a pressure varying at the nozzle from 0 to 294 lb per sq in. A total of 276 600 lb of cement was used, divided into the four groups of holes as shown in Table 4. The part of the dam under treatment had an area of about 11 300 sq ft. Therefore, more than 24 lb of cement was used per sq ft of dam surface.

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[3] Assoc. Engr., Tennessee Val. Authority, Pickwick Dam, Miss.

For Groups Nos. 1, 2, and 3, Table 4, a mixture of approximately equal parts of cement and water was used, or about 9.9 lb of cement per gal of water. A thin mixture (3.9 lb per gal) was used in Group No. 4. As shown in Table 4, an average of 113.7 lb of cement per ft of hole went into Group No. 1, where the distance between holes was 9.9 ft, whereas only 54.5 lb of cement per ft went into Group No. 2, which was placed midway between the holes of Group No. 1. There was considerable variation, however, between the cement use of the different holes. Injection in five of the six observation holes, which were placed between the holes of Groups Nos. 1 and 2, showed the largest average use of cement—127 lb per ft of hole. A thin mix of 3.6 lb of cement per gal of water, was used for these holes. The injected material had probably spread over a large area, especially up stream to the stone facing and down stream to the drainage system. For the most part, the cement was injected when the lake level was low, but some holes were treated when the water level was at its full height. The leakage water did not seem to indicate a severe loss of cement.

TABLE 4.—CEMENTATION RECORD, RINGEDALSVAND DAM, NORWAY

Group No.	Cement used, in pounds	Total length of bore holes, in feet	Average use of cement, in pounds per foot
1	70 600	621	113.7
2	29 900	548	54.5
3	126 900	1 126	112.7
4	49 200	390	126.0

The objective which was set up at the beginning of the work, respecting the clogging of the drainage system, was not attained. The cement paste did not reach the inspection tunnel except in a few places, but the drainage system was put out of operation at several places, as evidenced by the leakage which appeared on the down-stream face of the dam directly back of the injected section. Measurement of the leakage before and after the cement injection showed that about 76% of the leakage was stopped.

The following conclusions were drawn from the cement injection trial at the Ringedalsvand Dam:

(1) The result of the injection was not fully satisfactory. Even if a large part of the leakage was stopped, a relatively large part still remained. Removal of soluble material in the concrete would continue, therefore and it would probably only be a question of time until the injection must be repeated.

(2) Pores, holes, and passages in the concrete were partly filled with a dark organic slime, or had a coating of slime on the wall surfaces. During the injection the cement would mix with the slime, and the slime would possibly hinder or weaken the setting and hardening of the cement. In all cases there would be a film of slime between the concrete surface in the pores and holes, and the injected cement.

(3) For a dam with a drainage system immediately behind the tightening layer, it will be difficult with cement injected in the foregoing manner to

prevent partly filling the drainage system. Therefore, leakage will find a way through the lean concrete in the down-stream part of the section, and wash out the soluble portion.

Due to the foregoing disadvantages of the cement-injection method, it was not continued and other solutions were studied for the repair of the dam. The adopted method of reconstruction, begun in 1929, consisted of building a concrete flat-slab dam $6\frac{1}{2}$ ft in front of the up-stream face of the old dam and resting against it on struts. The new vertical deck was proportioned to take the full water load so that the only purpose served by the original dam is as a support. This work was completed in 1931.

F. F. FERGUSSON,[4] ASSOC. M. AM. SOC. C. E. (by letter).—Because it is a record of a difficult work, completed satisfactorily, and because its author has given such precise detail as to the method followed, this paper should prove of great practical value. It is particularly interesting and valuable to engineers in India because it draws attention to weaknesses in a type of construction which is probably peculiar to that country, namely, the use of lime instead of cement mortar or Portland cement concrete, in hydraulic structures.

The widespread use of lime is due to its general occurrence geologically in the form of "kankar", the ease with which it is quarried and prepared, and, consequently, its cheapness as compared with cement. The cost of masonry work in lime mortar in the region around Jodhpur is less than one-half that of masonry in cement mortar and less than one-quarter the cost of massive concrete work; these considerations force the engineer to use the lime prepared locally for almost all work, important or otherwise.

The weakness of lime when used for hydraulic works, such as dams, canal walls, and overflow weirs, is abundantly in evidence in many parts of India, but probably no where more so than in Rajputana, where the surface of the country is littered with masonry walls of all sizes built across stream beds and depressions for the purpose of arresting the flow of water and storing it for use during the few months immediately succeeding the monsoons, when wheat cultivation is undertaken. At some time or other during their life, a high percentage of these walls or dams have been breached, and the reason puzzled the writer because many of the structures observed were statically stable even with greater overtopping than could have occurred. Eventually, it was traced to the fact that water gains access through joints in the mortar and eventually passes through the wall, leaching out the lime in its passage. This action has been found to be especially drastic in a wall through which the writer had to cut recently in making some modifications. The wall, which is subjected to a pressure due to 25 ft of water, is 5 ft thick at the ground level and is backed by a bank of sand on both sides, the bank on the water side being pitched to protect it from wave action. The rock used in the masonry is a hard rhyolite that splits with smooth faces. It has been found that the lime has poor adherence to this stone and probably this may be said of the trap-rock, of which the dams described in Mr. Cole's paper, are constructed. It is also a well-known fact in Rajputana that the local masons prefer to work with sandstone and not with granite or rhyolite, because they find difficulty in making the lime "stick" to the stones.

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[4] Senior Executive Engr., Public Works Dept., Marwar State, Jodhpur, Rajputana, India.

The writer has constructed several small dams in sandstone or rhyolite, according to the geological formation of the locality, but in all cases the greatest care has been exercised to grind the lime fine and to use a mixture of 1 part lime to 1 part clean sand, to avoid regular courses in the masonry, and, finally, to rake out the mortar in both the up-stream and down-stream faces, and to point deeply with a mortar consisting of 1 part cement to 2 parts fine clean sand. This procedure has prevented leaks effectively and should form a rigid part of all specifications for this class of work.

The author's reflections on the stability of gravity dams is also of interest, as the writer has in his jurisdiction a 25-ft masonry dam, built about forty years ago, in which the resultant, due to water level with the crest, falls far outside the middle-third. The masonry is so deplorably poor that the dam is emptied through leaks within a few weeks of filling; it has had a 3-ft surcharge over its entire length and yet has shown no sign of failure. Notwithstanding these facts the writer agrees entirely with the author's final paragraph.

Joseph Wright,[5] M. Am. Soc. C. E. (by letter).—This is a most concise and thorough account of a novel and interesting job well done, wherein experience, meticulous care, and a full understanding of the difficulties encountered, and the hazards, involved, were conspicuously essential to the success attained. The profession is much indebted to the author for a full and clear description of the work.

The work on these important Indian dams indicates not only what may be done to save defective dams, but emphasizes more forcibly what must be done to avoid such defects. A dam that permits water to percolate through it cannot be classed as a safe, permanent structure, whether built of stone or concrete, and the remedy is to use a dense concrete or mortar well placed, with special care given to construction joints.

The close spacing and total length of drill holes required and the quantity of cement used is staggering, but is not surprising when it is considered that the dams are of uncoursed rubble masonry and that much of the grout would inevitably be lost in the reservoir or through the down-stream face. For a few of the most serious of these leaky holes the writer is inclined to think the use of asphalt by the "Christians'" process would have saved many tons of cement which found its way out of the dam section, and would have facilitated the cementation process.

Since no dimensions are given upon the dam sections shown, the author's comments concerning their refusal to fail as would be expected in accordance with accepted principles of instability, cannot be verified. The sections do seem light; and although the writer can agree fully that dam builders should take no "chances on the soundness of their materials, or the competence of their workmanship and field direction", he cannot agree that any chances should be taken in the provision for uplift. If all dams were founded upon basalt, or granite, one might reduce the allowance for uplift with safety. Dams fail more often by sliding upon strata beneath their bases; and where the foundation rock is stratified and horizontally bedded, it is believed that designers err more often in the other direction. In his closing discussion, it is hoped that the author will supply dimensions of dam sections, and will show the resultants of forces acting upon their bases.

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[5] Care, U. S. Engr. Office, Nashville, Tenn.

CHARLES W. COMSTOCK,[6] M. AM. SOC. C. E. (by letter).—The unusual and almost unprecedented character of the work described, together with the author's clear, complete, and interesting account of methods, difficulties, and results, make this paper unique and, therefore, worthy of something more than casual and perfunctory discussion.

In order fully to appreciate the situation which the author's work was intended to, and undoubtedly did, improve, and how such a condition of affairs could come about, it is necessary to cast a glance backward over the development of hydraulic construction in India.

Masonry dams exceeding 100 ft in height, and about a mile long, were undertaken by the Public Works Department prior to 1870 and pushed to completion within a few years. These were the early years of the theory of dam design developed by Sazilly, Delocre, and Rankine, and not much had materialized from it except the rule of the middle third. During the ensuing third of a century many dams between 100 and 200 ft in height, each comprising more than 500 000 cu yd of masonry, were designed on this principle and constructed in various parts of the country. Because the dams were long the sections were made as small as possible, and it was considered good practice to bring the lines of pressure as near as possible to the limits of the middle third. This resulted in sections rather more slender than are to-day thought conservative.

As to materials, they were those which the country afforded. The earlier works would not have been possible otherwise. Portland cement, of course, was known, but it was manufactured in but few localities, was costly, and was little understood. Concrete was practically unheard of. The origin of surkhi mortar is shrouded in the mysteries of the past, but its preparation and use are common knowledge among the Indian masons. It has unquestionable hydraulic properties, and tests have shown compressive strengths of 68 tons per sq ft at 12 months, and 79 tons at 18 months; respectively, 1 060 and 1 230 lb per sq in. Such material cannot be summarily dismissed from consideration.

The Indian people are essentially agriculturists. They are not mechanically minded. Elaborate and complicated machinery does not enter into their scheme of things, although their skill and patience in many handicrafts are revelations to the Westerner. Time is not important to them. They do not think in terms of the individual's span of life.

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[6] Jackson Heights, N. Y.

In these circumstances it is not surprising that the administrators of the Indian public works designed and built as they did—with a rudimentary theory, native materials, a minimum of mechanical equipment, and a maximum use of manual labor of the kinds to which the people were accustomed. After half a century without a major failure, notwithstanding the construction of a great number of large and important works, they were justified in considering their practice sound, and it is not strange that they were slow to adopt the later developments in materials and methods.

With this background the engineer who designed the Walwhan and Shirawta Dams, and who had spent most of his long professional life in India, naturally preferred to follow a tried and proven road rather than to pursue what might be only a will-o-the-wisp.

The Shirawta Dam contains about 650 000 cu yd of masonry. To construct such a volume, almost entirely of one-man stone, by hand labor within a few working seasons required an army of men and women. These literally swarmed all over the work. In fact, seen from an elevation and from a little distance the activity resembles nothing so much as a gigantic swarm of bees. The labor turnover is enormous, for these people will work on outside employment only during such times as their personal agricultural operations do not require attention.

Herein lies the weakness of the system. Really good masonry can be and has been executed by this somewhat primitive procedure, but with hundreds of masons distributed over a large area and served by thousands of coolies who bring the stone and mortar, adequate supervision and inspection are impossible. Here and there a careless or indifferent mason, an occasional batch of improperly ground mortar, a few stones which are dirty or cracked—the result is a weak spot, a discontinuity, to and through which water under pressure may find its way. Time and the never-ending erosion by running water do the rest.

Thus the slowly but steadily increasing leakage through these dams was regarded as symptomatic of a condition which should not be allowed to persist. The loss of water was unimportant, at least in any quantity which might be reached for some years. It was not believed that the dams were unsafe, but it seemed certain that they would become so if the deterioration should be permitted to continue. If that time should come, every one would be helpless. An edict of condemnation by Government would be a futile gesture, for no power on earth could prevent those reservoirs from filling. The waste-gates, designed only for use in the early stages of construction, are not large enough to empty the reservoirs in the period between monsoons.

To order the dams breached so that the reservoirs could not fill would deprive the City of Bombay and surrounding territory of tramway service, light, fans, and refrigeration. It would shut down about 80 cotton mills and deprive the railroads entering Bombay of power. In short, it would mean ruin and pestilence to a million and a half people. The Indian Government does not take a fiendish delight in destroying private capital, but even if it were indifferent to such destruction it could not face this possibility. On the other hand, failures of the dams would mean frightful havoc to agricultural regions below, and almost certainly much loss of life.

Clearly, the damage already suffered must be repaired and the structures must be rendered permanently immune to future deterioration, without interfering with the use of the reservoirs. The plan adopted after thorough investigation and careful study was endorsed by some of the most experienced engineers in India and received Government approval. The engineers for the contractor, François Cementation Company, Ltd., believed, in the light of their experience, that it promised success. They had not previously undertaken a precisely similar job, but no major engineering problem is an exact duplicate of any other. It was believed that cavities existing in the masonry, whether resulting from carelessness during construction or from subsequent washing out of mortar, could be filled with good cement grout, and that the old surkhi mortar in the neighborhood of these cavities might to some extent be impregnated with cement. It is the execution of this plan which has been so well described by the author.

It has been objected by some persons that the degree of betterment is uncertain since there is no visual evidence that all cavities are filled wholly or in part. No visual evidence, it is true, but the volume of cement retained in the structures (in the case of Shirawta 0.6% of the total volume of the masonry) is very convincing, especially as the pressures employed were sufficient to force the grout through minute openings. It is not claimed, of course, that the dams are of as high quality as though they had been originally laid up in cement mortar under close inspection, but they are certainly as good as (probably even better than) any of those built with surkhi mortar in the manner prevailing in India, and many of which have served for more than fifty years without manifesting any sign of distress.

It has been said that this work was almost unprecedented. The grouting of rock formations underlying dams prior to construction is a well-established practice, as is also subsequent grouting of parts of the structure through holes left for that purpose.

Grouting at the Camarasa Dam in Spain, completed in 1931, was not a parallel case to the Indian dams. At Camarasa the dam was absolutely tight and sound; leakage through the surrounding and underlying rocks was the fault to be remedied. This object was accomplished with marked success.

The Delta Barrage, in Egypt, consists of a series of piers, each 16 m long by 2 m thick and about 14 m high. These are spaced 7 m on centers, and rest on a continuous concrete mat 4 m thick. Construction was begun about the middle of the Nineteenth Century. As a result of political interference the foundation work was not properly done, cracks and settlements appeared almost immediately, it was impossible to maintain the pool level which had been intended—the dam was a failure. When Sir Colin Scott-Moncrieff became head of the Public Works Department in Egypt he took immediate steps to remedy the difficulties. The final stage of the repair job was to drill five vertical holes along the axis of each pier and to inject cement grout. No pressure was used except that due to static head; however, cement injected at one pier frequently appeared in the holes in adjoining piers. One pier took as much as 73 tons of cement; a total of 1 000 tons was used in 132 piers. This work was completed in 1898 or 1899; the dam has given no trouble since.

One work which in some respects parallels the Indian dams is the "barrage de l'Oued Fergoug" in Algeria. This dam, 34 m high, also was designed in the early days of the middle-third theory, construction completed about the beginning of 1872. Like the Indian dams, it was built of one-man rubble by native workmen using lime mortar, perhaps feebly hydraulic. From the beginning it leaked profusely at many places, and lime deposits on the down-stream face indicated serious destruction of the mortar. In December, 1881, a length of 125 m of the highest part of the structure was carried away, most of it clear down to the foundation. Between 1881 and 1885 the gap was filled by a new structure of slightly greater section than the old, but built in the same manner and with the same kinds of materials except that some hydraulic lime and some Portland cement were used in places regarded as especially critical. The dam continued to leak but in 1900 it withstood an unprecedented flood with no indication of weakness. In November, 1927, there was another major failure, carrying away a part of the old and a part of the later work, and resulting in heavy loss of life and property. For some time prior to the failure the authorities had been disturbed over the persistent leakage and the continual leaching of lime from the mortar, and had approved a project for drilling and the injection of cement. This work was in actual process near the left bank at the time of the failure, but had not progressed far enough to throw any light on the condition of the masonry. Whether this method of reinforcement and stabilization would have been effective if undertaken a few years earlier can only be conjectured.

The outstanding peculiarity of hydraulic construction in India is the general aversion to the use of Portland cement. This was understandable in the early days when cement would have had to be imported from Europe and would have been very costly. To-day, however, excellent Portland cement, meeting all standard specifications, is manufactured at several places in India and can be purchased at the factory at prices of about 35 rupees per ton—equivalent to about 9 shillings per bbl—so that this reason is no longer operative.

The late Robert Batson Joyner, who spent nearly a life time in India, and who designed Walwhan and Shirawta Dams and their appurtenant works, wrote[7] in 1919:

> "Indian engineers never use Portland cement in the construction of masonry dams, whether of stone or of concrete, except perhaps in wet foundations; not merely on account of its high cost—though good cement is now being made in the country more cheaply—but also because it is considered inferior to good hydraulic-lime mortar for dams. Cement work is too rigid, whilst the hydraulic-lime mortar, being somewhat elastic, withstands the variations of temperature without cracking. There is therefore not the somewhat alarming necessity of providing expansion joints. The range of temperature is probably not greater in India than in England, but it may occur more suddenly."

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[7] "The Tata Hydro-Electric Power-Supply Works, Bombay," by the late Robert Batson Joyner, *Minutes of Proceedings*, Inst. C. E., Vol. CCVII, p. 55.

These are arguments devised to justify an existing practice rather than reasons on which that practice is based. The reader may judge for himself of their validity.

With increasing education of Indian masons in the properties of Portland cement, and with the gradual change in personnel of the Public Works Department with the retirement of the seniors and the advent of younger men, it is probable that Indian practice in dam construction will eventually come to conform quite closely with that in other parts of the world. Tradition is strong in the East, and the inertia of 350 000 000 people is not easily overcome.

One of the most valuable features of this paper is that it emphasizes the dominant importance of execution, as compared with design, in the construction of masonry dams. Volumes of mathematical formulas and theoretical discussions in many languages have come and gone, each purporting to be the last word as to stresses in dams; yet sections have changed little in seventy-five years. On the other hand, with few exceptions, all important dam failures have been traced to defective materials or faulty execution. A dam is made safe, or otherwise, on the job—not in the office.

The author is entitled to great credit for his masterly handling of a difficult job. He is further to be commended for the exhaustive and incisive manner in which he has presented the subject in print. There is nothing superfluous; nothing has been omitted. Aspiring authors of engineering papers should be referred to this as a model.

V. L. Minear,[8] Assoc. M. Am. Soc. C. E. (by letter).—A notable contribution to the knowledge of the theory and practice of cement injection is contained in this paper. It has been read and re-read by those entrusted with pressure grouting at Boulder Dam and appurtenant works. The methods developed for this great dam correspond to a remarkable degree with the methods and practices adopted in India by the author, notwithstanding the vastly different conditions under which the two jobs were carried forward.

The author placed 64 000 bbl of cement at a maximum pressure of 80 lb per sq in. in inferior masonry structures in India, using native labor. At Boulder Dam, more than 100 000 bbl of cement have been placed at maximum pressures of 1 000 lb per sq in., and with high-grade American labor. Notwithstanding the wide variation in conditions and objectives sought, it is thought that the one job affords an independent check upon the methods of injection used and the conclusions reached upon the other. Recounting all the similarities would become tiresome, but the writer wishes to re-emphasize some of Mr. Cole's more important conclusions regarding methods, that experience at Boulder Dam tends to verify:

(1) The degree of hardness and strength (of injected grout) is proportional in general to the pressure of placement with accompanying extrusion of water.

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[8] Care U. S. Bureau of Reclamation, Denver, Colo.

(2) For cement injection into tight ground some better efficiency is attained by the use of extra finely ground and resifted cement. Many thousands of sacks of cement, 98% of which passed the 200-mesh screen, were injected at Boulder Dam, and it was definitely established that, other conditions being the same, a tight hole will admit as much as 10% more rescreened, than ordinary, cement in a given interval of time. However, rescreening costs are relatively high, and it is thought that the apparent advantage can be more than overcome by slightly increasing the pressure applied to ordinary cement (conditions permitting).

(3) Sand is not effective for general use. It is kept in suspension within the pipe with the greatest of difficulty. It cannot be carried very far in the rock passages due to their variable cross-section and the consequent variable velocity with which the grout flows through them. In many cases sand undoubtedly clogs not only the pipe and fittings, but also the hole.

(4) Isolated grouting effects no benefits but shows a tendency to hinder the flow of grout from subsequent holes. In effect, it places obstructions in the grout passages deliberately, making it difficult or impossible to vent the air, water, and thin grout that precedes the thicker grout stream. Theory indicates, and practice seems to confirm, that a channel that has been tapped should be filled in one operation, and holes that have given a return flow should be connected in the order in which they showed grout.

(5) There is always the danger of choking the hole and sacrificing the penetration of the finer connecting passages by too much cement, too much pressure, or too much speed of delivery. The water-cement ratio, pump speed, and induced pressure must be co-ordinated in conformity with the type of grouting being done and the requirements of the individual hole. Each hole is "a law unto itself." It seems probable that more holes have been lost through inexperienced men attempting to use a thicker grout than the hole will accommodate, than from any other single cause. A sudden increase in either the speed of delivery or the pumping pressure, frequently results in the loss of a hole, especially if a water-hammer effect is present. Changes should be made gradually.

(6) Subaqueous or other inaccessible leaks can often be sealed effectively by a manipulation of the water-cement ratio and the pressure applied. The method is superior to the introduction of a sanded mixture in most cases.

(7) All joints should be left open for the free escape of water, and caulking should be resorted to only when the grout that escapes is of the same approximate density as the mixture that is being injected.

(8) The routine of grout injection developed at Boulder Dam is almost identical with that described by Mr. Cole, except for the water-cement ratio and the pressures. At Boulder Dam, the engineers strove from start to finish of a hole to maintain the maximum permissible pumping pressure, but never more than this pressure. Speed was controlled by adjustment of the mixture.

(9) There is need of a safe and reliable means of sealing the voids in otherwise completed masonry structures, in order to make them virtually

impermeable by water. This is especially true of structures founded upon pervious rock. From the study of numerous cores recovered, the writer has reached the conclusion that the benefits to be derived from the injection of Portland cement grout into rock are limited to the filling of cavities and fissures within the rock, and that results are negligible so far as securing a cut-off in water-bearing rock itself is concerned.

James B. Hays,[9] M. Am. Soc. C. E. (by letter).—This is an unusually clear and detailed exposition of the general subject of grouting. The author has given, or outlined, a routine to follow in grouting that will apply to most conditions, particularly in foundation work. Variations in the behavior of different holes develop almost immediately since no two holes are alike. The operator or engineer must "feel" his way, depending on the pressure developed and the rate of flow of the grout which can be varied by changing the mixture.

Equipment for grouting has been developed gradually from makeshift apparatus into machinery specially suited for the work. Pumping grout has generally replaced the practice of forcing it in by compressed air. Engineers and operators who have worked with both systems prefer the continuous flow and positive drive as given by the pump, rather than the intermittent method with the possible danger of getting air into the hole. Mr. Cole has given an excellent description of the equipment, but the writer would like to know more about the pistons and cylinder liners.

Although regular cements are the most generally used, the writer, after handling approximately 40 000 bags of cement screened through a No. 200 screen, is inclined to think that it has some advantages, one of which is that the cement remains in suspension for a longer time and does not clog fittings as quickly. Certainly, small cracks and seams are more likely to be grouted with the coarse particles removed than if they are left in or, to carry it a step further, if sand is added. The writer is strongly opposed to the use of sand in grouting except in extreme cases and will admit that the leakage into the reservoirs through the up-stream faces could probably be stopped by no other method. Sand has a tendency to settle particularly when used with thin mixtures. Very little cement, if any, remains with it to form a tight filler or to bond the particles together. Instead, it will be found in a porous condition. Many grouting jobs have been only partly done by the inclusion of sand. When the top of a concrete-lined tunnel is filled, with sand in the grout, it will carry a load but cannot always be considered to be water-tight. The writer would suggest that before grout is used with sand, test cylinders be made using the proposed mix. Naturally, very fine sands will remain in suspension longer than coarse grades.

In the case cited by the author the leaks were low down, and the sand could be depended upon to settle to the low point. If leaks are higher, or if water is washing grout away as fast as it is pumped in, a floating substance such as sawdust often does the work of stopping the leak. Rolled oats, bran (mill feed), and other substances can be used until the leak is stopped, after which straight cement is used to the point of refusal.

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[9] Constr. Engr., Chickamauga Dam, T. V. A., Chattanooga, Tenn.

When leaks are on the surface and are known in advance, the writer has found it convenient to patch a considerable part of the leak before grouting is begun, leaving small outlets at frequent intervals for the escape of water, air, and thin grout. The final sealing of the patch can then be completed in short order. Various materials are used in patching, such as wedges, oakum, strips of cloth, lead wool, and quick-setting cements. The writer has had better results with a quick-setting cement mixed so as to give a flash set, and applied to the leak in a dry form. The leak furnishes sufficient moisture to cause the mix to set. Where cracks are wide, they are generally caulked with oakum, lead wool, or other material first, after which a quick-setting cement mixture is applied as soon as the leaks appear.

It is convenient, and a widespread practice, to designate the grout mixtures by the water-cement ratio (by volume). A water meter reading directly in cubic feet is useful for this purpose. Cement is generally handled and measured in bags. Grout can be mixed, using any number of full bags of cement, and the water can be metered to any fraction of a cubic foot.

Ordinarily, the writer favors isolated grouting and would prefer to drill a hole and grout it before the other holes in the vicinity were drilled. Usually, the specifications require that, when grout leaks through adjacent open holes, they be capped and also that the capped hole be grouted as soon as possible. The writer has seen grout leak from several holes, and it was absolutely impossible to start grout into any of the leaking holes before the cement had begun to set. As a consequence, the entire area was only partly grouted. Filling a group of holes simultaneously is not satisfactory since each hole takes grout at different pressures and each may require a different mixture. However, the writer wonders what the difference is between, say, four 25-ft holes and one 100-ft hole. Each of the 25-ft holes would take grout differently and, certainly, each 25-ft section of the 100-ft hole has different characteristics. This is an argument for "stage" grouting as well as for drilling single holes and grouting them before other near-by holes are drilled. Then, again, there are exceptions to this rule.

If the nature of the rock is such that there are seams carrying a material that could be washed out and replaced with grout, it would probably be advisable to drill about three holes and wash between them. The efficiency of the washing is generally doubtful. The writer has spent days trying to wash clay from a seam in the foundation of the Calderwood Dam, in Tennessee. After a few hours the water would run clear. If tested again the following day the seam would be found as dirty as ever. After several days of this experience, the holes were grouted regardless. The theory was that the clay was tight and solid in the joint and would resist leakage from the reservoir. This dam was completed in 1930 and five years later there was no increase in the originally very small leakage.

The writer has grouted more than one hole at the same time where underground connections required that it be done, by feeding small batches to each hole in turn, rotating from hole to hole until all were grouted to refusal. This method lacks the advantages of continuous flow that is one of the features of pumping grout. It can be compared with the intermittent batch system that has largely been abandoned. Considerable pains must be taken to obtain the proper mixture in each hole where variations in the mix are required.

The description of the author's experience in using the "lubrication" method and the chemical grouting is appreciated. It would seem that every effort was made to get the best possible results from these methods; hence the conclusions derived have the highest value.

D. W. COLE,[10] M. AM. SOC. C. E. (by letter).—In the several discussions of this paper there has appeared the recognition that the very special conditions and remedies exhibited in the case of the Indian dams have a rather general application to the entire field of "pressure grouting"; yet the discussions, in the main, relate to experiences in sealing by cementation the foundation rock under dams.

The experience most nearly like those treated of in the paper appears in the discussion by Mr. Reed, wherein he describes the attempt to stabilize the Ringedalsvand Dam. It is interesting to note that although some benefits were accomplished by the cementation process, the results were not entirely satisfactory and the final remedy was found in the construction of an auxiliary curtain dam strutted against the water face of the leaky dam. This must have involved unwatering the reservoir, or the construction of expensive coffer-dams, either of which expedients was to be avoided if at all possible in the Indian situation.

The discussion by Mr. Fergusson is particularly apposite to the local problem of masonry construction with surkhi mortar. Doubtless, Mr. Fergusson has become aware of the growing use of Portland cement in substitution for hydraulic lime in his territory. It has recently come to the writer's attention that the cement manufacturing industry of India is expanding, with much activity in the promotion of sales. In 1931 the writer visited the just completed largest masonry dam in Hyderabad, Deccan, which was constructed of ashler-faced rubble in surkhi mortar, and was there informed by the Chief Engineer that he proposed no more important dams of that character, but intended to use Portland cement in the next one. In this connection it may be suggested to Mr. Fergusson that useful possibilities may lie in the conversion of his kankar, or other local limestone, into natural cement rather than quick-lime, and then blending this cheaper form of cement with standard Portland cement for an economic mortar in dam construction. A lean mixture of this should prove better and but little more expensive than the surkhi.

Mr. Wright emphasizes the fundamental requirements for a safe and durable dam, a thesis which is affirmed by the experience described in the paper and which every generation of dam builders should take to heart.

The suggestion is pertinent that the "Christians" asphalt process might have been useful in staunching some of the worst runaway leaks to the lake as a preliminary to cementation. In retrospect, however, it would appear that the provision of all necessary equipment for this process in the distant and isolated locality would not have been justified for the limited requirement.

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[10] Civ. Engr., Phœnix Eng. Corporation, New York, N. Y.

No dimension drawings are available, but the scale diagram in Fig. 1 gives all cross-sectional dimensions of Shirawta Dam as built. From this Mr. Wright will be able to analyze the stresses, using the tabulated weight of 146 lb per cu ft for masonry and such data for uplift, sliding factor, and cantilever integrity, as he will derive from the text of the paper.

The illuminating discussion by Mr. Comstock supplies a background of history, statistics, Oriental color, and technical wisdom against which the paper is most favorably displayed. Mr. Comstock's closing observation that: "A dam is made safe, or otherwise, on the job—not in the office", touches a responsive chord in the writer's constitution.

In the discussion by Mr. Minear it is gratifying to read that the Boulder Dam program of foundation grouting developed a system, independently, which is remarkably in accord with the routine in India, notwithstanding the dissimilarity of conditions and objectives. The check of one system against the other with harmonious results is indeed striking, tending to prove, for example, that both programs followed sound procedure in the situations as each was developed.

Mr. Hays raises a number of interesting points which it might be profitable to discuss if space permitted. Suffice it to say that the writer is in agreement as to the general superiority of pumps over the air gun for controlled grout delivery. Answering direct questions by Mr. Hays: The pistons of the air-driven pumps in the beginning were of cold-rolled shafting, 2 in. and 3 in. in diameter, with gland packing of a good grade of square braided hemp. Excessive wear of the piston (plunger) and the packing led to the substitution of nickel steel plungers, polished and packed in the same manner. These plungers were quite satisfactory and gave many weeks of wear, although much packing was consumed. Cylinder lining was unimportant since all wear was on the plungers with full bore packing. The distinctive feature of the pumps was the valve assembly consisting of ring-seated steel balls, as described under "Plan of Operation" in the paper.

The re-sifting of cement is an important consideration for work in tight ground, and in dams where limitation to low pressure is demanded. If again confronted with similar conditions the writer would plan to put the commercial cement through a 200-mesh, rotating or vibrating sieve which could easily be rigged in the cement shed. This fine cement would be used in borings testing less than 10 gpm water loss. By this means the input of cement would be practically of colloidal texture and would probably accomplish about all that would result from application of the "chemical treatment".

The writer agrees that sand should be used sparingly, with caution, and only in large cavities, or in runaway leaks which must be staunched as a preliminary to effective pressure grouting with thin mixtures. Sawdust, rice chaff, or other perishable materials were deemed inappropriate substitutes for sand in choking this type of leak in a structure where density was the prime essential. The writer agrees with Messrs. Hays and Minear that the best results can be secured by holding the pressure on the hole from start to finish, with the cement mixtures adjusted to give a maximum total input of solids before the hole chokes. It is not always feasible, however, and this is where experienced judgment, patience, and conscience have application.

In the writer's view a vast amount of "grouting" has been done in a perfunctory manner with ineffectual results, although reported as "holes grouted to refusal"; all for lack of intelligent and diligent supervision.

As to the multiple connection of holes for simultaneous grouting, it has been the writer's practice to require or permit this when pump delivery is demonstrably capable of holding the specified pressure on each individual hole of the series, usually not more than four holes at a time.

The discussion by Mr. Hays evidences appreciation of the requirements for effective sealing of difficult ground. There is a great variety of problems in the field, worthy of expert study.

An early example of foundation grouting is well recorded in much detail and with pertinent discussions in the paper by Harold A. Rands, M. Am. Soc. C. E., entitled: "Grouted Cut-Off for Estacada Dam".[11] A classic in tunnel grouting is the paper by James F. Sanborn and M. E. Zipser, Associate Members, Am. Soc. C. E., entitled "Grouting Operations, Catskill Water Supply".[12] Students of the general subject would do well to include these papers in their research.

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[11] *Transactions*, Am. Soc. C. E., Vol. LXXVIII (1915), p. 447.
[12] *Loc. cit.*, Vol. LXXXIII (1919–20), p. 1058.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1938

LINE LOAD ACTION ON THIN CYLINDRICAL SHELLS

By HERMAN SCHORER,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. I. K. Silverman, W. Flügge, Anton Tedesko, U. Finsterwalder, F. W. Seidensticker, and Herman Schorer.

Synopsis

The object of this paper is the derivation of influence numbers for the longitudinal edge deformations of a thin, cylindrical, shell sector, rigidly held between transverse stiffening members. This problem is of practical importance in the design of modern shell structures, such as large pipe lines, self-supporting flume structures, sector gates, and the barrel type of roof described in this paper. The monolithic action of longitudinal stiffening members, as well as the analysis of discontinuous surface loads, leads to the consideration of corrective line loads, which restore the continuity of the structure.

The proposed method is based on simplifying assumptions, and, therefore, the solution is an approximation with definite limitations. As an illustrative example the secondary stress conditions in a half-full pipe line are investigated.

Introduction

Thin shells or bent plates are three-dimensional structures, extensively used by Nature as protective housings, on account of their great resistance to impact forces. They require a minimum of construction material and are adaptable to a great variety of possible shapes. The shells may be of the open or closed type.

The excellent structural properties of closed shells are explained by the fact that external surface loads are largely transmitted by direct stresses, which act in two principal stress planes. Although concentrated loads also induce bending stresses, Geckeler[2] has shown in the analysis of spherical

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Note.—Published in March, 1935, *Proceedings*.

[1] Gen. Mgr., Borsari Tank Corp. of America, New York, N. Y.

[2] "Zur Theorie der Elastizität flacher rotations-symmetrischer Schalen," von J. W. Geckeler, *Ingenieur-Archiv*, Vol. 1, 1930, p. 255.

shells that a direct stress system prevails within a short distance from the load concentration. The bending zone appears distributed over an imaginary, circular plate, elastically supported by the adjacent parts of the shell. The free span of this plate is comparatively small and, therefore, the induced bending stresses are likewise small.

In case of open shells, the free rims or edges represent zones of structural weakness, because a direct stress system is possible in one direction only, namely, parallel to the free boundary. Edge loads, therefore, must be transmitted to the interior of the shell by partial plate action; that is, bending stresses in planes normal to the boundary line. For instance, a closed eggshell represents a rather stiff structure. However, if the shell is cut into two halves, it will be observed that small edge forces cause relatively large distortions of the thin walls.

As a rule, Nature protects the free edges of thin shells by thickening the weak zone, as in the case of walnut shells. The same principle is used in the glass and sheet-metal industry, where thin edges are strengthened by means of flange-shaped stiffening members. In order to be most effective such members should be placed in planes nearly normal to the surface of the shell. The advantages of true shell action will then be realized also in the case of open shells.

The practical design of shell structures in steel and reinforced concrete has opened a wide field of interesting applications in industry and architecture. A description of the more recent developments in shell-type reinforced concrete structures is given by Dischinger and Finsterwalder.[3] Most of the practical problems lead to a consideration of edge loads in connection with the design of adequate stiffening members. For this reason the analysis of shell stresses due to edge loads becomes of practical importance. The following study is limited to the stresses and deformations in a circular, cylindrical, shell sector, with a uniform wall thickness and subject to edge loads distributed along a generatrix.

Closed cylindrical shells or tubes, subject to continuous surface loads, require circumferential stiffening members at the open ends, or over the supports, as in the case of large pipe lines,[4] horizontal storage tanks, culverts, etc. The actual stress condition in such a shell is determined by the superposition of two fundamental stress systems: First, the so-called membrane stresses (which represent a statically determinate, direct stress system, being in equilibrium with the surface loads); and, second, the so-called rim stresses (which are statically indeterminate) caused by the restraining action of the circumferential stiffeners. In the case of thin shells the rim stresses are distributed only over a comparatively narrow ring zone.

The analysis of the membrane stresses in symmetrical shells of revolution, due to continuous, symmetrical, and asymmetrical surface loads, has been well established and is presented in detail by Dischinger.[5]

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[3] "Neuere Entwicklungsformen der Schalenbauweise," von F. Dischinger und U. Finsterwalder, *Beton und Eisen*, Vol. 31 (1932).

[4] "Design of Large Pipe Lines," by Herman Schorer, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 101.

[5] "Handbuch für Eisenbeton," von H. J. Kraus und Franz Dischinger, Vol. 6, Fourth Edition, p. 269, Wilhelm Ernst & Sohn, Berlin, 1928.

A simplified method for the practical determination of rim stresses in symmetrical shells of revolution, subject to symmetrical rim loads, has been derived by Geckeler.[6] The rim load problem in circular, cylindrical shells, subject to asymmetrical loadings, has been fully treated by Miesel.[7]

Fig. 1 shows a cylindrical shell with a cross-section consisting of a thin, circular-ring sector, between two transverse end stiffeners, fixed in a plane normal to the axis of the shell. In the following derivations the term, "rim," always refers to the circumferential boundary of the shell and the term, "edge," refers to the longitudinal boundary, which coincides with a generatrix of the cylindrical surface.

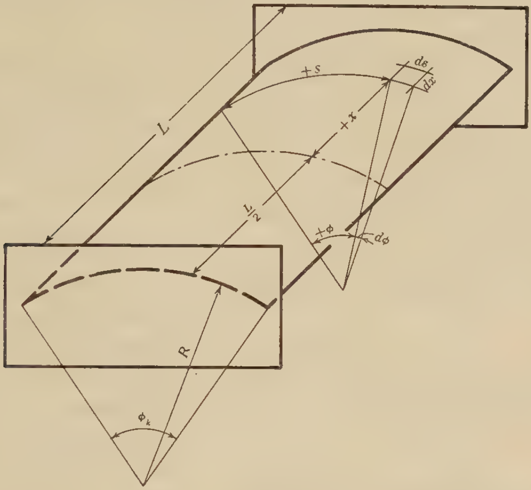

FIG. 1.

Line loads, or edge loads, are defined as loads acting on the two free edges of the shell. The present study is limited to continuous line loads only, such as those caused by the restraining action of a longitudinal edge-stiffening member. The shell stresses due to edge loads, then, represent a third stress system, designated as line load or edge stresses.

The edge load stresses in cylindrical shells, simply supported between stiff transverse members, have been investigated by Dr. Finsterwalder,[8] mainly for the purpose of analyzing a barrel, or vaulted, type of thin shell roof structure. Similar problems arise in the design of semi-circular, open flume structures, sector gates, decks of reinforced concrete dams, etc. The analysis of discontinuous surface loads, such as the partial fillings of a pipe line, partial submergence of submarines, lifting tanks, etc., also leads to the problem of line-load stresses.

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[6] "Ueber die Festigkeit achsensymmetrischer Schalen," von J. Geckeler, *Forschungsarbeiten,* Heft 276, VDI-Verlag, Berlin, 1926.

[7] "Ueber die Festigkeit von Kreiszylinderschalen mit nichtachsensymmetrischer Belastung," von Kurt Miesel, *Ingenieur-Archiv,* Vol. 1, 1929, p. 22.

[8] "Die Theorie der zylindrischen Schalengewölbe, System Zeiss-Dywidag," von Dr. Ing. Ulrich Finsterwalder, International Assoc. for Bridge and Structural Eng., Zurich, 1932; also, "Die querversteiften zylindrischen Schalengewölbe mit kreissegmentförmigen Querschnitt," von Ulrich Finsterwalder, *Ingenieur-Archiv,* Vol. 4, 1933, p. 43.

In contrast to the comparatively narrow zone of circumferential rim load stresses, the influence of edge loads may extend over the entire shell area, because such loads are largely transmitted to the stiff transverse members. At present, no exact solution of the line-load problem is available. The approximate method presented in this paper is based entirely on Dr. Finsterwalder's original investigation, which also introduces simplifying assumptions, justified by model tests and observation of full-sized structures.

The basic assumptions point out definite limitations, which the designer must keep in mind when considering structures in which plate action, rather than shell action, comes into play. For this reason the method is applicable mostly to the practical analysis of thin shells of considerable curvature and relatively large spans.

Notation.—The symbols introduced in this paper are defined where they first appear; and for the convenience of discussers, the complete notation is given in Appendix I.

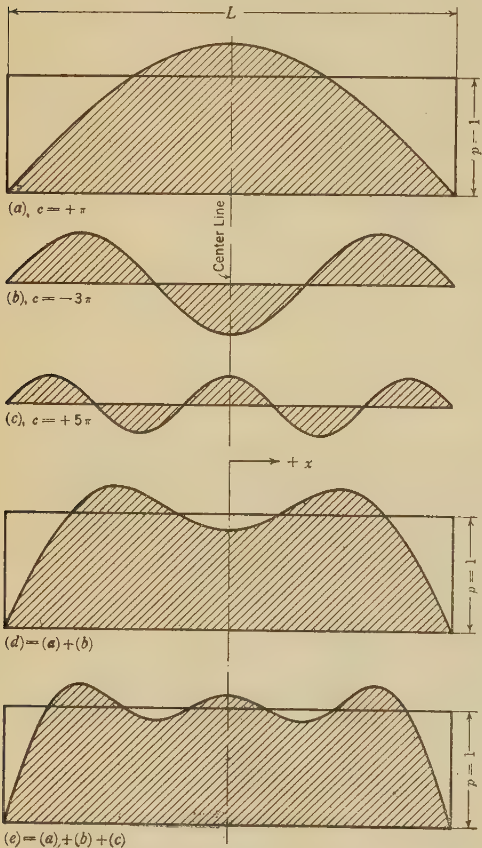

FIG. 2.—UNIFORM LOAD, $p = 1$, REPRESENTED BY FOURIER SERIES.

APPLICATION OF FOURIER SERIES

The introduction of trigonometric series in the theory of elasticity often yields decided analytical advantages because difficulties of integration are avoided and any desired degree of accuracy may be obtained by simply considering a sufficient number of terms. The solution of the flat slab problem by Lewe[9] and the derivation of the deflection curves of suspension bridges by Timoshenko,[10] are cited as well-known applications.

Consider a uniformly distributed unit load, p per unit length, acting on a simply supported beam of the span, L. This load distribution can be expressed by the sum of a trigonometric series. Assuming the origin of the co-ordinate system at the center of the span, then,

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[9] "Pilzdecken," von Dr. Lewe, pub. by Wilhelm Ernst & Sohn, Berlin, 1926.

[10] "The Stiffness of Suspension Bridges," by S. Timoshenko, *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 377.

$$p\,(x) = \frac{4}{\pi}\left(\cos\frac{\pi x}{L} - \frac{1}{3}\cos\frac{3\,\pi\,x}{L} + \frac{1}{5}\cos\frac{5\,\pi\,x}{L} - \dots\right) = (1) \quad \dots (1)$$

in which, $p(x)$ = the sum of a trigonometric series expressing intensity of load; x = a horizontal distance measured longitudinally, from the center of the span; and L = the span of a simply supported beam or cylindrical shell. The general term of this series (Equation (1)) is,

$$p_c = \frac{4}{c}\cos\frac{c\,x}{L} \quad \dots\dots (2)$$

in which, c = substitutions corresponding to the first, second, third, etc., term of a Fourier series. Fig. 2(*a*) illustrates the first term of the series, Equation (1), which is obtained by substituting $c = +\,\pi$ in Equation (2); Fig. 2(*b*), with $c = -\,3\,\pi$, represents the second term; and Fig. 2(*c*), with $c = +\,5\,\pi$, indicates the third term. The sum of the first and second terms is shown in Fig. 2(*d*); the sum of the first, second, and third terms is represented in Fig. 2(*e*); and the sum of an infinite number of terms gives the straight line, $p = 1$.

For example, at the center of the span, $x = 0$, Equation (1) results in the well-known expression,

$$p\,(0) = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) = \frac{4}{\pi} \times \frac{\pi}{4} = 1 \quad \dots\dots (3)$$

It will be noted from Fig. 2 that the sum of the first few terms in Equation (1) represents a close approximation to the uniform load distribution.

The bending moment, M, of a simply supported beam, subject to uniform loading, is obtained by double integration of Equation (1), which gives,

$$M = \frac{4\,L^2}{\pi^3}\left(\cos\frac{\pi\,x}{L} - \frac{1}{3^3}\cos\frac{3\,\pi\,x}{L} + \frac{1}{5^3}\cos\frac{5\,\pi\,x}{L} - \dots\right) \quad \dots (4)$$

The general term of the series, Equation (4), is,

$$M_c = \frac{4}{c}\,\frac{L^2}{c^2}\cos\frac{c\,x}{L} \quad \dots\dots (5)$$

At the center of the span, $x = 0$, the sum of the series, Equation (5), gives the well-known moment value for a uniformly distributed unit loading, $M_0 = \frac{L^2}{8}$.

Likewise, the deflection, η, of a uniformly loaded and simply supported beam, is obtained by double integration of Equation (4); thus,

$$\eta = \frac{4\,L^4}{\pi\ 5}\left(\cos\frac{\pi\,x}{L} - \frac{1}{3^5}\cos\frac{3\,\pi\,x}{L} + \frac{1}{5^5}\cos\frac{5\,\pi\,x}{L} - \dots\right) \quad \dots (6)$$

in which, η = the deflection of a simply supported, uniformly loaded beam, multiplied by $E\,I$. The general term of the series, Equation (6), is,

$$\eta_c = \frac{4}{c}\,\frac{L^4}{c^4}\cos\frac{c\,x}{L} \quad \dots\dots (7)$$

At the center of the span, $x = 0$, the sum of the series, Equation (6), gives the value, $\eta_0 = \frac{5\,L^4}{384}$, which is identical with that derived by the customary methods.

DEFORMATIONS OF A SMALL ELEMENT

Fig. 3(*a*) shows a small element of a cylindrical shell, loaded by internal forces and couples as indicated, which are defined as stress components, measured per unit width of the element. The position of the element is defined by the distance, x, from the center of the span (see Fig. 1), and the angle, ϕ, measured from the left-hand edge of the shell when the element faces in the positive direction of x. The direct stress components, T_1, acting in a longitudinal direction, and, T_2, acting in a transverse direction, are positive when they produce tension. The bending stress components, M_1, acting in a longitudinal plane, and, M_2, acting in a transverse plane, are positive when they produce tension on the inner surface of the element. The direct shearing stress component, S, acting in pairs of equal magnitude on all four sides in the plane of the element, is considered positive when it produces tension in a diagonal direction of increasing values of x and ϕ. The radial shearing stress components, N_1 and N_2, are considered positive when they act in an outward direction on the sides facing the origin of x and ϕ, respectively. The torsional moments, or twisting moments, M_t, are practically equal on all four sides, and are defined as positive when they produce tension on the outside surface in a diagonal direction of increasing values of x and ϕ. For reasons stated subsequently the influence of the longitudinal

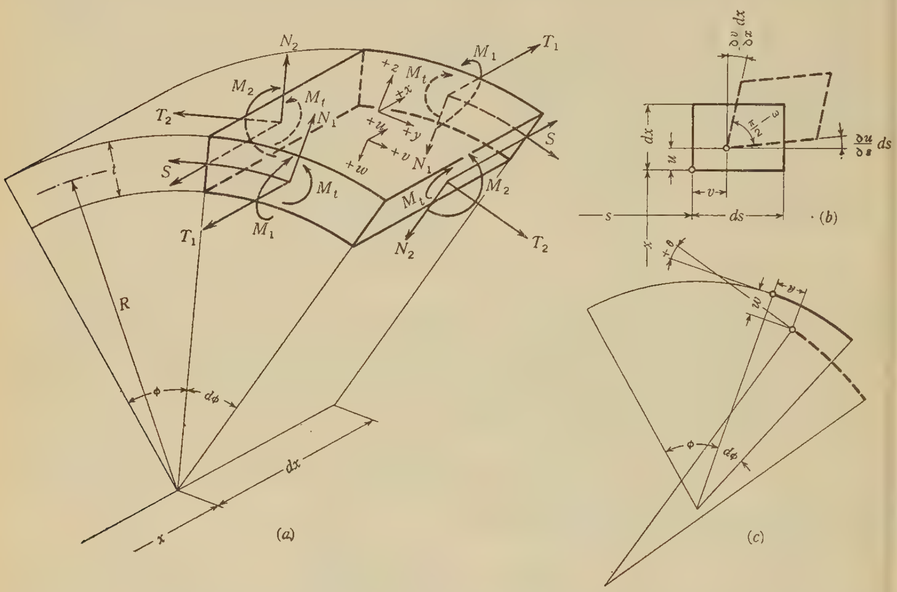

FIG. 3.

bending moment, M_1, the corresponding shearing stress component, N_1, and the influence of the torsional moments, M_t, will be neglected in the edge-load analysis.

The deflections of the small element are indicated in Figs. 3(*a*), 3(*b*), and 3(*c*). The longitudinal deflection, u, and the tangential deflection, v, are positive when the element moves in the positive direction of x and ϕ, respectively. The radial deflection, w, is positive when the element moves inward. The detrusion, ω, caused by the shearing stress component, S, is positive when S is positive. The angular deflection, θ, of the transverse tangent, is positive when the tangent is rotated in a clockwise direction.

Fig. 3 indicates the positive direction of all stress components and deflections. Love[11] has derived the following fundamental relations between the deformations, deflections, and stress components, applying to the small element of the length, dx; the width, $ds = R\, d\phi$; a constant thickness, t; and, a constant radius, R. Thus, the unit longitudinal elongation is expressed by:

$$e_1 = \frac{\partial u}{\partial x} = \frac{1}{E\,t}\,(T_1 - m\ T_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

the unit transverse elongation by,

$$e_2 = \frac{1}{R}\left(\frac{\partial v}{\partial \phi} - w\right) = \frac{1}{E\,t}\,(T_2 - m\ T_1) \quad \ldots\ldots\ldots\ldots (9)$$

the detrusion by,

$$\omega = \frac{\partial v}{\partial x} + \frac{\partial u}{R\,\partial \phi} = \frac{2\ S}{E\,t}\,(1 + m) \quad \ldots\ldots\ldots\ldots\ldots (10)$$

the angular deflection of the transverse tangent by,

$$E\,t\,\theta = \frac{1}{R}\left(\frac{\partial w}{\partial \phi} + v\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

the change of transverse curvature by,

$$\kappa_2 = \frac{1}{R^2}\left(\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right) = \frac{1}{R^2}\left(w + \frac{\partial^2 w}{\partial \phi^2}\right) = -\,\frac{12\,(1 - m^2)\,M_2}{E\,t^3} \ldots (12)$$

and the change of longitudinal curvature by,

$$\kappa_1 = \frac{\partial^2 w}{\partial x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

in which, e = unit elongation: e_1, longitudinal, and e_2, transverse; u, v, and w are, respectively, the longitudinal, tangential, and radial deflections of a shell element; E = modulus of elasticity; t = thickness of shell; T = direct stress components on a shell element: T_1, in a longitudinal direction, and T_2, in a tangential direction; m = Poisson's ratio; ω = angular detrusion of a shell element, caused by the shearing forces, S; R = radius of curvature of a cylinder element; ϕ = angular distance of a shell element from the left-hand edge; and $\kappa =$ change of curvature of a shell element: κ_1, longitudinal, and κ_2, transverse.

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[11] "Theory of Elasticity," by A. E. H. Love, Third Edition, Cambridge Univ. Press (1920), p. 546.

THE EDGE LOAD PROBLEM

Assume a thin cylindrical shell sector (such as that shown in Fig. 1)—of constant thickness, t; constant radius, R; a central angle, ϕ_k; and a span, L—between the two transverse end stiffeners. Assume, further, that no external loads are acting on the shell surface. Then consider a system of edge loads, such as thrusts, moments, shears, etc., continuously distributed along one or both of the free longitudinal boundaries, and resisted by the transverse stiffening members. Due to these edge loads the shell will be distorted by internal forces and couples (see Fig. 3), which transmit the loads to the end walls.

If the shell is thin, not extremely short, and has a fairly large central angle, experience and analysis both indicate that the internal work of the longitudinal bending moment, M_1, the corresponding shear, N_1, and the twisting moments, M_t, is rather small compared to the work done by the direct stress components, T_1, T_2, S, and the transverse bending moment, M_2, with the corresponding radial shearing stress component, N_2. Under these limiting conditions it will be permissible, for practical purposes, to neglect the influence of M_1, N_1, and M_t in considering the stresses and deformations of the shell due to edge loads. However, this statement does not apply to the ring zone adjacent to the end stiffeners, which is always subject to a supplementary stress system, defined by the rim stress zone, where M_1 and N_1 become relatively large, and M_2 and N_2 assume negligible values.

Based on the simplifying assumptions, $M_1 = N_1 = M_t = 0$, the five remaining stress components, M_2, N_2, T_2, T_1, and S, can then be derived from four equilibrium equations and one compatibility equation, as shown by Finsterwalder.[8] The edge load distribution is assumed to be symmetrical about the middle transverse plane, expressed by a cosine function, which can be considered as the general term of a trigonometric series. As suggested by Finsterwalder, the stress components are expressed conveniently by a stress function, $f(\phi)$, similar to Airy's presentation of the stress condition in a plane plate.

The stress function is introduced by the following definition of the bending stress component, M_2,

$$M_2 = -\cos\frac{c\,x}{L}\,f(\phi)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Fig. 3(*a*) gives the following equilibrium conditions on a small element:

Rotation in the transverse plane,

$$N_2 = \frac{\partial M_2}{R\,\partial\phi} = -\frac{1}{R}\,\frac{df}{d\phi}\cos\frac{c\,x}{L}\ldots\ldots\ldots\ldots\ldots(15)$$

projection on the z-axis,

$$T_2 = -\frac{\partial N_2}{\partial\phi} = +\frac{1}{R}\,\frac{d^2f}{d\phi^2}\cos\frac{c\,x}{L}\ldots\ldots\ldots\ldots\ldots(16)$$

projection on the y-axis, and by partial integration,

$$\frac{\partial S}{\partial x} = \frac{N_2}{R} - \frac{1}{R}\,\frac{\partial T_2}{\partial \phi} = -\frac{1}{R^2}\left(\frac{df}{d\phi} + \frac{d^3 f}{d\phi^3}\right)\cos\frac{c\,x}{L} \quad \ldots\ldots (17)$$

or,

$$S = -\frac{L}{R^2\,c}\left(\frac{df}{d\phi} + \frac{d^3 f}{d\phi^3}\right)\sin\frac{c\,x}{L} \quad \ldots\ldots\ldots\ldots\ldots (18)$$

and projection on the x-axis, and by partial integration,

$$\frac{\partial T_1}{\partial x} = -\frac{1}{R}\,\frac{\partial S}{\partial \phi} = \frac{L}{R^3\,c}\left(\frac{d^2 f}{d\phi^2} + \frac{d^4 f}{d\phi^4}\right)\sin\frac{c\,x}{L} \quad \ldots\ldots\ldots (19)$$

or,

$$T_1 = -\frac{L^2}{R^3\,c^2}\left(\frac{d^2 f}{d\phi^2} + \frac{d^4 f}{d\phi^4}\right)\cos\frac{c\,x}{L} \quad \ldots\ldots\ldots\ldots\ldots (20)$$

The following definition is introduced,

$$a = \frac{c^2\,R^2}{L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Substituting T_2 and T_1 from Equations (16) and (20), in Equation (8), the longitudinal deflection, u, is obtained by partial integration,

$$E\,t\,u = -\frac{1}{a\sqrt{a}}\left(\frac{d^2 f}{d\phi^2}\,(1 + m\,a) + \frac{d^4 f}{d\phi^4}\right)\sin\frac{c\,x}{L} \quad \ldots\ldots\ldots (22)$$

From Equation (10), by partial differentiation,

$$\frac{\partial^2 v}{\partial x^2} = \frac{2\,(1+m)}{E\,t}\,\frac{\partial S}{\partial x} - \frac{\partial^2 u}{\partial s\,\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

By partial differentiation of Equation (8), with respect to s,

$$\frac{\partial^2 u}{\partial s\,\partial x} = \frac{1}{E\,t}\left(\frac{\partial T_1}{\partial s} - m\,\frac{\partial T_2}{\partial s}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Substituting Equation (24) in Equation (23),

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{E\,t}\left(2\,(1+m)\,\frac{\partial S}{\partial x} - \frac{\partial T_1}{R\,\partial\phi} + m\,\frac{\partial T_2}{R\,\partial\phi}\right) \quad \ldots\ldots\ldots (25)$$

From Equations (16), (17), and (20), the expressions for the stress components are substituted in Equation (25), then, by partial integration,

$$E\,t\,v = \frac{1}{a}\left[2\,(1+m)\left(\frac{df}{d\phi} + \frac{d^3 f}{d\phi^3}\right) - \frac{1}{a}\left(\frac{d^3 f}{d\phi^3} + \frac{d^5 f}{d\phi^5}\right) - m\,\frac{d^3 f}{d\phi^3}\right]\cos\frac{c\,x}{L} \quad \ldots\ldots\ldots\ldots\ldots (26)$$

From Equation (9), with T_2 from Equation (16), T_1 from Equation (20), and $\frac{\partial v}{\partial \phi}$ from Equation (26),

$$E\,t\,w = \left[\frac{d^2f}{d\phi^2}\left(\frac{m}{a} - 1\right) + \frac{2}{a}\left(\frac{d^2f}{d\phi^2} + \frac{d^4f}{d\phi^4}\right) - \frac{1}{a^2}\left(\frac{d^4f}{d\phi^4} + \frac{d^6f}{d\phi^6}\right)\right] \cos\frac{c\,x}{L} \quad \ldots\ldots\ldots\ldots\ldots (27)$$

From Equation (11), with reference to Equations (26) and (27),

$$E\,t\,R\,\theta = \left[-\frac{d^3f}{d\phi^3} + \frac{2}{a}\left(\frac{df}{d\phi} + 2\,\frac{d^3f}{d\phi^3} + \frac{d^5f}{d\phi^5}\right) - \frac{1}{a^2}\left(\frac{d^3f}{d\phi^3} + 2\,\frac{d^5f}{d\phi^5} + \frac{d^7f}{d\phi^7}\right) + 2\,\frac{m}{a}\left(\frac{df}{d\phi} + \frac{d^3f}{d\phi^3}\right)\right] \cos\frac{c\,x}{L} \quad \ldots . (28)$$

The constants of integration are zero, provided the following conditions are fulfilled: First, the load distribution is symmetrical about the middle transverse plane, $x = 0$; second, the shell can deform freely at the end stiffeners in a longitudial and also in a radial direction. Since the present analysis is limited to edge loads expressed by a cosine function the first condition is met. The transverse stiffening members, as a rule, are relatively flexible in a longitudinal plane, so that free movement in this direction can be assumed for practical purposes. However, the movement in a radial direction is greatly resisted by the stiffness of the transverse members, requiring the consideration of rim stresses, which can always be treated as a separate stress system. For the present purpose, therefore, the loads can be assumed as transmitted to the end members by means of tangential shearing stresses only, in accordance with the membrane theory.

The compatibility equation is obtained by substituting M_2 from Equation (14), and w from Equation (27), in the expression for the transverse curvature, Equation (12), which gives the differential equation of the stress function,

$$\frac{d^8f}{d\phi^8} + 2\,(1 - a)\,\frac{d^6f}{d\phi^6} + [1 - (4 + m)\,a + a^2]\,\frac{d^4f}{d\phi^4} + [-(2 + m)\,a + a^2]\,\frac{d^2f}{d\phi^2} + 4\,a^2\,b^2\,(1 - m^2)\,f = 0 \quad \ldots\ldots (29)$$

The constant, b, is defined as follows,

$$b = \sqrt{3}\;\frac{R}{t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

The general solution of Equation (29) gives the following expression for the stress function,

$$f(\phi) = e^{(J1)\phi}\,(A \sin K_1\,\phi + B \cos K_1\,\phi) + e^{(J1)'\phi}\,(C \sin K'_1\,\phi + D \cos K'_1\,\phi) \quad \ldots\ldots\ldots\ldots\ldots (31)$$

Equation (31) indicates that the stress function consists of the sum of two damped waves, in which, $(J1)$ and $(J1)'$, represent the damping factors, and K_1 and K'_1 determine the wave length. These four values are constants, entirely governed by the dimensions of the shell (as exponents, $(J1)$ and $(J1)'$ correspond with J_1 and J'_1 in the text). The expressions, A, B, C, and D, are also constants, determined by the boundary conditions of the longitudinal edge, by which four stress components or deflections are assigned specified values, given by the edge load conditions or edge deformations.

The shell constants, J and K, are found by substituting the derivatives of Equation (31) in the differential formula, Equation (29). The first term of the stress function leads to the following relation:

$$\begin{aligned} e^{(J1)\phi} \sin K_1 \phi \left\{ J_8 + 2(1-a) J_6 + [1-(4+m)a+a^2] J_4 \right. \\ \left. + a(-2-m+a) J_2 + 4a^2 b^2 (1-m^2) \right\} + e^{(J1)\phi} \cos K_1 \phi \left\{ K_8 \right. \\ + 2(1-a) K_6 + [1-(4+m)a+a^2] K_4 \\ \left. + a(-2-m+a) K_2 \right\} = 0 \ldots\ldots\ldots\ldots(32) \end{aligned}$$

The expressions, J_2, J_4, etc., and K_2, K_4, etc., are defined by the general progressive functions,

$$J_n = J_1 J_{n-1} - K_1 K_{n-1} \ldots\ldots\ldots\ldots(33)$$

and,

$$K_n = J_1 K_{n-1} + K_1 J_{n-1} \ldots\ldots\ldots\ldots(34)$$

Equation (32) can be satisfied when the functions, F and G, defined by Equations (35) and (36), are zero,

$$\begin{aligned} F = J_8 + 2(1-a) J_6 + [1-(4+m)a+a^2] J_4 \\ + a(-2-m+a) J_2 + 4a^2 b^2 (1-m^2) = 0 \ldots\ldots(35) \end{aligned}$$

and,

$$\begin{aligned} G = K_8 + 2(1-a) K_6 + [1-(4+m)a+a^2] K_4 \\ + a(-2-m+a) K_2 = 0 \ldots\ldots\ldots\ldots(36) \end{aligned}$$

The derivatives of the second term of Equation (31), substituted in Equation (29), give an identical pair of equations for F and G.

The progressive functions, Equations (25) and (26), furnish the following expressions,

$$J_2 = J_1^2 - K_1^2 \ldots\ldots\ldots\ldots(37)$$

$$J_4 = J_2^2 - K_2^2 \ldots\ldots\ldots\ldots(38)$$

$$J_6 = J_2^3 - 3 J_2 K_2^2 \ldots\ldots\ldots\ldots(39)$$

$$J_8 = J_2^4 - 6 J_2^2 K_2^2 + K_2^4 \ldots\ldots\ldots\ldots(40)$$

$$K_2 = 2 J_1 K_1 \ldots\ldots\ldots\ldots(41)$$

$$K_4 = 2 J_2 K_2 \ldots\ldots\ldots\ldots(42)$$

$$K_6 = 3\, J^2_2\, K_2 - K^3_2 \ldots\ldots\ldots\ldots (43)$$

and,

$$K_8 = 4\, J^3_2\, K_2 - 4\, J_2\, K^3_2 \ldots\ldots\ldots\ldots (44)$$

By substituting Equations (38), (39), and (40), in Equation (35), and, likewise, by substituting Equations (42), (43), and (44), in Equation (36), it will be noted that two equations are obtained for the determination of the unknowns, J_2 and K_2. These simultaneous equations cannot be solved in explicit form. However, J_2 and K_2 may be determined by successive approximations and, for instance, by Newton's method.

For this purpose it will be advantageous to consider the typical properties of a damped wave system, such as that represented by the stress function, Equation (31). In case the damping factors, J_1 and J'_1 are larger than unity, each successive derivative will be larger than the preceding one. Since the present analysis is limited to thin shells, the damping factors will be comparatively large, so that, as a first approximation, the derivatives of a lower order may be neglected compared to the derivative of the highest order. This method has been used by Geckeler,[6] in arriving at a practical solution for the rim stresses in symmetrical shells of revolution, subject to symmetrical rim loads.

The same method can be applied to Equation (29) provided the coefficients of the derivatives of lower order are not too large. These coefficients are mainly a function of a (see Equation (21)) which contains the relative span, $\frac{R}{L}$, in connection with the general term, c, of the trigonometric series. Evidently, as a first approximation, the derivatives of lower order can be neglected if,

$$a = \left(\frac{c\,R}{L}\right)^2 \leqq 1 \ldots\ldots\ldots\ldots (45)$$

Finsterwalder has shown,[8] that the stress conditions in the shell are largely determined by considering the first term, $c = \pi$, of the Fourier series, only, so that the accuracy will not be greatly influenced by reasonable errors of the supplementary stresses due to the additional terms, $c = -3\,\pi$, $c = +5\,\pi$, etc. At the same time, it should be stated that the multiple wave loads, represented by the higher load terms (Fig. 2(*b*) and Fig. 2(*c*)), would also induce increasingly large proportions of work done by longitudinal bending stresses and torsional moments. Since M_1 and M_t have been neglected in the basic stress system it is evident that the stresses and deformations obtained from the higher load terms are too large in any case. For practical purposes, therefore, the first load term, $c = \pi$, may be substituted in the limiting condition, Equation (45), which can now be stated in the form,

$$\frac{L}{R} \geqq \pi \ldots\ldots\ldots\ldots (46)$$

The proposed approximation, therefore, is applicable to larger spans only, which, however, represent the most important field of practical applications.

With these simplifying assumptions the differential equation (Equation (29)) is now reduced to the following expression,

$$\frac{d^8 f}{d\phi^8} + 4\, a^2\, b^2\, (1 - m^2)\, f = 0 \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

The corresponding simultaneous equations, (Equations (35) and (36)), with J_3 and K_3 from Equations (40) and (44) respectively, become, therefore,

$$J^4_2 - 6\, J^2_2\, K^2_2 + K^4_2 + 4\, a^2\, b^2\, (1 - m^2) = 0 \ldots\ldots\ldots (48)$$

and,

$$4\, J^3_2 - 4\, J_2\, K^2_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

which gives,

$$J^2_2 = K^2_2 = a\, b \sqrt{1 - m^2} \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

The following definitions are introduced:

$$r = \sqrt{a\, b}\ \sqrt[4]{1 - m^2} = \frac{c\, R}{L} \sqrt{\frac{R}{t}}\ \sqrt[4]{3\,(1 - m^2)} \ldots\ldots (51)$$

$$n_1 = \left(\sqrt{0.5} + 0.5\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

and,

$$n_2 = \left(\sqrt{0.5} - 0.5\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

The solution of Equation (50) gives the following roots for the unknown, J_2:

$$J_2 = + r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

and,

$$J'_2 = - r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

The corresponding values for the unknown, K_2, are,

$$K_2 = - r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

and,

$$K'_2 = - r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57)$$

The values, J_1, J'_1, K_1, K'_1, as well as J_3, J'_3, K_3, K'_3, etc., are found by means of the progressive functions, Equations (33) and (34). Since only waves of decreasing amplitude can describe the stress conditions originating from a loaded edge, it is necessary to choose the negative sign for the damping factors, J_1 and J'_1. This requirement, in turn, determines the choice of the negative sign for K_2 and K'_2, as given by Equations (56) and (57). Table 1 (Appendix II), shows the values, J, K, J', and K', respectively, corresponding to the simplified differential formula (Equation (47)).

By Newton's method it is now possible to find corrections for J_2 and K_2, designated by ΔJ_2 and ΔK_2. For this purpose the approximate values of

J and K, as given by Table 1 (Appendix II), are substituted in Equations (35) and (36), whereby the errors, Δ_1 and Δ_2, are obtained:

$$\Delta_1 = -4\,r^3(1-a) - a\,r\,(2+m-a) \qquad (58)$$

and,

$$\Delta_2 = +4\,r^2\,(1-a) - a\,(2+m-a) + 2\,r\,(1-(4+m)\,a+a^2) \qquad (59)$$

By partial differentiation of Equations (35) and (36), with respect to J_2 and K_2, the functions, f_1, f_2, and g_1, g_2, result,

$$\frac{\partial F}{\partial J_2} = f_1\,(J_2,\,K_2) \qquad (60)$$

$$\frac{\partial F}{\partial K_2} = f_2\,(J_2,\,K_2) \qquad (61)$$

$$\frac{\partial G}{\partial J_2} = g_1\,(J_2,\,K_2) \qquad (62)$$

and,

$$\frac{\partial G}{\partial K_2} = g_2\,(J_2,\,K_2) \qquad (63)$$

The approximate values for J_2 and K_2, as given by Equations (54) and (56), are now substituted in Equations (60) to (63), whereby the following equations of differences are obtained:

$$\Delta F_1 = \Delta J_2 \times f_1(r) \qquad (64)$$

$$\Delta F_2 = \Delta K_2 \times f_2(r) \qquad (65)$$

$$\Delta G_1 = \Delta J_2 \times g_1(r) \qquad (66)$$

and,

$$\Delta G_2 = \Delta K_2 \times g_2(r) \qquad (67)$$

in which, r is a shell constant and $f(r)$ and $g(r)$ are functions of r. The desired corrections, ΔJ_2 and ΔK_2, can then be determined from the two simultaneous equations:

$$\Delta F_1 + \Delta F_2 = \Delta J_2 \times f_1\,(r) + \Delta K_2 \times f_2\,(r) = -\Delta_1 \qquad (68)$$

and,

$$\Delta G_1 + \Delta G_2 = \Delta J_2 \times g_1\,(r) + \Delta K_2 \times g_2\,(r) = -\Delta_2 \qquad (69)$$

The same method is applied in order to find corrections for the approximate values, J'_2 and K'_2, in Equations (55) and (57). Neglecting the terms of a lower order (which is justified when $a < r$) the corrected values are:

$$J_2 = +r + \Delta J_2 = +r + \frac{(a-1)}{2} \qquad (70)$$

$$K_2 = -r + \Delta K_2 = -r \dots\dots (71)$$

$$J'_2 = -r + \Delta J'_2 = -r + \frac{(a-1)}{2} \dots\dots (72)$$

and,

$$K'_2 = -r + \Delta K'_2 = -r \dots\dots (73)$$

If necessary, additional corrections could be obtained by repeated applications of Newton's method. Since the general expressions for the second correction are rather complicated it is preferable to substitute the numerical values in individual cases. The final values then are substituted in Equations (37) to (44), in such a way that the entire group can be determined with any desired degree of accuracy. However, the more accurate solution of the problem, as carried through by Finsterwalder, cannot be stated in explicit form. In addition, the numerical computations must be repeated for each individual load term, c, so that, for a complete solution, the work is considerable.

It seems desirable, therefore, to obtain approximate solutions in simplified form, which are entirely sufficient in many cases, especially for the purpose of preliminary designs and estimates. If the shell is comparatively thin $\left(\frac{R}{t} > 100\right)$ and the relative span remains within the approximate limits, $1 < \frac{L}{R} < 10$, it can be shown that the correction, $\frac{(a-1)}{2}$ in Equations (70) and (72), does not exceed about $0.1r$. Under these limiting conditions it is then permissible to use the first approximation of the values, J and K, given in Table 1 (Appendix II), as a basis for an approximate method.

Consistent with the reasoning stated in arriving at the simplified differential equation (Equation (47)), the derivations of a lower order may also be neglected in the expressions for the stress components and deformations, Equations (14) to (28). With the limiting assumption, $a < 1$, it will be noted that the derivative of the highest order in each case also has the largest coefficient.

In order to obtain identical conditions in a longitudinal direction, only the stress components and deflections containing the factor, $\cos \frac{c\,x}{L}$, will be considered in the simultaneous elastic equations. For this reason the stress component, $\frac{\partial S}{\partial x}$ (Equation (17)), is introduced instead of S (Equation (18)); likewise, the stress component, T_1 (Equation (20)), is introduced instead of the deflection, u (Equation (22)). Assuming that continuity of stresses and deformations has been established on any point of a generatrix, the same condition then applies throughout.

The equations for the stress components and deflections are written in the general form,

$$(A\ a^* + B\ b^* + C\ c^* + D\ d^*)\cos\frac{c\,x}{L} \ldots\ldots\ldots\ldots (74)$$

in which, with reference to Equation (31), the constants, A, B, C, and D, must conform to four edge conditions, at $\phi = 0$, for example, four known values of the stress components, M_2, N_2, T_2, and $\frac{\partial S}{\partial x}$, or, the three deflections, v, w, θ, and the stress component, T_1.

The expressions, a^*, b^*, c^*, and d^*, are functions of the angle, ϕ, generally defined as follows,

$$a^* = e^{(J1)\phi}\ (\alpha \sin K_1\,\phi + \beta \cos K_1\,\phi) \ldots\ldots\ldots\ldots (75)$$

$$b^* = e^{(J1)\phi}\ (\alpha \cos K_1\,\phi - \beta \sin K_1\,\phi) \ldots\ldots\ldots\ldots (76)$$

$$c^* = e^{(J1)'\phi}\ (\gamma \sin K'_1\,\phi + \delta \cos K'_1\,\phi) \ldots\ldots\ldots\ldots (77)$$

and,

$$d^* = e^{(J1)'\phi}\ (\gamma \cos K'_1\,\phi - \delta \sin K'_1\,\phi) \ldots\ldots\ldots\ldots (78)$$

Based on the simplifying assumptions regarding the derivatives of lower order, in case, $a < 1$, the constants, α, β, γ, and δ, are derived from Equations (14), (15), (16), (17), (20), (26), (27), and (28), in connection with Table 1 (Appendix II). The value of these constants, corresponding to the various stress components and deflections, is shown in Table 2 (Appendix II), where,

$$k = \sqrt{2} - 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

INFLUENCE NUMBERS FOR EDGE DEFORMATIONS

The practical analysis of edge load stresses is greatly facilitated by the use of influence numbers for the edge deformations, corresponding to elementary, unit loads, herein defined by the general term of the trigonometric series, Equation (2). All deformations must be given for the center of the span, $x = 0$, where the intensity of the unit load assumes the value, $\frac{4}{c}$, measured per unit length.

The four fundamental edge loadings to be considered herein, consisting of $M_2 = \frac{4}{c}$; $N_2 = \frac{4}{c}$; $T_2 = \frac{4}{c}$; and $\frac{\partial S}{\partial x} = \frac{4}{c}$, are preferably divided in symmetrical and contra-symmetrical load cases, as shown in Fig. 4. ("Contra-symmetrical" denotes "equal but in opposite directions.") In both cases the same absolute load is acting at the edges, $\phi = 0$ and $\phi = \phi_k$, with the positive direction at $\phi = 0$. The contra-symmetrical loading consists in reversing the directions of the loads at $\phi = \phi_k$. In all cases the deformations are given for the left-hand edge, $\phi = 0$.

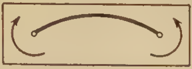
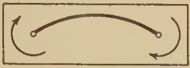
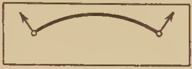
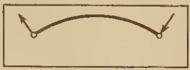
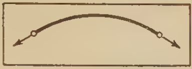
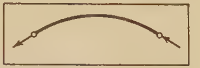
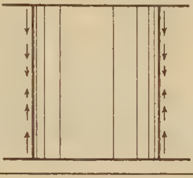
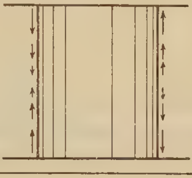
LOAD CONDITION					
M_2	N_2	T_2	$\frac{\partial s}{\partial x}$	Symmetrical	Contra-symmetrical
$\frac{4}{c}$	0	0	0		
0	$\frac{4}{c}$	0	0		
0	0	$\frac{4}{c}$	0		
0	0	0	$\frac{4}{c}$		

FIG. 4.

The stress components and deformations at any point of the shell sector, located by the angle, ϕ_1 (Fig. 5), are generally influenced by the stress waves from both the loaded edges. In Equations (75) to (78) the influence of the loads at $\phi = 0$ is found by means of the substitution, $\phi = \phi_1$. Likewise, the substitution, $\phi = \phi_2$, gives the influence of the loads acting at the edge, $\phi = \phi_k$. The algebraic sum of the two values represents the combined influence for the point, ϕ_1, and is designated by a^*_1, b^*_1, c^*_1, and d^*_1. Therefore,

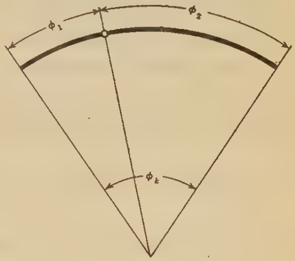


FIG. 5.

$$a^*_1 = a^*(\phi_1) \pm a^*(\phi_2) \tag{80}$$

$$b^*_1 = b^*(\phi_1) \pm b^*(\phi_2) \tag{81}$$

$$c^*_1 = c^*(\phi_1) \pm c^*(\phi_2) \tag{82}$$

and,

$$d^*_1 = d^*(\phi_1) \pm d^*(\phi_2) \tag{83}$$

With reference to Fig. 3(*a*), showing the positive directions of the stress components and deflections, it should be noted that, in case of symmetrical edge load conditions, the upper sign in Equations (80) to (83) applies to M_2, T_2, T_1, and w, and the lower sign to N_2, $\frac{\partial S}{\partial x}$, v, and θ. In case of anti-symmetrical loadings the corresponding signs must be reversed.

Considering the stress components and deflections on the left-hand edge, $\phi = 0$, the substitutions, $\phi_1 = 0$, $\phi_2 = \phi_k$, in Equations (80) to (83), furnish a group of values, designated by a_0, b_0, c_0, and d_0, which are shown in Table 3 (Appendix II) for the cases of symmetrical and anti-symmetrical edge loadings. The abbreviations of Table 3 are defined, as follows,

$$f = e^{(J_1)(\phi_k)} \sin K_1 \phi_k \quad \text{(84)}$$

$$g = 1 + e^{(J_1)(\phi_k)} \cos K_1 \phi_k \quad \text{(85)}$$

$$h = 1 - e^{(J_1)(\phi_k)} \cos K_1 \phi_k \quad \text{(86)}$$

$$f' = e^{(J_1)'(\phi_k)} \sin K'_1 \phi_k \quad \text{(87)}$$

$$g' = 1 + e^{(J_1)'(\phi_k)} \cos K'_1 \phi_k \quad \text{(88)}$$

and,

$$h' = 1 - e^{(J_1)'(\phi_k)} \cos K'_1 \phi_k \quad \text{(89)}$$

With reference to Table 1 (Appendix II) it will be noted that the edge values, a_0, b_0, c_0, and d_0, are mainly functions of the characteristic number, λ, defined as the shell ratio,

$$\lambda = \sqrt{r}\,\phi_k \quad \text{(90)}$$

The shell ratio contains all dimensions of the shell and expresses the law of similarity for the stress distribution in the case of edge loads.

With these preparations the values, A, B, C, and D in Equation (74), can be determined for any desired combination of four edge conditions. The present derivation is limited to the fundamental loadings, as shown in Fig. 4.

For instance, consider the case of a symmetrical unit load, $M_2 = \dfrac{4}{c}$, acting on both edges in combination with the specification, $N_2 = T_2 = \dfrac{\partial S}{\partial x} = 0$.

From Table 3(a) (Appendix II) (by using the subscripts, 1, 2, 3, etc., in reference to the edge values of M_2, N_2, T_2, etc.), the following equations can be written:

$$M_2 = A\,a_{01} + B\,b_{01} + C\,c_{01} + D\,d_{01} = \frac{4}{c} \quad \text{(91)}$$

$$N_2 = A\,a_{02} + B\,b_{02} + C\,c_{02} + D\,d_{02} = 0 \quad \text{(92)}$$

$$T_2 = A\,a_{03} + B\,b_{03} + C\,c_{03} + D\,d_{03} = 0 \quad \text{(93)}$$

and,

$$\frac{\partial S}{\partial x} = A\,a_{04} + B\,b_{04} + C\,c_{04} + D\,d_{04} = 0 \quad \text{(94)}$$

The solution of these four simultaneous equations furnishes the unknowns, A, B, C, and D, which can be stated in explicit form for any given numerical value of the shell ratio, λ.

Similar groups of equations are obtained for the unit loads, $N_2 = \frac{4}{c}$, $T_2 = \frac{4}{c}$, and $\frac{\partial S}{\partial x} = \frac{4}{c}$. In this manner Tables 4, 5, 6, and 7 (Appendix II) have been computed for symmetrical and contra-symmetrical unit loadings, using $\frac{K_1 \phi_k}{\pi}$ as argument, instead of λ.

As the argument increases, the functions, A, B, C, and D, approach constant values, which are also given. This condition is reached when the loads at the edge, $\phi = \phi_k$, do not influence the stresses and deformations of the opposite edge, $\phi = 0$ as, for instance, in the case of a very thin shell, combined with a large central angle or a short span.

The edge deflections, v, w, θ, and the edge stress component, T_1, are obtained from the general Equation (74), by substituting the values, A, B, C, and D, corresponding to a given loading, and the values, a_0, b_0, c_0, and d_0, corresponding to a desired deflection or stress component. For instance, consider again the case of a symmetrical unit loading, $M_2 = \frac{4}{c}$, for which the values, A, B, C, and D, are given by Table 4(a) (Appendix II). From Table 3(a) (Appendix II) with the subscripts previously defined, Equation (74) then furnishes the following expressions for the stress component, T_1, and the deflections at the center of the span, $x = 0$, defined as the influence numbers for M_2,

$$T_1 = A\, a_{O5} + B\, b_{O5} + C\, c_{O6} + D\, d_{O5} \quad \ldots\ldots\ldots\ldots (95)$$

$$v = A\, a_{O6} + B\, b_{O6} + C\, c_{O6} + D\, d_{O6} \quad \ldots\ldots\ldots\ldots (96)$$

$$w = A\, a_{O7} + B\, b_{O7} + C\, c_{O7} + D\, d_{O7} \quad \ldots\ldots\ldots\ldots (97)$$

and,

$$\theta = A\, a_{O8} + B\, b_{O8} + C\, c_{O8} + D\, d_{O8} \quad \ldots\ldots\ldots\ldots (98)$$

Similar expressions are obtained for the other unit loadings. The influence numbers, as shown in Table 8 (Appendix II), can be expressed in explicit form, except for a numerical coefficient, which is a function of the shell ratio only.

It is interesting to note that the influence numbers for the deflections, θ, w, and v, are symmetrical about the diagonal through the top left corner. The same symmetry applies to the coefficients, c_{21}, c_{12}, c_{31}, c_{13}, etc., throughout the diagonal, because the computations show that $c_{12} = c_{21}$, $c_{13} = c_{31}$, etc., in accordance with Maxwell's theorem of reciprocal relations between loads and deflections. Since T_1 represents a stress component, instead of a deflection, the symmetrical relations apply to the coefficients only, not to the influence numbers for T_1 as a whole.

Table 9 (Appendix II) gives the values of the influence number coefficients for symmetrical and contra-symmetrical loading, with $\frac{K_1 \phi_k}{\pi}$ as the argument. These coefficients, plotted on logarithmic paper, are also shown in Fig. 6. It will again be noted that, for large values of the shell ratio, the coefficients approach constant values, indicating that the deflections of the edge, $\phi = 0$, become independent from the loads acting at the opposite edge, $\phi = \phi_k$. For medium values of the shell ratio the coefficients have a pronounced damped-wave shape, caused by the interference of stress waves originating from the loaded edges, which indicates typical shell action.

For smaller values of the shell ratio the coefficients plot on straight lines, indicating that plate action, rather than shell action, prevails in this region. In order to illustrate this point, consider a comparatively long shell having a small central angle, ϕ_k, and comparatively thick walls. The behavior of such a shell approaches that of a narrow, flat plate, spanning between the end stiffeners. Assume that the shell is loaded by symmetrical edge moments, M_2, which loading represents an equilibrium system in a transverse plane. Since the shell is of such a shape that the restraining action of the transverse end stiffeners becomes negligible, it is evident that the cross-section of the shell can be considered as a slightly curved beam, loaded by two equal end moments. The angular deflection, θ, can then be computed by the usual beam theory. A comparison with the values obtained from the influence number of the shell shows close agreement in this case. The agreement is also fairly good for the deflections, w and v. The longitudinal stress component, T_1, becomes negligible, which is also in accordance with plate action. Of course, the reason for the satisfactory agreement under this extreme condition is due to the fact that the transverse moment, M_2, is included in the stress components acting on the small element, Fig. 3(a).

On the other hand, consider the same shell loaded by a system of antisymmetrical edge moments, M_2. The behavior of the narrow shell strip then approaches that of a plate or a rod, held at both ends, and subjected to torsional moments distributed over the entire length. Evidently, the torsional deflections will govern entirely in this case. However, since the torsional moments, acting on a small shell element, have been neglected in the derivation of the differential equation (Equation (29)), it cannot be expected that the influence numbers (Table 8 (Appendix II)) would give even a rough approximation in this case of extreme conditions. The actual deflections will be much smaller than those given by the influence numbers, because the torsional work comes into play.

Similar considerations apply to other unit loadings when the central angle becomes rather small. For instance, a symmetrical load system, N_2, would then induce comparatively large bending moments, M_1, in a longitudinal direction, which have also been neglected. For these reasons the influence

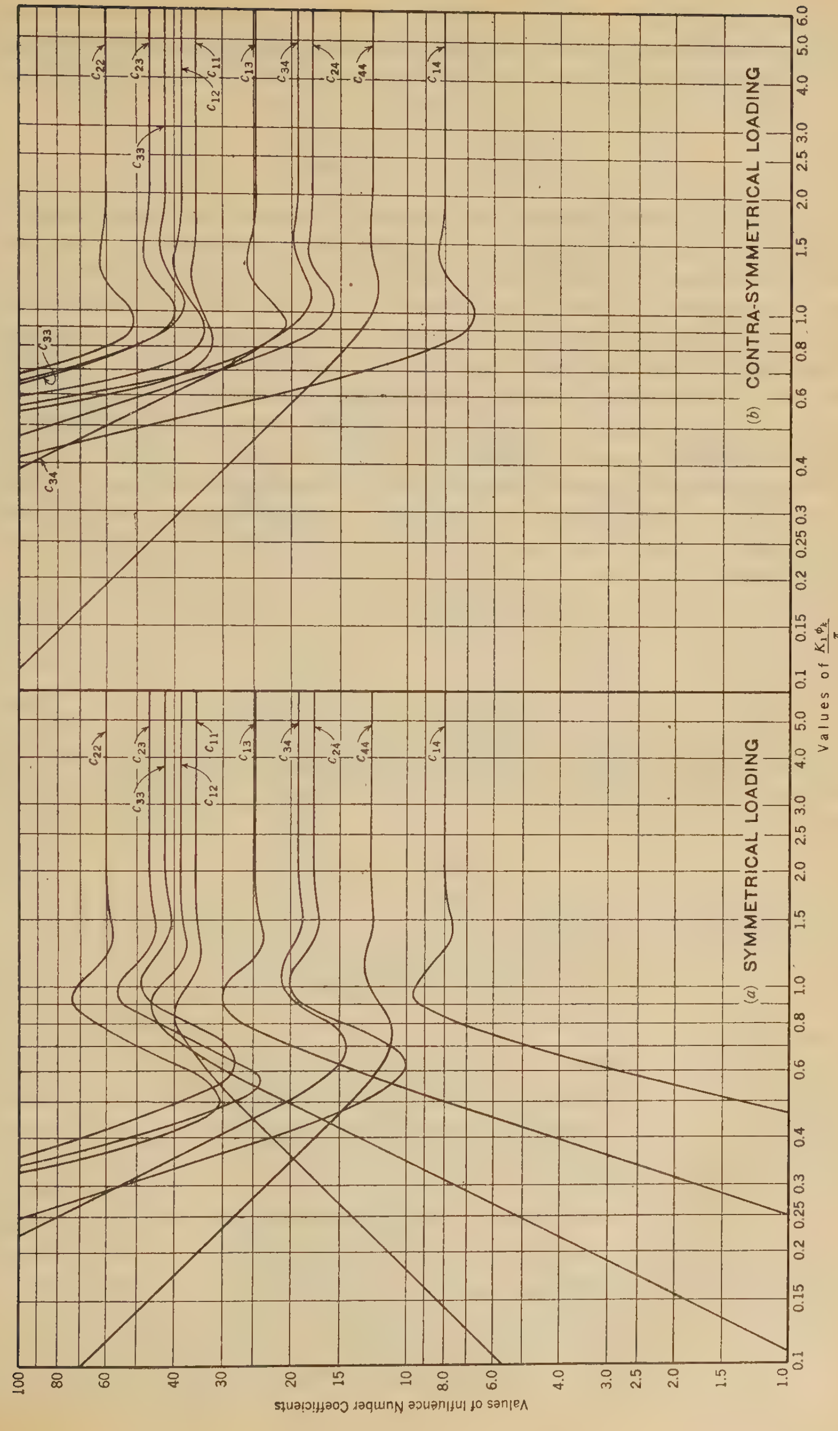


FIG. 6.—INFLUENCE NUMBER COEFFICIENTS.

numbers should not be used under such conditions, where the fundamental assumptions become entirely inadequate. It should be pointed out that the same exceptions also apply to Finsterwalder's more accurate solution of the differential equation (Equation (29)).

THE MEMBRANE SYSTEM

The theory of membrane stresses is based on the assumption that continuous surface loads are resisted by direct stresses only. Referring to Fig. 3(*a*), the stress components acting on a small element then consist of the direct stress components, T_1 and T_2, together with the shearing stress component, S, all acting in the plane of the element. The external load components, designated by $X_{x\phi}$, $Y_{x\phi}$, and $Z_{x\phi}$, with the positive direction as shown in Fig. 3(*a*), are generally expressed as functions of the angle, ϕ, and the coordinate, x. The longitudinal load distribution is assumed to be symmetrical about the middle plane, $x = 0$, and stated in the form of the general term of the trigonometric series, Equation (2). Assuming that $X_{x\phi} = 0$, and designating by $Y(\phi)$, $Z(\phi)$, the functions for the transverse load distribution, then, in case of uniform load distribution along a generatrix, from Equation (1),

$$Y_{x\phi} = Y(\phi) \times p\ (x) = Y\ p \qquad (99)$$

and,

$$Z_{x\phi} = Z(\phi) \times p\ (x) = Z\ p \qquad (100)$$

With reference to Fig. 3(*a*), the stress components, T_1, T_2, S, are obtained from the equilibrium conditions on a small element. By projection on the z-axis:

$$T_2 = -\ R\ Z\ p \qquad (101)$$

and by projection on the y-axis:

$$\frac{\partial S}{\partial x} = \left(-\ Y + \frac{\partial Z}{\partial \phi}\right) p \qquad (102)$$

Therefore, also,

$$S = \left(-\ Y + \frac{\partial Z}{\partial \phi}\right)\int_0^x p\ dx \qquad (103)$$

By projection on the x-axis:

$$\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial s} = 0 \qquad (104)$$

and by integration, from Equations (4) and (103):

$$T_1 = \frac{1}{R}\left(-\frac{\partial Y}{\partial \phi} + \frac{\partial^2 Z}{\partial \phi^2}\right) M \qquad (105)$$

The membrane stress deflections are obtained by substituting these stress components in Equations (8), (9), and (10). From Equations (8), (101), and (102), by partial integration,

$$E\,t\,u = \frac{1}{R}\left(-\frac{\partial Y}{\partial \phi} + \frac{\partial^2 Z}{\partial \phi^2}\right)\int_{-0.5\,L}^{x} M\,dx + m\,R\,Z\int_{-0.5\,L}^{x} p\,dx \ldots .(106)$$

From Equation (10), by partial differentiation:

$$\frac{\partial^2 v}{\partial x^2} = \frac{2\,(1+m)}{E\,t}\,\frac{\partial S}{\partial x} - \frac{\partial^2 u}{\partial s\,\partial x} \ldots\ldots\ldots\ldots\ldots .(107)$$

Substituting, $\frac{\partial S}{\partial x}$, from Equation (102), and, with the partial derivative from Equation (8), in connection with Equations (101) and (105),

$$E\,t\frac{\partial^2 v}{\partial x^2} = 2\,p\left\{-(1+m)\,Y + \left(1+\frac{m}{2}\right)\frac{\partial Z}{\partial \phi}\right\} - \frac{M}{R^2}\left\{-\frac{\partial^2 Y}{\partial \phi^2} + \frac{\partial^3 Z}{\partial \phi^3}\right\} ..(108)$$

By integration, in connection with Equations (4) and (6),

$$E\,t\,v = 2\,M\left\{(1+m)\,Y - \left(1+\frac{m}{2}\right)\frac{\partial Z}{\partial \phi}\right\} + \frac{\eta}{R^2}\left\{-\frac{\partial^2 Y}{\partial \phi^2} + \frac{\partial^3 Z}{\partial \phi^3}\right\} ..(109)$$

From Equation (9,) in connection with Equations (101), (105), and (109),

$$E\,t\,w = R^2 Z\,p + 2\,M\left\{\left(1+\frac{m}{2}\right)\frac{\partial Y}{\partial \phi} - \frac{\partial^2 Z}{\partial \phi^2}\right\} + \frac{\eta}{R^2}\left\{-\frac{\partial^3 Y}{\partial \phi^3} + \frac{\partial^4 Z}{\partial \phi^4}\right\} ..(110)$$

The angular deflection of the transverse tangent is negligible in the case of membrane stresses; therefore, $\theta = 0$.

EXAMPLE

Consider the partial filling of a simply supported, horizontal pipe line or liquid storage tank, provided with adequate transverse stiffening members over both end supports. Neglecting the weight of the shell, the surface load then consists of the liquid pressure, acting continuously over the wetted perimeter of the shell.

At the liquid level the surface load becomes discontinuous, which condition can be analyzed when the shell is cut along this plane and the membrane stress equations are applied to the loaded shell sector. It will be found that a direct stress component, T_1, and a longitudinal shearing stress component, S, are acting on the free edges of the lower shell. The latter force must be applied as a reaction on the edges of the upper shell, where it induces edge load stresses and deformations. The lower shell is deformed by membrane stresses and the combined effect will be a discrepancy in the geometrical continuity of the two shell sectors.

The membrane stress component, S, representing the main disturbing force, reaches a maximum value when the pipe is exactly half full. For this reason the following derivations will be limited to this loading, which is of practical importance, although any other filling may be analyzed in a similar manner. Referring to Fig. 7, and designating the unit weight of the liquid by q, the external load components are,

$$X = Y = 0 \quad \text{(111)}$$

and,

$$Z = -\,q\,R\,\sin\phi \quad \text{(112)}$$

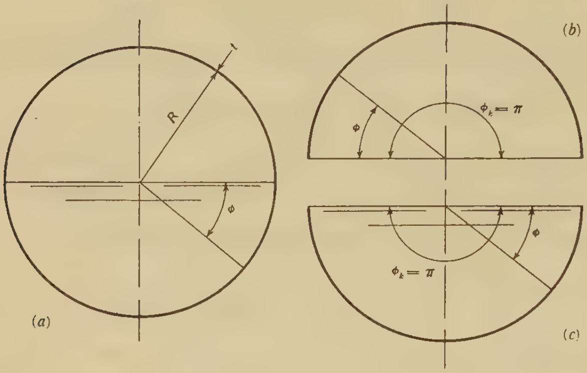


Fig. 7.

The membrane stress components and deformations at $x = 0$, $\phi = 0$, are obtained from Equations (101), (102), (105), (109), and (110):

$$T_2 = 0 \quad \text{(113)}$$

$$\frac{\partial S}{\partial x} = -\,\frac{4}{c}\,q\,R \quad \text{(114)}$$

$$T_1 = 0 \quad \text{(115)}$$

$$E\,t\,v = +\,\frac{4}{c}\left\{2\left(1 + \frac{m}{2}\right) + \frac{1}{a}\right\}\frac{q\,R^3}{a} \quad \text{(116)}$$

$$E\,t\,w = 0 \quad \text{(117)}$$

and, since the angular deflection is negligible,

$$\theta = 0 \quad \text{(118)}$$

Since $T_2 = 0$, the membrane stress component, $\dfrac{\partial S}{\partial x}$, Equation (114), represents the only reaction that must be applied as a symmetrical edge load on the upper shell. The corresponding deflections and the longitudinal edge stress of the upper shell are obtained from the influence numbers, Table 8

(Appendix II), which already contain the term, $\frac{4}{c}$, as a unit load. Therefore:

$$E\ t\ \theta = +\ c_{14}\ \frac{r^2}{c\ a^2}\ q\ R^2 \ldots\ldots\ldots\ldots\ldots\ldots (119)$$

$$E\ t\ w = -\ c_{24}\ \frac{r^{1.5}}{c\ a^2}\ q\ R^3 \ldots\ldots\ldots\ldots\ldots\ldots (120)$$

$$E\ t\ v = +\ c_{34}\ \frac{r}{c\ a^2}\ q\ R^3 \ldots\ldots\ldots\ldots\ldots\ldots (121)$$

and,

$$T_1 = -\ c_{44}\ \frac{r^{0.5}}{c\ a}\ q\ R^2 \ldots\ldots\ldots\ldots\ldots\ldots (122)$$

A comparison of the deflections, v, in Equations (121) and (116), indicates that the latter is relatively small in the case of thin pipes and, therefore, will be neglected.

A symmetric equilibrium system of unit edge loads, $M_2 = \frac{4}{c}$, $N_2 = \frac{4}{c}$, $T_2 = \frac{4}{c}$, and $\frac{\partial S}{\partial x} = \frac{4}{c}$, is now applied at the edges of both shells. The corresponding deformations, as given by Table 8, are shown in Table 10 (Appendix II), together with the deformations of the membrane stress system, Equations (116) to (122). Table 10 also gives the algebraic differences between the deflections of the upper and lower shell, including the differences between the longitudinal edge stresses. The continuity between the two shells is restored when the differential deformations and stresses are eliminated.

For this purpose four unknown edge force components, designated by $M_2 = X_1$, $N_2 = X_2$, $T_2 = X_3$, and $\frac{\partial S}{\partial x} = X_4$, are applied, so that the algebraic sum of the difference, Δ, becomes zero. Table 10 then gives the following elastic equations for the determination of the unknowns,

$$+\ X_1\, c_{11}\, r^{1.5} -\ X_3\, c_{13}\, r^{0.5}\, R + c_{14}\ \frac{q\ R^3}{2} = 0 \ldots\ldots\ldots\ldots (123)$$

$$-\ X_2\, c_{22}\, r -\ X_4\, c_{24}\, R - c_{24}\ \frac{q\ R^2}{2} = 0 \ldots\ldots\ldots\ldots (124)$$

$$+\ X_1\, c_{31}\, r^{1.5} -\ X_3\, c_{33}\, r^{0.5}\, R + c_{34}\ \frac{q\ R^3}{2} = 0 \ldots\ldots\ldots\ldots (125)$$

and,

$$-\ X_2\, c_{42}\, r -\ X_4\, c_{44}\, R - c_{44}\ \frac{q\ R^2}{2} = 0 \ldots\ldots\ldots\ldots (126)$$

The solution of Equations (123) to (126) gives the following values for the unknowns,

$$X_1 = C_2 \frac{q\,R^3}{2\,r^{1.5}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (127)$$

$$X_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (128)$$

$$X_3 = C_1 \frac{q\,R^2}{2\,r^{0.5}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (129)$$

and,

$$X_4 = -\frac{q\,R}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (130)$$

The coefficients, C_1 and C_2 are defined by the following equations,

$$C_1 = \frac{c_{13}\,c_{14} - c_{11}\,c_{34}}{c^2_{13} - c_{11}\,c_{33}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (131)$$

and,

$$C_2 = C_1 \frac{c_{33}}{c_{13}} - \frac{c_{34}}{c_{13}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (132)$$

The unit stresses at the liquid level, in the center of the span, may be determined by considering the stress condition of the upper shell, which is subject to edge loads only.

The total longitudinal edge stress component consists of the individual components due to the edge loads, X_1 to X_4, and the component of the membrane stress system. The latter is given by Equation (122) and the components caused by X_1, X_3, and X_4, are obtained from the influence numbers, Table 8 (Appendix II). Therefore, the resulting longitudinal edge stress component, designated by T_{1r}, becomes,

$$T_{1r} = -X_1 \frac{c_{41}\,r^2}{R\,c\,a} + X_3 \frac{c_{43}\,r}{c\,a} - X_4 \frac{c_{44}\,r^{0.5}\,R}{c\,a} - c_{44} \frac{r^{0.5}}{c\,a} q\,R^2 \ldots (133)$$

By substituting the corresponding values from Equations (127), (129), and (130), the following expression is obtained,

$$T_{1r} = -C_3 \frac{q\,R^2\,r^{0.5}}{2\,a\,c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (134)$$

in which,

$$C_3 = c_{44} - C_1\,c_{34} + C_2\,c_{14} \quad \ldots\ldots\ldots\ldots\ldots\ldots (135)$$

The unit stresses, corresponding to the stress components, X_1 X_3 and T_{1r}, in Equations (127), (129), and (134), are designated by σ_{M2}, σ_{T2}, and σ_{T1}, respectively, hence:

$$\sigma_{M2} = \pm \frac{6\,X_1}{t^2} = \pm C_2 \frac{3\,q\,R^3}{t^2\,r^{1.5}} \quad \ldots\ldots\ldots\ldots\ldots (136)$$

$$\sigma_{T2} = C_1 \frac{q\,R^2}{2\,t\,r^{0.5}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (137)$$

and,

$$\sigma_{T1} = -C_3 \frac{q\,R^2\,r^{0.5}}{2\,a\,c\,t} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(138)$$

The value of σ_{T2}, being positive, is a tensile stress; σ_{T1} is a compressive stress. The tangential stress, σ_{T2}, is relatively small. However, it is interesting to compare the transverse bending stress, σ_{M2}, and the longitudinal stress σ_{T1}, with the maximum beam stress of a simply supported, full pipe, designated by σ_B, which is,

$$\sigma_B = \frac{q\;L^2}{8\,t} = \frac{4}{c}\,\frac{q\;L^2}{c^2\,t} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(139)$$

Substituting r, from Equation (51), and using a value, $m^2 = 0.10$, for Poisson's ratio:

$$\frac{\sigma_{M2}}{\sigma_B} = 15.9\;C_2\,\sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}}\,\frac{1}{c^{1.5}} \quad\ldots\ldots\ldots\ldots\ldots(140)$$

and,

$$\frac{\sigma_{T1}}{\sigma_B} = 4.54\;C_3\,\sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}}\,\frac{1}{c^{2.5}} \quad\ldots\ldots\ldots\ldots\ldots(141)$$

Substituting C_2 and C_3, from Equations (132) and (135), respectively, it will be noted that the influence number coefficients are given as functions of $\frac{K_1\,\phi_k}{\pi}$, in Table 9(*a*) (Appendix II), or Fig. 6(*a*). Since, in the present example, $\phi_k = \pi$, and K_1, from Table 1 (Appendix II), is also a function of r, it will be possible, therefore, to write Equations (140) and (141) in the following form,

$$\frac{\sigma_{M2}}{\sigma_B} = C_{M2}\,\sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(142)$$

and,

$$\frac{\sigma_{T1}}{\sigma_B} = C_{T1}\,\sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(143)$$

The values of the coefficients, C_{M2} and C_{T1}, have been computed for the first term of the Fourier series, by substituting, $c = \pi$, and are shown in Fig. 8. As already stated, the consideration of additional terms of the series does not greatly influence the final result. Due to the approximations made in the derivation of the influence numbers it is doubtful whether the accuracy would be increased thereby.

By an entirely different method, Samsioe[12] has analyzed the same problem for a pipe half full, but with fixed ends, instead of the simply supported conditions assumed in the present example. Although the results for this reason are not directly comparable, the corresponding coefficients, derived from Samsioe's analysis, are also shown in Fig. 8.

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[12] "Die Spannungen in einem auf mehreren Stützen in gleicher gegenseitiger Entfernung aufgelegten und zur Hälfte mit Wasser gefüllten Rohr," von A. Frey Samsioe, pub. by Svenska Bokhandelscentralen, Stockholm, Sweden, 1926.

It will be noted that the maximum stresses due to the discontinuous surface load are of similar magnitude in both cases, except that the limitations of the present method are indicated in case of small values of $\sqrt{\frac{R}{L}\left(\frac{R}{t}\right)^{\frac{1}{2}}}$.

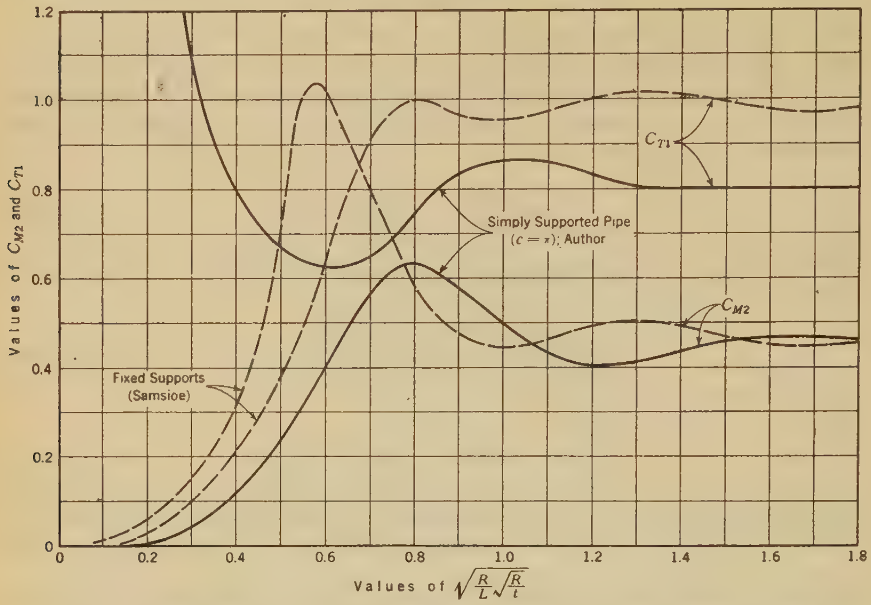

FIG. 8.—COEFFICIENT C FOR A PIPE HALF FULL OF WATER, WITH SIMPLE SUPPORTS AND FIXED SUPPORTS.

CONCLUSION

Thin shell structures will find an increasingly large field of useful application because of their economic advantages and their adaptability to architectural treatment in wide span construction. For practical purposes the analysis of such structures must be simplified so as to enable the designer to obtain a reasonably accurate stress picture in spite of the fact that exact solutions are not yet available on account of mathematical difficulties.

The proposed method is an approximation, but the comparison of the results with more accurate derivations indicates a satisfactory agreement, provided the given limitations are kept in mind.

APPENDIX I

NOTATION

The symbols introduced in this paper are defined where they first appear; and for the convenience of discussers the complete notation, in alphabetical order, is listed herein, as follows:

a = a factor, $\left(\frac{c\,R}{L}\right)^2$; also, a^*, b^*, c^*, and d^* = functions of the angle, ϕ; and, a_0, b_0, c_0, and d_0 = edge values for $\phi = 0$;

b = a shell constant $= \frac{R}{t}\sqrt{3}$; also, for b^* and b_0, refer to Symbol a;

c = substitutions corresponding to the first, second, third, etc., terms of a Fourier series; c_{11}, c_{12}, c_{13}, etc. = influence number coefficients for edge deflections; for c^* and c_0 refer to Symbol a;

d^* = function of the angle, ϕ (see Symbol a).

e = base of natural logarithms; also e = elongation: e_1, longitudinal, and e_2, transverse;

f = "function of": $f(\phi)$ = Finsterwalder's stress function for a cylindrical shell sector; f, g, and h = functions of J_1, K_1, and ϕ_k; f', g', and h' = functions of J', K', and ϕ'_k;

g = "function of": (for g and g', refer to Symbol f);

h = "function of": (for h and h', refer to Symbol f);

k = a constant $= \sqrt{2} - 1$;

m = Poisson's ratio;

n = a constant: $n_1 = (\sqrt{0.5} + 0.5)^{\frac{1}{2}}$ and $n_2 = (\sqrt{0.5} - 0.5)^{\frac{1}{2}}$;

p = load intensity per unit length of a simply supported beam; $p(x)$ = sum of a trigonometric series; p_c = general term of series;

q = unit weight of liquid;

r = a shell constant $= \sqrt{a\,b}\,\sqrt[4]{1 - m^2}$;

s = length of a circular arc corresponding to a central angle, ϕ; ds = length of a small element of arc, s;

t = thickness of shell; as a subscript, t, denotes "torsional";

u = longitudinal deflection of a shell element, positive in the direction of increasing values of x;

v = tangential deflection of a shell element, positive in the direction of increasing values of ϕ;

w = radial deflection of a shell element, positive in a direction toward the center of curvature;

x = a distance measured longitudinally along the arch barrel, from the center of the span; dx = an elementary small distance, x;

A = a constant; A, B, C, and D = constants conforming to given edge conditions of the shell;

B = a constant (see Symbol A); as a subscript, B, denotes "beam";

C = a constant (see Symbol A); C_1, C_2, and C_3 = functions of the influence number coefficients referred to in the example in this paper;

D = a constant (see Symbol A);

E = modulus of elasticity;

F = a function; F and G = functions determining J and K;

G = a function (see Symbol F);

I = moment of inertia of a beam section;

J = a shell constant; J_1 = a damping factor (as an exponent, written simply, $(J1)$); J'_1 = a damping factor of stress functions (as an exponent, written, $(J1)'$);

K = a shell constant; K_1 and K'_1 = wave length factors of stress function; K_n = a progressive function of K_1;

L = length; span of a simply supported beam or cylindrical shell;

M = moment; bending moment of a simply supported, uniformly loaded beam at the distance, x; M_0 = bending moment at the center of the span; M_1 = longitudinal bending stress component on a shell element per unit width (positive when tension is produced on the inner surface); M_2 = transverse bending moment corresponding to M_1; M_t = torsional moment acting on a shell element per unit width (positive when producing tension on the outer surface, in a diagonal direction of increasing values of x and ϕ); M_c = general term of a series;

N = radial shearing stress component per unit width (positive when acting in an outward direction on the sides facing the origin of x and ϕ);

R = radius of circular shell; radius of curvature of a cylinder element;

S = direct shearing stress component on a shell element per unit width, acting in the plane of the element (positive when producing tension in a diagonal direction of increasing values of x and ϕ);

T = direct stress component on a shell element per unit width; T_1, in a longitudinal direction, and T_2 in a tangential, or transverse, direction; T_{1r} = resulting longitudinal edge stress component in the design example of the paper;

X = force component: $X_{x\phi}$, $Y_{x\phi}$, and $Z_{x\phi}$ = external load components on a shell element per unit width; X_1, X_2, X_3, and X_4 = unknown edge force components in the design example of the paper;

Y = force component (see Symbol X);

Z = force component (see Symbol X);

α = a constant in Table 2 (Appendix II);

β = a constant in Table 2 (Appendix II);

γ = a constant in Table 2 (Appendix II);

δ = a constant in Table 2 (Appendix II);

Δ = a correction, or difference; ΔJ_2 = correction for J_2; ΔK_2 = correction for K_2; Δ_1 = errors in values of J; Δ_2 = errors in values of K;

η = deflection of a simply supported, uniformly loaded beam, multiplied by EI; η_0 = η at the center of the span; η_c = general term of a series;

θ = angular deflection of the transverse tangent of a shell element (positive when the tangent rotates in a clockwise direction);

κ = change of curvature of a shell element: κ_1 = longitudinal, and κ_2 = transverse;

λ = a typical shell constant defined as a ratio $= \sqrt{\;} \, r \, \phi_k$:

σ = unit stress; σ_B = maximum beam stress in a simply supported full pipe; σ_{M2}, σ_{T2}, and σ_{T1} = unit stresses corresponding to stress components, X_1, X_3, and T_{1r}, respectively;

ϕ = angular distance of a shell element from the left-end edge; $d\,\phi$ = angular width of a small shell element; ϕ_k = total central angle of a cylindrical shell sector; $f(\phi)$ = Finsterwalder's stress function for a cylindrical shell sector;

ω = angular detrusion of a shell element, caused by the shearing forces, S (positive when S is positive).

APPENDIX II

TABLES

The ten tables in Appendix II are introduced and explained in the text of the paper.

TABLE 1.—APPROXIMATE VALUES OF J, K, J', AND K'

Values	Indices							
	1	2	3	4	5	6	7	8
J	$-n_1 r^{0.5}$	$+r$	$-(n_1 - n_2) r^{1.5}$	0	$+2 n_2 r^{2.5}$	$-2 r^3$	$2 (n_1 + n_2) r^{3.5}$	$-4 r^4$
K	$+n_2 r^{0.5}$	$-r$	$+(n_1 + n_2) r^{1.5}$	$+2 r^2$	$+2 n_1 r^{2.5}$	$-2 r^3$	$2 (n_1 - n_2) r^{3.5}$	0
J'	$-n_2 r^{0.5}$	$-r$	$+(n_1 + n_2) r^{1.5}$	0	$-2 n_1 r^{2.5}$	$+2 r^3$	$2 (n_1 - n_2) r^{3.5}$	$-4 r^4$
K'	$+n_1 r^{0.5}$	$-r$	$-(n_1 - n_2) r^{1.5}$	$+2 r^2$	$-2 n_2 r^{2.5}$	$-2 r^3$	$2 (n_1 + n_2) r^{3.5}$	0

TABLE 2.—CONSTANTS α, β, γ, AND δ

Stress component or deflection (1)	α (2)	β (3)	γ (4)	δ (5)	Common factor (6)	Stress component or deflection (1)	α (2)	β (3)	γ (4)	δ (5)	Common factor (6)
M_2	-1	0	-1	0	1	T_1	0	$+1$	0	-1	$\frac{2 r^2}{a R}$
N_2	$+1$	$-k$	$+k$	-1	$\frac{n_1 r^{0.5}}{R}$	v	$-k$	-1	$+1$	$+k$	$\frac{2 n_1 r^{2.5}}{E t a^2}$
T_2	$+1$	-1	-1	-1	$\frac{r}{R}$	w	$+1$	$+1$	-1	$+1$	$\frac{2 r^3}{E t a^2}$
$\frac{\partial S}{\partial x}$	$+k$	-1	-1	$+k$	$\frac{(n_1 + n_2) r^{1.5}}{k^2}$	θ	-1	$-k$	$-k$	-1	$\frac{2 (n_1 + n_2) r^{3.5}}{E t R a^2}$

TABLE 3.—EDGE VALUES, a_0, b_0, c_0, and d_0 (AT THE EDGE, $\phi = 0$)

Stress component or deflection	a_0	b_0	c_0	d_0	Common factor
(*a*) SYMMETRICAL LOADING					
M_2.	$-f$	$-g$	$-f'$	$-g'$	1
N_2.	$-(k\,h+f)$	$+(h-k\,f)$	$-(h'+k\,f')$	$+(k\,h'-f')$	$\dfrac{n_1\,r^{0.5}}{R}$
T_2.	$-(g-f)$	$+(g+f)$	$-(g'+f')$	$-(g'-f')$	$\dfrac{r}{R}$
$\dfrac{\partial S}{\partial x}$	$-(h+k\,f)$	$+(k\,h-f)$	$+(k\,h'+f')$	$-(h'-k\,f')$	$\dfrac{(n_1+n_2)\,r^{1.5}}{R^2}$
T_1.	$+g$	$-f$	$-g'$	$+f'$	$\dfrac{2\,r^2}{a\,R}$
v.	$-(h-k\,f)$	$-(k\,h+f)$	$+(k\,h'-f')$	$+(h'+k\,f')$	$\dfrac{2\,n_1\,r^{2.5}}{E\,t\,a^2}$
w	$+(g+f)$	$+(g-f)$	$+(g'-f')$	$-(g'+f')$	$\dfrac{2\,r^3}{E\,t\,a^2}$
θ	$-(k\,h-f)$	$-(h+k\,f)$	$-(h'-k\,f')$	$-(k\,h'+f')$	$\dfrac{2\,(n_1+n_2)\,r^{3.5}}{E\,t\,R\,a^2}$
(*b*) CONTRA-SYMMETRICAL LOADING					
(1) M_2. . . .	$+f$	$-h$	$+f'$	$-h'$	1
(2) N_2.	$-(k\,g-f)$	$+(g+k\,f)$	$-(g'-k\,f')$	$+(k\,g'+f')$	$\dfrac{n_1\,r^{0.5}}{R}$
(3) T_2.	$-(h+f)$	$+(h-f)$	$-(h'-f')$	$-(h'+f')$	$\dfrac{r}{R}$
(4) $\dfrac{\partial S}{\partial x}$ . . .	$-(g-k\,f)$	$+(k\,g+f)$	$+(k\,g'-f')$	$-(g'+k\,f')$	$\dfrac{(n_1+n_2)\,r^{1.5}}{R^2}$
(5) T_1.	$+h$	$+f$	$-h'$	$-f'$	$\dfrac{2\,r^2}{a\,R}$
(6) v.	$-(g+k\,f)$	$-(k\,g-f)$	$+(k\,g'+f')$	$+(g'-k\,f')$	$\dfrac{2\,n_1\,r^{2.5}}{E\,t\,a^2}$
(7) w.	$+(h-f)$	$+(h+f)$	$+(h'+f')$	$-(h'-f')$	$\dfrac{2\,r^3}{E\,t\,a^2}$
(8) θ.	$-(k\,g+f)$	$-(g-k\,f)$	$-(g'+k\,f')$	$-(k\,g'-f')$	$\dfrac{2\,(n_1+n_2)\,r^{3.5}}{E\,t\,R\,a^2}$

TABLE 4.—CONSTANTS A, B, C, AND D $\left(M_2 = \frac{4}{c},\ N_2 = T_2 = \frac{\partial S}{\partial x} = 0\right)$

$\frac{K_1 \phi_k}{\pi}$	A_c	B_c	C_c	D_c
	(a) SYMMETRICAL LOADING			
0.000	.....	.....	.....	.....
0.125	—0.289	—1.559	—0.548	—1.104
0.250	—0.817	—2.306	—1.217	—0.978
0.375	—1.681	—3.272	—1.953	—0.550
0.500	—2.978	—4.474	—2.679	+0.245
0.625	—4.748	—5.867	—3.273	+1.421
0.750	—6.703	—7.167	—3.533	+2.762
0.875	—7.940	—7.768	—3.319	+3.619
1.000	—7.821	—7.453	—2.901	+3.529
1.125	—7.096	—6.901	—2.685	+3.017
1.250	—6.634	—6.641	—2.703	+2.690
1.375	....	....	....	....
1.500	—6.712	—6.777	—2.855	+2.744
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	—6.828	—6.828	—2.828	+2.828
	(b) CONTRA-SYMMETRICAL LOADING			
0.000	....	....	....	....
0.125	—186016	—289792	—270220	+16907
0.250	—2054	—2371	—1854	+957
0.375	—199.8	—184.7	—109.2	+120.5
0.500	—41.99	—33.79	—13.43	+26.20
0.625	—14.30	—11.51	—3.333	+7.936
0.750	—7.570	—6.946	—2.287	+3.327
0.875	—5.996	—6.154	—2.510	+2.237
1.000	—6.072	—6.383	—2.825	+2.295
1.125	—6.622	—6.799	—2.985	+2.684
1.250	—7.042	—7.035	—2.967	+2.980
1.375	....	....	....	....
1.500	—6.947	—6.879	—2.797	+2.912
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	—6.828	—6.828	—2.828	+2.828

TABLE 5.—CONSTANTS A, B, C, AND D $\left(N_2 = \frac{4}{c},\ M_2 = T_2 = \frac{\partial S}{\partial x} = 0\right)$

$\frac{K_1 \phi_k}{\pi}$	$A \frac{c r^{0.5}}{R}$	$B \frac{c r^{0.5}}{R}$	$C \frac{c r^{0.5}}{R}$	$D \frac{c r^{0.5}}{R}$
	(a) SYMMETRICAL LOADING			
0.000	....	....	....	....
0.125	—1348	—959	—450	+1255
0.250	—67.40	—36.32	—0.033	+55.42
0.375	—14.21	—6.288	+1.017	+10.06
0.500	—7.159	—3.654	—1.323	+4.372
0.625	—7.529	—4.906	—3.056	+4.504
0.750	—10.44	—7.232	—4.104	+6.511
0.875	—13.33	—8.951	—4.145	+8.526
1.000	—13.94	—8.914	—3.468	+8.951
1.125	—12.85	—7.958	—2.936	+8.174
1.250	—11.80	—7.289	—2.843	+7.431
1.375	....	....	....	....
1.500	—11.55	—7.308	—3.114	+7.257
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	—11.90	—7.502	—3.108	+7.502

TABLE 5.—(*Continued*)

$\frac{K_1 \phi_k}{\pi}$	$A \frac{c r^{0.5}}{R}$	$B \frac{c r^{0.5}}{R}$	$C \frac{c r^{0.5}}{R}$	$D \frac{c r^{0.5}}{R}$
(b) CONTRA-SYMMETRICAL LOADING				
0.000	....	....	....	....
0.125	—80256	—125028	—116585	+7295
0.250	—1773	—2044	—1600	+827
0.375	—258.3	—236.0	—141.1	+157.7
0.500	—71.36	—53.70	—22.05	+47.08
0.625	—28.49	—18.30	—4.901	+18.89
0.750	—15.31	—8.982	—2.217	+9.858
0.875	—11.08	—6.551	—2.297	+6.919
1.000	—10.38	—6.474	—2.866	+6.432
1.125	—11.13	—7.161	—3.293	+6.960
1.250	—12.04	—7.752	—3.388	+7.607
1.375	....	....	....	....
1.500	—12.23	—7.687	—3.089	+7.738
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	—11.90	—7.502	—3.108	+7.502

TABLE 6.—CONSTANTS A, B, C, AND D $\left(T_2 = \frac{4}{c},\ M_2 = N_2 = \frac{\partial S}{\partial x} = 0\right)$

$\frac{K_1 \phi_k}{\pi}$	$A \frac{c r}{R}$	$B \frac{c r}{R}$	$C \frac{c r}{R}$	$D \frac{c r}{R}$
(a) SYMMETRICAL LOADING				
0.000	....	....	....	....
0.125	+581	+414	+193	—542
0.250	+57.15	+32.17	—1.458	—48.02
0.375	+16.39	+8.264	—3.969	—12.70
0.500	+7.784	+4.327	—2.346	—5.748
0.625	+6.162	+4.300	+0.811	—4.432
0.750	+7.569	+5.816	+0.361	—5.379
0.875	+10.01	+7.495	+0.787	—7.083
1.000	+11.17	+7.973	+0.442	—7.897
1.125	+10.69	+7.421	—0.021	—7.559
1.250	+9.839	+6.820	—0.196	—6.956
1.375	....	....	....	....
1.500	+9.318	+6.614	—0.035	—6.592
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	+9.657	+6.741	0.000	—6.741
(b) CONTRA-SYMMETRICAL LOADING				
0.000	....	....	....	....
0.125	+13878	+21592	+20113	—1274
0.250	+615	+707	+549	—288
0.375	+134.4	+123.1	+70.91	—82.50
0.500	+48.97	+37.80	+13.24	—32.92
0.625	+23.51	+16.13	+1.992	—16.34
0.750	+13.77	+8.964	—0.472	—9.688
0.875	+9.802	+6.488	—0.724	—6.923
1.000	+8.579	+5.978	—0.373	—6.066
1.125	+8.805	+6.340	+0.018	—6.227
1.250	+9.516	+6.860	+0.197	—6.729
1.375	....	....	....	....
1.500	+9.973	+7.028	+0.025	—7.052
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	+9.657	+6.741	0.000	—6.741

TABLE 7.—CONSTANTS A, B, C, AND D $\left(\frac{\partial S}{\partial x} = \frac{4}{c}, M_2 = N_2 = T_2 = 0\right)$

$\frac{K_1 \phi_k}{\pi}$	$A \frac{c r^{1.5}}{R^2}$	$B \frac{c r^{1.5}}{R^2}$	$C \frac{c r^{1.5}}{R^2}$	$D \frac{c r^{1.5}}{R^2}$
(a) SYMMETRICAL LOADING				
0.000	....	....	....	....
0.125	−85.34	−59.33	−26.50	+77.30
0.250	−17.70	−8.861	+1.101	+13.12
0.375	−8.120	−2.998	+2.212	+4.676
0.500	−5.168	−1.514	+1.906	+2.241
0.625	−4.187	−1.227	+1.439	+1.453
0.750	−4.314	−1.573	+0.983	+1.513
0.875	−5.080	−2.182	+0.712	+2.043
1.000	−5.669	−2.515	+0.731	+2.455
1.125	−5.690	−2.445	+0.873	+2.470
1.250	−5.452	−2.255	+0.959	+2.301
1.375	....	....	....	....
1.500	−5.189	−2.117	+0.935	+2.117
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	−5.305	−2.197	+0.910	+2.197
(b) CONTRA-SYMMETRICAL LOADING				
0.000	....	....	....	....
0.125	−1005	−1553	−1439	+96.15
0.250	−91.84	−101.9	−76.78	+42.05
0.375	−31.18	−26.46	−13.96	+17.72
0.500	−15.79	−10.59	−2.829	+9.090
0.625	−9.866	−5.361	+0.090	+5.307
0.750	−7.068	−3.209	+0.949	+3.418
0.875	−5.665	−2.274	+1.139	+2.446
1.000	−5.061	−1.955	+1.080	+2.024
1.125	−4.982	−1.983	+0.953	+1.969
1.250	−5.171	−2.145	+0.866	+2.103
1.375	....	....	....	....
1.500	−5.410	−2.272	+0.889	+2.272
1.625	....	....	....	....
1.750	....	....	....	....
1.875	....	....	....	....
2.000	....	....	....	....
∞	−5.305	−2.197	+0.910	+2.197

TABLE 8.—INFLUENCE NUMBERS FOR EDGE DEFLECTIONS AT $x = 0$, $\phi = 0$

Item No.	LOAD CONDITION				EDGE DEFLECTIONS			
	M_2	N_2	T_2	$\frac{\partial S}{\partial x}$	θ (1)	w (2)	v (3)	T_1 (4)
(1)	$\frac{4}{c}$	0	0	0	$+\frac{c_{11} r^{3.5}}{E t R c a^2}$	$-\frac{c_{21} r^3}{E t c a^2}$	$+\frac{c_{31} r^{2.5}}{E t c a^2}$	$-\frac{c_{41} r^2}{R c a}$
(2)	0	$\frac{4}{c}$	0	0	$+\frac{c_{12} r^3}{E t c a^2}$	$-\frac{c_{22} R r^{2.5}}{E t c a^2}$	$+\frac{c_{32} R r^3}{E t c a^2}$	$-\frac{c_{42} r^{1.5}}{c a}$
(3)	0	0	$\frac{4}{c}$	0	$-\frac{c_{13} r^{2.5}}{E t c a^2}$	$+\frac{c_{23} R r^2}{E t c a^2}$	$-\frac{c_{33} R r^{1.5}}{E t c a^2}$	$+\frac{c_{43} r}{c a}$
(4)	0	0	0	$\frac{4}{c}$	$+\frac{c_{14} R r^2}{E t c a^2}$	$-\frac{c_{24} R^2 r^{1.5}}{E t c a^2}$	$+\frac{c_{34} R^2 r}{E t c a^2}$	$-\frac{c_{44} r^{1.5} R}{c a}$

TABLE 9.—INFLUENCE NUMBER COEFFICIENTS

$\frac{K_1 \phi_h}{\pi}$	c_{11}	c_{12} and c_{21}	c_{13} and c_{31}	c_{14} and c_{41}	c_{22}	c_{23} and c_{32}	c_{24} and c_{42}	c_{33}	c_{34} and c_{43}	c_{44}
					(a) SYMMETRICAL LOADING					
0.000	0	0	0	0	......	......	......	......	......	......
0.125	6.90	1.28	0.12	0.01	......	5185	746	2237	322	55.54
0.250	13.81	5.11	0.98	0.08	378.7	325.20	93.44	280.3	80.60	27.82
0.375	20.70	11.50	3.32	0.43	57.51	66.60	28.00	83.80	35.92	18.56
0.500	27.55	20.33	7.79	1.34	30.52	28.02	13.05	38.30	20.74	14.01
0.625	33.88	31.08	14.85	3.20	39.60	26.68	10.06	27.66	15.09	11.62
0.750	38.48	41.35	23.41	6.01	57.92	38.56	12.64	32.28	14.87	10.88
0.875	39.37	46.10	29.42	8.66	71.57	51.93	17.45	42.99	18.00	11.56
1.000	37.05	43.59	29.44	9.47	71.27	55.75	20.11	48.91	20.77	12.61
1.125	34.84	39.20	26.33	8.78	63.65	51.36	19.56	47.34	21.08	12.98
1.250	34.22	37.12	24.13	7.97	58.30	46.55	18.04	43.60	20.08	12.79
1.375	......	......	......	......	......	......	......	......	......	......
1.500	35.14	38.22	24.25	7.74	58.46	44.98	16.97	40.92	18.82	12.29
1.625	......	......	......	......	......	......	......	......	......	......
1.750	......	......	......	......	......	......	......	......	......	......
1.875	......	......	......	......	......	......	......	......	......	......
2.000	......	......	......	......	......	......	......	......	......	......
∞	35.16	38.63	24.86	8.00	60.02	46.63	17.58	42.44	19.31	12.43
					(b) CONTRA-SYMMETRICAL LOADING					
0.000	......	......	......	......	......	......	......	......	......	......
0.125	......	......	......	......	......	......	6788	......	1236	112.1
0.250	......	......	5244	754.3	......	4525	651	1581	240.7	46.20
0.375	1047	1343	693.9	149.6	1735	897.4	193.5	470.0	107.3	30.88
0.500	155.3	246.8	166.8	47.60	416.7	285.6	81.90	198.9	60.49	23.19
0.625	50.33	75.50	58.17	20.02	145.0	120.0	42.39	102.9	38.89	18.57
0.750	33.63	39.56	28.95	10.64	71.73	63.62	25.60	62.15	27.49	15.57
0.875	32.26	33.27	21.37	7.39	52.43	43.93	18.19	44.13	21.30	13.60
1.000	33.98	35.08	21.30	6.76	51.82	39.85	15.65	37.91	18.41	12.43
1.125	35.64	38.41	23.65	7.29	57.30	42.73	15.87	38.40	17.86	12.00
1.250	36.19	40.28	25.65	8.02	62.03	46.90	17.16	41.48	18.61	12.10
1.375	......	......	......	......	......	......	......	......	......	......
1.500	35.17	39.03	25.46	8.27	61.50	48.23	18.17	43.89	19.78	12.55
1.625	......	......	......	......	......	......	......	......	......	......
1.750	......	......	......	......	......	......	......	......	......	......
1.875	......	......	......	......	......	......	......	......	......	......
2.000	......	......	......	......	......	......	......	......	......	......
∞	35.16	38.63	24.86	8.00	60.02	46.63	17.58	42.44	19.31	12.43

TABLE 10.—EDGE DEFORMATIONS OF UPPER AND LOWER SHELL

Item No.	Edge deformations	LOAD CONDITION					Common factor
		$M_2 = \frac{4}{c}$	$N_2 = \frac{4}{c}$	$T_2 = \frac{4}{c}$	$\frac{\partial S}{\partial x} = \frac{4}{c}$	Membrane system	
1	θ, upper shell..	$+ c_{11} r^{1.5}$	$+ c_{12} r R$	$- c_{13} r^{0.5} R$	$+ c_{14} R^2$	$+ c_{14} q R^3$	$\frac{E t R c a^2}{r^2}$
2	θ, lower shell..	$- c_{11} r^{1.5}$	$+ c_{12} r R$	$+ c_{13} r^{0.5} R$	$+ c_{14} R^2$	0	
3	$\Delta\theta$ = Item No. 1. — Item No. 2.	$+ 2 c_{11} r^{1.5}$	0	$- 2 c_{13} r^{0.5} R$	0	$+ c_{14} q R^3$	
4	w, upper shell..	$- c_{21} r^{1.5}$	$- c_{22} r R$	$+ c_{23} r^{0.5} R$	$- c_{24} R^2$	$+ c_{24} q R^3$	$\frac{E t c a^2}{r^{1.5}}$
5	w, lower shell..	$- c_{21} r^{1.5}$	$+ c_{22} r R$	$+ c_{23} r^{0.5} R$	$+ c_{24} R^2$	0	
6	Δw = Item No. 4. — Item No. 5.	0	$- 2 c_{22} r R$	0	$- 2 c_{24} R^2$	$- c_{24} q R^3$	
7	v, upper shell .	$+ c_{31} r^{1.5}$	$+ c_{32} r R$	$- c_{33} r^{0.5} R$	$+ c_{34} R^2$	$+ c_{34} q R^3$	$\frac{E t c a^2}{r}$
8	v, lower shell...	$- c_{31} r^{1.5}$	$- c_{32} r R$	$+ c_{33} r^{0.5} R$	$+ c_{34} R^2$	0	
9	Δv = Item No. 7. — Item No. 8.	$+ 2 c_{31} r^{1.5}$	$+ 2 c_{32} r R$	$- 2 c_{33} r^{0.5} R$	0	$+ c_{34} q R^3$	
10	T_1, upper shell.	$- c_{41} r^{1.5}$	$- c_{42} r R$	$+ c_{43} r^{0.5} R$	$- c_{44} R^2$	$- c_{44} q R^3$	$\frac{c a R}{r^{0.5}}$
11	T_1, lower shell.	$- c_{41} r^{1.5}$	$+ c_{42} r R$	$+ c_{43} r^{0.5} R$	$+ c_{44} R^2$	0	
12	ΔT_1 = Item No. 10 — Item No. 11	0	$- 2 c_{42} r R$	0	$- 2 c_{44} R^2$	$- c_{44} q R^3$	

DISCUSSION

I. K. Silverman,[13] Jun. Am. Soc. C. E. (by letter).—That thin shells, properly supported, serve as very economical structures, has been known for some time. Mr. Schorer, for the first time, has introduced into American engineering literature the theory of such shells which has been developed largely by German and Swiss engineers.

Even with the simplifications introduced in this paper, the application of the theory requires considerable computation as, for example, when the structure analyzed is not a complete surface of revolution and the loading is not symmetrical. Such a condition occurs in the analysis of the structure such as the sector gate shown in Fig. 9.

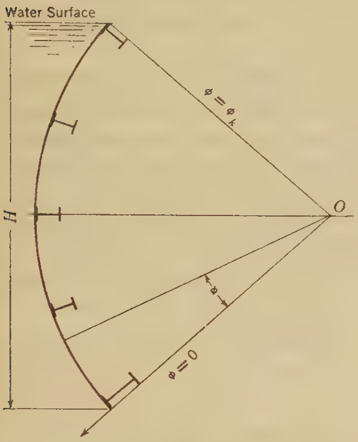

Fig. 9.

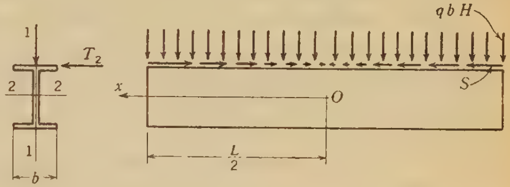

Fig. 10.

When applied to a structure of this type, the thin shell theory tends to show that large stresses may occur in the longitudinal supporting members, especially in the member at $\phi = 0$. These large stresses are due to the fact that the inevitable arch action developed in the curved skin-plate causes the member to bend about Axis 1–1, Fig. 10. Furthermore, stresses of the type, S, subject the longitudinal supporting members to loads which are axially eccentric. In combination with the direct water load, these stresses may reach large values, especially in gates of long spans.

In outline, the method of attacking such a structure as that presented in Fig. 9 is perfectly straightforward. The author has shown that the loads on a structure are carried by a primary system of stresses, the membrane system, and by a self-equilibrating system (secondary), the function of which is to preserve geometrical continuity. The stresses of the secondary system are applied along a line common to the shell part of the structure and its supporting members.

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[13] With U. S. Reclamation Bureau, Denver, Colo.

The stresses and deflections of the membrane system are obtained as if the shell were a complete surface of revolution by means of Equations (101) to (110). The stresses at $\phi = 0$ and $\phi = \phi_k$ (it is assumed now that intermediate longitudinal members have been removed) are applied to the supporting members as reactions, and the deflections and stresses in these members are obtained in the usual manner. In general, there will be differences in these stresses and deflections, and it is the purpose of the secondary system to annul such geometrical discrepancies. It is the object of the author to derive the stresses and deflections in the shell due to this secondary system. When these stresses and deflections have been obtained in the shell for the various conditions of loading (see Fig. 4), the corresponding stresses and deflections are obtained for the supporting members.

In general, there will be eight elastic equations for the indeterminate quantities in the secondary system. In a symmetrical structure under symmetrical loading, such as that treated by the author, these equations reduce to four.

Finsterwalder[14] has reduced the number of unknowns to six by assuming that $\theta = 0$ at the connection of shell to supporting member. This condition is quite possible because of the relative rigidities of shell and boundary member. Corresponding to Equations (91) to (94) for symmetrical loading there would result three sets of four simultaneous equations each for the determination of the constants, A, B, C, and D, namely: in Equation (92), $N_2 = \frac{4}{c}$; $N_2 = 0$; and $N_2 = 0$; in Equation (93), $T_2 = 0$; $T_2 = \frac{4}{c}$; and $T_2 = 0$; in Equation (94), $\frac{\partial S}{\partial x} = 0$; $\frac{\partial S}{dx} = 0$; and $\frac{\partial S}{\partial x} = \frac{4}{c}$; and in Equation (98), $\theta = 0$; $\theta = 0$; and $\theta = 0$.

In all cases, M_2 (Equation (91)), would be different from zero. With the constants known the method of procedure follows that of the author except that for a case of symmetrical loading and symmetrical structure, the equations of continuity are, as follows: For $\phi = 0$:

$$\Delta w = (w)_s - (w)_b = 0 \dots\dots\dots\dots (144)$$

$$\Delta v = (v)_s - (v)_b = 0 \dots\dots\dots\dots (145)$$

and,

$$\Delta \sigma = \left(\frac{T_1}{t}\right)_s - \sigma_b = 0 \dots\dots\dots\dots (146)$$

in which the subscripts, s and b, refer to shell and beam, respectively, and σ = the unit stress. When the loading is unsymmetrical or the edge members are not identical, six or eight elastic equations are necessary. The equation for w will be set up in general form using the following definitions: $w_1 =$

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[14] "Die Theorie der zylindrischen Schalengewölbe, System Zeiss-Dywidag," von Dr. Ing. Ulrich Finsterwalder, International Assoc. for Bridge and Structural Eng., Zurich, 1932.

deflection of shell due to membrane system at $\phi = 0$; w'_1 = deflection of shell due to membrane system at $\phi = \phi_k$; w_2 = deflection of beam due to membrane system at $\phi = 0$; w'_2 = deflection of beam due to membrane system at $\phi = \phi_k$; w_3 = deflection of shell due to secondary system at $\phi = 0$; w'_3 = deflection of shell due to secondary system at $\phi = \phi_k$; w_4 = deflection of beam due to secondary system at $\phi = 0$; and, w'_4 = deflection of beam due to secondary system at $\phi = \phi_k$. These deflections are broken up so that a symmetrical and contra-symmetrical system of secondary stresses may be applied.

For the symmetrical system:

$$\Delta w = \frac{(w_1 + w'_1)}{2} - \frac{(w_2 + w'_2)}{2} + \frac{(w_3 + w'_3)}{2} - \frac{(w_4 + w'_4)}{2} = 0 \quad ..(147)$$

and, for the contra-symmetrical system:

$$\Delta w = \frac{(w_1 - w'_1)}{2} - \frac{(w_2 - w'_2)}{2} + \frac{(w_3 - w'_3)}{2} - \frac{(w_4 - w'_4)}{2} = 0 \quad ..(148)$$

Corresponding equations are obtained for v and σ. Unsymmetrical loadings tend to cause large secondary stresses, and it is to be expected, in the case of the sector gate, that these high moments and stresses occur at $\phi = 0$.

DR. W. FLÜGGE[15] (by letter).—In general, the computations necessary for a numerical solution of shell problems have heretofore been very cumbersome. In the special case of a cylindrical shell of long span these computations may be simplified by applying the principles developed by J. W. Geckeler[6] and the author is to be congratulated on his analysis of the problem.

Following Equation (44) the author states:

> "By substituting Equations (38), (39), and (40), in Equation (35), and, likewise, by substituting Equations (42), (43), and (44), in Equation (36), it will be noted that two equations are obtained for the determination of the unknowns, J_2 and K_2. These simultaneous equations cannot be solved in explicit form."

It is possible to derive the explicit form by introducing the expression, $F(\phi) = e^{\mu\phi}$ into Equation (29) instead of Equation (31), thus[16]:

$$\mu^8 + 2\,(1 - a)\,\mu^6 + [1 - (4 + m)\,a + a^2]\,\mu^4$$
$$+ [-\,(2 + m)\,a + a^2]\,\mu^2 + 4\,a^2\,b^2\,(1 - m^2) = 0 \quad ....(149)$$

which is of the fourth order of μ^2 and may be solved by exact methods. The shell constants, J_1 and K_1, are the real numbers in the two terms of the complex number, μ; thus,

$$\mu_{1,\,2,\,3,\,4,} = \pm\,(J_1 \pm i\,K_1), \text{ and } \mu_{5,\,6,\,7,\,8,} = \pm\,(J'_1 \pm i\,K'_1)$$

The formula offered by the writer in 1934[16] is somewhat different from Equation (149):

$$\mu^8 + [-\,(2 + m)\,a + 2\,]\,\mu^6 + [\,(1 + 2m)\,a^2 - 2\,(2 + m)\,a + 1\,]\,\mu^4$$
$$+ [-\,m\,a^3 + (1 + m)^2\,a^2 - (2 + m)\,a\,]\,\mu^2 + 4\,a^2\,b^2\,(1 - m^2) = 0 \quad ..(150)$$

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[15] Privatdozent, Univ. of Göttingen, Göttingen, Germany.

[16] "Statik und Dynamik der Schalen", von W. Flügge, Berlin, 1934, p. 139.

The disparity is of no numerical importance, however, and is to be explained by the different treatment of some second-order terms in the fundamental equations of the shell theory. It disappears entirely if Poisson's ratio, m, is made equal to zero, an assumption that may simplify the numerical computations, particularly of reinforced concrete shells. The error due to this assumption will be less than that introduced by the basic simplifications of the paper.

ANTON TEDESKO,[17] ESQ. (by letter).—In presenting a rather complex subject in a clear and useful form, Mr. Schorer has performed a splendid service to the profession. The suggested method gives good approximations of practical value over a wide range of cases. It would be interesting to investigate its degree of exactness beyond the limits given by the author.

Working for Dr. Finsterwalder, the writer has used other practical simplifications of his theory, the results of which checked well with those of the accurate method. Deformations may be calculated for statically determinate membrane stress conditions, such as are impossible in manifold statically indeterminate systems. Consequently, these deformations are eliminated in the calculations for the latter system. The unknown and resultant stress values of the indeterminate system are obtained from elastic equations conforming to the edge conditions. Where the differences of numerical values of trigonometric functions of the angle, ϕ, give inexact results, the equations for deformations are expressed in power series with the angle, ϕ, as the argument. In these, and in his trigonometric series, Dr. Finsterwalder neglected terms of higher order. He decreased the degree of statical indeterminacy by pairing a vertical and a horizontal force component with a moment, M_2, in such a manner that full restraint is obtained at the edge; or, in other words, that the direction of the tangent of the cross-sectional curve at the edge remained fixed. In the second series of calculations the tangent is assumed to rotate freely. After the degree of actual restraint is determined, the values already obtained are re-adjusted accordingly. When considering roof shell problems the value, N_2, at the edge can often be neglected because of its rather small size as compared with loads acting on the edge.

The author's corrected values, J_2, K_2, J'_2, and K'_2, as given in his Equations (70), (71), (72), and (73), should be used successfully in Finsterwalder's accurate method.

A very clear and almost popular explanation of the edge problem was given by H. Rüsch,[18] whose name second to that of Finsterwalder is connected with the development of the bending theory of cylindrical shells. Valuable bibliography on the subject has been listed by Mr. A. Floris in his discussion[19] of a paper by D. C. Coyle, M. Am. Soc. C. E. The fundamental equations and principles of the author's paper were given and discussed by H. Savage.[20]

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[17] Structural Engr., Roberts & Schaefer Co., Chicago, Ill.

[18] "Theorie der Querversteiften Zylinderschalen für schmale, unsymmetrische Kreissegmente," von H. Rüsch, pub. by R. Noske, Borna-Leipzig, 1931.

[19] *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 1173.

[20] *Concrete and Constructional Engineering*, Vol. XXV, 1930, p. 490.

Using his accurate theory Dr. Finsterwalder[21] has computed the stress values, T_1, S, T_2, and M_2 for the roof of the Central Market, in Budapest,[22] Hungary. The roof is constructed of cylindrical barrels of a span, $L = 136$ ft, a width of 38 ft 9 in., and a radius, $R = 32$ ft 10 in. It consists of segmental shells in combination with edge members, the cross-section of which is shown in Fig. 11. A graphic picture of the meaning of the shell stress components, as discussed by the author, is given in Fig. 11 (*b*), 11 (*c*), 11 (*d*), and 11 (*e*), in which the solid-line curves denote values determined by the accurate theory (considering the first two terms of the Fourier series); and

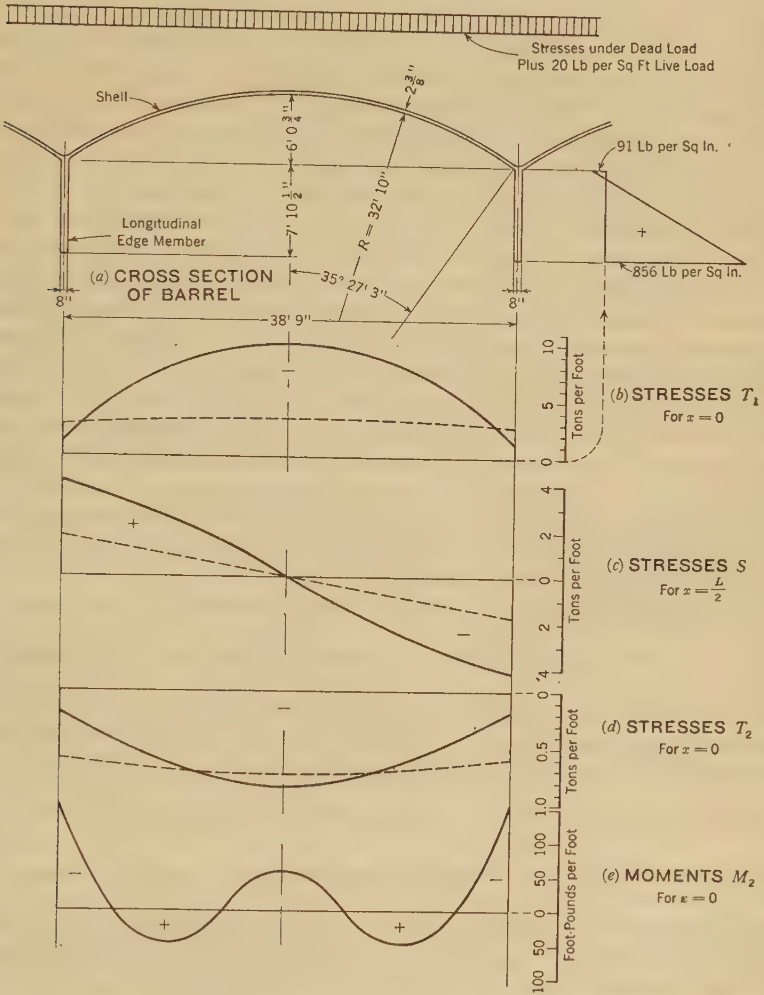
Stresses under Dead Load
Plus 20 Lb per Sq Ft Live Load
Shell
2 3/8"
6' 0 3/4"
7' 10 1/2"
R = 32' 10"
91 Lb per Sq In.
Longitudinal
Edge Member
+
856 Lb per Sq In.
(a) CROSS SECTION
OF BARREL
35° 27' 3"
8"
8"
38' 9"
10
5
0
Tons per Foot
(b) STRESSES T1
For x = 0
4
2
0
2
4
+
−
Tons per Foot
(c) STRESSES S
For x = L/2
0
0.5
1.0
−
Tons per Foot
(d) STRESSES T2
For x = 0
100
50
0
50
100
−
−
+
+
Foot-Pounds per Foot
(e) MOMENTS M2
For x = 0

FIG. 11.—STRESS COMPONENTS IN BARREL SHELL.

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[21] *Ingenieur Archiv*, Vol. IV, 1933, p. 57.

[22] *The Architectural Forum*, Vol. LX, May, 1934, p. 354.

the broken-line curves denote values determined for the membraneous condition. The stresses are given for $x = 0$ and $x = \frac{L}{2}$, with reference to Fig. 1 of the paper.

The thrust, T_2, in the transverse direction of the shell, according to Equation (101) for the membrane system, is a direct function of the radius and the load of the shell. These values of T_2 are indicated by a broken line in Fig. 11 and are nearly constant over the width of the barrel. The thrust, T_2, of the statically indeterminate system is indicated by a solid line. It has its maximum value at the crown of the barrel and decreases to a small value near the edge members which are too slender to take up a considerable amount of horizontal force. The resulting maximum thrust at the crown is greater than the statically determinate thrust required to carry the given load. The excessive crown thrust results in an increased carrying capacity of the crown portion where the shell acts as a support for loads from other parts of the shell. The supporting action at the crown and at the edge members is also expressed by the moment line, M_2, of the arched plate where negative restraining moments appear at the crown and at the edge members. The bending moments are small, however, because small spans only between narrowly spaced ideal supports have to be bridged by bending action. Furthermore, only a part of the load will cause bending as the other parts are carried by the thrust directly. The thrust, T_2, toward the edge members is decreased by shearing forces, S, transferring the loads to transverse end stiffeners. The T_1-diagram (Fig. 11 (*b*)) shows compression in the shell and tension in the edge member, a distribution which could be expected for such a "T-beam with curved flange."

The limiting equation for the author's method is Equation (46), $\frac{L}{R} \geqq \pi$. In the method to which the writer has referred as "Finsterwalder's accurate theory," the longitudinal bending moment, M_1, the corresponding shearing stress component, N_1, and the torsional moment, M_t (see Fig. 3), are neglected. This simplifying assumption does not influence the results in practical cases when $\frac{L}{R} \geqq 1$, or even when $\frac{L}{R}$ is slightly less than 1. For the example to which the stress diagrams in Fig. 11 apply, the value, $\frac{L}{R}$, is equal to 4.15 and the author's method will give practically the same curves. For cases in which a shell of very great radius, R, has transverse end stiffeners quite close together, even the results of Dr. Finsterwalder's ingenious theory will become inaccurate.

Dr. Ing. U. Finsterwalder[23] (by letter).—A clear and interesting contribution to the problems of cylindrical shells, and one that will be useful for the designing engineer, is contained in this paper. Various methods are in use to simplify cumbersome numerical computations in connection with the writer's theory.[8] Accurate stress values are obtained by the use of a simplifying hypothesis; namely, that for each case of unit loading the relationship between the work of deformation of the bending stress components and the direct stress components is constant for all proportions of a shell. With the help of this hypothesis it is possible to arrive at stress values for a given shell from stress values of cases that have been computed by the use of the rather cumbersome formulas of the writer's theory.

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[23] Chf. Engr., Dyckerhoff & Widmann, A. G., Berlin, Germany.

The writer neglected bending moments in the direction of the generatrix of the cylindrical shell. The solution as proposed by Mr. Schorer neglects, more or less, the bending resistance of the cylindrical shell in the direction of the generatrix, and also the influence of the stress components, S and T_2, on the work of deformation. For shells with widely spaced transverse stiffeners and of small widths the author's solution will be accurate. The rise of the cylindrical segment will then be comparatively small and the stress components, S and T_2, will not play an important part in the work of deformation. However, for very wide cylindrical shells, and although the distance between end stiffeners may be considerable, the influence of S and T_2 will be noticeable, and Mr. Schorer's solution may become inaccurate.

Professor F. Dischinger has worked on the problem, and has recently found the exact solution.[24] It is not more complicated than the writer's theory, although Professor Dischinger was able to eliminate the simplifying assumptions of the writer.

F. W. SEIDENSTICKER,[25] ESQ. (by letter).—Knowing from experience that many readers are reluctant to follow through a series of difficult mathematical derivations, the writer fears that, in many cases, only the "Synopsis" and "Conclusions" of this excellent paper will be read. The wording of the concluding remarks may be misleading to the average engineer unless they are read as a part of the whole subject.

The fact that the proposed method of shell design is termed an approximation does not mean that it will be unsafe to use this method for practical applications, especially if one considers that the established formulas of structural design for use in standard steel and reinforced concrete structures involve a much higher degree of inaccuracy than the method presented by Mr. Schorer. His results are within a small percentage of the elastic theory. They become less accurate in cases where the stiffening supports are narrowly spaced. However, an exact solution in that case will be less necessary; it would be interesting to mathematical physicists, but less so to engineers, since structures of small spans usually do not justify a great amount of intricate engineering computations.

Good agreement between theory and practice was recently found in the results of full-sized loading tests[26] conducted during the demolition of cylindrical barrels at the Century of Progress grounds, in Chicago, Ill.

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[24] "Die strenge Theorie der Kreiszylinderschale in ihrer Anwendung auf die Zeiss-Dywidag-Schallen", von Prof. Dr. F. Dischinger, *Beton und Eisen*, Vol. 34, August, 1935, *et seq.*

[25] Cons. Engr., Chicago, Ill.

[26] *Engineering News-Record*, November 7, 1935, p. 635.

HERMAN SCHORER,[27] ASSOC. M. AM. SOC. C. E. (by letter).—An interesting outline for the analysis of a sector gate is contained in the discussion by Mr. Silverman. In steel structures of this type, with comparatively heavy surface loads, it may become necessary to make provisions against buckling of the skin-plate in a longitudinal and radial direction, depending on the relative dimensions. As a rule, the shell is comparatively stiff in the longitudinal direction and the rim-bending stresses are limited to a narrow zone along the stiff end members. For this reason the intermediate longitudinal members, as shown in Fig. 9, could probably be eliminated. However, the danger of buckling in a radial direction would require closer investigation since the resistance of the arch is reduced by the large radial bending moments, originating from the heavy longitudinal edge member at $\phi = 0$. The necessary stiffness in a radial direction can be obtained by the provision of small radial ribs, welded to the skin-plate, perhaps combined with a few longitudinal ribs at the center zone of long spans.

Dr. Flügge's solution for J_2 and K_2 in explicit form represents a desirable improvement and will be welcomed by those who are interested in a more exact analysis of the problem. Mr. Tedesko contributes a practical illustration of the stress components in the roof of the Market Hall, in Budapest, one of the most outstanding shell structures.

The writer is greatly indebted to Dr. Finsterwalder for pointing out the exact solution of the entire problem, as recently published by Professor Dischinger[24], which enables the designer to include the stress components and simplifications neglected thus far. It also includes anisotropic shells (that is, shells with non-uniform bending resistance in different directions), such as those introduced by a series of radial or longitudinal stiffening ribs.

The discussion by Mr. Seidensticker is encouraging and his reference to the full-sized loading tests in Chicago is very timely. The excellent behavior of these structures under non-symmetrical loadings appears in striking contrast to the expectations derived from pure arch action.

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[27] Gen. Mgr., Borsari Tank Corp., of America, New York, N. Y.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1939

UNDERGROUND CORROSION

BY K. H. LOGAN,[1] ESQ.

WITH DISCUSSION BY MESSRS. THOMAS F. WOLFE, JOHN R. BAYLIS, E. P. FETHERSTONHAUGH, JOHN F. SKINNER, F. N. SPELLER, AND K. H. LOGAN.

SYNOPSIS

Underground corrosion is shown to be of economic importance by the fact that associations of pipe owners and pipe-owning corporations are spending considerable sums for the study of corrosion problems. The underground pipes in the United States have been estimated to have a length of 450 000 miles and a value of nearly $6 000 000 000. The annual loss due to underground corrosion of pipe lines in the oil industry alone has been estimated at $25 000 000.

Soil corrosion is characterized by the uneven distribution of the corrosion and the fact that most ferrous pipe materials corrode at nearly the same rates. The major cause of the corrosion seems to be non-uniformity in the distribution of oxygen and moisture along the surface of the pipe line.

Among the methods suggested for reducing corrosion losses are the use of copper and copper alloy pipe which corrode less rapidly than ferrous materials; increasing the thickness of the pipe walls; and the use of protective coatings.

Soil corrosivity surveys are suggested as a means for determining the need for protective coatings. Some of the weaknesses of coatings are noted and cathodic protection is suggested as a means of improving the effectiveness of coatings. In an Appendix are listed some of the more important recent articles on underground corrosion and related subjects.

NOTE.—Publication of this paper has been approved by the Director of the Bureau of Standards of the U S. Dept. of Commerce. Presented at the meeting of the Sanitary Engineering Division, New York, N. Y., January 18, 1934, and published in March, 1935. *Proceedings.*

[1] Chf., Underground Corrosion Section, National Bureau of Standards, Washington D. C.

The Importance of the Problem

A recent compilation gives the amount of underground trunk and distribution mains in the United States as 450 800 miles, having an estimated value of nearly $6 000 000 000. In addition, there are approximately 14 000 000 gas-service pipes and probably at least an equal number of water services. No estimate has been made of the mileage of underground pipe in use for waste disposal, but it will approximate that used for water services. The amount and value of the ferrous material exposed to the action of soil, therefore, are enormous, and even a moderate reduction in the rate of deterioration of underground pipe lines is important.

In 1931,[2] Carl R. Weidner, Assoc. M. Am. Soc. C. E., estimated the annual loss to the pipe industry on account of underground corrosion, at $50 000 000. Presumably, this value refers to oil lines, and if so, it should be multiplied by at least three to represent the losses on all underground pipe. In view of the large losses sustained by gas and oil companies on account of underground corrosion and the efforts which these organizations are making to reduce these losses, it is a little surprising that equal attention to the subject is not given by the owners of lines carrying water. Aside from the unlikely explanation that water lines do not corrode, several others might be offered. In general, the owners of water pipes are not as well organized to fight corrosion since they consist largely of independent municipalities and individuals. Water mains usually have greater wall thickness than gas mains of the same diameter and, as will be shown later, wall thickness is an important factor in pipe life. Moreover, water leaks of the same magnitude are less serious than leaks of oil or gas because of the damage which the latter cause to property and because of the fire and explosion hazard accompanying gas leaks in cities. The tendency to use thinner-walled pipe and to place expensive pavement over distribution mains makes it more important to reduce the corrosion losses to a minimum.

At first, it was customary to attribute all underground corrosion in cities to stray-current electrolysis. The improvement of operating conditions on some street railways and the abandonment of street-car lines have focused more attention on soil conditions.

Characteristics and Causes of Underground Corrosion

The findings of the National Bureau of Standards with respect to the action of soils have been presented in various ways and on numerous occasions. The repetition of four major conclusions herein is not for the purpose of imparting new information, but to establish a basis for the discussion of the reasons for the suggestions made in this paper.

Briefly, the Bureau has found: First, that the differences in the rates of corrosion of the commonly used ferrous pipe materials are of the same order of magnitude as those of different specimens of the same material; and, second, that the character and seriousness of soil action vary with the soil condition to which the material is exposed. This latter variation is so great that it appears to be a much more important factor than the choice of ferrous pipe material. Figs. 1, 2, and 3, are views of 6-in. lengths of $1\frac{1}{2}$-in. pipe of three different materials. (If the reader will face toward a source of direct light, preferably sunlight, so that the captions of Figs. 1, 2, and 3 are on his right, the pitted condition of the pipe surface will be apparent. If the illustrations are reversed the pits will appear as projections.) The similarity of the corrosion pattern of the three materials in the same soil is shown, as is the difference between the patterns in different soils. The wide difference which may occur in the rates of corrosion of the specimens of the same material in the same soil is illustrated by Fig. 1. The third characteristic of soil corrosion which must be considered is that the rate of corrosion after a pipe has been exposed to soil for several years is considerably less than for the first period of exposure. A fourth characteristic of underground corrosion (which not only is of great economic importance, but is suggestive of the cause of corrosion) is the unevenness of the distribution of the corrosion over the surface of the buried metal. Sometimes a length or section of pipe is penetrated at only one point, whereas almost no corrosion is found elsewhere on the section. The pitting is usually confined to the bottom of the pipe, but occasionally the most severe corrosion is found on the sides or top.

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[2] Opening Remarks to Group Session on "Corrosion of Oil Field Equipment," by C. R. Weidner, *Proceedings*, Am. Petroleum Inst., Vol. 12, Pt. 4, p. 26 (*Production Bulletin 308*).

The most interesting characteristic of underground corrosion is the irregular or spotty nature of the attack. The loss of material as such is usually too small to be of importance if it were uniformly distributed over the pipe surface. It will be helpful, in understanding these findings, to construct at least a tentative explanation of underground corrosion processes.

When the electrolytic theory of corrosion became generally accepted, the first explanation offered for the formation of pits in the pipe surface was impurities or segregations in the pipe material. This conclusion was supported by the fact that where impurities were known to exist, differences in electrical potential between the impurities and the metal could be measured. It was shown that there was a difference of potential between oxidized metal or mill scale and bright iron, and strains in metal were also shown to result in differences of potential. It was logical to assume that these potential differences resulted in corroding currents when the metal was exposed to an electrolyte. Of course, other differences of potential were observed when different metals, or two varieties of the same metal, were in metallic contact.

The mistake that has been made is not in reasoning but in evaluating the importance of the conclusion. There can be little doubt that some corrosion results as suggested, and that under some conditions one of these causes of corrosion is the controlling one. When materials are buried in soil, however, and under some other conditions as well, other causes of corrosion are more important.

The influence on corrosion of causes external to the pipe was brought forcibly to the attention of the observers at the Bureau of Standards when the first specimens of pipe were examined after approximately two years of exposure to widely different soils, and is illustrated in Figs. 1, 2, and 3. It was observed that in certain soils the corrosion took the form of isolated pits

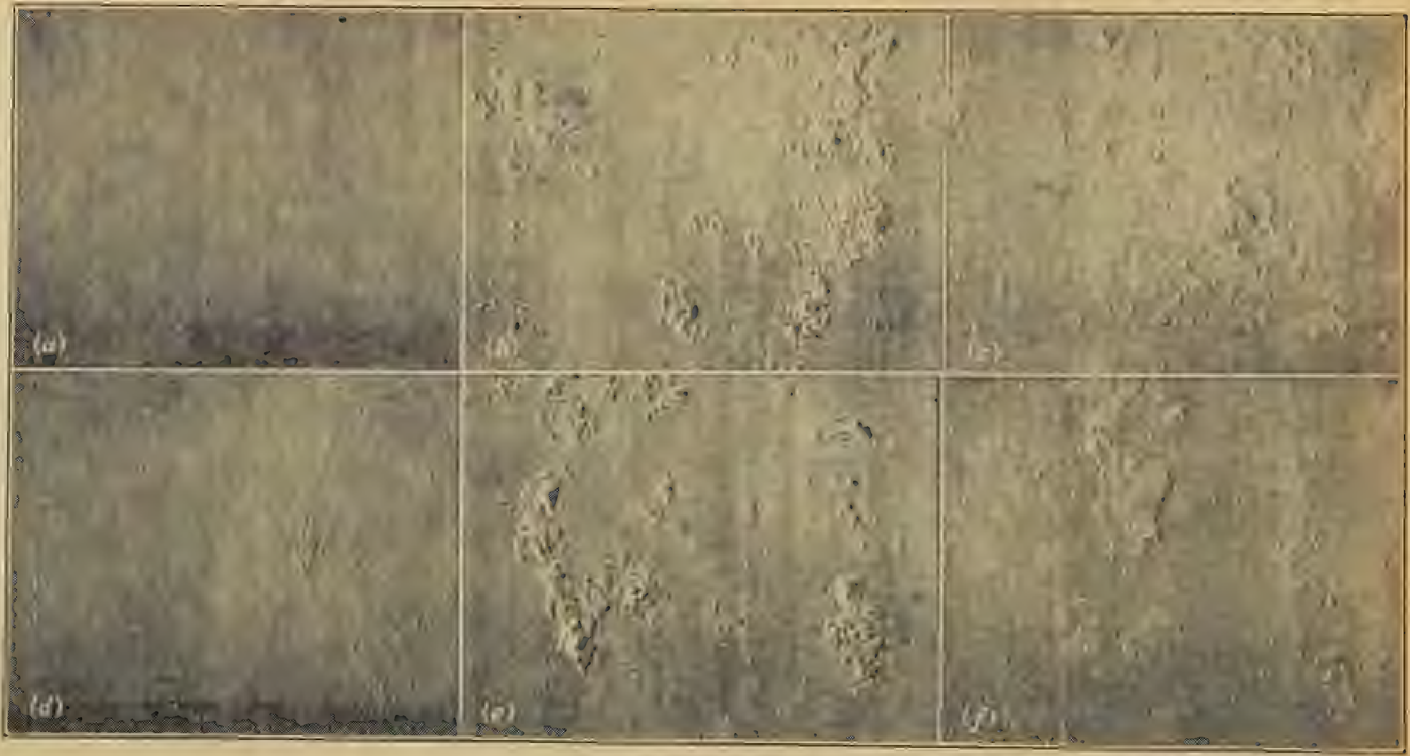

Fig. 1.—Specimens of Open-Hearth Iron Pipe Corroded by Ten Years of Contact with: (*a*) and (*d*), Wet Silt Loam; (*b*) and (*e*) Hempstead Silt Loam; and (*c*) and (*f*) Ontario Loam.

Fig. 2.—Specimens of Wrought-Iron Pipe. Corroded by Ten Years of Contact with: (*a*) Wet Silt Loam; (*b*) Hempstead Silt Loam; and (*c*) Ontario Loam.

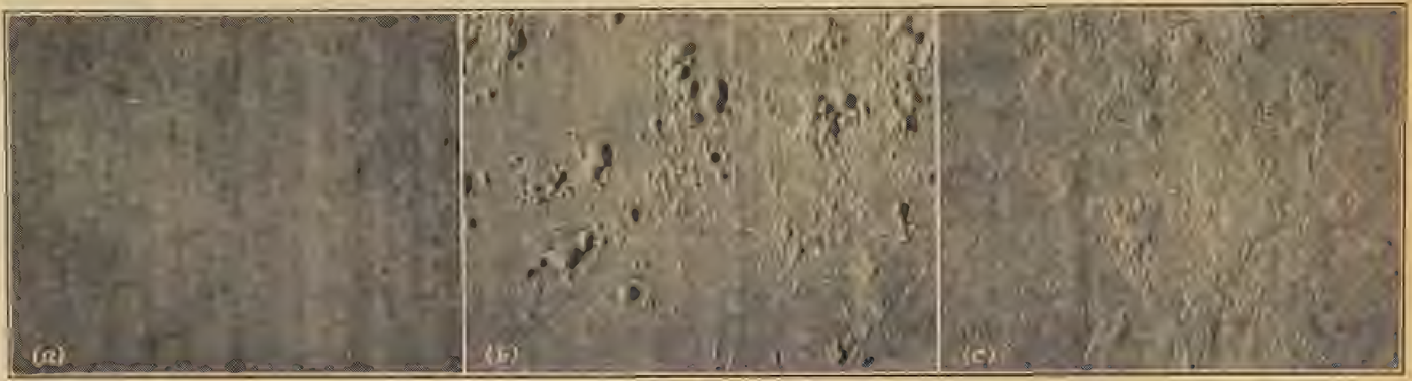

Fig. 3.—Specimens of Bessemer Steel Pipe. Corroded by Ten Years of Contact with: (*a*) Wet Silt Loam; (*b*) Hempstead Silt Loam; and (*c*) Ontario Loam.

with sharp margins. In other soils the corroded areas took the form of rather large shallow depressions with no well-defined boundaries. In one soil the removal of metal was very uniform over the entire exposed surface. Differences in the quantities and distribution of the impurities or strains might account for these differences in corrosion. That this was not the correct explanation was shown by the fact that, almost without exception, whenever Bessemer steel showed one form of corrosion, pure open-hearth iron, wrought iron, and specimens of Bessemer steel from another rolling mill showed similar corrosion patterns. Evidently, therefore, the corrosion pattern was primarily a function of the soil condition, and to a minimum extent only, a characteristic of the material. Although this observation was made public in 1927, and has been re-stated many times, its importance is still not fully appreciated.

Not infrequently the Bureau of Standards is asked what is the life of a certain type of pipe material. This will depend on the thickness of the pipe wall and the type of soil to which it is exposed. It will also depend on the area[3] exposed and on the definition of pipe life. Sometimes the inquirer has in mind the time until the first leak develops; others wish to know how long the pipe can be kept in service economically, which, of course, depends partly on the cost of repairs. In certain soils 8-in. steel oil lines have begun to develop leaks within six years after construction, although such early leaks are few. In other soils pipes have remained in service from 50 to more than 273 years. Only a small percentage of the soils of the United States are severely corrosive, but the individual corrosive areas are small in extent and widely distributed.

It is inadvisable therefore, to attempt to protect the entire length of a long line against corrosion unless the consequences of a few leaks would be very serious. Since underground corrosion is largely the result of certain soil characteristics many attempts have been made to correlate corrosion with soil properties and to develop methods for identifying corrosive soils. The location of corrosive soils along a pipe line is commonly known as a soil corrosivity survey or as a soil survey. The latter name is unsatisfactory because it has been previously used by the Department of Agriculture to indicate the identification and mapping of soils with respect to their physical appearance. Soil corrosivity surveys will be discussed in a subsequent section of this paper.

Soil Characteristics That Determine the Form and Rates of Corrosion.—In the course of an investigation of the relation of pipe-line currents to corrosion, Shepard[4] noted that the soils in which corrosion was most active were lower in electrical resistance than the adjacent soils. The lower resistance could be accounted for by higher percentages of water or of soluble salts. Of course, these differences would appear in complete soil analyses, but were obscured by other data.

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[3] "Adjustment of Soil Corrosion Pit-Depth Measurements for Size of Sample," by G. N. Scott, *Proceedings*, Am. Petroleum Inst., Vol. 14, Pt. 4, p. 204.

[4] "Pipe Line Currents and Soil Resistance, as Indicators of Local Corrosive Soil Areas," by E. R. Shepard, *Journal of Research*, Bureau of Standards, Vol. 6, p. 683, April, 1931.

It is not to be inferred from the fact that soils of low resistance are corrosive that the corrosivity is proportional to the conductivity of the soil or that the conductivity of the soil is in itself the cause of the corrosion. It appears, however, that the conductivity of the soil either is a factor or is associated with some soil property which influences and sometimes controls the corrosivity of the soil. Although soils of low electrical resistance are usually corrosive, exceptions are known, and some soils having high resistivity are quite corrosive. Therefore, it must be concluded that while in accordance with the electrolytic theory of corrosion the conductivity of the soil is a factor, in some instances, other factors are more important.

After a study of a wide variety of soils, Denison,[5] concluded that there was a relation between the acidity and the corrosiveness of soils. However, as in the case of the resistivity-corrosion relation, cases were found in which the correlation was unsatisfactory. Investigators are thus led to the conclusion that soil acidity is a factor affecting the rate of corrosion, but that it is not in all cases the controlling factor and, perhaps, not the fundamental one.

Consideration of underground corrosion phenomena occurring under widely different conditions leads to the conclusion that there are several, perhaps numerous, factors which modify the more fundamental causes of corrosion. In an unpublished communication to the Corrosion Committee of the American Petroleum Institute Denison introduces his discussion of corrosion in acid soils, as follows:

> "Corrosion of ferrous metals in soils may be assumed generally to originate in differences in potential between parts of the metal surface accessible to the oxygen of the atmosphere and other parts from which air is completely or partially excluded. According to this explanation, corrosion occurs at the points of contact between metal and soil which are anodic to adjacent areas more accessible to air. This general statement of the 'differential aeration' theory may be used as a basis for understanding the various forms of corrosion which are encountered in pipe line operation and to explain the marked differences which have been observed in the corrosiveness of soils."

The importance of oxygen concentration was first realized in 1916 by Aston,[6] although the creation of electric currents by variations in oxygen concentration was observed at a much earlier date. The importance of the phenomenon has been pointed out by Evans,[7] in a number of papers. Evans, however, has confined his attention largely to atmospheric and underwater corrosion which does not involve so many uncontrollable factors.

It now seems probable that the electrolytic theory will account for observed corrosion underground if it is assumed that the impelling differences of potential originate largely in differences of oxygen supply at various points on the surface of the buried metal, and that the potentials, or the conductance, of the path over which the corroding current flows, or both, are modified by secondary reactions such as polarization and the deposition of corrosion products.

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[5] "Corrosion of Ferrous Metals in Acid Soils," by I. A. Denison and R. A. Hobbs. *Journal of Research,* Bureau of Standards, Vol. 13, p. 125.

[6] *Transactions,* Am. Electrochemical Soc., Vol. 29 (1916), p. 449.

[7] "Corrosion of Metals," by U. R. Evans, Edward Arnold & Co., London, England.

The decrease in the rate of penetration with time manifested by the Bureau of Standards specimens and by a number of pipe systems for which records have been kept, can be explained by the protective effect of the corrosion products that are formed, or by the restriction of oxygen resulting from the settling of the soil in the trench and the increased uniformity of trench conditions. The determination of which change is the more effective is important since, as will be shown later, it is a factor in decisions as to the effectiveness of protective coatings.

The Reduction of Corrosion Losses

To be logical and complete, this paper should contain a section dealing with the estimation of rates of corrosion and their relation to the useful life of pipe. To be helpful, such a discussion would have to be long, since the problem is a complicated one. The writer has chosen, therefore, to avoid this subject for the present and to proceed directly to a discussion of the reduction of corrosion losses. The proposed methods follow from the ideas set forth in the preceding section. The recommendations follow the form of the ancient and well-known recipe for the cooking of a hare.

Soil Corrosivity Surveys.—As one should first catch the hare, so should the engineer first locate his corrosion. The most serious corrosion, of course, will locate itself, but as soon as the engineer is surprised by the first leak in his line, he begins to wonder how much of the line is about to fail. Experience indicates that many miles of line have been needlessly uncovered and reconditioned because the owner feared that the first few leaks would be followed by others with increasing frequency. As a result of investigations at the Bureau of Standards, based on the principles of underground corrosion which have just been enumerated, several practical methods have been developed by which the pipe-line operator can determine the point of most serious corrosion. The principles and limitations of the older and more frequently used resistivity test have been described by Shepard.[4] The method has been tested in widely separated sections of the United States under several soil conditions. In all cases it has been the means of locating some of the corrosive areas. Ewing made a study of the corrosiveness of soils along the right of way of an oil line in one of the Central States, using several methods for testing the soils and comparing the results with the replacement records for the pipe line.[8] He concluded that neither soil resistivity nor soil acidity alone is a satisfactory indicator of soil corrosivity and found that by combining the two tests with information as to soil types a useful result could be obtained. He also developed a formula for determining what part of a line should be protected against corrosion and a method for determining the economic value of soil corrosion surveys.

It is scarcely to be expected that any correlation of soil conditions at a specific time with the corrosion occurring at the selected point throughout an extended period of time can be perfect, since the action of the soil is modified by the effects of temperature, oxygen, and moisture supplies, and by local conditions which often cannot be recognized. Moreover, until methods for controlling corrosive conditions are much improved, the corrosiveness of the soil cannot be expressed by a single value. When such value is available it must be understood that it is an average and that in individual cases the observed value can be expected to be greater or less by a considerable percentage. Nevertheless, values for corrosiveness are quite useful and soil surveys which indicate even in a general way that the soil in question is not corrosive, mildly corrosive, or very corrosive, may more than justify their cost.

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[8] "Corrosion Surveys for Transmission Lines and Distribution Systems," by S. P. Ewing, *Proceedings*, Am. Gas Assoc., 1934, p. 846.

It seems probable that if a pipe line traversing open country is to be laid or reconditioned, a systematic soil survey by a competent engineer will be justified. If the line traverses territory in which the natural soil conditions have been seriously modified by Man, as city streets containing fills of foreign material, sewer leaks, or discharges from chemical plants, or if the pipe lies beneath pavement, the economy of a survey is somewhat more doubtful, although success under these adverse conditions has been reported by Smith.[9] For the benefit of those who wish to study the subject, a list of the more important recent articles on soil corrosivity surveys is included in the Appendix.

Although probably less than 10% of the underground pipes of the country is in corrosive soil, it is likely that any extensive corrosion survey will locate one or more places where a reduction of the possible corrosive effects of the soil seems desirable. The engineer is then faced with the choice of a method for maintaining his line in one or more localities. The preferable method will depend on local conditions. Since corrosion underground has been attributed to variations in the supply of oxygen at different points along the line, an obvious method of preventing it is to secure uniform conditions with respect to oxygen. Unfortunately, the application of this principle is limited. To bury a pipe deeply is helpful unless the change in soil character and moisture with depth is unfavorable. Likewise, uniform tamping of the soil is conducive to uniform conditions. It has been suggested that the laying of a pavement over the pipe would reduce the oxygen supply and, consequently, the corrosion. This may be true only if the entire line is under pavement; otherwise, one might expect galvanic action between the pipe beneath the pavement and that in unpaved streets, with the corrosion occurring under the pavement because of the smaller quantity of oxygen there. Refilling the trench partly with less corrosive soil has been tried, but no reports as to the effectiveness of the treatment are known to the writer.

Selection of Materials.—The demand for a less corrodible pipe material is perennial, and a variety of ferrous alloys have been offered to meet this demand. The results of the investigation by the Bureau of Standards do not encourage hope in this direction if the cost of the material is considered.

It is possible that a ply-metal pipe having a corrosion-resistant metal on the outside may be developed which will not be too costly for use under conditions in which the renewal of the pipe would be expensive. Copper and high copper alloys resist nearly all soils better than ferrous materials and are being used successfully for pipes of small diameter. However, very thin copper sheet used as a shield for bituminous coatings has failed under some soil conditions.

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[9] "A Practical Soil Corrosion Survey with Shepard Soil Corrosivity Rods," by A. V. Smith, *Proceedings*, Am. Gas Assoc., 1932, p. 790.

Those demanding more resistant pipe materials may do well to consider the significance of curves representing the change in rate of penetration with time as has been previously mentioned. Whatever the cause for the decline in the rate of pitting, there can be little doubt that in many soils this decline is so great that corrosion penetrates but slowly after the first few years. In such soils the failure of pipe lines could be minimized by providing a wall sufficient to withstand the initial attack and the subsequent slow penetration. In many instances the wall thickness required for an unusually long life before penetration, does not appear to be prohibitive. It may be better engineering to provide extra wall thickness rather than to invest in corrosion-resistant materials or protective coatings.

Experience has shown that the life of many protective coatings is rather limited. It is possible, however, that the usefulness of short-lived coatings has been under-estimated. If the decrease in the rate of corrosion previously referred to is largely the result of the settling of the back-fill of the trench (that is, the establishment of more nearly uniform conditions along the pipe with respect to its contact with the soil and to the supply of moisture and oxygen), a coating which will last until these conditions have been established will eliminate that part of the corrosion-time curve which represents the period of rapid corrosion and will greatly extend the life of the pipe.

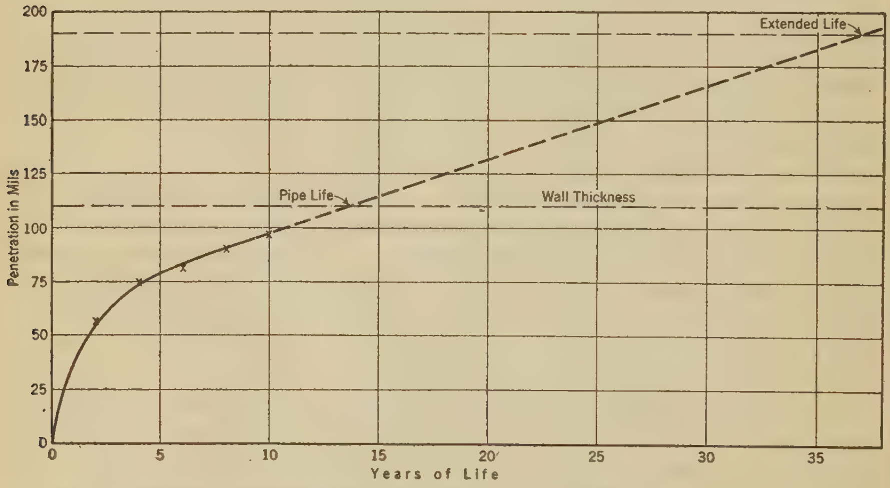

Fig. 4.—Penetration-Time Relation for a ¾-Inch Steel Service Pipe in Susquehanna Clay, at Meridian, Miss.

The effect of an increase in the thickness of the wall of a small pipe and the service performed by a temporary protective coating on such a pipe can both be seen by studying Fig. 4, the solid part of which shows the maximum penetration-time relation for wrought-iron and steel specimens in Susquehanna clay near Meridian, Miss. This curve is offered not as representative of corrosion in all soils, but only to illustrate the action of one soil with respect to ferrous metals. To be on the conservative side the curve has been extrapolated as a straight line, although it may bend downward because of the protective effects of corrosion products.

Consider the corrosion of a $\frac{3}{4}$-in. water service pipe the wall thickness of which is 0.11 in. It traverses a stratum of Susquehanna clay, a common Southern soil with stiff, plastic, red, heavy clay subsoil which becomes mottled in the lower part with gray, or bluish gray, red, and yellow. Fig. 4 indicates that the wall of this pipe will be punctured in a little less than 14 yr. If extra strong pipe, with a wall thickness of 0.154 in., or 40% greater, is used the life of the pipe should be approximately 26 yr, or nearly twice as long. If the standard pipe is galvanized or protected by a bituminous coating which will prevent corrosion for 5 yr (the period represented by the steep part of the curve), and if the corrosion decreases, only because of the settling of the trench as suggested previously, corrosion will begin at the end of the 5-yr period and progress as indicated by that part of the curve beyond the 5-yr point. Under this hypothesis, the application of the coating is equivalent (for corrosion-resisting purposes) to the addition of 80 mils (1 mil = 0.001 in.) to the thickness of the pipe wall, and the expected life of the pipe is extended from 14 to 37 yr, as is indicated by the intersection of the horizontal line representing 110 + 80 mils with the pitting-time curve. Thus, a coating lasting 5 yr may extend the life of a pipe 23 yr.

It must be remembered, however, that any conclusion is only as valid as the premises on which it is based. It is improbable that the decrease in the rate of corrosion is solely the result of the settling of the back-fill. Probably the formation of corrosion products plays an important part. If the settling of the trench has no effect, when the coating fails, corrosion begins as on a newly laid pipe and the extension of its life by the coating is equivalent only to the life of the coating. Probably the actual condition lies somewhere between the two extreme assumptions. The chief purpose of the present discussion is to call attention to the need for more complete and definite data on corrosion processes. Until such information is available the safe course is to provide as much protection for buried lines as can be justified.

Protective Coatings.—The idea is common that a moderately good coating is satisfactory for a moderately corrosive soil. There is no fundamental relation, of course, between the rates of deterioration of coating and of pipe since they are quite different materials and deteriorate for quite different reasons. The period of effectiveness necessary to justify the use of a coating is longer in a mildly corrosive soil than in a severely corrosive one. If a coating costs 10% of the cost of the line and if it prolongs the life of a line from 10 yr to a little more than 11 yr, it pays for itself; but if it lasts the same length of time on a line which would last 30 yr without the coating, it has increased the life of the line only 3% at a cost of 10% plus the interest on the additional investment. Recently, a water-works engineer in planning a new line, allowed a life extension of 5 yr for a coating to be applied to a line having an estimated life, bare, of 70 yr. A little thought will show that the interest paid on the initial investment for the coating will double that investment several times in the course of the life of the line, and that the cost per year of the protected line will be considerably higher than it would have been without the coating.

In 1924, the Bureau of Standards buried a considerable number of specimens of galvanized and lead-coated pipe in a variety of soils. These coatings have deteriorated slowly and until recently it has not been possible to make statements of practical value regarding their usefulness. The Bureau's data as to the corrosion of galvanized steel indicate that the protection afforded by the zinc coating is to be ascribed to the low corrodibility of that metal rather than to the fact that the zinc is anodic to steel. Tests of 2-oz coatings of zinc applied to pipe and sheet steel indicate that rust spots may be expected in a few soils within 4 yr, whereas in other soils more corrosive with respect to steel, rust will not appear for 6 or 8 yr.

At this time a definite relation between any soil characteristic and the rate of corrosion of zinc has not been recognized. Some acid soils appear to attack the coating, whereas other soils having higher acidity do not seriously affect it. In one severely corrosive alkali soil a 2-oz zinc coating failed within 2 yr. In less corrosive alkali soils, however, the same coating has shown much more satisfactory resistance.

A comparison of the rates of penetration of lead-coated steel and plain steel does not show, definitely, an increased rate of corrosion of the steel after puncture of the lead because of electrolytic action between the two metals. Sometimes, the punctured lead coating appears to corrode more rapidly than bare steel and sometimes less rapidly. Whether the differences in the rates of corrosion are accidental or whether they are the results of different electrolytes is not known.

Research on bituminous pipe coatings has been handicapped by the inability of the Bureau to keep the same man on the problem continuously. Most of the bituminous coating work at the Bureau has been done by research associates supported by the American Gas Association and the American Petroleum Institute. The results of their work will be much more valuable after later examinations of coatings which they have had exposed to soils since 1929 and 1930. These results are most directly applicable to steel pipe lines of moderate diameters. The recommended treatment for these lines may require some modification before they are suitable for service pipes, cast-iron pipe, or pipes of large diameter.

The work indicates that the failure of properly applied bituminous coatings is usually caused by stones or earth clods. The remedy indicated is to shield or reinforce the coating by a layer of rather stiff material, such as roofing felt. For soils that are destructive to organic fabrics, a metallic or asbestos reinforcing material is preferable. As yet, the lives of the better types of bituminous coatings have not been determined. However, a sufficient number of specimens have been buried to yield satisfactory data on coating life if the project can be continued over the period of time necessary for an adequate exposure of the coatings.

Several satisfactory methods for determining the condition of a pipe coating have been developed, but no entirely satisfactory accelerated test for coatings has been proposed. While the desirability of buying coatings on the basis of performance specifications has been recognized, no such specifications are in general use and the lack of convincing data makes the writing of such specifications, at this time, difficult and unsatisfactory.

In 1933 an electrical test[10] was developed whereby the application of a continuous coating could be insured. The writer had an opportunity to test a coating which had been applied so as to meet this continuity test. After the coated pipe had been laid in the ditch and covered, the test indicated that one or more bare spots had developed. In other words, the benefits of the careful application of the coating had been lost to a large extent because of careless treatment of the coated pipe.

The purchaser of a pipe coating should consider not only whether its fundamental properties are satisfactory but also the probability of getting it in the ground in a satisfactory condition. If the size or weight of the pipe section is so great that it cannot be handled commercially without injury to the coating, the advisability of the coating is at least debatable.

Cathodic Protection.—Fortunately, a method of treating coated pipes has been developed within the last few years which, under favorable conditions, protects the abraded parts. This is known as cathodic protection and has been well described by Kuhn[11] and by Thayer.[12] Essentially, the method consists in maintaining a negative charge in the pipe line with respect to the adjacent earth by means of a superimposed current supplied by a rectifier. The economic success of the method depends on the insulating properties of the protective coating. If the coating is impervious to moisture and has not been broken in too many places, an inexpensive amount of current may prevent corrosion. Cathodic protection of bare lines, or of coated lines in metallic contact with bare lines, is not feasible because of the amount of current required.

NEED FOR CORROSION RESEARCH

The purpose of this paper is primarily to show the status of the fight against corrosion. If the statements that have been made herein regarding rates of corrosion have been indefinite, and if the suggestions as to methods of reducing corrosion have been vague, it is because the collection of satisfactory corrosion data is necessarily slow and expensive. A definite plan of study has been outlined by students of corrosion, and definite progress has been made. The continuation of the work for a few more years should yield definite results.

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[10] "The Detection of Flaws in Pipe Line Protective Coatings before Burial," by G. W. Clarvoe, *Pipe Line News*, Vol. V. No. 8, p. 13, July, 1933.

[11] "Cathodic Protection of Underground Pipe Lines from Soil Corrosion," by R. J. Kuhn, *Proceedings*, Am. Petroleum Inst., 1933.

[12] "The Development and Application of a Practical Method of Electrical Protection for Pipe Lines," by Starr Thayer, *Proceedings*, Am. Petroleum Inst., 1933.

APPENDIX

CURRENT ARTICLES ON UNDERGROUND CORROSION

Rates of Corrosion

Logan, K. H., Ewing, S. P., and Yeomans, C. D. Soil Corrosion Studies: Soils, Materials and Results of Early Observations. *Technologic Papers,* Bureau of Standards, Vol. 22, p. 447, 1928 (T 368).

Logan, K. H., and Grodsky, V. A. Soil Corrosion Studies, 1930: Ferrous Materials. *Journal of Research,* Bureau of Standards, Vol. 7, p. 1, 1931 (RP 329).

Logan, K. H. Soil Corrosion Studies: Non-Ferrous Metals and Metallic Coatings. *Journal of Research,* Bureau of Standards, Vol. 7, p. 585, 1931 (RP 359).

Logan, K. H., and Taylor, R. H. Soil Corrosion Studies, 1932. *Journal of Research,* Bureau of Standards, Vol. 12, p. 119, 1934 (RP 638).

Scott, G. N. Pit Depths as a Measure of Soil Corrosion. *Proceedings,* Am. Petroleum Inst., Vol. 14, Pt. 4, p. 204, 1933. (Adaptation of soil corrosion pit depth measurements for size of sample.)

Soil Surveys

Shepard, E. R. Pipe Line Currents and Soil Resistivity as Indicators of Local Corrosive Soil Areas. *Journal of Research,* Bureau of Standards, Vol. 6, p. 683, 1931 (RP 298).

Denison, I. A. Correlation of Certain Soil Characteristics with Pipe Line Corrosion. *Journal of Research,* Bureau of Standards, Vol. 7, p. 631, 1931 (RP 363).

Denison, I. A. Methods for Determining the Total Acidity of Soils. *Journal of Research,* Bureau of Standards, 1933, Vol. 10, p. 413, 1933. (RP 539).

Smith, W. T. How Soil Corrosiveness Can Be Measured. *Gas Age Record,* Vol. 70, p. 129, 1932.

Logan, K. H. Soil Corrosion Surveys. *Production Bulletin No. 207,* Am. Petroleum Inst., p. 142, June, 1931; also, *Oil and Gas Journal,* Vol. 30, No. 3, p. 28, June 4, 1931.

Smith, A. V. A Practical Soil Corrosion Survey. *Proceedings,* Am. Gas. Assoc., 14th Convention, p. 790, 1932.

Weidner, C. R., and Davis, L. E. Causes of Pipe Line Corrosion Scanned. *Oil and Gas Journal,* Vol. 30, No. 26, p. 41, 1931; also, *Production Bulletin No. 208,* Am. Petroleum Inst., p. 36, 1931.

Cathodic Protection

Schneider, W. R. Cathodic Protection. *Western Gas,* Vol. 9, No. 4, p. 21, April, 1933.

Thayer, Starr. The Development and Application of a Practical Method of Electrical Protection for Pipe Lines. *Proceedings,* Am. Gas. Assoc., 1933 Convention; also, *Proceedings,* Am. Petroleum Inst., Vol. 14, Pt. 4, p. 143, 1933.

Kuhn, R. J. Cathodic Protection of Underground Pipe Lines from Soil Corrosion. *Proceedings,* Am. Petroleum Inst., Vol. 14, Pt. 4, p. 153, 1933.

Protective Coatings

Scott, G. N. API Coating Tests Progress Reports. *Proceedings,* Am. Petroleum Inst., 1930, 1931, 1932.

Ewing, S. P. AGA Studies of Coatings for Pipe Lines. *Proceedings,* Am. Gas Assoc., p. 774, 1931.

Distribution Sub-Committee Report on Inspection and Tests of Specimens Removed in 1932. *Proceedings,* Am. Gas Assoc., p. 741, 1933.

Boyd, G. H. Pipe Coatings and Corrosion. *Proceedings,* Am. Gas Assoc., p. 737, 1933.

Clarvoe, G. W. The Detection of Flaws in Pipe Line Protective Coatings before Burial. *Pipe Line News,* 1933.

Ewing, S. P. AGA Studies of Coatings for Pipe Lines. *Proceedings,* Am. Gas Assoc., p. 774, 1931; p. 741, 1933.

Fitzgerald, Charles. Corrosion and Pipe Line Coverings. *Production Bulletin No. 20,* Am. Petroleum Inst., p. 126, 1930.

Gill, S., and Karl, F. Evaluation of Production Corrosion Losses. *Production Bulletin No. 208,* Am. Petroleum Inst., 1931.

Scott, G. N. API Coating Tests. *Production Bulletin No. 208,* Am. Petroleum Inst., p. 53, 1931.

Scott, G. N. API Coating Tests. *Production Bulletin No. 210,* Am. Petroleum Inst., p. 114.

Weidner, C. R. Protecting and Reconditioning Pipe Lines. *Bulletin No. 6,* Am. Petroleum Inst., Vol. 8, p. 352, 1927.

DISCUSSION

THOMAS F. WOLFE,[13] M. AM. SOC. C. E. (by letter).—A concise description of the corrosion study being conducted at the National Bureau of Standards, is presented in this paper. In spite of the fact that metal pipes have been used underground in the United States for considerably more than a century, the investigation of the corrosive effects of soil had not been given the attention it deserved until investigators at the Bureau began their studies, which were based on a long-time plan and covered a number of different types of soil. In some of the tests conducted prior to this development, changes in management or in the personnel resulted either in discontinuing the test or in the loss of some of the data that would have been useful in interpreting the results. The tests at the Bureau of Standards have been based on a definite plan and from the very beginning have been under the direction of one man continuously thereafter.

The impetus responsible for the intensive study of soil corrosion was probably furnished by the fact that oil and gas lines installed in the early days of cross-country transmission were beginning to show signs of deterioration. Many of these lines were laid in soils that were more corrosive than soils in other sections of the country where pipes of considerably greater age were still giving service. These older pipes as a rule were cast-iron water mains and gas mains and had wall thicknesses in excess of that in the pipe used for gas and oil transmission lines.

The original idea behind the tests at the Bureau was not to develop information concerning relative merits of pipe materials when exposed to soil corrosion, but rather to determine the corrosive qualities of soils. However, the use of a number of types of ferrous pipe specimens in the work has led some engineers to use the data for the comparison of materials. Although the continuation of the tests described by Mr. Logan, at some future time, may warrant drawing conclusions as to relative lives of materials under similar soil conditions, it would seem wise at this time to reserve judgment on this phase of the problem until more data are available. It cannot be stressed too strongly that this work is really pioneer work and, although of considerable value, great care should be taken in using data that are necessarily based on short-time testing.

One of the dangers to be avoided in the use of these soil corrosion data is to attempt at this time to develop any hard and fast rule regarding pipeline depreciation. The best guide as to what depreciation actually occurs is to examine specimens removed from the property under consideration. The attempt to correlate any given soil with one of the soils presented in the paper, without having an intimate knowledge of soils, may lead one far astray. Even more serious error would result from attempting to predict depreciation according to the geographic location of pipe; for instance, one of the most corrosive soils under test is peat. The only test specimens buried in this particular soil in the original tests happens to be in Milwaukee, Wis. However, very few of the pipe lines in service in Milwaukee are actually laid in peat soil consequently, any one examining the data and predicting a life for Milwaukee pipe based on the specimens in Milwaukee would be very much in error.

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In addition to valuing pipe lines the data may finally come into use in the selection of materials for underground pipe construction. Under some conditions longevity is of much more importance than under other conditions. For instance, a gathering line in an oil field that has a limited life need last no longer than the field. On the other hand, a water line in a city street would properly be considered a permanent necessity. The economical angle varies also with location. Maintenance, reconditioning, or re-laying of cross-country lines at comparatively short intervals may be satisfactory from the standpoint of the operator of the lines, because of the fact that such lines are in open country and the excavation costs are relatively small. In the case of lines in city streets, conditions are reversed. The cost of the work is high because of the existence of pavements. As a rule the lines can only be shut down at great inconvenience to the consumers. In addition to the cost feature, the effect on public relations of repeatedly opening streets is very bad. The net result is that a greater expenditure to provide additional life is warranted for water and gas main construction within cities.

When further experience becomes available, the use of soil surveys to determine the amount and type of protective coating necessary may develop a regular procedure from which tangible results may be expected. It is doubtful if soil surveys within cities will be particularly useful. In many cities street grades have been changed by filling with cinders, ashes, slag, and various types of waste material of a more or less corrosive nature. In other cities, the construction of pavements and the installation of sewers, conduits, and other underground utilities may have disturbed the soil so that it no longer resembles the natural soil of that particular region.

The tests of coatings discussed in the paper have not been in process long enough to make it possible to draw any conclusions as to which type of coating would be suitable for all conditions. The general conclusion can be drawn, however, that it is difficult to predict what additional life may be expected when any of the ordinary coatings are used. Coatings which, in themselves, might appear satisfactory are often disturbed during installation, and although the coating that remains on the pipe may be effective, much of it may be removed, with the net result that corrosion may still continue its destructive work. The fact that the tests have shown that the rate of corrosion decreases with time would indicate that one of the best means of prolonging the life of a pipe is to provide additional wall thickness. The advantage of additional wall thickness over a coating is that the thicker pipe needs no special handling to avoid loss of its added protective qualities.

When one considers that hundreds of millions of dollars of pipe lines are now in service underground, one cannot help but be impressed by the necessity for developing information on soil corrosion. It is to be hoped that the

Bureau of Standards will be able to continue its investigations and that from time to time new specimens and new soils may be brought under the test to the end that, at some future time, soil corrosion data will be so extensive that they can be used by the Engineering Profession both in valuation and design work to the same extent as other scientific data.

JOHN R. BAYLIS,[14] ASSOC. M. AM. SOC. C. E. (by letter).—A brief but important factual summary of the principal findings in the field of underground storage, is contained in this paper. The writer is in full accord with Mr. Logan's interpretation of the facts as revealed by the soil corrosion study on practically all phases of the work. Certainly, this problem is now much better understood than before the work at the National Bureau of Standards was started.

The main comments that can be made by the writer, except to praise the work, is to urge that it be continued and that it be extended to include varying the conditions for the chief types of soils in which pipe are now being used. Mr. Logan has shown that there is marked similarity in the corrosion of the various ferrous metals buried in the same locality. In the localities where the samples of pipe are now buried, it is likely that such factors as moisture, temperature, penetration of oxygen into the soil, and acidity of the soil, are the same for samples buried at a certain point, although they vary for different localities. No one knows definitely the effect of varying the moisture and other factors that influence the rate of corrosion.

Another series of tests might well be undertaken with provisions for varying most if not all of the factors influencing the corrosion. Part of the soil should be covered to represent a pavement. Other parts should be subjected to artificial rainfall. An impervious stratum should be constructed under another section of the soil so that some samples may be permanently under water in the soil, others under water part of the time and out of water part of the time, and other samples out of the water all the time, except for the moisture received from rains or from capillary attraction. These are a few of the variables that should be studied.

It is likely that soils have been selected already which vary widely in moisture content, temperature, acidity, etc., but there are so many factors influencing the results that the effect of each factor for each type of soil should be studied. This implies costly experimentation, but the enormous value of pipe now buried in the ground and gradually depreciating in value, due to corrosion, justifies the expenditure of a very substantial sum to obtain full information on the subject.

E. P. FETHERSTONHAUGH,[15] ESQ. (by letter).—The gradual accumulation of information on soil corrosion and the growth of knowledge of the subject as a result of studying the data are well described in this paper. The theory of differential aeration as a probable major cause of soil corrosion is apparently supported by many of the observed and recorded facts and, if fully substantiated, may prove of great usefulness in forecasting what is likely to occur under a given set of conditions. It also suggests some important possibilities with regard to the value of protective coatings in prolonging the life of pipe buried in corrosive soils.

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In the references in the paper to the observed reduction in the rates of corrosion and the causes of this phenomenon, there seems to be an assumption that special abnormal conditions should be sought to account for the shape of the curves that show decreasing rates of penetration with increasing life. A consideration of an assumed set of ideal conditions would seem to indicate that the decreasing rate of penetration might be regarded as a normal, rather than as an abnormal, phenomenon.

The ideal conditions suggested are that: (*a*) The source of the electromotive force causing the pitting remains constant; (*b*) the resistance of the path of the current through the earth, the pipe, and the pit, remains constant; and (*c*) throughout its growth, the shape of the pit remains constant.

Conditions (*a*) and (*b*) would result in a constant current flowing from the pit to the earth and, therefore, equal weights of metal removed in equal intervals of time. With equal weights or volumes of metal removed in equal intervals of time, the volume of the pit will increase at a constant rate, but since its shape is assumed to remain unchanged, its depth will increase as the cube root of the time. These assumptions suggest a fundamental law governing the relation between penetration and time which may be expressed by the equation:

$$D = A\sqrt[3]{T} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which A is a constant, or the depth of the penetration at the end of the first year; T is the time of exposure, in years; and, D is the depth of the pit at the end of T yr.

It could not be expected that observed data from tests in which there are so many possible variables would follow any mathematical curve exactly, but the similarity between the curve plotted from Equation (1) and the graphs plotted from observed data is striking, and even in other graphs of observed data, in which there is a greater departure, their general form seems to be at least characterized by a fundamental law such as that expressed by the formula.

It seems, therefore, that it is not necessary to attribute the decreasing rates of penetration to any varying condition, such as the gradual settlement of back-fill, as a rapid decrease is to be expected when all the aforementioned conditions are constant. The decreasing rates of loss of weight may perhaps require such explanations, but it is interesting to note that this decrease is in many cases comparatively small, which might suggest that the conditions assumed herein do not differ greatly from those which, in some cases, obtain in and around a buried pipe.

The value of this method of approaching this intricate problem lies in the fact that it suggests a normal law for the rate of penetration with which observed results may be compared, in order that the importance of various abnormal or fluctuating influences, such as those suggested by the author, may be more closely evaluated. Equation (1) may also prove of some value in an attempt to calculate from a single set of observations the time in which a pipe might be penetrated if the conditions of exposure remain constant.

JOHN F. SKINNER,[16] M. AM. SOC. C. E. (by letter).—The information presented by Mr. Logan is interesting in the light of experience with a pipe line (designated Conduits I, II, and III) in Rochester, N. Y. Conduit I, about 28.3 miles long, is still in service although constructed in 1874. The first 10 miles of it, 36 in. in diameter, is made of wrought iron, $\frac{3}{16}$ in. thick. Then follows 3 miles of 24-in. wrought-iron pipe. The remainder of the conduit is cast iron, with a diameter of 24 in. Conduit II was completed in 1894, twenty years later. It consisted of 12 000 ft of 6-ft brick tunnel and a pipe line of which the greater part (nearly 26 miles) was made of riveted steel plate, $\frac{1}{4}$ in., $\frac{5}{16}$ in., and $\frac{3}{8}$ in., in thickness.

At the end of 1913, when the writer prepared a report[17] on the subject, there were 727 holes in Conduit II through approximately 450 of the steel sheets. In the 36-in. section of Conduit I (which was 20 yr older and made of thinner material), there were only 34 holes in 14 sheets. The next leak through the wrought-iron plate occurred on May 29, 1916, after a hiatus of 10 yr; and, to January 1, 1934, there had been a total of 81 holes in 29 sheets of Conduit I, which was then more than 58 yr old.

For several miles these conduits are parallel and only 20 ft apart, so that the soil conditions are practically identical. The first leak due to pitting the wrought-iron plate of the 36-in. Conduit I was discovered on July 12, 1894, after the pipe had been in the ground nearly 19 yr. The first leak through the steel plate of Conduit II was discovered on May 21, 1900, after the pipe had been in the ground nearly 6.5 yr, and the second on October 15, 1900; and 13 holes occurred in 1901 through 10 sheets scattered over 9.7 miles of conduit. All these leaks were through the $\frac{1}{4}$-in. plate. The first leak through the $\frac{5}{16}$-in. plate was in 1904; and the first through the $\frac{3}{8}$-in. plate was in 1908. By the end of 1913, when the aforementioned report was published, 31 978.7 ft of Conduit II had been excavated, scraped clean, and re-coated on the outside. This painting covered 23.5% of the conduit and included 77.5% of the holes discovered.

During the fall of 1933, a section of the conduit was uncovered upon which many leaks had been repaired since the last re-coating operations. To the time this work was begun, 2 260 holes in the steel pipe had been repaired. Including the foregoing work, a total of 2 532 holes had occurred prior to January 1, 1934.

When re-coating is in progress, a great many incipient leaks are scratched through in the process of scraping, thus anticipating holes which, if undisturbed, would occur in the following year or two. The first 200 ft of pipe re-coated in the fall of 1933 was laid through boulders which had injured the original coating. Along this section the plates showed an unusual amount of pitting, and the holes were more numerous than they were farther along the line where the trench was free from boulders. The coating of California asphalt had apparently given protection, where it was not abraded, for about ten years.

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[16] Cons. Engr., Los Angeles, Calif.

[17] Report on the Steel Plate Pipe Conduit II of the Rochester, (N. Y.), Water-Works, by John F. Skinner, M. Am. Soc. C. E.

Conduit III was constructed during the World War (1914–1918). About 17.6 miles of this line was a lock-bar steel pipe made of plates $\frac{1}{4}$ in and $\frac{5}{16}$ in. thick with riveted circular seams. A few leaks have occurred, generally where the painting was found to be defective. It has been suggested that the subsequent laying of the steel pipe parallel to the wrought-iron pipe may have had an inhibiting effect upon the corrosion of the wrought iron. The answer to this is that 58% of the holes in the wrought-iron pipe (and 52% of the sheets affected) occurred after the steel pipe was installed. The difference in the pitting of steel and wrought iron, in practically the same location, over a long period, is worthy of serious consideration.

In 1901, Professor F. L. Kortright investigated the causes of pitting in Conduit II. He examined samples of earth from twenty-five test pits along Conduit II, as well as rock from the vicinity, ground-water from the trench excavations, and the contents of pits in the steel, both outside and inside the pipe.

From his analysis, he concluded that the earth surrounding the conduit did not affect it except that, being more porous in some places than in others, it admitted the ground-water more freely to the pipe in the places where it was porous and in this way allowed carbon dioxide and oxygen freer access. He reported that the rock in the vicinity of the pipe had no effect on it; but that the injurious constituents in the water were dissolved oxygen and the carbon dioxide that was brought to the pipe in large quantities by the calcium carbonate, which is loosely combined with it.

Electrical measurements were made between Conduits I and II, and between the conduits and the surrounding earth, but neither current nor difference of potential could be detected.

From the foregoing remarks it will be noted that in nearly identical locations and exposure, the wrought-iron Conduit I resisted the first pitting through a $\frac{3}{16}$-in, plate nearly three times as long as the steel Conduit II and that, in later years, although it is twenty years older and of thinner material, it has developed far less pitting through the plate. Although the comparisons shown in the paper exhibit a marked uniformity of action for different pipe materials under the same exposure, it must be evident that a much longer period, commensurable with the life of such structures, may show very different results, such as have been observed in the different behaviors of different materials in places where one has replaced another and is subject to the same environment.

Mr. Logan is to be commended for the energy, imagination, and enthusiasm he has put into this work. It is hoped that he and his successors will be able to continue their observations over a period sufficiently long to insure reliable conclusions of lasting value.

F. N. SPELLER,[18] ESQ. (by letter).—The principal work on soil corrosion of metals and the study of protective coatings has been centered at the National Bureau of Standards. Considerable progress has been made in correlation of soil factors with corrosion, but much remains to be done before the practice of pipe-coating underground can be said to be on a sound engineering basis.

It is now probable that an economical and durable coating can be found for use in most of the corrosive soils. In many cases, as Mr. Logan points out, no coating is required. This is particularly true in the eastern part of the United States. In Philadelphia, Pa., the records of the local gas company over the thirty-six years, 1899 to 1935, indicate that small steel service pipes having a thickness of about $\frac{1}{8}$ in. have an average life of forty-seven years with no more than a thin coat of coal-tar paint which probably gave little advantage except to prevent more general distribution of pit-holes.

It seems very unlikely that a low-cost, rust-resisting metal will be found that is sufficiently durable in all corrosive soils, and (as Mr. Logan points out), there is no material difference between the corrosion resistance of the common ferrous metals; therefore, for the present, the work should be concentrated on improvements in protective coatings. Such coatings should be chemically resistant to soil water, mechanically strong or sufficiently reinforced to resist soil stresses, and thick enough to be durable and prevent contact between the soil and the metal. Reinforced bituminous coatings usually should not be less than $\frac{3}{32}$ in. in thickness.

Portland cement concrete coatings ($\frac{5}{8}$ in. to 1 in. thick) cost from 6 to 10 cents per sq ft, and are highly durable in many soils. They have given 35 to 40 yr of satisfactory protection on oil pipe lines; for instance, on lines crossing low salty marshes, and acid streams in coal-mining districts.

Research work on soil corrosion is of such general interest to tax-payers at large that it seems a portion of the expense of this work should be paid by the Government. Thus far, it has been carried out on this basis by the National Bureau of Standards with the co-operation of the American Petroleum Institute, American Gas Association, and pipe manufacturers, but has been seriously retarded by curtailment of appropriations due to the economy program. The general opinion of those who have followed this investigation is that it should by all means be continued in order to realize on the $300 000 already invested through the Bureau toward the solution of this problem.

K. H. LOGAN,[19] ESQ. (by letter).—As one who has been in close touch with the soil-corrosion investigation since its beginning in 1922, Mr. Wolfe is quite right in cautioning against the unrestricted use of the data for the comparison of materials or the prediction of pipe life. In many of the soils under consideration the corrosion is not sufficiently serious to permit a reliable estimate of pipe life. In a few soils, however, all or nearly all the thinner walled specimens have been punctured. It is expected that the data on the

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[19] Chf., Underground Corrosion Section, National Bureau of Standards, Washington, D. C.

12-yr old specimens, which will be available in the spring of 1936, will show the standard deviation for each material[20]. This will indicate whether or not there are significant differences between the rates of corrosion of the materials. As Mr. Baylis indicates, the causes and mechanism of underground corrosion need further investigation. To be of value any field investigation must involve a rather large number of specimens and, consequently, a very considerable expense for materials and labor. The outlook for the extension of the soil-corrosion investigation in the near future is not bright.

The relation between rate of penetration and time to which Professor Fetherstonhaugh calls attention, has been studied by several investigators. Dr. G. N. Scott[21] reached the conclusion that the relation for any soil could be represented by the equation, $P = \frac{U\,T}{B + T}$, in which P is the depth of the deepest pit at any time, T; B is a constant which is the same for all soils; and U is another constant different for each soil and equal to the ultimate pit depth.

Additional data recently obtained seem to indicate that some such an equation as that of Dr. Scott or Professor Fetherstonhaugh will represent the progress of corrosion in well-drained soils, but that in poorly drained soils the depth of penetration is nearly proportional to the duration of the exposure.

The interesting experience of the City of Rochester cited by Mr. Skinner deserves further investigation. Although the superior service rendered by Conduit I may be due to the superiority of the material used, two other explanations are possible. The map showing the locations of the conduits which the writer obtained from the City Engineer's Office indicates that, although the two older conduits lie quite near each other, the older line lies at a slightly higher elevation and, consequently, is somewhat better drained. It is possible also that the protective coating applied to the older line was superior either as to material or application. This is suggested by the fact that at least some of the failures of the older line occurred at points where the coating had been injured. It is probable that the troubles on the newer line have been accentuated by its rather frequent exposure for repairs. A thorough investigation of the causes of the differences between the performances of the two conduits might be of great value to those who have to construct similar lines.

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[20] "Soil Corrosion Studies, 1934—Rates of Loss of Weight and Pitting for Ferrous Specimens," by K. H. Logan, *Journal of Research,* National Bureau of Standards, Vol. 16, p. 431, May, 1936 (*Research Paper 883*).

[21] "A Preliminary Study of the Rate of Corrosion of Iron Pipe in Soils", by G. N. Scott, *Proceedings,* Am. Petroleum Inst., Vol. 14, Section 4, p. 212.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1940

THE ADJUSTMENT OF A LEVEL NET

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WITH DISCUSSION BY MESSRS. EARL F. CHURCH, W. H. RAYNER, AND GEORGE H. DELL.

SYNOPSIS

Two closely related methods of obtaining a "least squares" adjustment of the closing errors in a level net are described in this paper. The first method involves the writing, by inspection, of the appropriate normal equations and their subsequent solution by a process of converging increments; the second utilizes the principle of successive distributions, and is suggested by the Cross method of moment distribution.[2]

Both methods are well adapted to the use of the slide-rule and possess the advantages of simplicity and brevity, as compared with the conventional least squares solutions. They are intended to apply primarily to surveys of ordinary extent and accuracy, such as those executed by municipal and highway engineering organizations.

Beyond the consideration that, in practice, one method may appeal more than the other to a given individual, the reasons for presenting a dual treatment of the subject include the following: (*a*) The first method derives immediately from the principal hypothesis of the theory of least squares, and thus serves to substantiate the correctness of the second method; (*b*) the process utilized (in the first method) for solving the normal equations may be successfully applied to various other types of engineering problems (for example, to indeterminate structures); and (*c*) the parallel solutions show the inter-relations of the two methods and suggest a procedure that may be used to advantage in devising extensions of the distribution method to new types of problems.

NOTE.—Published in April, 1935, *Proceedings*.

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[2] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

INTRODUCTION

The precision that characterizes the planning and execution of present-day structures demands, in general, that reliable bench-marks be established and maintained throughout, at least, the period of construction. In towns and cities, bench-marks should be created with a view to permanence.

The value of a bench-mark depends as much upon the reliability of the elevation assigned to it as upon its ability to maintain itself at a fixed height. Much difficulty is encountered in attempting to arrive at the correct elevation of bench-marks by field work alone, due to unavoidable discrepancies which are found when closing in from various lines. It appears, at times, that the more one levels in a given locality the greater is the uncertainty in the bench-marks.

Much of this difficulty in the field can be avoided by adopting, in part, the methods that have been used by the United States Coast and Geodetic Survey in the execution of many thousands of miles of precise levels. Particularly commendable in this connection are the establishment of a definite standard of precision and the practice of running all level lines twice for the purpose of putting them to test; in case of satisfactory agreement, the mean of the observed differences in elevation is then used.

In spite of all precautions, however, discrepancies still persist, and some form of adjustment is necessary. Generally speaking, there are three methods of adjusting elevations, namely, (1) by "guessing"; (2) by using a "rule of thumb" (which may be worse than guessing); and (3) by calculating the most probable values in accordance with the laws of probability.

The latter method, under the title of "least squares," has been in use for more than a century, but its application has been restricted rather closely to problems in astronomy and geodesy. For such purposes, a high degree of precision is generally required, and, consequently, the treatises dealing with that method are likely to appeal to the layman as being too elaborate for his requirements, and the calculations overly precise and laborious.

The aim of this paper is to present methods whereby the closing errors in a level net can be adjusted with approximately the same time and effort as are required to solve the somewhat analogous problem of finding the bending moments in a framed structure. These methods are proposed for use principally in connection with work done with the ordinary engineer's level, in which instance it is doubtful whether the precision obtainable in the field warrants an attempt to deal with elevations or readings in units of less than 0.01 ft, or 0.005 ft. Nevertheless, with a little added effort, the precision of the calculations is capable of further extension.

The use of a "polyphase" type of slide-rule is of great assistance in performing the necessary calculations, as in both methods an entire series of values can then be read off with a single setting of the slide.

DEFINITIONS AND PRELIMINARY EXPLANATIONS

To facilitate description, the following terms will be used:

Control Bench-Mark: A bench-mark with an elevation which is to remain fixed, and to which the elevations of other bench-marks are to be adjusted.

Junction Point: A bench-mark at which three or more level lines meet.

Section: The length of level line connecting consecutive junction points or control bench-marks.

Intermediate Bench-Mark: A bench-mark located at some intermediate point of a section.

Sub-Section: A part of a section terminating at an intermediate bench-mark.

Circuit: A polygon or closed figure bounded by sections (but containing none within its interior).

Exterior Section: A section situated along the boundary of the net, and hence belonging to only one circuit.

Interior Section: A section situated within the interior of the net, and hence common to two circuits.

Positive Direction of a Section: The direction in which the correction is to be computed (regardless of whether the difference in elevation is positive or negative).

Direction Arrows.—"Main direction arrows" are those used to designate the positive directions of the sections. In general, these direction arrows serve also to indicate the direction in which the closing error of a circuit is calculated. In certain interior sections, however, the latter direction may be contrary to the former, and is then distinguished by means of an "auxiliary direction arrow," which is written beside the section and within the circuit in question, as shown in Fig. 1.

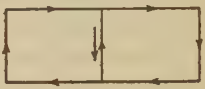

FIG. 1.

Weights.—The weight, p, of an observed difference in elevation in a given section depends upon the length of the section, the number of set-ups it contains, the number of independent measurements entering into the data, and the quality of instruments and survey methods used. On ordinary slopes, the number of set-ups is approximately proportional to the length of the line, and, therefore, it is customary to weight the observations in inverse proportion to the distance (which may be determined by stadia measurements, by pacing, or by scaling from a map). Moreover, the weight is directly proportional to the number of times the levels were run; that is, a difference in elevation which is determined from the mean of two separate runnings of a section is given twice as much weight as a difference in elevation obtained from a single running.

Effective Lengths.—It is convenient, in the problem of the level net, to deal with weights in terms of the "effective lengths" of sections. The effective length of a section is defined as the reciprocal of the weight; that is, $\frac{1}{p}$; consequently, it is directly proportional to the length of the section and inversely proportional to the number of measurements.

Closure Conditions.—Two types of closure will be considered, namely, the circuit closure and the route closure. By a "circuit-closure condition" is meant the requirement that the sum of the section corrections in a given circuit must be equal, and opposite in sign, to the closing error of that circuit.

If the level net contains two or more control bench-marks, conditions of route closure arise. By a "route-closure condition" is meant the requirement that the section corrections along an arbitrary route connecting two given control bench-marks must have a predetermined sum, equal (and opposite in sign) to the closing error along that route; this situation gives rise to the following term:

Conditioned Route: An arbitrary route, selected for purposes of adjustment, connecting two control bench-marks.

Condition Multipliers.—"Condition multipliers" are undetermined coefficients which are used to represent the closure conditions to which a section is subject. For example, Closure Conditions Nos. I, II, etc., are represented by the condition multipliers, C_{I}, C_{II}, etc., respectively. By this means, the correction for any section may be expressed as a linear function of the closure conditions.

Section Corrections.—The corrections to be determined for the observed differences in elevation along the various sections are called "section corrections." They are denoted by v_1, v_2, etc., the subscripts of which refer to Sections Nos. 1, 2, etc., respectively. By means of the theory of least squares it is found (see Appendix I) that the section correction for a given section is equal to the product of the effective length of the section by the sum of its condition multipliers. Expressed algebraically, the section correction, v_m, of Section No. m, with an effective length of L'_m, which is subject to Closure Conditions Nos. I, II, etc., is given by the equation,

$$v_m = L'_m \, (C_{\text{I}} + C_{\text{II}} + \ldots) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The number of closure conditions governing a given section is equal to the number of circuits or conditioned routes to which the section belongs.

Normal Equations.—Equations of the type obtained when the section corrections as expressed in Equation (1) are summed up around a circuit or along a conditioned route and placed equal, and opposite in sign, to the closing error, are known as "normal equations." The number of normal equations in any net is equal to the number of circuits, plus the number of conditioned routes.

DESCRIPTION OF METHOD 1; EXAMPLE NO. 1

For purposes of description, the solution will be divided into the following steps: (*a*) Preparation of plan for the adjustment, and calculation of closing errors; (*b*) formation and solution of normal equations; (*c*) calculation of section corrections and verification of circuit closures; and, (*d*) calculation of adjusted elevations of bench-marks.

To illustrate the detailed procedure, two examples will be given, the first involving only circuit-closure conditions, and the second entailing a route-closure condition.

The net chosen to exemplify Example No. 1 is illustrated in Fig. 2. It contains four circuits and nine bench-marks, five of which are junction points, and the remaining four, intermediate bench-marks. Point *A* will be assumed to be a control bench-mark, with an elevation of 416.15 ft. The approximate distances between bench-marks are shown in Fig. 2.

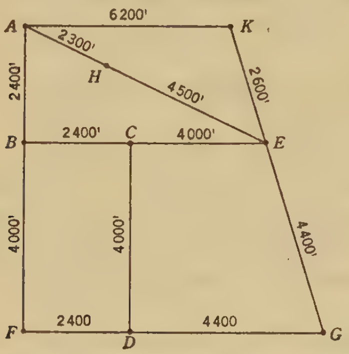

Fig. 2.—Sketch of Level Net.

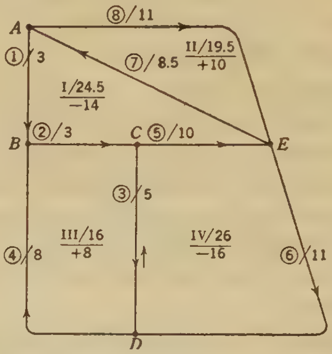

Fig. 3.—Plan for Adjustment.

The observational data are given in Table 1, Appendix II, and consist of field notes covering levels which were run over six routes, as follows: Route No. 1: *A* to *B* to *C* and return; Route No. 2: *C* to *D* to *G* to *E* and return; Route No. 3: *D* to *F* to *B* and return; Route No. 4: *C* to *E* (one way); Route No. 5: *A* to *H* to *E* and return; and, Route No. 6: *A* to *K* to *E* and return.

(a) Preparation of Plan for the Adjustment, and Calculation of Closing Errors.—The preliminary details necessary to put the data in readiness for performing the adjustment include the preparation of a plan and the determination of the closing errors, and may be performed in the following order:

(1) Prepare a plan showing the junction points and control bench-marks, and the connecting routes over which the leveling was executed;

(2) Assign a system of direction arrows;

(3) Number the sections, determine their effective lengths, and record these values on the plan;

(4) From the field data, calculate the observed difference in elevation over each section in the positive direction (that is, as shown by the main direction arrow); also, determine the observed differences in elevation over the sub-sections, which will be required in computing the adjusted elevations of the intermediate bench-marks; and,

(5) Number the circuits, calculate their total effective lengths and their closing errors, and record these values on the plan.

The plan for the adjustment for Example No. 1 is shown in Fig. 3, intermediate bench-marks being omitted. Direction arrows are assigned in such a way as to secure continuous routing through all the circuits with a minimum number of auxiliary direction arrows. This system of marking the sections possesses the following advantages: (1) It fixes the route to be taken in calculating the closing error of each circuit; (2) it determines the direction in which each section correction is to be calculated; and (3), it enables the normal equations to be formed by inspection.

In Fig. 3, the number of the section is shown by the figure enclosed within a circle, and the effective length is written to the right of the diagonal bar. Fig. 3 also contains, near the center of each circuit, a notation showing the number of the circuit, its total effective length, and its closing error (for example, $\frac{\text{I}/24.5}{-14}$ indicates Circuit No. I, with a total effective length of 24.5, and a closing error of − 0.14 ft).

The calculation of the effective lengths of sections and of the corresponding observed differences in elevation is contained in Table 2(*a*), Appendix II. It will be noted that the actual lengths are reduced to terms of relative lengths by dividing by 800. The use of the smaller units is a matter of convenience. The effective lengths are taken equal to the relative lengths of the corresponding sections, except for Section No. 5. Since this section was leveled only once (all other sections having been leveled twice), its effective length is twice its relative length.

In explanation of the calculation of the observed differences in elevation, consider Section No. 1, which extends from Point *A* to Point *B* in Fig. 2. The field notes, given in Table 1, Appendix II, show that this section is a part of Route No. 1 and was run both forward and back. In the forward running the observed difference in elevation was $422.59 - 416.15 = +6.44$; in the back running the result was $416.17 - 422.65 = -6.48$. In this instance the positive direction of the section (as shown by the direction arrow in Fig. 3) coincides with the forward running, and, consequently, the observed (mean) difference in elevation is + 6.46 ft (Column (8), Table 2(*a*), Appendix II). (In taking the arithmetic mean of an even number and an odd number, the even value is chosen for convenience; for example, in Section No. 7, Table 2(*a*), the mean of 9.20 and 9.07 is taken as 9.14). Table 2(*b*) contains the calculation of the observed differences in elevation over the sub-sections.

The calculation of the closing errors is contained in Table 3, Appendix II. The closing error of a given circuit is simply the sum of the observed differences in elevation of its component sections, taken in a continuous direction around the circuit in accordance with the direction arrows. The closure conditions corresponding to the various circuits are numbered in the same way as the circuits themselves, and, hence, the condition multipliers for Circuits Nos. I, II, III, etc. (Fig. 3), are C_{I}, C_{II}, C_{III}, etc., respectively.

(*b*) *Formation of Normal Equations.*—In accordance with Equation (1), the section corrections, taken in a continuous direction around the various circuits, are as shown in Table 4, Appendix II. When the sums of these corrections are placed equal, and opposite in sign, to the closing errors of the respective circuits (0.01 ft being taken, for convenience, as a unit), the following normal equations are obtained:

$$24.5\ C_{I} + 8.5\ C_{II} + 3\ C_{III} + 10\ C_{IV} = +\ 14 \qquad \text{(I)}$$

$$8.5\ C_{I} + 19.5\ C_{II} = -\ 10 \qquad \text{(II)}$$

$$3\ C_{I} + 16\ C_{III} - 5\ C_{IV} = -\ 8 \qquad \text{(III)}$$

and,

$$10\ C_{I} - 5\ C_{III} + 26\ C_{IV} = +\ 16 \qquad \text{(IV)}$$

In Equation (I) the greatest numerical coefficient is that of C_{I}, and is equal to the total effective length of Circuit No. I; likewise, in Equation (II) the greatest numerical coefficient is that of C_{II}, and it is equal to the total effective length of Circuit No. II; and, similarly, for Equations (III) and (IV). These quantities (namely, 24.5 C_{I} in Equation (I), 19.5 C_{II} in Equation (II), etc.) will be referred to as the "major" terms of their respective equations. The remainder of the quantities on the left-hand side of the equations will be called "minor" terms.

As a matter of fact, it is simpler to write the normal equations at once as in Table 5(*a*), Appendix II. They are written in columns which are numbered to correspond to the various closure conditions, the individual terms being entered in the lines with numbers that agree with the subscripts of the condition multipliers. The major terms (which are underscored) thus appear in diagonal formation. As already stated, their coefficients are merely the total effective lengths of the respective circuits (or conditioned routes).

Minor terms arise from sections which are subject to two (or more) closure conditions, and their coefficients are equal to the effective lengths of the sections involved. For example, in writing the minor terms of Equation (I), it is seen from Fig. 3 that Circuit No. I is tied to Circuit No. II by Section No. 7, with effective length, 8.5; to Circuit No. III by Section No. 2, with effective length, 3; and to Circuit No. IV by Section No. 5, with effective length, 10. Consequently, in Column (I), Table 5, Appendix II, 8.5 is entered in Line II, 3 in Line III, and 10 in Line IV.

In the case of sections marked with an auxiliary direction arrow the minor term is always prefixed with the negative sign. The constant, or right-hand, term of the equation is entered at the bottom of the appropriate column.

Checking the Normal Equations.—As the coefficients of the minor terms always appear in two of the normal equations, it is desirable to utilize this fact as a check. The coefficients should read the same vertically and horizontally.

Solution of Normal Equations.—From the following typical system of equations, in three unknowns, namely,

$$AC_{I} + bC_{II} + cC_{III} = k_1 \qquad \text{(I}a\text{)}$$

$$bC_{I} + BC_{II} + dC_{III} = k_2 \qquad \text{(II}a\text{)}$$

and,

$$cC_{\text{I}} + dC_{\text{II}} + CC_{\text{III}} = k_3 \ldots\ldots\ldots\ldots\ldots\ldots \text{(III}a\text{)}$$

Equations (Ib), (IIb), and (IIIb) may be written, thus:

$$AC_{\text{I}} = k_1 - (bC_{\text{II}} + cC_{\text{III}}) \ldots\ldots\ldots\ldots\ldots\ldots \text{(I}b\text{)}$$

$$BC_{\text{II}} = k_2 - (dC_{\text{III}} + bC_{\text{I}}) \ldots\ldots\ldots\ldots\ldots\ldots \text{(II}b\text{)}$$

and,

$$CC_{\text{III}} = k_3 - (cC_{\text{I}} + dC_{\text{II}}) \ldots\ldots\ldots\ldots\ldots\ldots \text{(III}b\text{)}$$

respectively. Let, $C_{\text{I}} = C_{\text{I-1}} + C_{\text{I-2}} + C_{\text{I-3}} +$ etc.; $C_{\text{II}} = C_{\text{II-1}} + C_{\text{II-2}} + C_{\text{II-3}} +$ etc.; and, $C_{\text{III}} = C_{\text{III-1}} + C_{\text{III-2}} + C_{\text{III-3}} +$ etc.

Upon substituting these values of C_{I}, C_{II}, and C_{III} into Equations (Ib), (IIb), and (IIIb), the following expressions, respectively, are obtained:

$$AC_{\text{I-1}} + AC_{\text{I-2}} + AC_{\text{I-3}} + \ldots = k_1 - (bC_{\text{II-1}} + cC_{\text{III-1}}) \\ - (bC_{\text{II-2}} + cC_{\text{III-2}}) - \ldots\ldots\ldots\ldots\ldots\ldots \text{(I}c\text{)}$$

$$BC_{\text{II-1}} + BC_{\text{II-2}} + BC_{\text{II-3}} + \ldots = (k_2 - bC_{\text{I-1}}) - (dC_{\text{III-1}} + bC_{\text{I-2}}) \\ - (dC_{\text{III-2}} + bC_{\text{I-3}}) - \ldots\ldots\ldots\ldots\ldots\ldots \text{(II}c\text{)}$$

and,

$$CC_{\text{III-1}} + CC_{\text{III-2}} + CC_{\text{III-3}} + \ldots = (k_3 - cC_{\text{I-1}} - dC_{\text{II-1}}) \\ - (cC_{\text{I-2}} + dC_{\text{II-2}}) - (cC_{\text{I-3}} + dC_{\text{II-3}}) - \ldots\ldots\ldots\ldots \text{(III}c\text{)}$$

A solution by parts (if convergent) may now be made, one term at a time being computed from each of the foregoing equations. Its value is then used in the succeeding equations to find the next term, in the following manner: $AC_{\text{I-1}} = k_1$; $BC_{\text{II-1}} = k_2 - bC_{\text{I-1}}$; $CC_{\text{III-1}} = k_3 - (cC_{\text{I-1}} + dC_{\text{II-1}})$; $AC_{\text{I-2}} = -(bC_{\text{II-1}} + cC_{\text{III-1}})$; $BC_{\text{II-2}} = -(dC_{\text{III-1}} + bC_{\text{I-2}})$; $CC_{\text{III-2}} = -(cC_{\text{I-2}} + dC_{\text{II-2}})$; $AC_{\text{I-3}} = -(bC_{\text{II-2}} + cC_{\text{III-2}})$; $BC_{\text{II-3}} = -(dC_{\text{III-2}} + bC_{\text{I-3}})$; $CC_{\text{III-3}} = -(cC_{\text{I-3}} + dC_{\text{II-3}})$; etc. Quantities of this type will be referred to as "principal terms;" they comprise both "preliminary values" (or terms found in the first set of computations, as, for example, $AC_{\text{I-1}}$, $BC_{\text{II-1}}$, $CC_{\text{III-1}}$) and "increments" (or terms similar to $AC_{\text{I-2}}$, $BC_{\text{II-2}}$, $CC_{\text{III-2}}$, etc.), which result from succeeding sets of computations.

This method appears to be applicable to the vast majority of normal equations encountered in practical engineering problems, and possesses the important advantage that, in dealing with equations containing four or more unknowns, the solution is usually obtained in appreciably less time than that required by a precise method. Although, generally, a small degree of approximation is involved, the results are practically exact in many cases.

The number of sets of computations required to obtain an accurate solution is usually from four to six; occasional examples may require one or two additional sets. If the convergence is rapid, the computations may be continued until the increments become negligible. The principal terms in each equation are then totaled, and the value of the unknown is found by dividing the resulting sum by the coefficient of the major term of the equation.

If the convergence is relatively slow, it generally happens that, after a few sets of computations are completed, the increments in the separate equations begin to converge with ratios that tend to be nearly equal. When this stage is reached, the solution can be expedited greatly, without appreciable sacrifice of precision, by summing up the remaining increments as geometrical series. This expedient may also be utilized in connection with the more rapidly converging equations, provided the ratios of convergence in the separate equations are approximately equal.

Before effecting the summation, in any case, it is desirable that the ratios of convergence in the separate equations be brought to agreement within fairly definite limits, depending upon the desired degree of precision. The writer's experience indicates that if the range of variation in these ratios does not materially exceed 0.05 (as will presently be illustrated by numerical examples), the remaining increments can be summed up as geometrical series with results sufficiently accurate to reduce the closing errors of the individual circuits to considerably less than 0.01 ft.

The process of summing up the remaining terms of the various series of increments consists of multiplying the last set of calculated increments by the summation factor. These summations are then added to the principal terms previously calculated, and, finally, the unknowns are found by dividing the resulting totals by the coefficients of the major terms of the respective equations. The summation factor of a geometrical series is equal to $\frac{r}{1 - r}$, in which r is the ratio of convergence of successive terms. For present requirements, the summation factor will be computed by means of an "average ratio of convergence."

If the increments in the separate equations attain the desired degree of uniformity of convergence in five or less sets of computations, the average ratio of convergence is given, with satisfactory precision, by:

$$r = \frac{s_n}{s_{n-1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which s_n is the absolute sum of the increments in the final set of computations, and s_{n-1} is the absolute sum of the increments in the preceding set.

If more than five sets of computations are required before the ratios attain the desired degree of uniformity, a better approximation of the average ratio of convergence is generally obtained by using the last three sets of results, as follows:

$$r' = \frac{r^2}{r_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which r is as defined in Equation (2), and $r_{n-1} = \frac{s_{n-1}}{s_{n-2}}$. The corresponding summation factor is then taken as $\frac{r'}{1 - r'}$.

Referring to Table 5(*a*), Appendix II, the calculations incident to the solution of the equations of Example No. 1 are as follows: In Line 1, the constant term of Equation (I) (namely, 14.00), is taken as the preliminary value of 24.5 C_I. It is entered as a "principal term," is underscored, and is extended horizontally in proportion to the coefficients in Line 1, Table 5(*a*), as follows:

$$\frac{8.5}{24.5}(14.00) = 4.85 \text{ (Column (II))}$$

$$\frac{3}{24.5}(14.00) = 1.71 \text{ (Column (III))}$$

and,

$$\frac{10}{24.5}(14.00) = 5.71 \text{ (Column (IV))}$$

For the preliminary value of 19.5 C_{II} in Line 2 the value found for Line 1, Column (II), is subtracted from the constant term of Equation (II), giving $-10.00 - 4.85 = -14.85$. This quantity likewise constitutes a principal term, and is underscored and extended horizontally in proportion to the coefficients in Line II, giving:

$$\frac{8.5}{19.5}(-14.85) = -6.48 \text{ (Column (I))}$$

The preliminary value of 16 C_{III} in Line 3 is determined by subtracting from the constant term of Equation (III) the value calculated for Line 1, Column (III) of Table 5(*a*), which gives the following result: $-8.00 - 1.71 = -9.71$. This quantity, in turn, is extended horizontally in proportion to the coefficients in Line III.

In a similar manner, the preliminary value of 26 C_{IV} in Line 4, Table 5(*a*), is calculated by subtracting the values in Line 1, Column (IV), and Line 3, Column (IV), from the constant term of Equation (IV), giving: $16.00 - (5.71 + 3.04) = 7.25$, which is then extended in the same fashion. This completes the first set of computations.

In succeeding sets (Lines 5 to 16), the principal terms are obtained by taking the negative sum of all the values which occur below the principal term in the preceding set; for example, in Line 5, the principal term, or increment of 24.5 C_I, is: $-(-6.48 - 1.82 + 2.79) = 5.51$.

At the end of the fourth set of computations, the ratios of convergence in the separate columns of Table 5(*a*) (obtained by dividing the increments of the fourth set by those of the third set) are as follows: $\frac{\text{I}}{0.429}$, $\frac{\text{II}}{0.431}$, $\frac{\text{III}}{0.404}$, and $\frac{\text{IV}}{0.423}$. From the foregoing quantities, the maximum range of variation is found to be $0.431 - 0.404 = 0.027$. Although these ratios are quite uniform, it is obvious that several additional sets of computations would be required in order to reduce the increments to negligible quantities; the remaining increments, therefore, are summed up as geometrical series.

The average ratio of convergence, as given by Equation (2), is $\frac{1.41}{3.33} = 0.423$, and the corresponding summation factor is 0.734. The resulting summations (obtained by multiplying the fourth set of increments by the summation factor) appear in Line 17, Table 5(*a*). Line 18 contains the sums of the principal terms (including the quantities in Line 17), and the values of the unknowns (the condition multipliers), are given in Line 19.

The calculations of Table 5(*a*) were performed with the aid of a 10-in. slide-rule. The results of this solution, when compared with values obtained from a precise solution,[8] are all found to be correct to three decimal places, as shown.

(c) Calculation of Section Corrections and Verification of Circuit Closures.—Table 6(*a*), Appendix II, contains the computation of the section corrections, which is developed in accordance with Equation (1). This step is facilitated by tabulating the condition multipliers in groups, according to the sections composing the successive circuits. For example (see Fig. 3), C_I is applied to Sections Nos. 1, 2, 5, and 7; C_{II}, to Sections Nos. 7 and 8; C_{III}, to Sections Nos. 2, 3, and 4; and C_{IV}, to Sections Nos. 3 (with sign reversed), 5, and 6. The sum of the condition multipliers for each section appears in Column (5), Table 6(*a*), and the section corrections (the products of Column (2) and Column (5) to the nearest tenth of a unit, corresponding to the thousandth of a foot), are given next. In Column (7) the adopted section corrections are recorded to the nearest 0.01 ft.

The verification of the circuit closures is given in Table 7(*a*), Appendix II, which indicates that the section corrections as calculated will reduce the closing errors to zero.

(d) Calculation of Adjusted Elevations of Bench-Marks.—The calculation of the adjusted elevations of the bench-marks is given in Table 8(*a*), Appendix II. In dealing with intermediate bench-marks, the adjusted difference in elevation over each sub-section is calculated by prorating the section correction in the ratio of the length of the sub-section to that of the section (see Table 2(*b*), Appendix II).

DESCRIPTION OF METHOD 1; EXAMPLE NO. 2

The data contained in Table 1, Appendix II, will also be used in connection with Example No. 2, with the added requirements that both Points *A* and *C* shall be treated as control bench-marks, with elevations of 416.15 and 424.01, respectively. For purposes of adjustment, it is theoretically immaterial which path connecting these points is taken as the conditioned route. The selected route is that consisting of Sections Nos. 1 and 2.

The closing errors of the four circuits are as given in Table 3, Appendix II; that of the conditioned route is found by subtracting the required difference in elevation from the observed difference in elevation, and is equal to $7.90 - 7.86 = +0.04$.

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[8] For a general method of obtaining a precise solution of normal equations, see "Application of the Theory of Least Squares to the Adjustment of Triangulation," by Oscar S. Adams, *Special Publication No. 28*, U. S. Coast and Geodetic Survey.

The new system of normal equations, together with their solution, is given in Table 5(*b*), Appendix II. Equation (V) expresses the closure requirements of the conditioned route. It will be noted that the inclusion of the added closure condition requires that the term, 6 C_V, be included in Equation (I), and the term, 3 C_V, in Equation (III).

At the end of the fifth set of computations, the increments in each column have begun to converge, with the exception of Column (III). By carrying the calculations through one additional line, it will be found that the next increment in that column is − 0.06, from which it may be concluded that the first five sets of computations are sufficient, provided the ratios of convergence are sufficiently uniform. These ratios are, as follows: $\frac{\text{I}}{0.557}$, $\frac{\text{II}}{0.554}$, $\frac{\text{III}}{0.600}\left(\text{using } \frac{0.06}{0.10}\right)$, $\frac{\text{IV}}{0.564}$, and $\frac{\text{V}}{0.540}$, from which it is seen that the maximum difference is approximately 0.06.

The final results given in Table 5(*b*), when compared with a precise solution, are found to be correct to three decimal places, except for C_V, which is in error by 0.003.

Table 6(*b*), Appendix II, contains the calculation of the section corrections, and Table 7(*b*), Appendix II, the verification of the closures. The calculation of the adjusted elevations of the bench-marks is given in Table 8(*b*), Appendix II.

DESCRIPTION OF METHOD 2

The second method of adjusting a level net differs from the first in the following respects: (1) Instead of the writing and solution of the normal equations, the errors in the various circuits are successively distributed, carried across, and re-distributed; (2) the section corrections are found by direct addition of the distributed quantities; and (3) all these calculations are entered directly on the plan for the adjustment.

In order to illustrate this method, the closing errors of the previous examples will be adjusted. The explanations will be confined to the determination of the section corrections, as the preliminary and final details are the same in both solutions. Moreover, the results are theoretically identical in the two methods.

Example No. 1.—The calculation of the section corrections is given in Fig. 4, and the errors requiring distribution in the different cycles are listed in Table 9(*a*), Appendix II.

Table 10, Appendix II, which has been especially prepared for the purpose of assisting in the explanation of the method, contains complete details of all the calculations involved in the first two cycles of the adjustment. The closing error of each circuit is distributed (with change in sign) among the several sections of the circuit in proportion to their effective lengths. Corrections thus applied to sections subject to two or more closure conditions constitute new errors in the adjoining circuits (or conditioned routes), and, therefore, are carried across at the first opportunity; in the first cycle of adjustments, they are added to the original closing error of the adjoining circuit, the combined closing error being then distributed.

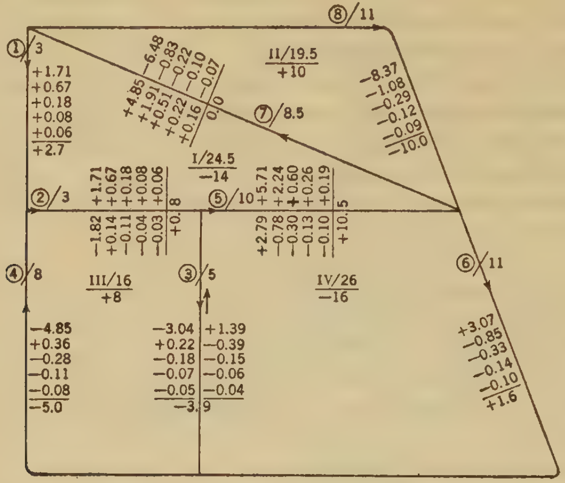
⑧/11
①/3
+1.71
+0.67
+0.18
+0.08
+0.06
+2.7
−6.48
−0.83
−0.22
−0.10
−0.07
+4.85
+1.91
+0.51
+0.22
+0.16
0.0
II/19.5
+10
⑦/8.5
−8.37
−1.08
−0.29
−0.12
−0.09
−10.0
I/24.5
−14
②/3
+1.71
+0.67
+0.18
+0.08
+0.06
8
−1.82
+0.14
−0.11
−0.04
−0.03
+0.
⑤/10
+5.71
+2.24
+0.60
+0.26
+0.19
5
+2.79
−0.78
−0.30
−0.13
−0.10
+10.
④/8
III/16
+8
③/5
IV/26
−16
⑥/11
−4.85
+0.36
−0.28
−0.11
−0.08
−5.0
−3.04
+0.22
−0.18
−0.07
−0.05
−3.
+1.39
−0.39
−0.15
−0.06
−0.04
9
+3.07
−0.85
−0.33
−0.14
−0.10
+1.6

FIG. 4.—ADJUSTMENT OF LEVEL NET BY SUCCESSIVE DISTRIBUTION OF THE CLOSING ERRORS, EXAMPLE NO. 1.

Referring to Table 10(a), the closing error of Circuit No. I (namely, −14.00), is distributed as follows: Section No. 1; $\frac{3}{24.5}$ (+14.00) = + 1.71; Section No. 2: $\frac{3}{24.5}$ (+ 14.00) = + 1.71; Section No. 5: $\frac{10}{24.5}$ (+ 14.00) = + 5.71; and, Section No. 7: $\frac{8.5}{24.5}$ (+ 14.00) = + 4.85.

In Circuit No. II, the foregoing correction to Section No. 7, is carried across and added to the original closing error, giving: +4.85 + 10.00 = +14.85; this quantity is then distributed as follows: Section No. 7: $\frac{8.5}{19.5}$ (− 14.85) = − 6.48; and, Section No. 8: $\frac{11}{19.5}$ (− 14.85) = − 8.37.

In Circuit No. III, the correction already applied to Section No. 2 is taken up, and the combined error (namely, +1.71 + 8.00 = +9.71) is distributed in the following proportions: Section No. 2: $\frac{3}{16}$ (−9.71) = −1.82; Section No. 3: $\frac{5}{16}$ (−9.71) = −3.04; and, Section No. 4: $\frac{8}{16}$ (−9.71) = −4.85.

In Circuit No. IV, the corrections previously assigned to Sections Nos. 3 and 5 are carried across, and the error requiring distribution is, therefore, +3.04 +5.71 −16.00 = −7.25, which, in turn, is distributed (with change in sign) in proportion to the effective lengths of the sections which compose this circuit.

The circuits are adjusted in a fixed sequence, and the same process of distribution is continued in succeeding cycles, the closing errors being found by bringing over and combining the corrections not previously transferred from adjacent circuits.

A record is kept of the errors requiring distribution in the successive steps (see Table 9(*a*)). These values are numerically identical with the "principal terms" found in the first method, and are used to determine: (1) The stage in the solution at which satisfactory convergence is established; (2) the average ratio of convergence; and (3) the summation factor, previously mentioned in connection with Equations (2) and (3).

When the errors in the separate circuits have attained the desired uniformity of convergence, the remainder of each series of corrections is computed by multiplying the last cycle of distributed corrections by the summation factor. These summations constitute the fifth set of values in Fig. 4.

It should be noted that in dealing with sections marked by the auxiliary direction arrow (Section No. 3, for example), the sign of the correction is always changed upon being carried over into the adjoining circuit; furthermore, the signs of all quantities on that side of the section which bears the auxiliary direction arrow are reversed when the individual corrections of a section are totaled.

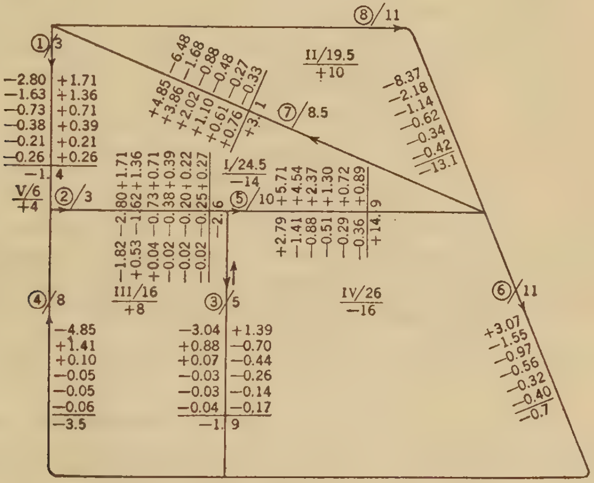
⑧/11
①/3
−2.80 −1.63 −0.73 −0.38 −0.21 −0.26 −1.4
+1.71 +1.36 +0.71 +0.39 +0.21 +0.26
−6.48 −1.68 −0.88 −0.48 −0.27 −0.33
+4.85 +3.86 +2.02 +1.10 +0.61 +0.76 +3.1
II/19.5 +10
⑦/8.5
−8.37 −2.18 −1.14 −0.62 −0.34 −0.42 −13.1
V/6 +4
②/3
−1.82 −2.80 +1.71
+0.53 −1.62 +1.36
+0.04 −0.73 +0.71
−0.02 −0.38 +0.39
−0.02 −0.20 +0.22
−0.02 −0.25 +0.27
−2.6
I/24.5 −14
⑤/10
+5.71 +4.54 +2.37 +1.30 +0.72 +0.89
+2.79 −1.41 −0.88 −0.51 −0.29 −0.36 +14.9
④/8
III/16 +8
③/5
IV/26 −16
⑥/11
−4.85 +1.41 +0.10 −0.05 −0.05 −0.06 −3.5
−3.04 +0.88 +0.07 −0.03 −0.03 −0.04 −1.9
+1.39 −0.70 −0.44 −0.26 −0.14 −0.17
+3.07 −1.55 −0.97 −0.56 −0.32 −0.40 −0.7

FIG. 5.—ADJUSTMENT OF LEVEL NET BY SUCCESSIVE DISTRIBUTIONS OF THE CLOSING ERRORS, EXAMPLE NO. 2.

Example No. 2.—The calculation of the section corrections for Example No. 2 is given in Fig. 5. The errors requiring distribution in the successive steps are shown in Table 9(*b*), Appendix II. Table 10(*b*), Appendix II, contains an explanation of the manner of determining the errors to be distributed in the first two cycles, together with their distributions (the first cycle in this case being the same for Examples Nos. 1 and 2).

In adjusting the conditioned route, the closing error is found by bringing across and combining the errors arising from the preceding distributions in the contiguous circuits (including, in the first cycle, the original closing error of the conditioned route), and is then distributed in proportion to the effective lengths of the sections composing the conditioned route. For exterior sections (Section No. 1 for example), such corrections are entered on the outer side of the section (see Fig. 5) whereas, for interior sections (Section No. 2), they are written astride the section.

In turn, these corrections are taken up into the adjacent circuits when the closing errors of the latter are next calculated. Since interior sections of a conditioned route are subject to three closure conditions, all corrections applied to such sections are carried across in two places. In Fig. 5, the sixth set of values are those found by summation of series.

CONCLUSIONS

As the result of the foregoing examples, the following conclusions appear to be justified: (1) The methods presented are sufficiently precise to meet the requirements of the majority of level nets, except those of large geodetic surveys; and (2), in view of the comparative ease with which a correct adjustment can be effected, and of the obvious advantages of thus eliminating the uncertainty and confusion which otherwise prevail, no carefully executed system of bench-marks should be considered complete until the elevations have been properly adjusted.

APPENDIX I

THEORY

According to the theory of least squares, the sum, Σpv^2, is to be made a minimum, subject to the given closure conditions. Using the data of Example No. 1 (Tables 1 to 3, inclusive, Appendix II), the function which is to be made a minimum may be expressed as follows:[4]

$$F = \frac{v^2_1}{3} + \frac{v^2_2}{3} + \frac{v^2_3}{5} + \frac{v^2_4}{8} + \frac{v^2_5}{10} + \frac{v^2_6}{11} + \frac{v^2_7}{8.5} + \frac{v^2_8}{11}$$
$$- 2\,C_{\text{I}}\,(v_1 + v_2 + v_5 + v_7 - 14) - 2\,C_{\text{II}}\,(v_7 + v_8 + 10)$$
$$- 2\,C_{\text{III}}\,(v_2 + v_3 + v_4 + 8) - 2\,C_{\text{IV}}\,(- v_3 + v_5 + v_6 - 16) \ldots\ldots(4)$$

When the partial derivatives of this function with respect to v_1, v_2, etc., are equated to zero, the following expressions for the section corrections are obtained: $v_1 = 3\,C_{\text{I}}$; $v_2 = 3\,(C_{\text{I}} + C_{\text{III}})$; $v_3 = 5\,(C_{\text{III}} - C_{\text{IV}})$; $v_4 = 8\,C_{\text{III}}$; $v_5 = 10\,(C_{\text{I}} + C_{\text{IV}})$; $v_6 = 11\,C_{\text{IV}}$; $v_7 = 8.5\,(C_{\text{I}} + C_{\text{II}})$; and, $v_8 = 11\,C_{\text{II}}$.

The foregoing results may be summarized in the statement that the section correction for a given section is equal to the product of the effective length of the section by the sum of its condition multipliers.

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[4] "Some Elementary Examples of Least Squares," by Oscar S. Adams, *Serial No. 250*, U. S. Coast and Geodetic Survey.

APPENDIX II

TABLES

The tables in Appendix II are each introduced and explained adequately in the text of the paper.

TABLE 1.—DATA FROM FIELD NOTES

ROUTE No. 1		ROUTE No. 2		ROUTE No. 3		ROUTE No. 4		ROUTE No. 5		ROUTE No. 6	
Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation
A	416.15	*C*	424.06	*D*	421.66	*C*	424.06	*A*	416.15	*A*	416.15
B	422.59	*D*	421.70	*F*	424.11	*E*	425.16	*H*	418.22	*K*	410.02
C	424.06	*G*	428.65	*B*	422.59	....	......	*E*	425.35	*E*	425.34
B	422.65	*E*	425.25	*F*	424.05	....	......	*H*	418.32	*K*	409.94
A	416.17	*G*	428.58	*D*	421.58	....	......	*A*	416.28	*A*	416.04
....	......	*D*	421.62	....	......	....	......	....	......	....	......
....	......	*C*	423.92	....	......	....	......	....	......	....	......

TABLE 2.—COMPUTATION OF EFFECTIVE LENGTHS AND CORRESPONDING OBSERVED DIFFERENCES IN ELEVATION

	(a) RELATING TO SECTIONS							(b) RELATING TO SUB-SECTIONS				
					Field Results *					Field Results *		
Section (1)	Between Points: (2)	Distance, *l*, in feet (3)	Relative length (4)	Effective length, *L′* (5)	Forward (6)	Back (7)	Observed difference in elevation † (8)	Sub-section: between Points: (9)	Ratio, length of sub-section to length of section (10)	Forward (11)	Back (12)	Observed difference in elevation † (13)
1	*A* and *B*.........	2 400	3	3	+6.44	−6.48	+6.46	....	......	......	......	......
2	*B* and *C*.........	2 400	3	3	+1.47	−1.41	+1.44	....	......	......	......	......
3	*C* and *D*.........	4 000	5	5	−2.36	+2.30	−2.33	....	......	......	......	......
4	*D* and *B* (*via F*)..	6 400	8	8	+0.93	−1.01	+0.97	*D* and *F*...	24 :64	+2.45	−2.47	+2.46
5	*C* and *E*.........	4 000	5	10	+1.10	......	+1.10	....	......	......	......	......
6	*E* and *D* (*via G*).	8 800	11	11	+3.55	−3.63	−3.59	*E* and *G*...	44 :88	−3.40	+3.33	+3.36
7	*E* and *A* (*via H*).	6 800	8.5	8.5	+9.20	−9.07	−9.14	*E* and *H*..	45 :68	+7.13	−7.03	−7.08
8	*A* and *E* (*via K*).	8 800	11	11	+9.19	−9.30	+9.24	*A* and *K*..	62 :88	−6.13	+6.10	−6.12

* In the direction actually run. † In the positive direction.

TABLE 3.—COMPUTATION OF CLOSING ERRORS (OBSERVED DIFFERENCES IN ELEVATION)

Section No.	Circuit No. I	Circuit No. II	Circuit No. III	Circuit No. IV
1..........................	+6.46	......	......	......
2..........................	+1.44	......	+1.44	......
3..........................	......	......	−2.33	+2.33*
4..........................	......	......	+0.97	......
5..........................	+1.10	......	......	+1.10
6..........................	......	......	......	−3.59
7..........................	−9.14	−9.14	......	......
8..........................	......	+9.24	......	......
	+9.00 −9.14	+9.24 −9.14	+2.41 −2.33	+3.43 −3.59
Summation..........................	−0.14	+0.10	+0.08	−0.16

* This value taken in the negative direction of Section No. 3; that is, as shown by the auxiliary direction arrow in Fig. 3.

TABLE 4.—FORMATION OF NORMAL EQUATIONS

Circuit No. I	Circuit No. II	Circuit No. III	Circuit No. IV
$v_1 = 3C_{I}$ $v_2 = 3\ (C_{I} + C_{III})$ $v_5 = 10\ (C_{I} + C_{IV})$ $v_7 = 8.5\ (C_{I} + C_{II})$	$v_7 = 8.5\ (C_{II} + C_{I})$ $v_8 = 11\ C_{II}$	$v_2 = 3\ (C_{III} + C_{I})$ $v_3 = 5\ (C_{III} - C_{IV})$ $v_4 = 8\ C_{III}$	$-v_3 = 5\ (C_{IV} - C_{III})$ $v_5 = 10\ (C_{IV} + C_{I})$ $v_6 = 11\ C_{IV}$ ....

TABLE 5.—SOLUTION OF NORMAL EQUATIONS

(a) EXAMPLE No. 1

Line	I	II	III	IV
I	24.5	8.5	3	10
II	8.5	19.5	....	....
III	3	....	16	−5
IV	10	....	−5.	26
...	....	....	....	....
k	14	−10	−8	16
1	14.00	4.85	1.71	5.71
2	−6.48	−14.85	....	....
3	−1.82	....	−9.71	3.04
4	2.79	....	−1.39	7.25
5	5.51	1.91	0.67	2.24
6	−0.83	−1.91	....	....
7	0.14	....	0.72	−0.22
8	−0.78	....	0.39	−2.02
9	1.47	0.51	0.18	0.60
10	−0.22	−0.51	....	....
11	−0.11	....	−0.57	0.18
12	−0.30	....	0.15	−0.78
13	0.63	0.22	0.08	0.26
14	−0.10	−0.22	....	....
15	−0.04	....	−0.23	0.07
16	−0.13	....	0.06	−0.33
17	0.46	−0.16	−0.17	−0.24
18	22.07	−17.65	−9.96	3.88
19	0.901	−0.905	−0.622	0.149

Example No. 1:

$s_4 = 1.41$; $s_5 = 3.33$; $r = 0.423$; and, $\frac{r}{1-r} = 0.734$

Example No. 2:

$s_5 = 3.64$; $s_4 = 6.47$; $r = 0.562$; and, $\frac{r}{1-r} = 1.28$

(b) EXAMPLE No. 2

I	II	III	IV	V
24.5	8.5	3	10	6
8.5	19.5	....	....	....
3	....	16.	−5	3
10	....	−5	26	....
6	....	3	....	6
14	−10	−8	16	−4
14.00	4.85	1.71	5.71	3.42
−6.48	−14.85	....	....	....
−1.82	....	−9.71	3.04	−1.82
2.79	....	−1.39	7.25	....
−5.60	....	−2.80	....	−5.60
11.11	3.86	1.36	4.54	2.72
−1.68	−3.86	....	....	....
0.53	....	2.83	−0.88	0.53
−1.41	....	0.70	−3.66	....
−3.25	....	−1.62	....	−3.25
5.81	2.02	0.71	2.37	1.42
−0.88	−2.02	....	....	....
0.04	....	0.21	−0.07	0.04
−0.88	....	0.44	−2.30	....
−1.46	....	−0.73	....	−1.46
3.18	1.10	0.39	1.30	0.78
−0.48	−1.10	....	....	....
−0.02	....	−0.10	0.03	−0.02
−0.51	....	0.26	−1.33	....
−0.76	....	−0.38	....	−0.76
1.77	0.61	0.22	0.72	0.43
−0.27	−0.61	....	....	....
−0.02	....	−0.10	0.03	−0.02
−0.29	....	0.14	−0.75	....
−0.41	....	−0.20	....	−0.41
2.27	−0.78	−0.13	−0.96	−0.53
38.14	−23.22	−7.00	−1.75	−12.01
1.557	−1.191	−0.437	−0.067	−2.002

TABLE 6.—COMPUTATION OF SECTION CORRECTIONS

Section No.	Effective length	(a) Example No. 1 Condition multipliers		Sum of Columns (3) and (4)	Section Corrections v = Column (2) × Column (5)	Section Corrections Final	(b) Example No. 2 Condition multipliers		Sum of Columns (8) and (9)	Section Corrections v = Column (2) × Column (10)	Section Corrections Fina
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1	3	+0.901	....	+0.901	+ 2.7	+0.03	+1.557	−2.002	−0.445	− 1.3	−0.01
2	3	+0.901	−0.622	+0.279	+ 0.8	+0.01	+1.557	{−2.002 −0.437}	−0.882	− 2.6	−0.03
3	5	−0.622	−0.149	−0.771	− 3.9	−0.04	−0.437	+0.067	−0.370	− 1.8	−0.02
4	8	−0.622	....	−0.622	− 5.0	−0.05	−0.437	....	−0.437	− 3.5	−0.03
5	10	+0.901	+0.149	+1.050	+10.5	+0.10	+1.557	−0.067	+1.490	+14.9	+0.15
6	11	....	+0.149	+0.149	+ 1.6	+0.02	−0.067	....	−0.067	− 0.7	−0.01
7	8.5	+0.901	−0.905	−0.004	0.0	0.00	+1.557	−1.191	+0.366	+ 3.1	+0.03
8	11	....	−0.905	−0.905	−10.0	−0.10	....	−1.191	−1.191	−13.1	−0.13

TABLE 7.—VERIFICATION OF CIRCUIT CLOSURES; SECTION CORRECTIONS

Section No.	(a) Example No. 1 Circuit No. I	Circuit No. II	Circuit No. III	Circuit No. IV	(b) Example No. 2 (I)	(II)	(III)	(IV)	(V)
1.............	+0.03	......	......	......	—0.01	......	......	......	—0.01
2.............	+0.01	......	+0.01	......	—0.03	......	—0.03	......	—0.03
3.............	......	......	—0.04	+0.04*	......	......	—0.02	+0.02*	......
4.............	......	......	—0.05	......	......	......	—0.03	......	......
5.............	+0.10	......	......	+0.10	+0.15	......	......	+0.15	......
6.............	......	......	......	+0.02	......	......	......	—0.01	......
7.............	0.00	0.00	......	......	+0.03	+0.03	......	......	......
8.............	......	—0.10	......	......	......	—0.13	......	......	......
Summation	+0.14	—0.10	—0.08	+0.16	+0.14	—0.10	—0.08	+0.16	—0.04

* This value taken in the negative direction of Section No. 3; that is, as shown by the auxiliary direction arrow in Fig. 3.

TABLE 8.—COMPUTATION OF ADJUSTED ELEVATIONS OF BENCH-MARKS

Section or sub-section	Observed difference in elevation	(a) Example No. 1 Correction	Adjusted difference in elevation	Initial point	Adjusted elevation	Terminal point	Adjusted elevation	(b) Example No. 2 Correction	Adjusted difference in elevation	Initial point	Adjusted elevation	Terminal point	Adjusted elevation
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1..........	+6.46	+0.03	+6.49	*A*	416.15	*B*	422.64	—0.01	+6.45	*A*	416.15	*B*	422.60
2..........	+1.44	+0.01	+1.45	*B*	422.64	*C*	424.09	......	......	......	......	......	......
3..........	—2.33	—0.04	—2.37	*C*	424.09	*D*	421.72	—0.02	—2.35	*C*	424.01	*D*	421.66
5..........	+1.10	+0.10	+1.20	*C*	424.09	*E*	425.29	+0.15	+1.25	*C*	424.01	*E*	425.26
D to *F*....	+2.46	—0.02	+2.44	*D*	421.72	*F*	424.16	—0.01	+2.45	*D*	421.66	*F*	424.11
E to *G*....	+3.36	+0.01	+3.37	*E*	425.29	*G*	428.66	0.00	+3.36	*E*	425.26	*G*	428.62
E to *H*....	—7.08	0.00	—7.08	*E*	425.29	*H*	418.21	+0.02	—7.06	*E*	425.26	*H*	418.20
A to *K*....	—6.12	—0.07	—6.19	*A*	416.15	*K*	409.96	—0.09	—6.21	*A*	416.15	*K*	409.94

TABLE 9.—ERRORS REQUIRING DISTRIBUTION

Cycle	(a) EXAMPLE NO. 1				(b) EXAMPLE NO. 2				
	(I)	(II)	(III)	(IV)	(I)	(II)	(III)	(IV)	(V)
1.....	−14.00	+14.85	+9.71	−7.25	−14.00	+14.85	+9.71	−7.25	+5.60
2.....	− 5.51	+ 1.91	−0.72	+2.02	−11.11	+ 3.86	−2.83	+3.66	+3.25
3.....	− 1.47	+ 0.51	+0.57	+0.78	− 5.81	+ 2.02	−0.21	+2.30	+1.46
4.....	− 0.63	+ 0.22	+0.23	+0.33	− 3.18	+ 1.10	+0.10	+1.33	+0.76
5.....	....	....	......	......	− 1.77	+ 0.61	+0.10	+0.75	+0.41
$s_4 = 1.41$; $s_3 = 3.33$; $r = 0.423$; and $\frac{r}{1-r} = 0.734$					$s_5 = 3.64$; $s_4 = 6.47$; $r = 0.562$; $\frac{r}{1-r} = 1.28$				

TABLE 10.—EXPLANATION OF ADJUSTMENT OF A LEVEL NET BY THE DISTRIBUTION METHOD

Circuit No. (1)	Section No. (2)	FIRST CYCLE; EXAMPLES NOS. 1 AND 2*		SECOND CYCLE			
				Example No. 1		Example No. 2	
		Closing error (3)	Distribution (4)	Closing error (5)	Distribution (6)	Closing error (7)	Distribution (8)
I	....	−14.0					
	1	....	$\frac{3}{24.5}\times +14.00 = +1.71$	....	$\frac{3}{24.5}\times +5.51 = +0.67$	−2.80	$\frac{3}{24.5}\times +11.11 = +1.36$
	2	....	$\frac{3}{24.5}\times +14.00 = +1.71$	−1.82	$\frac{3}{24.5}\times +5.51 = +0.67$	−1.82 −2.80	$\frac{3}{24.5}\times +11.11 = +1.36$
	5	....	$\frac{10}{24.5}\times +14.00 = +5.71$	+2.79	$\frac{10}{24.5}\times +5.51 = +2.24$	+2.79	$\frac{10}{24.5}\times +11.11 = +4.54$
	7	....	$\frac{8.5}{24.5}\times +14.00 = +4.85$	−6.48	$\frac{8.5}{24.5}\times +5.51 = +1.91$	−6.48	$\frac{8.5}{24.5}\times +11.11 = +3.86$
		−14.00	+13.98	− 5.51	+5.49	−11.11	+11.12
II	....	+10.00					
	7	+ 4.85	$\frac{8.5}{19.5}\times -14.85 = -6.48$	+1.91	$\frac{8.5}{19.5}\times -1.91 = -0.83$	+3.86	$\frac{8.5}{19.5}\times -3.86 = -1.68$
	8	....	$\frac{11}{19.5}\times -14.85 = -8.37$	....	$\frac{11}{19.5}\times -1.91 = -1.08$	....	$\frac{11}{19.5}\times -3.86 = -2.18$
		+14.85	−14.85	+1.91	−1.91	+3.86	−3.86
III	....	+ 8.00					
	2	+ 1.71	$\frac{3}{16}\times -9.71 = -1.82$	+0.67	$\frac{3}{16}\times +0.72 = +0.14$	+1.36 −2.80	$\frac{3}{16}\times +2.83 = +0.53$
	3	....	$\frac{5}{16}\times -9.71 = -3.04$	−1.39	$\frac{5}{16}\times +0.72 = +0.22$	−1.39	$\frac{5}{16}\times +2.83 = +0.88$
	4	....	$\frac{8}{16}\times -9.71 = -4.85$	....	$\frac{8}{16}\times +0.72 = +0.36$	....	$\frac{8}{16}\times +2.83 = +1.41$
		+ 9.71	−9.71	−0.72	+0.72	−2.83	+2.82
IV	....	−16.00					
	3	+ 3.04	$\frac{5}{26}\times +7.25 = +1.39$	−0.22	$\frac{5}{26}\times -2.02 = -0.39$	−0.88	$\frac{5}{26}\times -3.66 = -0.70$
	5	+ 5.71	$\frac{10}{26}\times +7.25 = +2.79$	+2.24	$\frac{10}{26}\times -2.02 = -0.78$	+4.54	$\frac{10}{26}\times -3.66 = -1.41$
	6	....	$\frac{11}{26}\times +7.25 = +3.07$	....	$\frac{11}{26}\times -2.02 = -0.85$	....	$\frac{11}{26}\times -3.66 = -1.55$
		− 7.25	+7.25	+2.02	−2.02	+3.66	−3.66
V*	....	+ 4.00					
	1	+ 1.71	$\frac{3}{6}\times -5.60 = -2.80$	....	....	+1.36	$\frac{3}{6}\times -3.25 = -1.63$
	2	+1.71 −1.82	$\frac{3}{6}\times -5.60 = -2.80$	....	....	+1.36 +0.53	$\frac{3}{6}\times -3.25 = -1.62$
		+5.60	−5.60	....	....	+3.25	−3.25

* Circuit No. V applies to Example No. 2 only.

DISCUSSION

Earl F. Church,[5] Assoc. M. Am. Soc. C. E. (by letter).—This interesting paper well deserves the attention of all engineers interested in the practical computations of surveys. It contains a description of two methods for adjusting engineering level nets and one of the most interesting features is that the two methods presented are identical. (The term, "engineering leveling," is used to denote ordinary leveling as distinguished from "geodetic leveling.") The first is the least squares method, the "condition multipliers" being the usual "correlatives." This part of the paper refers to a "converging increment" method of approximation for solving the normal equations to be used instead of the precise Gauss method. The lack of rigidity in the theory is implied by the author in his statement that the method "appears to be applicable to the vast majority of normal equations encountered in practical engineering problems." After all, the real test of any method of solving normal equations lies in obtaining values of the correlatives which satisfy the equations; and this method of approximation will undoubtedly give the desired results in most cases of engineering level nets, as it does in the specific examples shown. Regardless of whether the method possesses any marked advantages over the usual Gauss method, the author is entirely justified in presenting it on account of the fact that, by the identity of the results by the two methods and even of the steps in obtaining the results, it establishes the correctness of the second method.

The second method, called "successive distributions," shows a process of adjusting engineering level nets which is certainly a useful one. Requiring no knowledge of least squares and eliminating the necessity of setting up the normal equations themselves, it is easy to comprehend and easy to apply; and for both these reasons it should appeal to the average surveyor. The method appears at first glance to have no relation to least squares and to give results which, although they might render the level net consistent, would bear no relation to the "most probable values" and might even vary with the order arbitrarily chosen for closing the various circuits. This is not the case, however. The identity of the results of the two methods shows the elevations from the second method to be virtually the "most probable values." Furthermore, the identity of the steps in obtaining the results in the two methods establishes the correctness of the second method as virtually a least squares adjustment for cases in which the "converging increments" would solve the normal equations, regardless of the order chosen for closing the circuits in the second method.

The following notes may be of assistance in observing the identity in the successive steps in the two processes. Compare the values shown in Fig. 4 for the method of "successive distribution" with those shown in Table 5 for the solution of the normal equations by "converging increments": On Line 1 of Table 5, 4.85, 1.71, and 5.71 are the values "carried over" in Fig. 4 from Circuit No. I into Circuits Nos. II, III, and IV, respectively; on Line 2 of Table 5, –6.48 is the value in Fig. 4 "carried over" from Circuit No. II to Circuit No. I, and the two blank spaces on Line 2 show that in Fig. 4 there were no values "carried over" from Circuit No. II to either Circuit No. III or Circuit No. IV; on Line 3 of Table 5, –1.82 and 3.04, are the values in Fig. 4 "carried over" from Circuit No. III to Circuits Nos. I and IV, respectively; on Line 4 of Table 5, 2.79 and –1.39 are the values in Fig. 4 "carried over" from Circuit No. IV to Circuits Nos. I and III, respectively; on Line 5 of Table 5, 5.51 is the closure distributed in Circuit No. I, in Fig. 4 the second time; and 1.91, 0.67, and 2.24, on Line 5 of Table 5, are the values in Fig. 4 "carried over" the second time into Circuits Nos. II, III, and IV, respectively. This identity of the steps can be followed throughout.

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Undoubtedly, the use of series summations to obtain section corrections in the method of "successive distribution," will generally prove to be a useless refinement. In fact, in the specimen adjustment shown in Fig. 4, the correct values of the section corrections would have been obtained if the additions had been made after the third distribution, and both the fourth distribution and the series summations were actually unnecessary.

This discussion may be summarized somewhat as follows: The second method (that of "successive distribution"), gives a simple and useful means of adjusting engineering level nets. This method is particularly simple in the frequent cases in which one can dispense with the series summations. The desirability of the method depends entirely upon whether the resulting elevations are the "most probable values" regardless of the order of closing the circuits, and that they are approximately the most probable values is proved by the identity with the results from the least squares method. The only remaining question is whether the first method (with which the comparison is made) actually gives "most probable values" when the converging increment approximation is used to solve the normal equations; but it will be observed that it must do so provided the correlatives obtained actually satisfy the normal equations. Therefore, for any cases in which "converging increments" would solve the normal equations, the second method of "successive distribution" without the normal equations will give correct results.

In presenting to the average surveyor an excellent process for adjusting ordinary engineering level nets, this paper serves a useful purpose and is to be highly commended.

W. H. RAYNER,[6] ASSOC. M. AM. SOC. C. E. (by letter).—This presentation is an excellent example of that type of study which consists in the application of existing theory to new problems; and an especial reason for commendation is the fact that, for good measure, the author has given two equally good solutions of a complex and important problem.

The simplicity and clarity of treatment are excellent but a statement under "Solution of Normal Equations," needs a brief additional explanation, it seems, to make clear the computation of the "condition multipliers," C_{I}, C_{II}, etc. Referring to Table 5(*a*), Mr. Dell states that "Line 18 contains the sums of the principal terms (including the quantities in Line 17), and the values of the unknowns (the condition multipliers), are given in Line 19." The value of C_{I}, for Method 1, Example No. 1, is given by the relation, $C_{\text{I}} = \frac{\text{Major terms of Circuit No. I}}{\text{Total effective length of Circuit No. I}}$. Hence, $C_{\text{I}} = \frac{22.07\text{ (Line 18)}}{24.5} = 0.901$. Likewise $C_{\text{II}} = \frac{-17.65}{19.5} = -0.905$.

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[6] Asst. Prof, of Civ. Eng., Univ. of Illinois, Urbana, Ill.

Two approximate methods are sometimes used to adjust level nets, and may be designated: (1) The "successive-circuit" method; and (2) the Geological Survey method. The writer was interested to compare both the results and the labor involved in each of these approximate methods with those of Mr. Dell's methods.

Briefly stated, the successive-circuit method consists in adjusting the sides of connected circuits in proportion to the lengths of the sides, beginning with that circuit having the largest error of closure. The second circuit to be adjusted is the one having the second largest error of closure, etc. The method is described in detail, in the more complete surveying textbooks.[7]

The Geological Survey method consists in beginning with a fixed benchmark, from which the nearest junction points of connected circuits are given a preliminary adjustment. The weights assigned to the different elevations of such junction points, as found by the different connecting lines, are inversely proportional to the lengths of the lines. Using the preliminary adjusted values thus found, other junction points are similarly adjusted to include all points within the level net. From such tentative values, a second computation establishes the final elevations of the successive junction points.[8]

For purposes of comparison, a net including five circuits comprising nine lines from 5 miles to 30 miles in length, was adjusted by each of the approximate methods and by each of Mr. Dell's methods, with the results shown in Table 11.

TABLE 11.—COMPARISONS OF ADJUSTMENTS OF A LEVEL NET

Point	DELL	GEOLOGICAL SURVEY		SUCCESSIVE-CIRCUIT	
	Elevation	Elevation	Discrepancy with Dell	Elevation	Discrepancy with Dell
E	650.00	650.00	0.00	649.85	−0.15
B	810.21	810.25	+0.04	810.26	+0.05
A	601.39	601.35	−0.04	601.39	0.00
F	682.65	682.64	−0.01	682.56	−0.09
D	748.17	748.18	+0.01	748.14	−0.03
C	715.29	715.28	−0.01	715.36	+0.07

It is evident that, for this example, the discrepancies between the Geological Survey method and the Dell method are, in the average, about one-third as large as those of the successive-circuit method. Of course, the two Dell methods yielded identical results.

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[7] "Surveying," by Davis, Foote, and Rayner, Second Edition, p. 174.

[8] The method is described in detail in "Instructions to Topographers of the U. S. Geological Survey."

Each of the Dell methods and the Geological Survey method required about an equal amount of labor. However, the writer believes that the layman will find Method 2 the easier to apply. The successive-circuit method is much easier of application than the others; but it is also the least accurate.

It is evident, therefore, that for engineers who meet this problem, Mr. Dell has provided two methods that are mathematically correct and convenient, thus making unnecessary and inexcusable the use of approximate methods.

George H. Dell,[9] Assoc. M. Am. Soc. C. E. (by letter).—The discussions submitted on this subject by Professors Church and Rayner contain interesting and helpful comments. Both writers find the distribution method superior from the standpoints of simplicity and applicability.

The example analyzed by Professor Rayner is a typical indication of the errors inherent in "compromise" methods of adjustment. Professor Church has clarified an important item in pointing out that the corrections are substantially independent of the order of adjusting the separate circuits. He has also made noteworthy observations in regard to the elimination of undue refinements of precision.

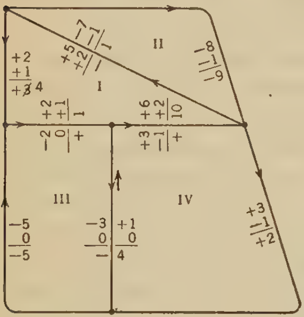

Fig. 6.—Simplified Adjustment of Level Net, Example No. 1.

The original solutions were prepared with a view to presenting methods capable of yielding results correct to approximately 0.001 ft, and therefore, applicable to precise surveys. For ordinary level nets, in which an inaccuracy of 0.01 ft in any section is permissible, two cycles will generally be sufficient if supplemented by occasional arbitrary corrections of 0.01 ft. Fig. 6 illustrates the solution of Example No. 1 under such conditions. It will be noted that at the end of the second cycle the only remaining error was in Circuit No. I; this was eliminated by changing the correction of Section No. 1 to + 0.04 ft.

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AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1941

STRUCTURAL BEAMS IN TORSION

By Inge Lyse,[1] M. Am. Soc. C. E.,
and Bruce G. Johnston,[2] Jun. Am. Soc. C. E.

With Discussion by Messrs. H. M. Westergaard and R. D. Mindlin, Joseph B. Reynolds, Harold E. Wessman, F. B. Seely and W. J. Putnam, P. Wilhelm Werner, F. L. Ehasz, W. P. Roop and John B. Letherbury, Alfred T. Waidelich, and Inge Lyse and Bruce G. Johnston.

Synopsis

Results of a study of the torsional properties of standard structural steel beams are offered for discussion in this paper. The purpose of the investigation was to furnish a reliable basis for the design of structural members subjected to torsional loads. The relation between torque and stress on the one hand, and between torque and twist on the other, for any piece subjected to torsion involves a constant the value of which is a function of the material and the shape of the cross-section. An accurate method is given for the evaluation of this torsion constant, K, for standard **H**-sections and I-sections, taking full account of all factors involved. This has been made possible by applying the "membrane analogy" to about sixty sections of widely varying flange, web, and fillet proportions.

The investigation included a study of the effect of end fixity in torsional design, and shows how it may be obtained effectively. The proposed formulas are applied to practical design problems, and are checked by torsional tests on structural steel sections ranging in size from a 3-in. I-beam weighing 7.5 lb per ft, to a 12 by 12-in. beam weighing 190 lb per ft.

Note.—Published in April, 1935, *Proceedings*.

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HISTORICAL FOREWORD

The problem of pure torsion as applied to non-circular sections was first treated correctly by Saint Venant (1)[3] in 1855, and his general solution is applicable to any cross-section. In 1903, Prandtl (2) showed that if a thin membrane were stretched across a hole having the shape of the cross-section in question and distorted slightly, the equation of its surface had the same form as the general differential equation involved in the torsion problem. Prandtl showed that by measuring the volume and slopes of the displaced membrane a direct measurement of the torsional rigidity and stress was obtainable. Prandtl's analogy, with a thin soap film as a membrane, was used in several torsion investigations, first in England by Griffith and Taylor (3) who studied the torsional strength of aeroplane sections in 1917, and, later, in the United States by Trayer and March (4), who, in 1930, made similar studies for the same purpose.

Important contributions to the torsion problem have been made by Timoshenko (5). He has shortened the pure torsional theory by slight modifications of Saint Venant's equations and by mathematical application of the principles of the membrane analogy. He was also among the first to consider the effect produced by preventing the warping of a cross-section. This problem has had the attention of numerous investigators in connection with problems of elastic stability and buckling during bending. Sonntag (7) treated the theoretical aspects of this problem in an article published in 1929.

INTRODUCTION

The investigation reported herein was undertaken as a study of all available information on the subject, both theoretical and experimental, supplemented by a considerable number of actual torsion tests of structural steel beams and soap film experiments on various cross-sectional shapes. The writers first considered testing beam sections 3 ft in length, welded to thick plates at the ends. A study of the problem, however, showed that such beams would be several times stronger than if they were tested free-ended, and that, unless the exact percentage of end fixity was known, it would not be possible to draw definite conclusions from such tests.

In order to study the effect of end fixity directly, tests were first made on eight sections of a 3-in. I-beam (7.5 lb per ft), varying from 3 in. to 4 ft 6 in. in length and cut from the same rolled section. The ends of each piece were welded to plates 1 in. thick, and the specimens were tested in a standard 26 000 in-lb torsion machine.

The results of these tests pointed the way to a revised general program, and a torsion testing rig capable of applying torsional load up to 750 000 in-lb was designed. Provision was made for testing beams either fixed or free at the ends, and with lengths of 1 ft 6 in., 3 ft, or 6 ft. Nineteen different tests were made, twelve on beams with the ends fixed by welding the side and end plates to form a box section at the ends, and seven with ends free. The beams ranged in sizes from the 3-in. I-beam, weighing 7.5 lb per ft, to a

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[3] The numbers in parentheses refer to references given in Appendix I.

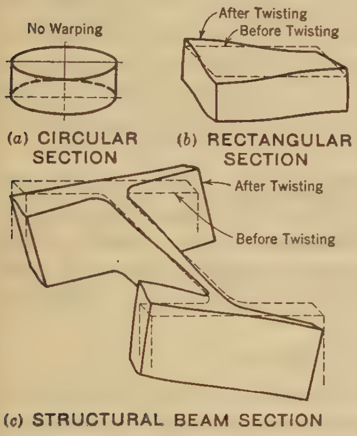
12 by 12-in. beam weighing 190 lb per ft. Tensile and shearing properties of the material in each type of beam were obtained by standard tensile tests, round bar torsion tests, and slotted plate shear tests. Soap film experiments on fifty-seven differently proportioned sections were made for the determination of the torsional rigidity.

This paper contains the final summary of all phases of the investigation. Use has been freely made of the findings of previous investigators, for which acknowledgment is made at appropriate points.

Notation.—The symbols in this paper are introduced in the text as they occur, and are summarized for reference in Appendix II.

THE TORSION THEORY

General Problem.—The solution of the torsional properties of a section of any shape consists primarily in determining the distribution of lateral shearing stresses over the cross-section. The shear components will be of uneven distribution, except in the case of the circular section, and as a result plane sections will be warped during twisting as shown in Fig. 1 (Note that, in Fig. 1(*c*), the web and each flange are warped as individual rectangles, in addition to the warping of the section as a whole.)



(*a*) CIRCULAR SECTION

(*b*) RECTANGULAR SECTION

(*c*) STRUCTURAL BEAM SECTION

FIG. 1.—TWISTING OF BARS OF VARIOUS CROSS-SECTIONS

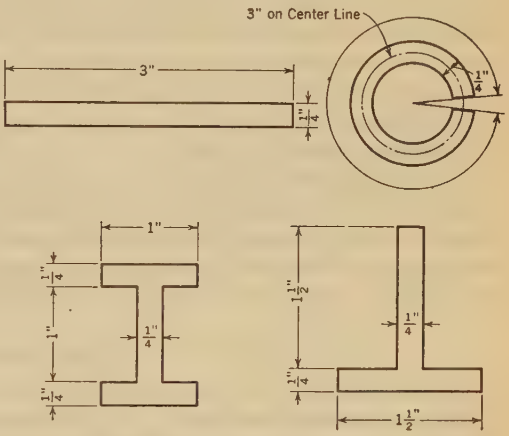


FIG. 2.—SECTIONS HAVING APPROXIMATELY EQUAL TORSIONAL RIGIDITY

It is assumed that the lateral displacements are proportional to the angular twist and to the distance from the twisting axis (as is the case in a circular section). The longitudinal displacements cause the warping, and the resulting distribution of shearing stress is taken care of by introducing a "stress-function," F, of x and y. This function must satisfy the differential equation:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -2 G \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which, G = the shearing modulus of elasticity, and θ = angle of twist, in radians per inch. It may be shown that the function, F, must be a constant along the boundary of the section for solid bars, and, therefore, may be chosen arbitrarily as equal to zero.

If the boundary conditions are such that Equation (1) may be solved and the value of F determined, it is possible to evaluate the torsion constant of the section and find the stress at any point in the cross-section. Formulas for the torsion constant and critical shearing stresses have been derived in this manner for such sections as the square, rectangle, ellipse, equilateral triangle, and sector of a circle (1).

In the case of the circular shaft the shearing stress components have a uniform distribution along each radius, and since the longitudinal shear is likewise evenly distributed, there is no longitudinal warping of first-order importance. The well-known simple theory using the polar moment of inertia is thus applicable to the case of the circular section.

If, in some way, warping which takes place in non-circular sections is restrained or prevented, longitudinal fiber stresses will be introduced and the beam stiffened and strengthened.

The Membrane Analogy.[4]—Equation (1) may be solved mechanically for any cross-section by means of Prandtl's membrane analogy, thereby overcoming the mathematical limitations of the theoretical derivation.

In the application of this analogy, a soap film is stretched across an opening having the same shape as the structural section under consideration. The bubble is distended slightly by a variation in pressure. Prandtl showed that the following relations obtain for this bubble: (1) The torsion constant, K, is proportional to the total volume of the displaced bubble; (2) the shearing stress at any point is proportional to the maximum slope of the film at that point; and (3) the contour lines on the bubble give the direction of maximum shearing stress.

The analogy is also useful as an aid in visualizing the rigidity and stress distribution in various sections, and makes evident why the four sections shown in Fig. 2 have approximately equal rigidities in pure torsion, since the volumes of the various soap bubbles are approximately the same in each case. This would not be the case if the ends of the beams were restrained.

Evaluation of the Torsion Constant

Definition.—The torsion constant, K, is the measure of the torsional rigidity and twisting deflections. It is also a part of any formula for torsional shearing stresses, and may be determined from test results by observing the ratio of torsional moment to unit twist, in radians per inch, at any place below the yield point of the beam, and dividing this ratio by the shearing modulus of elasticity.

The Relation Between K and J.—When a torsional couple, T, is applied to a circular shaft of radius, r, the maximum shearing stress, τ, at the surface, is given by:

$$\tau = \frac{T\,r}{J} \qquad \text{.............................(2)}$$

in which, J = polar moment of inertia. In terms of T Equation (2) may be re-arranged to read:

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[4] A detailed description of the soap film studies is given in a thesis by Bruce G. Johnston, Jun. Am. Soc. C. E., presented to Lehigh University in partial fulfillment of the requirements for the degree of Master of Science.

$$T = \frac{\tau J}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The torque, T, may also be expressed in terms of θ and G, thus:

$$T = J G \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

For non-circular sections the torsional resisting moment may again be expressed in terms of θ and G, with the substitution of K, the torsion constant, in place of J, thus:

$$T = K G \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The torsion constant, K, is equal to the polar moment of inertia for circular sections. Although for non-circular sections it is always less than the polar moment of inertia, there is no direct relation between the two factors.

The Rectangle.—In dealing with structural shapes, two principal types of section require consideration, the rectangle, and the rectangle modified by sloping sides, as in the flange of a standard I-beam. In the case of the rectangle an accurate formula was derived originally by Saint Venant (1):

$$K = \frac{n^3 b}{3} - 2 V n^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which n = the breadth of a rectangular section; b = the length of a rectangular section; and V = a factor depending upon the ratio, $\frac{b}{n}$, but practically constant for $\frac{b}{n} > 3$. Fig. 3 shows the values of V for $\frac{b}{n}$ from 1 to 3. For $\frac{b}{n}$ -ratios greater than 3, $V = 0.105$, and for $\frac{b}{n}$ greater than 4, $V = 0.10504$. Equation (6) finds a direct, qualitative interpretation in the

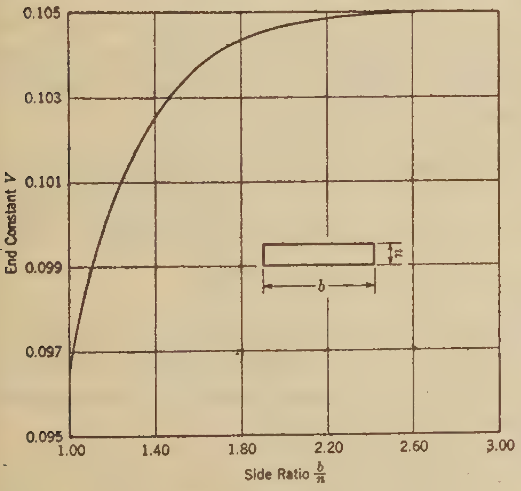
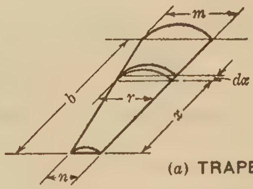
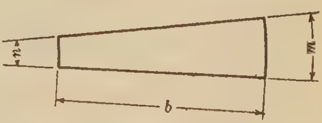


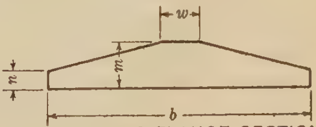
FIG. 3.—END CONSTANTS FOR RECTANGULAR SECTIONS, WITH $\frac{b}{n} < 3$. (SEE EQUATION (6)).



(a) TRAPEZOID



(b) PART OF A SECTOR



(c) SLOPING FLANGE SECTION

FIG. 4.

soap film analogy. It is evident that for long rectangular sections the bubble will be of constant cross-section along the central part, but at the two ends it will be contracted and brought down to meet the small side. The quantity, $-2\ V\ n^4$, then represents the "end loss," which for long sections is evidently a function of n only.

It also follows that if the ends were made discontinuous, as if they were parts of infinitely long rectangles, one might state without error:

$$K = \frac{1}{3}\ n^3\ b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and for any differential length, dx, along the section:

$$K = \frac{1}{3}\ n^3\ dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The Section with Sloping Sides.—Equation (8) provides a basis for evaluating K for the sloping flange section. Considering the section shown in Fig. 4(a), let the thickness at any point be taken as r. Then, if the ends are assumed discontinuous:

$$K = \frac{1}{3} \int_0^b r^3\ dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Evaluating r in terms of m and n, and integrating:

$$K = \frac{b}{12}\ (m + n)\ (m^2 + n^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which m = major flange thickness and n = minor flange thickness. A deduction must be made for end effects, as in the case of the simple rectangle, thus:

$$K = \frac{b}{12}\ (m + n)\ (m^2 + n^2) - V_L\ m^4 - V_S\ n^4 \ldots\ldots\ldots\ldots (11)$$

in which V_L and V_S are the end constants, V, for the large end and the small end, respectively, of the flange (see Fig. 5). The evaluation of these two constants was the work of Professor J. B. Reynolds, through an analysis of a section having the shape shown in Fig. 4(b)[5]:

$$V_L = 0.10504 - 0.10000\ S + 0.08480\ S^2 - 0.06746\ S^3 + 0.05153\ S^4 \ldots (12)$$

and,

$$V_S = 0.10504 + 0.10000\ S + 0.08480\ S^2 + 0.06746\ S^3 + 0.05153\ S^4 \ldots (13)$$

in which S = the total slope of the section; that is, $\frac{m - n}{b}$.

Torsion Constant for **H**-*Beams and* I-*Beams.*—The foregoing supplies a basis for evaluating the K-values of the component parts. Taking the section shown in Fig. 4(c) as a basis for the sloping flange section, the sum of two trapezoids and a small rectangular part is expressed by:

$$K_f = \frac{b - w}{12}\ (m + n)\ (m^2 + n^2) + \frac{1}{3}\ w\ m^3 - 2\ V_S\ n^4 \ldots\ldots (14)$$

---

[5] "Theory of Elasticity," by A. E. H. Love, Fourth Edition, p. 319.

in which K_f = the K-value for the flange. The web is considered as a discontinuous section between the flanges, giving:

$$K_w = \frac{1}{3}(d - 2\,m)\,w^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

in which K_w = the K-value for the web; d = total depth of beam; and, w = thickness of web (see Fig. 6). There still remains the evaluation of the added rigidity due to the connection of the flange and web and also due to the fillet at this point. It is evident that these will cause a considerable "hump" in the soap bubble.

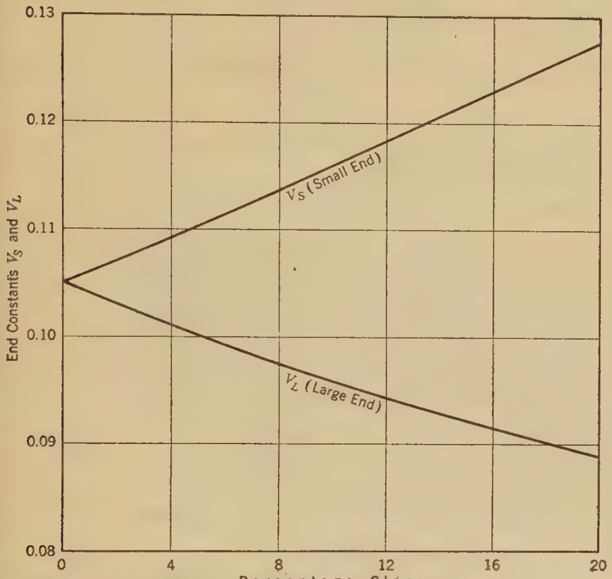
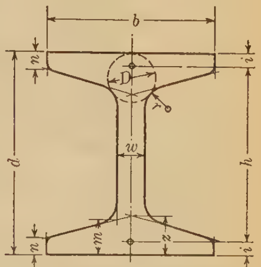
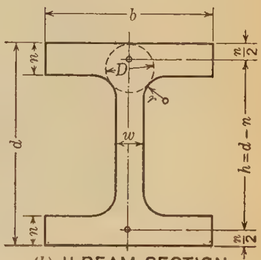


FIG. 5.—END CONSTANTS FOR K-VALUES OF FLANGES WITH SLOPING SIDES.



(a) I-BEAM SECTION



(b) H-BEAM SECTION

FIG. 6.

Trayer and March (4) in an investigation of aircraft strut sections assumed this addition to the torsion constant to be proportional to the fourth power of the diameter of the largest circle that can be inscribed at the juncture of the web and flange (see Fig. 6), giving, as an additional K-value:

$$K = \alpha\,D^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which D = the diameter of an inscribed circle; and α = a factor that depends on two ratios, $\frac{w}{m}$ and $\frac{r}{m}$.

Values of α for sections with parallel-sided flanges, and for sections with flange slopes of 1 on 6, are given in Fig. 7. These curves were determined experimentally as the result of soap film tests which will be described subsequently. It will be noted in Fig. 7(a) that for parts of the curves to the right of *a-a*, the lines are parallel and uniformly spaced. All standard rolled beams are in this area, in which case, for parallel flange sides:

$$\alpha = 0.094 + 0.070\,\frac{r}{m} \qquad (17)$$

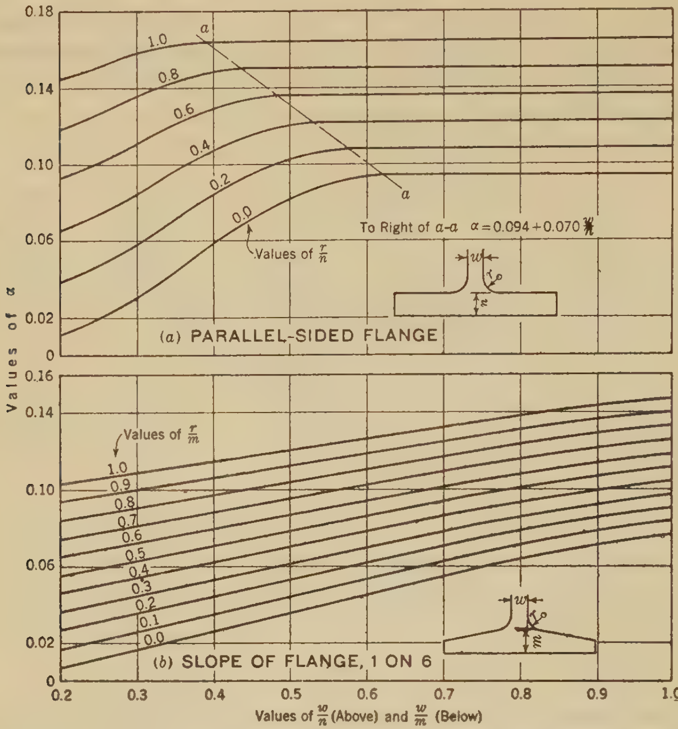


FIG. 7.—VALUES OF α FOR RIGHT-ANGLE JUNCTION OF FLANGE AND WEB.

For flange sections with side slopes of 1 on 20 and 1 on 50 the following formulas represent an interpolation between the curves in Fig. 7:

For a slope of 1 on 20:

$$\alpha = 0.066 + 0.021\,\frac{w}{m} + 0.072\,\frac{r}{m} \qquad (18)$$

and, for a slope of 1 on 50:

$$\alpha = 0.084 + 0.007\,\frac{w}{m} + 0.071\,\frac{r}{m} \qquad (19)$$

The various elements entering into the total K-values, can now be summarized as follows (refer to Fig. 6):

For sloping flange sections,

$$K = \frac{b - w}{6}\,(m + n)\,(m^2 + n^2) + \frac{2}{3}\,w\,m^3 + \frac{1}{3}\,(d - 2\,m)\,w^3 + 2\,\alpha\,D^4 - 4\,V_S\,n^4 \qquad (20)$$

Values of V_L and V_S for standard slopes are, as follows:

S	V_L	V_S
$\frac{1}{6}$	0.09045	0.12441
$\frac{1}{20}$	0.10026	0.11026
$\frac{1}{50}$	0.10307	0.10707
$\frac{1}{\infty}$	0.10504	0.10504

and, for sections with parallel-sided flanges,

$$K = \frac{2}{3}\, b n^3 + \frac{1}{3}\,(d - 2\,n)\, w^3 + 2\,\alpha\, D^4 - 0.42016\, n^4 \quad \ldots\ldots\ldots(21)$$

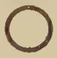
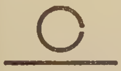
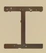
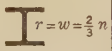
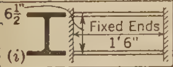
SECTIONS		RIGIDITY	STRENGTH
(a)		100.0	100.0
(b)		637.0	332.0
(c)		5.5	18.0
(d)		70.0	62.0
(e)		88.0	74.0
(f)		341.0 (Approx.)	280.0 (Approx.)
(g)		9.9 (Nearly Exact)	22.2 (Approx.)
(h)		11.6 (Nearly Exact)	22.8 (Approx.)
(i)		78.1 (Approx.)	38.3 (Approx.)

FIG. 8.—TORSIONAL RIGIDITY AND STRENGTH OF DIFFERENT SECTIONS OF EQUAL CROSS-SECTION AREAS.

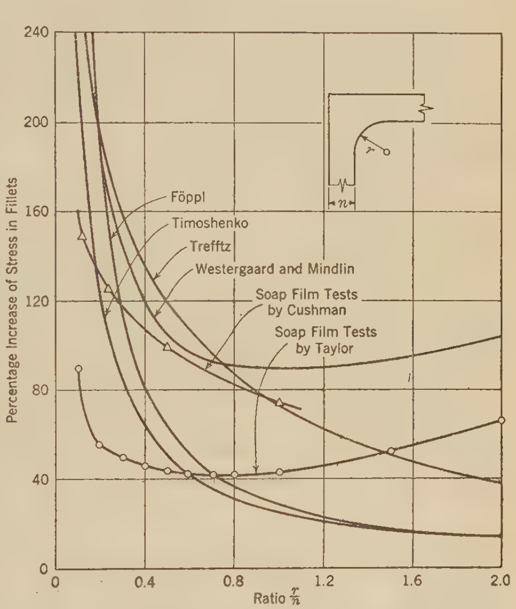


FIG. 9.—FORMULAS FOR STRESS CONCENTRATION IN FILLETS DUE TO TORSION.

Diameter D may be determined by a large scale layout, or by the following formulas:

For parallel-sided flange sections,

$$D = \frac{(n + r)^2 + w\left(r + \dfrac{w}{4}\right)}{2\,r + n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

for sloping-sided flange sections,

$$D = \frac{(B + z)^2 + w\left(r + \frac{w}{4}\right)}{B + r + z} \quad \dots\dots\dots\dots\dots\dots (23)$$

in which z = the maximum flange depth shown in Fig. 6, and,

$$B = r\,S\left[\sqrt{\frac{1}{S^2} + 1} - 1 - \frac{w}{2\,r}\right] \quad \dots\dots\dots\dots\dots\dots (24)$$

Comparative Efficiencies of Different Sections.—Fig. 8 shows the comparative torsional efficiencies of certain different shapes, illustrating the striking advantages of the hollow box, or tubular, construction. These advantages would not obtain entirely if the section were built up by use of bolts or rivets.

EXAMPLES OF DESIGN

General Statement

Structural members are often required to carry torsional loads, generally as a secondary factor combined with bending or direct stress. Problems involving the bending of unsymmetrical sections, and problems of elastic stability, such as the buckling of flanges during bending, also require a knowledge of torsional properties.

In the design of short beams to carry torsional loads considerable advantage may be obtained by fixing the ends, resulting in increased strength and rigidity, with corresponding decrease of angular deflection. External fixity is not needed—it is only necessary to box in the two flanges at each end and thereby prevent, as nearly as possible, their relative warping. The two flanges then act as two rectangular fixed-ended beams carrying a lateral displacement, mutually opposed in direction.

In designing long beams, the end effect tapers out rapidly toward the center, and the formulas for pure torsion, free-ended, will be adequate and simpler in their application. It would still be of practical advantage and a means of additional safety, however, to "box" the ends of the beam.

Free-Ended Torsion

If a beam is to be designed as free-ended, only shearing stresses need be considered. Regardless of the partial restraint that does exist as an incidental feature of the details, such a design will be on the safe side. The critical shearing stresses will occur along the outer surface of the beam where the material is thickest, generally along the outside center line of the flange and along the inside re-entry fillets.

The shearing stress is a function of the thickness of the material and the following formula is proposed for the maximum shearing stress in the flange of an **H**-beam or an **I**-beam in free-ended torsion:

For parallel-sided flange sections,

$$\tau_f = \frac{T\,(D + n)}{2K} \quad \dots\dots\dots\dots\dots\dots\dots\dots (25a)$$

for sloping flange sections,

$$\tau_f = \frac{T\,(D + m)}{2\,K} \quad \text{(25b)}$$

and for shear stress in the web,

$$\tau_w = \frac{T\,w}{K} \quad \text{(26)}$$

in which T = torsional moment. Although Equation (26) is in accord with torsional theory it gave results which proved low by comparison with actual tests. The following tentative formula was found to give better agreement:

$$\tau_w = \frac{T\,(w + 0.3\,r)}{K} \quad \text{(27)}$$

Equation (27) was adopted for use in the reported investigation. More accurate stress formulas might be developed by further use of soap film tests, taking slope measurements at critical points, and testing sections with various ratios of web, fillet, and flange as was done in determining the torsion constant. An equation similar to Equation (27) will give practically the same values for flange stress as Equation (25), thus:

$$\tau_f = \frac{T\,(n + 0.3\,r)}{K} \quad \text{(28)}$$

Equation (25) was used in the reported investigation.

Concentration of Shearing Stress at Re-Entry Fillet

At the re-entry fillet of an **I**-beam torsional shearing stress concentrations occur due to the sharp curvature of the fillet. These stresses are mostly of a local nature and do not greatly influence the yielding of the beam as a whole. Although they do not necessarily govern the design of the beam they are important in the study of adequate fillet sizes and in the determination of loads producing strain lines in the fillets. In a beam subjected to flexural as well as torsional stresses the total stress concentration at the fillet is the sum of the stresses due to these two causes.

The concentration of torsional stress at the fillet between flange and web is illustrated in Fig. 9 for the formulas developed by Föppl(9), Trefftz(8), Timoshenko(5), and Westergaard and Mindlin, and for soap film experiments conducted by Griffith and Taylor(3), and by Cushman(13). The analysis resulting in the Westergaard-Mindlin curve was communicated to the writers by its originators. It is noted that all curves indicate a rapid increase in stress concentration with the decrease of the ratio, $\frac{r}{n}$, between the radius of the fillet and the thickness of the section next to it, particularly for ratios of less than 1.0. Strain lines, therefore, will appear at relatively small torsional moment for sections having small fillets.

As the fillets become increasingly large another factor is introduced since the increased stress due to the greater thickness of material becomes of more importance. The rigidity of the beam is increased in proportion to the third power of the thickness, whereas the stress varies directly with the thickness. Hence, for any given moment the stress would actually decrease. The curve of soap film tests by Griffith and Taylor indicate this reverse effect, but further experiments along this line should be conducted to establish these relations more definitely.

Fixed-Ended Torsion

Assumptions.—If the ends of a non-circular section are fixed in some manner so as to prevent free warping of the end section, pure torsion no longer exists.

Various investigators have studied this problem—Föppl(9), Timoshenko(5), Sonntag(7), and others. In 1930 an investigation on channel sections was reported by Seely, Putnam, and Schwalbe (10). These investigators have generally been concerned with the problem of elastic equilibrium involved in the side buckling and twisting of a beam in bending without lateral support. Hence the torsion problem has been a secondary issue.

In the present tests both ends of structural beams have been fixed by welding on heavy end plates and additional side stiffening plates between the flanges at each end of the beam. The major stiffening effect produced is that of fixing the flanges relative to each other. (Refer to Fig. 10.) The prevention of the individual warping of the component rectangular parts would taper out so rapidly that it would be of negligible importance; but, by fixing the flanges with respect to each other, the effect of two opposed fixed-ended beams is produced, as illustrated in Fig. 10.

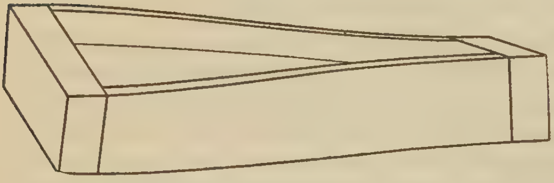


Fig. 10.—Beam with Fixed Ends.

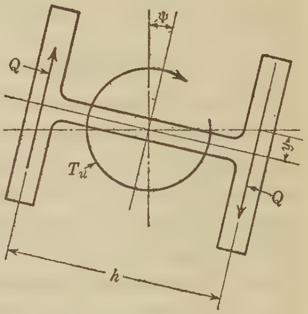


Fig. 11.—Twisting of a Structural Shape.

The following assumptions have been made: (1) The flanges remain at right angles to the web; (2) the angular deflection is small compared with the length of the beam; (3) the bending of each flange about its weaker axis is a negligible factor; (4) a point on the neutral axis of one flange can be located with sufficient accuracy by co-ordinates (x, y) measured along the perpendicular to the original position of the axes; (5) the two ends of the beam are held between mutually parallel planes as twisting takes place; (6) the displacement of the flanges due to beam action is due to bending only (that is, lateral shearing deflection is neglected; a correction is made afterward for shearing deflection in very short beams); and (7) $\frac{I_y}{2}$ = moment of inertia of one flange about the web axis.

Assumption (7) is quite accurate and of great convenience in the case of standard I-beams and H-beams for which I_y is given in all handbooks. The following symbols in addition to those previously given will be used:

$$a = \frac{h}{2}\sqrt{\frac{EI_y}{KG}}$$

or, for steel,

$$a = 0.806\, h \sqrt{\frac{I_y}{K}}$$

and for sloping flange sections,

$$h = d - \frac{2}{3}\left(n + \frac{z^2}{z + n}\right)$$

in which h = distance between flange centroids; or, approximately,

$$h = d - \frac{m + n}{2}$$

which is sufficiently accurate for practical purposes.

Derivation of General Equations.—Consider the equilibrium of any cut section as shown in Fig. 11. The outer torsional moment must be resisted by the internal moment of the resisting forces, thus:

$$T = T_u + Qh \quad \text{.......................(29)}$$

in which T_u = torque required to twist a beam in a free-ended condition; and Q = total lateral shearing force developed by one flange. In terms of the chosen co-ordinates, $\frac{h}{2}\,d\psi = dy$, approximately, and the twist, ψ, per unit length $= \frac{d\psi}{dx}$. Hence,

$$T_u = K\,G\,\frac{d\psi}{dx} = \frac{2\,K\,G}{h}\,\frac{dy}{dx} \quad \text{.....................(30)}$$

Furthermore, assuming that the larger moment of inertia of one flange is equal to $\frac{I_y}{2}$,

$$\frac{E\,I_y}{2}\,\frac{d^3y}{dx^3} = -\,Q \quad \text{.......................(31)}$$

Considering Equations (29), (30), and (31):

$$\frac{E\,I_y}{2}\,\frac{d^3y}{dx^3} - \frac{2\,K\,G}{h^2}\,\frac{dy}{dx} = \frac{-T}{h} \quad \text{................(32)}$$

Differentiating with respect to x, and substituting $a = \frac{h}{2}\sqrt{\frac{EI_y}{KG}}$:

$$a^2 \frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0 \qquad \text{(33)}$$

As a general solution of this differential equation:

$$y = A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + C + Dx \qquad \text{(34)}$$

Torsion with Both Ends Restrained.—By proper evalution of the constants for the conditions obtaining in a beam fixed at both ends:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh \frac{x}{a} \tanh \frac{l}{2\,a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{2\,a}\right) \qquad \text{(35)}$$

in which l = length of the beam. For $x = l$, the total deflection equals,

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\frac{l}{a} - 2 \tanh \frac{l}{2\,a}\right) \qquad \text{(36)}$$

The moment in each flange equals,

$$M = \frac{E\,I_y}{2} \frac{d^2 y}{dx^2} = \frac{T\,a}{h} \frac{\sinh u}{\cosh \frac{l}{2\,a}} \qquad \text{(37)}$$

in which $u = \frac{l}{2\,a} - \frac{x}{a}$; and, furthermore, the shear in each flange equals,

$$Q = \frac{E\,I_y}{2} \frac{d^3 y}{dx^3} = \frac{-T}{h} \frac{\cosh u}{\cosh \frac{l}{2\,a}} \qquad \text{(38)}$$

The longitudinal stresses along the outer fibers of the flanges will be given by:

$$\sigma = \frac{M\,c}{I} = \frac{M\,b}{I_y} = \frac{T\,a\,b}{h\,I_y} \frac{\sinh u}{\cosh \frac{l}{2\,a}} \qquad \text{(39)}$$

Critical stress will be either longitudinal stresses at the end of the beam, or the sum of the lateral and torsional shearing stresses at the center of the beam.

At the end of the beam, $u = \frac{l}{2\,a}$, and, consequently,

$$\sigma = \frac{T\,a\,b}{h\,I_y} \tanh \frac{l}{2\,a} \qquad \text{(40)}$$

and at the center of the beam, $u = 0$, giving,

$$Q_c = \frac{T}{h} \operatorname{sech} \frac{l}{2\,a} = \frac{T}{h \cosh \frac{l}{2\,a}} \qquad \text{(41)}$$

At the center of the beam both the lateral and torsional shearing stresses will be maximum in the middle of the flange.

By the simple beam theory, the lateral shearing stress will be: For parallel flange sections,

$$\tau = \left(\frac{3}{2}\,\frac{Q}{A_f}\right) = \frac{Q\,b^2}{4\,I_y} \quad \text{.....................(42)}$$

in which $A_f = A$ for one flange. For sloping flange sections,

$$\tau = \frac{Q\,b^2\,(2\,n + m)}{12\,m\,I_y} \quad \text{........................(43)}$$

For torsional shearing stress at the center of the beam, the approximate formulas proposed in Equations (25) and (27) are applied. Instead of K, the torsion constant, an equivalent constant must be introduced, which is measured by the rate of angular twist at the center of the beam.

To obtain this, differentiate Equation (35) with respect to x, giving the slope of the flange, $\frac{dy}{dx}$, at any point. The unit angular twist, $\frac{d\psi}{dx} = \frac{2\,dy}{h\,dx}$, and from this relation and the substitution of $x = \frac{l}{2}$, an expression is derived for $\theta_c = \frac{d\psi}{dx}$ at the center; thus,

$$\theta_c = \frac{T}{K\,G}\left[\frac{\cosh\left(\frac{l}{2\,a}\right) - 1}{\cosh\left(\frac{l}{2\,a}\right)}\right] \quad \text{.................(44)}$$

This derivation (Equation (44)) has been based on the assumption (6) that deflection is due to bending only. As the beam is shortened, however, shearing deflection becomes of increasing importance and should be considered.

Timoshenko(6) has indicated a strain energy method for calculating the deflection due to shear when cross-sections are constrained from warping. By combining his result for the simple cantilever beam the correction, $1 + 2.95\,\frac{b^2}{l^2}$, is obtained for a fixed-ended beam with point of inflection at the center; that is,

$$\theta_c = \frac{T}{K\,G}\left[\frac{\cosh\left(\frac{l}{2\,a}\right) - 1}{\cosh\left(\frac{l}{2\,a}\right)}\right]\left(1 + 2.95\,\frac{b^2}{l^2}\right) \quad \text{..........(45)}$$

and, denoting the "equivalent" torsion constant at the center by C_c:

$$C_c = \frac{T}{\theta_c\,G} = K\left[\frac{\cosh\left(\frac{l}{2\,a}\right)}{\cosh\left(\frac{l}{2\,a}\right) - 1}\right]\left[\frac{1}{1 + 2.95\,\frac{b^2}{l^2}}\right] \quad \text{.......(46)}$$

Using C_c as the measure of torsional shear developed at the center of the beam instead of K and combining Equations (25) with Equations (42) and (41), Equations (47) to (53) are derived for combined torsional and lateral shearing stresses, as follows:

Total Maximum Shearing Stress at Center of Beam (Along Center Line of Flange).—For parallel flange sections:

$$\tau = T\left[\frac{b^2}{4\,h\,I_y \cosh\left(\dfrac{l}{2\,a}\right)} + \frac{(D+n)}{2\,C_c}\right] \quad \text{.........(47}a\text{)}$$

for sloping flange sections:

$$\tau = T\left[\frac{b^2\,(2\,n+m)}{12\,h\,m\,I_y \cosh\left(\dfrac{l}{2\,a}\right)} + \frac{D+m}{2\,C_c}\right] \quad \text{.........(47}b\text{)}$$

and the stress in the web may be computed by:

$$\tau_w = \frac{T\,(w+0.3\,r)}{C_c} \quad \text{.........(48)}$$

Total Twisting Deflections of Fixed-Ended Beams.—Equation (46) provides an equivalent torsion constant based on the unit angular twist at the center of the beam. A measure of the total twisting deflection of the beam over the entire length is desired and can be obtained effectively by evaluating an average equivalent torsion constant, which will be denoted as C_A.

The expression for total angular twist is, then:

$$\psi = \frac{T\,l}{C_A\,G} \quad \text{.........(49)}$$

For very short beams most of the deflection is that producing shear and C_c approaches C_A in value. Equation (36) is an expression for the total deflection of the flanges due to bending only. Constant C_A may be evaluated from Equation (36) in so far as bending deflections are concerned.

The ratio of $\dfrac{C_A}{C_c}$ reduces to the following expression:

$$\frac{C_A}{C_c} = \frac{\left(\cosh\dfrac{l}{2\,a} - 1\right)}{\left(\cosh\dfrac{l}{2\,a}\right) - \left(\dfrac{2\,a}{l}\sinh\dfrac{l}{2\,a}\right)} \quad \text{.........(50)}$$

The graph of $\dfrac{C_A}{C_c}$ is given on Fig. 12, permitting the quick calculation of C_A after C_c is known. As the length approaches zero, C_A should approach C_c in value and the curve on Fig. 13 gives the reduction to be made in C_A for various $\dfrac{l}{b}$-ratios in terms of a factor to be multiplied by $(C_A - C_c)$.

As an example, let: $C_c = 6.00$ by Equation (46); $\dfrac{l}{2\,a} = 2.00$; and, $\dfrac{l}{b} = 5.00$.

From Fig. 12, $\frac{C_A}{C_c} = 1.42$; and, from Fig. 13, the reduction $= (0.42)\ (0.105) = 0.44$; or, $C_A = 6.00\ (1.42 - 0.044) = 8.28$.

Torsion with One End Fixed and One End Free.—Often, in cases of combined bending and torsion, one end of the beam will be relatively unrestrained while the other end is fixed. At the free end there will be no lateral shearing stresses in the flanges and the shearing stress formulas (Equations (25)) for free end torsion will apply. The evaluation of Equation (34) for these end conditions gives the following: For maximum bending moment in the flange at the unrestrained end:

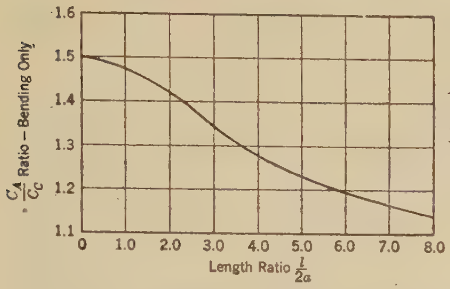


FIG. 12.—POSITIVE FACTOR TO OBTAIN C_C. TO BE USED IN CONJUNCTION WITH NEGATIVE FACTOR FROM FIG. 13.

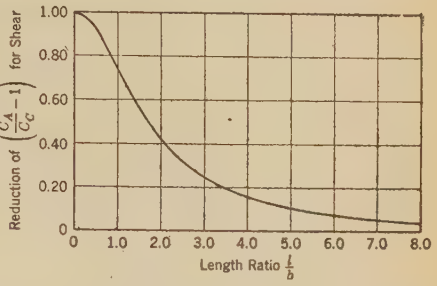


FIG. 13.—REDUCTION FACTOR FOR SHEAR (FOR USE WITH FIG. 12 TO OBTAIN C_A FROM C_C).

$$M_{\max} = \frac{T\,a}{h} \tanh \frac{l}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

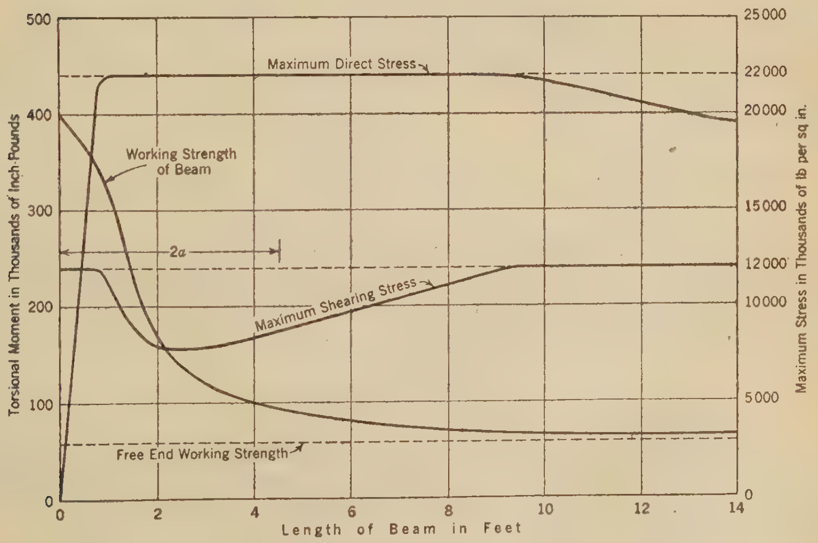


FIG. 14.—WORKING STRENGTH OF AN 8 BY 8-INCH BEAM WITH FIXED ENDS (SHOWING LIMITING STRESSES FOR DIFFERENT LENGTHS).

and for maximum direct fiber stress in the outer edges of the flanges at the restrained end:

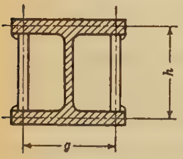
$$\sigma = \frac{T\,a\,b}{h\,I_y} \tanh \frac{l}{a} \qquad (52)$$

The total angle of twist is given by:

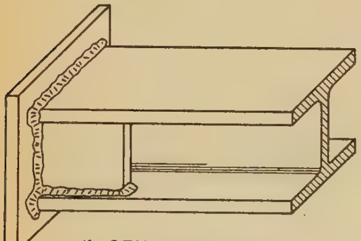
$$\psi = \frac{T\,a}{K\,G}\left[\frac{l}{a} - \tanh \frac{l}{a}\right]\left[1 + 0.74\,\frac{b^2}{l^2}\right] \qquad (53)$$

Equation (53) will be accurate for all except very short beams.

Fig. 14 shows the application of the proposed formulas for maximum longitudinal and shearing stresses to varying lengths of an 8 by 8-in. **H**-beam fixed at both ends. It is noted that for lengths ranging between about 1 ft and 9 ft the longitudinal stresses based on an allowable stress of 22 000 lb per sq in. determine the design. For very short lengths and for long lengths the shearing stresses are critical.



(a) END SECTION



(b) GENERAL VIEW

FIG. 15.—FIXED-ENDED BEAM.

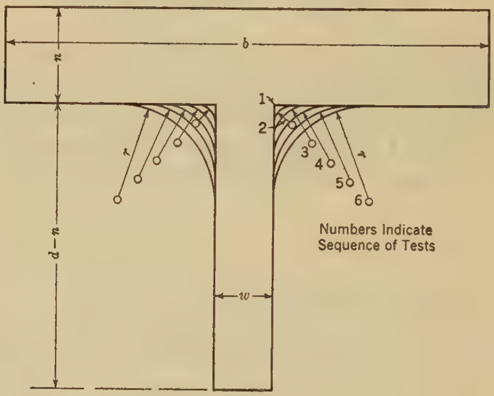


FIG. 16.—TYPES OF SECTIONS STUDIED BY SOAP FILM.

Design of End Connection.—It is suggested that the end connections be built as illustrated in Fig. 15. Connections of this type proved very satisfactory in the actual torsion tests and provided comparatively complete end fixity. The purpose of the end plates is to prevent relative warping of the flanges, and the following approximate analysis should serve as a guide in the design. The moment in one flange at the end is obtained from Equation (37)

$$M = \frac{T\,a}{h} \tanh \frac{l}{2\,a} \qquad (54)$$

The value of l should be measured as the over-all length. Let Q = the total shear of the beam in the stiffening plate; and s = the distance between the stiffening plates. Then, if the stiffening plates alone are assumed to fix the ends of the beam:

$$M = Q\,s \qquad (55)$$

Substituting in Equation (54):

$$Q = \frac{T a}{h s} \tanh \frac{l}{2 a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

The stiffener plate is welded to both flanges and to the end plate as well. The stiffener and the adjoining part of the end plate act as a short fixed-ended beam holding the flanges in place. No attempt was made to analyze the load distribution, and the design of the test beams was largely a matter of judgment.

The following tentative suggestions are made as the result of the tests:

(1) The length of the stiffener along the beam should be equal to about three-fourths the width of the flange for **H**-sections and to the full flange width for **I**-beams;

(2) The thickness of the stiffener plate material should be greater than that of the web thickness or greater than one-tenth the length of the stiffener plate along the beam;

(3) The stiffeners should be machined to a tight fit between the flanges and should be welded to flange and end plate continuously on the outer part;

(4) The end plates should have a thickness equal to twice the maximum thickness of the beam material; and

(5) The beam should be cut square and welded to the end plate with a continuous fillet weld about the entire beam end.

The stiffener plate and the weld between it and the flange should be designed to resist the shear as computed by Equation (52).

Design Examples

General Remarks.—The most economic structural **H**-beam or **I**-beam for torsional strength is one in which the material is most nearly of constant thickness throughout and is as thick and compact as obtainable. Column sections with parallel sided flanges and of the heaviest rolling in each series most nearly satisfy these requirements.

The torsional design should be made with ends assumed to be free in the case of riveted or bolted end connections; any percentage of end fixity incidentally present will simply provide an additional factor of safety. Only shearing stresses as computed by Equations (25) and (27) need be considered in free-end design.

Beams with boxed in and continuously welded end connections will be somewhat stiffer and stronger depending on the length. Both longitudinal stresses and shearing stresses must be considered (see Fig. 14). Tests indicate that the shearing stresses generally determine the yield point of the beam as a whole. The local direct stresses at the ends affect initially only a small part of the beam and are in the nature of secondary stresses. Shearing yield on the other hand occurs along the entire beam length. The allowable direct fiber stress will be made 22 000 lb per sq in. in the present discussion. It is suggested that allowable fiber stresses usual in secondary stress design be applied in general to these stresses.

General Data.—The following data apply to all the examples: Allowable working normal stress, $\sigma = 22\ 000$ lb per sq in., for secondary stresses due to fixed-end torsion; allowable working shear stress, $\tau = 12\,000$ lb per sq in.; $E = 29\ 000\ 000$ lb per sq in.; $G = 11\ 150\ 000$ lb per sq in.; Poisson's ratio, $\mu = 0.30$; l = over-all length of a beam, including stiffeners, along which a uniform torsional moment is assumed to act; $1° = 0.01745$ radian; and 1 radian = 57.30 degrees. Computations in these problems were made with a 10-in. slide-rule.

Design Example A.—A long beam with torsional deflection limited: Assume a beam 20 ft long designed to resist a total torsional moment of 20 000 in-lb with maximum total twist under the load limited to 1.2 degrees. The procedure for designing the beam as free ended involves three steps: (1) Determine the unit angle of twist, θ, in radians per inch, thus, $\theta = \dfrac{1.2 \times 0.01745}{20 \times 12} = 0.0000872$ radian per inch; (2) calculate the required K-value from Equation (5), thus, $K = \dfrac{T}{G\theta} = \dfrac{20\,000}{11\,150\,000 \times 0.0000872} = 20.6$ in.^{4}; and (3) refer to standard tables of K-values[6] and pick out the most economical section. In this manner, a Bethlehem section, 10 by 10 in., at 124 lb per ft (with $K = 20.37$) will be satisfactory. The end connection will provide additional rigidity and will allow a small tolerance in picking sections.

Design Example B.—Analysis of the torsional strength of a short beam (B8b, 8 by 8, at 67 lb per ft), with different end connections: The general data applying to this case are: $l = 66$ in., over-all; $K = 5.145$ in.^{4}; $I_y = 88.6$ in.^{4}; $n = 0.933$ in.; $b = 8.287$ in.; $D = 1.206$ in.; $h = 9.000 - 0.933 = 8.067$ in.; $a = 0.806\,h\sqrt{\dfrac{I_y}{K}} = 27.0$ in.; $\dfrac{l}{2a} = \dfrac{66}{54} = 1.22$; $\cosh\dfrac{l}{2a} = 1.8412$; and $\tanh\dfrac{l}{2a} = 0.8397$.

The free-ended working strength is computed by Equations (25) thus:

$$T = \frac{2K\tau}{D+n}\ \frac{(2)\,(5.145)\,(12\,000)}{1.206 + 0.933} = 57\,700 \text{ in-lb.}$$

To determine the fixed-ended working strength, based on shear, compute the equivalent torsion constant, C_c, for the center of the beam by Equation (46), thus:

$$C_c = 5.145\left(\frac{1.8412}{1.8412 - 1}\right)\left(\frac{1}{1 + 2.95\left(\dfrac{8.29}{66}\right)^2}\right) = 10.77 \text{ in}^4.$$

Then, from Equation (47),

$$T = \frac{12\,000}{\dfrac{8.29^2}{4 \times 8.067 \times 88.6 \times 1.841} + \dfrac{1.206 + 0.933}{2 \times 10.86}} = 108\,000 \text{ in-lb}$$

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[6] See, for example, Bethlehem Manual of Steel Construction, Catalog S-47, 1934, p. 285.

To determine the fixed-ended working strength, based on longitudinal fiber stresses at ends (tension or compression), apply Equation (40):

$$T = \frac{22\,000}{\dfrac{27.0 \times 8.287 \times 0.8397}{8.067 \times 88.6}} = 84\,000 \text{ in-lb}$$

The longitudinal stresses, therefore, determine the design of the beam, and the allowable torsional moment is 84 000 in-lb.

The shear to be resisted by the end plate is computed by Equation (56); thus, if s is assumed as 6.5 in.,

$$Q = \frac{84\,000 \times 27.0}{8.067 \times 6.5} \times 0.8397 = 36\,300 \text{ lb.}$$

A $\frac{5}{8}$-in. plate fitted into the 7.13-in. space between the flanges and 6 in. in length along the beam, will satisfy the requirements suggested. Assume a $\frac{5}{8}$-in. fillet weld between the stiffener and the flange. If $s = 8.827 - 3 \times \frac{5}{8} = 6.41$ in.,

$$Q = \frac{6.5}{6.41} \times 36\,300 = 36\,800 \text{ lb}$$

The stress in the plate is:

$$\tau = \frac{36\,800}{\dfrac{5}{8} \times 6} = 9\,800 \text{ lb per sq in.}$$

and in the weld:

$$\tau = \frac{36\,800}{\dfrac{5}{8} \times 0.707 \times 6} = 13\,900 \text{ lb per sq in.}$$

which is too high. Make the plate $7\frac{1}{2}$ in. long rather than 6 in. Then the stress in the weld is:

$$\tau = \frac{6}{7.5} + 13\,900 = 11\,100 \text{ lb per sq in.}$$

which is less than the limit of 11 300 lb per sq in. recommended by the American Bureau of Welding.

TEST RESULTS

Soap Bubble Tests: Purpose and Program

The purpose of the series of tests reported herein was to evaluate, accurately, the torsion constant of structural H-beam and I-beam sections. Specifically, this problem narrowed down to determining the added torsional rigidity introduced by the juncture of two rectangles, with fillets at the re-entry corners, in excess of the torsional rigidity of these rectangles treated as separate members. The problem, therefore, was to determine α in Equation

(16). In order to establish the value of α for any shape of section it was necessary to consider two variables, $\frac{w}{n}$, the ratio of web to flange thickness, and $\frac{r}{n}$, the ratio of fillet radius to flange thickness. Furthermore, it was essential to study sections with sloping flanges as well as those with parallel sides.

A program of tests was outlined to cover a wide range of the two variables, $\frac{w}{n}$ and $\frac{r}{n}$. Although it would have been desirable to measure the slopes of the bubbles and thereby study the stresses, particularly in the fillets, such a study would have greatly reduced the total number of tests possible. It was thought better to establish the torsion constant definitely, in which case it was only necessary to measure the volume of the displaced bubble. In each series a basic web and flange thickness was adopted and after testing the section with zero fillet radius, the various fillets were cut away in sequence as shown in Fig. 16.

The curves obtained from the test results are shown in Fig. 7. It was found that scattered points occurred along the lines of 0.0 and 0.2 fillet radius. These variations may be due to the difficulty in machining the plates with the small fillets, a slight inaccuracy causing considerable variation in the diameter, D, of the inscribed circle. A further difficulty is encountered in the plates of zero fillet radius due to the tendency of the soap film to jump across these sharp corners. Most of the structural beams actually rolled have ratios of $\frac{w}{n}$ and $\frac{r}{n}$ both greater than 0.5 and in this area the data were quite consistent.

Tests of Steel Beams: Purpose and Apparatus

Torsion tests were made on steel beams, with several objects in view. The sections themselves were chosen so as to give a range of shapes and sizes as great as possible, and tests of certain unusual shapes, which are not at present standard, were nevertheless valuable in the investigation.

Tests of free-ended conditions were made on a number of beams in order to check the results of the soap film experiments and the corresponding method of calculating the torsion constant. In these tests the distribution of shearing stress was studied, and a check was obtained on the proposed approximate formulas for stress. Tests of fixed-end conditions were made on different shaped beams and the effect of variations in length was studied. A type of end-connection design was developed to give a considerable degree of fixity.

A standard torsion machine of 26 000 in-lb capacity was used to test 3-in., fixed-end I-beams, of various lengths. It was also used for torsion tests of round bar samples of all material to determine the shearing modulus of elasticity.

The cable torsion rig, shown in Fig. 17, with an ultimate capacity of 750 000 in-lb was used in the major tests.[7] Most of the large beams were tested in lengths of 6 ft, but two tests each were made on beams 1 ft 3 in. and 3 ft long by means of the same sheaves and cables adapted for use with shorter top and bottom beams. During the tests of light beams, cables $\frac{5}{8}$ in. in diameter were used because of their flexibility and ease of handling, but in the tests of the heavier and shorter sections the cable was changed to 1 in. in diameter in order to develop the full capacity of the machine.

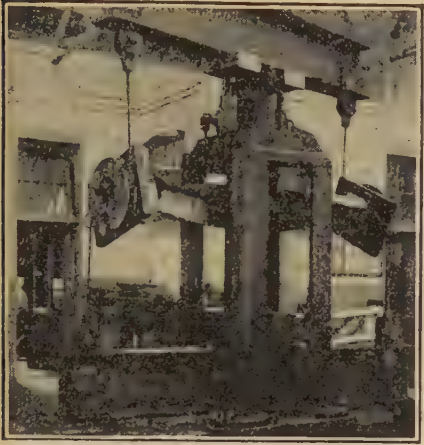

Fig. 17.—The Cable Torsion Rig.

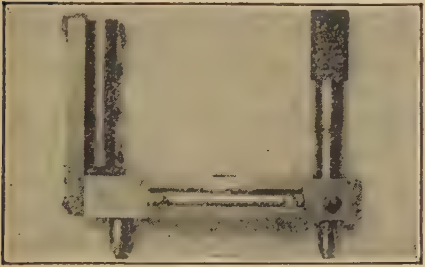

Fig. 18.—Level Bar.

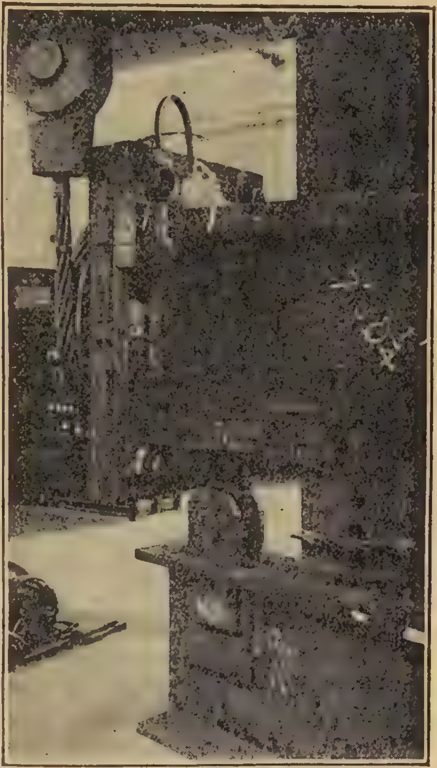

Fig. 19.—Connection for Free-Ended Beams.

The sheaves were made of material 2 in. thick and were machined to a minimum diameter of 17 in. A hole bored through one of the diameters allowed continuous action and reversing of the cable without fouling or introducing bending moment. This machine gave perfect satisfaction in every respect and was easily set up and dismantled. During tests the apparatus was in such a state of balance that the heavy pulling beams could be easily tilted either way by hand while maintaining a heavy torsional load of the test specimen.

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[7] "Torsion Testing Machine of 750 000-Inch-Pounds Capacity," by Bruce G. Johnston, Jun. Am. Soc. C. E., *Engineering News-Record,* February 28, 1935, p. 10.

Measuring Devices.—The level bar illustrated in Fig. 18 was built to measure the change in relative altitude of two points 3 in. apart. It was used to measure relative angle changes in all the beam tests. The micrometer vernier permitted readings to 0.0001 in. and the level bubble was sensitive to micrometer changes of about 0.0003 in. The total range of the instrument was about 9° 30′ ± from the original level position.

The torsion meter was used in the torsion tests of round bars in the 26 000-in-lb torsion machine to measure the angular twist over a 3-in. section of the bar. This device consisted of two steel collars attached by pointed thumb screws to the bar and provided with two 1:1 000 Ames dials for measuring the tangential displacements. The instrument was the same as that used and described in a previous investigation at the Fritz Laboratory.[8] Twenty tensometers were used to obtain the strains at all critical points during tests.

Test Procedure and Method.—In each torsion test, whether fixed or free-ended, there were three principal objectives: First, to learn as much as possible about the strain and stress distribution; second, to measure the torsional stiffness, or ratio of torsional load to angular twist; and, third, to learn the useful torsional load-carrying limit of the beam as a whole.

The strain and stress distribution was studied in two ways: First, by tensometers which were sensitive to changes in strain corresponding to from 150 to 300 lb per sq in. in stress, depending on the model type; and second, the beams were whitewashed with a mixture of water and hydrated lime which showed the distribution and location of the first surface strain-slip lines.

In computing the stresses from the observed strains the same values of E, G, and μ, as those used in Design Example A, were adopted. The torsional stiffness was gauged by measuring the relative angle changes between two points along the beam by use of the level bar. In the free-ended tests the angle changes were measured over a 36-in. length along the center part of the beam. In the fixed-ended beam the unit angle change varied along the length and a measure of the average stiffness was obtained by measuring the relative rotation of the end plates and of points a short distance from each end where the reinforcement ended.

The yield point of the beam as a whole was obtained by a study of the torque-twist diagrams and a knowledge of the load when the first strain-slip lines appeared. In a few cases a definite drop of the beam was noted and, in such cases, this was taken as the yield point. In most of the tests the yield point was taken as the torsional moment corresponding to the point on the torque-twist diagram where the co-tangent of the slope was 1.5 times the value of the co-tangent of the slope of the straight part.

Test Results.—Table 1, Appendix III, gives a general summary of all the tests made, including dimensions of beams and computed K-values based on actual dimensions. The dimensions were obtained by means of micrometers and calipers. Readings were taken at a number of different places on the beam, and averages from them were used to calculate the weight of the beam per foot of length. Furthermore, the beams were actually weighed and any discrepancy between computed and actual weight was taken care of by adjusting the average measured dimension to give the actual weight.

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[8] "Shearing Properties and Poisson's Ratio of Structural and Alloy Steels," by Inge Lyse, M. Am. Soc. C. E., and H. J. Godfrey, *Proceedings*, A.S.T.M., 1933, Vol. 33, Pt. II.

Table 2, Appendix III, presents the physical properties based on tests of samples taken from the test beams. The tensile values are based on the average of two tests of American Society of Testing Materials standard tension test specimens (2-in. gauge length). The torsion tests and slotted-plate shear tests were made in the same manner as described in a previous investigation at the Fritz Engineering Laboratory.[8] In most cases, the tensile specimens and round-bar torsion specimens were cut from the material where the flange and the web join, as it is at this point that critical torsional stresses develop. The material for the slotted-plate test specimens was cut from the webs of the beams.

Free-Ended Tests.—Free-ended tests were made on seven beams. Each specimen was held in the torsion rig by two bolts at each end which passed with a loose fit through the web and through the two angles, the angles being welded to end plates which, in turn, were bolted to the sheave plates of the testing rig. Torsional moment was applied by means of lugs welded to the end plates which engaged the flanges of the test specimens. The flanges were thus free to warp and the beams were almost entirely unrestrained at the ends. Fig. 19 shows the details at one end of the largest beam tested and is typical of all the free-end tests which were made. Figs. 20(a) and 21(a) show typical torque-twist diagrams of two of the tests. Figs. 20(b)

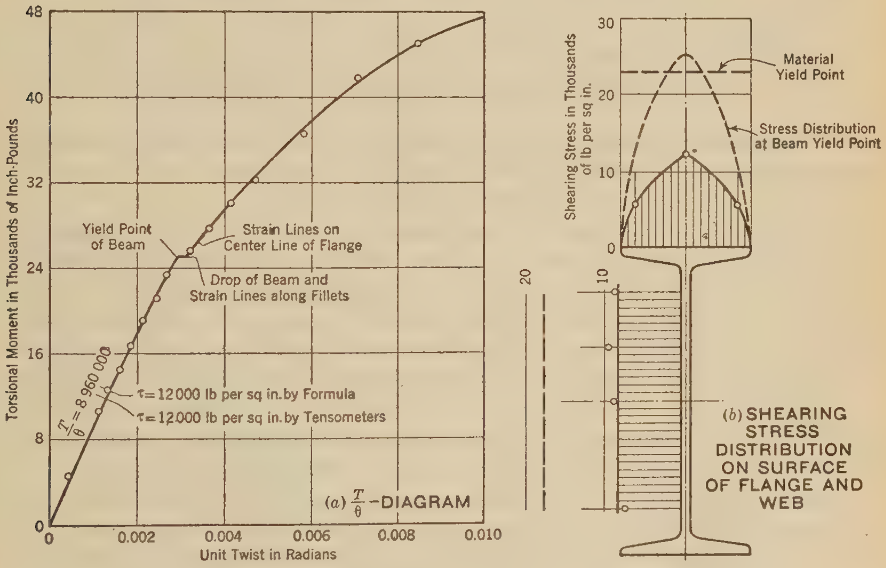


FIG. 20.—FREE-ENDED TORSION TEST T-26, A 12-INCH I-BEAM.

and 21(*b*) show the stress distribution based on tensometer readings taken in these same tests. The tensometers were placed on the flange and web surface at about the center cross-section of the beams and were set at an angle of 45° with the longitudinal axis of the beam, in order to measure one of the principal strains which, in the case of pure shear, will be equal in both directions. In Figs. 20(*b*) and 21(*b*) the shaded areas show the stress distribution for $\tau_m = 12\,000$ lb per sq. in.

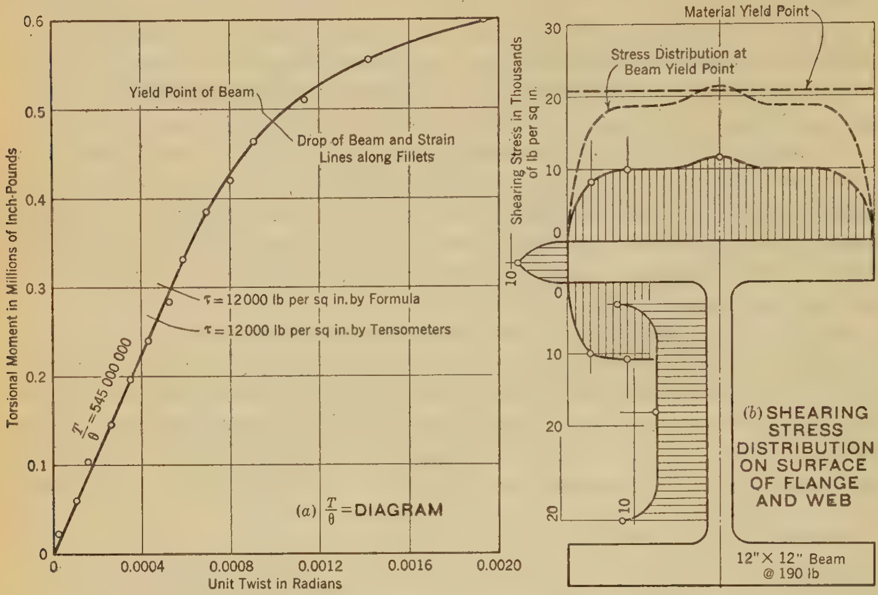


FIG. 21.—FREE-ENDED TORSION TEST *T*-31, A 12-INCH BETHLEHEM SECTION AT 190 POUNDS PER FOOT.

The data of actual stresses from tensometer readings were available at only a limited number of points. In drawing the curves of stress distribution these data have been supplemented by known facts, deducible from the general torsional theory, soap bubble tests, and from the actual beam tests; that is: (1) The shearing stress equals zero at outside corners; (2) there is a "hump" in the stress curve at the outside center of the flange, particularly if the fillets and the web thickness are relatively large as compared with the flange thickness; and (3) the shearing stress on the surface of the flange and web is approximately proportional to the thickness of the material.

Table 3, Appendix III, is a summary of the results obtained in the free-end tests. Good agreement is shown between the torsion constant computed from the measured dimensions and that obtained from the test results. The test value for *K* was obtained from the slope of the torque-twist diagram and from Equation (5); thus, $T = K\,G\,\theta$; or, $K = \dfrac{T}{G\theta}$.

The maximum variation for the *K*-value of test results was 6.7 and the average variation of seven tests was 2.26 per cent. It is noted that

K for the heaviest beam tested was about two hundred times greater than for the lightest beam and that the corresponding agreement for these tests was expressed by a variation of 0.9 and 0.0 per cent.

The shearing stress computed by Equation (25) gave average stresses 7% less than those based on tensometer readings. The stresses in the web by Equation (26) which, theoretically, should be correct, were much lower than as computed from tensometer readings. Equation (27) was suggested on the basis of these tests which, of course, are not complete enough to substantiate its adoption definitely. However, both Equations (25) and (27) are believed to be usable for practical design purposes with ordinary values of allowable shearing unit stress. The following special remarks apply to the individual free-end tests:

In Test T-12 strain lines appeared along the fillets at 13 500 in-lb; and along the outside center line of the flange at 15 200 in-lb. The yield point of the beam as determined by the slope of the torque-twist diagram was 15 900 in-lb.

In Test T-22 the freedom of the ends from restraint was checked by tensometers placed longitudinally near the ends. The strains were negligible. The first shear strain line along the center line of the flange appeared at 21 210 in-lb. At 25 610 in-lb, strain lines progressed rapidly along the flange and in the fillets, and a definite drop of the beam was noted.

In Test T-25 strain lines appeared along the fillets at 50 900 in-lb. Thereafter, the slope of the torque-twist diagram became nearly 50% greater than for lower loads, maintaining nearly the same slope up to 100 000 in-lb. The yield point was taken as 50 900 in-lb. Strain lines appeared along the outside of the flange at 77 340 in-lb.

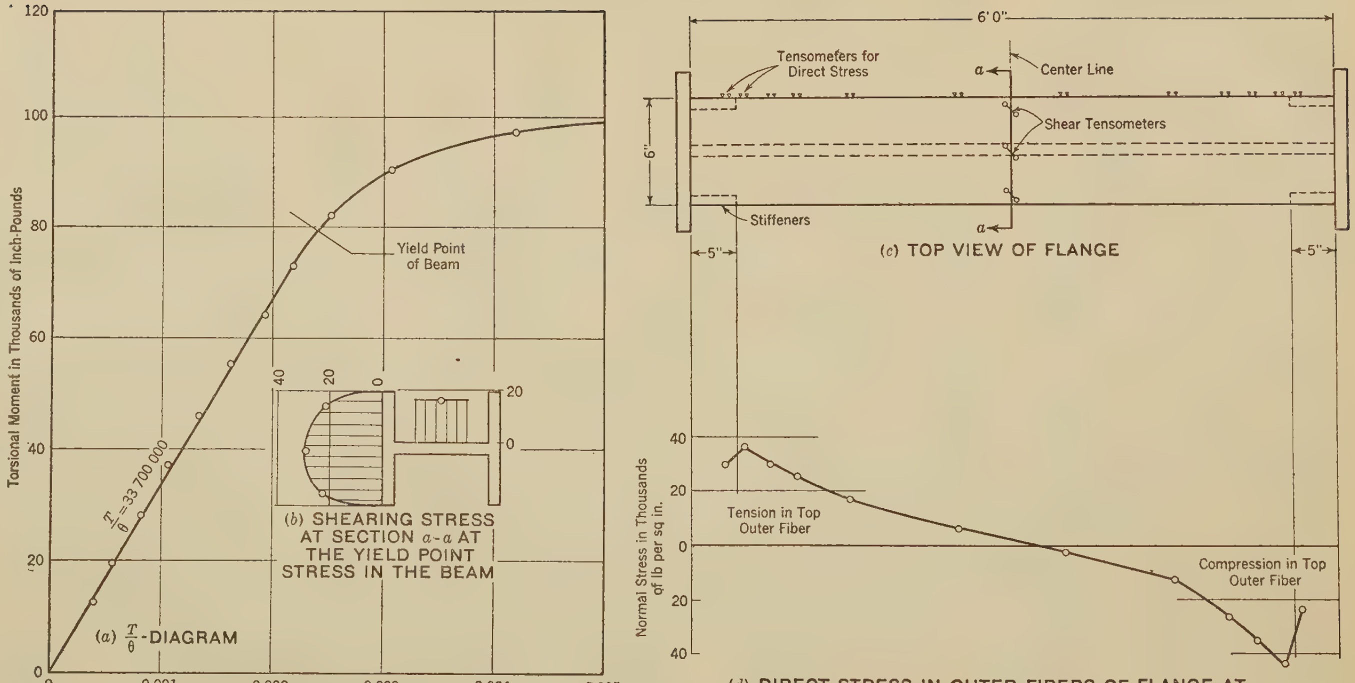
In Test T-26 (see Fig. 20) a slight checking in the fillets was noted at 23 000 in-lb, with a drop-of-beam yield point at 25 000 in-lb; strain lines progressed along the outside of the flange at 26 500 in-lb.

In Test T-30 the yield of the beam was noted by the 1.5 on 1.0 slope of the torque-twist diagram at 64 000 in-lb. The first strain lines appeared over the web at an indeterminate load due to the presence of heavy scale on the section.

In Test T-31 (see Fig. 21) the beam and dropped strain lines occurred along the fillets at 48 800 in-lb. The yield point was noted by the 1.5 on 1.0 slope at 480 000 in-lb.

In Test T-33 strain lines occurred along the fillets at 65 600 in-lb. Yield point and strain lines appeared along the outside of the flange at 76 400 in-lb.

Fixed-End Tests.—Twenty-two fixed-end tests were made on beams of different sizes. Tests T-4 to T-13, inclusive, were made with different lengths of a 3-in., 7.5-lb I-beam. These specimens were welded to end plates, 1 in. thick, with a continuous $\frac{3}{8}$-in. fillet weld. The end plates were bolted to $1\frac{1}{4}$-in. plates which were attached to the jaws of the standard 26 000 in-lb torsion machine. The relative rotation of the end plates was measured by level-bar readings taken at each end. The remainder of the beams were tested in the cable-testing rig. All these beams were larger than the 3-in. I-beams



(b) SHEARING STRESS AT SECTION a-a AT THE YIELD POINT STRESS IN THE BEAM

(c) TOP VIEW OF FLANGE

(d) DIRECT STRESS IN OUTER FIBERS OF FLANGE AT YIELD POINT OF THE BEAM

FIG. 22.—FIXED-ENDED TORSION TEST T-17, A 6-INCH H-BEAM AT 41 POUNDS PER FOOT.

and were welded to $1\frac{1}{4}$-in. end plates with additional end stiffeners, fitted and welded between the flanges, as was illustrated in Fig. 15.

Relative rotation of the end plates was observed on all beams by means of level-bar observations, and the twist of the large beams having stiffener plates was also measured at points just inside those plates. Strain readings for longitudinal and shearing strains were observed wherever feasible. Fig. 22(*a*) shows a typical fixed-end torque-twist diagram, and Fig. 22(*b*), 22(*c*), and 22(*d*) show the computed stresses from strain readings at the yield point of the beam. These observations are typical of all the fixed-end tests.

Table 4, Appendix III, gives the summary of the test results for fixed-end beams. In computing the values of C_0 and C_A the question arose as to the correct length to be used. If 100% end fixity were possible the correct length would be slightly less than the over-all length and somewhat greater than the length between the end stiffener plates. However, the over-all length is the simplest approximation, and it gives the best results by comparison with the tests, except in the case of very short beams with end stiffeners. In these two tests (*T*-19 and *T*-27, Table 4) the apparent percentage of end-fixity seems inconsistently high. Two tests have unusually low percentages of end fixity (see Tests *T*-16 and *T*-24). The explanation for this is given under the special remarks. The average percentage of end fixity with Tests *T*-16 and *T*-24 omitted is 88.3 and all the 6-ft beams, except *T*-24, have an end efficiency greater than 85 per cent.

In most cases the yield points of the beams were determined from the slope of the torque-twist diagrams and the theoretical direct stresses computed on the basis of this yield-point torque are given in Column (14), Table 4 (Appendix III). It is noted that, in spite of incomplete end fixity, these stresses, in every case, are above the tensile yield-point strength of the material as given in Table 2 (Appendix III). Hence, all the beams would have been designed safely on the basis of working, direct, fiber stresses. The average of Column (14), Table 3 (Appendix III), is 55% more than the average yield-point strength of the material in the test beams.

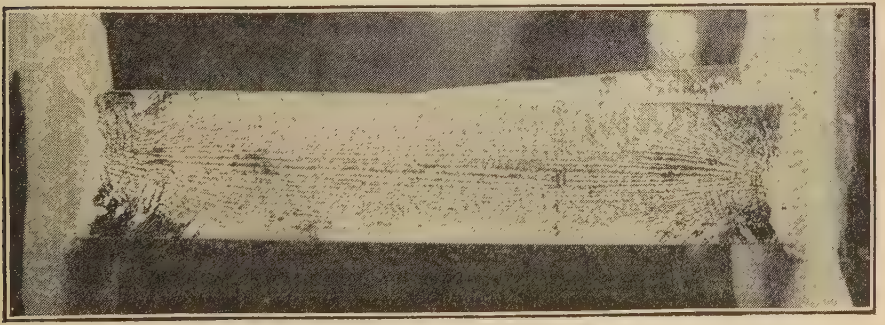


FIG. 23.—ILLUSTRATION OF STRAIN PATTERN.

The computed and measured shearing stresses in the flange agree well for all the 6-ft beams, with the exception of *T*-16 and *T*-24 in which the low

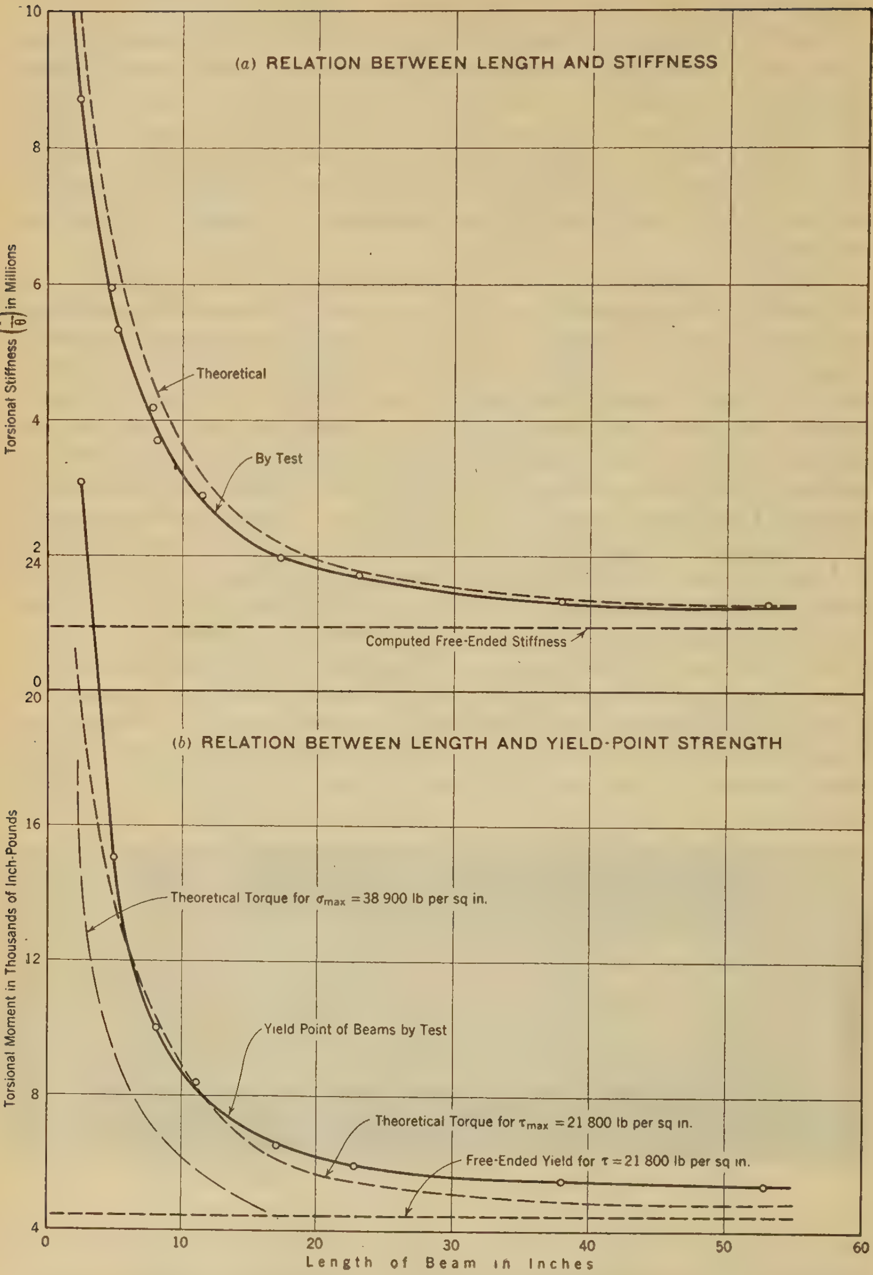


FIG. 24.—TEST OF 3-INCH I-BEAM AT 7.5 POUNDS PER FOOT, FIXED AT BOTH ENDS.

end fixity affects, directly, the shearing stress agreement. On the short 18-in. beams with stiffener plates (*T*-19 and *T*-27), the discrepancy between the computations and the test results is high, as might be expected. The computed stresses in the web give an approximate check on the stresses indicated by tensometers. Special remarks on fixed-end tests are, as follows:

Tests *T*-4 to *T*-12 were of different lengths of the same 3-in. **I**-beams and were well adapted to show the influence of end fixity on the strength and stiffness with the length of beam the only variable. The flange of each beam was whitewashed so that the appearance of first strain lines might be noted. Fig. 23 shows the strain line pattern on the flange of one of these beams after yield had taken place. Fig. 24(*a*) gives a graph of test results for this series showing the influence of length upon the rigidity, and Fig. 24(*b*) illustrates the influence of length upon the yield-point strength of the beams Special attention in this series of tests was given to Tests *T*-8 and *T*-10. Stress measurements were taken along the extreme fiber of the flanges at short intervals of length and the lateral bending moment in the flanges for a definite torque load was computed from these readings. The bending moments along the beam were plotted and the curves were differentiated to give the lateral shear in the flanges. These results are compared in Fig. 25 with the theoretical variation in shear by Equation (38).

Tests *T*-15 and *T*-16 should be compared with the free-end test, *T*-14, of the same section. Test *T*-16 was a special run with additional stiffeners placed midway along the section. These stiffeners were of the same type as the end stiffeners and were parallel in a plane with the web. No outstanding stiffener was provided. Although this additional stiffener gave the beam an average stiffness 42% greater than Test *T*-15, it provided only 40.1% of the theoretical stiffness of a beam 3 ft in length, rigidly fixed at each end. The design of this beam would have been safe for strength, however, if based on the 3-ft length and designed for the proper longitudinal working stress.

Test *T*-17, in contrast to Test *T*-15, was of the heaviest 6-in. section rather than the lightest. It should be noted that an effective fixity of 96.8% was attained in this test. The end stiffeners were $\frac{5}{8}$ in. thick and 5 in. long. The torque-twist diagram and data on stress distribution are given in Fig. 22 as typical of the results for fixed-end tests.

Test *T*-18, of a subway column, provided an opportunity to observe a section of extreme proportions.

Tests *T*-19 to *T*-21, together with free-end Test *T*-22 on the same sized section, provided a series of different lengths of 8-in. **H**-section. It is noted that the shortest beam tested showed an end-fixity efficiency of 101%, whereas the general trend for shorter beams should be less end fixity because of the greater strains placed upon the end connection. This effect is explained by the fact that the over-all length was used in the computations. Although this is good approximation for the longer beams, the stiffness changes rapidly in the short length range and the correct length is some-

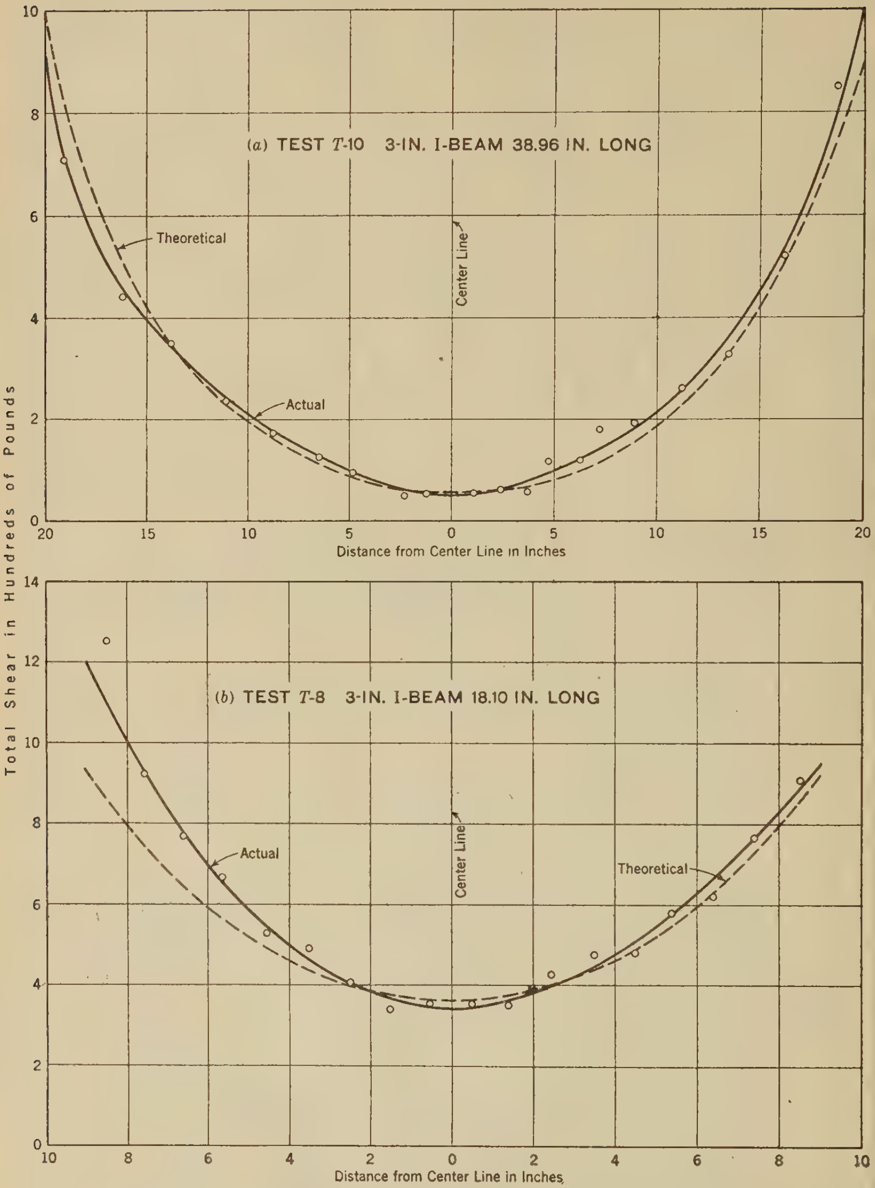


FIG. 25.—SHEAR DISTRIBUTIONS IN FLANGES OF FIXED-ENDED BEAMS FOR A TORSIONAL MOMENT OF 2 500 INCH-POUNDS.

value less than that used. Fig. 26 shows the strain-line distribution near the end of Beam *T*-21 after yielding had occurred.

Test *T*-23 is of the heaviest section of the 8-in. **H**-sections, whereas Tests *T*-19 to *T*-22 were of the lightest weight section.

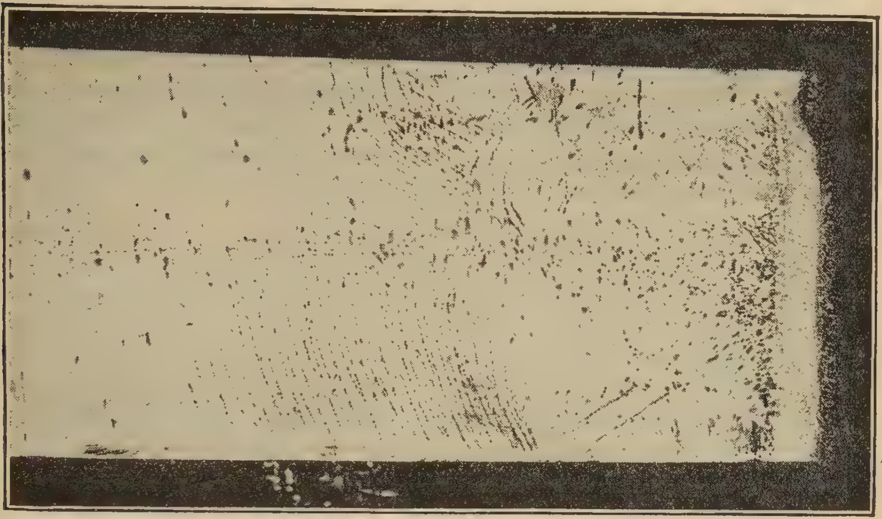


FIG. 26.—STRAIN LINES AT THE FIXED END OF BEAM.

Test *T*-24 is of interest because it had the lowest end-fixity efficiency of the 6-ft beam sections, with an end fixity of only 45.5 per cent. The remaining 6-ft sections were more than 85% efficient in end fixity. The explanation is found in the fact that the end stiffeners in this base were not properly designed, and the beam not only pulled loose at the weld from the end plate, but the end plate was badly warped during the test. The stiffeners were $\frac{9}{16}$ in. thick and 6 in. long. Had they been 9 in. long and 1 in. thick, as provided in the tentative Design Suggestions (1) and (2) (see heading, "Design of End Connection"), it is believed that much higher efficiency would have resulted.

Tests *T*-27 to *T*-29 are of three different lengths of 12-in. **I**-beam. The remarks concerning Tests *T*-19 to *T*-21 appear to apply in the case of these beams also.

SUMMARY AND CONCLUSIONS

Torsional Theory and the Torsion Constant.—The material presented in this paper may be outlined under eight divisions, as follows:

(1) The essential features of the general torsion problem are outlined;

(2) The application of Prandtl's membrane analogy is presented;

(3) An accurate and detailed method of evaluating the torsion constant of structural **H**-beams and **I**-beams is presented (this method is based on a known theoretical evaluation combined with factors determined by experiment from the membrane analogy);

(4) Formulas are proposed for the shearing stress in the flange and web due to pure torsion;

(5) The effect of shearing stress concentration in the fillets due to both torsion and bending is discussed;

(6) The problem of torsion with either one or both ends of the beam restrained is studied in detail and formulas for maximum shearing and longitudinal fiber stresses are given;

(7) Specifications are suggested for the design of a welded end connection which will give a high degree of end-fixity efficiency in torsion; and,

(8) Detailed design examples are presented to illustrate the application of the formulas to torsional design.

Test Results.—The purpose of the soap film tests was to determine, experimentally, the additive portion in the K-factor due to the added rigidity accruing from the juncture of the web and flange with the corresponding fillets. To this end, soap film experiments were made on fifty-seven differently proportioned sections both with parallel and sloping flanges. The application of the membrane analogy as developed by this investigation was restricted to volume measurement only. The method used was rapid, and, it is believed, gave results having an error considerably less than ± 1 per cent.

In computing the K-values of structural sections the experimentally determined part of K amounts, in the most extreme case, to 10% of the total K. Hence, an experimental error of ± 1% would give a possible error of only 0.1% in the total K-value.

Free-end torsion tests were made on seven different beams ranging in size from a 6-in. **H**-beam @ 20 lb per ft to a 12-in. **H**-beam @ 190 lb per ft. The heaviest beam had a torsion constant about two hundred times as great as the lightest beam.

The method of applying the torque to the ends of the beam provided a high degree of end freedom. By measuring the unit twist of the free-ended beams and obtaining the slope of the torque-twist diagram, the free-ended torsion tests provided (through Equation (5)) a definite check on the torsion constant as computed by the proposed method. Using $G = 11\,150\,000$ lb per sq in., which is theoretically correct for $E = 29\,000\,000$ lb per sq in. and $\mu = 0.30$, the test results checked well with computed K-values. The maximum variation was 6.7% and the average for the seven tests was 2.26 per cent.

The yield point of the beams in both free-ended and fixed-ended tests was determined by a study of the torque-twist diagram and in some cases by a drop of the beam. In the free-ended tests the distribution of shearing stress across the flange was studied by measuring, with tensometers, the principal stress at an angle of 45° around the center section.

The fixed-ended tests provided a means of studying the additional rigidity and strength over the free-ended tests, due to fixing the ends of the beams. Twenty-two various lengths and sizes of beams were tested ranging from a 3-in. I-beam at 7.5 lb per ft, to a 10 by 12-in. beam at 62 lb per ft. The heaviest beam had a K-factor fifty-seven times greater than the least, and with ends fixed the most rigid fixed-end beam had a rigidity three hundred and ten times as great as the least rigid with ends fixed.

The distribution of longitudinal direct stresses in the extreme fibers of the flanges was studied by strain measurements taken along the outer edges of the flanges by tensometers. The total shearing stress distribution at the center section was studied by measurement of principal strains at an angle of 45 degrees.

The fixed-ended tests furnished information as to the proper design of welded end connections for high torsional rigidity.

The tensometers in both the free-ended and fixed-ended tests were of value in studying the relative distribution of stress. Some of the results are erratic. The warping of the section during twist made it difficult to obtain a steady set-up for the tensometers, but in most cases the test results checked fairly well with the formulas. In spite of variations in result and incomplete end fixity it is noted that every beam tested would have been amply strong if designed on a basis of working longitudinal stresses at the ends.

The present investigation has covered, accurately, the question of torsional rigidity and the evaluation of the torsion constant. The formulas for stresses which are proposed are not exact but will be satisfactory for practical design purposes.

Acknowledgment

The torsional investigation was conducted as a co-operative project, the McClintic-Marshall Corporation, subsidiary to the Bethlehem Steel Company, furnishing the steel beams and the material for constructing the test apparatus, and Lehigh University bearing all other expenses.

Through the courtesy of the Baldwin-Southwark Corporation, of Philadelphia, Pa., sixteen tensometers were loaned to the University (without charge), during these tests.

Special acknowledgment is due Jonathan Jones, M. Am. Soc. C. E., Chief Engineer; C. H. Mercer, M. Am. Soc. C. E., Consulting Engineer, and Sterling Johnston, M. Am. Soc. C. E., Engineer, all of the McClintic-Marshall Corporation, and V. E. Ellstrom, Manager of Sales Engineering, Bethlehem Steel Company, for their co-operation in the investigation.

The study was conducted as a regular research project of the Fritz Engineering Laboratory, Lehigh University. C. C. Keyser, Laboratory Assistant, and all others associated with the Laboratory contributed valuable help throughout. Professor J. B. Reynolds, of the Department of Mathematics, gave his assistance to the theoretical studies.

APPENDIX I

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APPENDIX II

NOTATION

The following symbols, adopted for use in this paper, are presented as a guide to discussers:

a = a torsional factor in fixed-ended beams.

b = length of a rectangular section.

d = depth; total depth of beam.

e = eccentricity.

f = a subscript denoting "flange."

h = distance between flange centroids.

$i = \frac{1}{2}(d - h)$.

m = major flange thickness.

n = breadth of a rectangular section; also, where defined, n = minor flange thickness.

p = pressure per unit area.

r = radius of a fillet (authors' duplicate); also, where defined, r = variable thickness of a section.

s = distance between stiffening plates.

t = thickness; as a subscript, t, denotes "due to bending."

u = a substitution factor in Equation (37); as a subscript, u, denotes "free ended."

w = web thickness; as a subscript, w, denotes "web."

y = deflection.

A = area; cross-section area; also, where specifically defined, A, B, C, and D are coefficients in a general equation; as a subscript, A, denotes "average."

B = a substitution factor in Equation (23) (see, also, Symbol A).

C = constant; torsion constant equivalent to K for a fixed-ended beam; $C_c = C$ at the center; C_A = average value of C (see, also, Symbol A).

D = diameter; diameter of an inscribed circle (see, also, Symbol A).

E = elasticity; modulus of elasticity in tension and compression.

F = torsion stress function.

G = shearing modulus of elasticity.

I = rectangular moment of inertia; I_x and $I_y = I$ for a cross-section with respect to the X-axis and the Y-axis, respectively.

J = polar moment of inertia.

K = a torsion constant; K_f and K_w = values of K for flange and web, respectively.

L = length; as a subscript, L, denotes "large end."

M = bending moment.

Q = total shear over a cross-section.

S = total slope of a flange section $= \frac{m - n}{b}$; as a subscript, S, denotes "small end."

T = torque; torsional moment; T_a = torque required to twist a beam in a free-ended condition.

V = a factor depending on the $\frac{b}{n}$-ratio, but practically constant for $\frac{b}{n}$ greater than 3; V_L and V_S = value of V for large end and small end, respectively, of the flange.

α = a factor that depends on two ratios, $\frac{w}{m}$ and $\frac{r}{m}$.

ε = unit elongation.

θ = angle of twist, in radians per unit length; $\theta_c = \theta$ at the center.

μ = Poisson's ratio.

ρ = radius of curvature.

σ = unit stress; normal stress; σ_t = longitudinal fiber stress in a flange due to bending.

τ = unit shearing stress; $\tau_f = \tau$ for flange sections; $\tau_w = \tau$ for web sections.

ψ = total angle change between two cross-sections.

APPENDIX III

TABLES

The tables presented herewith are introduced and adequately discussed in the text of this paper.

TABLE 1.—GENERAL SUMMARY OF ALL TESTS

Test No.	Nominal size	Weight, in Pounds per Foot		Length, in inches	Measured Dimensions, in Inches, Adjusted to the Actual Weight (See Fig. 6)						Measured diameter, D, of inscribed circle, in inches	Torsion constant, K
		Nominal	Actual		Total depth, d	Length, b	Web thickness, w	Flange thickness		Fillet radius, r		
								m	n			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
T-4*	3-in. I-beam.......	7.5	7.4	3.2	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-5*	3-in. I-beam.......	7.5	7.4	6.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-6*	3-in. I-beam.......	7.5	7.4	9.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-7*	3-in. I-beam.......	7.5	7.4	12.1	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-8*	3-in. I-beam.......	7.5	7.4	18.1	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-9*	3-in. I-beam.......	7.5	7.4	23.9	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-10*	3-in. I-beam.......	7.5	7.4	39.0	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-11*	3-in. I-beam.......	7.5	7.4	53.9	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-12*	3-in. I-beam.......	7.5	7.4	5.5	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-13*	3-in. I-beam.......	7.5	7.4	8.8	2.98	2.51	0.342	0.342	0.162	0.27	0.532	0.0865
T-14†	6 by 6-in..........	20	19.6	72	6.06	6.01	0.257	.....	0.362	0.28	0.566	0.243
T-15*	6 by 6-in..........	20	19.6	72	6.06	6.01	0.257	.....	0.362	0.28	0.566	0.243
T-16*	6 by 6-in..........	20	20.9	72	6.08	6.06	0.303	.....	0.372	0.24	0.581	0.281
T-17*	6 by 6-in..........	40.5	39.0	72	6.73	6.25	0.501	.....	0.699	0.35	0.928	1.738
T-18*	6 by 10-in.........	40	38.1	72	6.20	9.91	0.431	.....	0.447	0.35	0.735	0.802
T-19*	8 by 8-in..........	31	29.4	18	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-20*	8 by 8-in..........	31	29.4	36	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-21*	8 by 8-in..........	31	29.4	72	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-22†	8 by 8-in..........	31	29.4	72	8.05	8.04	0.290	.....	0.401	0.41	0.676	0.463
T-23*	8 by 8-in..........	67	66.1	72	9.05	8.29	0.606	.....	0.907	0.42	1.208	4.912
T-24*	10 by 12-in........	62	60.9	72	10.05	11.92	0.390	0.659	0.546	0.45	0.919	2.101
T-25†	10 by 12-in........	62	60.9	72	10.05	11.92	0.390	0.659	0.546	0.45	0.919	2.101
T-26†	12-in. I-beam......	31.8	31.3	72	12.06	4.97	0.350	0.705	0.345	0.42	0.882	0.833
T-27*	12-in. I-beam......	55	53.6	18	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-28*	12-in. I-beam......	55	53.6	36	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-29*	12-in. I-beam......	55	53.6	72	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-30†	12-in. I-beam......	55	53.6	72	12.07	5.74	0.730	0.900	0.449	0.54	1.244	3.318
T-31†	12 by 12-in........	190	186.7	72	14.39	12.56	1.069	.....	1.714	0.62	2.186	48.437
T-33†	12 by 12-in........	65	65.6	72	12.14	12.03	0.409	.....	0.604	0.58	0.953	2.227

* Fixed end. † Free end.

TABLE 2.—PHYSICAL TESTS OF MATERIAL IN TEST BEAMS

Samples; taken from Test Beam No.	TENSILE PROPERTIES						SHEARING PROPERTIES, IN POUNDS PER SQUARE INCH				Shearing modulus, *G*, in thousands of pounds per square inch
	Stresses, in Pounds per Square Inch			Modulus of elasticity *E*, in thousands of pounds per square inch	Percentage elongation in 2 inches	Percentage reduction in area	Slotted Plates		Round Bar Torsion Tests		
	Upper yield point	Lower yield point	At ultimate				Shearing yield point	Shearing ultimate	Shearing yield point	Apparent shearing ultimate	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
T-4 *T*-13	38 900	......	63 550	29 100	32.0	62.1	21 800	52 850	......	......	....
T-14 *T*-15	41 000	37 860	60 450	29 200	35.8	66.4	25 600	48 500	26 400	80 600	12 080
T-16	39 710	35 790	58 130	29 150	35.5	66.7	22 000	45 100	22 000	50 700	11 280
T-17	43 020	37 000	60 220	28 500	37.0	67.3	23 700	49 900	24 000	64 100	12 020
T-18	34 180	31 570	56 570	28 250	40.0	65.4	22 600	47 900	22 200	65 600	11 500
T-19 *T*-22	42 880	38 290	61 280	29 450	36.0	66.1	25 500	51 000	22 300	64 700	10 780
T-23	34 890	31 580	61 180	28 600	38.8	66.1	21 200	50 700	20 200	65 700	11 780
T-24 *T*-25	40 860	35 530	59 600	29 150	36 3	64 9	23 700	49 800	24 400	64 100	12 350
T-26	38 310	33 900	56 330	29 400	38.0	69.3	24 600	52 800	21 400	71 000	11 920
T-27 *T*-30	32 380	30 990	59 850	29 400	34.5	65.9	18 200	47 700	20 900 18 700	66 200 64 500	11 770 10 780
T-31	32 910	28 620	60 000	29 200	38.0	63.7	21 200	51 600	20 300	64 600	11 260
Average..	38 090	34 110	59 740	29 040	36.5	65.8	22 740	49 800	22 070	65 620	11 590

TABLE 3.—TESTS ON FREE-ENDED BEAMS

Test No.	Nominal size	Nominal weight, in pounds per foot	TORSION CONSTANT, *K*			Yield point of beam, in inch-pounds	SHEARING STRESSES, IN POUNDS PER SQUARE INCH				
							Maximum, at the Yield Point				Yield point strength of material
							In the Flange		In the Web		
			From measured dimensions	By test	Percentage variation		By Equation (25)	By tensometers	By Equation (27)	By tensometers	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
T-14.......	6 by 6-in.....	20	0.243	0.243	0.0	15 900	30 400	29 700	22 400	19 700	26 000
T-22.......	8 by 8-in.....	31	0.463	0.451	—2.6	25 610	29 800	31 400	22 600	22 200	23 900
T-25.......	10 by 12-in...	62	2.101	1.960	—6.7	50 900	19 100	23 400	12 600	17 900	24 050
T-26.......	12-in I-beam..	31.8	0.833	0.804	—3.5	25 000	23 800	25 400	14 900	17 700	23 000
T-30.......	12-in. I-beam..	55	3.318	3.380	+1.9	64 000	20 700	20 200	17 100	18 500	19 000
T-31.......	12 by 12-in...	190	48.440	48.870	+0.9	480 000	19 300	21 400	12 400	12 600	20 750
T-33.......	12 by 12-in...	65	2.227	2.222	—0.2	76 400	26 600	29 300	......	......	......
Average..	....	....	......	......	2.26	......	24 240	25 830	17 000	18 100	22 780

TABLE 4.—Tests of Fixed-Ended Beams

Test No.	Nominal size	Nominal weight, in pounds per foot	Length of beam, l, in inches	Moment of inertia, I_y, in inches^4 (from measured dimensions)	Torsion constant, K, from measured dimensions	Values of coefficient, $a = 0.806\,h\sqrt{\frac{I_y}{K}}$	Values of ratio, $\frac{l}{2a}$	Torsion Constants		
								C_s', computed by Equation (46)	C_A	
									Computed	By test
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
T -4......	3-in. I-beam. .	7.5	3.2	0.57	0.0865	5.67	0.282	0.7998	0.943	0.787
T -5......	3-in. I-beam. .	7.5	6.0	0.57	0.0865	5.67	0.529	0.4558	0.605	0.474
T -6......	3-in. I-beam. .	7.5	9.0	0.57	0.0865	5.67	0.794	0.2823	0.394	0.330
T -7......	3-in. I-beam. .	7.5	12.1	0.57	0.0865	5.67	1.067	0.1996	0.284	0.258
T -8......	3-in. I-beam. .	7.5	18.1	0.57	0.0865	5.67	1.596	0.1341	0.190	0.174
T -9......	3-in. I-beam. .	7.5	23.9	0.57	0.0865	5.67	2.108	0.1102	0.154	0.153
T-10......	3-in. I-beam. .	7.5	39.0	0.57	0.0865	5.67	3.439	0.0913	0.120	0.116
T-11......	3-in. I-beam. .	7.5	53.9	0.57	0.0865	5.67	4.753	0.0875	0.109	0.111
T-12......	3-in. I-beam. .	7.5	5.5	0.57	0.0865	5.67	0.485	0.5004	0.654	0.534
T-13......	3-in. I-beam. .	7.5	8.8	0.57	0.0865	5.67	0.776	0.2904	0.404	0.377
T-15......	6 by 6-in.....	20	72.0	13.1	0.2433	33.7	1.068	0.6186	0.905	0.777
T-16......	6 by 6-in.....	20	36.0*	13.8	0.2810	32.4	0.556	1.896	2.761	1.106
T-17......	6 by 6-in.....	40.5	72.0	28.5	1.7375	19.7	1.872	2.431	3.445	3.333
T-18......	6 by 10-in....	40	72.0	72.5	0.8018	44.1	0.816	2.917	4.264	3.944
T-19.....	8 by 8-in.....	31	18.0	34.8	0.4631	53.5	0.168	20.927	27.55	27.90
T-20......	8 by 8-in.....	31	36.0	34.8	0.4631	53.5	0.336	7.496	10.75	7.43
T-21......	8 by 8-in.....	31	72.0	34.8	0.4631	53.5	0.673	2.346	3.45	3.30
T-23......	8 by 8-in.....	67	72.0	86.2	4.9122	27.4	1.314	9.479	13.67	12.00
T-24......	10 by 12-in...	62	72.0	162.7	2.101	67.0	0.537	15.104	22.03	10.02
T-27......	12-in. I-beam.	55	18.0	18.9	3.318	21.8	0.413	32.059	44.30	34.50
T-28......	12-in. I-beam.	55	36.0	18.9	3.318	21.8	0.826	11.636	16.89	11.23
T-29......	12-in. I-beam.	55	72.0	18.9	3.318	21.8	1.651	5.1675	7.39	6.58

Test No.	End fixity (percentages)	Yield point of beam, by test in inch-pounds	Unit Stresses, in Pounds per Square Inch					
			Maximum direct stress at end, by Equation (40)	Direct stress near end by tensometers	Maximum Shearing Stress at the Yield Point			
					In the Flange		In the Web	
					By Equation (47)	By tensometers	By Equation (48)	By tensometers
(1)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
T -4......	83.5	26 000	58 700	....	30 500	....	....	....
T -5......	78.3	16 000	70 500	....	24 500	....	....	....
T -6......	83.8	10 000	60 300	....	20 360	....	....	....
T -7......	90.8	8 500	61 100	....	22 000	....	....	....
T -8......	91.6	6 500	54 600	....	22 800	....	....	....
T -9......	99.4	6 160	54 500	....	25 400	....	....	....
T-10......	96.8	5 500	50 100	....	26 400	....	....	....
T-11......	101.8	5 400	49 300	....	27 000	....	....	....
T-12......	84.2	8 000†	32 900†	....	11 600†	....	....	....
T-13......	93.4	6 000†	35 600†	....	12 000†	....	....	....
T-15......	85.8	33 980	72 600	66 100†	28 000	29 600	18 700	13 500
T-16......	40.1	35 000	44 100	61 500†	11 100	23 200	....	....
T-17......	96.8	79 000	54 000	40 000	27 800	29 000	19 700	16 400
T-18......	92.5	69 000	48 700	30 300	17 000	16 700	....	....
T-19......	101.0	151 600	40 700	49 000	13 000	25 800	....	....
T-20......	69.1	115 000	60 000	56 700	14 900	19 100	....	....
T-21......	95.7	65 000	61 600	56 300†	18 100	19 500	....	....
T-23......	87.8	194 000	53 300	37 400	24 000	24 800	15 000	14 500
T-24......	45.5	223 000	57 000	48 100	16 200	21 600	7 800	13 450
T-27......	77.9	310 000	71 000	29 400	17 700	30 000	8 600	5 920
T-28......	66.6	162 000	64 300	68 500	17 900	24 400	12 400	13 800
T-29......	89.0	105 000	57 100	65 800	22 700	20 200	18 100	23 740

* 36 in. = one-half total length on account of additional stiffener in middle.

† Not yield point.

DISCUSSION

H. M. WESTERGAARD,[9] M. AM. SOC. C. E., and R. D. MINDLIN,[10] JUN. AM. SOC. C. E. (by letter).—In Fig. 9 the authors show curves which represent different formulas for the concentration of torsional shearing stresses at fillets. The curve which is identified by the names of the writers was derived as a result of correspondence with the authors, and for this reason its derivation is given herein.

The process of approximate analysis is closely related to those used by A. and L. Föppl[11] and by S. Timoshenko,[12] but differs in the following respects:

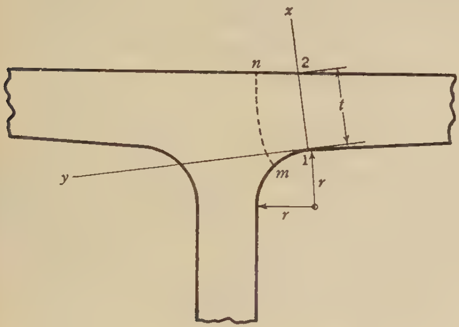

FIG. 27.—I-BEAM IN TORSION.

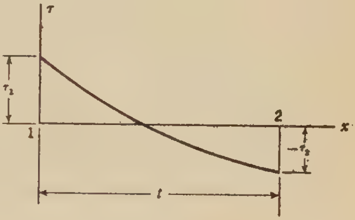

FIG. 28.—SHEARING STRESSES AT SECTION 1-2, IN FIG. 27.

First, consideration of the influence of the straight edge of the cross-section opposite the fillet led to a slightly greater concentration factor near the beginning of the fillet, at Point 1 in Fig. 27, than that obtained by Timoshenko's formula; and, second, the writers had before them the detailed data obtained by P. A. Cushman[13] in tests with soap films. These data show clearly that the maximum shearing stress occurs at a point such as m in Fig. 27, and that there is a notable increase of stress from Point 1 to Point m. Special consideration was given to this increase.

Let τ_0 denote the shearing stress that would exist at Point 1 in Fig. 27 if the edge of the cross-section at that point were straightened out by moving the beginning of the fillet toward the left, when, at the same time, the thickness of the flange, t, the torsion factor, K, and the total twisting moment, T, are left unchanged. The stress, τ_0 is defined by the formula:

$$\tau_0 = \frac{T\,t}{K} \qquad \text{.........................(57)}$$

Let τ_1 and τ_m denote the actual shearing stresses at Points 1 and m, τ_m being the maximum stress at the fillet. Then $\frac{\tau_1}{\tau_0}$ is the concentration factor at Point 1 and $\frac{\tau_m}{\tau_0}$ is the desired concentration factor for the fillet.

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[11] "Drang und Zwang," by A. and L. Föppl, Vol. 2, Second Edition, 1928, p. 73.

[12] "Theory of Elasticity," by S. Timoshenko, 1934, p. 259.

[13] "Shearing Stresses in Torsion and Bending by Membrane Analogy," by P. A. Cushman, Doctoral Dissertation, Univ. of Michigan, 1932.

Prandtl's soap film analogy, which the authors have used advantageously, furnishes the key to the solution. It is noted that the soap film is stretched over an opening shaped like the cross-section and is inflated a small amount by an excess air pressure on one side. The shearing stresses on the original cross-section follow the contour lines on the film, the edge being one of the contour lines. Furthermore, the shearing stresses are proportional to the slopes of the film, or proportional to the density of the contour lines.

The contour lines at Section 1–2 in Fig. 27 must be approximately perpendicular to that section. Accordingly, the shearing stresses, τ, at that section are approximately in the direction of y. Since the slope of the film is $c\tau$, c being a constant, the curvature of the film at Section 1–2 in the direction of x becomes $c\,\frac{d\tau}{dx}$. The curvature of the film in the direction of y at Section 1–2 is accounted for by the curving of the contour lines. If the radius of curvature of the contour lines at a particular point is R, then the curvature of the film in the direction of y at the same point becomes $c\,\frac{\tau}{R}$. The curvature of the surface is the sum of the curvatures in the directions of x and y; that is, $c\left(\frac{d\tau}{dx} + \frac{\tau}{R}\right)$.

The equilibrium of the film requires that this combined curvature be constant. Since $R = r$ at Point 1 and $R = \infty$ at Point 2, it follows that,

$$\left[\frac{d\tau}{dx}\right]_1 + \frac{\tau_1}{r} = \left[\frac{d\tau}{dx}\right]_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58)$$

Since Points 1 and 2 are on the same contour line, it is also required that,

$$\int_0^t \tau\, dx = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(59)$$

The diagram of the shearing stresses at Section 1-2 must be shaped about as shown in Fig. 28. Equation (60) has been constructed so that it satisfies this general requirement of shape as well as the specific requirements in Equations (58) and (59), and, therefore, it may be assumed to represent the shearing stress approximately:

$$\tau = \tau_1\left[1 - \left(2 + \frac{t}{3r}\right)\frac{x}{t} + \frac{x^2}{2\,tr}\right] \quad \ldots\ldots\ldots\ldots\ldots(60)$$

The constant curvature of the film may be computed as:

$$c\left[\frac{d\tau}{dx}\right]_2 = -\frac{2c\,\tau_1}{t}\left(1 - \frac{t}{3r}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(61)$$

If the fillet were some distance away, this curvature would remain the same, τ_1 would be replaced by τ_0, and the term containing r would disappear. Consequently,

$$\tau_0 = \tau_1 \left(1 - \frac{t}{3r}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (62)$$

which gives the concentration factor at Point 1,

$$\frac{\tau_1}{\tau_0} = \frac{1}{1 - \dfrac{t}{3r}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63)$$

The important indications of Equation (63) are preserved when the following simpler formula is substituted:

$$\frac{\tau_1}{\tau_0} = 1 + \frac{t}{3r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (64)$$

As was mentioned, the maximum stress, τ_m, occurs at a point such as m in Fig. 27. Equation (64) can be used to obtain an estimate of this stress by making the following replacements: t is replaced by a distance, t_m, measured along a curved section as drawn from m to n; and τ_0 is replaced by a stress corresponding to straight edges and a thickness, t_m: that is, according to Equation (57), by $\tau_0 \dfrac{t_m}{t}$. Thus, the concentration factor becomes,

$$\frac{\tau_m}{\tau_0} = \frac{t_m}{t} \left(1 + \frac{t_m}{3r}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

A reasonable estimate of t_m is,

$$t_m = t + 0.3\,r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66)$$

This value, substituted in Equation (65), leads directly to the formula represented graphically by the authors,

$$\frac{\tau_m}{\tau_0} = 1.2 + \frac{1}{3}\left(\frac{t}{r} + \frac{r}{t}\right) \ldots\ldots\ldots\ldots\ldots\ldots (67)$$

JOSEPH B. REYNOLDS,[14] ESQ. (by letter).—The problem of the twisting of I-beams has been studied thoroughly by the authors for special cases. It is the purpose of this discussion to compare the authors' results with a more general theory for the twisting of I-beams under differing types of loading. Only the effect of twisting is considered. The assumptions and the notation are the same as those used in the paper.

Consider the requirements for equilibrium at a section of the beam at a distance, x, from the fixed end as shown in Fig. 29. The inner moments at

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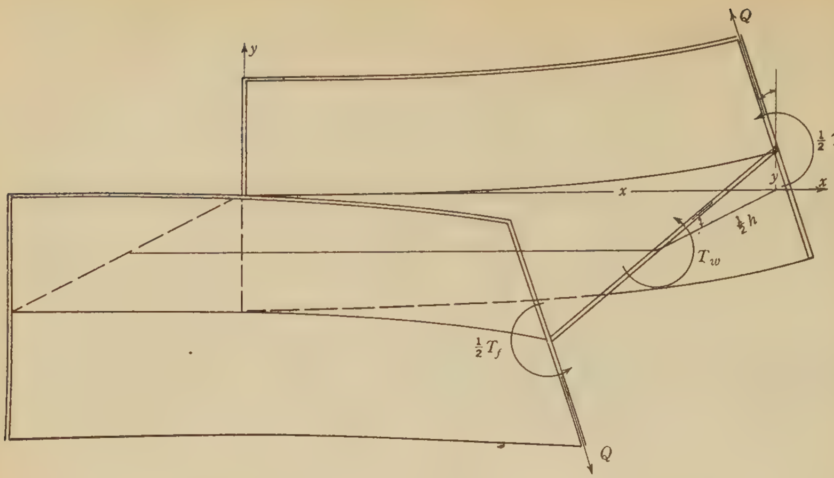

FIG. 29.

this section, together with the moment of the shears, must balance the outer moment applied to the beam to the right of the section. This outer twisting moment, T_x, will vary in general with the position of the section; that is, with x. For equilibrium at this section one must have:

$$T_f + T_w + Q\,h = T_x \ldots\ldots\ldots\ldots(68)$$

in which T_f, T_w, and T_x indicate twisting moments of flanges, web, and total of section, respectively.

In terms of the co-ordinates of points on the neutral axis, $y = f(x)$, the following approximate relation may be written:

$$\frac{1}{2}\,h\,d\theta = dy \ldots\ldots\ldots\ldots(69)$$

and the twist per unit length is $\frac{d\theta}{dx}$, hence:

$$T_f + T_w = KG\,\frac{d\theta}{dx} = \frac{2\,K\,G}{h}\,\frac{dy}{dx} \ldots\ldots\ldots\ldots(70)$$

From the relation between shear and bending moment:

$$\frac{1}{2}\,E_y\,I\,\frac{d^3 y}{dx^3} = -\,Q \ldots\ldots\ldots\ldots(71)$$

By means of Equations (70) and (71) and the value of a, Equation (68) can be written:

$$a^2\,\frac{d^3y}{dx^3} - \frac{dy}{dx} + X = 0 \ldots\ldots\ldots\ldots(72)$$

in which, $X = \dfrac{h\,T_x}{2\,K\,G}$.

The differential in Equation (72), has the general solution:

$$y = A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + C + y_1 \ldots\ldots\ldots\ldots\ldots (73)$$

in which y_1 is the particular solution satisfying Equation (72) and the values of A, B, and C are determined by the boundary conditions for the beam. Equation (73) is the same as the authors' Equation (34) for the case considered. The value of y_1 is:

$$y_1 = \int X\,dx - \frac{1}{2}\,a\,e^{\frac{x}{a}} \int e^{\frac{-x}{a}}\,X\,dx - \frac{1}{2}\,a\,e^{\frac{-x}{a}} \int e^{\frac{x}{a}}\,X\,dx \quad \ldots (74)$$

For the usual types of loading, X can be written in the form:

$$X = X_0 + X_1\,x + X_2\,x^2 + \ldots + X_n\,x^n \ldots\ldots\ldots\ldots (75)$$

in which, X_0, X_1, X_2, . . . X_n, are constants. When Equation (75) holds successive integration by parts shows that Equation (74) may be written in the form:

$$y_1 = \int X\,dx + a^3 \left(\frac{dX}{dx} + a^2\,\frac{d^3X}{dx^3} + a^4\,\frac{d^5X}{dx^5} + \ldots \right) \ldots (76)$$

If n is even in Equation (75) the last term in Equation (76) is $\frac{d^{n-1}X}{dx^{n-1}}$; if n is odd, the last term is $\frac{d^nX}{dx^n}$. With the value of y_1 thus determined one may, by Equation (73) and the relation, $M = \frac{1}{2}\,EI\,\frac{d^2y}{dx^2}$, write:

$$M = \frac{2\,K\,G}{h^2}\left[A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + a^2 \left(\frac{dX}{dx} + a^3 \frac{d^3X}{dx^3} + a^5 \frac{d^5X}{dx^5}, \ldots \right)\right] (77)$$

and, since the shear is given by $Q = \frac{1}{2}\,EI\,\frac{d^3y}{dx^3}$:

$$Q = \frac{2\,K\,G}{a\,h^2}\left(A \cosh \frac{x}{a} + B \sinh \frac{x}{a} + a^3\,\frac{d^2X}{dx^2} + a^6\,\frac{d^4X}{dx^4} + \ldots \right) .. (78)$$

The longitudinal stresses along the outer fibers of the flange will be given by $\sigma = \frac{M\,b}{I_y}$, or by:

$$\sigma = \frac{Eb}{2\,a^2}\left(A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + a^2 \frac{dX}{dx} + a^5 \frac{d^3X}{dx^3} + a^7 \frac{d^5X}{dx^5} + \ldots \right) .. (79)$$

and the lateral shearing stresses in the flanges by $\tau = \frac{Qb^2}{4\,I_y}$, or by:

$$\tau = \frac{Eb^2}{8\,a^3}\left(A \cosh \frac{x}{a} + B \sinh \frac{x}{a} + a^3\,\frac{d^2X}{dx^2} + a^6\,\frac{d^4X}{dx^4} + \ldots \right) \ldots (80)$$

The angle through which the beam twists comes from the relation, $\theta = \frac{2\,y}{h}$, and has the value:

$$\theta = \frac{2\,A}{h} \sinh \frac{x}{a} + \frac{2\,B}{h} \cosh \frac{x}{a} + \frac{2\,C}{h} + \frac{2}{h} \int X\,dx + \frac{2\,a^3}{h} \times \frac{dX}{dx} + \frac{2\,a^5}{h} \times \frac{d^3 X}{dx^3} + \frac{2\,a^7}{h} \times \frac{d^5 X}{dx^5} + \ldots\ldots\ldots\ldots \quad (81)$$

In this manner, general values are derived for the principal variables of interest in the case of twisted beams. The maximum values of these quantities will occur for differing values of x, depending upon the conditions surrounding the strained beam. Example (a) demonstrates that Equations (73) to (81) reduce to those given by the authors when the proper limitations are applied.

Example (a).—Beam Twisted by Constant Torque, T, Applied at Its Ends, with Both Ends Restrained.—In this case, $T_x = T$, a constant, and, therefore, $X = \frac{h\,T}{2\,K\,G}$. The constants A, B, and C are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$. Furthermore, for $x = \frac{l}{2}$, $\frac{d^2 y}{dx^2} = 0$. Since the derivations of X are all zero, Equation (76) yields $y_1 = \frac{h\,T\,x}{2\,K\,G}$; and, by Equation (73):

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh \frac{x}{a} \tanh \frac{l}{2\,a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{l}{2\,a}\right) \ldots (82)$$

which is the same as that presented by the authors in Equation (35). The maximum displacement, y_m, occurs where $x = l$ and has the value given in the authors' Equation (36).

Similarly, Equation (77) reduces to the authors' Equation (37) for the moment in each flange, which has (where $u = \frac{1}{2}\,\frac{l}{a}$, or $x = 0$), a maximum value:

$$M_m = \frac{T\,a}{h} \tanh \frac{l}{2\,a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (83)$$

Equation (78) reduces to Equation (38) and Equation (79) becomes Equation (39).

By use of Equation (80):

$$\tau = \frac{T\,b^2}{4\,I_y}\,\frac{\cosh u}{\cosh \frac{l}{2\,a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (84a)$$

and,

$$\tau_{\max} = \frac{T\,b^2}{4\,I_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (84b)$$

which corresponds to the authors' Equation (42), and by Equation (81):

$$\theta = \frac{T\,a}{G\,K}\left(\cosh\frac{x}{a}\tanh\frac{l}{2\,a} - \sinh\frac{x}{a} + \frac{x}{a} - \tanh\frac{l}{2\,a}\right) \ldots (85)$$

from which,

$$\theta_{\max} = \frac{T\,a}{G\,K}\left(\frac{l}{a} - 2\tanh\frac{l}{2\,a}\right) \ldots\ldots\ldots\ldots\ldots (86)$$

Example (b).—Beam with One End Fixed, the Other Free, Under Constant Torque, T.—In this case, $X = \dfrac{h\,T}{2\,K\,G}$ as in Example (*a*). The conditions determining A, B, and C are that for $x = 0$, $y = 0$, and $\dfrac{dy}{dx} = 0$, and for $x = l$, $\dfrac{d^2 y}{dx^2} = 0$. The value of the deflection obtained is:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\cosh\frac{x}{a}\tanh\frac{l}{a} - \sinh\frac{x}{a} + \frac{x}{a} - \tanh\frac{l}{a}\right) \ldots (87)$$

From this all the other principal variables are readily found as in Example (*a*).

Example (c).—Beam with Both Ends Fixed and a Uniform Eccentric Load Along Its Length Producing an External Moment, T.—In this case, $T_x = T\left(1 - \dfrac{2\,x}{l}\right)$, and, $X = \dfrac{h\,T}{2\,K\,G}\left(1 - \dfrac{2\,x}{l}\right)$, giving,

$$y_1 = \frac{h\,T}{2\,K\,G}\left(x - \frac{x^2}{l}\right) - \frac{a^3\,h\,T}{K\,G\,l} \ldots\ldots\ldots\ldots\ldots (88)$$

The values of A, B, and C are determined by the requirements that for $x = 0$, $y = 0$, and $\dfrac{dy}{dx} = 0$, and for $x = \dfrac{l}{2}$, $\dfrac{dy}{dx} = 0$. The deflection found is:

$$y = \frac{T\,h\,a}{2\,K\,G}\left(\frac{\cosh u}{\sinh\dfrac{l}{2\,a}} - \coth\frac{l}{2\,a} + \frac{x}{a} - \frac{x^2}{a\,l}\right) \ldots\ldots (89)$$

Example (d).—Beam with One End Fixed and the Other Free, Uniform Eccentric Load Along Its Length.—Here, $Tx = T\left(1 - \dfrac{x}{l}\right)$ and,

$$y_1 = \frac{h\,T}{2\,K\,G}\left(x - \frac{x^2}{2\,l}\right) - \frac{T\,h\,a^3}{2\,K\,G\,l}$$

The constants, A, B, and C, are determined by the requirements that for $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, and for $x = l$, $\frac{d^2 y}{dx^2} = 0$. The deflection proves to be:

$$y = \frac{T\,h\,a}{2\,K\,G}\left[\frac{\sinh\left(\frac{l}{a} - \frac{x}{a}\right)}{\cosh\frac{l}{a}} - \tanh\frac{l}{a} + \frac{a}{l}\operatorname{sech}\frac{l}{a}\left(\cosh\frac{x}{a} - 1\right) + \frac{x}{a} - \frac{x^2}{2\,a\,l}\right]. \quad \text{.........................(90)}$$

It is thus shown that the formulas developed and substantiated by test results in the authors' investigation, are identical with those obtained by a general mathematical treatment of the subject of torsional resistance.

HAROLD E. WESSMAN,[15] ASSOC. M. AM. SOC. C. E. (by letter).—Structural engineers seem to be displaying an increasing interest in the effects of torsional resistance on stresses in structural members. This curiosity does not restrict itself solely to the member in torsion, but also embraces study of the resulting effects on bending moments of connecting members. The paper is a timely contribution to the subject. Although the problem is more apparent in the monolithic reinforced concrete building frame, it occurs also in steel structures where some connections insure full continuity and practically all types of connections develop at least a modicum of restraint.

In most structural units torsion may prove to be quite unimportant. One is not justified in dismissing it, however, simply because he considers it unimportant. Every one is prone to do this at times with knotty problems, because it is the easiest course. It is better to be an "agnostic" rather than an "atheist" in such matters, however, until a basis of scientific research, both analytical and physical, justifies conclusions. In the matter of torsion, that basis will be provided by papers such as this one and additional researches involving other structural cross-sections and their relation to the structure as a whole.

The writer has suggested that torsion may fall into the category of knotty problems. It is complex, both in its broad aspects and in its details. A thorough consideration of torsion in the field of structures takes the problem of analysis away from planar confines and puts it in the space realm. This will discourage many from further consideration of the subject because, as a rule, structural engineers prefer to be "two-dimensional analysts," despite the fact that they live in a space world.

The details of the problem, likewise, are not simple of solution. One needs only to look at the disagreement apparent in Fig. 9 to confirm this statement. Various difficulties are involved in the study of torsion stresses in a single twisted unit, quite apart from those which come to light when one considers related effects in connecting members.

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As the authors note, only a few simple cross-sections, such as the circle, ellipse, rectangle, square, equilateral triangle, and certain hollow sections, are susceptible to pure mathematical analysis. The membrane analogy, or soap bubble, is called upon to serve as an aid in the solution of other sections, such as the familiar rolled steel shapes used in structural practice. It is difficult, however, to get accurate measurements of the slopes of the soap bubble at the boundaries because of the curving or "meniscus" effect of the film. Since the slope of the film at the boundary is a measure of the unit shear stress at corresponding points, the modification in curvature makes accurate interpretation somewhat difficult. The materials testing laboratory is another aid for the determination of the torsion constant by twisting actual specimens; but it is difficult to "look inside" the beam to find out what happens, and at points on the exterior at fillets, it is hard to measure strains.

This paper emphasizes the co-operation possible between mathematical analysis, mathematical aids such as the membrane analogy, and the physical testing laboratory. The preceding comments by the writer indicate briefly that this co-operation is necessary in securing an answer to torsion problems. The authors are to be congratulated for using all the available tools at their command.

It is unfortunate, however, that they did not present their soap film tests in more detail. It would be interesting to know precisely how they determined the volume under the film and the coefficients, α, in Fig. 7. Was any attempt made to obtain an independent curve for stress concentration at fillets similar to those in Fig. 9 (which exhibit a considerable lack of agreement both in values and trends)? At the ratio, $\frac{r}{n} = 0.40$, the percentage increase of stress in fillets ranges from about 45 to 140. At the ratio, $\frac{r}{n} = 1.00$, the range is from 25 to 90 per cent. For ratios of $\frac{r}{n} > 1$, Westergaard's and Mindlin's curve and Taylor's soap film tests,[16] portrayed in Fig. 9, show increasing values of stress concentration. All other curves continue the downward trend. In other words, they conform to the thesis that an increase in the radius of the fillet always reduces the stress concentration. Cushman's soap film tests[13] were not carried far enough to give values in the range, $1 < \frac{r}{n} < 2$, although the general trend is downward.

Incidentally, the following question may be raised at this point: "Are the ordinates in Fig. 9, 'Percentage of Increase of Stress in Fillets,' or do they actually give 'Percentage of Increase of Stress Concentration Factor'"? This question is pertinent, because, as the fillet size increases, the torsion constant also increases; hence, for a given twisting moment, the stress at the fillet might actually decrease, even if the ratio of stresses, $\frac{\tau_m}{\tau_o}$, increases. It is to be noted that Equation (67) is expressed in terms of this ratio. Föppl, who derives an equation from a consideration of Stokes' theorem, expresses results in terms of a similar ratio.[11]

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[16] Technical Repts., Advisory Committee for Aeronautics, No. 33, June, 1917 (Rept. covers experimental work with soap films by Messrs. A. A. Griffith and G. I. Taylor).

If the authors plotted contours to determine the volume under each film for different fillet radii, the following procedure would give an approximate check on the curves in Fig. 9 with little additional labor. Even if contours were not plotted, the determination of a slope at one point on the fillet near the edge, but far enough away to eliminate edge influence, should not take much additional time, considering the extent of the laboratory testing that was done.

In Fig. 30(*a*), Section *OA* bisects the fillet, and Fig. 30(*b*) shows a profile of the section. The differential equation of curvature in terms of the stress function for torsion is given by the authors in Equation (1). This equation, of course, follows directly from the differential equation of equilibrium of forces expressed in terms of displacement rather than stress. Moreover, it is the same as the one for curvature of the soap bubble surface, except that the constant, $2\,G\,\theta$, is replaced by the constant, $\frac{p}{S}$, in which p is the unit pressure normal to the surface of the film, and S is the constant tension in the film.

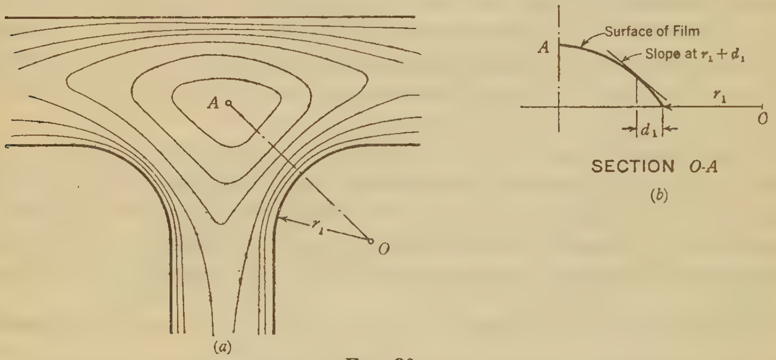

FIG. 30.

In polar co-ordinates, Equation (1) becomes,

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\,\frac{\partial F}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 F}{\partial \theta^2} = C \;\ldots\ldots\ldots\ldots\ldots\ldots (91)$$

in which C is the constant, $-2\,G\,\theta$, or $-\frac{p}{S}$.

For a surface of revolution, $\frac{\partial F}{\partial \theta} = 0$ and Equation (91) takes the simpler form,

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\,\frac{\partial F}{\partial r} = C \qquad (92)$$

The solution of this differential equation is,

$$F = \frac{C r^2}{4} + A \log r + B \qquad (93)$$

The equation for the slope, which measures the unit shear is,

$$\frac{\partial F}{\partial r} = \frac{C r}{2} + \frac{A}{r} \qquad (94)$$

At the fillet of a rolled section, part of the soap film is approximately a surface of revolution with its axis passing through the center of the fillet; contours over this part will be parallel to the edge of the fillet, as in Fig. 30(*a*). Radius r in Equation (94) is, therefore, the radius relative to the center of the fillet. The value of the constant, C, has already been noted. It only remains for Constant A to be evaluated. In one approximate solution proposed, A is found by assuming that the slope is zero at a distance, $\frac{n}{2}$, in from the edge of the fillet,[12] and that the surface is one of revolution for at least this same distance in from the edge. The assumption of zero slope at the distance noted is open to question.

By an inspection of contours, or by one or two measurements, either of slope or of elevation, F, of the surface, at a point a small definite distance in from the edge, the authors could have obtained a finite value for the slope, $\frac{\partial F}{\partial r}$, at a finite distance from the center of the fillet. By substituting this in Equation (94), the constant, A, can then be evaluated more closely.

The writer realizes that this involves fairly precise measurement. One might also raise the question, "Why not measure the slope right at the edge and get the shear stress at the critical point directly"? If one can eliminate edge influence on curvature, this would be the logical solution. The writer suggests the preceding steps, however, to get away from edge influence and to obtain a more accurate determination of the constant, A. He ventures to predict that the resulting curve for the stress concentration factor will exhibit a rising trend for values of $\frac{r}{n} > 1$.

F. B. SEELY,[17] ESQ., AND W. J. PUTNAM,[18] ESQ. (by letter).—It may be of value to study the results of the experiments discussed in this paper, on the torsional properties of H-beams and I-beams, in connection with the results of somewhat similar, but less exhaustive, tests on several rolled steel channels conducted by the writers in 1930.

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For a bar or shaft having a circular cross-section, the relation between the twisting moment, T, and the angle of twist per unit of length, θ, as given by Equation (4), is $T = J\ G\ \theta$. Similarly, the authors point out that for members having non-circular cross-sections, Equation (4) becomes $T = K\ G\ \theta$ (see Equation (5)), in which K is the torsion constant for the given section. For a narrow rectangle Saint Venant gave the approximate value of $K = \frac{1}{3}\ n^3\ b$ (see Equation (6)) in which n is the short dimension and b the long dimension of the rectangle. Likewise, for a section made up of narrow rectangles, such as a ship channel, the approximate value of K for the entire section is $K = \Sigma(\frac{1}{3}\ n^3\ b)$; but for a (trapezodial) section with sloping sides, such as the flanges of most rolled steel channels, the value of K is expressed by Equations (9) and (10) of the paper.

For a channel section, then, the expression for K, would be,

$$K = \frac{1}{3}\, w^3\, d + 2\ \frac{b}{12}\ (m + n)\ (m^2 + n^2) \qquad \text{.............}(95)$$

in which the symbols are identified by reference to Fig. 31. As noted by the authors, the value of K is proportional to the volume under a soap film that is stretched across an opening of the same shape as that of the cross-section, the soap film being inflated slightly by a difference of pressure on the two sides of the film. In Equation (95) the factor corresponding to the "hump" or "hill" that occurs in the soap film at the junctions of the flanges and web is omitted. The "end loss" corresponding to the "sloping down" of the soap film to meet the edges of the short dimensions of the rectangular or trapezodial sections, is also omitted. In their studies, the writers concluded that corrections for these two effects could be omitted in view of the deviation of the actual dimensions of rolled sections from the tabular handbook values. Furthermore, these effects in a channel section are relatively less than in an I-section or in an H-section.

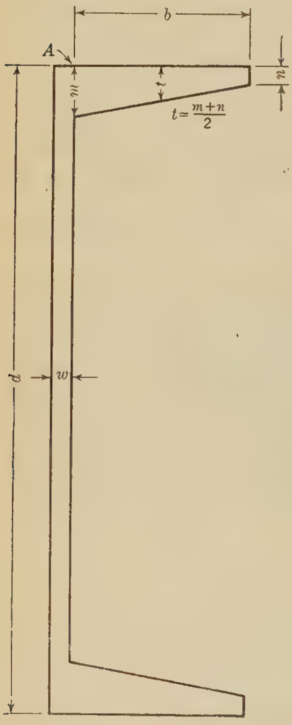

FIG. 31.

Comparison of Test and Calculated Values of K.—Four channels ranging in depth from 6 to 15 in., as listed in Table 5, were tested as free-ended members in pure torsion to determine the torsional constant, K, from Equation (4), namely,

$$K = \frac{1}{G} \times \frac{T}{\theta} \qquad \text{............}(96)$$

The value of G was taken to be 12 000 000 lb per sq in.; T was measured in a 23 000 in-lb torsion machine, and the angle of twist, $\phi = \theta L$, in a given length, L, was measured by means of a 20-in. level bar on flat bars attached to the top flange of the channel, as shown in Fig. 32.

TABLE 5.—TEST RESULTS AND COMPUTED VALUES FOR PURE TORSION IN CHANNEL SECTIONS

Channel	Torsion Constant, K		Shearing Stresses in Web and Flange			
	Test value (Equation (96))	Computed value (Equation (95))	Test value of $E\epsilon$	$\tau = \sigma = 0.8 \times$ Column (4) (see Equation (97))	Computed value of τ (Equation (98))	Approximate maximum value of in flange (Equation (99))
(1)	(2)	(3)	(4)	(5)	(6)	(7)
6-In., 15.3-Lb:*						
Web	0.206	0.201	$2.00T$	$1.60T$	$1.69T$	......
Flange			$2.43T$	$1.94T$	$1.93T$	$2.18T$
6-In., 15.5-Lb:†						
Web	0.400	0.405	$1.72T$	$1.38T$	$1.38T$	......
Flange			$1.48T$	$1.18T$	$0.847T$	$1.20T$
10-In., 15.3-Lb:						
Web	0.256	0.203	$1.54T$	$1.23T$	$1.18T$	......
Flange			$2.25T$	$1.80T$	$2.15T$	$3.11T$
15-In., 33.9-Lb:						
Web	1.08	0.951	$0.593T$	$0.474T$	$0.420T$	......
Flange			$0.838T$	$0.670T$	$0.685T$	$0.947T$

* Ship channel. † Heavy channel.

The relations between T and ϕ for several lengths along a 10-in., 15.3-lb channel are shown in Fig. 33(a). The values of K found from the curves

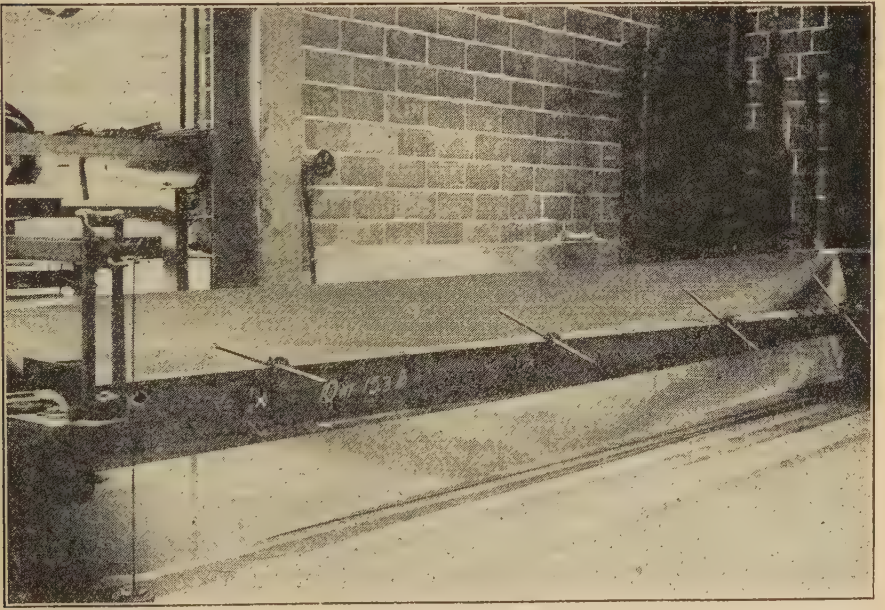

FIG. 32.—VIEW OF CHANNEL IN TESTING MACHINE.

similar to those shown in Fig. 33(a), together with the values of K computed from Equation (95), are given in Table 5, Column (3) for each of the four channels tested. It will be observed that the values computed from Equation (95) are not in close agreement with the values obtained from test results (Column (2), Table 5), but they indicate that Equation (95) may be used satisfactorily for approximate results. The computed values in Column (3) that deviate most from the test values apply to the 10-in. and the 15-in. channels, which have the sections with relatively thin webs and relatively thick and tapering flanges. In computing the values of K, the tabular or handbook values of the dimensions of the channel sections were used. All the channels tested checked closely the tabular values of the average weights per foot of length, but some of the linear dimensions deviated considerably from the tabular dimensions.

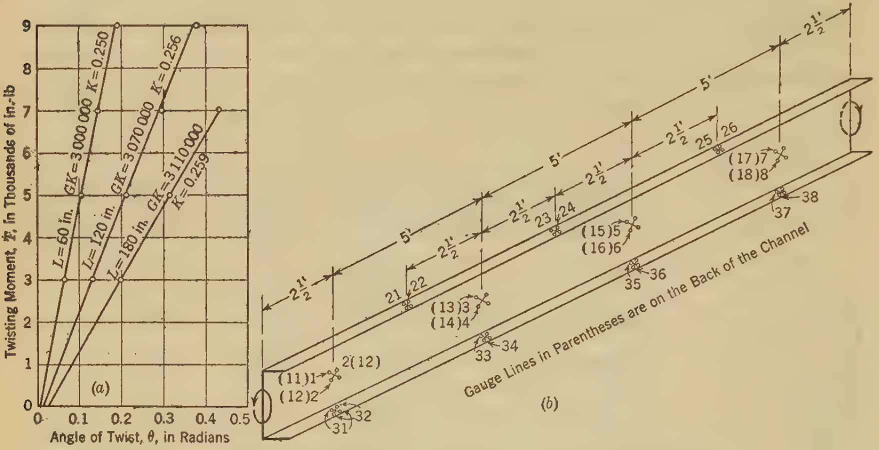
Twisting Moment, T, in Thousands of in.-lb
0
1
2
3
4
5
6
7
8
9
0
0.1
0.2
0.3
0.4
0.5
Angle of Twist, θ, in Radians
L=60 in.
GK=3 000 000
K=0.250
L=120 in.
GK=3 070 000
K=0.256
L=180 in.
GK=3 110 000
K=0.259
(a)
2½'
5'
2½'
5'
2½'
2½'
5'
2½'
2½'
2½'
21
22
23
24
25
26
(11)1
(12)2
2(12)
(13)3
(14)4
(15)5
(16)6
(17)7
(18)8
31
32
33
34
35
36
37
38
Gauge Lines in Parentheses are on the Back of the Channel
(b)

FIG. 33.—LOCATION OF 2-INCH GAUGE LINES ON 10-INCH, 15.3-POUND CHANNEL SUBJECTED TO PURE TORSION.

Comparison of Test and Calculated Values of Shearing Stresses.—The approximate values of the elastic shearing stresses at the center of each flange and at the center of the web on each channel at several sections were obtained by measuring the elastic strains in the channels caused by twisting moments at the ends. The locations of the gauge lines are shown in Fig. 32 and Fig. 33(b). The stresses were determined by measuring the unit strains, ϵ, on 45° gauge lines by means of a 2-in. strain-gauge. From these strain readings the tensile and compressive unit stresses in the directions of the gauge lines can be found from the expression, $\sigma = \dfrac{E\,\epsilon}{1+\mu} = 0.8\,E\,\epsilon$, in which E is the tensile or compressive modulus of elasticity and μ is Poisson's ratio (which is assumed to be 0.25 for steel). Since the tensile and compressive stresses at any point in the directions of the 45° gauge lines are equal to the shearing stresses on the vertical and horizontal planes at the same point,

$$\sigma = \tau = 0.8\,E\,\epsilon \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (97)$$

The relations between the twisting or torsional moment, T, and the values of ϵ (or of $E\ \epsilon$, in which E is assumed to be 30 000 000 lb per sq in.) for several gauge lines are shown in Fig. 34. Similar curves were found for all the gauge lines on all the channels tested. The average of the values of $E\ \epsilon$ and the corresponding values of τ and σ in the web and flanges of each of the four channels tested, are shown in Table 5, together with the calculated values of the shearing stresses from the equation,

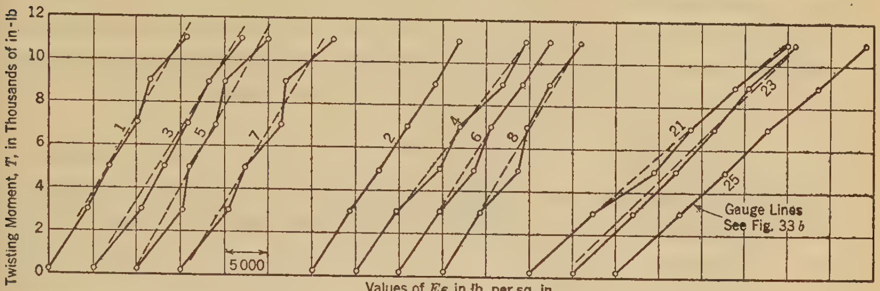

FIG. 34.

$$\tau = \frac{T\ t}{K} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (98)$$

in which t is the thickness of the section at the location of the gauge line (see Fig. 31). It will be observed that Equation (98) is consistent with the soap film analogy. The larger t becomes, the more will the soap film puff up; the greater will be the slope of the film; and, therefore, the greater the shearing stress. Equation (98) yields only approximate values, but leads to useful and reliable results as may be seen from Table 5, in which it is shown that the calculated and test results are in reasonably satisfactory agreement (compare Column (5) and Column (6)). It should be noted that since the strain-gauge used had a 2-in. gauge length, the stresses obtained from the measured strains are the average in the region of the point considered rather than the stress at the given point.

The computed values in Column (6), Table 5, that deviate most from the test values apply to the 6-in. 15.5-lb, and the 10-in. 15.3-lb channels. For these sections the 2-in. gauge line included too much of the relatively short flange width for satisfactory stress determinations at the point considered.

The maximum value of τ in the flange (not considering the stress at the fillet) would be (on the basis of the soap film analogy) at the point where the largest inscribed circle is tangent to the face of the flange, since the soap film would have the greatest slope at this point; the maximum shearing stress would occur, therefore, at about the point, A, Fig. 31. The approximate value of the maximum shearing unit stress on the outside face of the flange, should, then, be given by the equation,

$$\tau = \frac{T\ m}{K} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (99)$$

The values of τ found from Equation (99) are given also in Table 5, Column (7).

The results of these tests of rolled steel channels in pure torsion are in substantial agreement with the results of the similar but more exhaustive tests of I-beams and H-beams presented in the paper under discussion, in showing that reasonably reliable values of the angles of twist of the members and of the shearing stresses in the web and in the flanges may be calculated by means of a relatively simple formula based on the soap film analogy.

P. Wilhelm Werner,[19] Assoc. M. Am. Soc. C. E. (by letter).—Torsion in structural beams has been studied, theoretically and experimentally, by several investigators, the most important of whom have been cited by the authors. The names of C. Weber,[20] C. Schmieden,[21] and H. Engelmann.[22] should be added for completeness. Weber[20] has treated the double-flanged type of section in general, and obtains the H-beam and I-beam sections as a special case of the general solution. In this connection he also explains the apparent relative weakness of channels in bending when compared with beams of symmetrical cross-section with the same moment of resistance.

Perhaps the most important result of the reported investigation is the method of evaluating the torsion constant, which is defined theoretically and substantiated by the tests. In view of the extensive tests presented in the paper, it would be of interest to know whether some data are also available regarding the influence of rivet holes upon the torsional rigidity and stresses in structural beams. As far as the writer is aware, this question has hitherto been neglected entirely in problems dealing with torsion. From the hydrodynamical analogy in the torsion theory,[23] however, it is known that a small hole, or cavity, near the edge of a twisted circular shaft will cause a considerable increase of the shearing stresses around the hole. It may be expected that a similar stress concentration will occur also around rivet holes in structural beams, although the holes may not materially influence the yielding of the beam as a whole.

With reference to the theoretical analysis, the paper deals exclusively with the case of a beam twisted by a constant torque, T, applied at its ends. It is of interest to note, that the results thus obtained may also be derived as a special case of a more general problem—a beam twisted by a concentrated torque, T_p, applied at an arbitrary point on the span. This general case is of considerable importance for practical design purposes.

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[19] Civ. Engr., A. B. Vattenbyggnadsbyran, Stockholm, Sweden.

[20] "Uebertragung des Drehmomentes in Balken mit doppelflanschigem Querschnitt," by C. Weber, *Zeitschrift für angew. Math. Mech.*, Vol. 6, 1926, p. 85.

[21] "Ueber die Torsion von Walzeisen-Profilen," by C. Schmieden, *Zeitschrift für angew. Math. Mech.*, 1930, p. 251.

[22] Experimentelle und theoretische Untersuchungen zur Drehfestigkeit der Stäbe," by H. Engelmann, *Zeitschrift für angew. Math. Mech.*, 1929, p. 386.

[23] "Theory of Elasticity," by S. Timoshenko, 1934, p. 264.

Referring to Fig. 35, the general expression for the twisting moment is,

$$T_x = \frac{2\,K\,G}{h}\left(\frac{dy}{dx} - a^2\,\frac{d^3y}{dx^3}\right) \ldots\ldots\ldots\ldots\ldots\ldots (100)$$

which is also the differential equation for the deflection of the flanges. It should be observed, however, that Equation (100), like the author's Equation (33), is strictly applicable only when the deflection is due to bending alone, which implies small ratios of $\beta = \frac{b}{L}$, or, that the flange width may be considered as small in relation to the span. From the general theory of beams in flexure it may be concluded that the ratio should be $\beta \leqq 0.25$ in order to keep the error in the calculated stresses within about 5 per cent.

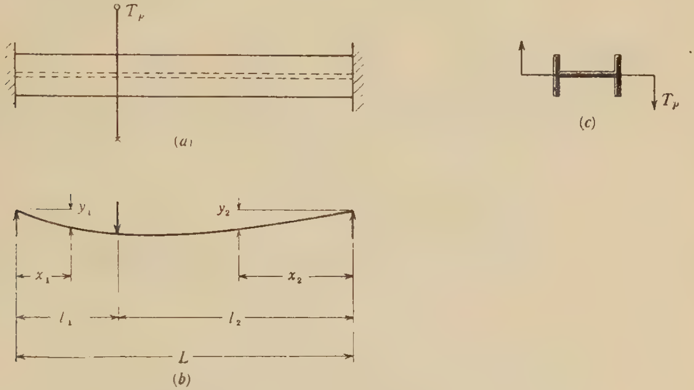

FIG. 35.—DIAGRAMMATICAL REPRESENTATION OF TWISTED I-BEAM.

If T_1 and T_2 denote the constant torque on the left and right-hand side of T_p, respectively:

$$T_1 = \frac{2\,K\,G}{h}\left(\frac{dy_1}{dx_1} - a^2\,\frac{d^3y_1}{dx^3_1}\right) \ldots\ldots\ldots\ldots\ldots\ldots (101a)$$

and,

$$T_2 = \frac{2\,K\,G}{h}\left(\frac{dy_2}{dx_2} - a^2\,\frac{d^3y_2}{dx^3_2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (101b)$$

The solution of Equations (101) is,

$$y_1 = A_1 \sinh \frac{x_1}{a} + B_1 \cosh \frac{x_1}{a} + D_1\,x_1 + C_1 \ldots\ldots\ldots (102a)$$

and,

$$y_2 = A_2 \sinh \frac{x_2}{a} + B_2 \cosh \frac{x_2}{a} + D_2\,x_2 + C_2 \ldots\ldots\ldots (102b)$$

By differentiating Equations (102) and substituting in Equations (101):

$$T_1 = \frac{2\,K\,G}{h}\,D_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (103a)$$

and,

$$T_2 = \frac{2\,KG}{h} D_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (103b)$$

From the equilibrium condition, $T_p = T_1 + T_2$:

$$T_p = \frac{2\,KG}{h} (D_1 + D_2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (104)$$

The integration constants may be derived for two typical examples, as shown subsequently in Cases (*a*) and (*b*).

Case (*a*).—*Both Ends Free.*—For this case the boundary conditions are, for $x_1 = x_2 = 0$: $y = 0$ and $\frac{d^2y}{dx^2} = 0$; and, for $x_1 = l_1$ and $x_2 = l_2$: $y_1 = y_2$; $\frac{dy_1}{dx_1} = -\frac{dy_2}{dx_2}$; and, $\frac{d^2y_1}{dx_1^2} = \frac{d^2y_2}{dx_2^2}$.

According to these requirements and Equation (104), the coefficients, *A*, *B*, *C*, and *D*, are determined, as follows:

$$A_1 = -\frac{a\,h\,T_p}{2\,KG} \frac{\sinh\frac{l_2}{a}}{\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}} \ldots\ldots (105a)$$

$$A_2 = -\frac{a\,h\,T_p}{2\,KG} \frac{\sinh\frac{l_1}{a}}{\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}} \ldots\ldots (105b)$$

$$B_1 = B_2 = C_1 = C_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (105c)$$

$$D_1 = +\frac{h\,T_p}{2\,KG}\frac{l_2}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (105d)$$

and,

$$D_2 = +\frac{h\,T_p}{2\,KG}\frac{l_1}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (105e)$$

By making $l_1 = l_2 = l = \frac{L}{2}$ and $T_p = 2\,T$:

$$A_1 = A_2 = -\frac{a\,h\,T}{2\,KG}\frac{l}{\cosh\frac{l}{a}} \ldots\ldots\ldots\ldots\ldots (106a)$$

and,

$$D_1 = D_2 = +\frac{h\,T}{2\,KG} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (106b)$$

Then the deflection function for this case may be written,

$$y = \frac{a\,h\,T}{2\,K\,G}\left[\frac{x}{a} - \frac{\sinh\dfrac{x}{a}}{\cosh\dfrac{l}{a}}\right] \qquad \text{.................(107)}$$

Equation (107) also represents the special case of a beam with one end fixed, the other free, and twisted by a constant torque applied at its ends—which case has been treated by Professor J. B. Reynolds,[24] although his solution is given in a somewhat different form. The very simple expression obtained in Equation (107) is due to the suitable choice of the origin of co-ordinates.

Case (b).—Both Ends Fixed.—For this case the boundary conditions are, for $x_1 = x_2 = 0$: $y = 0$ and $\dfrac{dy}{dx} = 0$; and, for $x_1 = l_1$, and $x_2 = l_2$: $y_1 = y_2$; $\dfrac{dy_1}{dx_1} = -\dfrac{dy_2}{dx_2}$; and $\dfrac{d^2y_1}{dx^2_1} = \dfrac{d^2y_2}{dx^2_2}$.

These requirements, together with Equation (104) give the following values of the coefficients, A, B, C, and D,

$$A_1 = -D_1\,a = -\frac{a\,h\,T_p}{2\,K\,G}\left[\sinh\frac{l_2}{a}\left(\sinh\frac{l_1}{a} + \sinh\frac{l_2}{a}\right)\right.$$
$$\left. + \left(\cosh\frac{l_1}{a} - \cosh\frac{l_2}{a}\right)\left(\cosh\frac{l_2}{a} - 1\right) - \frac{l_2}{a}\left(\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}\right)\right]$$
$$\div \left(\sinh\frac{l_1}{a} + \sinh\frac{l_2}{a}\right)^2 - \left(\cosh\frac{l_1}{a} - \cosh\frac{l_2}{a}\right)^2$$
$$-\frac{L}{a}\left(\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}\right) \qquad \text{..........(108}a\text{)}$$

$$A_2 = -D_2\,a = -\frac{a\,h\,T_p}{2\,K\,G}\left[\sinh\frac{l_1}{a}\left(\sinh\frac{l_1}{a} + \sinh\frac{l_2}{a}\right)\right.$$
$$\left. - \left(\cosh\frac{l_1}{a} - \cosh\frac{l_2}{a}\right)\left(\cosh\frac{l_1}{a} - 1\right) - \frac{l_1}{a}\left(\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}\right)\right]$$
$$\div \left(\sinh\frac{l_1}{a} + \sinh\frac{l_2}{a}\right)^2 - \left(\cosh\frac{l_1}{a} - \cosh\frac{l_2}{a}\right)^2$$
$$-\frac{L}{a}\left(\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}\right) \qquad \text{..........(108}b\text{)}$$

$$B_1 = -C_1 = -\frac{h\,T_p}{2\,K\,G}\left[\sinh\frac{l_2}{a}\left(l_2\sinh\frac{l_1}{a} - l_1\sinh\frac{l_2}{a}\right)\right.$$
$$-\left(\cosh\frac{l_1}{a} - 1\right)\left(a\sinh\frac{l_2}{a} - l_2\cosh\frac{l_2}{a}\right)$$
$$\left.-\left(\cosh\frac{l_2}{a} - 1\right)\left(a\sinh\frac{l_1}{a} - l_1\cosh\frac{l_2}{a}\right)\right] \div \left(\sinh\frac{l_1}{a} + \sinh\frac{l_2}{a}\right)^2$$
$$-\left(\cosh\frac{l_1}{a} - \cosh\frac{l_2}{a}\right)^2 - \frac{L}{a}\left(\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}\right) \text{.(108}c\text{)}$$

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[24] See p. 903.

and,

$$B_2 = -C_2 = -\frac{h\,T_p}{2\,K\,G}\left[\sinh\frac{l_1}{a}\left(l_1\sinh\frac{l_2}{a} - l_2\sinh\frac{l_1}{a}\right)\right.$$
$$-\left(\cosh\frac{l_1}{a} - 1\right)\left(a\sinh\frac{l_2}{a} - l_2\cosh\frac{l_1}{a}\right)$$
$$\left.-\left(\cosh\frac{l_2}{a} - 1\right)\left(a\sinh\frac{l_1}{a} - l_1\cosh\frac{l_2}{a}\right)\right]$$
$$\div\left(\sinh\frac{l_1}{a} + \sinh\frac{l_2}{a}\right)^2 - \left(\cosh\frac{l_1}{a} - \cosh\frac{l_2}{a}\right)^2$$
$$-\frac{L}{a}\left(\sinh\frac{l_1}{a}\cosh\frac{l_2}{a} + \sinh\frac{l_2}{a}\cosh\frac{l_1}{a}\right) \quad \ldots\ldots\ldots\ldots (108d)$$

By making $l_1 = l_2 = l = \dfrac{L}{2}$ and $T_p = 2\,T$:

$$A_1 = A_2 = -D_1\,a = -D_2\,a = -\frac{a\,h\,T}{2\,K\,G} \quad \ldots\ldots\ldots\ldots (109a)$$

and,

$$B_1 = B_2 = -C_1 = -C_2 = +\frac{a\,h\,T}{2\,K\,G}\tanh\frac{l}{2\,a} \quad \ldots\ldots\ldots (109b)$$

and the deflection function for this case may be written,

$$y = \frac{a\,h\,T}{2\,K\,G}\left(-\sinh\frac{x}{a} + \tanh\frac{l}{2\,a}\cosh\frac{x}{a} + \frac{x}{a} - \tanh\frac{l}{2\,a}\right) \quad \ldots (110)$$

Equation (110) also represents the special case treated by the authors, namely, a beam twisted by a constant torque, T, applied at its ends, with both ends restrained.

In the "General Remarks" under "Design Examples" the authors state that "the local direct stresses * * * are in the nature of secondary stresses" and suggest that "allowable fiber stresses usual in secondary stress design be applied in general to these stresses." Referring, for instance, to Loading Cases (*a*) and (*b*) in this discussion, which represent typical torsion problems in practical design, the writer would not feel inclined to accept a specification that the direct fiber stresses obtained in the flanges should generally be classed as secondary stresses. As a matter of fact, the twisting of the beam may be conceived as pure bending of the two flanges, which are supported by an elastic medium, the reacting forces of which are reflected by the torsional rigidity of the twisted elements of the beam.

Consider, now, a beam which is simply supported in both ends and subjected to a twisting moment, T_p, applied at the center. The deflection function for this case, according to Equation (107), may be written,

$$y = \frac{a\,h\,T_p}{4\,K\,G}\left(\frac{x}{a} - \frac{\sinh\dfrac{x}{a}}{\cosh\dfrac{L}{2\,a}}\right) \quad \ldots\ldots\ldots\ldots\ldots (111)$$

The bending moment in each flange is,

$$M = -\frac{E I_y}{2}\frac{d^2y}{dx^2} = \frac{a\,T_p}{2\,h}\frac{\sinh\frac{x}{a}}{\cosh\frac{L}{2\,a}} \quad\ldots\ldots\ldots\ldots\ldots(112)$$

and thus the maximum moment (at the center of the beam), is,

$$M_0 = \frac{a\,T_p}{2\,h}\tanh\frac{L}{2\,a} \quad\ldots\ldots\ldots\ldots\ldots(113)$$

If, momentarily, the beam is assumed to have no torsional rigidity (that is, if the flanges are required to take the entire superimposed torsional load) the corresponding maximum bending moment in each flange would be.

$$M_f = \frac{T_p}{h}\frac{L}{4} \quad\ldots\ldots\ldots\ldots\ldots(114)$$

The ratio between the two moments is,

$$\kappa = \frac{M_0}{M_f} = \frac{\tanh\lambda}{\lambda} \quad\ldots\ldots\ldots\ldots\ldots(115)$$

in which, $\lambda = \frac{L}{2\,a}$. The value of κ is given in Fig. 36 for various ratios, λ.

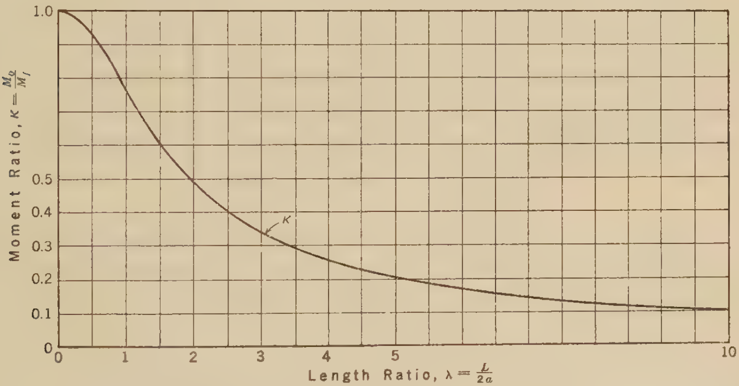

FIG. 36.—GRAPHICAL REPRESENTATION OF EQUATION (115)

As indicated, the writer's derivations are applicable only for $\beta \leqq 0.25$, and the inaccuracy increases as β increases. Equation (115), however, gives a clue as to how the problem may be approached for $\beta > 0.25$. Consider, for example, a beam with the same dimensions as in the authors' "Design Example *B*"—that is, with $b = 8.29$ in. and $2\,a = 54$ in. For $\beta = 0.25$ in this case, $L = 33.2$ in. and $\lambda = 0.615$. According to Fig. 36 this corresponds to $\kappa = 0.9$, or M_f varies as 1.11 M_0. This means that for $\beta > 0.25$ one may compute—at least for the beam under consideration—approximately (the inaccuracy is maximum 11% on the safe side) the stresses in the flanges under the assumption that each flange acts as a beam, or rather as a narrow plate, the depth of which cannot be considered as small in relation to the span width and which is stressed by a force, $P = \frac{T_p}{h}$, in its own plane.

F. L. EHASZ,[25] JUN. AM. SOC. C. E. (by letter).—In Fig. 9 the authors demonstrate the fact that there has been considerable diversity of opinion regarding the problem of stress concentration in the fillets of twisted bars. Several analytical and experimental solutions giving widely varying results have been proffered. Consequently, the effect of size and shape of structural members on torsional behavior was investigated in an attempt to help in the final solution of this problem. More than eighty experimental solutions involving various structural shapes were found. The membrane analogy provided a fairly rapid method for determining the increase of shearing stress in fillets by means of soap films. Torsion constants were also obtained for most of the sections analyzed.

The following variables were studied: The radius of fillets, length and thickness of the components of the member, and the shape of the member, angle, or I-section. In each series, the fillet varied from about $\frac{1}{8}$ in. to $1\frac{1}{2}$ in. Eight series of angles having $\frac{1}{2}$-in. to 1-in. thicknesses and 3 to 6-in. lengths; and three series of I-beams with wide flanges, corresponding to three of the angle series, were analyzed to determine the change of the hump at the junctures. An angle formed by taking half the I-section (that is, a T-section), and clipping one wing of the flange with the web intact will be said to correspond to that particular I-section. For the sake of clarity it is mentioned that an angle having long sides of 4 in. and 3 in., and a constant thickness of $\frac{3}{4}$ in., corresponds to a wide-flanged I-section having a depth of 8 in., a flange width of $5\frac{1}{4}$ in., and flange and web thicknesses of $\frac{3}{4}$ in. Torsion constants were evaluated for all the I-sections and for six of the angle series.

The radius of the circular hole that is used for comparison in soap-film analysis must be approximately equal to twice the area divided by the perimeter of the test hole. Volume-correction factors allowed for a slight deviation from this precaution. The errors due to the assumption that the sine equals the tangent and that the pressure acts vertically instead of normal to the surface of the film are largely eliminated, and pressure correction is achieved by resorting to the volume-correction curve which was procured by testing several sections of known torsional properties against each other, two at a time. The initial height of water in the flask, an essential part of the volume-displacement apparatus, was kept, at all times, approximately equal to that prevailing in the preliminary volume-correction tests.

In contour work, the height of the circular film was checked after every five or six points to insure that conditions remained constant. Elevations were correct to within 0.001 in. It was found that a boundary angle ranging between 15° and 25° was advisable for the circular film in order to obtain a precision of 2% in the stress concentration tests. For satisfactory volume measurements the boundary angle for the test film should not exceed 35 degrees. The minimum radius of the circular section in this investigation was $\frac{1}{2}$ in., a value which, according to Taylor,[36] should be considered the minimum allowable for reliable results. In the case of a symmetrical shape, such as the I-beam, only one-half the section was studied. A vertical septum passing through the axis of symmetry provided the essential continuity. It was necessary to double the torsion constant of the reduced shape in order to obtain the desired constant.

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[25] Lawrence Calvin Brink Research Fellow in Civ. Eng., Lehigh Univ., Bethlehem, Pa.

After a few complete contour diagrams were made, it was thought best to diminish the work required for a section by adopting the so-called cross-section method wherein points spaced from 0.03 to 0.1 in. apart along a normal to the boundary were located and their elevations recorded. Especial care and smaller spacings were taken immediately near the edge, since the crux of the slope determination lies there. Tangents to enlarged drawings of the films at the boundaries gave a direct measure of the border shearing stresses. Cross-sections were taken in the case of angles at the center of the fillet and at the center of the inner straight portion of the flanges. For I-sections additional slope data were recorded for the web. The shearing stress in the fillet could thus be compared with that in the arms of angles and with the shear in the flange and web of I-sections.

The effect of size and shape of members on stress concentration may be observed from Fig. 37, which gives the results of a few series. It is apparent from Fig. 37(a) that the stress concentrations of I-sections are considerably greater than for their corresponding angles. An angle, 4 by 3 in. by $\frac{1}{2}$ in., is one having long sides, 4 in. and 3 in., with the thickness of both legs $\frac{1}{2}$ in. An I-section, 8 by $5\frac{1}{4}$ in. by $\frac{3}{4}$ by $\frac{3}{4}$ in., has a depth of 8 in., a flange width of $5\frac{1}{4}$ in., with web and flange thicknesses of $\frac{3}{4}$ in. That the stress varies directly with the thickness may be concluded from the fact that

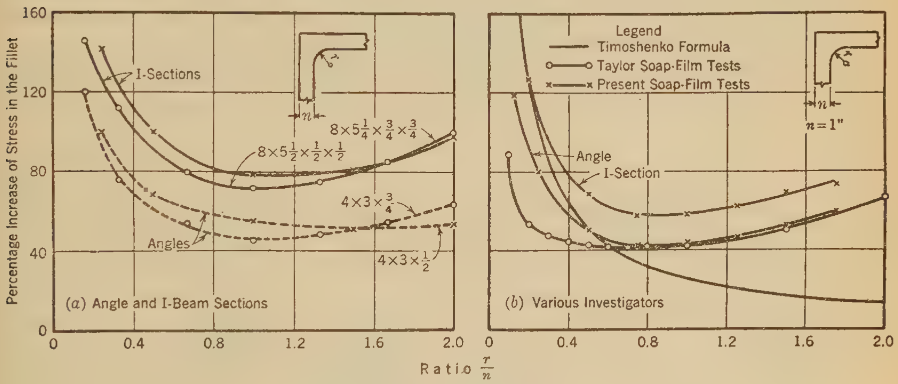

FIG. 37.—COMPARISONS OF STRESS CONCENTRATION.

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[36] "The Mechanical Properties of Fluids: A Collective Work," 1924, p. 236.

the fillet-stress concentration factors are inversely related to the thickness. The lowest points on the curves are peculiar in two respects. As the thickness of the straight part increases, the ratio of fillet to thickness for the minimum concentration decreases linearly according to Fig. 38, which also shows a fairly straight-line variation of the magnitude of the minimum concentration factor. This was obtained by taking averages of all the angle series.

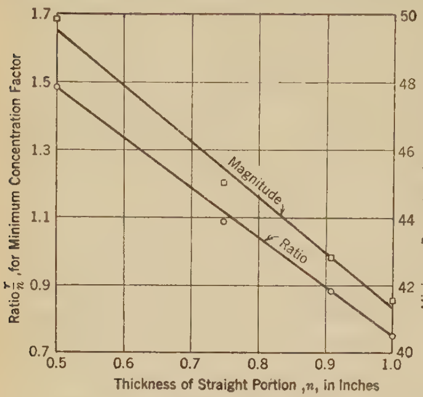

FIG. 38.

A comparison of the results of this investigation with those of Timoshenko's analytical solution and Taylor's soap-film tests is found in Fig. 37(*b*). For small fillets, including the usual radii, Timoshenko's formula proved to be superior to all others for angles. Timoshenko obtained his approximate theoretical solution[27] from the membrane analogy by assuming that the shearing stress in the fillet becomes zero at a point, $\frac{n}{2}$, from the boundary, in which n is the thickness of flange. Soap-film tests showed that this assumption is true for small radii of the fillet, r. For larger fillet sizes there is fair agreement between the tests of Taylor and those of the present investigation. For I-sections the results follow approximately the general curve given by Westergaard and Mindlin, in Fig. 9.

Torsion constants are found in Figs. 39 and 40, from which it is evident that the thickness of angle has a considerable effect on the rigidity, whereas the effect of length of arm is comparatively small. Experimental results were checked analytically and a fair precision was noted.

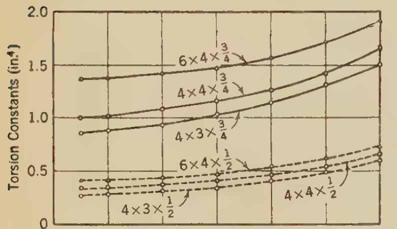

FIG. 39.—TORSION CONSTANTS FOR ANGLES

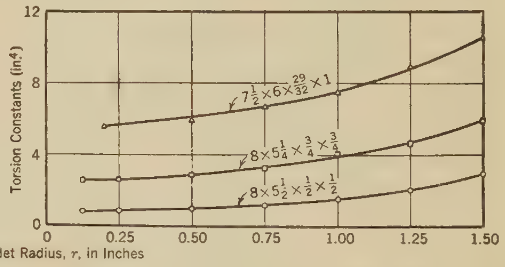

FIG. 40.—TORSION CONSTANTS FOR I-SECTIONS.

The following are the principal deductions drawn from these tests: (1) The inherent relations of the membrane analogy have proved especially useful in determining the stress concentration factors at the fillets of a twisted rod; (2) Timoshenko's formula for stress concentration proved suitable for the usual fillet radii in the case of angles, whereas that proposed by Westergaard and Mindlin (see Fig. 9) gave results having similar tendencies manifested in the present tests on **I**-sections; (3) observations of the soap films in the case of **I**-sections confirmed the fact mentioned by the authors that the critical shearing stresses prevail at the fillets and at the center of the outer side of the flange; (4) there is a linear relation between the minimum percentage increase of stress in the fillets of angles and the thickness of the straight portion (the ratio of fillet size to flange thickness, $\frac{r}{n}$, for minimum concentration factor also shifts linearly with variation in thickness); (5) the shearing stress, in general, increases linearly with the thickness (in **I**-sections, however, the stress in the web is slightly greater than in the flange for the same web and flange thickness. The stress falls off rapidly at the fillet, the rate being greater for small radii); and, (6) for ductile materials not subjected to alternating stress, the use of small fillets does not involve danger because of the redistribution of stress that follows local yielding. In the case of brittle materials, however, the weakening effect of the stress concentration should be mitigated by the use of greater fillet radii.

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[27] "Theory of Elasticity," by S. Timoshenko, First Edition, 1934, p. 258.

W. P. ROOP,[28] M. AM. SOC. C. E., and JOHN B. LETHERBURY,[29] JUN. AM. SOC. C. E. (by letter).—Although in this paper no procedure is outlined for estimating torsional rigidity of tubular sections, when the writers recently had occasion to deal with such a section, an effort was made to apply the methods of this paper. Interest was attached to a tube of square section, and some tests were made on a section about 1 in. on the side with walls $\frac{1}{8}$ in. thick. The polar moment of inertia is,

$$J = \frac{1}{3} b^2 A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (116)$$

A being the sectional area and b the side of the square.

The authors' Equation (6) applies to a solid section, but by removing a small square from the interior of a large one, the value,

$$K = 0.28\, b^2 A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (117)$$

is found. Such a procedure could be justified only if the walls were thick enough to act as if the interior square was not removed. This did not seem a severe assumption, as the tube contained heavy transverse diaphragms spaced about $\frac{3}{2} b$. Nevertheless, when such a tube was made and an experimental value of K was found by measuring torsional deflection under known load, it proved to be only about one-fourth the value calculated by Equations (116) and (117).

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[28] Lt. Comdr. (C.C.), U. S. Navy; Office of Superintending Constructor, New York Shipbuilding Co., Camden, N. J.

[29] Hull Draftsman, New York Shipbuilding Co., Camden, N. J.

The next step was to make up another tube using more care. Both tubes were welded up, but in the second case the welding was continuous and more uniform. However, when allowance was made in the calculations for the area of the weld fillets, the result was almost the same as at first, that the effective value of K was only about one-third to one-fourth the calculated value.

One objective in this work was to obtain verification of a formula for torsion of a tube of toroidal form. When the toroid was constrained to lie in a plane, the angular deflection was found by calculation to be,

$$\theta = \frac{\pi}{8} \times \frac{M\,R}{K\,G} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (118)$$

in which θ is the angle of rotation, in radians, caused by a couple, M, applied in a radial plane, R being the ring radius and G the shearing modulus.

Equation (118) was rather closely verified by test, provided the experimental value of K from the straight tube was used. This afforded some additional evidence for the correctness of the value of K, being about one-fourth the calculated value.

It is noted that a square tube is shown in Fig. 8(f), but it is not clear how the data given can be applied to the writers' problem, as it appears that tubes of equal sectional area would differ in torsional rigidity according to the wall thickness and the corresponding size of the square chosen. Comparison between cases, Fig. 8(f) and Fig. 8(h), suggests that the closed section has much greater rigidity, but in this case also the basis of the comparison is uncertain.

Application of the methods described under "Torsion Constant for H-beams and I-beams," to the closed section very naturally leads to a value that is too low by a margin greater than that by which Equation (6) is too high. It is evident that the square tubular section has a rigidity intermediate between the values as calculated by these two quite different procedures.

As compared with the open section, two powerful influences work to give the tube increased rigidity. The first is that of closing the section, as shown in comparing Cases (b) and (c), Fig. 8. The benefit obtained by the short longitudinal plates at the ends, as in Fig. 15, may be only partly in the improved end constraint; part of the benefit may consist in the partial closure of the section. The question arises whether additional benefit might not be easily obtainable by introducing additional shear connections between toes of flanges at sections other than the ends.

The other influence is that of the diaphragms. Their function is like that of the fillets, but they are naturally much more effective than fillets. This suggests that the tripping brackets now commonly fitted in certain I-sections may be of considerable benefit for torsional rigidity.

For specific comparisons, the numerical data, referring to the sections tested, are given in Table 6.

In general agreement with the developments in this paper, under the heading "Fixed-Ended Torsion," the writers have had some success in devising a set of artificial assumptions that will give a value of K for a tubular section in fair agreement with observations. This is only hindsight, however, and the writers feel rather certain that such precarious assumptions as those which have thus been found to work (that is, flat sides are beams which remain flat under load) would not be very highly regarded if the effective value of K were not known from test data. Possibly, a solution of some merit might be found in this way, but it would require further verification by test and, therefore, would become no rational solution, but only a somewhat rational expression of empirical data.

TABLE 6.—NUMERICAL DATA.

Description	First bar	Second bar	Ring
Sectional area, A, in inches ^{2}	0.58	0.61	0.703
Equivalent side of Square b, in inches	1.06	1.08	1.06+
Polar moment of inertia, J, in inches ^{4}	0.218	0.237	0.307
K by Equation (6), in inches ^{4}	0.184	0.199	0.30 (?)
K by Equation (16), in inches ^{4} *	0.009	0.015	0.015—
Observed K, in inches ^{4}	0.053	0.080	0.128

* Ignoring webs.

In the light of these considerations the decision was made to use Equation (6) and calculate K for a tubular section by subtracting the value for the inner profile from that for the outer profile; but to apply a factor of from 2 to 3 to account for the deficiency of the effective value as compared with the value thus calculated.

The models referred to were made by the New York Shipbuilding Corporation and tested in its laboratory.

ALFRED T. WAIDELICH,[30] JUN. AM. SOC. C. E. (by letter).—An important research, for which there was a very real need, is described in this paper. An obvious application is in the design of spandrel beams and other beams with eccentric loading. Not only has this research indicated the method of analysis in such a case, but already it has made available for the designer one practical set of formulas and tables[31] for the two most common cases: A beam with a full uniform load applied with constant eccentricity; and a beam with a single eccentric concentrated load at the center of the span. It should be only a short time until the standard textbooks and handbooks will treat adequately the subject of torsional loads.

The dual practical problem of designing apparatus capable of twisting the larger steel sections and, at the same time, preserving a free end connection has been given a simple solution. This problem has prevented previous investigators from testing any but the smaller and lighter sections, with the result that the tests were inconclusive.

Definitely, the authors have settled the problem of determining the torsion constant for structural I-beams and H-beams. This constant is far more useful than the scant attention it has received would indicate. An early structural use of it was contained in a small book on the bow girder[32] which describes some torsion tests on small English joist beams. About the same time the constant appeared in aircraft design and in the study of the elastic stability of structures.

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[30] Philadelphia, Pa.

[31] Bethlehem Manual of Steel Construction, 1934, pp. 279–289.

The principles of elastic stability are of increasing importance as the use of high-strength alloy steels increases, and as structural layouts become simpler. Thus, the modern catenary-supporting bent for railway electrification is a simple H-frame, consisting of three rolled shapes lying in one plane. Designers of such structures have been handicapped by the uncertainty of the value of the torsion constant. As a result, the few available tentative tabulations of the torsion constant[33] were considerably in error on the safe side, so that the full economy of this method of analysis could not be realized.

The measurement of fiber stresses in the beams under torsion furnishes a much needed check on the theoretical stresses. It is interesting that for the shear stress in the web, Equation (26) gives values that are definitely lower than those measured. The statement by the authors that Equation (26) is in accord with torsional theory is probably based on the assumption that the web acts as a part of an infinitely long rectangle undergoing the same degree of twisting as the flanges. Apparently, the proportions of the wide-flanged shapes (a relatively thin web tying together heavy flanges) prevents the web from playing such a simple rôle. It is unfortunate that the web stresses were not measured in the tests of the beams with web and flanges of nearly the same thickness. Since the governing shear stress will rarely occur in the web, Equation (27) is acceptable, although the introduction of the radius of the fillet does not seem logical—especially since this stress occurs at the center of the side of the web. Fig. 21 is interesting in that it shows an increase in the web shear stress as the fillet is approached (but before there is any increase in the web thickness). This is confirmed by the slope of the soap film, which increases on approaching the fillet because of the steep slope at the fillet and the "hump" at the junction of the web and the flanges.

INGE LYSE,[34] M. AM. SOC. C. E., AND BRUCE G. JOHNSTON,[35] JUN. AM. SOC. C. E. (by letter).—The original analysis by Professor Westergaard and Mr. Mindlin with regard to stress concentration in fillets is a most worthy contribution on this subject, which was subsequently expanded and studied in the light of experimental tests with soap films on both I-beams and angle sections by Mr. Ehasz, and so reported in his discussion. Mr. Ehasz found that their analysis applied well for the stress concentration in I-beams. Professor Reynolds gave close attention to the investigation and aided liberally in the analytical studies presented in the paper. His discussion of the more general study of I-beams subjected to torsion is a worthy addition to the typical cases considered by the writers. Mr. Werner has furnished additional bibliography on the subject of torsion and has contributed general formulas similar to those given by Professor Reynolds, for the case of a beam either fixed or free at the ends and twisted by a couple at any point along the beam.

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[32] "The Circular Arc Bow Girder," by Gibson and Ritchie, 1914.

[33] For example, "Elastic Equilibrium in the Theory of Structures," by H. S. Richmond, *Transactions*, Am. Soc. C. E., Vol. 94 (1930), Table 1, p. 860.

[34] Research Associate Prof. of Eng. Materials, in Chg., Fritz Eng. Laboratory, Lehigh Univ., Bethlehem, Pa.

[35] Instructor in Civ. Eng., Columbia Univ., New York N. Y.

The writers still believe that the use of higher unit stresses for the direct longitudinal stresses at the extreme corners of the flanges of beams at their fixed ends is justifiable. All the fixed-end tests, as originally reported, show that at the torsional yield point of the beam both the theoretical and measured stresses at the ends of the beam were roughly 50% higher than the yield-point strength of the material in direct tension.

In connection with the more practical problems of combined bending and torsion, with the use of ordinary fabrication processes and present assumptions accepted in building design, the outer ends of beams at points where they are riveted to a column or girder will probably be considered unrestrained. Even in a heavily built-up welded end connection it would not be possible to rely upon 100% fixity. At some section along the beam, however, a plane of torsional fixity is maintained, but the accompanying direct stresses are localized in the outer edges of the flanges. Since each flange acts as a separate rectangular beam section, its reserve strength after plastic action commences is much greater than in the usual case of an I-beam in bending. Another factor, which does not lend itself readily to mathematical calculation, is the restraining action of walls, brickwork, intermediate floor-beams and the "giving" of the columns or girders into which the beam in combined bending and torsion is framed.

The questions raised by Professor Wessman have been answered in detail in the discussion by Mr. Ehasz and in a later descriptive paper.[36] The record of experimental data on the torsional properties of channel section as furnished by Professors Seely and Putnam is a valuable addition to the writers' experiments on I and H-sections. Mr. Ehasz's continuation of the writers' soap film experiments shows that this method serves well for the study of the concentration of torsional stress in structural sections as well as for the evaluation of the torsional rigidity. The remarkable uniformity in results obtained by Mr. Ehasz gives confidence in the experimental work and indicates how successfully the membrane analogy may be applied to analytical problems. The data given in Fig. 39 and Fig. 40 might have been presented also in a more general form with the known accurate formula (Equation (6)) for rectangles as a basis. The additional torsional rigidity due to the angular shape would then be represented as a definite function of fillet radius and the thickness of the legs of the angle.

The writers are interested in the results reported by Messrs. Roop and Letherbury on closed sections. Unfortunately, the writers did not investigate such sections and, therefore, are unable to account for the apparent loss in torsional rigidity shown in their tests. However, since the same effect was noted in the case of the ring, the torsional properties of which may readily be analyzed precisely and which have been corroborated by numerous tests, it is possible that the welded seam did not provide complete continuity of material.

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[36] "Torsional Rigidity of Structural Sections", by Bruce G. Johnston, Jun. Am. Soc. C. E., *Civil Engineering*, November, 1935, pp. 698–701.

Professor Waidelich has raised questions which should rightly be the subject of further study. In conclusion, the writers are very grateful for the general interest evidenced in the paper by all the discussers. It is particularly gratifying to record the interest in more exact methods of analysis of structural members shown by the steel industry.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1942

PHOTO-ELASTIC DETERMINATION OF SHRINKAGE STRESSES

BY HOWARD G. SMITS,[1] ESQ.

WITH DISCUSSION BY MESSRS. THOMAS H. EVANS, I. K. SILVERMAN, J. H. A. BRAHTZ, ARSHAG G. SOLAKIAN, L. N. G. FILON, AND HOWARD G. SMITS.

SYNOPSIS

Stresses due to shrinkage of concrete in masonry dams are of interest and importance, but the determination of these stresses is complex and difficult. In practice, the large masses of concrete used in gravity dams are not homogeneous; nor are they isotropic, as the structure is poured in blocks which present many planes of discontinuity. The indeterminate factors on these planes of discontinuity present such formidable difficulties that an accurate determination of shrinkage stresses is impossible. If the problem is idealized, however, by assuming a homogenous, isotropic, simple gravity dam subject to uniform shrinkage, it becomes possible to predict these stresses with some assurance. Although this idealized case is not truly consistent with actual practice, it creates a philosophical "picture," nevertheless, of the strained condition that should be of value to the judgment of the designing engineer.

One approach to the mathematical solution of this problem has been made by J. H. A. Brahtz,[2] of the United States Bureau of Reclamation. Considering the applicability of the solution of this problem, a complete diagrammatic representation of the stress distribution may be of more interest than the mathematical description. The former is herein presented. This problem was studied at the California Institute of Technology in the photo-elastic laboratory of the Guggenheim School of Aeronautics. The writer wishes to express his appreciation for the continued interest in this problem expressed by Theodor von Kármán, M. Am. Soc. C. E., who is the Director of the Guggenheim School.

NOTE.—Published in May, 1935, *Proceedings*.

[1] Structural Designer, O. G. Bowen, Los Angeles, Calif. Mr. Smits was elected Jun. Am. Soc. C. E., on July 8, 1935.

[2] "Stress Distribution in Wedges with Arbitrary Boundary Forces," *Physics*, Vol. 4, No. 2, February, 1933.

METHOD OF SOLUTION

The photo-elastic method is particularly adapted to this problem of determining shrinkage stresses as only a two-dimensional analysis is required. A complete description of the theoretical background of the photo-elastic method has been given by Messrs. L. N. G. Filon and E. G. Coker.[3]

The most desirable procedure would be to stretch the dam, *CDE* (Fig. 1), longitudinally, by heat or direct loading; then, in this elongated condition, fasten it securely along the base to the stress-free, elastic ground, *AB*. This is a true representation of uniform shrinkage in a theoretical gravity dam section. A line of discontinuity in the stress pattern is encountered along Line *AB*, which is evident from the discontinuity in the elongation on either side of that line. The following general conclusions are evident: (*a*) The crest of the dam, *E*, is free from stress; (*b*) horizontal compression occurs in the ground in the region near the center of the base of the dam; and (*c*) tension gradually dissipates in the regions from Point *G* to Point *A* and from Point *H* to Point *B*.

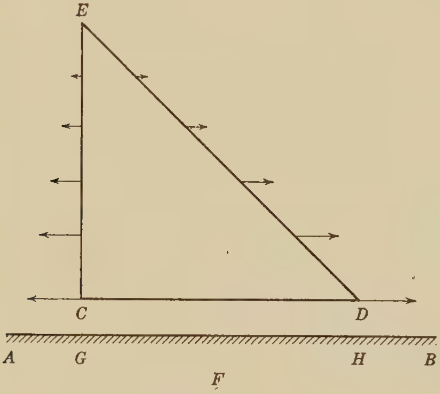

FIG. 1.

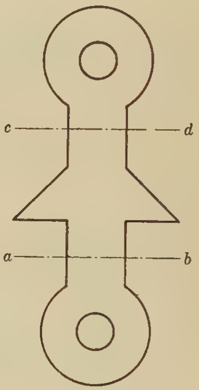

FIG. 2.

To obtain, experimentally, the effect of dam shrinkage upon an elastic foundation, M. Biot[4] suggested reversing the operation by elongating the base. A homogeneous model of the dam and ground was made in one piece, thus taking advantage of the fact that rock and concrete have approximately the same modulus of elasticity. The "ground" was then stretched with a direct tensional load, which produced a high tension of, say, *p* lb per sq in., in Regions *A* and *B* (Fig. 1), which is contrary to the foregoing description of the problem. Since this tension had to be corrected, a blanket horizontal compression of *p* lb per sq in. was applied to the ground, arbitrarily, after the complete stress analysis had been determined. The result was a stress of zero in the regions, *A* and *B* (Fig. 1); tension in the restricted regions,

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[3] "Photo-Elasticity," by E. G. Coker and L. N. G. Filon. Cambridge Press, 1931. (A working summary of the theory, together with some laboratory technique, is given by Mr. Coker in a series of five articles published every two months, in the *General Electric Review*, beginning with the issue for November, 1920.)

[4] "Contribution à la Technique Photo-élastique," *Annales* de la Société Scientifique de Bruxelles, LIII B., 1933.

G and *H*; and compression in the region, *F*. A decided discontinuity results along the base line, *CD*. (It is obvious that the addition of a blanket compression is not mathematically exact as (1) the principal tension stress in the ground is not exactly parallel to the ground surface except at infinity, and (2) the ground is partly restrained against movement perpendicular to the ground surface so that some relatively small stress perpendicular to this blanket compression should be added. In this case, however, the error introduced by not considering these two items is within the experimental error and, therefore, their omission is justified.)

The model was made so that the tension could be induced in the ground by means of pins as shown in Fig. 2. To be certain of pure axial loading the tension strap was made symmetrical about its axis; thus, the symmetry of the optical pattern was always a gauge of the degree of axial loading. The loading frame is shown with the model in place in Fig. 3. A load of 1 220 lb per sq in. was used in the strap of the model.

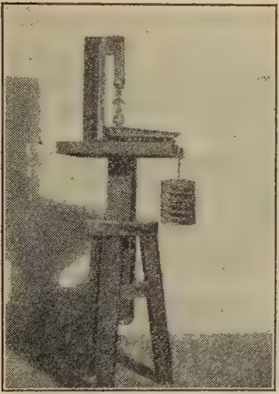

FIG. 3.

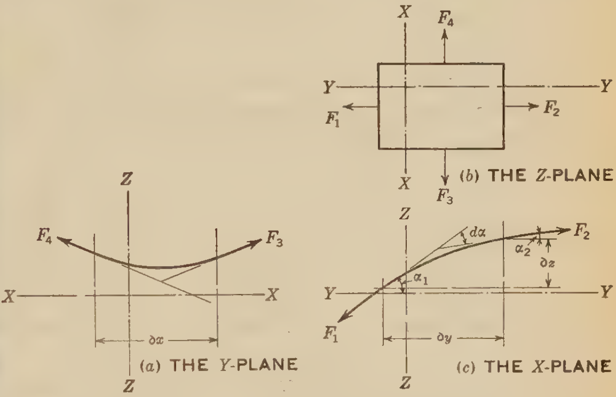

FIG. 4.

The lines of constant shear and the direction lines of principal stress were determined by the standard photo-elastic method. It was desirable furthermore to find the principal stresses. Three methods were open: (*a*) Graphical integration; (*b*) an electrical analogy; and (*c*) a membrane analogy. Graphical integration is the standard method of obtaining the principal stresses, but it is a long and tedious process. No record was found of the electrical analogy having been used in this work. The Prandtl membrane analogy has been used in the solution of Laplace's equation found in problems of shafts carrying torque. At approximately the same time that this problem was solved, E. E. Weibel, at the University of Michigan, was using a soap membrane to find values of $p + q$.[5] Dr. Biot[4] has generalized this analogy theoretically to make it applicable to the photo-elastic method.

THEORY OF NEW METHOD

Desiring a practical and relatively short convenient method of finding the principal stresses, it was decided to test the applicability of the membrane analogy. A rubber membrane was used with success. The analogy is based on the fact that both the sum of the principal stresses and the displacement of the membrane obey Laplace's equation. (This statement is true only if the slope of the membrane at every point is small. (See assumption under heading, "The Membrane.") In this experiment the slope of the membrane was excessive only in the restricted regions of the fillets at the toe and heel of the dam. It is in these two regions that the largest experimental errors occurred.)

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[5] See his paper presented at the Annual Meeting of the Am. Soc. Mech. Engrs., December, 1933.

The Model.—For any given point let p and q be the principal stresses, p_x and q_y, the stresses acting parallel to the respective axes, and ψ, the unknown Airy stress function. Then,

$$\nabla^4 \psi = 0 = \nabla^2 \times \nabla^2 \psi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which ∇ = the differential operator. Furthermore, $p_x = \frac{\partial^2 \psi}{\partial y^2}$, and,

$$q_y = \frac{\partial^2 \psi}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

At any given point, however, the sum of the stresses at right angles to each other, having any direction, is equal to the sum of the principal stresses at that point. Therefore,

$$p + q = p_x + q_y = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = \nabla^2 \psi \quad \ldots\ldots\ldots\ldots (3)$$

It follows that,

$$\nabla^2 (p + q) = \nabla^2 \times \nabla^2 \psi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

which demonstrates that the sum of the principal stresses satisfies Laplace's equation.

The Membrane.—Let the tension in the membrane be a constant, T, induced by a uniform stretch. Fig. 4 shows any element of the membrane which is in equilibrium, although deformed. (Forces F_1, F_2, F_3, and F_4, are components of the total tension, T; and α = the angle with the horizontal at a given point.) In the X-plane (Fig. 4(c)), the angle that the force makes with the Z-axis is $\tan^{-1} \frac{\partial z}{\partial y}$. If the angle is small, the tangent may be taken equal to the angle; then:

$$F_1 = T \frac{\partial z}{\partial y} dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

and,

$$F_2 = T \frac{\partial z}{\partial y} dx + T \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) dy\ dx \quad \ldots\ldots\ldots\ldots (5b)$$

This gives a resultant component of force parallel to the Z-axis,

$$\frac{\partial}{\partial y} \left(\frac{\partial x}{\partial y} \right) T\ dy\ dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Similar reasoning in the Y-plane leads to,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) T\ dy\ dx \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

which is the component of force parallel to the Z-axis in the Y-plane. To have equilibrium the sum of the two components (Equations (6) and (7)) must equal zero; thus,

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) T\ dy\ dx = 0 \ldots\ldots\ldots\ldots\ldots (8)$$

or, $\nabla^2 z = 0$, from which it is to be noted that z and $(p + q)$ are interchangeable.

The difference, $p - q$, between the principal stresses at any point is determined from the isochromatic lines, a sample of which is shown in Fig. 5. Around the free boundaries of the dam and on the sides of the strap one principal stress must be zero, as there is no stress perpendicular to the model edge. If p or q is zero for any point, the difference, $p - q$, between the principal stresses, and the sum, $p + q$, of the principal stresses, are equal. Along Lines ab and cd in Fig. 2, the total tension is distributed uniformly across the strap so that the sum of the principal stresses is again known. Therefore, the sum, $p + q$, is known completely around the boundaries of the model.

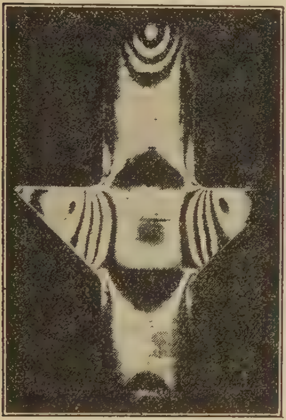


Fig. 5.

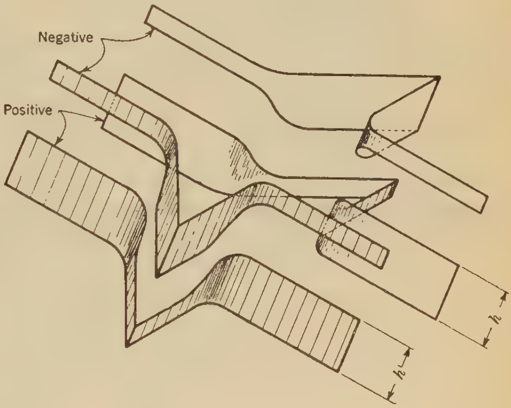
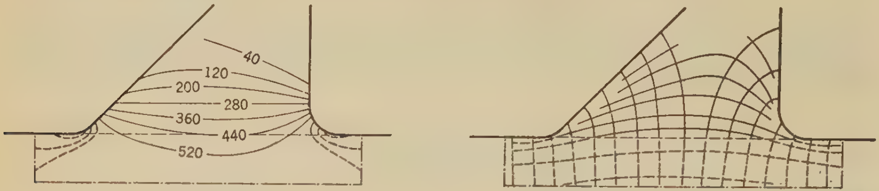


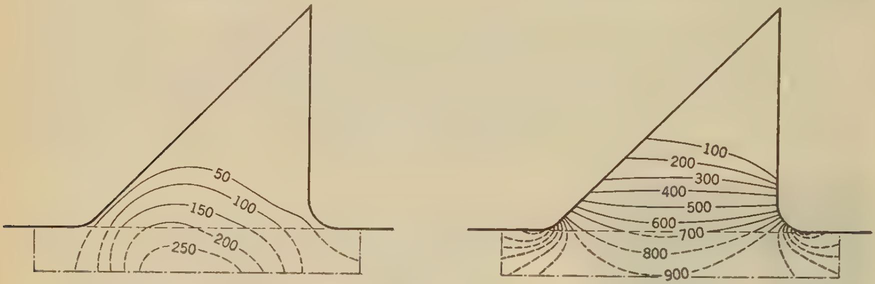
Fig. 6.

Application of New Method

Strips were cut from a thin aluminum plate to form an outline of the model (see Fig. 6). The height, h, of these strips was made proportional at every point to $p + q$ which, as explained previously, was known. Another series of strips was cut to form a negative such that the two fit closely when set one on the other. A rubber sheet, $\frac{1}{64}$ in. thick, was stretched to a uniform tension, T (see Equation (5)), and clamped between the negative and positive dies. A survey of the height of the membrane at every point was made with a micrometer screw on a two-way carriage. This gave a contour map of the value of $p + q$ for every point. With both $p + q$ and $p - q$ determined (the latter having been found by the usual photo-elastic method), the individual stresses for every point in the model were known.


40
120
200
280
360
440
520

(a) SHEAR (b) DIRECTION OF MAXIMUM STRESSES


50
100
150
200
250
100
200
300
400
500
600
700
800
900

(c) COMPRESSION (d) TENSION

FIG. 7.—STRESSES DUE TO 1.65 × 10^{-3} INCH PER INCH ELONGATION OF BAR.

Fig. 7 is a summary of the results, in pounds per square inch, the correction for the difference between the modulus of elasticity of bakelite and concrete having not been made. Checks against equilibrium were made to

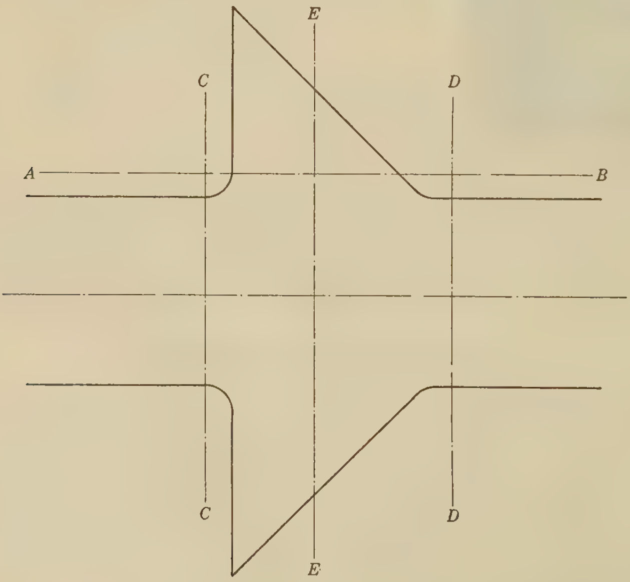
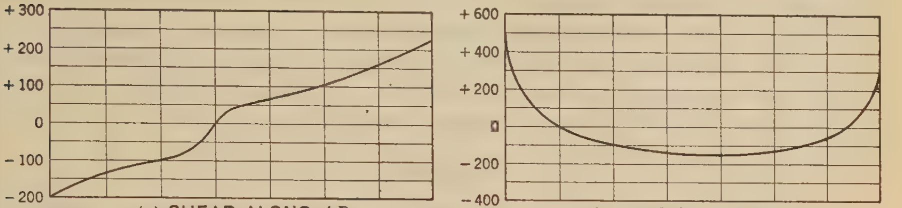
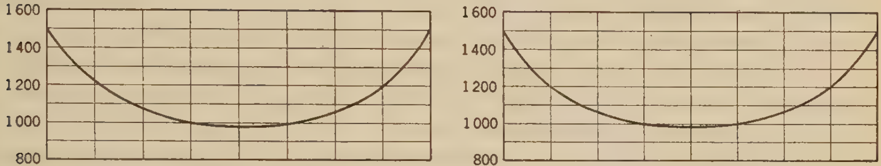


FIG. 8.

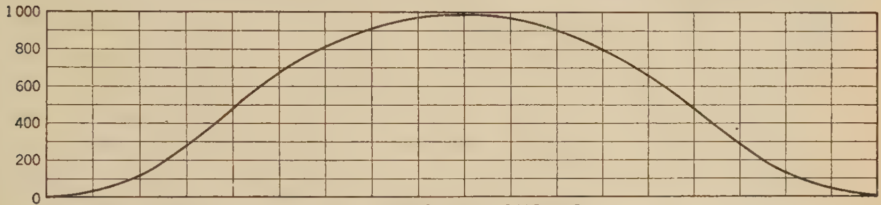
determine the experimental accuracy, by obtaining the forces perpendicular to, and the shears parallel to, various sections taken through the model. The results, shown in Figs. 8 and 9, are as follows: Along Section *AB* (Fig. 8), the error in shear indicated by Fig. 9(*a*), is 13%; the error in tension along Section *AB* (Fig. 9(*b*)), is 8%; and the respective errors in tension along Sections *CC*, *DD*, and *EE* (Fig. 8), were 5, 5, and 2 per cent.



(*a*) SHEAR ALONG *AB* (*b*) TENSION ALONG *AB*



(*c*) TENSION ALONG *CC* (*d*) TENSION ALONG *DD*



(*e*) TENSION ALONG *EE*

FIG. 9.—CHECKS AGAINST EQUILIBRIUM (FOR LOCATING OF SECTIONS, SEE FIG. 8).

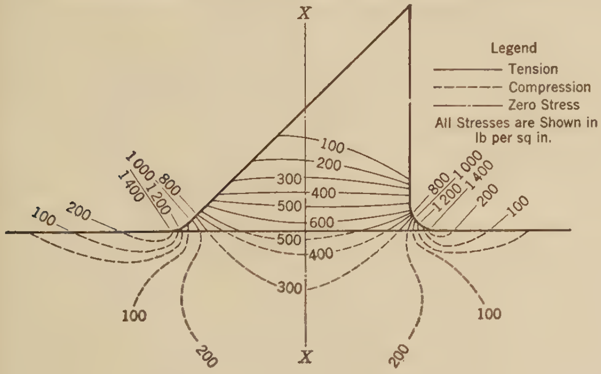

X
X
Legend
Tension
Compression
Zero Stress
All Stresses are Shown in lb per sq in.
100
200
300
400
500
600
500
400
300
1000
800
1400
1200
100
200
800
1000
1200
1400
200
100
100
200
200
100

FIG. 10.—STRESSES IN A GRAVITY DAM AND ITS FOUNDATION DUE TO A UNIFORM SHRINKAGE OF 0.0006 INCH PER INCH ($E = 2\,000\,000$).

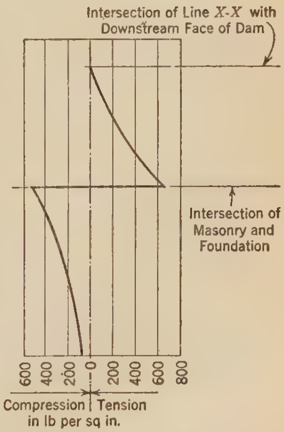


FIG. 11. — DISTRIBUTION OF MAXIMUM TENSION AND COHESION ALONG LINE *X-X*, FIG. 10.

Finally, a uniform compression is added to that part of the model representing the ground in accordance with the foregoing discussion. Figs. 10 and 11 show the complete stress diagrams for the theoretical condition of a unit shrinkage of 0.00165 in. per in. The line of discontinuity of stresses along the base of the dam is to be particularly noted. The relation between the stresses at any point and the unit shrinkage is a simple ratio. Thus, if the unit shrinkage is taken as 0.000825 in. instead of 0.00165 in., the stresses at every point are only one-half those shown.

Conclusion

The model test was that of a theoretical condition that does not occur in practice, and for that reason the results shown are only of philosophical interest; nevertheless, they are of value to the designer. Points of high stress are indicated and a general "feeling" for the behavior of shrinkage stresses can be obtained from the results. Of special interest is the high unit stress between the dam and the ground at the heel and the toe of the dam. In general, the larger the fillet at these points, the lower will be the shearing stress.

DISCUSSION

THOMAS H. EVANS,[6] JUN. AM. SOC. C. E. (by letter).—The problem of determining, with some exactness, the stresses in a concrete dam due to shrinkage seems well adapted to the photo-elastic method. Although the writer disagrees with the author's contention that the stress problem is only two-dimensional, it will not greatly affect the "philosophical picture" to consider it as such. The principal inaccuracy in the determination is not in the method itself but, as Mr. Smits points out, in the physical inconsistency between the prototype and the necessarily idealized model. However, the method does provide a rational first approximation to an extremely complex stress distribution. It should be a distinct aid in supplementing the engineering experience and judgment of the designer.

As indicated by the author, the "membrane analogy" as a supplement to photo-elastic analysis appears to have decided advantages over the method of graphical integration for the determination of the principal stress sum. There was also another excellent paper by Supper and McGivern[7] that exhibited in great detail the use of the membrane analogy for the determination of the quantity, $p + q$. The accuracy of this method decreases, however, as the slope of the membrane increases. This occurs in regions of high stress concentration, such as the heel and toe of the dam illustrated in the paper. Since a designer might have more interest in such regions than in others, it would be desirable to avoid the effects of this limitation of the membrane analogy.

Another supplementary procedure which seems to be adaptable to all cases is that proposed[8] by Professor V. Tesar. This is a purely optical method requiring even less additional apparatus than the membrane analogy, and its accuracy is independent of the concentration of stress in the model. Its fundamental principle is the interference of light waves as they are reflected from the bounding surfaces of a thin wedge of air. The surfaces in this case are a face of the model and an optically flat glass. Changes in thickness in the model (which are a function of the quantity, $p + q$), cause corresponding variations in the interference fringe pattern across the image of the model on a screen. These variations can be calibrated, and, therefore, $(p + q)$ can be found for a given load.

I. K. SILVERMAN,[9] JUN. AM. SOC. C. E. (by letter).—What would be the stress pattern in the bakelite model of a triangular dam resting on a continuous elastic foundation and subjected to shrinkage? In this paper, Mr. Smits has described a conception of this phenomenon. His description of how the stresses might vary seems logical. Based on this preconceived picture, he proceeds to load his model so as to obtain this possible distribution of stress and, in Figs. 10 and 11, presents curves for the resulting stress distribution near the base. He has failed to show, however, that this is the only way this stress pattern could have been produced and, for this reason, and for others cited herein, no great value can be attached to the numerical results shown in Fig. 10.

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[6] Asst. Prof. of Civ. Eng., Dept. of Eng., Univ. of Virginia, Charlottesville, Va.

[7] *Journal*, Franklin Inst., April, 1934.

[8] *Revue d'Optique*, March, 1932.

[9] With U. S. Bureau of Reclamation, Denver, Colo.

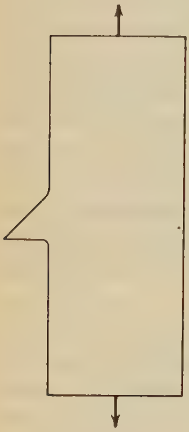

FIG. 12.

The author states that the form of the test specimen was chosen so that the stress pattern would be a gauge of the degree of axial loading. It is evident that in choosing this type of specimen, he has clouded the stress distribution in the very region that he is examining. The stress distribution in any model is dependent to a high degree on its geometry. From Fig. 8 the width of the strap is of the same order of dimension as that of the base of the dam. For this reason the stress distribution of Fig. 7 may be far different from that in a model which is of the form shown in Fig. 12.

By means of Fig. 9 Mr. Smits has shown the accuracy obtained in measuring the stresses shown in Fig. 7. It would be interesting if, in his closing discussion, he would present curves similar to those of Fig. 9 for the resultant stress distribution shown in Fig. 10. Evidently, the total areas under these curves should be zero.

J. H. A. BRAHTZ,[10] ESQ. (by letter).—The sections of the paper entitled "Method of Solution" and "Theory of New Method" are valuable contributions to the theory of photo-elasticity in that they demonstrate how to evaluate the stresses within the boundaries of a stressed body.[11] The membrane method described by the author has been successful in the photo-elastic laboratory of the U. S. Bureau of Reclamation. The over-all error in the experiments on heavy monolithic structures is estimated to be from 5 to 10%, depending on the type of structure and loading.

The section of the paper entitled "Application of New Method," gives numerical values of the shrinkage stresses in a gravity dam which are in good agreement with those obtained by the writer based on theoretical considerations and photo-elastic experimentation. For example, for the horizontal stresses along the bisector of the top angle of a 45° triangular dam (Poisson's ratio = 0.2), set on a rigid foundation, the writer finds the following approximate solution (taking the two first terms of an infinite series):

$$\sigma = \left\{ \left(\frac{r}{h}\right)^{4.39} \left[1.812 \cos\left(2.7 \log \frac{r}{h}\right) - 0.178 \sin\left(2.7 \log \frac{r}{h}\right) \right] - \left(\frac{r}{h}\right)^{12.66} \left[0.838 \cos\left(3.8 \log \frac{r}{h}\right) + 0.062 \sin\left(3.8 \log \frac{r}{h}\right) \right] \right\} K E \quad ..(9)$$

in which r = the distance from the vertex; h = the height of vertex above the foundation (measured along the bisector); E = the elastic modulus of elasticity; and, K = the shrinkage per unit length. If $r = h$, the horizontal stress at the foundation becomes:

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[10] Engr., U. S. Bureau of Reclamation, Denver, Colo.

[11] See, also, *Zeitschrift angew. Math.*, Vol. II, 1931, p. 156.

$$\sigma_1 = 0.974 \, K \, E \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

which value is estimated to be correct within 5% due to the omission of the higher order terms in Equation (9). Using the value of K in Fig. 10, that is $K = 0.0006$, the stress becomes, $\sigma_1 = 0.0005844 \, E$.

Assuming E to be 2 000 000 lb per sq in., the stress would be, $\sigma_1 = 1\,169$ lb per sq in.; and, with $E = 3\,000\,000$ lb per sq in., the stress would be, $\sigma_1 = 1\,753$ lb per sq in.

These quantities are not directly comparable with the author's stresses in Figs. 10 and 11, because he assumes an elastic foundation of the same modulus as the dam. Consequently, when the dam shrinks, there will be a compromise between the foundation and the dam, the former going into compression in the region directly below the dam and the latter into tension. It will not be an even division, however, because the foundation does not deform quite as readily as the dam with its much smaller mass; and, therefore, it imposes the greater part of the shrinkage as an elongation of the elements of the dam adjacent to the foundation.

Using the stresses given in Fig. 11 at the foundation (630 in the dam and 460 in the foundation) as a basis of the compromise, the computed stress in the dam would be, $\frac{630}{630 + 460} \sigma_1$, or 676 and 1 014, respectively, depending on the value of E introduced in Equation (10). The first value compares closely with the value (630) given in Fig. 11, corresponding to $E = 2\,000\,000$ lb per sq in. which the author adopted for his calibration.

Based on the writer's theory and experiments, he concludes that the average horizontal tensile stress along the base of a dam on an elastic foundation varies from 5 to 15 lb per sq in. per degree Fahrenheit of decrease in temperature, depending on the ratio of the elastic moduli in the dam and the foundation. This range of stress shows the dire necessity of providing against shrinkage effects during construction by cooling and pressure grouting.

Arshag G. Solakian,[12] Esq. (by letter).—The experimental method used by Mr. Smits, namely, of stretching the base of the model to introduce shrinkage stresses in the body of an "idealized" dam, does not well represent the actual conditions prevailing in Nature, as will be demonstrated presently.

Shrinkage stresses in concrete dams of the gravity type are produced under several conditions, the most important of which are: (1) The setting of the cement in concrete, in a dam resting on a clay foundation or loose rock; (2) the setting of the cement in concrete, in a dam having a perfect bond with a solid rock foundation (Mr. Smits' paper deals only with this case); and (3) the daily (or weekly) and seasonal changes of atmospheric variations of temperature, in a dam having a perfect bond with a solid rock foundation.

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[12] Lecturer in Civ. Eng., Columbia Univ., New York, N. Y.

Condition (1).—Stresses in this catagory are due to the premature setting of the cement at the two exposed surfaces of the dam, before such action begins to develop in the interior regions of the cross-section. As a result, a relatively harder crust is formed around the structure in Nature's attempt to resist subsequent shrinkage effects of the interior region. Such a phenomenon and the resulting stress distribution will be similar in Nature to that which occurs in every day foundry practice when dealing with cast metals.

Stresses of the intensity and direction produced under Condition (1) can be obtained by photo-elastic methods from a model cast under proper curing conditions. In a previous investigation[13] bakelite specimens revealed the existence of such stresses with a sufficient number of interference fringes. A model was prepared by filling with bakelite powder a metal form 1 in. deep, having the outlines of the profile of the dam with extended footings. The material was heated for several days at a constant temperature and then gradually cooled to room temperature. After being machined to a thickness of about ⅝ in. for plane parallel surfaces, the cast plate was examined in circularly polarized light for the fringe pattern in Fig. 13. In this view each fringe line is a contour of the constant difference between the principal stresses, σ_1 and σ_2 acting at a point, with a relative intensity for each fringe line equal to its order as marked. Since the maximum shear stress intensity is known to be equal to one-half the difference of the principal stresses, the fringe pattern will also show the distribution of these types of stresses which is quite useful for actual design purposes under certain circumstances. The variation of these stresses across three typical sections, *A–A*, *B–B*, and *C–C* (Fig. 13)—coinciding, respectively, to the base, one-third, and two-thirds the height of the dam—are shown in Fig. 14(*a*). From Figs. 13 and 14(*a*) it is evident that sharp corners and fillets of small radii are regions of high stress concentrations under the conditions of structure as discussed (Condition (1)). Furthermore, the stress gradient increases from the center of the dam toward the outer surfaces, for all sections considered.

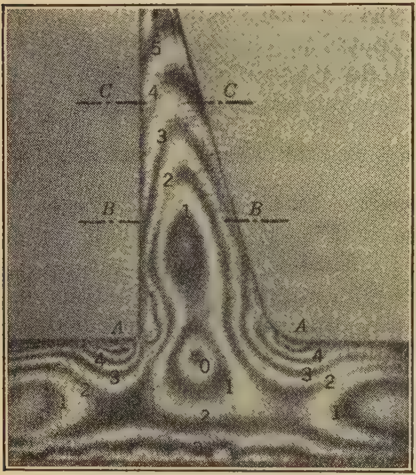

FIG. 13.—FRINGE PATTERN.

The orientation of the principal stresses, σ_1 and σ_2, is found by using plane polarized light at constant directions of vibration from 0° to 90°, with 10° intervals. The resulting isoclinic bands that are contours of constant orientation of these stresses are traced as in Fig. 14(*b*). From these curves, the stress trajectories of Fig. 14(*c*) are obtained by a graphical process. The existence of a singular point (known also as an isotropic or neutral point) near the base of the dam is clearly evident from Figs. 13, 14(*b*), and 14(*c*). At such a point the two principal stresses are at all possible orientations, and are of equal magnitude and sign (both tension or both compression). Note, for example, that $\sigma_1 - \sigma_2 = 0$, represents the zero order fringe in Fig. 13.

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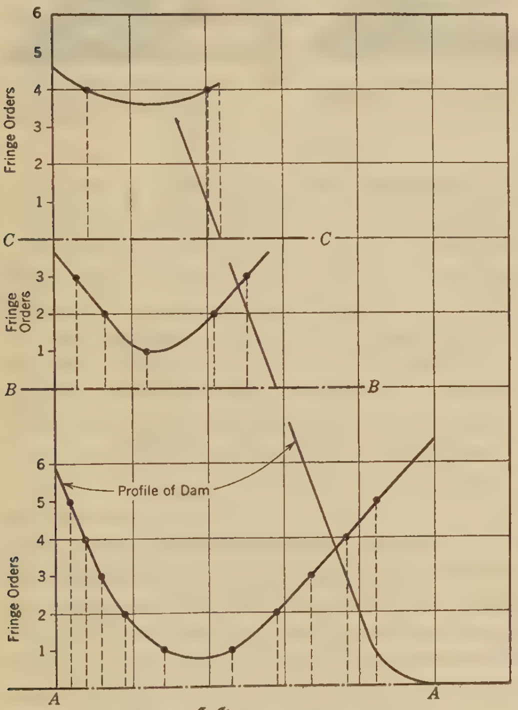
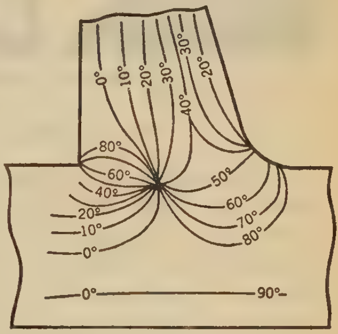
[13] "Photo-Elastic Analysis of Stresses in Composite Materials", by A. H. Beyer, M. Am. Soc. C. E., and A. G. Solakian, Assoc. M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E. Vol. 99 (1934), p 1196.

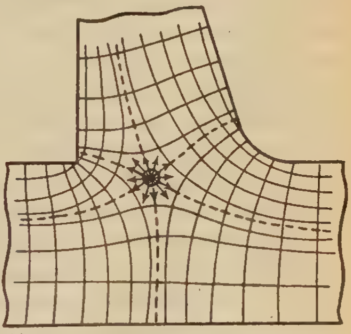
Condition (2).—When the dam has a perfect bond with an underlying foundation, the method outlined under Condition (1) does not apply. In this case a mass of infinite extent which forms an integral part with the base of the dam, is free from shrinkage phenomena, since it is underground and at a constant temperature. Such action, however, is developing within the entire body of the dam. The method proposed by Mr. Smits serves to introduce stresses only in the base region of the dam, whereas the larger part of the structure above this region is not affected. A logical method would be to make a model consisting of two pieces, a large block to represent the massive foundation and a small triangular plate to represent a thin section of the dam. The block is kept in a bath of hot water, whereas the thin plate is kept in dry ice, both long enough to establish a uniform temperature distribution throughout their entire mass. The two pieces then would be cemented together along the base of the dam, taking care that each section was kept at



(*a*) Maximum Shear ($\frac{\sigma_1-\sigma_2}{2}$) Stress Distributions



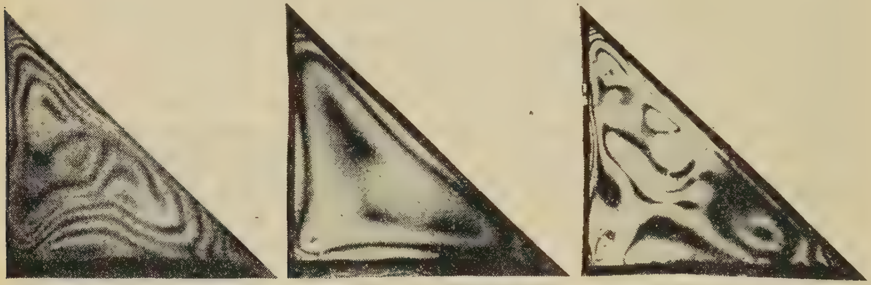
(*b*) Isoclinics



(*c*) Stress Trajectories

FIG. 14.—STRESSES IN A DAM ON CLAY FOUNDATION, DUE TO SETTING OF CEMENT IN THE CONCRETE.

its original uniform temperature during the final setting of the cement. The cemented model would then be introduced into a cell with glass walls and containing hot water, to raise the temperature of the dam section to that of its foundation and thus to make it expand, the block as a whole being unaffected. This action is similar to that which occurs under actual conditions in the structure considered. Fig. 15 is a fringe pattern of the stresses produced in a model examined by the method as described. A new photo-elastic material with a greater stress-optical sensitivity, was used for the two parts of the cemented model. A special quick-setting cement was also used.[14] An inspection of Fig. 15 makes evident the existence of shrinkage stresses all over the cross-section of the dam, in variance with that of Fig. 5 in the paper.



(*a*) Due to the Setting of Cement. (*b*) Due to Daily Variations of Atmospheric Temperature. (*c*) Due to Seasonal Variations of Atmospheric Temperature.

Fig. 15.—Stresses in a Dam on Rock Foundation.

Condition (3).—The effect of temperature variation producing shrinkage stresses in concrete dams can be reproduced by a process similar to that discussed under Condition (2). The model, however, can be made from a single plate if desired. The large base plate is kept in a bath of hot water, whereas the dam section is enclosed in a cell with glass walls. The glass plates are in contact with the polished surfaces of the dam section, and the two compartments thus formed on either side of the dam are filled with dry ice, or the up-stream cell with hot water and the down-stream cell with dry ice, as it is desired to duplicate the cases of the reservoir filled or empty. With this arrangement, flow of heat through the body of the dam will be restricted across the two edge surfaces and the base of the dam.

When the effects of a sudden change of temperature (daily or weekly atmospheric variations) are considered, the fringe pattern in Fig. 15, taken a few minutes after the model is put in the baths of hot water and dry ice, will give a qualitative picture of the resulting momentary temperature-stress potential that exists over the cross-section of the dam. To duplicate the effects of seasonal variations of atmospheric temperature, a sufficient time of heat flow should be allowed in the test, to produce a more or less constant flow of heat from one part of the dam section to the other, thus affecting the interior points of the dam as well. Under these conditions, then, the fringe pattern will be quite steady provided the temperature of the source of heat and the sink is kept constant. Fig. 15(*c*) represents the latter situation.

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[14] "A New Photoelastic Material", by A. G. Solakian, *Mechanical Engineering*, December, 1935.

Due to the necessity of certain improvements in the details of conducting the test, as described under Conditions (2) and (3), further analysis with curves showing the intensities and orientations of the stresses for the last two cases is disregarded. As a matter of fact, because of the small size of the models used in this discussion (3-in. legs of dam section) and by Mr. Smits, the experimental results should be considered only from a qualitative point of view.

The suggestions described herein, for the investigation of shrinkage stresses arising from various causes, combined with the centrifuge-polariscope method[15] of analyzing the stresses in dams due to gravity and hydrostatic pressure, should give the investigator sufficient tools for a complete and exact experimental solution of the dam problem.

L. N. G. FILON,[16] ESQ. (by letter).—In view of the importance of the problem treated in Mr. Smits' paper, the writer has studied a xylonite model made similar to Fig. 16, with a broad strap, there being a projecting "dam" on one side only. If the loads at the ends of the strap are properly centered the longitudinal tension, Q, is sensibly uniform throughout a region where the distance from the base line, $A\,B$, of the dam (Fig. 17) exceeds one-fifth the height of the dam.

However, when the stresses are measured at the point, O, of Line $A\,B$ (Fig. 17), where the shear stress parallel to $A\,B$ vanishes, Q is found to be (0.65) T and ($-$ 0.35) T, respectively (after applying the corrective pressure, T, below $A\,B$), on the upper and lower sides of the line, $A\,B$, of discontinuity. For the case taken by the author (T = 1 220 lb per sq in.), these forces would produce a tension of 793 lb per sq in., and a pressure of $-$ 472 lb per sq in., results which differ appreciably from those deduced from Fig. 11. Apparently, this gives 650 lb per sq in., for the tension above, and 550 lb per sq in., for the pressure below, leading to a lesser tension in the dam proper than that

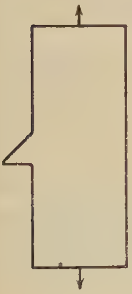

FIG. 16.

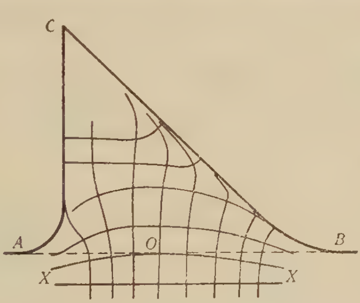

FIG. 17.—LINES OF PRINCIPAL STRESS IN DAM SUBJECT TO UNIFORM SHRINKAGE.

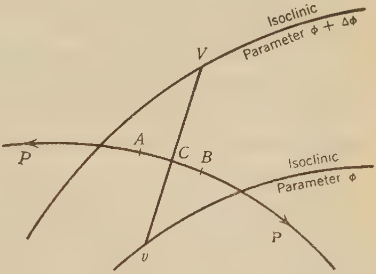

FIG. 18.—STEP-BY-STEP METHOD OF EXPLORATION OF PRINCIPAL STRESS.

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[15] "Centrifuge Method of Testing Models", by P. B. Bucky, A. G. Solakian, and L. S. Baldin, *Civil Engineering*, May, 1935.

[16] Prof. of Applied Mathematics and Mechanics, University Coll., London, England.

now found. Furthermore, the stress appears to extend deeper into the base of the dam in Mr. Smits' experiment than in the present case.

The writer has also traced the isoclinics and lines of principal stress in this case. The latter are shown in Fig. 17 and appear to be different, in the upper part of the dam, from those given by the author in Fig. 7(*b*). Fig. 7(*b*) is probably incorrect, since the lines of principal stress should approach a free boundary tangentially and normally.

The device used by Mr. Smits to solve Laplace's equation empirically is an interesting application of a well-known method originally suggested by L. Prandtl[17] in connection with the torsion of rods, and further developed by A. A. Griffith and G. I. Taylor.[18] A soap film is preferable to an india-rubber membrane, owing to the difficulty of being certain that the tension in the latter is uniform at all points and in all directions. Even if this condition is satisfied initially, it is likely to be disturbed when the india-rubber sheet is clamped between templates as described, any rapid variations of height of the edge involving appreciable pinching, which must modify the tensions postulated. This seems to be a more serious source of error than the exaggeration of the slope, as it may well not be localized.

This method assumes, of course, that the equation to be solved is Laplace's equation; in other words, that the material under consideration is perfectly elastic. The same assumption is made in applying those methods which depend on measurement of lateral extension or contraction. So far as the writer's experience goes, bakelite is a material which is very far from being perfectly elastic, and it exhibits considerable strain creep and optical creep with time. The author does not state whether any precautions have been taken to eliminate them.

Mr. Smits is to be congratulated on having been able to obtain specimens of bakelite of adequate size apparently free from large initial double refraction. The bakelite obtainable in England is badly affected in this way, and comparatively useless for photo-elastic work.

It is not quite clear why the author (following Fig. 4) describes what he calls the method of graphical integration (but which the writer would prefer to call the step-by-step method) as "long and tedious." A similar statement appears in another paper.[19] It would seem that engineers are under a misapprehension as to the work involved in this method.

If A and B (Fig. 18) are two near points on a line of principal stress, P, through the middle point, C, of $A\,B$, draw a perpendicular to Line $A\,B$, meeting at U, V, respectively, any two neighboring isoclinics which correspond to inclinations, ϕ, $\phi + \Delta\,\phi$, of the polarizer and analyzer to any standard directions of reference. Then,

$$P_B - P_A = \frac{A\,B}{U\,V}\,\Delta\,\phi\;(Q - P)_C \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

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[17] *Physicalische Zeitschrift*, Vol. 4, 1903.

[18] *Engineering* (London), Vol. 104, 1917, pp. 658, 699.

[19] "The Stress Function and Photo-Elasticity Applied to Dams", by J. H. A. Brahtz, Esq., *Proceedings*, Am. Soc. C. E., September, 1935, p. 1015.

in which $\Delta \phi$ is usually a small whole number of degrees, which must be multiplied by 0.01745 to yield $\Delta \phi$, in radians.

Now, in any photo-elastic investigation, both the isoclinics and the lines of principal stress are fundamental and will have to be drawn in any case; $Q - P$ at any required point can be obtained in a variety of ways: (1) By interpolation between isochromatics; (2) by direct measurement with a compensator; or (3) by varying all loads on the model together in such a ratio that an isochromatic of a given order passes through the point. Thus, the data entering into the right-hand side of Equation (11) are immediately to hand; the value of $P_B - P_A$ is then calculated in a few moments with a slide-rule.

In this manner it is possible to work along a line of principal stress and obtain the value of P, step by step, at a number of points. If the investigator suspects inaccuracies, owing to an accumulation of errors, there is an important check, which should be applied frequently. For example, the investigator may reach the same point, using different rectangular zigzag paths along lines of principal stress, and, if the values agree (as they do when the work has been done carefully), the stresses found can be accepted with considerable confidence and used as a basis for further exploration.

There is no necessity for "graphical" integration in the ordinary sense; nor need the steps, in general, be inconveniently small. What is essential, however, is that the isoclinic lines shall have been obtained accurately in the first instance. For this, visual observation, with bright illumination and a good graticule of reference, is necessary. An ordinary photograph is usually not adequate for this purpose.

This step-by-step method[20] is entirely independent of the elastic properties of the material and is not based on the assumption that the substance used is perfectly elastic. It does, however, involve the assumption that stress difference is proportional to optical retardation, and this law generally holds well beyond the elastic limit.

HOWARD G. SMITS,[21] JUN. AM. SOC. C. E. (by letter).—The writer is gratified with the variety of viewpoints expressed by the discussers of his paper, particularly since the mathematical solution of the problem is complex enough to make the approach extremely difficult and, at best, only approximate.

Mr. Silverman has questioned the shape of the tested model, feeling that the strap was too narrow in relation to the size of the dam. Fortunately, when Professor Filon performed a similar test, he used a model similar to that suggested by Mr. Silverman. The difference in results obtained by Professor Filon and the writer would indicate that this geometric change in the model has bearing on the quantitative result. It is unfortunate that a relatively wider strap was not used in the original experiment.

The agreement between the experimental results and the results obtained from the mathematical expression derived by Mr. Brahtz is very gratifying. Mr. Brahtz estimates that Equation (10) gives a value which is correct within

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[20] Rept. of the British Assoc. for the Advancement of Science, 1923; also, *Engineering* (London), Vol. 116 (1923), pp. 511–512.

[21] Structural Designer, O. G. Bowen, Los Angeles, Calif.

5 per cent. Thus, when $K = 0.0006$ and $E = 2\,000\,000$, Equation (10) yields: $\sigma_1 = 1\,169$ lb per sq in. This is comparable to the sum of the maximum stresses on either side of the line of discontinuity at the ground line. From Fig. 11 these stresses are shown to be 525 lb per sq in. and 650 lb per sq in., giving a total of 1 175 lb per sq in. Professor Filon evaluated these two stresses at 793 lb per sq in. and 472 lb per sq in., for a total of 1 265 lb per sq in. The three values obtained (1 169, 1 175, and 1 265) fall in a narrow region, showing very good agreement between the three approaches.

Professor Filon takes exception to the lines of direction of principal stress as drawn by the writer in Fig. 7(*b*). At no time did the writer observe isoclinic lines which, when plotted, would give the sudden change of direction in the lines of principal stress, indicated in Professor Filon's Fig. 17, near the faces of the dam. It is to be noted that in all cases the lines of principal stress approach the free boundaries normally and tangentially in Fig. 7(*b*).

The advisability of using a rubber membrane is questioned by Professor Filon. Concentric circles were drawn on the rubber before stretching, and these were found to remain true circles in plan, both after stretching and after placing between the aluminum dies. It was only after observing this precaution that the writer felt justified in assuming a uniform tension in the membrane.

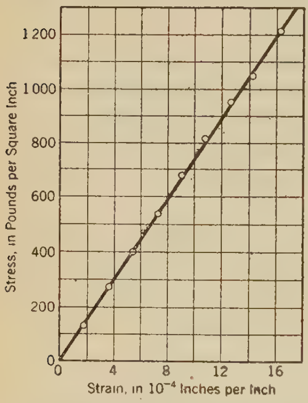

Fig. 19.

The bakelite used in this work was not free from initial double refraction until after the specimen had been carefully annealed. The annealing operation presented several difficulties. However, once properly heated and cooled, the model remained free of initial stresses for some time. In performing the experimental work on the model, the element of time was always considered to decrease the effect of optical creep.

Fig. 19 represents a stress-strain diagram for the bakelite used in the model. The relationship indicates a remarkably straight line, which showed little tendency to creep with time.

Mr. Solakian presented some very interesting material in the three conditions considered by him. It is unfortunate that more detailed information was not given, both quantitatively and qualitatively, particularly in relation to Condition (3). It is noted with particular interest that a bakelite cement has been perfected with physical qualities such that it is adaptable to photo-elastic work. The writer has long sought such a cement. The next step in approximating actual conditions with the photo-elastic method is now possible. Bakelite blocks can be cemented together simulating the manner in which they are now actually cast in the field and the entire model tested as a unit. Engineers interested in dam construction will look forward to the publication of the results of photo-elastic tests made on models built with construction joints.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 1943

THE SHEAR-AREA METHOD

By Horace B. Compton,[1] Assoc. M. Am. Soc. C. E., and
Clayton O. Dohrenwend,[2] Jun. Am. Soc. C. E.

With Discussion by Messrs, George E. Large, Samuel T. Carpenter, Roland H. Trathen, A. W. Fischer, J. Charles Rathbun, Harold R. Kepner, Fred L. Plummer, Albin H. Beyer, John M. Beatty, R. B. Peck, Ralph W. Stewart, C. W. Johnson and H. W. Birkeland, Garrett B. Drummond, Harold E. Wessman, Fang-Yin Tsai, David A. Molitor, F. Muls-Guinotte, and Horace B. Compton and Clayton O. Dohrenwend.

Synopsis

The shear diagram may be used in a manner similar to the moment diagram for the solution of the elastic functions of loaded beams. This procedure is not generally known although the moment-area methods are of common practice and are taught in most schools of engineering. Use of the shear diagram simplifies the solution of many beam problems, and, therefore, should be of interest to engineers. This type of solution is applicable to all types of beams, and this paper includes the analysis of statically determinate, and statically indeterminate, beams with concentrated loads, uniform loads, and couples.

Notation.—The symbols used throughout the paper are given in the Appendix. An effort has been made to conform as nearly as practicable with "Symbols for Mechanics, Structural Engineering, and Testing Materials," advanced by the American Standards Association.[3]

General

The use of the moment diagram to solve for slopes and deflections in statically determinate beams and for redundants in statically indeterminate

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Note.—Published in May 1935, *Proceedings*.

[1] Asst. Prof. of Mechanics, Rensselaer Polytechnic Inst., Troy, N. Y.

[2] Instr., Dept. of Civ. Eng., Rensselaer Polytechnic Inst., Troy, N. Y.

[3] A. S. A.—*Z*10*a*—1932.

beams, is well known. Since corresponding values can be obtained by a similar treatment of the shear diagram (and in some cases, more easily), it is suggested that this method of using the shear diagram should be given more general consideration.

The shear-area method has the following advantages:

(*a*) It eliminates the moment diagram and, therefore (especially with uniform loading), it provides a simpler figure to work with than that of the moment-area method;

(*b*) It simplifies the solution for slope and deflection at any section and affords an easy means of locating the section of maximum deflection; and

(*c*) It provides a ready solution for the derivation of the "Theorem of Three Moments" for any type of loading.

For the student this method will bring into play certain "Mathematical Beams" which will aid his general knowledge of beam action. In the solution of the beam problems the shear diagram is used in the same manner as the moment diagram in the "Conjugate Beam Method."

In considering the shear-area method the following assumptions should be kept clearly in mind:

(1) The modulus of elasticity is considered constant in all the problems;

(2) The shear diagram divided by the moment of inertia is used as the loading on a mathematical beam if the moment of inertia is constant;

(3) The shear at any section of this mathematical beam represents the bending moment at the same section of the real beam;

(4) The bending moment at any section of this mathematical beam represents the slope at the same section of the real beam;

(5) The slope at any section of the mathematical beam represents the deflection at the same section of the real beam (this deflection can be expressed in terms of the moment of inertia of the shear diagram); and,

(6) The end conditions of the mathematical beam are determined by the given beam.

Referring to Assumption (2), when the moment of inertia is not constant the ratio, $\frac{V}{I}$, is replaced by,

$$E \frac{d^3y}{dx^3} = \frac{V}{I} - \frac{M}{I^2}\left(\frac{dI}{dx}\right) \qquad \text{(1)}$$

Equation (1) is derived as follows:

$$E \frac{d^2y}{dx^2} = \frac{M}{I} \qquad \text{(2)}$$

and,

$$E \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{M}{I}\right) = \frac{I\,dM - M\,dI}{I^2\,dx} = \frac{V}{I} - \frac{M}{I^2}\left(\frac{dI}{dx}\right) \qquad \text{(3)}$$

in which, E = modulus of elasticity; y = deflection measured parallel to the Y-axis; x = a distance measured parallel to the X-axis; M = bending moment; I = moment of inertia; and V = shear. Assumptions (1) to (6) are concerned with the relations between beam loading, shear, moment, slope, and deflection. Their significance may be reviewed by reference to Fig. 1 in which:

For shear (Fig. 1(*b*)),

$$V_x = R - \int w_x \, dx \dots\dots\dots\dots \tag{4}$$

for moment (Fig. 1(*c*)),

$$M_x = \int V_x \, dx = \int dA \dots\dots\dots\dots \tag{5}$$

for slope (Fig. 1(*d*)),

$$\phi_x = \phi_L - \frac{1}{EI}\int M_x \, dx = \phi_L - \frac{1}{EI}\int x \, dA + C_1 \dots\dots \tag{6}$$

and, for deflection (Fig. 1(*e*)),

$$y = \int \phi_x \, dx = \phi_L \, x - \frac{1}{EI}\int\int x \, dA \, dx = \phi_L x - \frac{I_x}{2\,EI} + C_2 \dots \tag{7}$$

in which R = a reaction; A = the area of a loading diagram on a mathematical beam; and $C_1 = C_2$ = a constant (= 0).

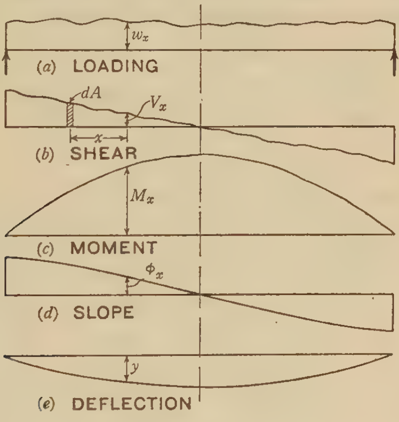
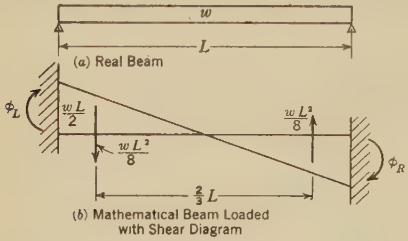


FIG. 1

ANALYSIS OF STATICALLY DETERMINATE BEAMS

The following typical cases are given as a development of the shear-area method of solving problems applied to beams that are statically determinate.

Case 1.—Simple Beam with Uniform Load Over Entire Span.—In Case 1, the real beam (Fig. 2(*a*)) has slope at each end and, therefore, the mathematical beam (Fig. 2(*b*)) has a moment at each end. The real beam has no bending moment at the ends and, therefore, the mathematical beam has no shear at the ends. Hence, the reactions of this mathematical beam are couples. The loading is also a couple and is placed symmetrically about the center and, consequently, one-half the total couple is carried by each end reaction. This is evident since in this case the real beam has equal end slopes.



(a) Real Beam

(b) Mathematical Beam Loaded with Shear Diagram

FIG. 2

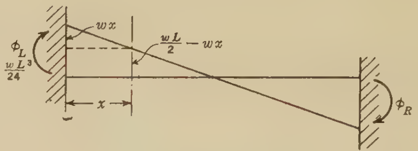


FIG. 3

The total moment of the loading is expressed by,

$$\phi_T = \frac{wL}{2} \times \frac{L}{2} \times \frac{1}{2} \times \frac{2}{3} L \times \frac{1}{EI} = \frac{wL^3}{12\,EI} \quad \text{...........(8)}$$

With one-half assigned to each end, the end slope of the real beam is:

$$\phi_R = \phi_L = \frac{wL^3}{24\,EI} \quad \text{.......................(9)}$$

Since the moment of the shear diagram gives the change in slope, the slope at any point can be computed by measuring a distance, x, from one end and then taking moments at the desired section, including the end reactions in the moment equations; thus (see Fig. 3):

$$EI\,\phi_x = \frac{wL^3}{24} - w \times x \times \frac{x}{2} \times \frac{2x}{3} - \left(\frac{wL}{2} - wx\right)\frac{x^2}{2}$$

or,

$$\phi_x = \left(\frac{wL^3}{24} + \frac{wx^3}{6} - \frac{wLx^2}{4}\right)\frac{1}{EI} \quad \text{................(10)}$$

To find the deflection, y, at any section integrate Equation (10):

$$EI\,y = \int_0^x \frac{wL^3\,dx}{24} + \int_0^x \frac{wx^3\,dx}{6} = \int_0^x \frac{wLx^2\,dx}{4}$$

$$= \frac{wL^3x}{24} - \left(\frac{wLx^3}{12} - \frac{wx^4}{24}\right) = \frac{wL^3x}{24} - \frac{I_x}{2} \quad \text{.........(11)}$$

The maximum deflection occurs at the section where $x = \frac{L}{2}$:

$$y_m = \frac{5}{384}\,\frac{wL^4}{EI} \quad \text{.......................(12)}$$

When concentrations occur in the shear diagram each value is multiplied by the square of its distance and included in Equation (11).

Case 2.—Simple Beam with Concentrated Load at Mid-Span.—For this case (see Fig. 4) the total moment of load is,

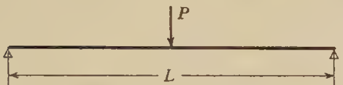
$$\phi_T = \frac{PL^2}{8\,EI} \quad \text{.........................(13)}$$

and the end slope is,

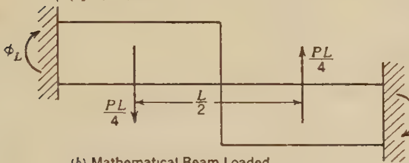
$$\phi_L = \frac{P\,L^2}{16\,E\,I} \qquad (14)$$

Taking moments for slope at any point:

$$\phi_x = \frac{P\,L^2}{16\,E\,I} - \frac{P\,x^2}{4\,E\,I} = \frac{P}{4\,E\,I}\left(\frac{L^2}{4} - x^2\right) \qquad (15)$$

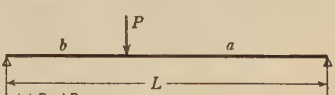


(a) Real Beam

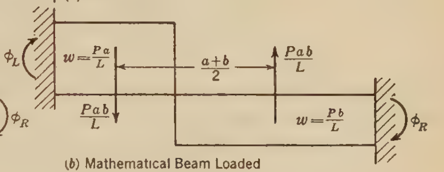


(b) Mathematical Beam Loaded with Shear Diagram

FIG. 4



(a) Real Beam



(b) Mathematical Beam Loaded with Shear Diagram

FIG. 5.

The deflection at any point is equal to:

$$E\,I\,y = \frac{P\,L^2\,x}{16} - \frac{I_x}{2} \qquad (16)$$

and the maximum deflection occurs at the section where $x = \frac{L}{2}$:

$$y_m = \frac{P\,L^3}{48\,E\,I} \qquad (17)$$

Case 3.—Simple Beam with Concentrated Load Not at Mid-Span.—To determine the end conditions of the mathematical beam for this case (see Fig. 5), write the general moment equation with the left end as the origin, thus, the moment of the mathematical beam is,

$$M_m = \phi_L - \frac{P\,a\,x^2}{2LEI} \qquad (18)$$

and the slope of the mathematical beam is,

$$Q_m = \phi_L\,x - \frac{P\,a\,x^3}{6LEI} + (C = 0) \qquad (19)$$

The moment equation of the mathematical beam, as derived from the right end is,

$$M_m = \phi_R + \frac{P\,b\,x^2}{2LEI} \qquad (20)$$

or, the slope of the mathematical beam is,

$$Q_m = \phi_R\,x + \frac{P\,b\,x^3}{6LEI} + (C = 0) \qquad (21)$$

Equate the two tangents of the mathematical beam when x in Equation (19) is equal to b, and x in Equation (21) is equal to a, since the deflection of the real beam can be expressed by both equations at that point, thus:

$$-\phi_L\, b + \frac{P\, a\, b^3}{6\, L\, EI} = \phi_R\, a + \frac{P\, b\, a^3}{6\, L\, EI} \quad\ldots\ldots\ldots\ldots\ldots (22)$$

Furthermore, since $\phi_R = \phi_L - \phi_T$:

$$EI\,\phi_L = \frac{P\, a\, b}{6\, L}\,(L + a) \quad\ldots\ldots\ldots\ldots\ldots (23)$$

and,

$$EI\,\phi_R = -\frac{P\, a\, b}{6\, L}\,(L + b) \quad\ldots\ldots\ldots\ldots\ldots (24)$$

The equations for deflection and slope for any section can now be readily written: For slope,

$$EI\,\phi_x = \frac{P\, a\, b}{6\, L}\,(L + a) - \frac{P\, a\, x^2}{2\, L} \quad\ldots\ldots\ldots\ldots\ldots (25)$$

and for deflection,

$$EIy = \phi_L x - \frac{I_x}{2} \quad\ldots\ldots\ldots\ldots\ldots (26)$$

Since at the section of maximum deflection the slope is equal to zero in the real beam, the moment equation of the mathematical beam need only be set to zero and solved for the value of x; thus:

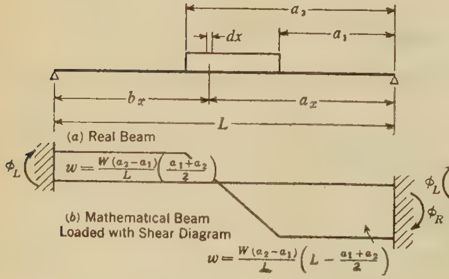
$$-\frac{P\, a\, b}{6\, L}\,(L + b) + \frac{P\, b\, x^2}{2\, L} = 0 \quad\ldots\ldots\ldots\ldots\ldots (27)$$

or,

$$x = \sqrt{\frac{a}{3}\,(L + b)} \quad\ldots\ldots\ldots\ldots\ldots (28)$$

Substituting Equation (28) in the deflection equation,

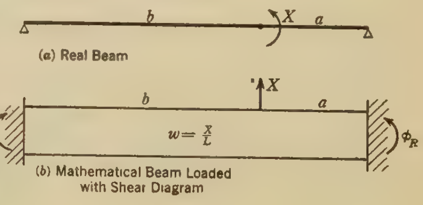
$$y_m = \frac{P\, a\, b}{9\, E\, I\, L}\sqrt{\frac{a}{3}\,(L + b)^3} \quad\ldots\ldots\ldots\ldots\ldots (29)$$


a_2
dx
a_1
b_x
a_x
L
ϕ_L
$w = \frac{W(a_2 - a_1)}{L}\left(\frac{a_1 + a_2}{2}\right)$
ϕ_R
$w = \frac{W(a_2 - a_1)}{L}\left(L - \frac{a_1 + a_2}{2}\right)$

(a) Real Beam

(b) Mathematical Beam Loaded with Shear Diagram

FIG. 6.



(a) Real Beam

(b) Mathematical Beam Loaded with Shear Diagram

FIG. 7.

It is interesting to note that, as the value of b changes from zero to $\frac{1}{2} L$, the value of x varies from $\frac{1}{\sqrt{3}} L$ to $\frac{1}{2} L$. This means that x_m will always be such as to bring the section of maximum deflection within 8% of L, distant from the middle of the beam.

Case 4.—Simple Beam with Segment of Uniform Load at Any Position.—Equations (23) and (24) may be used to determine the end condition of the mathematical beam in the case of a simple beam with a segment of uniform load at any position (see Fig. 6). Then (see Fig. 6(*a*)),

$$EI\,\phi_L = \int_{a_1}^{a_2} \frac{w\,dx \times a_x\,(L - a_x)}{6\,L}\,(L + a_x)$$

$$= \frac{w}{6\,L}\left[\frac{L^2}{2}\,(a^2_2 - a^2_1) - \frac{1}{4}\,(a^4_2 - a^4_1)\right] \dots\dots\dots\dots (30)$$

The slope at any point can now be obtained from the moment equation of the mathematical beam; thus, for values of x from zero to $(L - a_2)$ with the origin at the left support,

$$\phi_x = \phi_L - w\,(a_2 - a_1)\left(\frac{a_1}{2} + \frac{a_2}{2}\right)\frac{x^2}{2\,L\,EI} \dots\dots\dots\dots (31)$$

The deflection at any point can be expressed in the same manner as in Case 3 (see Equation (26)).

Case 5.—Simple Beam with Couple Applied at Any Section.—Considering the real beam first, it will be noted that at the section where the couple is applied there is an abrupt change in the bending moment. Since the shear of the shear diagram is to represent the bending moment of the real beam, this means that a concentrated load must be placed on the shear diagram at that point equal to the magnitude of the couple (see Fig. 7).

The end conditions of the mathematical beam are determined as in Case 3. Taking moments from the left end (see Fig. 7), the moment of the mathematical beam is: $M_m = \phi_L - \frac{X\,x^2}{2\,L\,EI}$; and,

$$Q_m = \phi_L\,x - \frac{X\,x^3}{6\,L\,EI} \dots\dots\dots\dots (32a)$$

in which, X, Fig. 7, is the couple in question. Taking moments from the right end, the moment of the mathematical beam is, $M_m = \phi_R + \frac{X\,x^2}{2\,L\,EI}$; and, the slope of the mathematical beam is,

$$Q_m = \phi_R\,x + \frac{X\,x^3}{6\,L\,EI} \dots\dots\dots\dots (32b)$$

Equate tangents when $x = b$ in Equation (32*a*), and $x = a$ in Equation (32*b*), remembering that the sum of the reaction couples must equal the couple due to the loading. In other words, $\phi_L + \phi_R = \frac{X}{L\,EI} \times L\left(\frac{L}{2} - a\right)$; or,

$$\phi_L + \phi_R - \frac{X\,L^2}{L\,2\,E\,I} + \frac{Xa}{E\,I} = 0; \text{ and,}$$

$$\phi_L = \left(\frac{X\,b^3}{6\,L^2} + \frac{X\,a\,b^2}{2\,L^2} - \frac{X\,a^3}{3\,L^2}\right)\frac{1}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots (33a)$$

and,

$$\phi_R = \left(-\frac{X\,a^3}{6\,L^2} - \frac{X\,b\,a^2}{2\,L^2} + \frac{X\,b^3}{3\,L^2}\right)\frac{1}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots (33b)$$

The slope and deflection at any point are quickly computed, as before (in b segment):

$$\phi_x = \phi_L - \frac{X\,x^2}{2\,L\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots (34)$$

and,

$$y = \phi_L\,x - \frac{I_x}{2\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots (35)$$

Case 6.—Cantilever Beam with Concentrated Load at the End.—For Case 6, refer to Fig. 8:

$$\phi_A = \frac{M\,L}{E\,I} - \frac{P\,L^2}{2\,E\,I} = \frac{P\,L^2}{2\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots (36)$$

$$\phi_x = \frac{M\,x}{E\,I} - \frac{P\,x^2}{2\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots (37)$$

and,

$$E\,I\,y = \frac{I_x}{2} = \frac{Mx^2}{2} - \frac{Px^3}{6} \quad \ldots\ldots\ldots\ldots\ldots (38)$$

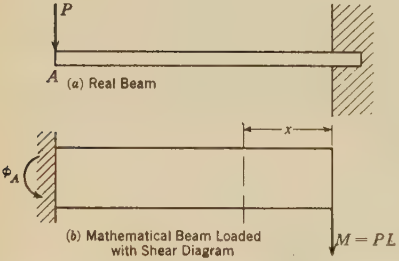

P
A
(a) Real Beam
x
ϕ_A
(b) Mathematical Beam Loaded with Shear Diagram
$M = PL$

FIG. 8

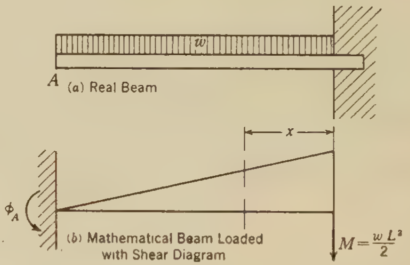

w
A
(a) Real Beam
x
ϕ_A
(b) Mathematical Beam Loaded with Shear Diagram
$M = \frac{w\,L^2}{2}$

FIG. 9

The maximum deflection occurs when $x = L$, or,

$$y_m = \frac{P\,L^3}{3\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots (39)$$

Case 7.—Cantilever Beam with Uniform Load Over Entire Span.—For this case, (see Fig. 9):

$$\phi_A = \frac{M\,L}{E\,I} - \frac{w\,L^2}{2\,E\,I} \times \frac{2}{3}\,L = \frac{w\,L^3}{6\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots (40a)$$

$$\phi_x = \frac{w\,L^2x}{2\,E\,I} - \frac{w\,(L-x)}{E\,I}\,\frac{x^2}{2} - \frac{w}{E\,I} \times x \times \frac{x}{2} \times \frac{2\,x}{3} \quad \ldots\ldots\ldots (40b)$$

and, $E\,I\,y = \frac{I_x}{2}$ (see Equation (38)) $= \frac{M\,x^2}{2} - \frac{w\,x^4}{8} - \frac{(w\,L - w\,x)}{6}\,x^3$. The maximum deflection occurs where $x = L$, or,

$$y_m = \frac{w\,L^4}{8\,E\,I} \quad \dots\dots\dots\dots\dots\dots\dots\dots (41)$$

Case 8.—Simple Beam with Variable Moment of Inertia and with Concentrated Load at Mid-Span.—The concentrated load in Fig. 10 is due to the abrupt change in the moment of inertia which makes the shear curve of the mathematical beam drop directly at that point. Its value is determined by the extra area added, due to the difference in I. Since the mathematical beam is loaded symmetrically one-half the total loading moment will go to each end, or,

$$\phi_L = \phi_R = \left[\frac{P}{2} \times \frac{L}{2} \times \frac{L}{4} + \frac{P}{2} \times \frac{L}{4} \times \frac{3}{8}L - \frac{P\,L}{8} \times \frac{L}{4}\right] \times \frac{1}{E\,I} = \frac{5}{64}\,\frac{P\,L^3}{E\,I} \;.(42)$$

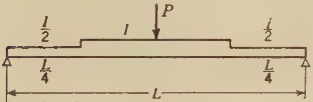
The maximum deflection can be found in the usual manner (see Equation (35)). It occurs at the section where $x = \frac{L}{2}$, or,

$$y_m = \frac{3}{128}\,\frac{P\,L^3}{E\,I} \quad \dots\dots\dots\dots\dots\dots\dots\dots (43)$$

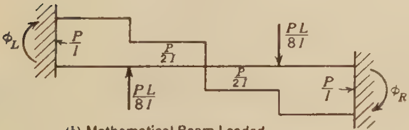
EXAMPLE OF STATICALLY DETERMINATE BEAM

As an example, consider the loading in Fig. 11. By Equation (23),

$$E\,I\,\phi_A = \sum \frac{P\,a\,b}{6\,L}\,(L + a) = \frac{2000 \times 8 \times 10 \times 26 + 1000 \times 14 \times 4 \times 32}{108} = 55\,111$$

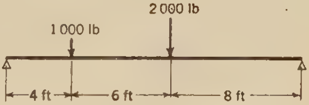


(a) Real Beam

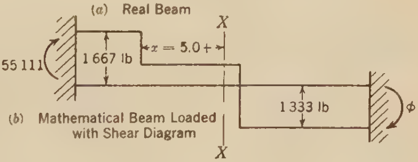


(b) Mathematical Beam Loaded with Shear Diagram, Divided by I

FIG. 10



(a) Real Beam



(b) Mathematical Beam Loaded with Shear Diagram

FIG. 11

The moment of the mathematical beam being equal to zero at the section of maximum deflection: $M = 0 = 55\,111 - 1\,667 \times 4\,(2 + x) - 667\,\frac{x^2}{2}$; and $x = 5.0 +$ (use $x = 5$).

By Equation (26),

$$y = \phi_a x - \frac{I_x}{2\,E\,I}$$

$$E\,I\,y = 55\,111 \times 9 - \frac{1}{2}\left(1\,667 \times \frac{4^3}{12} + 1\,667 \times 4 \times 7^2.0 + 667 \times \frac{5^3}{3}\right)$$

and,

$$y_m = \frac{314\,300}{E\,I}$$

Analysis of Statically Indeterminate Beams

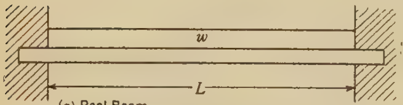
The following typical problems are given as a development of the shear-area method of solving problems applied to beams that are statically indeterminate.

Case 9.—Fixed Beam with Uniform Load Over Entire Span.—The mathematical beam for this case (see Fig. 12) has a symmetrical load and is held in equilibrium by end shears, each of which is the end moment of the real beam. In other words,

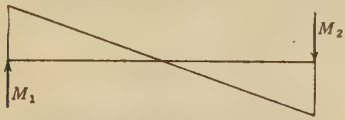
$$M_1 = \frac{w\,L}{2} \times \frac{L}{2} \times \frac{1}{2} \times \frac{2\,L}{3} \times \frac{1}{L} = \frac{w\,L^2}{12} \qquad \text{(44)}$$

The moment at the center of the beam is,

$$M_m = \frac{w\,L^2}{12} - \frac{w\,L^2}{8} = -\frac{w\,L^2}{24} \qquad \text{(45)}$$

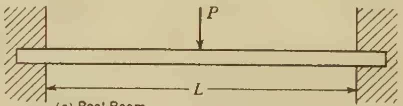


(*a*) Real Beam

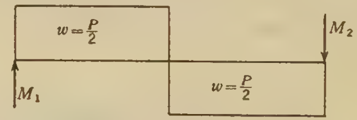


(*b*) Mathematical Beam Loaded with Shear Diagram

FIG. 12



(*a*) Real Beam



(*b*) Mathematical Beam Loaded with Shear Diagram

FIG. 13

the slope at any point is,

$$E\,I\,\phi_x = \frac{w\,L^2}{12}\,x - \frac{w\,x^3}{3} - \left(\frac{w\,L}{2} - w\,x\right)\frac{x^2}{2} = \frac{w}{12}\,(L^2\,x + 2\,x^3 - 3\,L\,x^2) \qquad \text{(46)}$$

the deflection at any point is,

$$E\,I\,y = \frac{I_x}{2} \qquad \text{(47)}$$

and the deflection is a maximum at the section where $x = \frac{L}{2}$, or,

$$y_m = \frac{1}{2\,E\,I}\left[\frac{w\,L^2}{12}\left(\frac{L}{2}\right)^2 - \frac{w\,L}{2}\left(\frac{L}{2}\right)^3\frac{1}{4}\right] = \frac{w\,L^4}{384\,E\,I} \qquad \text{(48)}$$

The points of inflection will occur where the shear of the mathematical beam is equal to zero; that is, $\frac{w\,L^2}{12} - \frac{w\,x^2}{2} - \frac{w\,L\,x}{2} + w\,x^2 = 0$; or, $x = 0.211\,L$, or $0.789\,L$.

Case 10.—Fixed Beam with Concentrated Load at Mid-Span.—For this case (see Fig. 13), the end moment is:

$$M_1 = \frac{P\,L^2}{8\,L} = \frac{P\,L}{8} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49)$$

the slope at any point is,

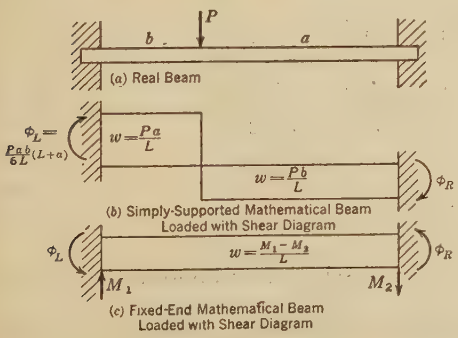
$$E\,I\,\phi_x = \frac{P\,L\,x}{8} - \frac{P\,x^2}{4} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50)$$

the deflection at any point is equal to $E\,I\,y = \frac{I_x}{2}$; and this deflection is a maximum at the section where $x = \frac{L}{2}$; that is,

$$y_m = \frac{1}{2\,E\,I}\left[\frac{P\,L}{8}\left(\frac{L}{2}\right)^2 - \frac{P}{2}\left(\frac{L}{2}\right)^3 \frac{1}{3}\right] = \frac{P\,L^3}{192\,E\,I} \quad\ldots\ldots\ldots(51)$$

A point of inflection of the real beam will occur at the section (see Fig. 13) where the shear of the mathematical beam is zero. Consequently, $\frac{P\,L}{8} - \frac{P\,x}{2} = 0$; and, $x = \frac{L}{4}$.

Case 11.—Fixed Beam with a Concentrated Load Not at the Mid-Span.—Consider the shear diagram in two parts; the first (Fig. 14(*b*)) as if the span were simply supported, and the second (Fig. 14(*c*)) with constant shear produced by the fixing moments. Since the slope at each support due to the combined effect is zero, one may solve end slopes due to a general beam carrying end moments only and equate to the end slopes of the same beam as a simply supported one carrying the load. This method then permits solving the fixing moments of the real beam.

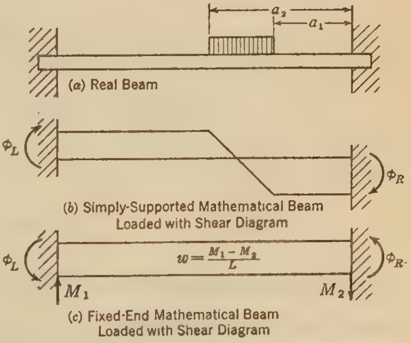


(*a*) Real Beam

(*b*) Simply-Supported Mathematical Beam Loaded with Shear Diagram

(*c*) Fixed-End Mathematical Beam Loaded with Shear Diagram

FIG. 14



(*a*) Real Beam

(*b*) Simply-Supported Mathematical Beam Loaded with Shear Diagram

(*c*) Fixed-End Mathematical Beam Loaded with Shear Diagram

FIG. 15

A derivation of the end moments of a mathematical beam loaded with constant shear (see Fig. 14(*c*)) is as follows: The moment of the mathematical beam is expressed by,

$$M_m = \phi_L - \frac{M_1 x}{E I} + \left(\frac{M_1 - M_2}{L E I}\right) \frac{x^2}{2} \quad \ldots\ldots\ldots\ldots\ldots (52)$$

and the slope of the mathematical beam is:

$$Q_m = \phi_L x - M_1 \frac{x^2}{2 E I} + \left(\frac{M_1 - M_2}{L E I}\right) \frac{x^3}{6} \quad \ldots\ldots\ldots\ldots\ldots (53)$$

Since Equation (53) equals zero when $x = L$:

$$E I \phi_L = \frac{M_1 L}{3} + \frac{M_2 L}{6} \quad \ldots\ldots\ldots\ldots\ldots (54)$$

and,

$$E I \phi_R = \frac{M_1 L}{6} + \frac{M_2 L}{3} \quad \ldots\ldots\ldots\ldots\ldots (55)$$

The combined end slope of the real beam is zero; therefore:

$$\frac{M_1 L}{3} + \frac{M_2 L}{6} = \frac{P a b}{6 L} (L + a) \quad \ldots\ldots\ldots\ldots\ldots (56)$$

$$\frac{M_1 L}{6} + \frac{M_2 L}{3} = \frac{P a b}{6 L} (L + b) \quad \ldots\ldots\ldots\ldots\ldots (57)$$

and, therefore,

$$M_1 = \frac{P b a^2}{L^2} \quad \ldots\ldots\ldots\ldots\ldots (58)$$

and,

$$M_2 = \frac{P a b^2}{L^2} \quad \ldots\ldots\ldots\ldots\ldots (59)$$

To determine the section of maximum deflection equate the moment equation of the complete mathematical beam to zero and solve for x, thus:

$$M_2 x + \left(\frac{M_1 - M_2}{L}\right) \frac{x^2}{2} - \left(\frac{P b}{L}\right) \frac{x^2}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (60)$$

$$\frac{P a b^2 x}{L^2} + \left(\frac{P a^2 b}{L^3} - \frac{P a b^2}{L^3}\right) \frac{x^2}{2} - \frac{P b}{L} \frac{x^2}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (61)$$

and,

$$x = \frac{2 a L}{2 a + L} \quad \ldots\ldots\ldots\ldots\ldots (62)$$

As in the case of the simple beam (Fig. 14(*b*)) it is interesting to note that as b varies from zero to $\frac{1}{2} L$ that x varies from $\frac{2}{3} L$ to $\frac{1}{2} L$. This means that the section of maximum deflection in this case can be displaced only 16.7% of L from the middle of the beam (compare with Case 3).

To determine the points of inflection of this beam write the general shear equation of the mathematical beam and equate to zero; solve the resulting equation for x:

$$\frac{P\,a\,b^2}{L^2} + \left(\frac{P\,a^2 b}{L^3} - \frac{P\,a\,b^2}{L^3}\right) x - \frac{P\,b\,x}{L} = 0 \;\ldots\ldots\ldots\ldots (63)$$

or,

$$x = \frac{a\,L}{L + 2\,a} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (64)$$

It is to be noted that in Equation (64), x is equal to one-half the distance to the point of maximum deflection (see Equation (62)).

Case 12.—Fixed Beam with Segment of Uniform Load at Any Position.—Applying the method in Case 11, and using the slope of the simply supported beam with a segment of uniform load as in Case 4:

$$\frac{M_1\,L}{3} + \frac{M_2\,L}{6} = \frac{w}{6\,L}\left[\frac{L^2}{2}(a^2_2 - a^2_1) - \frac{1}{4}(a^4_2 - a^4_1)\right] \ldots\ldots (65)$$

and,

$$\frac{M_1\,L}{6} + \frac{M_2\,L}{3} = \frac{w}{6\,L}\left[\frac{L^2}{2}\left\{(L - a_1)^2 - (L - a_2)^2\right\}\right.$$
$$\left. - \frac{1}{4}\left\{(L - a_1)^4 - (L - a_2)^4\right\}\right] \;\ldots\ldots\ldots\ldots\ldots\ldots (66)$$

Solving for M_1 and M_2:

$$M_2 = \frac{w}{3\,L^2}\left[\frac{L^2}{2}\left\{2\,(L - a_1)^2 - 2\,(L - a_2)^2 - a^2_2 + a^2_1\right\}\right.$$
$$\left. - \frac{1}{4}\left\{2\,(L - a_1)^4 - 2\,(L - a_2)^4 - a^4_2 + a^4_1\right\}\right] \ldots\ldots\ldots\ldots (67)$$

$$M_1 = \frac{w}{3\,L^2}\left[\left(\frac{L^2}{2}\left\{2\,(a^2_2 - a^2_1) - (L - a_1)^2 + (L - a_2)^2\right\}\right.\right.$$
$$\left.\left. - \frac{1}{4}\left\{(2\,a^4_2 - 2\,a^4_1) - (L - a_1)^4 + (L - a_2)^4\right\}\right]\right. \ldots\ldots\ldots (68)$$

and,

$$M_1 = \frac{w}{L^2}\left[\frac{L}{3}\,(a^3_2 - a^3_1) - \frac{1}{4}\,(a^4_2 - a^4_1)\right] \ldots\ldots\ldots\ldots (69)$$

Case 13.—Fixed Beam with a Couple Applied at Any Point.—Applying the method developed in Case 11, and using the slope at the end of a simply supported beam loaded with a couple, which was developed in Case 5 (see, also, Fig. 16):

$$\frac{M_1 L}{3} + \frac{M_2 L}{6} = \frac{X b^3}{6 L^2} + \frac{X a b^2}{2 L^2} - \frac{X a^3}{3 L^2} \quad \ldots\ldots\ldots\ldots (70)$$

and,

$$\frac{M_1 L}{6} + \frac{M_2 L}{3} = - \frac{X a^3}{6 L^2} - \frac{X b a^2}{2 L^2} + \frac{X b^3}{3 L^2} \quad \ldots\ldots\ldots\ldots (71)$$

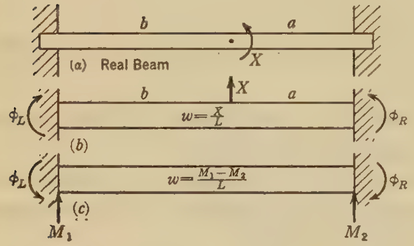


FIG. 16

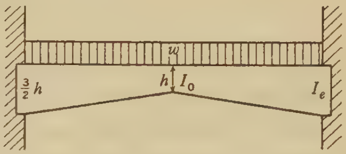


FIG. 17

From Equations (70) and (71),

$$M_2 = \frac{X}{L^3} (b^3 - 2 b a^2 - a b^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (72)$$

and,

$$M_1 = \frac{X}{L^3} (- a^3 + 2 a b^2 + a^2 b) \quad \ldots\ldots\ldots\ldots\ldots\ldots (73)$$

Case 14.—Fixed Beam with Variable Moment of Inertia and with Uniform Load Over Entire Span.—To solve the fixing moments, the beam in Fig. 17 is considered simply supported and the end slopes are found. The end moments (unknown) are then applied, bringing the end slopes to zero. The moment of inertia at any section (with the origin at the center) is,

$$I_x = \frac{I_o (L + x)^3}{L^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

The loading is expressed by the formula:

$$\frac{V}{I} - \frac{M}{I^2} \left(\frac{d I}{d x} \right) = - \frac{w x L^3}{I_o (L + x)^3} - \frac{3 w L^5}{8 I_o (L + x)^4} + \frac{3 w L^3 x^2}{2 I_o (L + x)^4} \quad \ldots (75)$$

in which I_o = moment of inertia at the center of the beam (see Fig. 17). The end slope of a simply supported beam is:

$$E \phi_L = - \frac{w L^3}{I_o} \int_0^{\frac{L}{2}} \frac{x^2 \, dx}{(L + x)^3} - \frac{3 w L^5}{8 I_o} \int_0^{\frac{L}{2}} \frac{x \, dx}{(L + x)^4} + \frac{3 w L^3}{2 I_o} \int_0^{\frac{L}{2}} \frac{x^3 \, dx}{(L + x)^4}$$

$$= \frac{w L^3}{I_o} (- 0.0167 - 0.0162 + 0.00645) = - 0.0265 \frac{w L^3}{I_o} \quad \ldots\ldots (76)$$

For a beam loaded with Moment M at the ends, the intensity of loading equals $\frac{M}{I^2}\left(\frac{dI}{dx}\right)$. Then the end slope is:

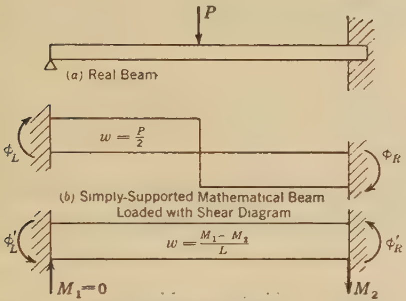
$$E\,\phi_L = \frac{M\,L}{I_e^2} + \int_0^{\frac{L}{2}} \frac{M}{I^2}\,\frac{dI}{dx}\,x\,dx = \frac{4\,M\,L}{27\,I_o} + \int_0^{\frac{L}{2}} \frac{3\,M\,x\,dx\,L^3}{I_o\,(L+x)^4} \quad \ldots\ldots(77)$$

and,

$$E\,I_o\,\phi_L = M\,L\left(\frac{4}{27} + \frac{7}{54}\right) = \frac{5}{18}\,M\,L \quad \ldots\ldots\ldots\ldots\ldots(78)$$

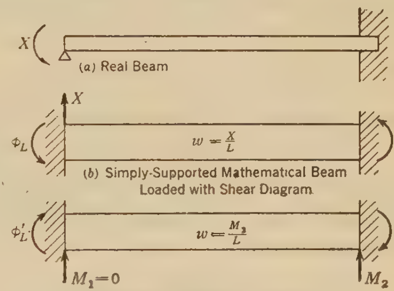
Equate Equations (76) and (78) for ϕ_L: $\frac{5}{18}\,M\,L = -\,0.0265\,w\,L^3$; and,

$$M = -0.0954\,w\,L^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(79)$$


P
(a) Real Beam
ϕ_L
$w = \frac{P}{2}$
ϕ_R
(b) Simply-Supported Mathematical Beam Loaded with Shear Diagram
ϕ'_L
$w = \frac{M_1 - M_2}{L}$
ϕ'_R
$M_1 = 0$
M_2

(c) Fixed-End Mathematical Beam Loaded with Shear Diagram

FIG. 18


X
(a) Real Beam
X
ϕ_L
$w = \frac{X}{L}$
ϕ_R
(b) Simply-Supported Mathematical Beam Loaded with Shear Diagram
ϕ'_L
$w = \frac{M_2}{L}$
ϕ'_R
$M_1 = 0$
M_2

(c) Fixed-End Mathematical Beam Loaded with Shear Diagram

FIG. 19

Case 15.—Propped Cantilever with Concentrated Load at Mid-Span.— Applying the same method as in Case 11 where $M_1 = 0$ (Fig. 18): $\frac{M_2 L}{3} = \frac{P\,L^2}{16}$; $M_2 = \frac{3}{16}\,P\,L$; and,

$$\phi_L = \frac{P\,L^2}{16\,E\,I} - \frac{M_2\,L}{6\,E\,I} = \frac{P\,L^2}{32\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots(80)$$

Case 16.—Propped Cantilever with Uniform Load Over Entire Span.— For this case: $\frac{M_2 L}{3} = \frac{w\,L^3}{24}$; $M_2 = \frac{1}{8}\,w\,L^2$; and,

$$\phi_L = \frac{w\,L^3}{24\,E\,I} - \frac{M_2\,L}{6\,E\,I} = \frac{w\,L^3}{48\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots(81)$$

Case 17.—Propped Cantilever with Concentrated Load Not at Mid-Span.— When the concentrated load in Fig. 18 is off center: $\frac{M_2 L}{3} = \frac{P\,a\,b}{6\,L}\,(L + b)$; and $M_2 = \frac{P\,a\,b}{2\,L^2}\,(L + b)$.

Case 18.—Propped Cantilever with Couple Applied at End.—Using Equations (33*b*) and (55): $\frac{M_2 L}{3} = -\frac{X L^3}{6 L^2}$, when $a = L$ and $b = 0$. Then,

$$M_2 = -\frac{X}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (82)$$

and,

$$\phi_L = \frac{M_2 L}{2 EI} + \frac{X L}{2 EI} = \frac{M_2 L}{4 EI} \quad \ldots\ldots\ldots\ldots\ldots (83)$$

Case 19.—Continuous Beam with Concentrated Loads in Each Span.—Using the general equations derived in Case 11 (Equations (54) and (55)), with the simple beam end slopes, the end slopes of any two adjoining spans (Fig. 20) are equated as follows:

$$\frac{P_1 a b}{6 L_1}(L_1 + b) - \frac{M_1 L_1}{6} - \frac{M_2 L_1}{3} = -\frac{P_2 c d}{6 L_2}(L_2 + c) + \frac{M_2 L_2}{3} + \frac{M_3 L_2}{6} \quad .(84)$$

and,

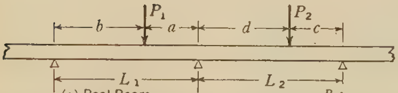
$$-M_1 L_1 - 2 M_2 (L_1 + L_2) - M_3 L_2 = -\frac{P_1 a b}{L_1}(L_1 + b) - \frac{P_2 c d}{L_2}(L_2 + c) \quad ..(85)$$

Case 20.—Continuous Beam with Uniform Load Over Each Span.—For the case of an uniformly distributed load over the entire length of a continuous beam:

$$\frac{w_1 L^3_1}{24} - \frac{M_1 L_1}{6} - \frac{M_2 L_1}{3} = -\frac{w_2 L^3_2}{24} + \frac{M_2 L_2}{3} + \frac{M_3 L_2}{6} \quad \ldots\ldots (86)$$

and,

$$-M_1 L_1 - 2 M_2 (L_1 + L_2) - M_3 L_2 = -\frac{w_1 L^3_1}{4} - \frac{w_2 L^3_2}{4} \quad \ldots\ldots (87)$$


P_1
P_2
b
a
d
c
L_1
L_2

(a) Real Beam

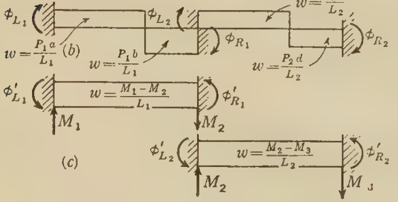
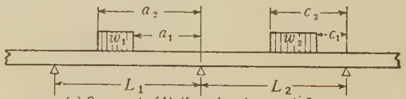
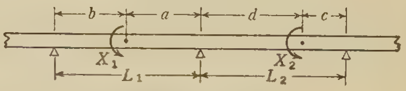

ϕ_{L_1}
ϕ_{L_2}
$w = \frac{P_2 c}{L_2}$
ϕ_{R_1}
ϕ_{R_2}
$w = \frac{P_1 a}{L_1}$
(b)
$w = \frac{P_1 b}{L_1}$
$w = \frac{P_2 d}{L_2}$
ϕ'_{L_1}
$w = \frac{M_1 - M_2}{L_1}$
ϕ'_{R_1}
M_1
M_2
(c)
ϕ'_{L_2}
$w = \frac{M_2 - M_3}{L_2}$
ϕ'_{R_2}
M_2
M_3

FIG. 20


a_2
c_2
w_1
a_1
w_2
c_1
L_1
L_2

(*a*) Segment of Uniform Load on each Span


b
a
d
c
X_1
X_2
L_1
L_2

(*b*) Couple Acting in each Span

FIG. 21

Case 21.—Continuous Beam with Section of Uniform Load in Each Span (see Fig. (21a)).—Applying the same reasoning as in Case 19, using Equations (30), (54), and (55),

$$\frac{w_1}{6\,L_1}\left[\frac{L^2_1}{2}\left\{(L_1-a_1)^2-(L_1-a_2)^2\right\}-\frac{1}{4}\left\{(L_1-a_1)^4-(L_1-a_2)^4\right\}\right]-\frac{M_1L_1}{6}$$

$$-\frac{M_2L_1}{3}=-\frac{w_2}{6\,L_2}\left[\frac{L^2_2}{2}(c^2_2-c^2_1)-\frac{1}{4}(c^4_2-c^4_1)\right]+\frac{M_2L_2}{3}+\frac{M_3L_2}{6}\;..(88)$$

Therefore:

$$-M_1L_1-2\,M_2(L_1+L_2)-M_3L_2=-\frac{w_1}{L_1}\left[\frac{L^2_1}{2}\left\{(L_1-a_1)^2-(L_1-a_2)^2\right\}\right.$$

$$\left.-\frac{1}{4}\left\{(L_1-a_1)^4-(L_1-a_2)^4\right\}\right]-\frac{w_2}{L_2}\left[\frac{L^2_2}{2}(c^2_2-c^2_1)-\frac{1}{4}(c^4_2-c^4_1)\right].(89)$$

Case 22.—Continuous Beam with Couple, X, Applied in Each Span (see Fig. 21(b)).—Applying the same method as in Case 19, using Equations (33*a*), (33*b*), (54), and (55),

$$\frac{1}{6}\left(-\frac{X\,a^3}{L^2_1}-\frac{3\,X\,a^2\,b}{L^2_1}+\frac{2\,X\,b^3}{L^2_1}\right)-\frac{M_1L_1}{6}-\frac{M_2L_1}{3}$$

$$=-\frac{1}{6}\left(\frac{X\,d^3}{L^2_2}+\frac{3\,X\,c\,d^2}{L^2_2}-\frac{2\,X\,c^3}{L^2_2}\right)+\frac{M_2L_2}{3}+\frac{M_3L_2}{6}\;......(90)$$

Therefore,

$$-M_1L_1-2\,M_2(L_1+L_2)-M_3L_2$$

$$=+\frac{X_1}{L^2_1}(a^3+3\,a^2\,b-2\,b^3)-\frac{X_2}{L^2_2}(d^3+3\,c\,d^2-2\,c^3)\;.....(91)$$

CONCLUSIONS

The shear-area method, which uses a type of elastic load, is not suggested as the shortest method for solving slopes and deflections for all problems; but it is suggested as particularly adapted for those problems involving distributed load. In the latter case the static moment and one-half the moment of inertia of the shear area are used more easily than the shear and the static moment of the curved moment area. For the concentrated loads the functions of the shear area are obtained quite as easily as those of the moment area. For a beam with varying moment of inertia the solutions by the shear and moment areas require about the same effort even though the loading for the former, $\frac{V}{I}-\frac{M}{I^2}\left(\frac{dI}{dx}\right)$, appears to be more cumbersome than the loading of the latter, $\frac{M}{I}$.

Any one who is interested in using a simpler elastic load than the shear area, or the shear area modified, can investigate the loading directly. In this case the functions of the slope and the deflection are given by the following equations,

For slope:

$$\phi_x EI = -\frac{1}{2}\int_0^x dA\, x^2 + \frac{Rx^2}{2} + M_e x + \phi_e EI \quad \ldots\ldots\ldots (92)$$

and,

$$y EI = -\frac{1}{6}\int_0^x dA\, x^3 + \frac{Rx^3}{6} + \frac{M_e x^2}{2} + \phi_e EI x + y_e EI \quad \ldots\ldots (93)$$

or,

$$\phi_x EI = -\frac{1}{2} I_x + M_e x + \phi_e EI \quad \ldots\ldots\ldots\ldots\ldots (94)$$

and,

$$y EI = -\frac{1}{6} Q_x + \frac{M_e x^2}{2} + \phi_e EI x + y_e EI \quad \ldots\ldots\ldots\ldots (95)$$

in which $\int dA = \int w dx$, or P; M_e = end moment; ϕ_e = end slope; and y_e = end deflection.

APPENDIX

NOTATION

In the following notation, presented for the convenience of reference, an effort has been made to conform as nearly as practicable with "Symbols for Mechanics, Structural Engineering, and Testing Materials" advanced by the American Standards Association[8]:

a = distance of a section from the right end of a beam ($= L - b$).
b = distance of a section from the left end of a beam ($= L - a$).
c = distance of a section (corresponding to a) from the right end of Span 2 of a continuous beam ($= L_2 - d$).
d = distance of a section (corresponding to b) from the left end of Span 2 of a continuous beam ($= L_2 - c$).
e = a subscript denoting "at the end."
m = a subscript denoting "maximum."
o = a subscript denoting "origin," or "at the center."
w = load per unit distance; load intensity per foot on a given beam.
x = a distance measured parallel to the X-axis; as a subscript, x refers to Section X–X;
y = deflection at any section, X–X, in a given beam (= slope at any section, X–X, of a mathematical beam); y_m = maximum deflection.
A = area of a loading diagram on a mathematical beam.
C = constants (see Equation (6)).
E = modulus of elasticity.

I = rectangular moment of inertia; I_x = moment of inertia of the shear diagram at any section, X–X; I_o = moment of inertia at the center of a beam; I_e = moment of inertia at the ends of a beam.

L = length; span of a given beam; as a subscript, L denotes "left end."

M = moment of force; bending moment in a given beam; M_T = total moment of force.

P = concentrated load on a given beam.

R = reactions, or resultants; as a subscript, R denotes "right end."

T = a subscript denoting "total."

V = total shear in a given beam.

X = a force couple.

ϕ = slope of the elastic curve; ϕ_L = slope at the left end of a given beam (= bending moment at the left end of a mathematical beam); ϕ_R = slope at the right end of a given beam (= bending moment at the right end of a mathematical beam); ϕ_T = total change in slope; ϕ_x = slope at any section, X–X, in a given beam (= bending moment at Section X–X, in a mathematical beam).

DISCUSSION

George E. Large,[4] Assoc. M. Am. Soc. C. E. (by letter).—The writer has often noticed that many engineers do not fully understand the relationship of the beam diagrams so often drawn. The slope diagram shown in Fig. 1 is almost a total stranger to them, yet a recognition of it as a member of the family of five curves is necessary to a full understanding of the very useful general relationships about to be mentioned. It so happens that the order of the diagrams, correct in Fig. 1, is also important, especially to the beginner, yet new textbooks are still appearing with cuts faulty in this respect. Instructors have been too facile with their calculus and too sparing with correct and complete sketches showing the geometry of the operations being performed.

It will be demonstrated herein that no new principles or devices are necessary for solving the authors' examples quickly if advantage is taken of a simple geometric relation which exists between the slope and shear diagrams, as well as between the deflection and moment diagrams.

The moment-area principles referred to by the authors are not usually applied to any diagrams except the moment and deflection diagrams. Note the restricted usage required by the usual statement of these principles:

(1) When a member is subjected to flexure, the change in slope of the elastic curve between any two points is equal in magnitude to the area of the $\frac{M}{EI}$ - diagram for the part of the member between the two points.

(2) When a member is subjected to flexure, the ordinate from any point, Q (Fig. 22), on the elastic curve, to a tangent drawn to the elastic curve at any other point, P, is equal in magnitude to the moment of the area of the $\frac{M}{EI}$ - diagram between the two points, about Point Q.

The generalized statement of these principles is much more useful because they may then be applied to any of the five beam diagrams, including the slope and shear diagrams so much used by the authors.

General Relationship of Beam Diagrams.—When the five diagrams, or curves, of any beam are arranged in the ascending order shown in Fig. 22, each curve is the integral of the one immediately below it. (The ascending order is somewhat preferable especially if the user thinks of successive integration as an ascending process.) An equation may be written for the lowest curve, successive integrations of which will yield the equations of all the higher curves, including the deflection curve. This procedure has long been used with conspicuous success by C. C. More, M. Am. Soc. C. E., in teaching structures at the University of Washington, Seattle, Wash. The relationship thus established makes it possible to state the moment-area principles in a more general form, so that their assertions may be utilized to determine all the curves completely. As stated by Professor More they are:

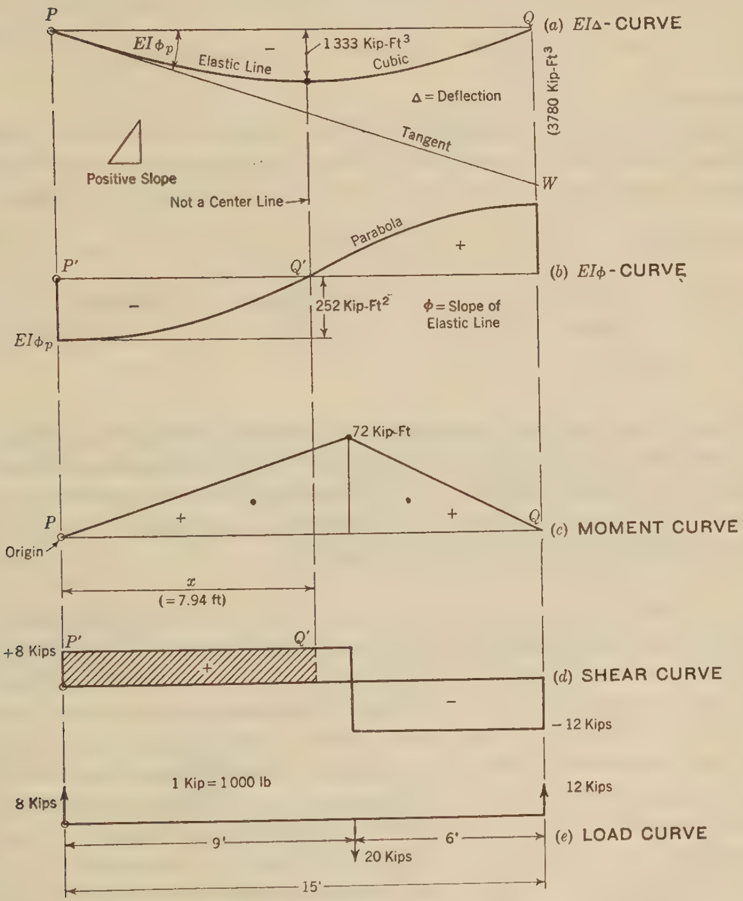
---

[4] Associate Prof., Civ. Eng., Ohio State Univ., Columbus, Ohio.

I.—The ordinate at any point on any curve equals the slope at the corresponding point in the next higher curve;

II.—The area between any two ordinates on any curve equals the difference of the corresponding ordinates in the next higher curve; and,

III.—The ordinate from any point (Point Q, Fig. 22) on any curve, to a tangent drawn to the curve at any other point (Point P) is equal in magnitude to the moment of the area of the second lower curve between the two points, about Point Q.



P
Q
(a) EIΔ-CURVE
EIφp
−
Elastic Line
1 333 Kip-Ft³
Cubic
Δ = Deflection
(3780 Kip-Ft³
Tangent
W
Positive Slope
Not a Center Line
Parabola
+
P′
Q′
(b) EIφ-CURVE
−
252 Kip-Ft²
φ = Slope of Elastic Line
EIφp
72 Kip-Ft
P
+
+
Q
(c) MOMENT CURVE
Origin
x
(= 7.94 ft)
+8 Kips
P′
Q′
+
(d) SHEAR CURVE
−
− 12 Kips
1 Kip = 1 000 lb
12 Kips
8 Kips
(e) LOAD CURVE
9′
6′
20 Kips
15′

FIG. 22.—COMPLETE BEAM SOLUTION BY MOMENT-AREA PRINCIPLES.

In accordance with the foregoing laws, if the ordinates of one curve are positive and increase from left to right, the slope of the next higher curve will also be positive at that point and will increase from left to right. Corresponding statements may be made for negative ordinates, and for ordinates that decrease from left to right.

These principles enable one to sketch the curves of any beam in their correct relation, one after the other, and to evaluate their maximum points. In Fig. 22, a numerical example of the authors' Case 3 (Fig. 5), the uniform ordinates of the shear curve determine the uniform slopes of the moment curve (Law I). By Law II, the maximum bending moment may be found from the shear curve. The increasing positive ordinates of the left portion of the moment curve call for an increasing positive slope to the $E\,I\,\phi$-curve directly above it (Law I), $E\,I$ being constant. Upon drawing the $E\,I\,\Delta$-curve by inspection it is seen that the slope of the elastic line is negative at the left end, or that the Y-intercept on the $E\,I\,\phi$-curve is negative. The value of this intercept is determined as usual, by applying Law III to the moment and deflection curves; Thus, $Q\,W = 36 \times 6 \times 4 + 36 \times 9 \times 9 = 3\,780$ kip-ft^3; and $E\,I\,\phi_P = \frac{3\,780}{15} = 252$ kip-ft^2.

The point of maximum deflection will be where the $E\,I\,\phi$-curve crosses the X-axis and this point is easily found without resort to any new principles by applying Law III to the slope and shear curves. Taking moments of the shear area about the unknown point, Q': $8\,(x)\left(\frac{x}{2}\right) = E\,I\,\phi_P = 252$; and $x = 7.94$ ft. Furthermore, by Law II, the maximum deflection is the area of the $E\,I\,\phi$-diagram: $E\,I\,\Delta_{\text{max.}} = \frac{2}{3} \times 252 \times 7.94 = 1\,333$ kip-ft^3; and, $\Delta_{\text{max.}} = \frac{1\,333 \text{ kip-ft}^3}{E\,I}$.

Statically indeterminate beams can also be solved with corresponding facility, as compared to the authors' expressions.[5] The writer cannot refrain from suggesting that the introduction of an equivalent "mathematical beam" device may easily be a source of confusion, which is quite unnecessary, once the general relationship of all curves is recognized.

SAMUEL T. CARPENTER,[6] JUN. AM. SOC. C. E. (by letter).—Moment areas have proved their usefulness and shear areas are no doubt a unique application of moment-area principles. However, the writer doubts if "mathematical beams" are to be preferred to real beams. Engineers have used area moments as a tool, the principles being easily retained and recalled when the need arises; only two laws must be remembered. The shear-area method has at least five relations which must be remembered or reviewed before using. The authors suggest that for a student the "mathematical beam" will aid his knowledge of beam action; the writer would make an exception to this since area moments must necessarily be understood before shear areas could be grasped. It will be granted that teachers of applied mechanics have been lax in the past by not calling to the attention of students the relation of the third, second, and first derivatives of the elastic line, or the shear, moment, and slope curves as depicted in Fig. 1. C. C. More, M. Am. Soc. C. E., has taught those relations successfully at the University of Washington, at Seattle, Wash., without the aid of the "mathematical beam."

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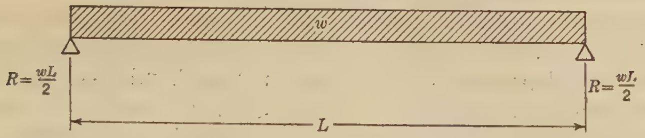
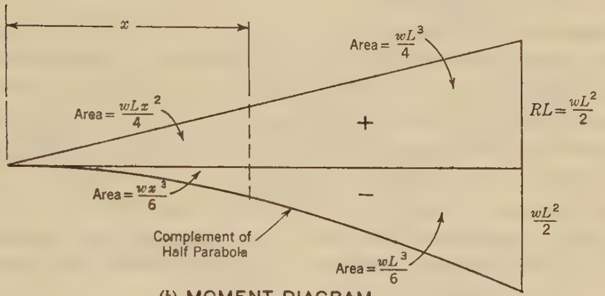
[5] See Appendix of *Bulletin 66*, Ohio State Univ. Eng. Experiment Station, "The Moment Distribution Method of Structural Analysis Extended to Lateral Loads and Members of Variable Cross-Section" (Revised Edition), by George E. Large, Assoc. M. Am. Soc. C. E., and Clyde T. Morris, M. Am. Soc. C. E.

[6] Instr., Civ. Eng., Swarthmore Coll., Swarthmore, Pa.



(a) LOAD DIAGRAM



(b) MOMENT DIAGRAM

FIG. 23.

The authors suggest that the shear-area method is particularly adaptable to problems involving a distributed load, stating that the shear areas are used more easily than the curved moment areas. This is true with the moment diagrams as ordinarily used. For some time the writer has followed the method suggested by Mr. James E. Boyd[7] for treating uniform loads, which places the parabola involved in a form easier to apply. To illustrate, take the authors' Case 1. Fig. 23 indicates how the moment diagram would be drawn, the diagram being divided into positive and negative areas; the curved negative area is the complement of a half parabola. This is convenient when finding deflections at any point on the beam. The end slopes are: $2\,EI\phi = \frac{wL^3}{4} - \frac{wL^3}{6} = \frac{wL^3}{12}$; and $\phi = \frac{wL^3}{24\,EI}$. The deflection at any point a distance, x from the left end is: $EIy = \frac{wL^3}{24}\,x - \left(\frac{wLX^2}{4} \times \frac{X}{3} - \frac{wX^3}{6} \times \frac{X}{4}\right)$; and $y = \frac{wX}{24\,EI}\,(L^3 - 2\,LX^2 + X^3)$.

This means of building up the moment curves is equally applicable to cases of concentrated loads and continuous beams, the component curves always being easily drawn by starting from the left end and considering all the loading elements.

---

[7] "Strength of Materials," by James E. Boyd, McGraw-Hill Publishing Co., 1924.

ROLAND H. TRATHEN,[8] ESQ. (by letter).—The "shear-area method," presented by Professor Compton and Mr. Dohrenwend, supplies a new and novel procedure for the solution of the elastic functions of beams. There are many cases, of course, in which the method is not advantageous. This, in itself, in no way detracts from the usefulness of the concept.

The writer looks upon the method as a new tool to facilitate structural analysis. No one method of analysis is universal in scope and simplicity. The methods of "slope deflection", "moment areas", "the column analogy", "moment distribution", "internal work", etc., are all available to the structural engineer. Each of them is peculiarly adapted to the solution of particular problems. The structural engineer who is familiar with all methods is usually able to decide before beginning an analysis, which is likely to prove to be the most expedient.

In presenting, "the shear-area method", to the profession the authors have dealt quite thoroughly with beams. In the space of a few pages they have derived equations for slope and deflection, for a variety of beams, each under several conditions of loading. Such a wealth of beam functions if available for immediate reference may prove to be very useful. The same information may be found in any standard textbook on the strength of materials, but the writer has never been able to find all the given data in such condensed form. Students as well as structural engineers may well look upon the paper as a convenient reference table of slopes and deflections to be used in the solution of more complex problems.

The following simple problems are included to illustrate the usefulness of the paper when applied in this manner. The notation of the original paper is preserved throughout, with the following exceptions: ϕ_{BC} = partial slope at End B of Member BC; and ϕ_{TBC} = total slope at End B of Member BC.

Problem 1,—An L*-Frame with Concentrated Load Applied at Mid-Point of Deck.*—The structure shown in Fig. 24 is indeterminate to the first degree. In Fig. 25 it has been reduced to a determinate structure by cutting it at

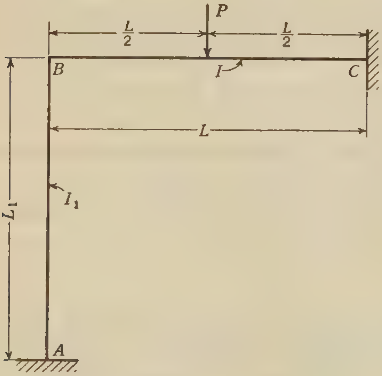


FIG. 24.

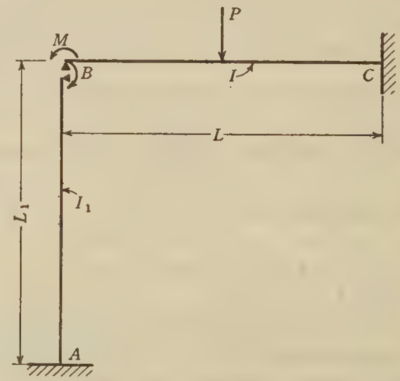


FIG. 25.

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[8] Instr. in Civ. Eng., Rensselaer Polytechnic Inst., Troy, N. Y.

Point B. The determination of the moment, M, will be sufficient to make the structure statically determinate. The solution for the unknown moment at Joint B is as follows, using directly the slopes as derived in the paper: By Equation (80), the slope, ϕ_{BC}, due to the concentrated load, P, is $\frac{P L^2}{32 E I}$; by Equation (83), the slope, ϕ_{BC}, due to the bending moment, M, is $-\frac{M L}{4 E I}$; and the total slope, ϕ_{TBC} equals $\frac{P L^2}{32 E I} - \frac{M L}{4 E I}$.

By Equation (83) again, the total slope, ϕ_{TBA}, due to the bending moment $= -\frac{M L_1}{4 E I_1}$. Since $\phi_{TBC} = -\phi_{TBA}$, it follows that $\frac{P L^2}{32 E I} - \frac{M L}{4 E I} = \frac{M L_1}{4 E I_1}$; and,

$$M = \frac{P L^2}{8}\left(\frac{I_1}{L_1 I + L I_1}\right) \dots\dots\dots\dots\dots\dots (96)$$

Problem 2.—Two-Span Continuous Beam with Uniform Load on One Span and Concentrated Load at Center of Other Span.—The structure in Fig. 26 is statically indeterminate to the first degree. The solution for the moment over the support at Point B will make it statically determinate. The beam is reduced to a simple structure by cutting it as shown in Fig. 27. Then, by Equation (14), the slope, ϕ_{BA}, due to a concentrated load, P, on Span L, is $\frac{P L^2}{16 E I}$; by Equation (33) (when $a = 0$ and $b = L$), the slope, ϕ_{BA}, due to the bending moment, M, on Span L, is $-\frac{M L}{3 E I}$; and, as in Problem 1, the total slope, ϕ_{TBA}, is the sum, or $\frac{P L^2}{16 E I} - \frac{M L}{3 E I}$. Finally, by Equation (8), the slope, ϕ_{BC}, due to a uniform load, w, on Span L_1, is $\frac{w L^3_1}{24 E I}$; by Equation (33) (when $b = 0$ and $a = L$), the slope, ϕ_{BC}, due to a moment, M, on Span L_1, is $-\frac{M L_1}{3 E I}$; and, the total slope, ϕ_{TBC}, is $\frac{w L^3_1}{24 E I} - \frac{M L_1}{3 E I}$. Since $\phi_{TBA} = -\phi_{TBC}$, it follows that: $\frac{P L^2}{16 E I} - \frac{M L}{3 E I} = -\frac{w L^3_1}{24 E I} + \frac{M L_1}{3 E I}$ and,

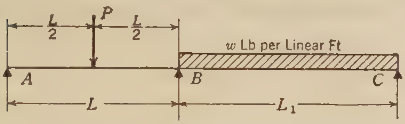

FIG. 26.

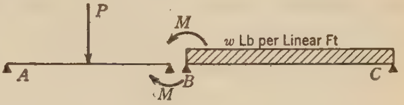

FIG. 27.

$$M = \frac{3 P L^2}{16 (L + L_1)} + \frac{w L^3_1}{8 (L + L_1)} \dots\dots\dots\dots\dots\dots (97)$$

Problem 3.—Continuous Beam of Two Equal Spans, with Concentrated Load at Center of One Span.—The structure in Fig. 28(*a*) is indeterminate to the first degree. In this case the reaction at Point *B* will be chosen as the redundant. With the reaction at this point known, the structure is statically determinate, as shown in Fig. 28(*b*) and the shear diagram for this case is shown in Fig. 28(*c*). By Equation (17), the deflection, y_B, due to a unit load at Point *B*, is $\frac{L^3}{6\,EI}$; and, by Equation (24), the slope, ϕ_R, due to a concentrated load, *P*, is,

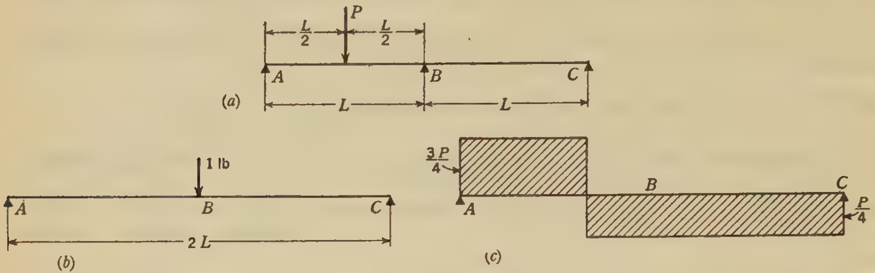

FIG. 28.

$$\phi_R = \frac{P\,a\,b}{6\,E\,I\,L}(L+b) = \frac{P \times \frac{3}{2}\,L \times \frac{1}{2}\,L}{12\,E\,I\,L}\left(2\,L + \frac{1}{2}\,L\right) = \frac{5\,P\,L^2}{32\,E\,I} \quad .(98)$$

Then, by Equations (7), and (9) the deflection, y_B, due to a concentrated load, *P*, is,

$$y_B = \phi_R\,x - \frac{1}{2}\,I_x = \frac{5\,P\,L^2}{32\,E\,I} \times L - \frac{1}{2} \times \frac{P}{4} \times L^3 \times \frac{E\,I}{3}$$

$$= \frac{5\,P\,L^3}{32\,E\,I} - \frac{P\,L^3}{24\,E\,I} = \frac{11\,P\,L^3}{96\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots(99)$$

and, the reaction is,

$$R_B = \frac{y_B\text{ (due to Load }P)}{y_B\text{ (due to load of unity)}} = \frac{11\,P\,L^3}{96\,E\,I} \times \frac{6\,E\,I}{L^3} = \frac{11\,P}{16} \quad \ldots\ldots(100)$$

A. W. FISCHER,[9] ESQ. (by letter).—An interesting method of calculating the deflection of beams is offered in this paper, but as stated in the "Conclusions," it is not suggested as the shortest method for solving slopes and deflections for all problems. This is a true statement because there are many problems in which slopes and deflections can be solved in less time than is required by the shear-area method.

As an example the writer will calculate the maximum deflection of the shaft shown in Fig. 29 by the shear-area method, to demonstrate that it is not the shortest. Furthermore, it is more complex than a comparable method presented by the writer[10] in that a simple differential equation must be used to obtain the required results, which may not appeal to every engineer.

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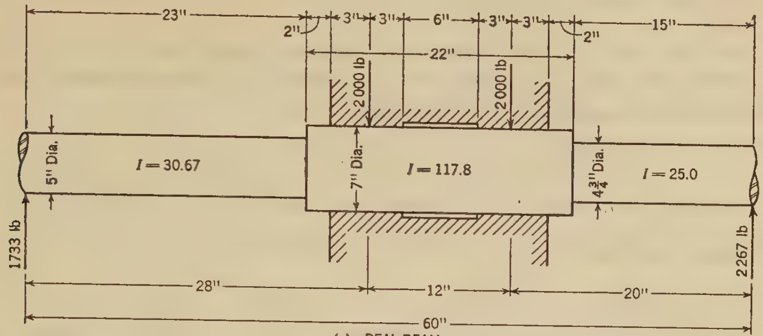
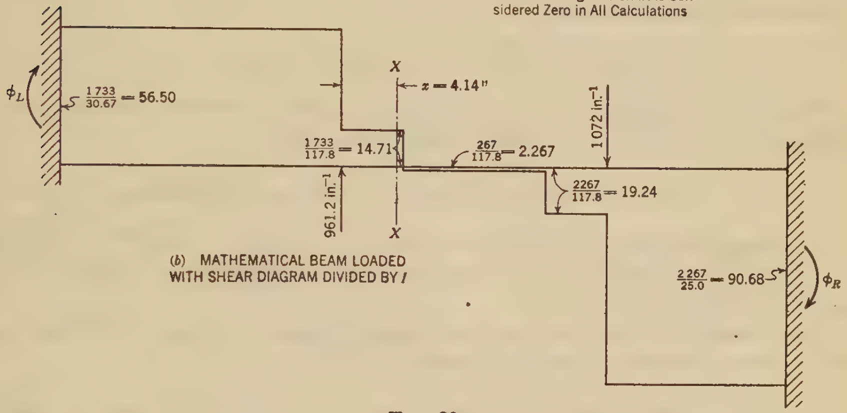
[9] Care, Pennsylvania Sugar Co., Philadelphia, Pa.


23"
2"
3"
3"
6"
3"
3"
2"
15"
22"
2 000 lb
2 000 lb
5" Dia.
I = 30.67
7" Dia.
I = 117.8
4¾" Dia.
I = 25.0
1733 lb
2267 lb
28"
12"
20"
60"

(a) REAL BEAM

Note: The Weight of Shaft is Considered Zero in All Calculations


φ_L
1733/30.67 = 56.50
X
x = 4.14"
1733/117.8 = 14.71
267/117.8 = 2.267
1 072 in.⁻¹
961.2 in.⁻¹
X
2267/117.8 = 19.24
2 267/25.0 = 90.68
φ_R

(b) MATHEMATICAL BEAM LOADED WITH SHEAR DIAGRAM DIVIDED BY I

FIG. 29.

As E is a constant it will be assumed equal to unity which will somewhat simplify the equations. To determine the end conditions of the mathematical beam the general moment equation with the left end as the origin, is,

$$\frac{d^2y}{dx^2} = \phi_L - \frac{1}{2}(56.50\,x^2) - \frac{1}{2}(14.71\,x^2) + 961.2\,x \quad \ldots\ldots.(101)$$

Integrating Equation (101):

$$6\,\frac{dy}{dx} = 6\,\phi_L\,x - 56.50\,x^3 - 14.71\,x^3 + 3\,(961.2\,x^2) \quad \ldots\ldots.(102)$$

The moment equation as derived from the right end of the mathematical beam, is:

$$\frac{d^2y}{dx^2} = \phi_R - \frac{1}{2}(90.68\,x^2) - \frac{1}{2}(19.24\,x^2) - \frac{1}{2}(2.267\,x^2) + 1\,072\,x \quad .(103)$$

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[10] "Shaft Deflections by the Method of Elastic Weights", by A. W. Fischer, *Product Engineering*, November, 1933, p. 428.

Integrating Equation (103):

$$6\,\frac{dy}{dx} = 6\,\phi_R\,x - 90.68\,x^3 - 19.24\,x^3 - 2.267\,x^3 + 3\,(1\,072\,x^2)\quad ..(104)$$

Equate, six times, the two tangents of the mathematical beam at a point 28 in. from the left end, since the deflection of the real beam can be expressed by both equations at that point, thus:

$$6\,\phi_L\,x\Big]_0^{28} - 56.50\,x^3\Big]_5^{28} - 14.71\,x^3\Big]_0^5 + 3\,(961.2\,x^2\Big]_0^5$$

$$= 6\,\phi_R\,x\Big]_0^{32} - 90.68\,x^3\Big]_{17}^{32} - 19.24\,x^3\Big]_{12}^{17} - 2.267\,x^3\Big]_0^{12} + 3\,(1\,072\,x)^2\Big]_0^{17}\quad .(105)$$

Substituting the limits of x and reducing Equation (105):

$$28\,\phi_L - 206\,700 + 1\,177 - 306 + 12\,020$$

$$= 32\,\phi_R - 495\,200 + 74\,250 - 15\,750 + 5\,541 - 653 + 154\,900.(106)$$

Reducing Equation (106):

$$28\,\phi_L - 193\,809 = 32\,\phi_R - 276\,912..............(107)$$

From the mathematical beam the total, ϕ_T, equals 33 477, and using the notation, $\phi_R = \phi_T - \phi_L$; then, from Equation (107):

$$28\,\phi_L - 193\,809 = 32(33\,477) - 32\,\phi_L - 276\,912........(108)$$

Reducing Equation (108), $\phi_L = 16\,470$; and $\phi_R = 33\,477 - 16\,470 = 17\,007$. (All computations in this discussion were made with a 20-in. slide-rule.)

The moment of the mathematical beam is equal to zero at the section of maximum deflection, thus, at Section X–X:

$$M = 0 = 16\,740 - 56.50 \times 23\,(11.5 + x) - \tfrac{1}{2}\,(14.71\,x^2) + 961.2\,x.(109)$$

From Equation (109) $x = 4.14$ in., which means that the point of maximum deflection of the real beam occurs 27.14 in. to the right of the left reaction (see Fig. 29).

Using the notation of the paper: $y_{\text{max.}} = \phi_L x - \frac{1}{2}\,I_x$, in which $x = 27.14$ in.; or, $y_{\text{max.}} = 16\,470(27.14) - \frac{1}{2}\left[(56.50)\,\frac{23^3}{12} + 56.50\,(23)\,(15.64^2) - 961.2\,(4.14^2) + \frac{14.71\,(4.14^3)}{3}\right] = 447\,000 - 179\,500 = 267\,500$ in. when E equals unity. When E equals 29 000 000, the maximum deflection equals $\dfrac{267\,500}{29\,000\,000} = 0.009224$ in.

The deflection at a point 28 in. to the right of the left reaction from the left-hand part of Equation (107), is $28(16\,470) - 193\,809 = 267\,391$ in. when E equals unity, and when E equals $29(10^6)$, the deflection equals 0.009220.

From the foregoing computations the deflection at the point of zero shear of the real beam is found to be just a little less than the maximum deflection. On comparison with the solution of this problem which is preferred by the writer[10] it does not seem that the foregoing solution of the maximum deflection by the shear-area method is the shortest.

If Fig. 29 had been loaded with uniformly distributed loads of different values, then to solve for the maximum deflection the shear-area method would have been the shortest, and as the authors state it is suggested as particularly adapted to those problems involving distributed loads.

The shaft as given in the example actually has uniformly distributed loads of different values, but for the purpose of comparison it was considered as having no weight.

J. Charles Rathbun,[11] M. Am. Soc. C. E. (by letter).—In elementary textbooks on strength of materials the development of the beam theory is given, usually starting with the authors' Equation (2) and deriving the slope and deflection curves by direct integration. It is also shown that the loading, shear, and moment curves are related in the same manner as the moment divided by EI, the slope and deflection curves. It follows at once, with the usual assumptions of the beam theory and for a constant moment of inertia, that the five curves in Fig. 30 form a series in which each succeeding curve may be obtained by direct integration.

If the method of obtaining the equation of the $n + 1$-curve of this series from the n-curve is the same whether n is 1, 2, 3, or 4, one can obtain the fifth curve from the second one in the same way that he obtained the fourth from the first; or, he may obtain the fourth from the second in the same way that the third is obtained from the first. In each case the constants of integration must be taken care of by other means than those of pure mathematics. However, as shown by the authors in Equation (1), if the moment of inertia is not constant, the curves of Fig. 30(a) and Fig. 30(b) of the series do not have the same physical meaning when divided by the variable, I, as they have in the series of the previous paragraph. They must be considered as the first and second derivatives of the $\frac{M}{EI}$-equation. The (d)-curve is the slope, whereas the (e)-curve is the deflection. With this

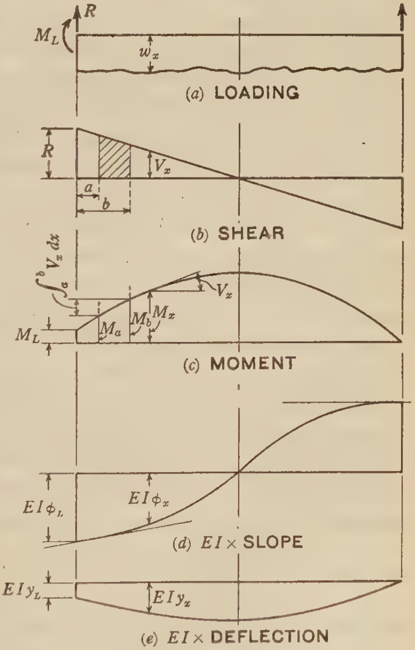

Fig. 30.

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[11] Associate Prof., Civ. Eng., Coll. of the City of New York, New York, N. Y.

second series equations are related to each other in the same way that they are in the first.

In order to take advantage of these well-known facts, one must keep the signs consistent as well as the units. The writer feels that inconsistencies have occurred in the several basic formulas derived by the authors and he has taken the liberty to re-state Equations (6), (7), (92), (93), (94), and (95), giving the assumptions under which the re-stated formulas are derived. It is expected that some of the differences between the two sets of formulas can be explained by differences of definitions both of terms and signs. It is hoped that the authors will clarify this phase of their paper in their closing discussion and will state definitely their assumptions as to signs. The writer has used the convention of analytic geometry that a dimension is positive if measured to the right or upward from the origin, whereas a force is positive if it acts in a positive direction—that is, upward. The origin is assumed at the left end of the beam. Curves (*a*) and (*d*) of Fig. 30 differ from those of Fig. 1 in sign possibly because of these assumptions. In the case of a sudden change in the formula of the load, or in the moment of inertia, the origin can often be shifted to advantage.

In Fig. 30 the area bounded by the curve and any two ordinates is equal to the difference between the corresponding ordinates in the next lower curve. This is the result of the method of obtaining the equation of the curve, by integration. It also follows that the units used in measuring the ordinate of any curve are those of the curve from which it is derived multiplied by feet; whence, if the ordinates of the loading curve (Fig. 30(*a*)) are in pounds per foot, those of the shear curve (Fig. 30(*b*)) are in units of pounds, and those of the moment curve (Fig. 30(*c*)) are in pound-feet; the $E\,I\,\phi$-curve (Fig. 30(*d*)) are in pound-feet^2; and, those of the $E\,I\,y$-curve (Fig. 30(*e*)) are in pound-feet.^{3} If one elects to divide by $E\,I$ and so use the second series, as the y-curve ordinates must be in feet, the units for the other curves follow at once with the (*d*)-curve ordinates being non-dimensional, the (*c*)-curve in feet^{-1}, the (*b*)-curve, or mathematical curve, in feet^{-2}, and the (*a*)-curve in feet^{-3}. Each term of the corresponding equation derived should conform to one or other of these systems of units, or the reason for not doing so should be evident. In any case each equation should be self-consistent in its units.

It has been shown that the relationship between Fig. 30(*b*) and Fig. 30(*d*) is the same as that between Fig. 30(*a*) and Fig. 30(*c*). In the first series the equation for Fig. (30*c*) is obtained (from the definition of moment) from that of Fig. 30(*a*) for the moment, $M_x = R\,X$, plus the moment of the loads about Point X (the load, w, is negative and this load moment, therefore, is negative in value). If x is taken as the distance from the load, $w\,dx$, to the point, X: $M_x = R\,X + \int_0^x w\,x\,dx$. It follows, then, from Fig. 30(*b*), that the equation for Fig. 30(*d*) is:

$$E\,I\,\phi_x = E\,I\,\phi_L + \int M\,dx = E\,I\,\phi_L + I\,A\bar{x} \ldots\ldots\ldots (110)$$

or $\phi_x = \phi_L + \frac{A\bar{x}}{E}$, in which $A\bar{x}$ is the first moment of the area of the shear, divided by the I-curve (or mathematical beam curve) between $x = 0$ and $x = X$ about Point X. In this case, M_L is assumed to be zero. In the example shown in Fig. 30, ϕ_L, w, and, therefore, $\int_0^x w\,x\,dx$, are all negative in value whereas $A\bar{x}$ and $R\,X$ are positive. Equation (110) the writer feels should be used instead of the authors' Equation (6). Equations (6) and (7) apply only when $E\,I$ is a constant. If it is desired to consider I as a variable, Equation (110) becomes:

$$\phi_x = \phi_L + \frac{1}{E}\int_0^x \frac{M}{I}\,dx \ldots\ldots\ldots\ldots\ldots\ldots (111)$$

or $\phi_x = \phi_L + \frac{A\bar{x}}{E}$. Integrating Equation (110) gives,

$$E\,I\,y = E\,I\,y_L + E\,I\int_0^X \phi_x\,dx = E\,I\,y_L + E\,I\,\phi_L X + I\int_0^X A\bar{x}\,dx$$

$$= E\,I\,y_L + E\,I\,\phi_L X + I\iint x\,dA\,dx$$

and dividing by the constant, $E\,I$,

$$y = y_L + \phi_L\,X + \frac{I_x}{2\,E} \ldots\ldots\ldots\ldots\ldots\ldots (112)$$

In Equation (112), I_x is taken as the second moment or moment of inertia of the area under the mathematical-beam curve between the ordinates, $x = 0$ and $x = X$, about the ordinate throught Point X. The value of the authors' $\int A\bar{x}\,dx = \frac{I_x}{2}$ presents an interesting and a novel idea. The proof is quite simple. Equation (112) corresponds to the authors' Equation (7) when y_L is zero. If I is a variable, $y = y_L + \phi_L X + \frac{1}{E}\iint \frac{M}{I}\,dx^2$ $= y_L + \phi_L X + \frac{I_x}{2\,E}$, as in Equation (112).

Apparently, the authors have assumed that EI is a constant in deriving Equations (92) to (95). The writer has obtained the following equations under the same assumption:

$$E\,I\,\phi_x = E\,I\,\phi_L + M_L X + \frac{R\,X^2}{2} + \iiint w\,dx^3$$

$$= E\,I\,\phi_L + M_L X + \frac{R\,X^2}{2} + \frac{1}{2}\int x^2\,(w\,dx)$$

and,

$$E I y = E I y_L + E I \phi_L X + M_L \frac{X^2}{2} + R \frac{X^3}{6} \int\int\int\int w \, dx^4$$

$$= E I y_L + E I \phi_L X + M_L \frac{X^2}{2} + \frac{R X^3}{6} + \frac{1}{6} \int x^3 \, (w \, dx)$$

in which such terms as $\int x^2 \, (w \, dx)$ and $\int x^3 \, (w \, dx)$ may be interpreted as the second and third moments of the area under the load curve between $x = 0$ and $x = X$ about the ordinate through Point X. As R may be considered as a part of the load, and writing dW to represent the load whether concentrated as R, or distributed as $w \, dx$:

$$E I \phi_x = E I \phi_L + M_L X + \frac{1}{2} \int x^2 \, dW$$

and,

$$E I y = E I y_L + E I \phi_L X + \frac{M_L X^2}{2} + \frac{1}{6} \int x^3 \, dW$$

As dA and I_x are given definitions relative to the mathematical and shear curves and Q has not been defined by the authors these notations have not been incorporated in these equations. The subscript, L, for "left end" used by the authors is used in this discussion instead of the less definite, e, as the origin is always placed at the left end.

The writer takes issue with the authors in the case of the equations in a number of the examples particularly as to their inconsistency in units and signs. This inconsistency makes it practically necessary to re-work the examples in order to connect their application with the ordinary beam theory and through them to connect the area-moment method with this theory. It is hoped that in their closing discussion the authors will clear up these points so that their work can be followed more readily.

Harold R. Kepner,[12] Assoc. M. Am. Soc. C. E. (by letter).—Another analogy is presented in this paper, whereby structural problems involving the solution of the elastic curve equation may be simplified. Like the moment-area method and the column analogy it makes use of a dummy loading on a dummy structure, certain functions of which loading equal the desired functions of a real load on a real structure.

So far as the writer knows the analogy of the shear-area method has not been presented before, although the difficulties involved in the use of the moment area for distributed loads have indicated the desirability of some method involving the simpler shear curve.

A study of this paper has suggested a slightly different statement of the method as presented by the authors which seems more nearly to approach the "conjugate beam" method in its procedure. In spite of its limitations, the writer believes a brief presentation of this procedure may be worth while.

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[12] Associate Prof. of Eng., Utah State Agri. Coll., Logan, Utah.

Since the advantages of this analogy over those of the conjugate beam analogy seem to disappear for beams of varying cross-section, this procedure will be presented for the case of prismatic beams only, although its use is perfectly general if not always simple of computation.

Statement.—If a dummy beam is loaded with the shear diagram from some real loading on a real (prismatic) beam, the following relations of load functions exist:

(1) The shear at a given section of the dummy beam equals the bending moment at the corresponding section of the real beam;

(2) The bending moment at a given section of the dummy beam, divided by EI (modulus of elasticity and moment of inertia of a section of the real beam, respectively), equals the slope at the corresponding section of the real beam; and

(3) The second moment of the forces on one side of a given section of the dummy beam, divided by $2\,EI$, equals the deflection at the corresponding section of the real beam.

That these relations are true may be seen by a study of Equations (1) to (7). By "second moment" is meant what is commonly called "moment of inertia" and is computed as follows:

(*a*) For an area, the expressions for the moment of inertia of an area are applicable, are available in handbooks and textbooks, and, for simple figures, have been memorized by most engineers;

(*b*) For a single force, it is the product of the force by the square of the perpendicular distance from the force to the point in question; and

(*c*) For a couple, it is the product of the moment of the couple by twice the distance from its point of application to the point in question. That this is true for a couple may be shown as follows: Referring to Fig. 31, the second moment of the couple, $M = F\,a$ about Point O equals,

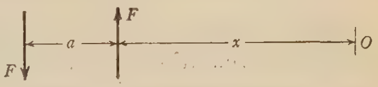


FIG. 31.

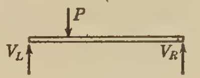
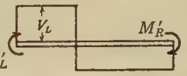
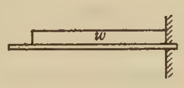
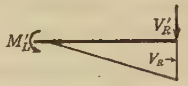
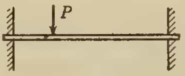
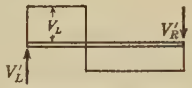
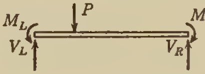
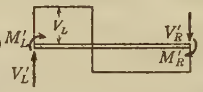
$$I_0 = F\,(a + x)^2 - F\,x^2 = F\,a^2 + 2\,F\,a\,x + F\,x^2 - F\,x^2 = F\,a\,(a + 2\,x)$$
$$= 2\,x\,F\,a = 2\,x\,M \qquad (113)$$

(Making a very small and F very large, without changing the moment, $M = F\,a$.)

The reactions at the supports of the dummy beam are either forces, couples, or a combination of forces and couples, and are determined by the conditions at the corresponding supports of the real beam. Table 1 shows how they are determined for several common types of beam. The symbols, V, M, ϕ, and y, represent the shear, bending moment, slope, and deflection at the given section of the real beam, and the primed symbols, V', M', and I', represent the shear, bending moment, and second moment, respectively, at the corresponding section of the dummy beam.

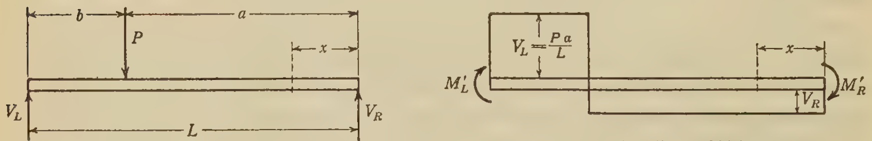
The signs used for both beams correspond to the convention in which upward forces on the left produce positive shear, and a beam bent concavely upward has positive bending moment. With a positive shear diagram taken as a downward load on the dummy beam, the sign of the second moment is the product of the sign of the bending moment by the sign of x and corresponds to a positive slope on the real beam, down to the right, and to positive deflection downward. In order to illustrate the procedure several of the cases presented by the authors will serve as examples.

TABLE 1.—COMPUTATIONS OF REACTIONS FOR DUMMY BEAM

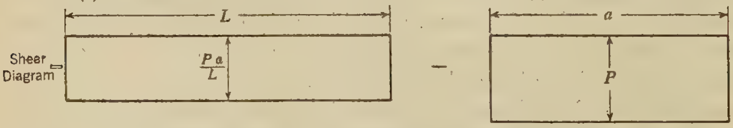
Type of Real Beam	REAL BEAM			DUMMY BEAM		
	Conditions at Left End		Conditions at Right End	Conditions at Left End		Conditions at Right End
Simple	$M_L = 0$ $\phi_L \neq 0$ $y_L = 0$		$M_R = 0$ $\phi_R \neq 0$ $y_R = 0$	$V'_L = 0$ $M'_L \neq 0$ $I'_L = 0$	 Reactions = 2 Couples	$V'_R = 0$ $M'_R \neq 0$ $I'_R = 0$
Cantilever	$M_L = 0$ $\phi_L \neq 0$ $y_L \neq 0$		$M_R \neq 0$ $\phi_R = 0$ $y_R = 0$	$V'_L = 0$ $M'_L \neq 0$ $I'_L \neq 0$	 Reactions = 1 Force, 1 Couple	$V'_R \neq 0$ $M'_R = 0$ $I'_R = 0$
Fixed	$M_L \neq 0$ $\phi_L = 0$ $y_L = 0$		$M_R \neq 0$ $\phi_R = 0$ $y_R = 0$	$V'_L \neq 0$ $M'_L = 0$ $I'_L = 0$	 Reactions = 2 Forces	$V'_R \neq 0$ $M'_R = 0$ $I'_R = 0$
Continuous. Supports Level	$M_L \neq 0$ $\phi_L \neq 0$ $y_L = 0$		$M_R \neq 0$ $\phi_R \neq 0$ $y_R = 0$	$V'_L \neq 0$ $M'_L \neq 0$ $I'_L = 0$	 Reactions = 2 Forces, 2 Couples	$V'_R \neq 0$ $M'_R \neq 0$ $I'_R = 0$

Case 3.—Simple Beam with Concentrated Load Not at Mid-Span.—Making use of the condition of zero deflection at the right support and using the equivalent rectangles of Fig. 32(c) (the moment of inertia, I, of the rectangle about its base equals $\frac{b\,h^3}{3}$):

$$I'_R = 0 = (M'_L)\,(2\,L) - \left(\frac{Pa}{L}\right)\left(\frac{L^3}{3}\right) + \frac{P\,a^3}{3} = 0 \ldots\ldots (114)$$



(a) REAL BEAM (b) DUMMY BEAM



(c) RECTANGLES EQUIVALENT TO SHEAR DIAGRAM

FIG. 32.

$$M'_L = \left(\frac{Pa}{6L}\right)(L^2 - a^2) = \left(\frac{Pab}{6L}\right)(L + a) \qquad (115)$$

$$\phi_L = \frac{M'_L}{EI} = \left(\frac{Pab}{6EI}\right)\left(\frac{(L+a)}{L}\right) \qquad (116)$$

and,

$$\phi_R = \left(\frac{-Pab}{6EI}\right)\left(\frac{(L+b)}{L}\right) \qquad (117)$$

The deflection, y, at a distance, x, from the right end is given by $\frac{I'_x}{2EI}$, as follows (when x is measured to the left the sign is negative):

$$I'_x = -(M'_R)(-2x) - \left(\frac{Pb}{L}\right)\left(\frac{x^3}{3}\right) = \left(\frac{2Pabx}{6L}\right)(L + b) - \frac{Pbx^3}{3L} = \left(\frac{Pbx}{3L}\right)(L^2 - b^2 - x^2) \qquad (118)$$

and,

$$y = \frac{I'_x}{2EI} = \left(\frac{Pbx}{6EIL}\right)(L^2 - b^2 - x^2) \qquad (119)$$

Since maximum deflection occurs where the slope, ϕ, is zero:

$$M'_x = 0 = -M'_R + \left(\frac{Pb}{L}\right)\left(\frac{x^2}{2}\right) = 0 \qquad (120)$$

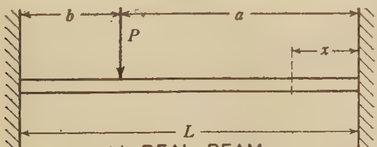
$$\left(\frac{Pab}{6L}\right)(L + b) = \left(\frac{Pb}{L}\right)\left(\frac{x^2}{2}\right) \qquad (121)$$

$$x = \sqrt{\frac{a}{3}(L + b)} \qquad (122)$$

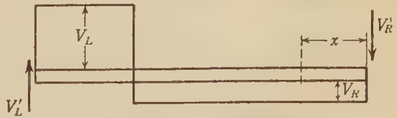
and,

$$y_m = \left(\frac{Pab}{9EIL}\right)\sqrt{\frac{a}{3}(L + b)^3} \qquad (123)$$

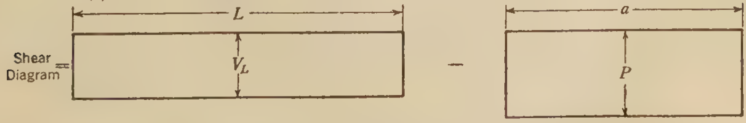
Case 11.—Fixed Beam with a Concentrated Load Not at the Mid-Span.—Making use of the conditions of zero slope and zero deflection at the right



(a) REAL BEAM



(b) DUMMY BEAM



(c) RECTANGLES EQUIVALENT TO SHEAR DIAGRAM

FIG. 33.

support and the equivalent rectangles (see Fig. 33): When $\phi_R = 0$, $M'_R = 0$,

$$V'_L\,L - \frac{V_L\,L^2}{2} + \frac{P\,a^2}{2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(124)$$

and, when $y'_R = 0$, $I'_R = 0$,

$$V'_L\,L^2 - \frac{V_L\,L^3}{3} + \frac{P\,a^3}{3} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(125)$$

Eliminating V_L between Equations (124) and (125),

$$M_L = V'_L = \frac{P\,a^2\,b}{L^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(126)$$

and,

$$M_R = V'_R = \frac{P\,a\,b^2}{L^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(127)$$

Then, by substitution in Equation (124),

$$V_L = \left(\frac{P\,a^2}{L^3}\right)(L + 2\,b) \quad\ldots\ldots\ldots\ldots\ldots\ldots(128)$$

and,

$$V_R = \left(\frac{P\,b^2}{L^3}\right)(L + 2\,a) \quad\ldots\ldots\ldots\ldots\ldots\ldots(129)$$

Points of inflection occur where $V' = 0$; hence:

$$\frac{P\,a^2\,b}{L^2} - V_L\,x = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(130)$$

and,

$$\left(\frac{P\,a^2}{L^3}\right)(L + 2\,b)\,x = \frac{P\,a^2\,b}{L^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(131)$$

If measured from the left end, $x = \dfrac{L\,b}{(L + 2\,b)}$; and, if measured from the right end, $x = \dfrac{L\,a}{(L + 2\,a)}$. Maximum deflection occurs where the slope, ϕ_x, is zero, hence:

$$M'_x = 0 = \frac{-\,P\,a\,b^2\,x}{L^2} + \left(\frac{P\,b^2}{L^3}\right)(L + 2\,a)\left(\frac{x^2}{2}\right) = 0 \quad\ldots\ldots(132)$$

If measured from the right end, $x = \dfrac{2\,a\,L}{(L + 2\,a)}$, and,

$$I'_x = V'_R\,x^2 - V_R\,\frac{x^3}{3} = \frac{P\,a\,b^2\,x^2}{L^2} - \left(\frac{P\,b^2}{L^3}\right)(L + 2\,a)\left(\frac{x^3}{3}\right)$$

$$= \left(\frac{P\,b^2\,x^2}{3\,L^3}\right)(3\,a\,L - 3\,a\,x - b\,x) \quad\ldots\ldots\ldots\ldots\ldots(133)$$

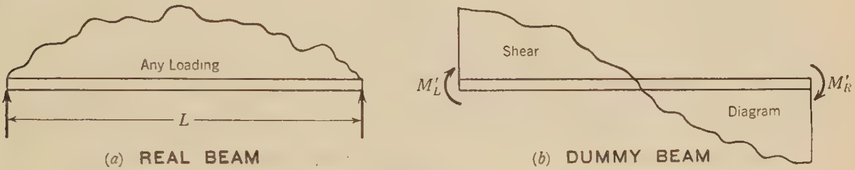
and the deflection equation, $y = \frac{I'_x}{2\,EI}$, becomes:

$$y = \left(\frac{P\,b^2\,x^2}{6\,E\,I\,L^3}\right)(3\,a\,L - 3\,a\,x - b\,x) \qquad (134)$$

Substituting the value of x for maximum deflection, the expression becomes:

$$y_m = \frac{2\,P\,a^3\,b^2}{3\,EI\,(L + 2\,a)^2} \qquad (135)$$

Cases 19 to 22.—Continuous Beam; Three-Moment Equation.—The right-hand side of the three-moment equation is the difference of two terms. The first equals the slope at the right end of a simple span of dimensions and loading of the left span of the continuous beam, multiplied by 6 E, and



(a) REAL BEAM (b) DUMMY BEAM

FIG. 34.

the second term equals the slope at the left end of a simple span of the dimensions and loading of the right span of the continuous beam, multiplied by 6 E. Now, for a simple beam with any loading (see Fig. 34):

$$I'_L = 0 = (-M'_R)\,(2\,L) + I'_{LS} = 0 \qquad (136)$$

$$M'_R = \frac{I'_{LS}}{2\,L} \qquad (137)$$

$$\phi_R = \frac{I'_{LS}}{2\,E\,I\,L} \qquad (138)$$

and,

$$\phi_L = \frac{I'_{RS}}{2\,E\,I\,L} \qquad (139)$$

in which, I'_{RS} and I'_{LS} are the second moments of the shear diagram only, about the right and left supports, respectively.

Substituting these values for the slope, the general equation for three moments reads as follows, in terms of the second moments of the shear diagram for any loading:

$$\frac{M_1\,L_1}{I_1} + 2\,M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + \frac{M_3\,L_2}{I_2} = 6\,E\,\phi_{R1} - 6\,E\,\phi_{L2} = \frac{3\,I'_{L1}}{L_1\,I_1} - \frac{3\,I'_{R2}}{L_2\,I_2} \qquad (140)$$

and for equal values of moment of inertia in both spans:

$$M_1\,L_1 + 2\,M_2\,(L_1 + L_2) + M_3\,L_2 = \frac{3\,I'_{L1}}{L_1} - \frac{3\,I'_{R2}}{L_2} \qquad (141)$$

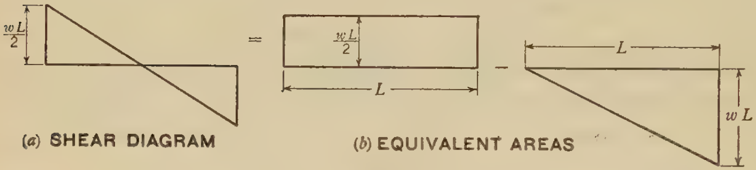
in which I'_{L1} and I'_{R2} are the second moments of the shear diagrams only, in the two spans, about the left and right outside supports, respectively.

Case 20.—Continuous Beam with Uniform Load Over Each Span.—Referring to Fig. 35 $\left(\text{the moment of inertia, } I, \text{ of a triangle about its apex, is } \frac{b\,h^3}{4}\right)$:

$$I'_{LS} = \left(\frac{w\,L}{2}\right)\left(\frac{L^3}{3}\right) - (w\,L)\left(\frac{L^3}{4}\right) = -\frac{wL^4}{12} \quad \text{........(142)}$$

and,

$$I'_{RS} = \frac{w\,L^4}{12} \quad \text{........................(143)}$$



(a) SHEAR DIAGRAM (b) EQUIVALENT AREAS

FIG. 35.

Equations (142) and (143) could also have been obtained by treating the shear diagram as a couple. Substituting these values for I'_{RS} *and* I'_{LS} in the three-moment equation, it becomes:

$$\frac{M_1 L_1}{I_1} + 2\,M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + \frac{M_3 L_2}{I_2} = -\frac{w_1 L^3_1}{4\,I_1} - \frac{w_2 L^3_2}{4\,I_2} \quad \text{...(144)}$$

or,

$$M_1 L_1 + 2\,M_2\,(L_1 + L_2) + M_3 L_2 = -\frac{w_1 L^3_1}{4} - \frac{w_2 L^3_2}{4} \quad \text{.....(145)}$$

It is hoped that the foregoing examples will serve to show the general procedure in applying the shear-area method in this manner and will also reveal its chief advantage, as stated by the authors; namely, the relative simplicity of calculating the second moment of shear areas composed of triangles and rectangles as compared with the first moment of parabolic segments of moment areas when uniformly distributed loads are involved. By the proper choice of equivalent areas a distributed load of any extent, or any arrangement of concentrated loads, may be handled simply. For loads of variable intensity and for beams of variable cross-section the dummy loading becomes quite as complex as that of the moment-area method. For beams of variable cross-section the loading curve is that of Equation (1) of the paper, and Statements (1), (2), and (3) of this discussion must be modified to take this into account. However, it is the writer's opinion that, for these more complicated cases, the classical methods involving integration, or approximate solutions involving the $\frac{M}{E\,I}$-diagram, might as well be used. It is worth mentioning, perhaps, that the shear-area method has the pedagogical value inherent in all such analogies in that it provides the student with excellent drill in the fundamental relations and quantities of mechanics.

FRED L. PLUMMER,[13] ASSOC. M. AM. SOC. C. E. (by letter).—Engineers, in general, are not adequately familiar with the relations existing between load, shear, moment, slope, and deflection for continuous members subjected to lateral forces. Although most engineers may have had occasion to use Equations (146) and (147), at some time, relatively few have made any use of the companion Equations (148) and (149). In these equations all the symbols for slope:

$$\frac{dy}{dx} = \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (146)$$

for moment:

$$EI\,\frac{d^2y}{dx^2} = M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (147)$$

for shear:

$$EI\,\frac{d^3y}{dx^3} = V \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (148)$$

and, for load:

$$EI\,\frac{d^4y}{dx^4} = w \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (149)$$

have their usual significance. The authors indicate these relationships graphically in Fig. 1.

The wider use of the methods of "slope deflection" and "moment distribution", together with their many modifications, has created keen interest in methods of structural analysis during the past few years. Since these methods of analysis are based on a study of distortions, it is natural that interest in methods of determining distortions and deflections should be equally stimulated. The writer has been amazed by the extent of this interest as evidenced by the attendance of designing engineers and architects in the vicinity of Cleveland, Ohio, at special classes devoted to the study of the various methods of determining stresses and deflections for statically indeterminate structures.

The authors have performed a double service to the profession, therefore, first by describing the successive relationships between load, shear, moment, slope, and deflection, and then demonstrating how these relationships may be used to develop alternates to the "moment-area" method of determining slopes and deflections. The writer has solved several problems by both the "shear-area" and the "moment-area" methods, and finds that he prefers the latter method. This decision may be due, however, to the fact that he has been familiar with this method for a much longer period.

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[13] Assoc. Prof., Structural Eng., Case School of Applied Science, Cleveland, Ohio.

In all such "shorthand" methods of analogy, a set of rules may be established and followed blindly, thereby "grinding out" the required results with little mental effort and possibly with little appreciation of the physical significance of the process. It is quite proper that such methods be developed. The routine labor required for many similar designs may thus be lightened materially. It is also important, however, that the young engineer first familiarize himself with the physical meaning of his process so that he may use it when, and only when, it is entirely applicable.

The writer would have preferred that the authors omit several of the routine examples, substituting in their place a more complete development of the theoretical principles illustrated.

ALBIN H. BEYER,[14] M. AM. SOC. C. E. (by letter).—The claim is advanced by the authors that the shear-area method as described in their paper has the advantage over the standard moment-area method, in that it provides a simpler figure to work with, and that its use simplifies the solution for slope and deflection. This claim would be true if the authors had not introduced the computations of moments and shears in their so-called mathematical beam, loaded by the shear-area loading and supported in a hypothetical manner. The shear-area method of computation can be applied readily to any beam and loading without introducing the hypothetical beam and loading, which complicates rather than simplifies the solution.

To use any fundamental formula intelligently, the engineer must be familiar with its derivation and the meaning and sign of every symbol used. Therefore, the writer presents briefly, first the derivation of the fundamental formulas involved and, second, their application to two comparatively simple problems.

Derivation of Fundamental Formula for the General Case of a Loaded Beam.—Fig. 36(a) shows the beam and loading; the end restraining moments, M_A and M_B, are indicated as negative.

Fig. 36(b) shows the straight beam, $B\,O\,A$, deflected under the loads to the position, $B\,O'A$, where $O\,O' = y_o$, is the desired deflection of the beam at any point, O, distant x_A from A; θ_o is the slope of the beam at this point; and θ_A and θ_B are the slopes of the beam at A and B, respectively.

Select a Cartesian system of co-ordinates with the origin at O, where the deflection and slope are desired, distant x_A, a variable, from A; x is measured positive to the right of O, and y is measured positive upward. With this system of co-ordinates, the signs which apply are indicated in Fig. 37.

The differential equation of the deflected beam is given by the classical formula,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{E\,I} \quad \text{.....................(150)}$$

Multiplying the last two terms by $x\,dx$ gives,

$$x\,d\left(\frac{dy}{dx}\right) = \frac{M\,x\,dx}{E\,I} \quad \text{.....................(151)}$$

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[14] Prof. of Civ. Eng., Columbia Univ., New York, N. Y. Professor Beyer died on April 19, 1936.

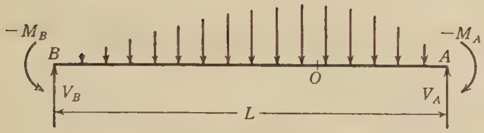
Integrating both sides between the limits, O and A, using the method of parts for the first term:

$$\left[x \frac{dy}{dx} - y \right]_0^{x_A} = \int_0^{x_A} \frac{M\,x\,dx}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots(152)$$

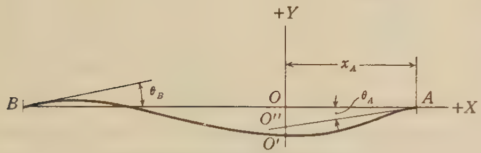
or,

$$y_O = y_A - x_A\,\theta_A + \int_0^{x_A} \frac{M\,x\,dx}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots(153)$$

Equation (153) is the fundamental general formula of the moment-area method.



(a) LOADING DIAGRAM



(b) DEFLECTION DIAGRAM

FIG. 36.

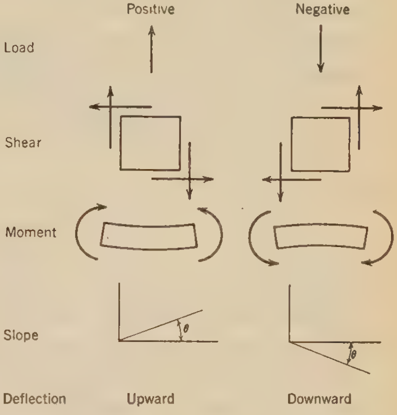


FIG. 37.

Referring to Fig. 36(b), $y_A = 0$ when the support at A is fixed in space; $x_A\theta_A = O\,O''$; $\int_0^{x_A} \frac{M\,x\,dx}{E\,I} = O'O''$; and $O\,O'' + O''O' = O\,O'$, the downward deflection of the beam at Point O.

The authors' contribution to this well-known theory consists in expressing M, the bending moment at any point, in terms of the shear area. In general, $M = M_A - \int_x^{x_A} V\,dx$, in which the last term is the shear area from x to x_A.

The equation for y_O expressed in terms of the shear area, O to A, then becomes,

$$y_O = y_A - x_A\theta_A + M_A \int_0^{x_A} \frac{x\,dx}{E\,I} - \int_0^{x} \left(\int_x^{x_A} \frac{V\,dx}{E\,I} \right) x\,dx \ldots.(154)$$

a very much more complicated expression than Equation (153). Only when E and I are constant can Equation (154) be simplified to,

$$y_O = y_A - x_A\theta_A + \frac{M_A\,x^2_A}{2\,E\,I} - \frac{I_x}{2\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots(155)$$

in which I_x is the moment of inertia of the shear area (sign to be determined in accordance with Fig. 37) to the right of the origin of co-ordinates taken about the Y-axis through the origin. Taking the first derivative of y_o with respect to x_A (noting that $\frac{dy_o}{dx_A}$ is negative):

$$\theta_o = + \theta_A - \frac{M_A x_A}{EI} + \frac{A\bar{x}}{EI} \ldots\ldots\ldots\ldots\ldots\ldots (156)$$

In Equation (156) Ax is the static moment of the shear area between O and A about the Y-axis through O, using the signs for the shear area as determined by Fig. 37.

Application of Formula to Beams with Constant Moment of Inertia.—The general equations to be used are,

$$y_o = y_A - x_A\theta_A + \frac{M_A x_A^2}{2\,EI} - \frac{I_x}{2\,EI} \ldots\ldots\ldots\ldots\ldots (157)$$

and,

$$\theta_o = + \theta_A - \frac{M_A x_A}{EI} + \frac{A\bar{x}}{EI} \ldots\ldots\ldots\ldots\ldots\ldots (158)$$

The general formulas, Equations (157) and (158), give the deflection, y_o, and the slope, θ_o at any point, O, distant x_A from A, in terms of the shear area. They can be applied directly to any beam or loading when the end shears and moments have been computed. For statically indeterminate beams, they can also be used for determining the end moments and shears. In solving these equations, the analogy of a hypothetical beam and loading is not necessary and is not recommended by the writer as the analogy does not contribute to the directness or simplicity of the solution, as can be seen from the following typical examples.

Analysis of Statically Determinate Beams (Authors' Case I).—The general formulas required are:

$$y_o = y_A - x_A\theta_A + \frac{M_A x_A^2}{2\,EI} - \frac{I_x}{2\,EI} \ldots\ldots\ldots\ldots\ldots (159)$$

and,

$$\theta_o = \theta_A - \frac{M_A x_A}{EI} + \frac{A\bar{x}}{EI} \ldots\ldots\ldots\ldots\ldots\ldots (160)$$

In this case, y_A and M_A are each equal to zero. The value of θ_A can be computed by means of Equation (159) or Equation (160). Equation (160) is used because of its simple form; also, from symmetry, $\theta_B = - \theta_A$.

Using Equation (160) (making $x_A = L$ (see Fig. 38)):

$$- \theta_A = \frac{A\bar{x}}{2\,EI} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (161)$$

In Equation (161), $A\bar{x}$ is the static moment of the shaded shear area about a vertical axis through B and is equal to $-\frac{W L^3}{12}$; therefore, $\theta_A = \frac{W L^3}{24 E I}$.

From Equation (159), therefore, the elastic curve is expressed by:

$$y_O = -x_A\theta_A - \frac{I_x}{2 E I} \quad \text{.......................(162)}$$

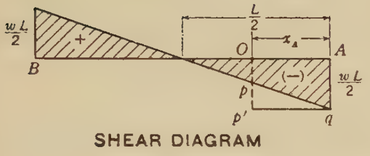


FIG. 38.

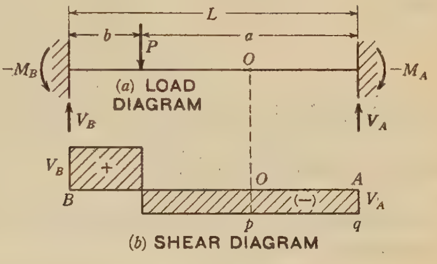


FIG. 39.

in which, I_x is the moment of inertia of the trapezodial shear area, $O\,p\,q\,A$ (Fig. 38) about $O\,p$, and is equal to the static moment of the rectangle, $O\,p'\,q\,A$, minus that of the triangle, $p\,p'\,q$, all of which is equal to $-\frac{1}{6}\,w\,L x^3_A + \frac{w x^4_A}{12}$.

The equation of the elastic curve then becomes,

$$y_O = -\frac{w L^3 x_A}{24\ E\ I} + \frac{w\,L\,x^3_A}{12\ E\ I} - \frac{w x^4_A}{24\ E\ I} \quad \text{...............(163)}$$

which, for $x_A = \frac{L}{2}$, gives:

$$y_O = -\frac{5}{384}\,\frac{w\,L^4}{E\,I} \quad \text{.......................(164)}$$

The slope at O can be found by differentiating y_O with respect of x_A, keeping in mind that the origin is shifted as x_A changes, which makes $\frac{dy_O}{dx_A}$ minus; thus:

$$\theta_O = +\frac{w\,L^3}{24\,E\,I} - \frac{w\,L\,x^2_A}{4\,E\,I} + \frac{w\,x^3_A}{6\,E\,I} \quad \text{...............(165)}$$

The same result for θ_O can be obtained directly from Equation (160), in which $A\bar{x}$ is now the static moment of the shear area, $O\ p\ q\ A$, about $O\ p$, and is equal to $-\frac{w\,L\,x^2_A}{4} + \frac{w\,x^3_A}{6}$, which gives,

$$\theta_O = \frac{w\,L^3}{24\,E\,I} - \frac{w\,L\,x^2_A}{4\,E\,I} + \frac{w\,x^3_A}{6\,E\,I} \quad \text{...............(166)}$$

Analysis of Statically Indeterminate Beams (Authors' Case II).—The end reactions, M_A, M_B, V_A, and V_B (Fig. 39), can be computed in the usual manner. They can also be found readily by means of Equation (159) and Equation (160), and are as follows: $M_A = -\frac{P\,a\,b^2}{L^2}$; $M_B = -\frac{P\,a^2\,b}{L^2}$; and, $V_A = -\frac{P\,b^2}{L^3}(3a+b)$. Note that θ_A, θ_B, and y_A are each equal to zero. Then,

$$y_o = +\frac{M_A\,x^2_A}{2\,E\,I} - \frac{I_x}{2\,E\,I} \ldots\ldots\ldots\ldots\ldots\ldots (167)$$

in which, I_x is the moment of inertia (see Fig. 39), of the rectangle, $O\,p\,q\,A$, about $O\,p$, and is equal to $\frac{V_A\,x^3_A}{3}$.

Expressing M_A and V_A in terms of the load, P, and a and b, the equation of the elastic curve is:

$$y_o = -\frac{P\,a\,b^2\,x^2_A}{2\,E\,I\,L^2} + \frac{P\,b^2}{6\,E\,I\,L^3}(3\,a+b)\,x^3_A + \ldots\ldots\ldots (168)$$

in which y_o is a maximum when $\frac{dy_o}{dx_A} = 0$, provided a is $\lesseqgtr \frac{L}{2}$. This gives $x_A = \frac{2\,a\,L}{2\,a+L}$. The solution becomes simpler when numerical values can be substituted for M_A, V_A, I_x, and $A\bar{x}$ in Equations (159) and (160).

JOHN M. BEATTY,[15] JUN. AM. SOC. C. E. (by letter).—The "shear-area method" for the solution of the elastic functions of loaded beams as presented in this paper outlines another interesting and valuable method of analysis for both statically determinate and statically indeterminate beams. As the authors have stated it is not "the shortest method for solving slopes and deflections for all problems," but its advantages in those cases involving uniformly distributed loads become quite apparent when the method is put to the test.

The parabolic moment-curve areas involved in the solutions by the moment-area methods are always disagreeable and sometimes difficult to handle. On the other hand, the shear-curve areas for the same loadings are usually simply a combination of triangles, rectangles, and trapezoids, the properties of which are well known to every engineer and designer.

There are many cases, also, in which the shear-area method may be used to advantage in combination with some other method, such as the conjugate beam method. In the case of simply supported beams, for instance, the end slopes many times may be found much more readily by the shear-area method and then these slopes may be applied as end reactions for the "conjugate beam." The problem may then be continued to completion by the latter method if an inspection seems to indicate an easier solution by that method.

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[15] Instr., Civ. Eng., Rensselaer Polytechnic Inst., Troy, N. Y.

For example, consider the following simple beam loaded with two patches of uniform load symmetrically placed with respect to the center of the beam (see Fig. 40(*a*)). An inspection of the shear and moment diagrams immediately suggests an easier and quicker solution of the end slopes by the shear-area method. The end slopes, which are numerically equal to each other are also equal to one-half the moment of the couple produced by the shear-area loadings. The value of the shear-area couple is equal to the value of the shear area above or below the base line, multiplied by the distance between their centers of gravity. The value of one-half this shear area couple may then be found immediately taking moments about the center line of the mathematical beam of the shear area to one side of the section. This principle may be used for all simple beams with loadings that are symmetrical with respect to the center line, providing, of course, that the beam itself is also symmetrical about the center line; thus (see Fig. 40(*b*)):

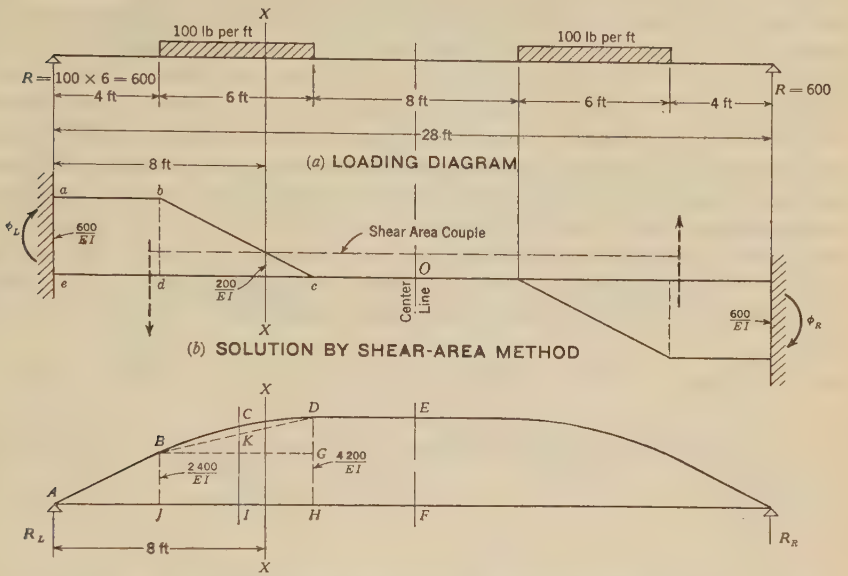
X
100 lb per ft
100 lb per ft
R = 100 × 6 = 600
R = 600
4 ft
6 ft
8 ft
6 ft
4 ft
28 ft
8 ft
(a) LOADING DIAGRAM
a
b
φL
600/EI
Shear Area Couple
O
e
d
200/EI
c
Center Line
X
600/EI
φR
(b) SOLUTION BY SHEAR-AREA METHOD
X
C
D
E
B
K
G
4 200/EI
2 400/EI
A
J
I
H
F
RL
RR
8 ft
X
(c) SOLUTION BY CONJUGATE-BEAM METHOD

FIG. 40.—COMPARATIVE SOLUTION FOR END SLOPES

$$\phi_L = \frac{\text{Moment of Shear Area, } a\ b\ c\ d\ e\text{, About Point } O}{E\,I} \times 144$$

or,

$$\phi_L = \frac{600 \times 4 \times 12 + 0.5 \times 600 \times 6 \times 8}{E\,I} \times 144 = \frac{6\,220\,800}{E\,I}$$

To find the same end slope by the conjugate beam method involves the solution for the end reaction, R_L, of the conjugate beam. As the moment diagram is symmetrical with respect to the center line, this reaction, R_L, must equal one-half the total moment-area loading. Of this total area, $A\ C\ E\ F\ A$, that under the curved portion (that is, Area $J\ B\ C\ D\ H$, Fig. 40(c)), presents the greatest difficulty. This area, of course, may be found by means of the calculus by integrating the general expression for the moment area between the limits of J and H, but this introduction of the calculus is troublesome to many and is often avoided wherever possible. Another method of finding the value of Area $J\ B\ C\ D\ H$ is to divide it into the two areas, $J\ B\ D\ H\ J$ and $B\ C\ D\ B$, and then to find each area separately. The area of the trapezoid, $J\ B\ D\ H\ J$, is readily found. The area under the curve, $B\ C\ D\ B$, may be determined by finding the value of the ordinate, $C\ K$, at the mid-point between B and D and, considering this as the mid-ordinate of a parabolic curve spanning the horizontal distance, $B\ G$, the area will then be equal to two-thirds the product of $C\ K$ times $B\ G$ (see Fig. 40(c)). Referring to Fig. 40(c): $C\ K = C\ I - K\ I = 3\,750 - 3\,300 = 450$; $M_{BJ} = 600 \times 4 = 2\,400$; $M_{CI} + 600 \times 7 - 300 \times 1.5 = 3\,750$; $M_{BF} = 600 \times 14 - 600 \times 7 = 4\,200$;

$$\text{and } R_L = \text{Areas } \frac{A\,B\,J + B\,D\,H\,J + B\,C\,D\,B + D\,E\,F\,H}{E\,I} \times 144$$

$$= \frac{1}{E\,I}\left[\frac{2\,400 \times 4}{2} + \left(\frac{2\,400 \times 4\,200}{2}\right) 6 + \frac{2}{3}(450 \times 6) + 4\,200 \times 4\right] \times 144$$

$$= \frac{6\,220\,800}{E\,I}.$$

Regardless of which method is used to find the total area, $A\ C\ E\ F\ A$, it is apparent that the work involved is considerably more than that incurred in the solution of ϕ_L by the shear-area method. As the values of ϕ_L and R_L are certain to be identical, both representing the end slope of the same simple beam, the shear-area method is obviously much the easier and quicker method and, at the same time, because of its simplicity, much less liable to error than the solution by the conjugate beam method.

If the values of the moment areas are known (as they now are in this problem) there is very little to choose from between the two methods in so far as the determination of the maximum deflection at the center of the beam is concerned. Each method involves about an equal amount of numerical work, although the determination of the static moments of the moment areas in the case of the conjugate beam may be preferred by some to the solution for the moments of inertia of the shear areas in the case of the shear-area method. For example, by the shear-area method:

$$y_{\text{max.}} = \phi_L \times x - \frac{I_x}{2} = \frac{1}{E\,I}\left\{6\,220\,800 \times 14 \times 12 \right.$$

$$\left. - \left[\frac{\dfrac{600 \times 4^3}{12} + 600 \times 4 \times 12^2 + \dfrac{600 \times 6^3}{36} + \dfrac{600 \times 6}{2} \times 8^2}{2}\right] 1\,728\right\}$$

$$= \frac{641\,088\,000}{E\,I}$$

and, by the conjugate beam method:

$$y_{\text{max.}} = \frac{1}{EI}\left\{6\,220\,800 \times 14 \times 12 - \left[\frac{2\,400 \times 4}{2} \times \frac{34}{3} + 2\,400 \times 6 \times 7 + \frac{1\,800 \times 6}{2} \times 6 + \frac{2}{3}(450 \times 6) \times 7 + 4\,200 \times 4 \times 2\right]1\,728\right\}$$

$$= \frac{641\,088\,000}{EI}$$

It is in the determination of the deflection at any intermediate point, such as Point X, Fig. 41, that the advantage of the shear-area principle becomes apparent.

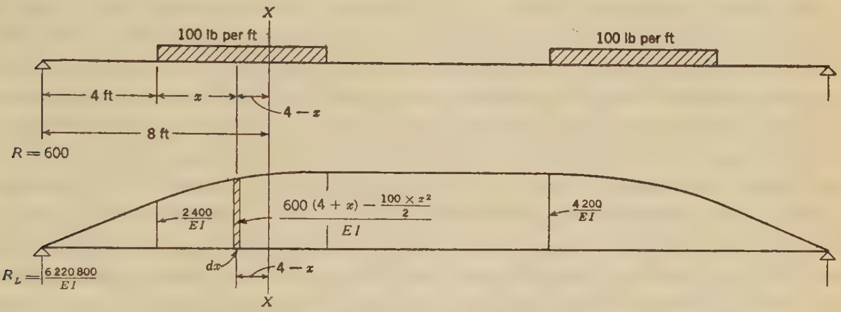

FIG. 41.—DEFLECTION AT ANY POINT BY THE CONJUGATE BEAM METHOD

In finding the static moments about Point X, of the moment areas in the conjugate beam method, recourse must again be made to the calculus for the moment of the area between Points J and X (see Fig. 40(c)). The general expression for the moment of the moment area must be integrated between the limits of J and X (see Figs. 40(c) and 41). An inspection of the following computations plainly shows the relative simplicity of the solution by the shear-area method over that by the conjugate beam method:

By the shear-area method (see Fig. 40(b)):

$$y_x = \phi_L \times x - \frac{I_x}{2};\ \phi_L = \frac{6\,220\,800}{EI};\ \text{and, } y_x = \frac{1}{EI}\left\{6\,220\,800 \times 8 \times 12 - \left[\frac{\frac{600 \times 4^3}{12} + 600 \times 4 \times 6^2 + \frac{200 \times 4^3}{3} + \frac{400 \times 4^3}{4}}{2}\right]1\,728\right\} = \frac{510\,566\,400}{EI}$$

By the conjugate beam method:

$$y_x = \frac{1}{EI}\left\{6\,220\,800 \times 8 \times 12 - \left[\frac{2\,400 \times 4}{2} \times \frac{16}{3} + \int_0^4 \left(600\,(4 + x) - \frac{100\,x^2}{2}\right) dx\,(4 - x)\right]1\,728\right\} = \frac{1}{EI}\left\{6\,220\,800 \times 96 - \left[25\,600 + 50\left(192\,x - \frac{16\,x^3}{2} + \frac{x^4}{4}\right)_0^4\right]1\,728\right\} = \frac{510\,566\,400}{EI}$$

No single method of analysis can be claimed as superior to all others for the solution of all problems. Certain methods are more readily applied to certain types of problems and, therefore, are to be preferred over all others for these specific types. The shear-area method thus supplies one more possible tool which, when used judiciously, can be of real service. The authors are to be commended for bringing this method of analysis to the attention of the Engineering Profession.

R. B. Peck,[16] Jun. Am. Soc. C. E. (by letter).—The use of the shear-area method is most advantageous, as the authors state, for problems involving distributed load. Where the load is uniformly distributed over entire span lengths—that is, where the moment diagram includes a full parabola—it appears that the method offers no advantages over the moment-area methods because the properties of parabolic half-segments are well known. For uniformly distributed loads occurring in patches, however, so that the moment diagram includes parts of parabolas, the location of the vertices of which is not apparent by inspection, the shear-area method affords a simpler solution.

The equation for deflection is direct and simple in use. If the beam is fixed-ended and the end slopes are known, the solution is rapid. If the beam is simply supported so that the end slopes are not known, it should be emphasized that, although they are easily determined for specific cases by the formulas proposed by the authors, their derivation is inherently less direct than the process of obtaining the reactions of the conjugate beam. Much of the advantage of the shear-area method is lost in this case because of this difficulty in obtaining end slopes, particularly if the computer does not have the aforementioned formulas available.

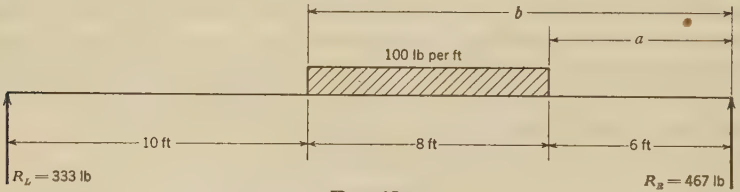

Fig. 42.

It will be noticed, however, that a very direct means of computing the end slopes for simply supported beams is found in the method, suggested by the authors, of investigating the loading directly. The deflection formula, Equation (95), will reduce to:

$$y\,E\,I = -\frac{1}{6}Q_x + \phi_x x + y_e \qquad (169)$$

for simple supports. When written for a point of support, where $y = 0$, it becomes, since $x = L$,

$$0 = -\frac{1}{6}Q + \phi_e L + y_e$$

which may be solved immediately for the end slope, ϕ_e. A numerical example is given for illustration.

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[16] Troy, N. Y.

In order to solve for the deflection at any point in the beam shown in Fig. 42, it is necessary to know one of the end slopes, say ϕ_L. By Equation (30), $\phi_L = 25\,500$. If, however, instead of Equation (30) the loading is used directly, the expression for the deflection at the right-hand end, using Equation (169), is $0 = -\frac{1}{6} Q + \phi_L L$, in which Q is taken for the entire span and equals $\frac{R x^3}{6} - \int x^3\, dA$. For an area, $\int x^3\, dA = 3\, x_0\, I_{cg} + A\, x^3_0$. Then,

$$0 = \phi_L \times 24 - \frac{333}{6} \times 24^3 + \frac{3}{6} \times 10 \times \frac{100 \times 8^3}{12} + \frac{100 \times 8 \times 10^3}{6}$$

from which $\phi_L = 25\,500$.

Ralph W. Stewart,[17] M. Am. Soc. C. E. (by letter).—An interesting mathematical exercise is presented in this paper. The principle which makes the shear-area method possible is the same as that which makes the conjugate beam or elastic weight method possible, namely, in Fig. 1, each curve is the integral of the one above it. This principle is illustrated more simply by drawing the loading, shear, moment, slope, and deflection curves for a cantilever beam using the free end as the origin, thereby eliminating the effect which the reaction has on the shear curve.

In connection with these integrations, as usually performed for specific loading, it is of interest to note that the shear and moment in either the cantilever beam or the simple beam are governed by forces and distances on one side only of the point at a distance, x, from the origin. The slope and deflection, however, are affected by the length of the entire beam and by the loading on both sides of the point at a distance, x. In line with this physical condition the integrations from load to shear and shear to moment are not affected by the valuation of constants of integration, these constants being zero, but the integrations from moment to slope and from slope to deflection require the evaluation and use of constants of integration which are dependent on the entire length of the beam. The authors' treatment of Fig. 1 involves loading which is general rather than specific, thereby leading to two term expressions for the slope and the deflection with constants of integration that equal zero.

The one obvious advantage which the shear-area method offers in comparison with methods of analysis based on moment areas is that, for beams with uniformly distributed loads, the shear areas are bounded by straight lines, whereas with moment areas a parabolic boundary line is involved. To a computer with meager knowledge of geometry the straight-line figure offers an advantage. The properties of parabolic moment diagrams are so simple, however, and so easily remembered, that any one whose professional practice includes structural engineering can hold them in mind, and by their use can solve beam problems with less work than by the methods set forth in this paper.

As an illustration, the fact that the parabola cuts the rectangle in Fig. 43 into an upper and lower area equal, respectively, to two-thirds and one-third of the area of the rectangle, and that the centers of gravity of these areas are located as shown, should not tax the memory of an engineer. Even if forgotten, the integrations to obtain these areas and distances are easier than such integrations as that of Equation (10) and various subsequent integrations in the paper.

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[17] Engr. of Structural Design, City of Los Angeles, Los Angeles, Calif.

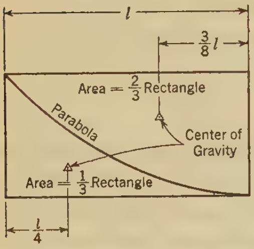
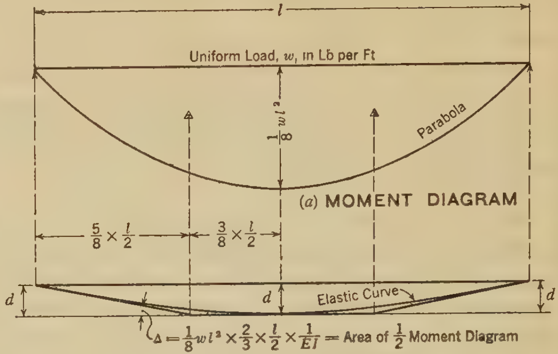

l
$\frac{3}{8}l$
Area = $\frac{2}{3}$ Rectangle
Parabola
Center of Gravity
Area = $\frac{1}{3}$ Rectangle
$\frac{l}{4}$

FIG. 43.


l
Uniform Load, w, in Lb per Ft
$\frac{1}{8}wl^2$
Parabola
(a) MOMENT DIAGRAM
$\frac{5}{8}\times\frac{l}{2}$
$\frac{3}{8}\times\frac{l}{2}$
d
d
Elastic Curve
d
$\Delta=\frac{1}{8}wl^2\times\frac{2}{3}\times\frac{l}{2}\times\frac{1}{EI}$ = Area of $\frac{1}{2}$ Moment Diagram

(b) TRAVERSE DIAGRAM

FIG. 44.

Using these parabolic areas and the principle that the change in direction of the tangent between two points on the elastic curve is equal to the area between the two corresponding points on the $\frac{M}{E\,I}$-diagram, the analysis of beams with uniformly distributed loads can be performed more rapidly and more satisfactorily than by the method set forth by the authors. For example, by the moment-area method, or by the traverse method as illustrated by Fig. 44(*b*),[18] the deflection at the center of a simple beam uniformly loaded is:

$$\frac{5}{16}\,l\left(\frac{1}{8}\,w\,l^2\times\frac{2}{3}\times\frac{l}{2}\times\frac{1}{E\,I}\right)=\frac{5\,w\,l^4}{384\,E\,I}\quad\ldots\ldots\ldots(170)$$

This is certainly simpler treatment than the procedure indicated by the authors' Equations (10) and (11). In fact, their derivation of this maximum deflection appears more tortuous than the usual double integration of the bending-moment equation as taught in most treatises on applied mechanics.

An inherent difficulty in the general use of the shear-area method for the analysis of continuous frames is that the shear curve is one step too far removed from the curve of deflections, and the analysis of continuous frames is dependent to a great extent upon equating deflections at certain points to each other, or to zero. The slope is one step closer to the shear curve, and since the theorem of three moments is derived by equating slopes over supports, it is practicable to use the shear-area method to derive it as set forth in the paper.

As a practical tool for the analysis of structures, it appears that the prospect of the shear-area method to gain recognition is not encouraging.

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[18] For explanation of Fig. 44(*b*) see "An Improved Method of Finding Beam Deflections", *Civil Engineering*, February, 1934.

C. W. JOHNSON,[19] and H. W. BIRKELAND,[20] JUNIORS, AM. SOC. C. E. (by letter).—Upon close examination and by comparison with other methods, the writers have been unable to discover any advantages in favor of the so-called "shear-area method." With a little effort it is easy to understand why the modified moment diagram was chosen as an elastic load diagram in solving for deflections. As one moves down the list of curves from the moment toward the actual load curve, the modification becomes more and more complex. For example, for the moment diagram, $\frac{M}{EI}$ is used; for the shear diagram, $\frac{V}{EI} + \frac{M}{(EI)^2} \frac{d(EI)}{dx}$; and, for the load diagram, $\frac{w}{EI} + \frac{M}{(EI)^2} \frac{d^2(EI)}{dx^2} - \frac{2M}{(EI)^3} \left[\frac{d(EI)}{dx}\right]^2$. If the reader doubts the increasing difficulty of using these expressions, let him try an example and be convinced.

Another misleading inference in the paper suggests that the second moment of the modified shear area is easier to find than the first moment of the moment area. This conclusion is inaccurate, of course, because in the two cases the integrations are comparable. For use with the conjugate beam method the writers have long used a simple table of areas and locations of centroids in cases where only deflections at particular points are desired. Such an expedient would be equally necessary and useful in regard to the shear-area method.

The authors have placed great emphasis upon the ease of finding the slopes by this method and the merit of having "eliminated" the moment curve. Compare the usefulness of these two curves and then decide which should be omitted. In the conjugate beam method the most used curve is employed whereas the least used curve is not used. In the shear-area method the valuable moment curve is not used and apparent emphasis is laid on the less important slope curve.

The method does not seem to be readily adaptable to any but the most simple beam problems such as statically determinate, constant section cases with uniformly distributed load. The writers have been unable to find one case in which the procedure could be applied more easily than some other one. To the practicing engineer the method offers nothing that is of sufficient value to investigate as it does not improve upon the methods now in use. However, if the paper were more complete, and included a genuine derivation of the principles, it would be of interest to students, since it calls attention to the second-moment and third-moment principles.

In Fig. 1 the authors have ostensibly set out to show the relation between beam diagrams. In doing so they start with the load diagram as a calculus curve and attempt to proceed from one curve to another by means of integration.

Since an extremely simple case was chosen, it is unfortunate that the equations were not shown mathematically complete. Some of the constants of integration are shown and others are omitted. It is necessary to show a constant of integration in each equation involving the solution of an integral, such as $y = \int f(x)\, dx$.

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[19] Junior Engr., U. S. Bureau of Reclamation, Denver, Colo.

[20] Junior Engr., U. S. Bureau of Reclamation, Denver, Colo.

In deriving the equations of the shear and moment curves the origin is at the left end; this is likewise true in the first parts of Equations (6) and (7), expressing the slope and deflection. The second parts of Equations (6) and (7) are based on the first-moment and second-moment principles, respectively, and the origins are at any variable point, x. Thus, to avoid confusion, the x occurring with dA under the integral sign might well be expressed as another variable since its origin is not at the left end. Space does not permit a discussion of all the authors' examples, but the same types of omissions and inconsistencies, as mentioned in connection with Fig. 1 and Equations (4) to (7), inclusive, appear throughout.

Although it is not the chief topic of the authors' paper, the writers present a simple and consistent method of showing beam diagrams. The successive curves bear the proper mathematical relationships and the sign conventions are logical. The calculus relationships can be expressed in two simple rules: (*a*) The slope at any point on a curve is equal to the ordinate at the corresponding point in the derived curve; and, (*b*) the difference between the ordinates at any two points on a curve is equal to the area under the derived curve between the two corresponding points.

These two rules do not establish the starting point of the higher degree curve; this point is determined from the constant of integration which is fixed by physical considerations and is often referred to as the initial condition. The equations of the beam diagrams and the sign conventions are, as follows:

(1) Loading (positive when acting upward):

$$w_x = f(x) \dots\dots\dots\dots\dots\dots\dots\dots (171)$$

(2) Shear (positive when acting upward on the part of the beam on the right of the cut; an imaginary cut helps to visualize internal shears):

$$V_x = R_L + \int_0^x w_x\, dx \dots\dots\dots\dots\dots\dots (172)$$

(3) Moment (positive when the lower fibers are in tension):

$$M_x = M_L + \int_0^x V_x\, dx \dots\dots\dots\dots\dots\dots (173)$$

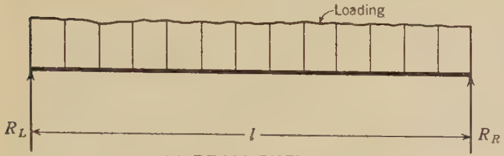
(4) Slope (positive when deflection curve rises from left to right):

$$\theta_x = \phi_L + \int_0^x \frac{M_x}{EI}\, dx \dots\dots\dots\dots\dots\dots (174)$$

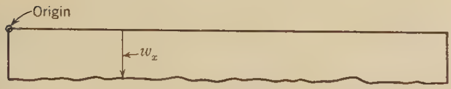
(5) Deflection (positive when upward from the original position):

$$\delta_x = \delta_L + \int_0^x \theta_x\, dx \dots\dots\dots\dots\dots\dots (175)$$

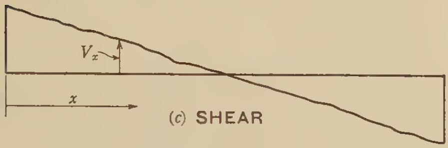
In order to keep Equations (171) to (175) general, all terms are shown positive. Therefore, algebraic signs must be included in numerical substitutions, which is the usual practice in mathematics.


Loading
R_L
l
R_R

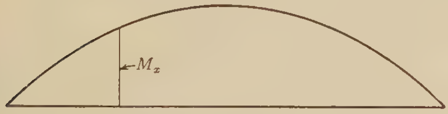
(*a*) BEAM SKETCH



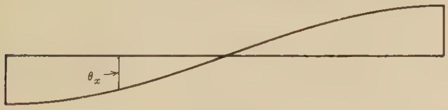
(*b*) LOAD



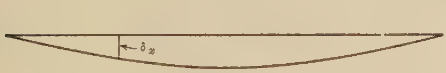
(*c*) SHEAR



(*d*) MOMENT



(*e*) SLOPE



(*f*) DEFLECTION

FIG. 45.—BEAM DIAGRAMS.

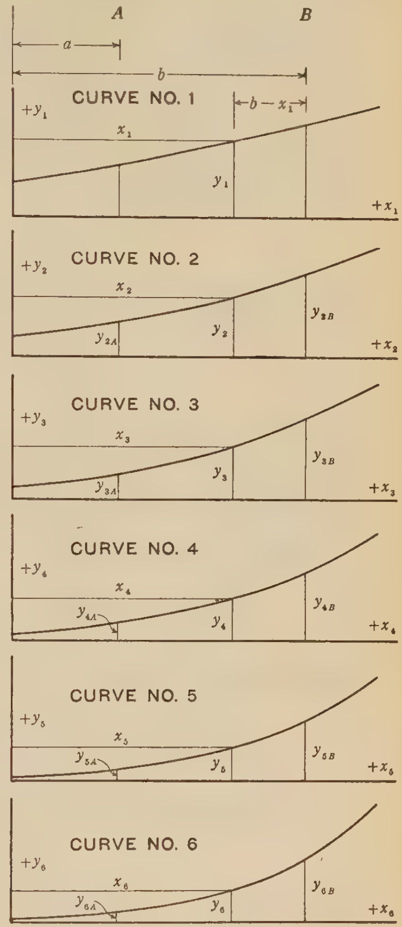

A
B
a
b
CURVE NO. 1
$+y_1$
$b-x_1$
x_1
y_1
$+x_1$
CURVE NO. 2
$+y_2$
x_2
y_{2A}
y_2
y_{2B}
$+x_2$
CURVE NO. 3
$+y_3$
x_3
y_{3A}
y_3
y_{3B}
$+x_3$
CURVE NO. 4
$+y_4$
x_4
y_{4A}
y_4
y_{4B}
$+x_4$
CURVE NO. 5
$+y_5$
x_5
y_{5A}
y_5
y_{5B}
$+x_5$
CURVE NO. 6
$+y_6$
x_6
y_{6A}
y_6
y_{6B}
$+x_6$

FIG. 46.—CONSECUTIVE CALCULUS CURVES.

This procedure, as applied to the example of Fig. 1, is shown in Fig. 45. Fig. 45(*a*) is a sketch of the beam showing external loads and restraints. Figs. 45(*b*) to 45(*f*) show a series of beam diagrams from load curve to deflection curve, inclusive. Note that, consistent with the foregoing sign convention, the load curve, Fig. 45(*b*), is drawn below the *X*-axis, since a downward acting load is negative. The reason for such a sign convention is to show better the mathematical relationships between the load and shear curves. It is important that the sketch of the beam, showing loading and restraints, is not confused with the load curve. The Cartesian co-ordinate system, with origin at the left end throughout, is used in this method. Values are positive when measured above, and negative when measured below, the X-axis.

Since Equations (4) to (7), inclusive, are incomplete, and do not contain a proof of the principles involved, the writers will derive them as mathematical laws that are not limited to beams alone.

The First-Moment Principle.—A special application of the first-moment principle to beams is called Mohr's second law; conjugate beam method; and moment-area principle. Another special application to beams may be called the shear-area method, Part I. A general application, using different terms, is sometimes called the third law of derived curves. Referring to Fig. 46, let the equation of Curve No. 1 be:

$$y_1 = f(x_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (176)$$

Integrating Equation (176), the equation of Curve No. 2 is obtained; thus:

$$y_2 = y_{2A} + \int_a^{x_2} y_1 \, dx_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (177)$$

Note that instead of taking the constant of integration at the origin and starting the integration from there, the constant is taken at Point A (Fig. 46), and the integration starts from that point. This is done in order to remove any doubt about the universal truth of the law to be derived.

Integrating again, the equation of Curve No. 3 is obtained:

$$y_3 = y_{3A} + \int_a^{x_3} y_{2A} \, dx_2 + \int_a^{x_3} \int_a^{x_2} y_1 \, dx_1 \, dx_2 \ldots\ldots\ldots\ldots (178)$$

Rewriting Equation (178) with the order of integration reversed:

$$y_3 = y_{3A} + \int_a^{x_3} y_{2A} \, dx_2 + \int_a^{x_3} \int_{x1}^{x_3} y_1 \, dx_2 \, dx_1 \ldots\ldots\ldots\ldots (179)$$

It is now possible to perform the first integration::

$$y_3 = y_{3A} + y_{2A} \, (x_3 - a) + \int_a^{x_3} y_1 \, (x_3 - x_1) \, dx_1 \ldots\ldots\ldots (180)$$

Substituting b for x_3,

$$y_{3B} = y_{3A} + y_{2A} \, (b - a) + \int_a^{b} y_1 \, (b - x_1) \, dx_1 \ldots\ldots\ldots (181)$$

which is the first-moment principle, so-called because the integral involves the first moment of an area. It is comparable to the second part of Equation (6).

Equation (181) may be translated into words, as follows: If there are three curves, Nos. 1, 2, and 3, that are so related that Curve No. 2 is obtained by integration of Curve No. 1, and Curve No. 3 by integration of Curve No. 2 (or *vice versa* by differentiation), then Curves Nos. 1, 2, and 3, may be treated in a manner similar to load, shear, and moment curves; that is, the ordinate at any point of Curve No. 3 may be obtained from Curve No. 1 by cutting it at that point and setting the sum of moments equal to zero. Of course, Curve No. 1 must be placed in equilibrium, consistent with known conditions in the other related curves. This is illustrated in Fig. 47. The various quantities occurring in Equation (181) should then be interpreted as follows: y_{2A} = shear at Point A; y_{3A} = moment at Point A; y_1 = load intensity at Point x_1; $y_1\,dx_1$ = differential load at Point x_1; $(b - x_1)$ = lever arm of the differential load about Point B; and y_{3B} = moment at Point B.

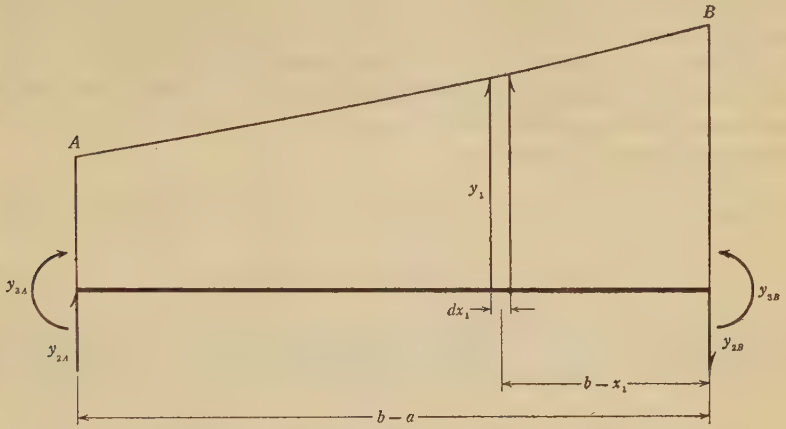


FIG. 47.—EQUILIBRIUM SKETCH FOR THE FIRST MOMENT PRINCIPLE (CONJUGATE BEAM).

The Second-Moment Principle.—A special application of the second-moment principle to beams may be called shear-area method, Part II (moment of inertia of shear diagram). Referring to Fig. 46, the equation of Curve No. 4 is obtained by integration of Equation (180):

$$y_4 = y_{4A} + \int_a^{x_4} y_{3A}\,dx_3 + \int_a^{x_4} y_{2A}\,(x_3 - a)\,dx_3 + \int_a^{x_4}\int_a^{x_3} y_1\,(x_3 - x_1)\,dx_1\,dx_3 \quad \ldots\ldots(182)$$

Rewriting Equation (182) with the order of integration reversed:

$$y_4 = y_{4A} + \int_a^{x_4} y_{3A}\,dx_3 + \int_a^{x_4} y_{2A}\,(x_3 - a)\,dx_3 + \int_a^{x_4}\int_{x_1}^{x_4} y_1\,(x_3 - x_1)\,dx_3\,dx_1 \quad \ldots\ldots(183)$$

It is now possible to perform the first integration; thus:

$$y_4 = y_{4A} + y_{3A}\,(x_4 - a) + \frac{1}{2}\,y_{2A}\,(x_4 - a)^2 + \frac{1}{2}\int_a^{x_4} y_1\,(x_4 - x_1)^2\,dx_1 \quad .(184)$$

Substituting b for x_4:

$$y_{4B} = y_{4A} + y_{3A}\,(b - a) + \frac{1}{2}\,y_{2A}\,(b - a)^2 + \frac{1}{2}\int_a^b y_1\,(b - x_1)^2\,dx_1 \text{ .(185)}$$

which is the second-moment principle, so-called because the integral involves the second moment of an area. It is comparable to Equations (7), (92), and (94). It must be remembered that the four curves are consecutive.

The Third-Moment and Fourth-Moment Principles.—Continuing the derivation beyond the second-moment principle, and using the same procedure, the following equation is obtained,

$$y_{5B} = y_{5A} + y_{4A}\,(b - a) + \tfrac{1}{2}\,y_{3A}\,(b - a)^2 + \tfrac{1}{6}\,y_{2A}\,(b - a)^3$$
$$+ \frac{1}{6}\int_a^b y_1\,(b - x_1)^3\,dx_1 \text{..................(186)}$$

which is the third-moment principle, and which is comparable to Equations (93) and (95). It is understood that the five curves are consecutive. Similarly,

$$y_{6B} = y_{6A} + y_{5A}\,(b - a) + \frac{1}{2}\,y_{4A}\,(b - a)^2 + \frac{1}{6}\,y_{3A}\,(b - a)^3$$
$$+ \frac{1}{24}\,y_{2A}\,(b - a)^4 + \frac{1}{24}\int_a^b y_1\,(b - x_1)^4\,dx_1 \text{..........(187)}$$

which is the fourth-moment principle. The six curves involved must be consecutive. This procedure may be extended to obtain any desired moment principle.

Discussion of the Moment Principles.—From the first, second, third, and fourth-moment principles, represented by Equations (181), (185), (186), and (187), respectively, the following points may be noted:

(1) The first-moment principle is the only one of the group that lends itself to use in connection with an equilibrium sketch, because the numerical coefficients of all terms are unity (1.0) in this one case only; and,

(2) A problem, which in ordinary procedures would involve a multiple integration, has been reduced to one with a single integration. It is well to note that the intermediate curve or curves are not entirely eliminated since the constants of integration of all these curves are used, and, therefore, must be evaluated.

In solving a problem by beam diagrams, a choice must be made between the following procedures: (1) To apply a series of successive, straight integrations to obtain the equations of all curves; (2) to apply the first moment principle, repeatedly if necessary and desired, along with such successive integrations as are necessary to determine the constants of integration of the intermediate curves; and, (3) to apply the second or higher moment principle, along with the first-moment principle and successive integrations to determine the constants of integration of the intermediate curves.

It is futile, of course, to say that one procedure is better than the others, for one must bear in mind the possible complications accompanying the moment principles.

Garrett B. Drummond,[21] Esq. (by letter).—There will be many who recognize in this paper another approach to the solution of beams; but it is certainly not remiss to search deeper and recognize another emphasis of the mathematical fundamentals of elasticity.

Consider the familiar expression of the calculus:

$$\int f(x)\ dx = y = F(x) \ldots\ldots\ldots\ldots\ldots\ldots (188)$$

which, with limits, becomes,

$$\int_a^b f(x)\ dx = F(b) - F(a) \ldots\ldots\ldots\ldots\ldots\ldots (189)$$

but $F(b)$ is the ordinate on the integral curve at $x = b$, and $F(a)$ is the ordinate at $x = a$; which leads to the statement which is the basis of graphical integration:

The area under the derived curve between any two ordinates is equal numerically to the difference between the corresponding ordinates on the integral curve.

Assumptions (3), (4), and (5) of the paper are a consequence of the foregoing mathematical statement.

Papers such as this contribute to a broader viewpoint of the geometry of statics. It is not sufficient that designers be able to apply the so-called laws of mechanics without the facility of visualizing just what happens to a beam or other structure. The shear-area method provides another approach to that visualization.

Thus, the structural engineer is able to express the so-called mathematical principles in ordinary language. Shear being the sum of all the forces and the moment the sum of these forces times their distances from any given point, if the point is moved a short distance the moment will be increased by the shear times the distance. In addition, the rate of change of the shear is the unit intensity of the load.

In other words, if there are no loads on a beam, the shear across the beam is constant and the moment curve is a straight line. If all the loads on any span are in the same direction the shear in that span either has the same sign at all points or changes sign at one point in the span. Furthermore, the moment curve has a continuous slope in one direction or changes direction of slope at one point.

This geometrical conception of the action of any structure is fundamental, not only to statically determinate, but to statically indeterminate, structures. It has been stated that if a structure does not fail, it holds together. If that is true, continuity requires that there be certain geometric relations between its parts.

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[21] Asst Prof. of Mathematics, Oklahoma Agri. and Mech. Coll., Stillwater, Okla.

These physical interpretations of mathematical statements should be stressed more and more to students. It is true that, often, a teacher of mechanics uses his integral calculus with a disregard for the true mathematical correctness of his juggling; but teachers of mathematics far too often advance mathematical laws without any effort to establish a link in the mind of the student with the more concrete evidences of their universal application. The authors have presented another demonstration of the relationship between derived curves and their integrals, and also have added to the geometrical analysis of the elastic functions of beams.

Harold E. Wessman,[22] Assoc. M. Am. Soc. C. E. (by letter).—Several writers of "engineering problems" texts have stated the general mathematical law of which the moment-area method and the shear-area method are special cases. In one text[23] appears the statement that:

> "In any curve, the length of the ordinate from the tangent to the curve at any point (called the first point) to any other point on the curve (called the second point) equals the algebraic sum of the moments of the areas between the ordinates of the corresponding points in the second lower curve, moments being taken about the ordinate through the second point."

Because this general law has such significance for engineers, one might expect to find it stated in some calculus text also. The writer has never seen an explicit statement of it, however, in the texts with which he is familiar. He is curious to know, simply for purposes of information, whether the authors have ever encountered it in any mathematical text. Teachers of calculus for engineers could well afford to give the general law more emphasis.

The special application of this law in the moment-area theorems of Mohr and Green is familiar to many engineers. The application *via* the shear-area concept of the authors is not so well known. That is not strange when one realizes that, in college, the embryo engineer is usually introduced to the double integration method of finding beam deflections, a method which begins by expressing the relationship between curvature and moment in the familiar equation, $\frac{d^2y}{dx^2} = -\frac{M}{EI}$. He performs the calculus operations involved in solving this equation to find slopes and deflections. He is then, in many places, exposed to the moment-area method (often cluttered up with an imposing title) which parallels the calculus method, but which fortunately places the emphasis upon geometrical concepts.

In teaching the subject of deflections, the writer always portrays, geometrically, the five curves of load, shear, moment (or curvatures), slope, and deflection similar to those shown in Fig. 1, but he supplements these with a picture of a deformed segment of a beam. It is logical to reason geometrically from curvature of beams to deflections of beams. Since curvature at a point is expressed in terms of the moment at the same point, the moment-area method has some physical significance for the student. It does not seem quite so satisfactory to reason from the rate of change of curvature to the slope of the beam and then to deflection, even though in some cases it may be simpler to evaluate a triangular shear area rather than a curved moment area.

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[22] Assoc. Prof. of Structural Eng. and Mechanics, Coll. of Eng., Univ. of Iowa, Iowa City, Iowa.

[23] "Engineering Problems Manual" by F. C. Dana and E. H. Willmarth, p. 46.

In the classroom, every teacher is confronted with the task of selecting the minimum essentials in this age of voluminous theory and with the importance of making those essentials clear to the student; hence, it is advisable, particularly in undergraduate work, to give the student the best possible tool and to make him familiar with that tool, rather than give him a nodding acquaintance with a variety of tools which have the same purpose.

This is not to be construed as a depreciation of the authors' paper. They have performed a service in presenting this special concept. Such contributions give the members of the profession an opportunity to weigh and then select that which is best suited for particular needs.

FANG-YIN TSAI,[24] ASSOC. M. AM. SOC. C. E. (by letter).—In applying the principle of moment areas to determine the slope and deflection in beams under flexure, two different methods are available. One is the so-called slope-deviation method introduced by Professor Charles E. Greene[25], of the University of Michigan, and the other is the elastic-weight or conjugate-beam method presented by H. H. Westergaard[26], M. Am. Soc. C. E., and by Professors Otto Mohr[27], and H. Müller-Breslau.[28]

In the slope-deviation method, the following two well-known theorems apply when a beam with a varying moment of inertia is subjected to bending:

1.—The angle between the tangents drawn at any two points on the elastic curve is numerically equal to the area of the $\frac{M}{EI}$-diagram between the ordinates at the two points; and,

2.—The deviation of any point from the tangent drawn at any other point on the elastic curve is numerically equal to the moment of the area of the $\frac{M}{EI}$-diagram between the ordinates at the two points about the first point.

The foregoing two theorems have proved very useful for determining the slope and deflection in beams, particularly when the point of zero slope on the elastic curve is known or easily found as in the case of a cantilever beam or a beam supported and loaded symmetrically. In the shear-area method, no such simple and elegant relationships exist between the shear-area and the slope and deflection as in the case of the moment-area method. Such relationships as these are prone to be so complicated and cumbersome as to have no practical value. The authors are to be commended for developing the shear-area method, not along the line of the slope-deviation theory, but along the line of the conjugate-beam theory (or "mathematical" beam, as named by the authors).

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[24] Prof. of Structural Eng., Dept. of Civ. Eng., National Tsing Hua Univ., Peiping, China.

[25] *Michigan Technic*, June, 1910.

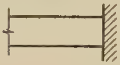
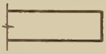
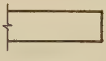
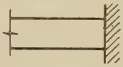
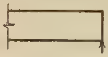
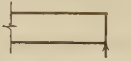
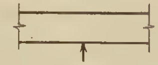
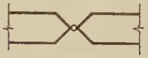
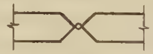
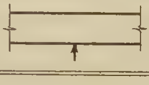
[26] "Deflection of Beams by the Conjugate Beam Method", *Journal*, Western Soc. of Engrs., Vol. 26 (1921), p. 369

[27] "Beitrag zur Theorie der Holz-und Eisen-Construktionen", *Zeitschrift* d. Arch. u. Ing. Vereines z. Hannover, Vol. 14 (1868), p. 19.

[28] "Beitrag zur Theorie des Fachwerks", *Zeitschrift* d. Arch. u. Ing. Vereines z. Hannover, Vol. 31 (1885), p. 418.

Although the foregoing discussion illustrates one of the disadvantages of the shear-area method as compared with the moment-area method, the use of the mathematical beam is not entirely satisfactory. In the conjugate-beam method of moment areas, the slope and deflection at any section of a given beam corresponds to (and also equals), the shear and moment, respectively, at the same section of the conjugate beam loaded with the $\frac{M}{EI}$- diagram of the given beam, provided the supporting conditions of the conjugate beam have been properly chosen in accordance with those of the given beam. For instance, if the given beam has a fixed support at a certain end, at which both the slope and deflection equal zero, the same end of the conjugate beam must be supported so as to have both the shear and moment equal to zero; that is, the end must be free. Table 2 gives the proper supporting conditions for a conjugate beam corresponding to those of the given beam. (The notation is that of the paper.) With the supporting conditions for the conjugate beam properly chosen in accordance with Table 2, the shear and bending moment curves of the conjugate beam will be identical with the slope and deflection curves, respectively, of the given beam, not only in magnitude, but also in sign if a proper sign convention has been adopted.

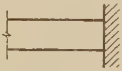
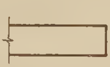
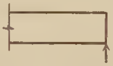
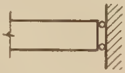
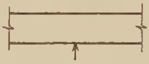
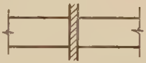
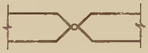
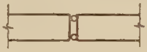
TABLE 2.—SUPPORTING CONDITIONS FOR THE CONJUGATE BEAM (THE MOMENT-AREA METHOD)

CASE	REAL BEAM			CONJUGATE BEAM		
1	Fixed		$\phi = 0$ $y = 0$		$V = 0$ $M = 0$	Free
2	Free		$\phi \neq 0$ $y \neq 0$		$V \neq 0$ $M \neq 0$	Fixed
3	Simple		$\phi \neq 0$ $y = 0$		$V \neq 0$ $M = 0$	Simple
4	Continuous		$\phi \neq 0$ $y = 0$		$V \neq 0$ $M = 0$	Hinge
5	Hinge		$\phi \neq 0$ $y \neq 0$		$V \neq 0$ $M \neq 0$	Continuous

The solution is very simple and straightforward, and the same procedure is equally applicable to beams with any supporting conditions and with the moment of inertia varying in any manner. No special devices such as the arbitrary introduction of two concentrated loads for the abrupt change in moment of inertia (Fig. 10) in the shear-area method are necessary.

In the shear-area method, the shear, bending moment, and slope at any section of the mathematical beam represent, respectively, the bending moment, slope, and deflection at the section of the real beam. As stated by the authors in Assumption (6), the supporting conditions of the mathematical beam must be determined in accordance with those of the real beam in the same manner as in the moment-area method. The writer has made a study of this phase of the problem the results of which are presented in Table 3.

TABLE 3.—SUPPORTING CONDITIONS FOR THE MATHEMATICAL BEAM (THE SHEAR-AREA METHOD)

CASE	REAL BEAM			MATHEMATICAL BEAM		
1	Fixed		$M \neq 0$ $\phi = 0$ $y = 0$	?	$V \neq 0$ $M = 0$ $\phi = 0$	?
2	Free		$M = 0$ $\phi \neq 0$ $y \neq 0$	?	$V = 0$ $M \neq 0$ $\phi \neq 0$	?
3	Simple		$M = 0$ $\phi \neq 0$ $y = 0$		$V = 0$ $M \neq 0$ $\phi = 0$	Two Rollers
4	Continuous		$M \neq 0$ $\phi \neq 0$ $y = 0$		$V \neq 0$ $M \neq 0$ $\phi = 0$	Fixed at Support
5	Hinge		$M = 0$ $\phi \neq 0$ $y \neq 0$		$V = 0$ $M \neq 0$ $\phi \neq 0$	Two Rollers

It is to be noted that the writer has failed to find the proper supporting condition for the mathematical beam (indicated by interrogation points in Table 3) to correspond with both the fixed and free ends of the real beam, since it seems impossible to devise an end-supporting condition with any physical meaning at which both the bending moment and slope will be equal to zero or not equal to zero at the same time. The authors have used a free end for the mathematical beam to correspond to the fixed end of the real beam, and *vice versa*, as shown in Figs. 8 and 9. This is incorrect. According to Assumption (5) of the paper, "the slope at any section of the mathematical beam represents the deflection at the same section of the real beam." At the free end of the mathematical beam the slope is not equal to zero; therefore, at the fixed end of the real beam, the deflection likwise must not be equal to zero, which is just opposite to the actual condition of a fixed end.

Similar inconsistencies may be cited against the authors' use of the fixed end for the mathematical beam to correspond to both the free and the simply supported end in the real beam. When the real beam has a simply supported end, at which $M = 0$, $\phi \neq 0$, and $y = 0$, the same end of the mathematical beam must be supported by two rollers in a vertical plane, as shown in Table 3, so that the corresponding values are $V = 0$, $M \neq 0$, and $\phi = 0$. Although, any value can be assigned, arbitrarily, to the shear, moment, and slope at any support of the mathematical beam, such a procedure will not only make meaningless the various types of conventional supports for the mathematical beam and the real beam, but will also cause serious confusion in the application of the method.

The conjugate-beam method of moment areas is based upon the similarity between the following two sets of well-known equations:

$$w = \frac{dV}{dx}\text{; and, }\frac{M}{EI} = \frac{d\phi}{dx}$$

and,

$$w = \frac{d^2 M}{dx^2}\text{; and, }\frac{M}{EI} = \frac{d^2 y}{dx^2}$$

Hence, if $w = \dfrac{M}{EI}$, then $V = \phi$ and $M = y$, the loading of the conjugate beam for finding the slope and deflection being the same. Similarly, the mathematical beam method of shear areas is based upon the similarity between the following three sets of well-known equations:

$$w = \frac{dV}{dx}\text{; and, }V = \frac{dM}{dx}$$

$$w = \frac{d^2 M}{dx^2}\text{; and, }\frac{V}{EI} = \frac{d^2 \phi}{dx^2}$$

and,

$$w = EI\,\frac{d^3 \phi}{dx^3}\text{; and, }V = EI\,\frac{d^3 y}{dx^3}$$

Hence, the loading of the mathematical beam for finding the bending moment and deflection is the V-diagram only, and that for finding the slope is the $\dfrac{V}{EI}$ -diagram, the loading of the mathematical beam for the three cases being thus not the same. In this connection, it may be also noted that Assumption (2) is not well founded, and that, if the V-diagram only is used as loading, the shear and slope at any section of the mathematical beam will not only "represent", but will also be numerically equal, respectively, to the bending moment and the deflection at the same section of the real beam.

Thus, despite the slight advantage in that the shear diagram is usually a somewhat simpler figure than the moment diagram for the same loading, the shear-area method has many inherent disadvantages as compared with the moment-area method.

David A. Molitor,[29] M. Am. Soc. C. E. (by letter).—In their attempt to popularize the shear-area method of analyzing beams, the authors have not presented a very convincing claim for simplicity. Although it is true that shear diagrams are more easily drawn than moment diagrams, it does not follow that the former afford any material advantages over the latter in solving structural problems. Slopes and deflections are more directly derivable from moments than from shears, for which reason the shear-area method has not received much consideration.

In the stress analysis of simple beams or frames involving redundancy, the stresses are always evaluated from the moments at the critical sections to be analyzed. Therefore, the simplest and most direct method of approach is to deal with moments first, last, and all the time. The derivation of shears from known moments is relatively easy, whereas the reverse operation is attended with more difficulty. The slope of the elastic curve can be deduced from the moment curve by the method of area moments, without resorting to shear areas, by remembering that the slope at any point of a moment curve is represented by the moment derivative for that point. This is expressed in mathematical terms by the formula,

$$\frac{dM}{dx} = \tan \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (190)$$

in which M is the moment at any point, m, acting over a small increment of beam length, dx, and β is the slope angle of the moment curve; but $\frac{dM}{dx} = V$ is the shear at the point, m, where the moment is taken.

By area moments, the deflection, y_m, at the point, m, is equal to the moment of the elastic weights, $M\ dx$, about the point, m, divided by $E\,I$. Therefore, the shear of the elastic weights on one side of the point, m, divided by EI, must equal the tangent of the slope angle to the elastic curve at the point, m. It should be noted that the elastic weights for negative moment areas are likewise negative.

By the same reasoning, the tangent of the slope angle of the elastic curve at any end support, A, must equal the end shear of the moment area, or the end reaction of the elastic weights divided by EI. Hence, calling V_m the shear of the elastic weights on one side of the point, m, of any beam, and R_a, the end reaction of the elastic weights at the support, A, the tangent of the slope angle, ϕ_m, of the elastic curve at the point, m, becomes:

$$\tan \phi_m = \frac{V_m}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (191)$$

and the end slope at the support, A, becomes:

$$\tan \phi_a = \frac{R_a}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (192)$$

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[29] Structural Engr., Procurement Div., Public Works Branch, U. S. Treasury Dept., Washington, D. C.

Equations (191) and (192) are applicable to either the graphic or the analytic methods of area moments. When the moment of inertia is variable, the elastic weights should be chosen as $\frac{M\,dx}{I}$ instead of $M\,dx$.

It is regrettable that the authors introduced the term "mathematical beam" without giving a precise definition, and also that they referred to what they term the conjugate beam method instead of simply the method of elastic weights. It seems inconsistent to attach qualifying terms to the beam itself, merely because it may be loaded either with actual loads, with moment areas, or with shear areas treated as elastic weights. More properly, the distinction should relate to the method of loading and not to the nature of the beams, which remains unchanged.

The authors have wisely limited their application of the shear-area method to beams. In dealing with rigid frames, the method could scarcely find favor when compared with the universal and elegant solution afforded by the use of Mohr's work equation.

F. MULS-GUINOTTE,[30] ESQ. (by letter).—The well-known theorem which states that the shear diagram may be obtained by the successive derivatives of the elastic curve, is the basis of this paper. The theory of beams on elastic foundations is based on the principle that a fourth derivative of the elastic curve gives the loading diagram. The authors give the value of the successive derivatives in Fig. 1 of the paper, making the implicit assumption that the moment of inertia is constant. This assumption is not necessary. It is quite easy to obtain the elastic curve starting from the loading diagram when the rule of successive derivatives is written, as follows: $\frac{dw}{dx} = \phi_x$; $\frac{d\phi_x}{dx} = \frac{M_x}{E\,I_x}$ $\left(\text{or } M_x = E\,I_x\,\frac{d\phi_x}{dx}\right)$; $\frac{dM_x}{dx} = T_x$; and, $\frac{dT_x}{dx} = p_x$.

This suggestion for developing the p_x-diagram indicates how the equation of the elastic curve may be solved in the general case of a variable moment of inertia.

The variable loading of a tapered beam is shown in Fig. 48(*a*); the first integration yields the shear diagram (Fig. 48(*b*)); and, the second integration results in the moment diagram (Fig. 48(*c*)). If the latter curve is divided by the product, $E\,I$, at successive ordinates the resulting curve represents the M-function (Fig. 48(*d*)) which, when integrated, produces, successively, the slope curve (Fig. 48(*e*)) and the deflection curve (Fig. 48(*f*)).

It is also of interest to show the results of deriving the diagram of moments when the elastic beam is a curved one. Suppose, first, that there are no forces applied on the element, ds (Fig. 49), which is then kept in equilibrium by the forces, M, T, N, $M + d\,M$, $T + d\,T$, and $N + d\,N$. For the condition, $\Sigma\,M = 0$:

$$d\,M + T\,\Delta\,s\,\cos\,\Delta\,\theta + N\,\Delta\,s\,\sin\,\Delta\,\theta = 0$$

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[30] Asst. in Univ. of Liége, Liége, Belgium.

for $\Sigma N = 0$:

$$N - (N + dN) \cos \Delta \theta - (T + dT) \sin \Delta \theta = 0$$

and, for $\Sigma T = 0$:

$$T - (T + dT) \cos \Delta \theta + (N + dN) \sin \Delta \theta = 0$$

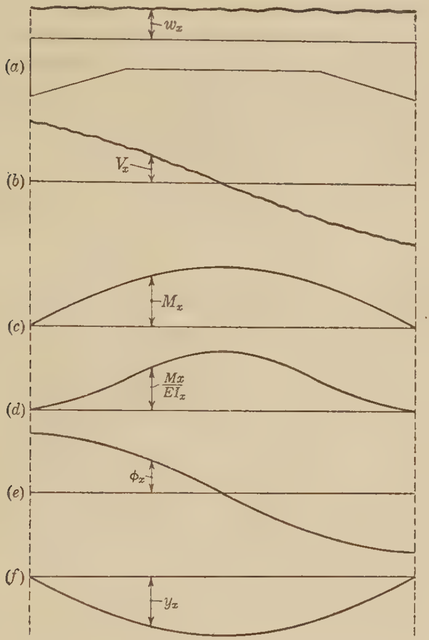

FIG. 48.

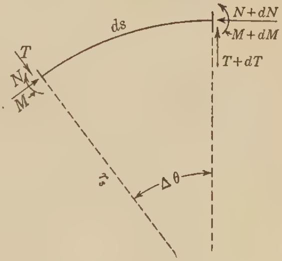

FIG. 49.

Since the element, *ds*, is very small: $\cos \Delta \theta = 1$; $\sin \Delta \theta = \Delta \theta$; and, $\Delta \theta = \frac{\Delta s}{r_s}$. When *ds* tends to become zero; $\frac{dM}{ds} = T_s$; $\frac{dT}{ds} = + \frac{N}{r_s}$; and, $\frac{dN}{ds} = - \frac{T}{r_s}$.

The coefficient, $\frac{1}{r_s}$, may be eliminated by using a derivative with respect to the variable, θ, instead of the variable, s. With this elementary change: $\frac{dN}{d\theta} = - T$; and $\frac{dT}{d\theta} = N$. Consequently, it is possible to define a rule by which the moment curve can be derived successively. Assuming that $T = \psi_s$: $\frac{dM}{ds} = \psi_s$; $\frac{d\psi_s}{d\theta} = N$; $\frac{d^2\psi_s}{d\theta^2} = - T$; $\frac{d^3\psi_s}{d\theta^3} = - N$; $\frac{d^4\psi_s}{d\theta^4} = T$; $\frac{d^5\psi_s}{d\theta^5} = N$; etc., and $\frac{d^{4n}\psi_s}{d\theta^{4n}} = T$; $\frac{d^{4n+1}\psi_s}{d\theta^{4n+1}} = N$; etc.

The foregoing properties of the successive derivatives may yield results of particular interest in the study of influence lines.

When external forces occur on the section of beam analyzed the components along the normal or the tangent appear in the equations of equilibrium. These components are easily introduced, and, in the case of radial forces, special and simple results are obtained.

HORACE B. COMPTON,[31] ASSOC. M. AM. SOC. C. E., AND CLAYTON O. DOHRENWEND,[32] JUN. AM. SOC. C. E. (by letter).—The shear-area method was presented to enlarge on the usual methods for the solution of slopes and deflections in beams. It was stated that the method was particularly adapted to those problems which involve distributed loads (when compared with solutions by the moment-area method) and still would solve other problems readily. Solutions which consider variation in the moment of inertia of the beam sections are probably obtained with a greater degree of safety by the moment-area method.

Since it appears that the engineer remembers the procedure for the solution by the conjugate beam for slopes and deflections better than by the slope-deviation method, the mathematical beam was introduced.

In most cases the tendency of the commentators has been to follow the very well known and common solutions of moment area and not to discuss the subject in question. Discussions of simple problems by the moment-area method need no further comment. However, in contrast, Professor Kepner with a few others has added materially to the subject.

The end slopes of the shaft indicated by Mr. Fischer may be solved by writing deflection equations about the ends of the shaft; thus, about the right end, $y = 0 = \phi_L L - \frac{1}{2} I_L$ by Equation (7). Therefore:

$$E \phi_L 60 - \frac{1}{2} \left[56.5 \times \frac{23^3}{12} + 56.5 \times 23 \times 48.5^2 + 14.71 \times \frac{5^3}{12} - 961.2 \times 37^2 \right.$$

$$+ 14.71 \times 5 \times 34.5^2 - 2.27 \times \frac{12^3}{12} - 2.27 \times 12 \times 26^2 + 1\,072 \times 15^2$$

$$\left. - 19.24 \times \frac{5^3}{12} - 19.24 \times 5 \times 17.5^2 - 90.68 \times \frac{15^3}{3} \right] = 0$$

and $\phi_L = \frac{16\,474}{E}$. Mr. Fischer has indicated a similar procedure by letter. This method shortens the problem by the shear-area method considerably. The difficulty of Mr. Peck's problem can be overcome in the same manner.

The writers wish to express their appreciation for the constructive criticism offered by Professor Kepner, especially for demonstrating, by the use of Equation (7), a short method for obtaining end slopes. In correspondence submitted directly to the writers, Mr. Stewart suggested the following for the problem solved by the conjugate beam method in Mr. Beatty's discussion:

$$\phi_L = \frac{1}{E\,I} \left[2\,400 \times 12 + 1\,800 \times 8 \right] 144 = \frac{6\,220\,800}{E\,I}$$

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[31] Asst.. Prof. of Mechanics, Rensselaer Polytechnic Inst., Troy, N. Y.

[32] Instr., Dept. of Civ. Eng., Rensselaer Polytechnic Inst., Troy, N. Y.

It is evident that the solution for the reaction can be given in a somewhat simpler manner than that indicated by Mr. Beatty by considering the area of *BCDGB* (Fig. 40(*c*)) as two-thirds of a rectangle with dimensions, *BG* and *GD*.

In the discussion by Messrs. Johnson and Birkeland it is inferred that there is no difference in the work necessary to solve the first moment of the moment area, or the second moment of the shear area, because in the two cases the integrations are comparable. This inference is not supportable except in deriving the two cases, since the use of the cases is not by integration directly, but by replacing the integration with equivalent expressions. The mathematical demonstrations given, showing the relations between the functions is very complete, but the following formula, due to Lejeune Dirichlet:

$$\int_a^b dx \int_{y_2}^{y_1} f(x, y)\, dy = \int_c^d dy \int_{x_1}^{x_2} f(x, y)\, dx$$

necessary in the solutions shown, is not common enough to engineers to use without some explanation. If more emphasis had been given to specific cases the following statement would have been considerably modified: "The method does not seem to be readily adaptable to any but the most simple beam problems such as statically determinate, constant section cases with uniformly distributed load."

Most of the apparent difficulties encountered by Professor Tsai will be clarified if it is recalled that the mathematical beam is hypothetical and has slope with no deflection. The loading of the mathematical beam is the shear diagram; the first moment yields the slope of the given beam only when divided by *E I*, and the second moment or moment of inertia gives deflection of the original beam only when divided by 2 *E I*. This means that for the two cases mentioned, the diagram on the beam can be considered as the shear divided by *E I*.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1944

FLOOD-STAGE RECORDS OF THE RIVER NILE

By C. S. Jarvis,[1] M. Am. Soc. C. E.

With Discussion by Messrs. Halbert P. Gillette, R. W. Davenport, H. E. Hurst, Thomas H. Means, J. W. Beardsley, J. C. Stevens, Jesse W. Shuman, Kamel Osman Ghaleb, and C. S. Jarvis.

Synopsis

The need for longer hydrographic records than are now available on American streams, as disclosed by recent investigations dealing with floods and their various characteristics, has led the writer to a study of the records of foreign rivers. The paper affords only a brief summary of outstanding records or evidences regarding stream-flow trends of European and Asiatic rivers, and then undertakes the presentation of stages of the Nile River at the Roda gauge, in Cairo, Egypt, covering a period of thirteen centuries. Additional data, such as those procured at Aswan, dealing with both stages and volumes, and various versions of divergent data, are added to make the graphic representation as nearly complete as is practicable on one continuous chart. No attempt is made at this time to evaluate the data, as they are too numerous and complex to permit ready appraisals. The main purpose is a presentation of records, or some kind of derivation from actual observations, as a preliminary to further study by the profession.

General Considerations

The study of flood frequencies and magnitudes by the United States Geological Survey in collaboration with, and under the sponsorship of, the Mississippi Valley Committee and its successor, the National Resources Board (Water Planning Committee), has advanced some interesting information relating to long-period records of river stages. These records were investigated with a view to furnishing a background for the interpretation of available hydrographic records in the United States, most of them covering relatively short periods.

Note.—Published in August, 1935, *Proceedings*.

[1] Hydr. Engr., Soil Conservation Service, Washington, D. C.

A few American flood records extend intermittently into the Colonial era —notably the Ohio River, at Pittsburgh, Pa., and the James, Roanoke, Delaware, Mississippi, and other rivers at or near tide-water. In this manner a sketchy and somewhat uncertain notion may be derived as to the comparative river stages, or the corresponding discharges, where the channel sections and controls have not changed materially; but at best a period of only about 200 yr may be thus covered in this country from written records or satisfactorily authenticated marks.

The study of drift or débris deposits, the heights of mounds distributed along the flood-plains of rivers and presumably intended as islands of refuge (as on the Lower Mississippi and in the Nile Delta), the character of trees and other vegetation along various contours, the evidences of either channel scouring or filling, and consideration of other prominent geologic and geomorphic features may add significant items of information relating to the occurrence of floods in the past, possibly within a few hundred years.

What can be said more definitely of such floods as have left their marks, and of similar contemporaneous events of equal importance that have left no physical effects or record marks clear enough for recognition and separate interpretation? Some investigators have expressed the opinion that a continuous record of stages on a large stream, over a period of 500 or 1 000 yr, for example, might indicate more definitely the trends and cyclic or other recurrent periods of stages above or below normal than several shorter records of equal or greater aggregate length. Before such long-period records of foreign streams can serve a useful purpose most effectively, account must be taken of the differences in physical factors as affecting run-off habits. This applies particularly in the present study, which concerns flood frequencies and magnitudes, to those influences that would either retard, accelerate, augment, dissipate, or otherwise change, the discharge habits of the stream or modify the discharge capacity of the channel. Thus, the regimen of a river may be affected by: (*a*) Changes in the vegetative cover of a drainage area; (*b*) erosion; (*c*) increase or decrease of natural or artificial storage either in basins or on flood-plains; (*d*) diversion to other drainage basins by either natural or artificial means; (*e*) restriction of the river channel by levees; (*f*) facilitation of flow by dredging and removal of obstructions; or (*g*) formation or abandonment of auxiliary channels and by-pass areas. Furthermore, any one of Items (*a*) to (*g*) is likely to interfere with satisfactory interpretation of records.

What is more important, available human experience is often capable of making fairly intelligent approximations of such influences, as, for example, by recognizing that one factor may be additive, another may be subtractive, and the algebraic sum may be of small amount. Thus, it has been shown for several stations on large foreign rivers, and on the Upper Mississippi River, that the dredging of bars, removal of snags and other obstructions, rectification of alignment, and improvement of the hydraulic elements of the channel, tend to lower the river surface; whereas channel encroachments and the development and protection of great areas among the natural flood-plains cause a corresponding rise of stage. The actual effect is the resulting difference of these two factors, normally lowering the stages of low water and raising the stages of high water, the net effect being to increase the range of variation in stages, except where regulation works are used to produce opposite effects.

Flood Records of Rivers in Europe

Among the rivers of Europe, the Thames, the Seine, the Rhine, the Rhône, the Loire, the Po, and the Danube seem to afford the best information. It is realized that the progressive changes which these river channels and drainage systems have undergone may vitiate, more or less, the close comparability of stages as indexes of flood magnitudes over long periods; nevertheless, the study of outstanding flood catastrophes should afford a valuable record of experience. The floods of the Rhône, the Loire, and the Seine Rivers in the years 563, 572, and 583 A. D., respectively, have been described briefly in available records. Thereafter, until about 1 000 yr later, the published fragmentary data are qualitative rather than quantitative; certain outstanding inundations, which were equal to or worse than others, are designated by date or by year of occurrence. The usual standards for comparison seem to have been based on loss of life and damage to property, and in this connection the suddenness of flood rise without warning played a prominent part; therefore, the floods at unusual or unexpected seasons would naturally be mentioned.[2]

Within the last 300 yr the quantitative or magnitude relations of floods have entered into the records of European streams. Thus, the foregoing references, together with those listed by Kuichling,[3] show that the flood in the Seine River Valley on January 28, 1910, and that of February 27, 1658, at Paris, were of approximately the same stage (8.40 m and 8.81 m, or 27.6 ft and 28.9 ft, respectively, above the zero of the Pont de Tournelle); and that the maximum estimated discharges were of a similar order (88 300 and 84 500 cu ft per sec, respectively). The greater volume and lesser gauge height thus applied to the flood crest of 1910. No other floods on the Seine during a period of nearly 300 yr approached within 15% of the maximum recorded for 1910. The catastrophic occurrences during the period, 683 to 1658, A. D., probably included several of nearly the same order, to judge from descriptive accounts of complete submergence of homes, undermining of bridges and other structures, and general devastation, including the demolition of fortification walls during the flood and the accompanying earthquake. The question naturally arises as to whether the recorded earth tremors were the cause or the effect of the structural collapse.

The year 1012 A. D., witnessed intense rainfall and the "great flood" of the Danube River at Vienna; likewise, along the River Rhine, homes were inundated, and a multitude of people and their live stock were destroyed. The margins of forests intruding into the flood-plains were wiped out during a catastrophe that was without parallel in all the preceding history of that region. Prior to 1012 A. D., as in present times, the sub-normal river stages of successive years created a false sense of security that left the people unprepared for the eventual torrent.

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[2] "Materiaux pour l'etude des calamities." 5e annee, nos. 17–20, Genéve, Soc. Géographic, 1928–29; "Les inondations en France," par Maurice Champanion, 6 vol., Paris, 1858–64 (deals with flood events from the Sixth Century to 1862); "Forests and Floods," by the late H. M. Chittenden, M. Am. Soc. C. E., *Engineering News*, October 29, 1908, p. 467 (discusses report of Ernst Lauda, Chief of the Hydrographic Bureau of the Austrian Government (Hydrographischen Dienst in Osterreich: Beitrage Hydrographic Osterreichs, Heft 4, pp. 155–162, Wien, 1900)); "Le Seine," par E. Belgrand, Inspector General Ponts et Chaussées, 1869. (Includes a discussion of notable floods beginning with 1649); and *Engineering News*, Vol. 63, p. 327, 1910.

[3] "Flood Flows," by W. E Fuller, M. Am. Soc. C. E.; discussion by the late Emil Kuichling, M. Am. Soc. C. E, *Transactions*, Vol LXXVII (1914), p. 643.

The Danube flood of 1342 was reported to have caused the loss of 6 000 human lives. In the flood of 1501 on the Danube several marks were established on monuments and are still preserved as permanent records. Comparison of this event with another devastating flood, in 1899, at fourteen stations in the same valley, shows that the range above low-water stage was 7.24 to 16.58 m (23.8 to 54.4 ft) in 1501 and only 6.52 to 13.18 m (21.4 to 43.2 ft) in 1899, the lower stages perhaps being due in part to channel improvements in the interest of navigation.

The scattered references in early records show that flood heights above either normal or low-water stages are so nearly comparable with those of later centuries as to give support to the opinion that no radical progressive changes of intensities or frequencies of floods and droughts have taken place in historic times on any extensive river systems, except as logical consequences of regulation, diversion, utilization, or other activities of Man. On the other hand, no direct proof is available regarding the trends of climate and rainfall throughout historical periods, such evidence as exists being based on considerations other than physical measurements. On extending the period considered into geologic time, however, it is indisputably established that marked changes in both temperature and rainfall have taken place by gradual processes.

Flood Records of Rivers in Asia

Traces of irrigation, regulation, channel-training, and flood-protection works have survived from early centuries of historical epochs in various civilizations, or even from times before the earliest records now available for some countries, particularly in Asia. These remnants of a past civilization seem to supply indisputable proof of a long-continued similarity of climatic, rainfall, and run-off habits. The only possible explanation and justification for such engineering works must be found in deficient local rainfall and in reliable stream flow, even when the discharge was flashy and irregular. This combination of circumstances presumably reflects the occurrence of storms among the head-waters and intervening catchment areas, or the thawing of snow and ice accumulations.

The records of several Asiatic rivers, although enveloped in vagueness, extend back to remote periods of history. Thus, the series of irrigation canals in Mesopotamia, with combined original capacities much greater than the reliable flow of the Euphrates and Tigris during recent times, do not necessarily betoken a shrinkage of water supply since they were built. According to opinions of engineers who have studied the situation on the ground,[4] new canals were established as the irrigated tracts under the older projects became water-logged and overcharged with alkali, for want of systematic drainage, as often happens under modern practice. The important facts to note, however, are that the rainfall was deficient throughout the historical period of Babylon, Nineveh, and neighboring cities, and that the rivers were both fairly stable and reasonably tractable, to permit such bridging, navigation, and diversion for irrigation at that time. It is plain that irrigation was found necessary to raise crops and to sustain life in that country, and that the rivers were fairly reliable sources of supply. During the festival in November, about 539 B. C., Cyrus, the Persian, captured Babylon by the simple expedient of diverting the Euphrates River into the desert through one or more canals and marching his troops through the walls along the channel thus made available. This feat is quite consistent with the known habits and volume of discharge of the Euphrates as observed during the Nineteenth Century.

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[4] "Euphrates River and Valley," by Francis Rawdon Chesney, Lond., 1868

Confirmatory evidence concerning both the variations and the stabilization of water resources lies in the rainfall records extending back more than 200 yr at a few stations, mainly in Europe, and considerably more than 100 yr at many other places. These records parallel the meager hydrographic records, and both sources furnish valuable interrelated and, therefore, mutually supporting data.

Tradition and history combine to give descriptive accounts of notable floods and droughts in India and in China. The 1930 report of the Huai River Commission, Bureau of Engineering,[5] lists the floods of 1649, 1741, 1879, and 1921 as the outstanding ones on this important northern tributary of the Yangtze River. They were measured indirectly by the extent of the inundation on populated land, the total number of "districts" inundated being reported as 5, 11, 6, and 10, respectively, whereas the usual damaging inundations affect only 3 or 4 such districts. With all available data before it, the Commission adopted the flood of 1921 as the basis for designing the Huai River regulation. Presumably, this decision was not entirely due to the high discharge rate, but also to the sustained volume, which was reported as having inundated 11 740 sq km (4 533 sq miles) to a depth of 1 to 4 m (3.3 to 13.1 ft). It is noteworthy that the expected 100-yr flood on the Huai derived from a study of rainfall, is 15 500 cu m per sec, whereas Fuller's formula gives only 11 700 cu m per sec. Reduced to English units, these rates become 548 000 and 414 000 cu ft per sec, respectively, or 8.55 and 6.46 cu ft per sec per sq mile. The maximum recorded flood discharge, which occurred in 1916, was 12 900 cu m per sec (455 000 cu ft per sec, or 7.10 cu ft per sec per sq mile). perhaps representing a 30 or 35-yr average frequency. The failure of levees during the more severe flood of 1921 prevented even approximate measurement of the maximum discharge.

THE RIVER NILE IN EGYPT

It is freely admitted that the foregoing brief references to available long-period river-stage records and to what these data seem to show, should be supplemented by further research in this inviting field perhaps in a separate paper. The presentation of fairly complete river-stage data, so far as they are now available and capable of interpretation, is limited herein to the Nile River, in Egypt. (See Fig. 1.)

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[5] "Projects of Flood Control and Irrigation for the Huai River System," Official Tech. Rept. 1, p. 23, Huai-Yin, China, 1930.

The nilometer records are most notable with respect to antiquity, continuity, and permanence. The annual inundation of the Nile Valley in the late summer generally supplied enough moisture to the soil to insure fair crops in the fall and winter. A subnormal flood meant restricted plantings and deficient harvests; and well-sustained high water, at or above "wafa"—the stage insuring plenty—meant bounteous crops on more and more of the fertile alluvial land, as long as the stage did not overtop the protective works and islands of refuge. Thus, it happened that the most important annual event in Egypt was the Nile flood; and, therefore, records were engraved on the cliff walls in various places, notably at Semna, a section of the Second Cataract. In this locality, 179 distinct markings have been discovered dating back to about the 12th Dynasty; that is, 1 750 to 1 800 yr B. C., or earlier. Of these markings 18 on the west bank are about 8 m, or nearly 27 ft, above the flood range of recent centuries, but the difference probably reflects to a considerable degree the erosion in both channel walls and floor and does not necessarily indicate any pronounced or progressive change in discharge from century to century.

On that fragment of an ancient inscribed monument known as the "Palermo stone," because it now reposes in the Palermo Museum, in Sicily, there are notations regarding the successive ruling monarchs of the first five dynasties, dating back to between 3 000 and 3 500 yr B. C.; there are also records of such important events as military campaigns and public festivals. These notations include mention of the rise of the Nile during several consecutive years, apparently recording a range between 1 and 8 cubits (2 to 14 ft) somewhere in the Delta,[6] but with gauge readings usually about midway. Although it is difficult to tie these records to any particular section of the Nile Valley, they establish the importance of the Nile flood in making Egypt the never-failing storehouse of grain for export, especially in periods when droughts and famine prevailed in neighboring countries. Likewise, they indicate the irregular distribution of flood heights over a range quite comparable with that observed in the past century, prior to the construction of large regulative works.

Fortunately, several sections of nilometers, at various points along the stream channel, have been discovered, and their inscriptions have been deciphered and correlated. Capt. H. G. Lyons[7] states that the height above sea level of corresponding points of the ancient nilometers in Egypt and Nubia have been determined with the aid of bench-marks of the Irrigation Department, and that the zero points of the nilometer scales were found to lie in a line inclined to the north. In Nubia, according to Lyons, the line is sensibly parallel to the water slope, but north of Aswan it is considerably less inclined than the flood slope. As a consequence, a flood was indicated by a higher reading on the nilometer at Aswan than on that of Roda, at Cairo. This is offered as an explanation of the statement found in the Egyptian inscriptions and the works of the Greek and Roman authors that, in very good years, the Nile River rose 28 cubits (46.5 ft) at Elephantine, 21 cubits (36.12 ft) at

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[6] "Egypt," by James H. Breasted, Vol. 1, p. 51 *et seq.*, Univ of Chicago Press, 1906.

[7] "Physiography of the River Nile," by H. G. Lyons, pp. 316 *et seq.*, Cairo, 1906.

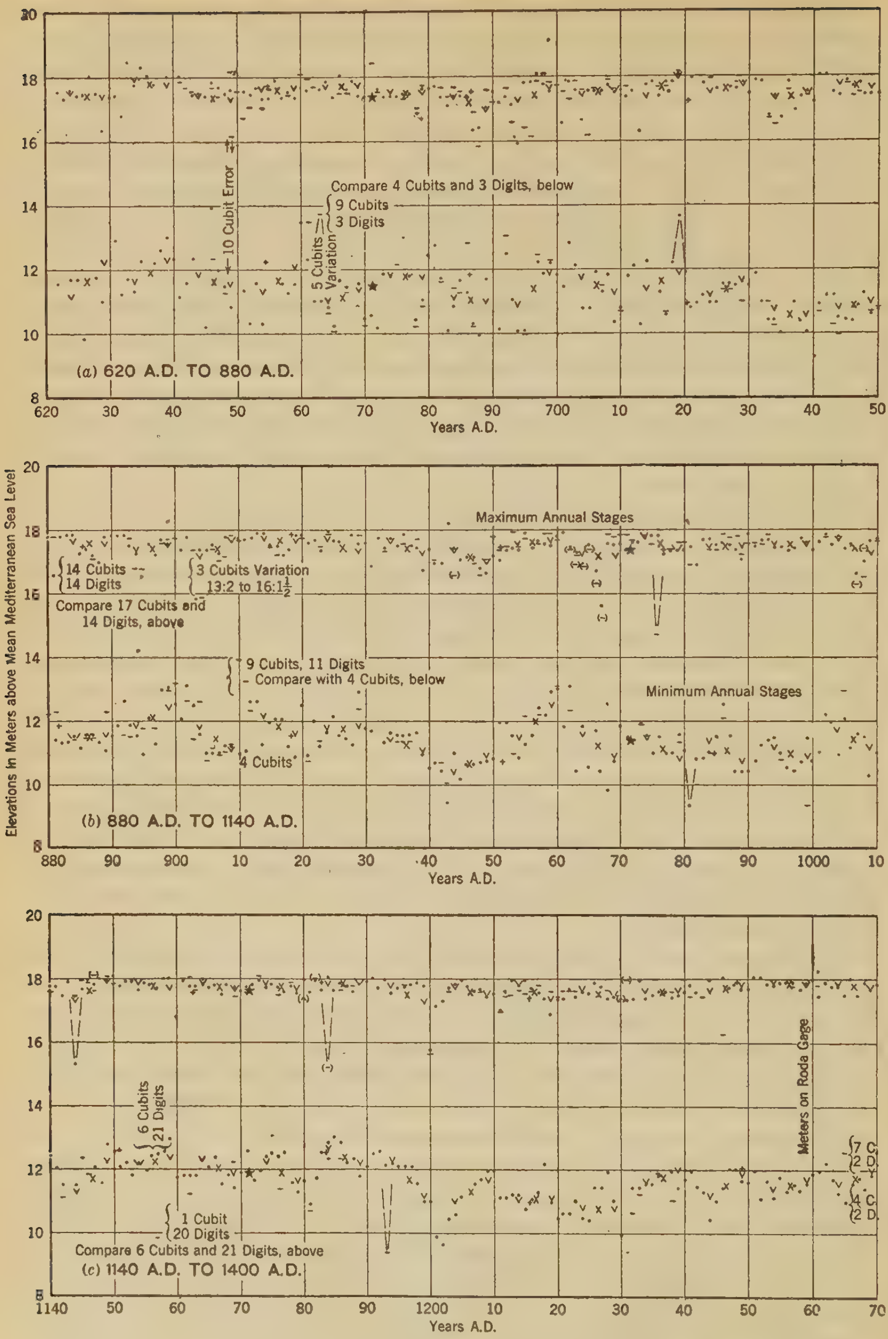
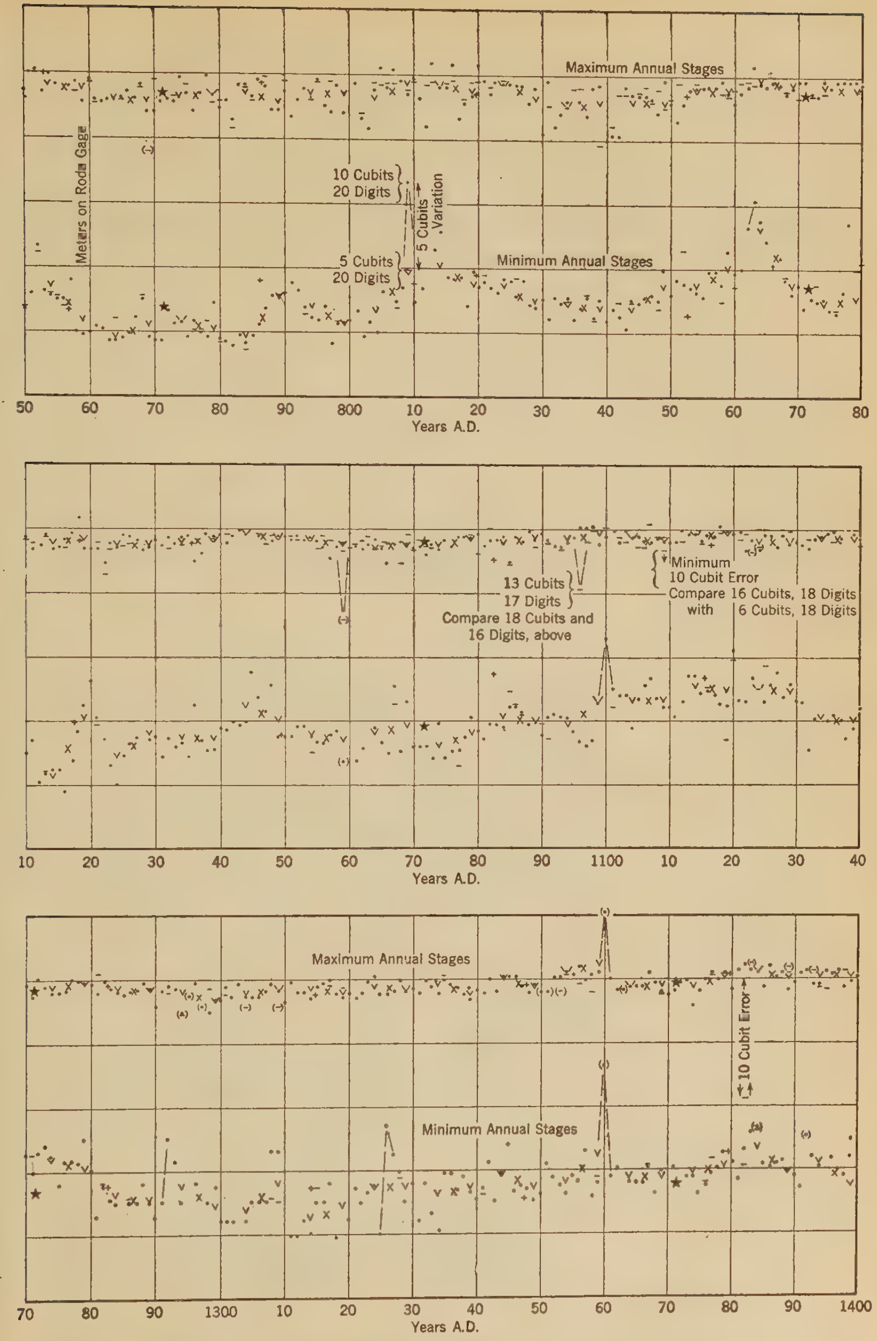


FIG. 1.—MAXIMUM AND MINIMUM STAGES OF THE



Nile River in Egypt, from All Available Records

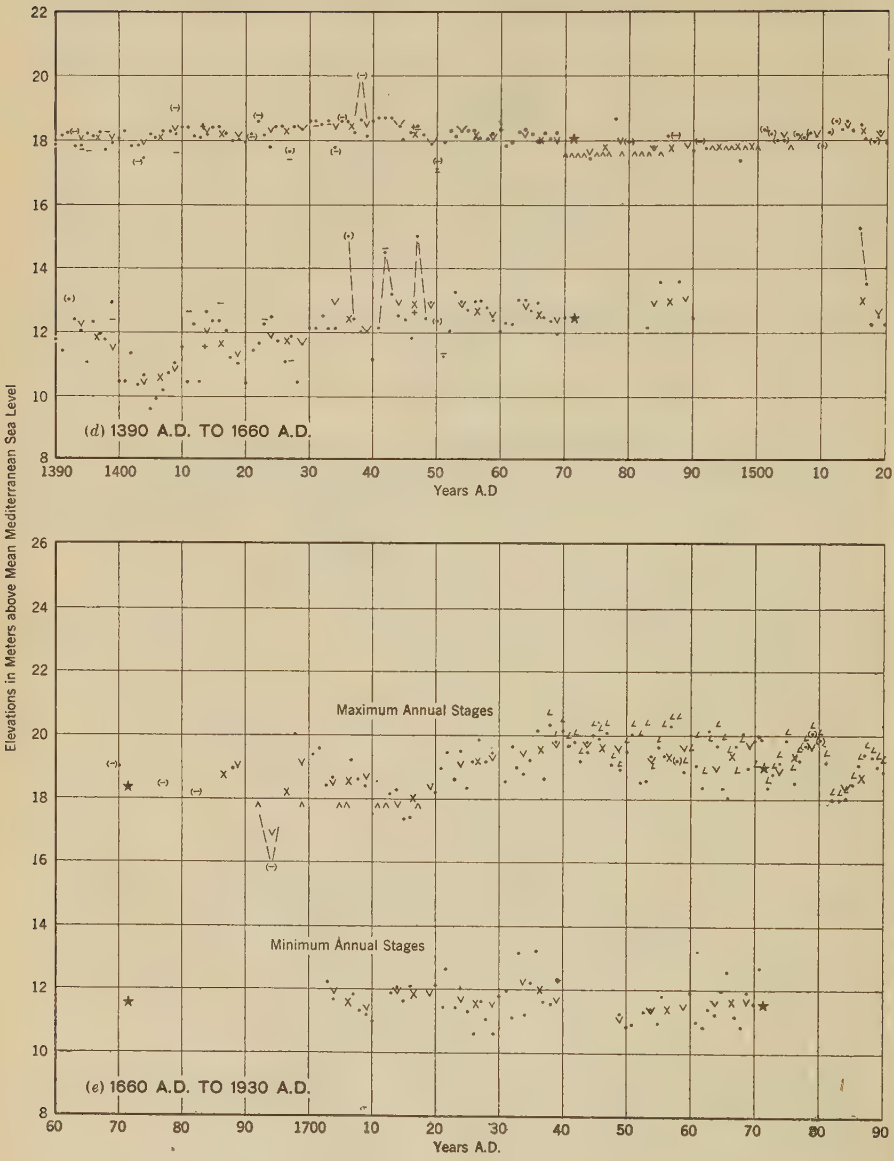
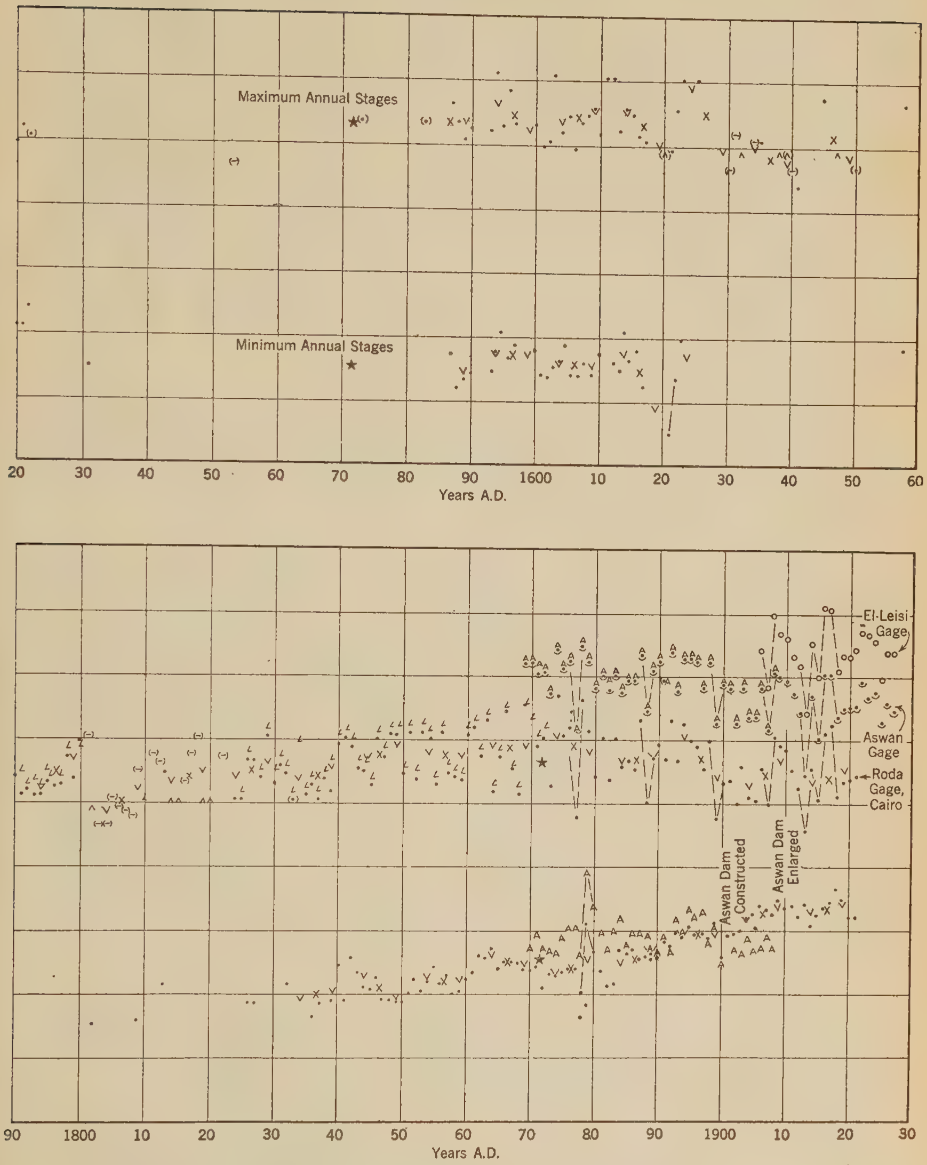


FIG. 1 (*Continued*).—MAXIMUM AND MINIMUM STAGES OF



- THE NILE RIVER IN EGYPT, FROM ALL AVAILABLE RECORDS

Koptos, 14 cubits (24 ft) at Roda, and 7 cubits (12 ft) in the Delta. The altitude of the zero point of the ancient nilometer at Roda may be obtained approximately from the slope of the line determined by the zero points of the remaining nilometers. After computing the secular rise of the bed one can conclude that the flood levels about 3 000 B. C. (which are recorded on the Palermo stone) and the later data concerning Nile floods given by the Greek and Roman authors agree well with those of to-day.

From the evidence he studied, Lyons was convinced that about 100 A. D. the Nile often rose to 24 cubits (41 ft) and sometimes to 25 cubits (43 ft) on the nilometer scale on Elephantine Island, at Aswan, below the First Cataract so that the high floods of that time reached the level of 91 m (299 ft) above sea level. In 1874 they reached 94 m (308 ft), or 3 m (10 ft) above the level of about 1 800 yr earlier, which corresponds to a rise of the bed of 0.17 m (6.7 in.) per century at this point. If the mean flood level of the period, 1870 to 1906, is taken, the height becomes 93 m (305 ft) and the rise, 0.11 m (4.3 in.) per century. Quoting Lyons:[7]

> "At Karnak in 1895 M. LeGrain found a series of 40 high Nile levels marked on the quay walls of the great temple. They date from about 800 B. C., and the mean altitude given by them for a high Nile is 74.25 meters [243.60 ft] above sea-level, while that of today is 74.93[8] [1906], showing a rise of the river-bed of 2.68 meters [8.8 ft] in 2 800 years, or at the rate of 0.096 meters [3.8 in.] per century."

In 1935, 76.93 m (252.3 ft) seems to be a better elevation than that reported by Lyons (74.93 m, or 245.8 ft) in 1906. If 0.15 m (5.9 in.) in a century is assumed as the average rate of rise of the Nile flood-plain, due to overflow and sedimentation, the 10-m (33-ft) average thickness of observed alluvial deposits might represent about 67 centuries of growth. Other methods of reckoning based, for example, on the maximum thickness of sediment, or the extension of the delta seaward, almost invariably indicate much longer periods of geomorphic development.

Similar conclusions regarding the rate of sedimentation follow the comparison of maximum flood height during the reign of the Roman Emperor Severus (about 200 A. D.), and the maximum flood mark noted by the French savants during the Napoleonic Invasion, in 1800 A. D., was 2.11 m (6.9 ft) higher.[9] If those two floods and their available channels were comparable, the rate of aggradation of the Delta was $\frac{2.11}{16} = 0.132$ m (5.2 in.) per 100 yr. Incidentally, that rate seems to be slightly greater than the rate of erosion at the First Cataract, about 600 miles above Cairo, derived by a comparison of maximum flood crests, on the assumption that they were of almost equal magnitude near the beginning and the end of the last 1 900 yr.

The durable outcropping ledge of Syene granite (syenite) forming the First Cataract, at Aswan, from which great obelisks and other monuments were quarried for Lower Egypt and eventual transfer to other countries, accounts for the relatively slight erosion here compared with 8 m (26 ft) in about 3 700 yr, or an average of 0.22 m (8.7 in.) per century, at Semna, on the Second Cataract, where the outcropping stone is not so durable and where sandstone or other sedimentary formations predominate along the canyon walls.

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[8] 74.25 + 2.68 = 76.93 m which presumably was intended.

[9] "The Nile in 1904," by Sir William Wilcocks, p. 48.

The only long-period continuous records now available for the Nile relate to the Roda gauge, at Cairo. There is fair assurance that the gauge datum has been maintained throughout the thirteen centuries of record without change, unless the Temple of Roda, in which the nilometer column is located, has settled into the alluvium. The depth of the temple foundation, and the tradition that the gauge which was used throughout most of the Arabian epoch and thereafter, was referred to the datum and site of a much older structure, are items of evidence tending to insure the stability of the foundation.

Incidentally, the graduation of the masonry column into plainly marked cubits, with the zero of the scale somewhat lower than was ever attained (according to the available records) in the 1 300 yr following 622 A. D., is indirect evidence of its antiquity, if the zero represented the lowest previously observed and recorded stage. During the first 1 000 yr of this record the lowest annual stages approached within about 2 ft of the zero only five or six times, depending on which tabulation is adopted; but in 1531 and in 1621 the recorded low stage was only 9 digits, or 8 in., above the zero. Moreover, the graduations between the sixteenth and twenty-second cubit marks on the column are half cubits, whereas above 22 cubits (38 ft) they resume the full measure. This is explained by Lyons on the ground that the canals were opened at 16 cubits (27.5 ft), as was customary at the beginning of the Arabic epoch, about 641 A. D., and a half-unit rise at Roda meant a full-unit rise above the canal diversions; but at 22 cubits, the irrigation basins were ordinarily full, and no further diversion was necessary. As a further assistance in visualizing the units of the paper, roughly, in terms of their English equivalents, Fig. 2 includes three general conversion scales.

It is understood that most of the records had a common origin, but the originals are not known to exist at the present time. There are various versions of parts of the records, transcribed by different authors either from the original documents or from other transcriptions. Volumes 4 and 9 of the *Memoirs* of the Institute of Egypt (published in French by Prince Omar Toussoun during the period, 1923 to 1925) afford three different sources of data, all more or less interrelated, yet not identical. Ibn Iyas and others have compiled voluminous textual notes, which appear in part in Volume 9 of the *Memoirs* and from which many data may be derived for various years from 769 to 1878 A. D. Some of these data confirm similar figures in tabulated form; others differ; and still other significant notes make cross-references that agree with previous records and supply data for many years which are omitted in the tabulations.

The record of Aboul Mehasin in Volume 4 of the *Memoirs* extends from the year 20 to 855 of the Hegira, or 641 to 1451 A. D., but all according to Mohammedan years of 12 lunar months, which are thus 11 days shorter than the solar years. Therefore, there are 103 yr per century of the present-day mode of reckoning. The longer tabulation published in Volume 9, in

1925, or two years later than that of Aboul Mehasin, gives 1 108 maximum annual stages and 1 025 corresponding annual minima during the 1 300-yr period from 622 to 1921 A. D.

There would be definite advantages in publishing these tabular data, in pics and kirats, coudees and doights, or cubits and digits, according to the various transcriptions, with their equivalents, in meters, above mean sea level; but they are too voluminous to present in this manner, especially for all three main sources of data mentioned herein. The use of a chart on which all the data could be shown, and their trends compared, century by century, by connecting successive century or other averages of maximum or minimum stages, has seemed to be a more acceptable method of presentation. Naturally, the longer tabulation, representing the latest and most nearly complete version of available data regarding Nile River stages, was adopted as the one to be given preference.

In Fig. 1, the various plotted points may be identified as follows:

Applying to All the Records:

v = five-year average for the data shown by dots, supplemented by — or (—) when the location of the dot is not given;
× = ten-year average for data shown by dots, supplemented by — or (—) when the location of the dot is not given; and,
★ = one hundred-year average.

Applying to the Roda gauge, at Cairo:

• = records compiled by Omar Toussoun[10] covering the 1 300-yr period from 622 to 1921 A. D.;
(•) = confirmation from textual notes for the records compiled by Omar Toussoun; when — is lacking an agreement is indicated between all three sources;
_ = records compiled by Aboul Mehasin,[11] covering the period, 20 to 855 of the Hegira, or 641 to 1451 A. D., a total of 811 years. A small + indicates extra data, representing 25 surplus years of Mohammedan reckoning;
(—) = records from notes compiled by Ibn Iyas and others,[12] for the period 769 to 1878 A. D.
ʟ = records by Lyons;[13] and,
∧ = records indicating "wafa," or the stage that assures plenty, at which the canals were opened to supply the basins; the maximum flood stage, ordinarily, was somewhat higher.

Applying to the gauge down stream from Aswan Dam:

A = maximum annual river stages at the Aswan gauge above the assumed datum, 71.0 m, or 232.9 ft, above mean Mediterranean Sea level;
•̮ = maximum ten-day, average gauge heights, and, therefore, somewhat below the actual maximum.

Applying to the El-Leisi gauge, 37 miles up stream from Cairo:

○ = maximum ten-day average gauge heights, and, therefore, somewhat below the actual minimum.

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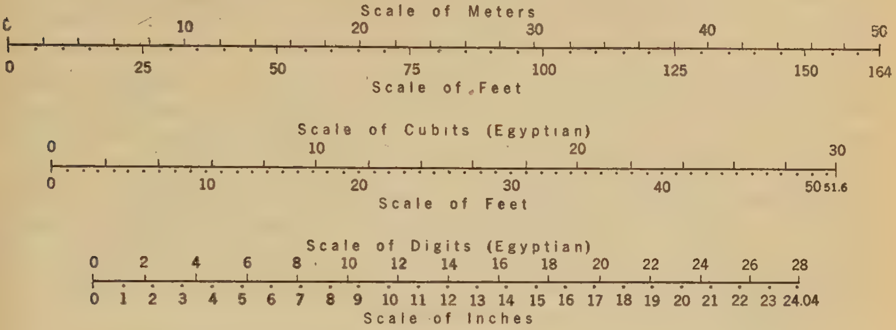
[10] *Memoirs*, Inst. of Egypt, 1925, Vol. 9.

[11] *Loc. cit.*, 1923, Vol. 4.

[12] *Loc. cit.*, 1923 and 1925, Vol. 4 and 9.

[13] "The Nile Flood in 1905," by Capt. H. G. Lyons.

The gauge heights plotted as ordinates in Fig. 1 are readings from the Roda gauge on the Nile River, at Cairo. The exceptions, marked *A* or *J*, refer to readings of the Aswan gauge and those marked *O* refer to readings of the El-Leisi gauge, 37 miles above Cairo. In all cases the readings are the mean Mediterranean Sea level elevations, at Alexandria.



10 Digits = 8.5875 Inches
20 Digits = 17.1750 Inches
28 Digits = 24.045 Inches

FIG. 2.—CONVERSION SCALES

Fig. 1 probably includes many items that represent river stages at the time of the festival attending the opening of canal gates, after which the flood ordinarily continued to rise to varying heights. According to tradition and historical references, furthermore, some of the flood heights were increased falsely in the records for the purpose of adding to the revenues from land taxation. There appear to be good reasons for expecting fully as wide ranges of variation for the maximum annual gauge heights as for the minima, except as the former may be affected by valley storage and flood-plain discharge.

The data from Aboul Mehasin's tabulation that differed by measurable amounts (say, 3 in., or more, on the adopted scale) are shown as short dashes crossing the grid lines; and those derived from or confirmed by textual notes are shown within parentheses. In this manner, it was found possible to supply about 65 additional maxima, so that a total of 1 173 out of a possible 1 300 are plotted, some with more than one value owing to the unexplained discrepancies among the records. These differences may reach 3, 4, 5, or even 10 cubits, but usually are less than 1 cubit. The data for the extra 25 yr of Mohammedan reckoning are shown as short dashes between grid lines.

For the 811 yr of concurrent record covered by the aforementioned two tabulations, about one-fourth the maxima and one-half the minima coincide, but if those in practical agreement or those differing by only a few inches are included, the proportions increase to about one-half and three-fourths, respectively. The textual notes of Ibn Iyas, El Gabarti, and others provide several coincidences of both maxima and minima, as well as at least an equal number of measurable divergences. Furthermore, they supply the sixty-five yearly maxima not found in either of the long tabulations. All these new data are shown in Fig. 1 as short dashes enclosed in parentheses. Agreement of textual notes with the data from the longer tabulation is indicated by parentheses around the individual dots or small groups of dots. Such a symbol for any year between 641 and 1451 A. D., without a dash, indicates agreement of all three sources, as explained in the foregoing list of symbols.

It seems to be fairly well established, by tradition as well as by historic references, that the records of some of the sub-normal Nile flood stages were increased arbitrarily by order of unscrupulous rulers whose prospects of land revenues had been adversely affected by the drought. In years of assured plenty, or abundant water supply, such sordid motives for falsification of records would seem to be lacking. However, there is considerable evidence that a record was sometimes made of the stage at which the canal gates were opened, upon the assurance of a satisfactory water supply, after which the river may have continued its rise in varying unrecorded amounts. Furthermore, a sudden or early recession of the flood from what was at first considered to be a stage assuring plenty, or "wafa," was occasionally followed by a partial famine, due to the brief duration of high flood. Obviously, any neglected maximum stages would tend to counterbalance the artificial increase of sub-normal stages as expressed in trends.

With full recognition of the obvious discrepancies in the records and the likelihood that many other errors and inaccuracies were incorporated which may never be detected (or at least are not detectable at this time) it is truly remarkable, nevertheless, how nearly the computed trends of rise apparently ascribable to sedimentation, ranging from 0.10 to 0.15 m (3.9 to 5.9 in.) per century, are reflected in Table 1.

TABLE 1.—COMPUTED TREND OF RISE, FLOOD STAGES OF THE NILE, ASCRIBABLE TO SEDIMENTATION

YEARS OF RECORD (A. D.):		Cumulative interval, in years	AVERAGE ANNUAL LOW-WATER AND FLOOD STAGES				AVERAGE RISE FOR THE CUMULATIVE INTERVAL			
From	To		In Meters Above: Mean Mediterranean Sea Level at Alexandria, Egypt		In Feet Above: Mean Mediterranean Sea Level at Alexandria, Egypt		In Meters per Century		In Inches per Century	
(1)	(2)	(3)	Minimum (4)	Maximum (5)	Minimum (6)	Maximum (7)	Minimum (8)	Maximum (9)	Minimum (10)	Maximum (11)
622	721	100	11.51	17.50	37.8	57.4	......	......	......	....
1 422	1 521	900	12.52	18.21	41.1	59.7	0.13	0.09	5.12	3.54
1 522	1 621	1 000	11.09	18.71	36.4	61.3	—0.05	0.13	—1.97	5.12
1 522	1 721	1 100	11.66	18.34	38.3	60.2	0.015	0.084	0.59	3.23
1 722	1 821	1 200	11.66	19.06	38.3	62.5	0.014	0.14	0.55	5.51
1 822	1 921	1 300	13.19	19.33	43.3	63.4	0.14	0.15	5.51	5.91

The year 1834 marks the construction of a main barrage below Cairo and the beginning of perennial irrigation in Egypt. The effect of this development soon after the severe drought that culminated in 1833, is more marked in the trend of minima than in that of maxima. The barrages raised the river bed, the release of storage water from large reservoirs far up stream augmented the minimum flow, and the storage of a portion of the flood flow in these distant reservoirs tended to decrease the peak stages. A study of the trends on the graphic composite record covering

1 300 yr, especially between successive century averages (marked * in Fig. 1), shows unmistakably why the flood stages that indicated an assurance of plenty progressed gradually from 16 to 20 cubits (28 to 35 ft) on the Roda gauge and then, during the final century of record, increased nearly another 6 cubits (10.3 ft), owing to increased usage of water and to the construction of the main barrage or diversion dam about 15 miles below. This was designed to raise the low-water surface about 3.2 m, or 10.5 ft, or 6 cubits (10.3 ft), and it actually raised the flood crests about 0.75 m (29.5 in.) at the barrage, even with all gateways open; or 0.25 to 0.75 m (9.8 to 29.5 in.) at the Roda gauge.[14]

The first notable effort toward providing a reliable perennial irrigation supply for Egypt was evidently the Lake Moeris Reservoir project, described by Herodotus after his personal visit to Egypt about 450 B. C. The site has been identified definitely as the fertile Fayum Basin of the present day, with its saline lake occupying the lowest part of the valley, 140 ft below sea level, in a manner similar to the Salton Sea in Imperial Valley, California. The width of the canal supplying the reservoir was given as the equivalent of about 300 ft. With an average depth of 25 ft during high flood and a mean velocity of 4 ft per sec, the discharge would have been 30 000 cu ft per sec, or 60 000 acre-ft per day, thus requiring approximately 17 days of continuous flow to deliver 1 000 000 acre-ft. It is plain that both the diversion and the storage quantities might have been considerably greater or less than these values, but they afford at least an idea of the possibilities in the light of modern practice.

Sir R. Hanbury Brown[15] derived the available storage depths and capacities for Lake Moeris which are used in the following comments: Taking into account the present contours of the Fayum Basin, enclosing an area of fully 630 sq miles under the level of ancient lake shells, or 22.5 m (74 ft) above mean sea level, and assuming the depth of usable storage for return to the Nile to augment the low flow, as 10 ft, intermediate between the high-water and low-water stages of the adjacent Nile channel, the net available volume is found to have been about 3 000 000 acre-ft, or nearly the usable capacity of the Aswan Reservoir from 1912 to 1930, after the first enlargement and before the recent one. The storage and release of this water, therefore, must have modified the Nile discharge at Cairo, about 80 miles below the diversion canal leading to the Fayum Basin; but it appears that before 622 A. D., the beginning of the nilometer record herein presented, Lake Moeris had been abandoned as a storage project, and the rich alluvial lands of its bed had been devoted to agricultural use, such as prevails to this day. In addition to this storage, the chains of natural lakes were doubtless operating as effectively in those ancient times as at the present, if not more so. Presumably, the natural processes of erosion and sedimentation would have tended to enlarge the lake outlets and to convert the shallower lakes or portions of them into marsh land, such as now prevails on the White Nile above Khartoum.

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[14] "The Delta Barrage," by Sir R. Hanbury Brown, Cairo, 1902.

[15] "The Fayum and Lake Moeris," by Sir R. Hanbury Brown, Lond., 1892, p. 80.

From what seems to be the best information and opinions thus far published and discovered in the present research, it appears that the storage and regulative features of the Nile were not greatly changed during the first 1 200 yr of the record plotted herewith; but that the developments during the final century of this record at least restored the quantity of storage and regulation provided by Lake Moeris for possibly sixteen centuries before, and more than a century after, the visit of Herodotus, about 450 B. C.

In spite of all the changing, uncertain, and erroneous factors that must be considered in connection with the records of stages of the Nile River, it is believed that they disclose some important information; and there is a fair prospect that they may yield more data with further study and the cumulation of ideas of various students.

It is readily apparent that the use of records covering only a single early century might yield an unreliable estimate of the frequency characteristics of the maximum stages for the total period. On the other hand, it is believed that, after eliminating the obvious errors in Fig. 1, such as 3, 5, or 10 cubits (5, 8.5, or 17 ft), and some lesser ones, and then applying an average rate of rise per century to indicate the increase of sedimentation on the flood-plains and the resultant rise of flood stages, any one of the first twelve centuries of record may be found to qualify as a rough indicator of what should be expected during a longer period, such as 500 or 1 000 yr. Furthermore, the comparison of the Toussoun data (plotted as dots in Fig. 1) and the Lyons data (marked ∠ in Fig. 1) for the two centuries ending in 1902, shows several measurable differences, some of them systematic and nearly constant, and others of irregular amounts and sign; that is, alternating as positive and negative variations. In view of such discrepancies, of the ease with which a "3" may be read as an "8," or a single unit changed in the 10's column in Arabic numerals, and of the chance of other errors of transcription, it is not surprising that about 5 or 10 digits or cubits or other differences exist in the two tabulations covering 811 yr concurrently. The greater occasion for surprise seems to lie in the possibility of synchronizing the two time scales (Mohammedan and solar years), which has been done in such a way that a majority of the items are in actual or substantial agreement, for both maximum and minimum annual stages.

This presentation of what purport to be records or data derived in some way from actual records, is made with the hope that others may be spared the tedious work of computation, co-ordination, and plotting of the various items. Perhaps it may serve as a kind of groundwork on which comparison may be made between segments of this record, or between these data and others for rivers in the United States, proper account being taken, of course, of differences in drainage-basin characteristics and river regimen.

It seems important to observe that the site of the Roda gauge, at Cairo, is virtually at the apex of the Nile Delta cone and that the opportunity of the stream to spread through a wide angle resulted in various conditions regarding the number of outlet channels. Thus, some of the early maps outline six distinct outlet branches; others show three or four; and now there are only two well-established natural channels. Perhaps the combined capacities of the various canals in Lower Egypt, or even those restricted to the Delta itself, would account for the equivalent of one or more of the abandoned channels. It is quite probable that under such conditions as are known to have existed some of the annual flood heights were affected more by channel changes than by variations in rates of flow. For example, a devastating flood may have lowered the river heights temporarily as a result of scouring, or an obstruction may have increased them. The transfer of the main gauging operations, about 1870, to the Aswan site, below the First Cataract, avoided some of the disturbances that have been felt at Cairo from new irrigation developments since the British occupation. These Aswan gauge heights, above an assumed datum, are superimposed on Fig. 1 with Roda gauge readings.

Attention is also invited in Fig. 3 to the plotted records of both annual volumes and 4-month flood volumes at Aswan, beginning with 1870, as determined from the records of the Ministry of Public Works, of Egypt.[16] These values may be read from the various scales in almost any desired unit of measure in common use. The decrease from the first to the latest decade of record seems to be due, in part, to climatic trends over the period of record, but perhaps it may reflect also the increased use of water in Upper Egypt and the increased losses in storage or in transit due to recent works.

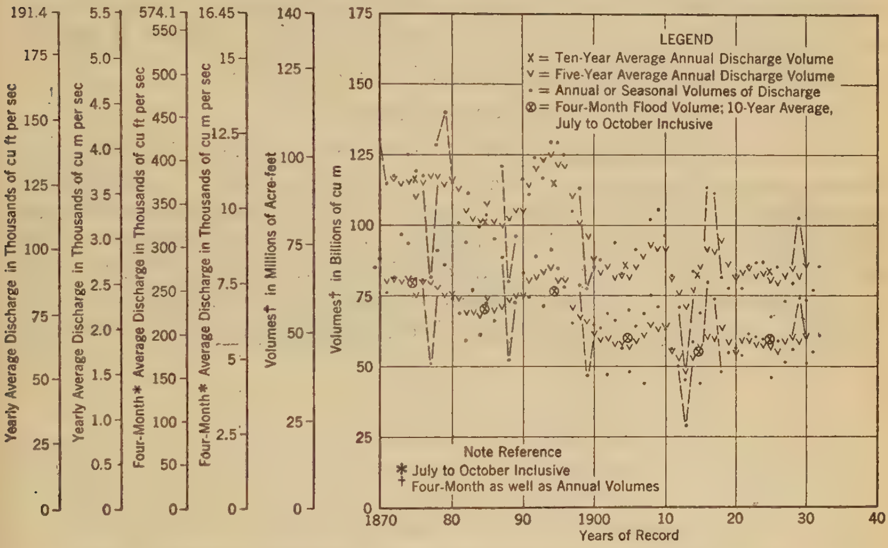

FIG. 3.—FLOOD DISCHARGE AND VOLUMES OF THE NILE RIVER AT ASWAN

Probably the most authoritative appraisal is that of the Physical Department, Egyptian Ministry of Public Works, as expressed[17] by H. E. Hurst, Director General, and P. Phillips, Director of the Hydrological Service:

"One series of maxima and minima at Cairo extending from 641 A. D. to 1450 A. D. is fairly complete. The outstanding feature of these is that over considerable periods, sometimes as long as 50 years, floods are above the average, while over other periods they are below the average. It is also the case that very low floods may occur amongst a high series and *vice versa*. These records have been analyzed for periodicities, and periods of small amplitude have been found. They are, however, so masked by irregularities as to be useless for forecasting purposes."

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[16] Physical Dept. Papers, Nos. 29 and 31; also "The Nile Basin," Vol. III and IV (Supplement).

[17] "The Nile Basin," Vol. 1, p. 20, 1931.

Despite the obscuring effects of errors, uncertainties, and difficulties attending these records of Nile River stages, it seems probable that the concentrated attention of members of the Engineering Profession who are so inclined may yet bring into view many facts and relationships to build up a better background of experience and to afford perspective covering long-time trends of river behavior. Throughout the notable long-period records recently investigated on both American and foreign rivers, aggregating thousands of years in length, it appears that no single flood event is completely at variance or out of character with what appear to be the natural and reasonable potentialities of that particular basin or river system, as indicated by other similarly outstanding events of record.

ACKNOWLEDGMENTS

A wide circle of associates and advisers have been most helpful in the conception and development of the plan for this paper, and in reviewing or contributing ideas to be included. Especial mention should be made of the personnel of the U. S. Geological Survey and that of the Society's Committee on Flood Protection Data; and, particularly, to N. C. Grover, H. T. Cory, R. W. Davenport, J. C. Hoyt, Gerard H. Matthes, and Thorndike Saville, Members, Am. Soc. C. E., for their direct contributions and co-operation.

DISCUSSION

Halbert P. Gillette,[18] M. Am. Soc. C. E. (by letter).—In a paper on "The Cycles That Cause the Present Drought"[19] the writer has compared curves of mean flood levels of the Nile River from 1735 to 1919 with those of mean annual rainfall at or near Boston, Mass., from 1750 to 1934. The two curves are quite similar, each having a minimum for the 5-yr. period, 1790–94, and a maximum at or near the period, 1865–69.

Another diagram[20] gives the mean thickness of annual tree rings by decades, for Arizona pines, from 1390 to 1909, indicating three cycles of about 152 yr. Each cycle has less amplitude than the one preceding because tree rings vary less in thickness as the roots attain greater depth. The California sequoias also show the 152-yr cycle clearly, particularly when they are young. The first great valley in the curve of their annual rings shows a maximum drought about 1255 B. C.

The annual silt layers, or varves, in the ancient glacial lakes of New England and Canada show the 152-yr cycle; but the most impressive evidence of this "grand cycle" is found in the recessional moraines left by the retreating ice sheet, and in the ancient beaches around the Great Lakes and Lake Winnipeg.

From the foregoing evidence the writer has concluded that the exact length of the cycle is one month longer than 152 yr and that its amplitude is cyclic, reaching a maximum every 1 825 yr. T. W. Keele, an Australian civil engineer, was the first to discern any evidence of the existence of the cycle[21] and he found it in the Nile flood-level records subsequent to 1736, as well as in the records of Australian rainfall. Unfortunately, he erred 12% in determining its length.

By means of tree rings the writer estimates that the year of maximum drought due to this cycle will be 1939; but since there are a dozen other cycles longer than 2 yr, not to mention several shorter than that, there may be years even dryer than 1939, in the near future. This is illustrated by reference to Table 2,[22] in which the mean amplitude (Column (2)) is the percentage departure of the cyclograph peak or valley from the mean. The cyclographs were those derived from Arizona pines (1391–1910) except for the $100\frac{2}{3}$-yr cycle, for which early sequoias were used. The sequoias usually have amplitudes about two-thirds as great as Arizona pines; therefore, the 13% in Column (2), Table 2, should be increased about one-half. The amplitude varies periodically through the cycle in Column (3), Table 2; and Column (4) gives a year in which a peak of the basic cyclograph had its greatest amplitude, which is to be used as an epoch date.

Four principles seem to be indicated by Table 2, in its relation to the paper by Mr. Jarvis, namely, (1) the amplitude of a rainfall cycle tends to increase with the length of the cycle; (2) the amplitude itself is cyclic; (3) cycle lengths approximate a geometrical progression series, with a ratio equal to $\sqrt{2}$; and (4) cycle lengths are either an integral number of months, or of two-thirds of a month.

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[18] Editor, *Roads and Streets*, Chicago, Ill.

[19] *Water Works and Sewerage*, August, 1935, p. 289.

[20] *Loc. cit.*, Fig. 2, p. 290.

[21] "The Great Weather Cycle," by T. W. Keele, *Proceedings*, Royal Soc. of New South Wales, Vol. 44, 1910, p. 25.

[22] *Water Works and Sewerage*, August, 1935, p. 292.

TABLE 2.—THIRTEEN RAINFALL CYCLES

Basic cycle, in years (1)	AMPLITUDE		Year (A. D.) of maximum rainfall (4)	Basic cycle, in years (1)	AMPLITUDE		Year (A. D.) of maximum rainfall (4)	Basic cycle, in years (1)	AMPLITUDE		Year (A. D.) of maximum rainfall (4)
	Mean (percentage) (2)	Cycle, in years (3)			Mean (percentage) (2)	Cycle, in years (3)			Mean (percentage) (2)	Cycle, in years (3)	
$1\frac{17}{18}$	4	35	1911	$13\frac{11}{12}$	8	167	1816	$48\frac{5}{6}$	12	293	1756
$3\frac{1}{6}$	6	19	1900	$18\frac{5}{6}$	11	113	1889	$69\frac{2}{3}$	15	209	1760
$4\frac{1}{9}$	7	37	1922	$23\frac{5}{6}$	12	143	1859	$100\frac{2}{3}$	13	302	1912
$5\frac{13}{18}$	6	103	1919	$39\frac{2}{3}$	10	119	1921	$152\frac{1}{12}$	25	1 825	1711
$7\frac{5}{9}$	6	68	1921	.....	.....	.....	......	.....	.....	.....	......

Since sequoia tree rings have been measured spanning 3 200 consecutive years, it follows that they supply a means of determining lengths of cycles with great precision. A total of one hundred and thirty-three 24-yr cycles can be traced by the sequoia data. Therefore, if the peak of the cycle is determined near the beginning and near the end of the 3 200 yr, the error in the length of the cycle cannot exceed $\frac{1}{130}$ yr.

For the foregoing reasons the writer finds that tree rings are preferable to level records of the River Nile in deriving the lengths of cycles. Furthermore, the low-water levels of this river are more variable than the flood-stage records cited by Mr. Jarvis and, therefore, are preferable for cycle analysis. A simple method of cycle analysis has been recommended by the writer[23] which is similar to that devised by Balfour Stewart.[24] It should be noted that this method or Schuster's[25] harmonic analysis modification of it will yield pseudo-cycles of two-thirds, one-half, one-third, or one-fourth the length of the true cycle. No writer has previously called attention to this fact. It arises from the shape of the rainfall curve of a cycle; and, consequently, meteorological literature abounds in pseudo-cycles.

Except for the records of the River Nile, man-kept records by weather cover such few years that they are of little value in determining cycles longer than about 6 yr. It often requires data covering more than 30 cycles to establish the existence of a cycle with a high degree of probability, because there are so many cycles that they tend to hide each other. Two rainfall cycles (approximately 152 yr and 101 yr) often are very clearly evident in spite of the other obscuring cycles. Occasionally, at intervals of 35 yr, the cycle of approximately 2 yr is well defined in winter temperatures and in laminated silt or varves.[26]

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[23] *Water Works and Sewerage*, August, 1935, p. 289.

[24] "World Weather," by H. H. Clayton, p. 376.

[25] "The Periodogram of Magnetic Declination," by Arthur Schuster, *Transactions*, Cambridge Philosophical Soc., Vol. XVIII (1899), p. 107.

R. W. DAVENPORT,[27] M. AM. SOC. C. E. (by letter).—The occurrence of a flood is determined by the conjunctive operation of a variety of meteorologic, hydrologic, topographic, and other factors and conditions in a manner favorable to the production of abnormally excessive flows. As knowledge of these fundamental factors and conditions has become more plentiful through the collection of records at rainfall and river measurement stations, topographic surveys, etc., it has been possible to introduce more science into the analysis of flood occurrence and thereby to advance somewhat in relating floods to their constituent elements. Notable advances of this kind have been made in recent years by different individuals and organizations along more or less closely related lines, with resulting improvement in the technique of flood study and the promise of further improvement as the supply of essential base data increases and the knowledge of practical and theoretical hydrology progresses.

The flood record of the Nile River, as presented in the paper by Mr. Jarvis, is the result of a thorough and painstaking endeavor to reconcile conflicting evidence of Nile flood heights as derived from various sources and to convert the information into a form that can be readily understood and discussed. Future students of the long Nile record may have the advantage of starting from the advance point to which Mr. Jarvis has progressed in his researches. Although numerous uncertainties and difficulties tend to place the record somewhat in a cloud of doubt as regards its reliability of detail, full knowledge of the facts as developed will be of great value to future investigators. The writer believes with Mr. Jarvis that despite these unfortunate defects of the record the Engineering Profession may be able to derive from it valuable knowledge in regard to long-time characteristics of river behavior, and particularly in regard to the changes in behavior with respect to floods within the centuries covered by the record.

The proposition that the characteristics of flood behavior of a stream may be best understood and dealt with by analyzing them in relation to their fundamental constituent elements has the direct corollary that such an analysis of these characteristics for a certain stream may also best contribute to knowledge of corresponding characteristics for other streams, neighboring or distant. Obviously, therefore, if the flood characteristics of the Nile and other foreign rivers, as displayed by long records, are to be made rationally applicable to streams elsewhere in the world they require to be analyzed and translated, so far as practicable, into common fundamental terms. In undertaking to determine, in a superficial way at least, something of the significance of the

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[26] "Recession of Last Ice Sheet in New England," by Ernst Antevs, Plates I to V, p. 121 *et seq.*

[27] Hydr. Engr., U. S. Geological Survey, Washington, D. C.

Nile record in relation to other streams, the writer has been interested to review some of the outstanding features of the Nile floods.

The Nile drainage basin has an area of 1 119 737 sq miles, and the river has a waterway length of 3 728 miles.[28] The corresponding values for the Mississippi-Missouri System are 1 240 000 sq miles and about 4 200 miles.

The main Nile is formed by the junction of the White Nile and the Blue Nile at Khartoum, 1 913 miles above the mouth. Through this 1 913-mile reach below the junction the river flows through a very arid country and except for the River Atbara, which enters from the east, 159 miles below the junction, the tributary area is small and the inflow practically negligible. The flow is subject to substantial diminution by evaporation and seepage; moreover, for thousands of years each flood has supplied substantial volumes of water, largely from the lower part of this reach, for the irrigation of the fertile Nile Valley. The flow of the river is derived from two main sources: (1) The lakes and great swamp areas of Central Africa drained by the White Nile and its tributaries, with a heavy tropical rainfall distributed through the greater part of the year; and (2) the highlands of Ethiopia, drained by the River Sobat, tributary to the White Nile from the East, the Blue Nile, and the River Atbara—a region marked by an extraordinary concentration of rainfall and run-off in the period from July to September. Thus "reduced to its simplest expression, the Nile system may be said to consist of a great steady-flowing river fed by the rains of the tropics, controlled by the existence of a vast head reservoir, and annually flooded by the accession of a great body of water with which its eastern tributaries are flushed."[29]

TABLE 3.—DRAINAGE AREAS OF ETHIOPIAN EFFLUENTS

River	Drainage area, in square miles	Distance of junction from mouth of the Nile, in miles
Sobat	94 560	2 429
Blue Nile	127 998	1 913
Atbara	85 216	1 754
Total	307 774	......

Table 3 shows, in down-stream order, the drainage areas of the Ethiopian effluents that are the sources of the annual Nile flood and the relative locations of their junctions with the main river. The Blue Nile rises above Lake Tsana, about 2 800 miles from the mouth of the Nile.

Some understanding of the seasonal distribution of the rainfall that is the source of the Nile floods may be obtained from Table 4, which gives the mean monthly rainfall at Addis Ababa for a period of 13 yr and 2 months.[30]

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[28] These and other statistical data of the Nile presented herein are obtained from "The Physiography of the River Nile and Its Basin", by Capt. H. G. Lyons, Director Gen., Survey Dept., Finance Ministry, Egypt, Cairo, National Printing Dept., 1906.

[29] "Encyclopædia Britannica, 14th Edition, Vol. 16, p. 454.

[30] "The Rains of the Nile Basin and the Nile Flood of 1912", by J. I. Craig, Director, Meteorological Service, Survey Dept., *Survey Department Paper 32*, Ministry of Finance, Egypt.

The hydrographs of flow at stations in the upper reaches of the tributaries[31] show a most pronounced flood rise extending from July to the late autumn and culminating in a maximum stage in August or September (averaging about September 1). The hydrographs at the upper stations follow an irregular saw-tooth line, thus displaying direct sensitiveness to the variations of rainfall distribution over the upper part of the drainage basin.

TABLE 4.—MEAN MONTHLY RAINFALL AT ADDIS ABABA, ETHIOPIA

Units	January	February	March	April	May	June	July	August	September	October	November	December	Total
Millimeters	15	39	70	82	64	142	283	302	163	16	19	3	1 198
Inches	0.59	1.54	2.76	3.23	2.52	5.60	11.13	11.90	6.42	0.63	0.75	0.12	47.19

The flow of the White Nile above the mouth of the Sobat is extraordinarily regular. The influx of the Sobat tends to swell the flow materially, beginning with June, but apparently the river continues to be characterized by unusual regularity until it approaches the mouth of the Blue Nile, where it shows greater domination by the influences that create the Nile floods. Below the junction of the Blue Nile, the flow assumes the definite pattern of the Nile floods.

Farther down the main river the hydrographs become continuously more smooth, until at the Roda gauge, at Cairo, where the long record has been kept, the incisions in the hydrograph that are due to natural causes are undoubtedly very small. Moreover, under natural conditions, the time of maximum stage at Cairo is about October 1 on the average, or about a month later than on the upper reaches of the tributaries.

From the available information it is apparent that:

(1) The rainfall that creates the Nile floods is concentrated largely in a 90-day period;

(2) The collecting system of the flood waters integrates the variable flows originating from the heavy rainfall over a wide area in a peculiarly effective manner, so that the flood at the place where it enters upon its long course to the sea through the main Nile channel is characterized by a remarkable uniformity of pattern from year to year; and,

(3) The passage through about 1 900 miles of capacious flood channel, with the resultant regulating influence of a great volume of channel storage, still further smooths the hydrograph of flow, except as artificial regulation and diversion have produced irregularities, usually of minor character.

The only drainage basin in the United States that is comparable in area and length to that of the Nile is the Mississippi Basin, but its hydrologic characteristics are quite dissimilar. The Colorado Basin is similar to the Nile Basin in respect to the fact that its flood is peculiarly seasonal in its occurrence, but it is much smaller and quite unlike the Nile Basin in various other ways. It is evident, therefore, that the application of the Nile experience to the rivers of this country is not direct and easy, but requires an appropriate analytical treatment of fundamental factors.

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[31] As pub. in the *Survey Department Papers* of the Ministry of Finance, Egypt.

In reviewing the evidence of past centuries, therefore, it appears that the annual maximum stage at the Roda gauge represents the culminating height attained by a flood rise of large volume and unusual regularity of pattern. Outstanding features of the flood are its regular seasonal recurrence and its obscuring of erratic tendencies, both in distribution of rainfall and in the characteristics of drainage basins in shedding flood waters. Any application of the facts disclosed by the Nile record to streams elsewhere, of course, should take these and other essential characteristics appropriately into account.

H. E. HURST,[32] ESQ. (by letter).—The Roda nilometer situated in Cairo is said to be the oldest existing Arab monument in Egypt, and consists of a well of about 5 m (16.4 ft) square communicating by tunnels with the river and having a vertical column at its center on which the scale is cut. The column is held at the top by a cross-piece of masonry spanning the well, and the water levels were read from a staircase that winds around the outer edge. Its records dating back to A. D. 622, although not complete, form perhaps the longest written record of any meteorological phenomenon.

The originals as far as known do not exist at present, and the form in which the records were kept is also unknown. Colonel Sir Henry Lyons who came to Egypt in the Nineties was told that the original records were written in Coptic and existed in the Ministry of Public Works in the time of Aly Pasha Mobarak, a few years before his arrival in the country, but he was never able to trace them. The writer has never heard of a record with any claims to antiquity, and if any such existed in Egypt, H. H. Prince Omar Toussoun who has collected so much old material about the Nile would probably have discovered it.

In assessing the value of these records many points need consideration (some of which will be mentioned subsequently), and the investigator must divest himself of the idea that they are comparable in accuracy with present-day records. The scientific spirit of recording exactly what is observed, without personal bias and with every effort to eliminate sources of error, is a recent growth and is not to be expected in records the most complete portion of which is from 600 to 1400 A. D.

The Roda nilometer has been repaired on various occasions recorded by historians, in some of which the repairs have been so extensive as to leave a doubt as to whether it was not entirely reconstructed. There is little doubt that the list of repairs is incomplete and in no case is there any account of what steps, if any, were taken to preserve the datum level during the repairs.

Apart from changes of datum of an accidental nature one must consider the effects of changing topography and varying regime of the river such as have occurred in historical times. An example of this mentioned in the paper is the varying number and position of channels through the Delta, and a further example is that the position of the western bank of the river near Cairo has changed by nearly 1 km (0.6 mile) since Napoleon's time.

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[32] Director-General, Physical Dept., Ministry of Public Works, Cairo, Egypt.

A point that may be mentioned is the effect of the basin system of irrigation on the river levels at Cairo. In this system, which was the only one practised in Egypt until the Nineteenth Century, the country is divided into sections by banks transverse to the course of the river, and the flood waters of the Nile are admitted to these sections or basins by means of short canals. After a period of forty days or more the water is allowed to escape back to the river and the basins are planted with crops. In recent times, basin irrigation has been well controlled by means of masonry regulators on the canals and escapes, and the water is admitted to and escaped from the basins according to a program. The effect of this is to alter the natural flow of the river at Cairo very considerably, lowering the natural peak and the succeeding levels to raise them again when water begins to escape. In some cases the second peak has been higher than the first one and, in others, it has not been noticeable, the actual regime depending largely on the nature of the flood. Since 1900, the basin area of Upper Egypt has been diminishing gradually, owing to the conversion of the basins to the perennial system.

Previous to the Nineteenth Century basin irrigation must have been a much more haphazard process, as lack of communications would prevent its regulation on an organized plan directed from headquarters. Moreover, there would be no masonry regulators; the only control of entrance to, and escape from, basins would be by earthen banks; and after these were once cut the water would take its course. Possibly the water merely flowed through breaches in the river bank as the flood rose, and back through the same breeches as the river fell, making the process more or less a natural one, which would be modified occasionally by the bursting of cross-banks. In very early times the cross-banks probably did not exist, and the annual inundation of Upper Egypt was a still simpler phenomenon.

Unfortunately, no details of the form of the Roda gauge curve previous to the Nineteenth Century have been handed down, and only readings purporting to be the maximum and minimum, are available. The minimum readings would be much more likely to be affected by changes of topography, such, for example, as the closing of a main channel near Cairo by a silt bar or an earthen bank (as must frequently have happened), or by the gauge itself being shut off from the main stream.

The possible exaggeration of low floods for revenue purposes has been mentioned in the paper. Another source of error is that the gauge pillar must always have been difficult to read as the observer cannot approach it closely; furthermore, the scales became worn and often no doubt were incrusted with mud. This led to a practice which was followed by some of the gauge readers of recent times and was referred to by Col. J. C. Ardagh in 1889,[33] of reading not on the gauge, but on private marks either on the wall of the gauge-well or on a flight of steps outside, leading down to the Nile.

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[33] "Nilometers", by Col. J. C. Ardagh, *Proceedings*, Royal Geographical Soc., Vol. XI, No. 1, January, 1889.

Some of the sources of error that affect the records are of an accidental nature and will be largely eliminated by the number of observations, which is approximately 1 100 both of flood stage and of low stage.

Systematic changes such as the gradual rise of the river bed can be eliminated for some purposes by dividing the observations into groups of, say, 100 yr and dealing with the departures from the mean of the century. This procedure helps also to eliminate the effects of topographical changes. Deliberate exaggeration of low floods cannot be eliminated, but it is quite likely that it is a practice which was only used for a time and does not affect all the records, and, therefore, its effect is minimized when the entire mass of material is discussed.

As a whole, these records represent observations extending over a very long period of the rise and fall of the Nile. They have not the precision of modern observations, but are probably as reliable as many of the statistics collected at the present day about less well-defined phenomena, such as the health and social conditions of mankind.

Two uses to which these records have been put in recent times are: (1) Analysis to discover the existence of periodicities; and (2) construction of a curve giving the frequency of floods of different heights.

The old records of the Roda gauge given by various authors were scrutinized by Mr. J. I. Craig, of the Egyptian Ministry of Finance, and converted to a metric scale, and it is this collection, with certain minor changes, that has been used in the Physical Department, Egypt, or communicated by the Department to inquirers.

Periodic analyses have been made by the late Professor H. H. Turner[34], and by Messrs. J. I. Craig, and C. E. P. Brooks[35], and periods varying from 2 to 240 yr have been found. The period of greatest amplitude so far is one found by Professor Turner of 240 yr, with an amplitude of 15 cm (6 in.) for the maxima and 46 cm (18 in.) for the minima.

The analysis by Dr. Brooks does not extend beyond a period of 76.8 yr, but he finds a number of periods of average amplitudes of the order of 10 cm (4 in.). His best defined periodicity is the one of 76.8 yr, with a mean amplitude of 17 cm ($6\frac{5}{8}$ in.). The average standard deviation of the flood levels is 56 cm (22 in.); this makes apparent the relative smallness of any periodic effects, which although of theoretical interest, are of no use to the forecaster. A glance at the records when plotted on a fairly large scale shows that there is no period which is directly evident to the eye, and that the principal features are the existence of fairly long terms of years when, on the whole, the floods have been high and others when floods have been low. This fact is well illustrated by the Aswan gauge records for the period, 1869–1933. By dividing this period into two parts, 1869–1898 and 1899–1933, an example is afforded of a high term of 30 yr followed by a low term of 35 yr, with the results indicated by Table 5.

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[34] "On a Long Period (240 Years) in Chinese Earthquake Records", by H. H. Turner, Monthly Notices, Royal Astronomical Soc., May, 1919, Vol. LXXIX.

[35] "Periodicities in Nile Floods", by C. E. P. Brooks, *Memoirs*, Royal Meteorological Soc., Vol. II, No. 12.

The most striking feature of Table 5 is that in the high term, one year in every two was higher than the highest flood of the following low term. No regularity in the occurrence of these high and low terms has been discovered so far; nor are floods uniformly high in a high term or low in a low term.

TABLE 5.—RECORDS OF ASWAN GAUGE DIVIDED INTO TWO PERIODS

Description	1869–1898			1899–1933		
	In meters	In feet	Year	In meters	In feet	Year
Maximum gauge reading of highest flood..........	94.15	308.9	1878	93.30	306.1	1908
Maximum gauge reading of lowest flood..........	91.40	300.9	1877	90.11	295.6	1913
Mean flood maximum for the period..............	93.26	306.0	....	92.39	303.1	....
Number of years with maxima greater than 93.30 m..	15		....	0		....
Number of years with maxima less than 91.40 m....	0		....	2		....

A frequency curve was constructed by the writer from all available Roda gauge maxima. This curve was used to estimate the frequency of occurrence of very low floods such as that of 1913.[36]

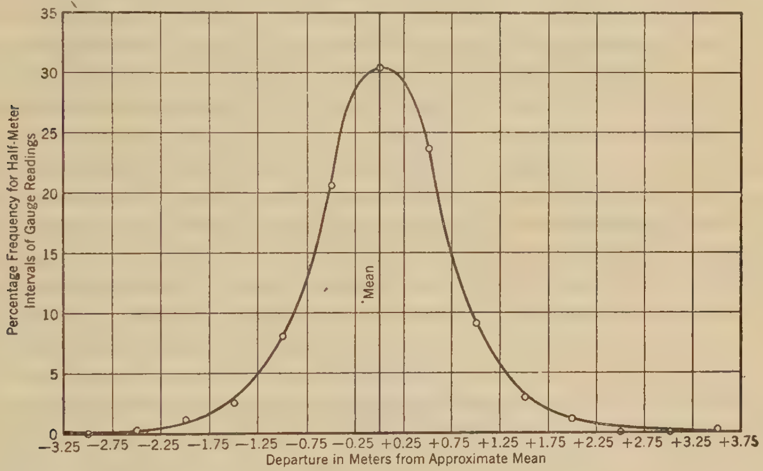

FIG. 4.—FREQUENCY CURVE, MAXIMUM READINGS, RODA GAUGE, CAIRO, EGYPT.

In Fig. 4 the effect of progressive change and discontinuities has been eliminated to some extent. The ordinates give the percentage of cases in which the reading falls within the half-meter indicated by the abscissa and, therefore, an ordinate is a measure of the probability of occurrence. The plotted points are derived from observations from 640 to 1451 A. D., together with observations from 1737 to 1917 A. D., making a total of 961 observations. It will be seen that the frequency distribution of Nile floods approximates closely that given by the Gaussian law of errors, and the regularity of the curve tends to show that systematic misrepresentation of the records of low floods did not occur to any great extent, and that perturbing effects act mainly in an accidental manner.

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[36] "Nile Control", by Sir M. MacDonald, Cairo, Govt. Press, 1921.

In the paragraph following Table 1 of the paper there seems to be a misconception of the effect of the Delta Barrage below Cairo. The function of this structure is purely to raise the river level so that the three main canals taking off at the Barrage can be fed at all times of the year. The storage behind the Barrage is negligible and, therefore, so is its effect in lowering the flood-peak or augmenting the minimum flow.

In recent years considerable archæological research has been done with regard to Lake Moeris and the Fayum Basin and the results have been published.[37] A few references to old records of the Nile are: (1) "Nile Gauge at Roda", by Mohammed Kasim[38]; (2) "The Nilometer at Cairo", by Dr. Reiss[39]; and (3) sundry items of correspondence on nilometers.[40]

THOMAS H. MEANS,[41] M. AM. SOC. C. E. (by letter).—Attention is drawn, in this paper, to the long record of the Nile. Many references have been made to such records and there have been a number of summaries and brief discussions of the data, but the records themselves have not been available to many engineers in America. Mr. Jarvis has presented in pictorial form about as much as most engineers will want to see, while the specialist can go back to the more detailed data. The author has done another "good turn" in his studies of the reconciliation of several sources of data, thereby supplying some of the omissions in the Roda record, and in his explanation of the conditions under which the records were collected and of the probable reasons for some of the errors that may be included.

These records are of great importance because, aside from tree rings, they are the longest known continuous record of any kind of information that bears upon the changes in river flow or variations in climate. Briefly, they show the yearly high and low level of the Nile for 1 300 yr—information of great interest to hydraulic engineers the world over. The gauge from which these records were obtained is located on the Island of Roda, at Cairo, and no doubt has been inspected by many engineers who have been fortunate enough to visit Egypt. The substantial character of the structure and of the many repairs and improvements that have been made upon it, and the care with which records have been kept through the more than thirteen centuries, speak eloquently of the importance of the level of the Nile to the people of Egypt. No nation has been so dependent upon one source of water and in no other place in the world is there so much interesting material for profitable study by the engineer.

The recent series of dry and hot summers in many parts of the United States has attracted more than usual attention to matters of weather. In parts of the Pacific Coast there have been seventeen years with precipitation below the average—below the normal. The question is, "Do we know what is normal?" "Is this dry period normal or were the earlier years of relatively high precipitation more nearly normal?" News items have told of railroads hauling drinking water for stock in Imperial Valley. Cities and towns in the Mississippi Valley have placed restrictions on the use of water, and thousands of fertile acres are suffering from lack of rain. An "unprecedented drought" is the explanation. Such disasters occurred not only in 1934, but for several preceding years. Are they "unprecedented"? Are they likely to occur soon again? The climate is changing, say some.

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[37] "The Desert Fayum", by G. Caton Thompson and E. W. Gardner, The Royal Anthropological Inst., 1934; see, also, the review of this book in the *Geographical Journal*, August, 1935.

[38] "Nile Gauge at Roda", by Mohammed Kasim, Cairo, Govt. Press.

[39] "The Nilometer at Cairo", by Dr. Reiss, *Zeitschrift für Messungewesen*, 1889, pp. 439–445 (Konrad Wittwer, Stuttgart).

[40] Correspondence on Nilometers, *Proceedings*, Royal Geographical Soc., Vol. XI, No. 4, April, 1889, pp. 245 and 246.

[41] Cons. Engr., San Francisco, Calif.

The belief that the climate is changing has been adopted as probable by many people and sensational broadcasts have told of the necessity of abandoning large areas of the great plains. The people are told that they have overdeveloped their water resources and must now take a backward step, consolidating their dwindling resources to fit the true conception of the water supply or rainfall.

The memory of human beings is notoriously unreliable as to weather. The deep snows of the past generation, the freedom from floods, or greater floods, the "big freeze", or the "big wind" are phenomena which, to human recollection, no longer occur. Examinations of records of weather, when they are available, seldom support these beliefs. Actual records seldom show any progressive change within historic times. Long-term records, such as these from the Nile, enable the student to carry facts a little further back, to compare with beliefs and theories.

When one reads of abandoned cities in now desert places, or sees wave-cut shore lines made by prehistoric lakes now long evaporated and either gone or greatly reduced in size; when one sees in the abandoned trails of glaciers proof of changed climate, one wonders, if such changes are going on to-day.

There are no weather records in America of any high degree of accuracy extending over 200 yr; in Europe, of not more than 400 yr, which is a very short time when compared with the 4 000 or 5 000 yr in which Man has gathered some historical facts, and insignificantly small when compared with the length of the most recent geological period. Estimating the length of geological time in years is by no means accurate, but from all the evidence available it appears that the time since the last glacial period is from 25 000 to 100 000 yr. The longest rainfall record, therefore, is in the order of one-twenty-fifth of the historical period, or one-five-hundredths of the most recent geologic period. If the rainfall in the eastern part of the United States was reduced one-half in quantity since the last glacial epoch the rate of reduction would be about 0.1 in. per century. There is no way of ascertaining with certainty any such small average change in rainfall. So far as local records alone show, this country may be in an increasing or decreasing cycle—no man knows which. The records of the Nile, therefore, are of great interest in extending the period of observation. Perhaps if it were 4 000 or 5 000 yr long, the trend could be predicted.

The height of the Nile at flood time, in a general way, is dependent upon rains in the Abyssinian highlands; the low flow depends on rains in Central

Africa which reach the main stream through the White Nile. There are many factors other than rain that may effect both these flows but, on the whole, rain is the most important single influence, and in a long period, such as this, its effects must have over-shadowed all other factors.

The Nile is an unusual stream. It flows out of a tropical region including in its water-shed humid, semi-arid, and desert areas of great size. The total area of water-shed approximates 1 200 000 sq miles, or nearly one-half the area of the United States. Its yearly flow in volume approximates 75 000 000 acre-ft, or about as much as all streams in California combined, or one-half of the Columbia River at The Dalles, Ore. The flow in Egypt, ordinarily, is at a minimum in May when it begins to rise from rains in Ethiopia; first the Atbara, then the Blue Nile rise in flood. The flood increases, reaching a maximum in September or October to fall as the Ethiopian summer rains diminish. The White Nile maintains the flow during the winter until the summer rains come again. In round numbers, two-thirds of the flow comes from the Blue Nile, one-fourth from the White Nile, and one-eighth from the Atbara.

In this manner the Nile has been rising and falling each year for several thousand years. In ancient times the cause of this regular procedure was a mystery satisfactorily explained only by introducing the supernatural. Then, nearly the entire irrigated area in Egypt was flooded in basins, under more or less control, as the river rose. In years of very high floods the control was ineffective, and disaster and famine followed; in years of low flood, the supply became inadequate and equally great disaster resulted. The reports accompanying the records published in the *Memoirs* of the Institute of Egypt give the harrowing story of these periods. On the whole, however, the rise and fall of the stream were so regular and dependable that Egypt became one of the richest countries in the world and was the storehouse of food from which neighboring countries were supplied in time of drought and famine.

It was the custom of the rulers to announce each year the time when the river had reached the proper level for the dikes to be opened and the basins flooded. This "wafa" was the occasion of rejoicing and feasting, the important period of the year, for on it depended the prosperity of the country and the amount of taxes which could be collected.

The records of flood level—presumably the highest level on the Roda gauge—and the low-water stage at the same place then became one of the most important records in Egypt and probably epitomizes the economic history of the country. It is no wonder such care was taken in their preservation.

The Roda gauge was in its present location as early as the Seventh Century although there is reason to believe that earlier gauges existed either here, or at Memphis, a few miles up stream. The gauge was built in a temple, or mosque, and the collection and preservation of records was the duty of the priest or sheikh of the temple. The record now existing begins in 622 A. D.

A summary of the thirteen centuries of record is given in Fig. 5. This includes only the Roda gauge readings and not the additional data recorded in Mr. Jarvis' charts. This record of two readings of river level—the highest and the lowest— in each year does not give information from which the yearly flow may be computed but, in a stream like the Nile which goes through a regular cycle, such figures are very significant and should indicate any progressive change that may be taking place.

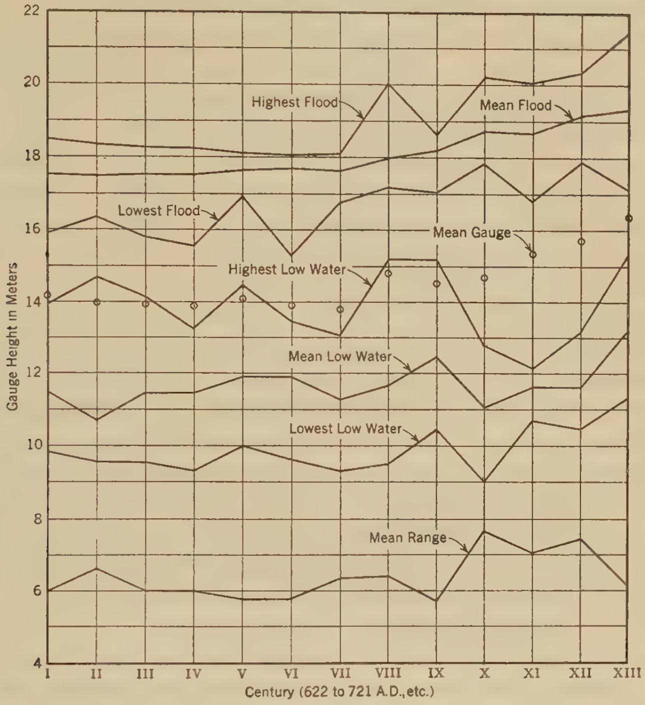

FIG. 5.—RIVER NILE AT RODA GAUGE, CAIRO, EGYPT.

Little information is available concerning changes in the size of the channel, but such as there is indicates that there has been no notable variation in the width of the waterway. From Aswan to the head of the Delta near Cairo the stream seems to have occupied much the same channel since the Egyptians began to build dikes and cities and temples. The location of prominent objects and many historical references all indicate no great change in channel width.

The river levels at Cairo might have been effected by changes in the irrigated area in Upper Egypt, but so far as is known from historical records there was no important change in either area or method of irrigation in the thirteen centuries, until 1833, when Mehemet Ali began the construction of barrages at the head of the two branches of the Nile a few miles below Cairo. This structure effected the levels at Roda by back-water. Prior to 1833, the stream flowed past Cairo substantially as it has done for many centuries. In ancient times Lake Moeris, one of the "seven wonders of the world", may have effected the levels. This lake, which occupied what is now known as the Fayum Basin, a depression in the Lybian plateau west of the Nile Valley, was filled at high water and drained as the flood receded, thus affording both flood protection and water supply to the valley below. It is believed that this lake was not used after 300 A. D. Certainly, it has not influenced the river during the period for which the Roda records are available.

The silt deposited by the Nile has been building up the level of the valley since the dawn of history. This effect is very noticeable throughout the length of the valley. The earliest scientific study of this, of which there is a record, was made by the savants who accompanied Napoleon to Egypt in 1798. They estimated from the depth to which structures were buried that 10 to 12 cm (4 to 4¾ in.) of soil were deposited per century. More exhaustive research, with other approximate information considered, has confirmed this figure and, to-day, it is generally accepted that the mean rise is about 13 cm, or 5 in. per century.

The gauge heights at Roda, for the thirteen centuries show nearly the same rise, both in high water and low water, disregarding the period since 1833. This fact is very significant and indicates that there has been only one change in conditions and that corresponds to the rise from silt deposition.

In the ages in which historical records have been preserved the number and location of the arms of the Nile have been changed several times, and there has been a constant building out of the Delta, extending the channel length. The swamps which once made up the Delta, have been built up by silt, dikes have been constructed to control overflow, and the entire character of the country has been changed; all of which is in agreement with what one would expect from a silt-bearing stream.

The entire Nile Valley, from the first Cataract at Aswan to the Mediterranean, is formed of sediment. The river encounters no rock after leaving the granite at Aswan. The slope of the water surface as far as Cairo is remarkably uniform both at low flow and in flood; it is about 1 on 13 000, or a little less than 7 in. per mile. The cross-section of the stream bed is also very uniform, increasing slightly down stream. On the whole, the result is what might be expected from a silt-bearing stream flowing through a channel of its own deposition. As the valley level rose through deposition, the stream has maintained its channel in depth, width, and slope.

Since the change in average levels on the Roda gauge corresponds so closely to the general average change in valley levels there is no reason to believe that there has been any progressive change in volume of flow.

These records from the Roda gauge indicate the following conclusions:

(1) There has been a fairly constant rise in levels, which may be attributed to the sediment building up the valley lands and the river bed at the same rate. Disregarding the last century of records (which since 1833 has been effected to some extent by work on the barrage at the head of the Delta),

there has been an average in flood levels of about 14 cm, or 5½ in. per century, in the 1 200 yr.

(2) The mean flood, taken as the average of high flood and low flood, has increased at about the same rate.

(3) The mean height of low water has apparently increased at a smaller rate, although the greater variation in low flow renders determination of the change uncertain. This may have been caused by the regulating effect on low flow by diversions, or by the dams from vegetation in the Sudd or swamp region on the White Nile.

(4) The mean of all records, averaging mean flood and mean low water, has changed about as the mean flood level, an increase of about 5½ in. per century. The change in the first seven centuries was less than in the latter five centuries.

(5) The mean range showed little change for nine centuries; then it increased about a meter for three centuries, but on the whole, it changed little in the 1 300 yr. This seems to indicate little change in the rise and fall of the river as a whole and points to a comparatively uniform channel during that period.

(6) There is nothing in this 1 300-yr record that shows any progressive change which may be attributable to change in climate.

J. W. Beardsley,[42] M. Am. Soc. C. E. (by letter).—Based on extensive records, this paper is complicated by the need of changing units and the interpretation of various languages. The Italian, Lombardini, stated in 1865 that "no river in the world lends itself to hydrological studies on so majestic a scale as the Nile." The Nile is Egypt. British engineers have estimated that it built up the Delta at the rate of about 6 cm per century (see Fig. 6 (*a*)). It furnishes ample water for fertile lands in a region practically rainless. In 1908, Sir William Willcocks, referring to the old basin irrigation, wrote:

> "It will be an evil day for Egypt if she forgets that, though basin irrigation with its harvest of corn has given way to perennial irrigation with its cotton fields, the lessons which basin irrigation has taught for 7 000 years can be unlearned with impunity. The rich muddy water of the Nile flood has been the mainstay of Egypt for many generations, and it can no more be dispensed with today than it could be in the past. * * * The basin irrigation of the ancient Egyptians may well be likened to the path of the eagle in its boldest flight, while the perennial irrigation of our day finds its true simile in the laborious task of the working bee."

Exclusive of Fayum and a few square miles of irrigable patches along the Nile between Aswan and Cairo, the productive area of Egypt is an equilateral triangle with sides about 240 km (see Fig. 6 (*b*)) long and with its apex at Cairo, approximately one-fifth of the area of New York State.

The White Nile rises in Lake Victoria under the equator, about 5 600 km from the sea, flows through long stretches of the Sudd with its papyrus, its swamps, and marshes, and delivers clear water into the Nile at its junction with the Blue Nile at Khartoum, 3 036 km from the sea. The Atbara joins the Nile about 320 km below Khartoum, and these two tributaries are the only ones between Khartoum and the Mediterranean Sea—a solitary course of 2 768 km to the sea.

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[42] Cons. Engr., Syracuse, N. Y.

The total drainage area of the Nile is about 2 867 600 sq km (501 000 sq miles), of which the White Nile drains about 60%, the Blue Nile and the Atbara about 10% each, and these last two rivers bring down the floods of muddy waters during August and September; otherwise, their discharge is nominal. A table of monthly discharges prepared by Mr. E. M. Dawson[43] covers the years 1890 to, and including, 1908, the discharges being taken at Sarras 390 km south of Aswan. The mean low flow for May, the lowest month, was 800 cu m (see Fig. 6 (*c*)) per sec, ranging from 540 cu m in 1907 to 1 410 cu m in 1894. The high stage of September, the highest month, averaged 10 470 cu m, varying from 6 870 cu m in 1900 to 14 590 cu m in 1893.

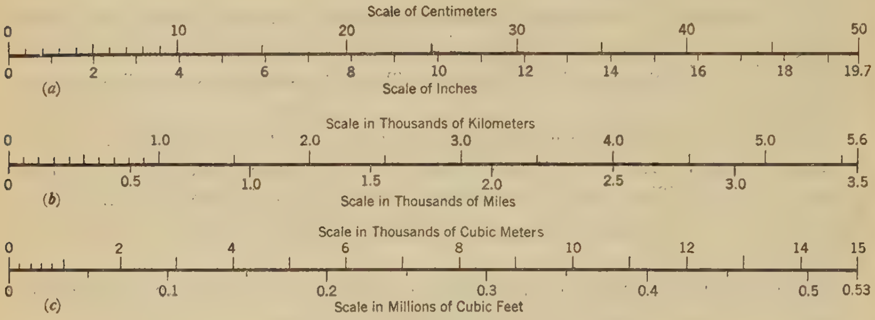
Scale of Centimeters
0
10
20
30
40
50
0
2
4
6
8
10
12
14
16
18
19.7
(a)
Scale of Inches
Scale in Thousands of Kilometers
0
1.0
2.0
3.0
4.0
5.0
5.6
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
(b)
Scale in Thousands of Miles
Scale in Thousands of Cubic Meters
0
2
4
6
8
10
12
14
15
0
0.1
0.2
0.3
0.4
0.5
0.53
(c)
Scale in Millions of Cubic Feet

FIG. 6.—CONVERSION SCALES.

An inspection of Fig. 1, covering maximum and minimum stages for 1 310 yr, suggests that equatorial cycles of rainfall may be shown by a study of minimum stages, the clear waters of which are derived almost entirely from the White Nile. A casual examination indicates cycles between 30 and 40 yr, or multiples thereof. Such an opinion needs verification based on original records. An expression of opinion by Mr. Jarvis regarding such verification would be of considerable interest.

The writer's notes are based on investigations made during the spring of 1909, when the height of the Aswan Dam was increased by 5 m. At that time a photograph was taken of the nilometer on Roda Island, near Cairo (see Fig. 7). This old gauge of Nile floods, a granite-like column with an ornamental capital, is located in a well about 16 ft square. It is said that when waters flooded the capital, benefits were a maximum and so were taxes. When flood waters did not reach the ornamental capital, or when they covered it entirely, taxes were reduced proportionately. Insufficient water reduced the normal crops. The excessive flood waters were harmful and taxes were reduced accordingly. The diameter of the gauge column is now (1935) estimated to be from 18 to 20 in. This gloomy well in a neglected garden, covered with a crude roof of galvanized iron supported by four posts, with its narrow steps growing slimy opposite the beautifully carved capital of the gauge column, and with a black bottom of unknown depth, was not specially inviting, but the view obtained was well worth the trouble. The date of the construction of this gauge was not determined. It was said that when floods reached the capital, agricultural conditions were perfect. Higher floods caused damage and lower floods were deficient. Land taxes were levied proportional to the stages above or below the capital, and if such stages were excessive, no taxes were levied. A maximum tax was due when the flood-stage was on the capital, a range of about 0.75 m. The low stages of May, and the rapidly increasing floods during July and August, and the September peak[44], are clearly shown on Fig. 8.

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[43] *Cairo Scientific Journal*, October, 1908.

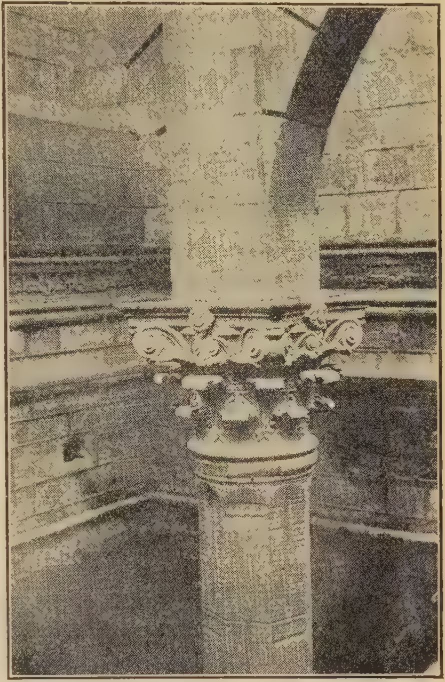

FIG. 7.—THE NILOMETER ON RODA ISLAND, NEAR CAIRO, EGYPT.

Possibly, the Yellow River of China has as many features in common with the Nile as any other river in the world. It flows through a fertile delta built up during past centuries by its load of silt. It flows through a region of scant precipitation insufficient for sprouting spring plantings, and the dense

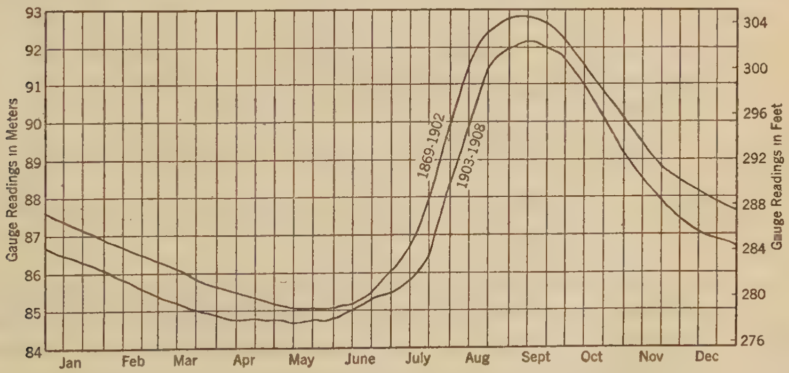

FIG. 8.—GAUGE READINGS AT ASWAN DAM, 1869–1909.

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[44] Pub. by the Survey Dept., Cairo, Egypt, March, 1909.

population is dependent upon irrigation. Its tributaries are small and few throughout its 900-mile course through its deltaic area from its western mountain barriers to the sea. It also has long-time records of floods, which are complicated, however, by the occasional destruction of controlling dikes. Its seasonal floods lack the regularity in time and volume of the Nile floods. In contrast with the Nile, which flows north and is entirely free from ice, the Yellow River flows easterly and through some regions of intensely cold winters.

A study of the Yellow River might develop hydrological facts of value somewhat similar to the author's study of the Nile.

J. C. Stevens,[45] M. Am. Soc. C. E. (by letter).—These records constitute some of the most interesting long-time river data that have been published, and the author certainly deserves the thanks of the profession for presenting them.

The writer is particularly interested in the rate at which the Lower Valley of the Nile has been raised by sedimentation as disclosed by these records. He is also curious to know whether any cyclic variations are in evidence.

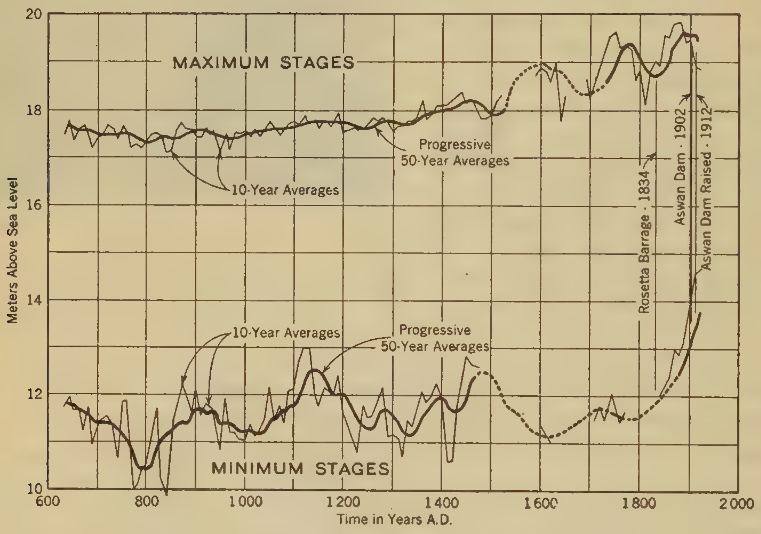

Fig. 9.—Maximum and Minimum Heights of River Nile at Cairo, Egypt, 622 to 1933 A. D.

Upon request, Mr. Jarvis kindly supplied large-scale prints of the data in Fig. 1. From these the average gauge height for each decade was read and tabulated. Progressive 5-decade averages were then computed. The results are shown in Fig. 9.

The light lines are 10-yr averages read directly from the prints. The heavy lines are the progressive 5-decade averages—that is, each point on the heavy line is an average of the preceding five decades. The upward slope of the heavy lines may be taken to be an index of the rate at which the Lower Valley of the Nile is being raised by sedimentation. Neglecting the increased stages caused by the construction of the Rosetta Barrage and Aswan Dam the average rate of rise is practically 4 in. per century for both the maximum and minimum stages.

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[45] Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.

The highs and lows are prominent in the minimum stages and are nicely demonstrated by the progressive 5-decade line. No periodic cycle is in evidence, but high cycles alternate with low cycles of irregular duration.

Another significant point is that as evidenced by this nearly continuous record and from the sporadic records of isolated periods running back over 5 000 yr, there appears to have been little or no climatic changes that can be detected. In other words, it appears that the regimen of the Nile at present is very much as it was about 5 000 yr ago.

JESSE W. SHUMAN,[46] M. AM. SOC. C. E. (by letter).—The labor and patience involved in preparing a paper of this kind, are generally appreciated only by those who have made similar attempts, and the author deserves the thanks of the entire profession for his industry. As stated in the paper, the discharge records of rivers in the United States and in most other places extend back relatively only a few years. Although the Mississippi River, in length, is on a par with the Nile, the continuous discharge records run back only to about 1885; the records for the Tennessee River extend back about fifteen years more; so that if one wishes to study the behavior of a river, its mean, minimum, and flood flows, the available data upon which to base performance are really quite meager. In some cases the analyst finds that rainfall records over the drainage area of a river run back considerably farther than the discharge records, and through correlation between rainfall and run-off, he can "piece out" flows somewhat, and thus lengthen the record for statistical analysis. Although one might not believe in the theory of probability as applied to the study of river discharge, the longer the period embraced by the observations, the nearer to truth the results derived from them should be.

The Nile, alone, stands out as one river in the world on which flow data have been recorded for centuries. The author cites the various sources from which he has gathered the data depicted graphically in his paper. Doubtlessly, there are errors in the data—departures from the absolutely true data; but considering the nature of recording and bringing together these data in the first place, attended by the plus and negative errors of personal observation, it is quite likely that these, as well as the errors arising from copying and translation, would tend to neutralize one another, and thus leave an array of data that was fairly homogeneous, and suitable for any type of analysis. The Nile records, therefore, extend from the present back to about 620 A. D., with gaps here and there, but fairly continuously from 620 to 1521, and from 1700 to date, for the maximum stage of the river.

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[46] Secy.-Treas., Power Eng. Co., Minneapolis, Minn.

Except for a casual glancing over of the graphs of dots and various check marks, and drawing from them only general conclusions, no type of analysis can be applied to the data of the paper unless the values are "picked off" with a magnifying glass, which again introduces another error. The general conclusions that can be drawn are that the maximum annual stages were confined to elevations between 16 and 18.5 m for the entire period, 620 to 1500 A. D., with occasional wide departures; and that from about 1600, the stages were higher, culminating in high values about 1860 to 1880. The chart of minimum stages shows more fluctuation and depicts the secular rise in the river bed, as pointed out by Mr. Jarvis.

Using the magnifying glass, then, and a suitable scale (in this case, 40), one can determine the ordinate values of the charted data, and the information is ready for statistical treatment. The question arises, as to what can be accomplished.

There are many adherents of the practice of treating river-flow data as chance numbers, and the problem is left to them for their exposition. The writer is convinced that there is some law and order in river discharge (that the data are related numbers), and will endeavor to give his reasons therefor, followed by the type of analysis of river flow that results from such reasoning.

Consider the Nile River. It has a drainage area of about 1 107 227 sq miles. Although a record of the stages of the river is given over a long period, there is uncertainty, nevertheless, as to when any departures, increases, or decreases, may be expected or how long an apparent trend of values will continue. A search is made, therefore, for all available data that might possibly bear on the subject. Not much is really known about the Nile and its behavior, except that its source is now considered to be the Victoria Nyanza, with contributions from the Albert and Albert Edward Nyanzas, and the surrounding swamp countries close to the equator; that as it proceeds northward, it receives the discharge of the Sobat River, which drains a region of heavy rainfall; that farther north, the Blue Nile, rising in Abyssinia, pours its turbulent flows into the main stream, giving rise to the floods that have been recorded. The base, or steady flow, is provided by the White Nile, and the floods by the Blue Nile. Rainfall records are either lacking entirely or are scant, being for only a few years; so that little can be done by correlation with it as a factor. The River Atbara, rising in Northern Abyssinia, discharges into the Nile about 200 miles north of, and into the Blue Nile at, Khartoum.

An analysis of a river's behavior involves one of three methods: (1) Treating discharge data as purely fortuitous; (2) using harmonic analysis; or (3) using what is finally becoming known as cycle analysis. The disadvantages of the first method are that conclusions are drawn from data embraced only in the available record, for possible performance beyond that period; and that several eminent mathematicians have proved that the data are not chance numbers. The disadvantage of the second method is that, although one can break up into component elements a given curve of river flow, by means of the Fourier series, the equations developed by the data will not necessarily reproduce the actual observed data at a later date. The disadvantages of the third or cycle analysis are, that it is as yet not completely developed, there being much to learn; that no treatise is yet available that covers the subject; and that, due to these causes, members of the profession have not become acquainted with it. It has some advantages over the other types of analyses, however, as the writer hopes to demonstrate in what follows.

The principles of cycle analysis will be developed as the writer proceeds with the treatment of the Nile data, but it is deemed necessary first to give some background to show that it is not extravagant nor lacking in consistency.

Briefly, cycle analysis consists of breaking up data into component elements, by suitable reducing means, quite differently than in harmonic analysis, and deriving certain pattern curves or cycles, which are similar to curves derived from sunspot data, giving thereby something to which terrestrial phenomena can be tied, and thus an inkling of what is to come and of what has gone before, beyond the periods of known data.

In order to make any progress, a research worker is obliged to set up certain postulates, for without his imagination, he would be merely an observer of phenomena and a gatherer of observational data. Therefore, he accepts the best that is offered at the moment. Clayton[47], Abbott[48], and others may be all wrong in their contention that the trigger action of the varying heat output of the sun is responsible for the changing weather. Clayton's work as head of the Weather Forecasting Department of the Argentine Republic, some years ago (where he successfully forecasted rainfall and temperature a week in advance) may not have been what it was purported to be, and the weather may be purely fortuitous or governed entirely by terrestrial causes; but even granting these points, it seems to the writer that there should be no objection to utilizing the fact that a certain cycle can be derived statistically from the sunspot numbers, varying in periodicity depending upon the length of two consecutive solar cycles, and which is named the Bruckner cycle[49]; and that also a similar cycle derived by similar processes in river run-off, rainfall, temperature, lake levels, etc., is identical in its general pattern, although varying in phase from place to place. This fact of phase variation, it seems to the writer disposes of much of the previously found confusing results.

It is found that the pattern of weather, as defined by the Bruckner cycle, is general, but with modifications as to intensity and fortuitous occurrence of yearly values, everywhere about the globe, and that the phase varies, depending upon the location, continentality, etc. For example, the Bruckner cycle detectable in the flow of the River Nile, is exactly opposite in its swings up and down, to the Bruckner cycle in the Mississippi River.

The question is frequently asked: "How can the changes in weather, which is different at various places, be brought about by any slight variation in the solar constant, when the sun is more than 90 000 000 miles distant?" There are certain groups who maintain that it is not necessary to depart from this planet to find suitable agents that can cause weather to change. So far, however, this group has failed to bring forward any workable procedure for long-range forecasting, and it is certainly this very thing that the engineer is attempting when he analyzes the flow of a river which has records for only, say, fifty years; and yet he is trying to deduce its performance over a longer period of time in the future. Consequently, whether or not it is recognized, engineers are very much interested in long-range forecasting, in studying a river, designing a bridge, or laying out a great highway. The problem involved, is to ascertain and use, as key factors in design, not only the values in the past and present, but also what will be probable for some time to come in the future.

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[47] "World Weather", The Macmillan Co., N. Y., 1923.

[48] *Smithsonian Miscellaneous Collections*, Vol. 85, No. 1, and Vol. 87, No. 18.

[49] "Klimaschwankungen seit 1700", Vienna, 1890; also, *Monthly Weather Review*, Washington, D. C., July, 1926, and March, 1928.

Studying the data of the Nile River as presented in this paper, with a view to utilizing them for cycle analysis, after several preliminary trials, it was decided to use the Lyons data[13] as charted from 1738 to 1872, and the Roda gauge data from 1873 to 1921, for the maximum annual stages, as well as the records taken from Fig. 3, giving the annual or seasonal volume of discharge at Aswan, from 1870 to 1932.

TABLE 6.—MAXIMUM ANNUAL STAGES AT RODA GAUGE

Date (1)	0 (2)	RESIDUALS		Scale and phase (5)	RESIDUALS		Scale and phase (8)	$0-R_4=S_1$ (9)	M (10)	R_4+M_1 = Double Wolf cycle (11)
		R_1 (3)	R_2 (4)		R_3 (6)	R_4 (7)				
1738	2.70									
1739	2.00	4.70	....	2.29						
1740	2.45	4.45	915	2.21	4.50	....	2.20	+0.25	+0.04	2.24
1741	1.95	4.40	885	2.10	4.31	8.81	2.07	−0.12	+0.04	2.11
1742	2.05	4.00	840	1.86	3.96	8.27	1.87	+0.18	−0.04	1.83
1743	1.40	3.45	745	1.66	3.52	7.48	1.75	−0.35	−0.15	1.60
1744	1.80	3.20	665	1.82	3.48	7.00	1.86	−0.06	−0.11	1.75
1745	2.30	4.10	730	2.14	3.96	7.44	2.15	+0.15	0.00	2.15
1746	2.15	4.45	855	2.50	4.64	8.60	2.30	−0.15	+0.10	2.40
1747	2.40	4.55	900	2.06	4.56	9.20	2.03	+0.37	+0.10	2.13
* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *

In performing the computations to secure the cycles, the data are arranged, and manipulated as outlined, in Table 6. Values of the data are included only for 10 yr, in order to illustrate the method. Use paper ruled with vertical and horizontal lines, and enter in Column (1) the dates for the complete record to be analyzed. In Column (2) are the data from the chart after deducting 18.00 (so as to yield as small a number as possible with which to work); call these data 0. Now, add the first two values in Column (2), entering the sum in Column (3) opposite the second value as shown. This Column (3) is headed "Residual No. 1", or R_1. Now, repeat this operation, adding the two consecutive values of R_1, and writing the sum in Column (4), opposite the second value as before, these values being called R_2. R_4 can be derived by repeating the addition twice, and then dividing by 16, but after considerable work in this kind of manipulation, the writer prefers to make only two additions, as shown, to R_2, and then to divide

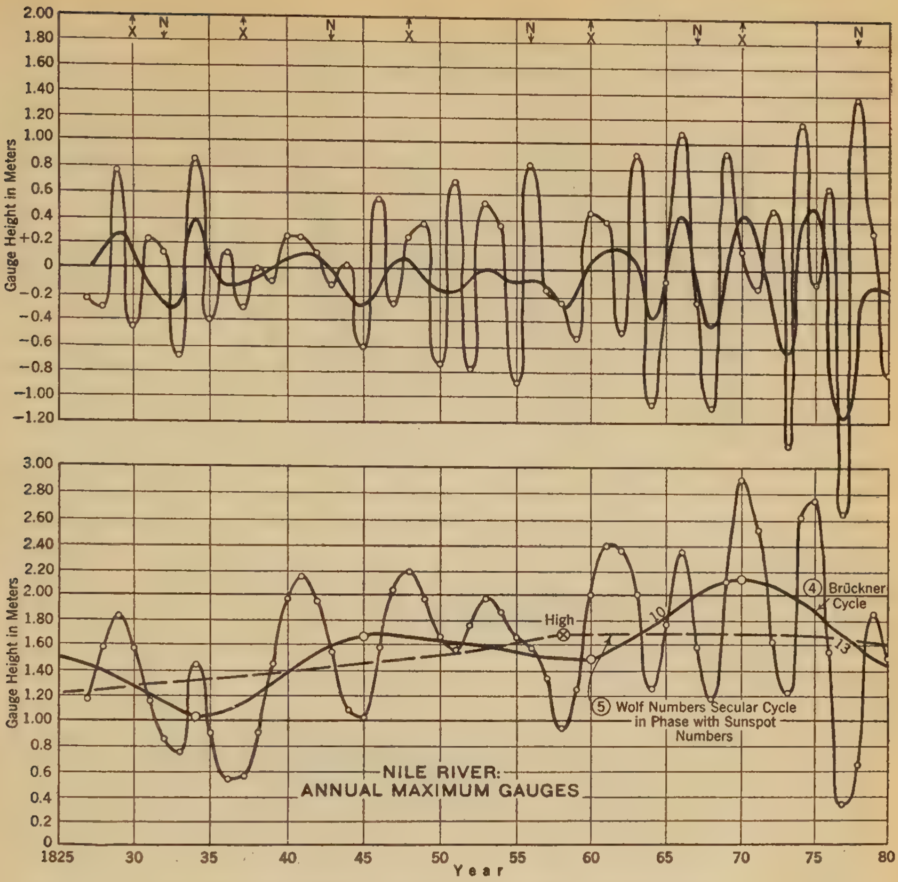

Fig. 10.—Annual Maximum Gauges, Nile River.

R_2-values by 4 and place the quotient one step upward in Column (5), headed "Scale and Phase". After R_4 is derived, it is restored to "Scale and Phase", and its values are entered in Column (8). Now, subtract the values of R_4 from the original value in Column (2), and enter the results, plus or minus, in Column (9), the values of which are called "Stratum No. 1", or S_1.

After the values of S_1 are secured for the length of the record, they are plotted as shown, for a representative part of the record, in Figs. 10 and 11. Although the original data began at 1738, the earliest value the residual yields is for 1740. Some care must be exercised in adopting proper vertical and horizontal scales, in order that the oscillating values of S_1 will show clearly, and also that the median line, M_1, drawn through the various loops of S_1, will be of sufficient scale that its ordinate values can be read with reasonable accuracy.

After S_1 has been plotted, as shown in Figs. 10 and 11 (Curve 1), one looks over the data of R_4 in Column (8), Table 6, and checks the maximum and minimum values which will alternate through the record. Next, the median

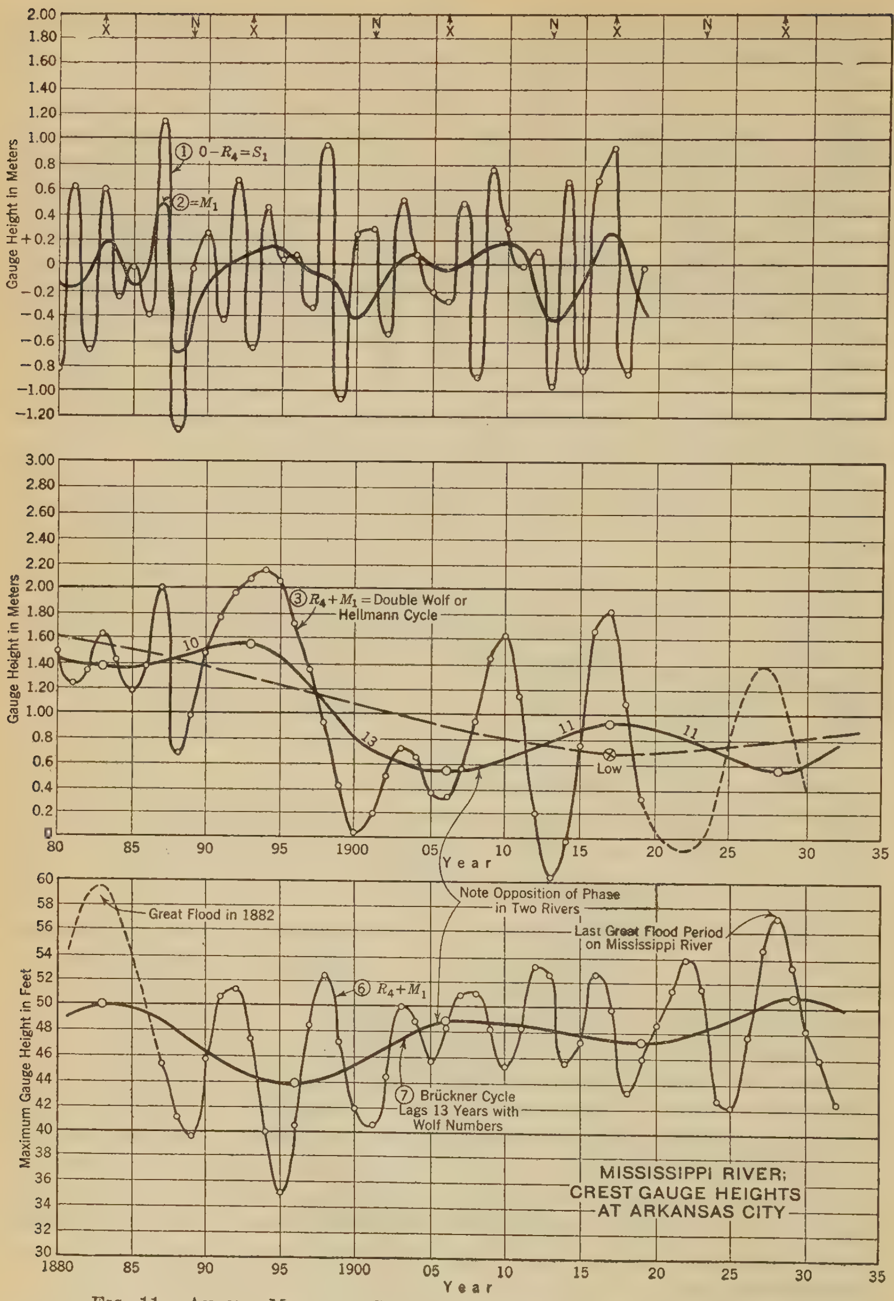

FIG. 11.—ANNUAL MAXIMUM GAUGES, NILE RIVER AND MISSISSIPPI RIVER.

line, M_1 is drawn through the curve, S_1, following as closely as possible, the "ups and downs" of R_4. The ordinate values of M_1, as now derived, are picked from the drawing, and entered as plus or negative values in Column (10), Table 6. Now, if M_1 is subtracted from S_1, the smallest cycle is secured, called the Clough cycle[50]. If M_1 is added to R_4 and the values are entered in Column (11), Table 6, the result is the Double Wolf, or Hellmann, cycle.

Before proceeding further it might be wise to offer a series of definitions, due to the undeveloped state of this type of analysis. The following cycles have all been discussed by various writers, but the notation has not yet been established.

Clough Cycle.—This is a small cycle of 2 to 3-yr periodicity, averaging about 28 months. It is very prevalent and persistent in all meteorological data, and was discovered by Clough, Clayton, and Abbot.

Wolf, or Wolfer Numbers.—These are provisional sunspot numbers; a Wolf number is an empirical index number, derived from the number and area of sunspots. The numbers are usually published after observational data have been slightly smoothed.

Double Wolf, or Hellmann, Cycle.—A cycle that crests about every 5.5 yr, generally just before, at, or after, sunspot maximum and minimum. It is variable and does not always appear consistently, although in some locations it is very clear, consistent, and definite. Hellmann was one of the first to point out the maximum rainfall that occurred twice in a Solar cycle. Horton also referred to it later. Some writers call it the Double Wolf, and some the Hellmann, cycle.

Solar Cycle.—This is the well-known cycle of the sunspot periodicity. From maximum to maximum sunspots, the period averages a little greater than 11 yr. This is commonly called the 11-yr cycle. It is detectable in a great many terrestrial phenomena and is especially prominent in lake levels.

Bruckner Cycle.—This is a cycle of twice the periodicity of two consecutive solar cycles, about 22 to 23 yr in length. It was first discovered by Bruckner[49], but by him given a varying length of 25 to 50 yr, the average value of which was set at ± 35 yr, due to the fact, as shown by Streiff[51] later, that his cycle was a composite of the secular and the cycle of two solar years.

Double Wolf Secular Cycle.—A cycle of twice the periodicity of the Bruckner cycle. First mentioned by Streiff in personal correspondence with the writer, and prominent in many meteorological data, especially in lake levels[52].

Secular Cycle.—The long swing in Wolf numbers is of varying length: The last high point was about 1856, the three previous highs were at 1798, 1704, and 1646. It is a cycle that is apparent in Wolf numbers, in tree-ring growth, varves, rainfall, temperature, lake levels, etc. Sometimes it is of such small amplitude as to be unrecognizable, in certain locations.

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[50] *Monthly Weather Review*, April, 1933.

[51] *Loc. cit.*, July, 1926, and March, 1928.

[52] "Notes on Lake Levels", by Jesse W. Shuman, *Monthly Weather Review*, March, 1931.

There are still longer and shorter cycles, but the foregoing are of the most use in analyzing the immediate past and future. Finally, looking back over the foregoing array of cycles, the observational data oscillate above and below the Double Wolf cycle, which, in turn, oscillates about the Solar cycle. This, in turn, oscillates about the Bruckner cycle, which, in turn, oscillates about the Double Wolf secular cycle; and, finally, the latter oscillates about the secular cycle of the Wolf numbers.

The values of $R_4 + M_1$, the Double Wolf cycle in the Nile River data, from Column (11), Table 6, are now plotted in Figs. 10 and 11, on the same time scale, as Curve 3. The break in the records from 1800 to 1825 is shown on the charts.

Now, after the $R_4 + M_1$ cycle is drawn, as in Figs. 10 and 11, it is sometimes necessary to reduce it further by two to four additional operations, before drawing in the median line, Curve 4, which is the Bruckner cycle. In this case, however, the further reduction seems to leave the relative vertical positions of the loops with respect to each other, unchanged. Therefore, the median line (Curve 4, Figs. 10 and 11) is drawn through the plotted data of $R_4 + M_1$. The median line, Curve 5 (the Secular cycle) is now dotted through the loops of the Bruckner cycle, the high point being at about 1860 and the low point, in Figs. 10 and 11, at about 1915 to 1920.

At the top of Figs. 10 and 11, under the scale of years, are placed arrows marked X or N, pointing upward or downward, to identify the times of sunspot maximum and minimum. The analysis is now complete and the analyst is free to examine what has been developed. It must be kept in mind that each plotted point of $R_4 + M_1$ (Curve 3), is a residual derived from five annual numbers, and that the actual yearly value may be more, less, or the same. In general, a peak of the Double Wolf cycle indicates that the average value of the five years' data—two years before and two years afterward—combined by residuation with the data for the plotted year, was at a maximum, and the troughs indicate the reverse.

Examining the Double Wolf cycle (Curve 3), it is noted that a succession of peaks and troughs, oscillate about the heavily drawn Bruckner cycle. At some places on the record they occur quite consistently with respect to sunspot maxima and minima, but by no means sufficiently so that one could rely upon the recurrence. The peak at 1861 occurs one year after sunspot maxima; at 1870, it occurs exactly on that date, as well at at 1883; but it is one year delayed at 1893, etc.; so that in the case of the Nile River, the Double Wolf cycle can be used for purposes of forecasting only in a general sense.

It is to be noticed, however, despite the secular rise in the river's bed, as depicted by the records of the minimum stages in Fig. 1 of the paper, that the period centering around 1870 represented a high point, or increased flood conditions, and that the period about 1905 represented a decreased condition of flow. The Bruckner cycle derived in the flow records of the River Nile, are just about, if not exactly, in phase with the Bruckner cycle in sunspot numbers.[53] The graph shows that the behavior of the Nile River is very strongly influenced by the Bruckner cycle. The data ended at 1921, and the last plotted date of $R_4 + M_1$ was at 1919. The question arises, can any of these cycles be extended into the future, with any dependability? The answer would have to be negative, if these data were all that were available. Fortunately, however, there are many other data on which to perform similar analyses, and results that can be compared with those of the Nile River. After a turning point has been developed in a cycle, it can be extended with considerable accuracy; hence the necessity of cross-identification with data from elsewhere, the phase of which may differ, and upon which a certain cycle may have reached a turning point.

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[53] For Bruckner cycle in sunspot numbers, see, the writer's Fig. 8 in discussion of "Rainfall Characteristics and Their Relation to Soils and Run-off", by C. S. Jarvis, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 95 (1931), p. 379.

The writer has taken data of tree-ring growth from Douglass[54], for a certain tabulation named "Flagstaff High (F.L.H.) 10 trees", and by using the same type of analysis, to derive the Double Wolf cycle, and, subsequently, drawing a median line through the various loops, the Secular cycle shown diagrammatically in Fig. 12 was secured. It will be noted that the wet periods were at 1646, 1704, 1798, and 1856; that the dry periods were at 1670, 1766, and 1822, and that the last dry period centered around the date 1918. The tree-ring data did not extend far enough to get this last date, and until about 1932, the writer thought that this last date should be about 1905 to 1910; but examination of data from varied and widely placed sources, has led him to believe that the date must be about 1918. Then, too, another confirmation has only recently been given this date. A study[55] of Halbert P. Gillette, M. Am. Soc. C. E., on certain cycles, chiefly the 70 and 152-yr cycles, was presented at Los Angeles, Calif., in June, 1935. The analysis as used herein will not develop this 152-yr cycle, but in examining Fig. 12, giving the Secular cycle in tree rings, it will be noted that from 1646 to 1798 (two Secular cycles) is a lapse of 152 yr; from 1704 to 1856 (again two cycles) is 152 yr; furthermore, from 1670 to 1822 is 152 yr. This remarkable consistency, if genuine, would set the last low of the Secular cycle at 1766 plus 152 yr = 1918, and the next high at 1798 plus 152 yr = 1950. Every record of meteorological data examined recently by the writer seems to confirm this last low date as about 1918. This low is the turning point of the Secular cycle, which is threaded through the Bruckner cycles, so that there will be a low of the Bruckner cycle before, and after, the low of the Secular cycle.

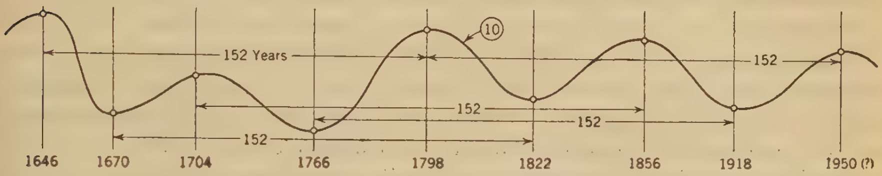

FIG. 12.—SECULAR CYCLE.

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[54] "Climatic Cycles and Tree-Growth", by A. E. Douglass, Vol. II, The Carnegie Institution of Washington, Washington, 1928.

[55] "The Cycles That Cause the Present Drought", by Halbert P. Gillette, presented before the Annual Meeting of the Am. Meteorological Soc., Los Angeles, Calif., June 26, 1935.

Now, having some conception of what the Secular cycle is like, and the relative magnitude of its previous highs and lows, the student is in a position to draw some conclusions about the behavior of the Nile River, and to make some extensions beyond the times of the recorded data. Fig. 13 is a diagram, showing the Bruckner cycle of the Nile River from 1740 to 1960. The actual data analyzed ran from 1738 to 1921, with a break from 1800 to 1825. The diagram shows the highest period at 1870, the low period that must have occurred about 1800 (before and afterward), and the low period centering around the low turning point of 1918, that is, the low of the Bruckner cycle at 1905 and again the probable low of the Bruckner cycle at 1930. The next probable high period of maximum stages will center around 1940 to 1942 and conditions may equal those of the 1892–94 period; or, if the behavior is erratic, even those of the 1870 period. These ideas are indicated by the trends of the Bruckner cycle oscillating on the Secular cycle.

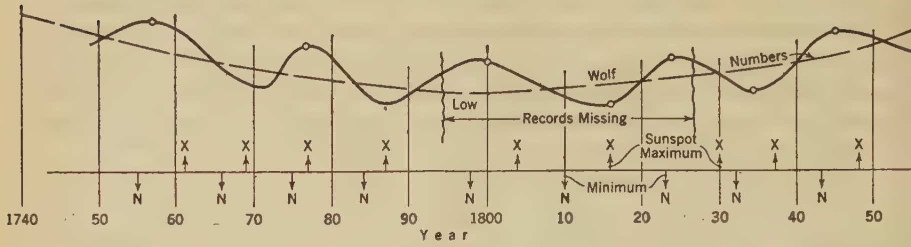
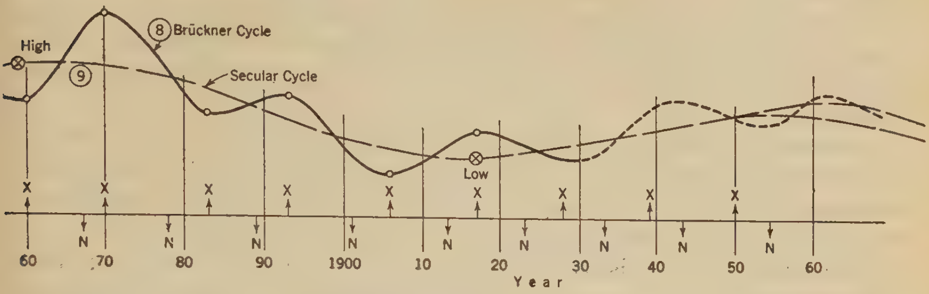


FIG. 13.—BRUCKNER AND SECULAR CYCLES IN

On Fig. 11, at the bottom, with the same time scale, is shown a graph of the Double Wolf cycle and the Bruckner cycle, in the Mississippi River data, taken at Arkansas City, Ark., for annual maximum gauges. The data on this river run back to only about 1885. Note that the Bruckner cycle in this case, is just opposite in phase to the Bruckner cycle in the Nile River. During the early Eighties, the Mississippi River had severe floods; whereas the Nile River was lower then than before or afterward. The last great flood of the Mississippi River was in 1927, a period that must have had relatively moderate stages on the Nile River. In other words, the Bruckner cycle in the Nile River, is ahead of (leading) the Bruckner cycle in the Mississippi River by one-half the Bruckner cycle period—about 11 to 13 yr. The performance of the Mississippi River, in general, can be forecast from the behavior of the Nile. For example, the high stages on the Nile in 1870 were followed in the Mississippi River about 1882 and 1883; the low stage of the Nile in 1883 was followed on the Mississippi River at about 1895, etc. Judging from past performance, the comparison of the Nile River and Mississippi River cycles, indicates that the high stages of the Nile in 1917 would be followed on the Mississippi River about 11 to 13 yr later, with real flood stages. The great flood of 1927 on the Mississippi River, came just ten years later. The low condition of the Nile in 1927 would lead one to expect a downward trend of the Mississippi River flood conditions until about 1940 (bearing in mind, however, that frequently a real flood occurs close to the low point of the Bruckner cycle, as at 1897, etc.)



MAXIMUM ANNUAL GAUGES, NILE RIVER.

Fig. 14 has been prepared to show a comparison between the Nile River and the Mississippi River, using for the Nile, the data from Fig. 3, of seasonal discharge at Aswan, and for the Mississippi River, the mean annual flow at Arkansas City. In the Nile graphs (Fig. 14(*a*)) the Bruckner cycle lags about

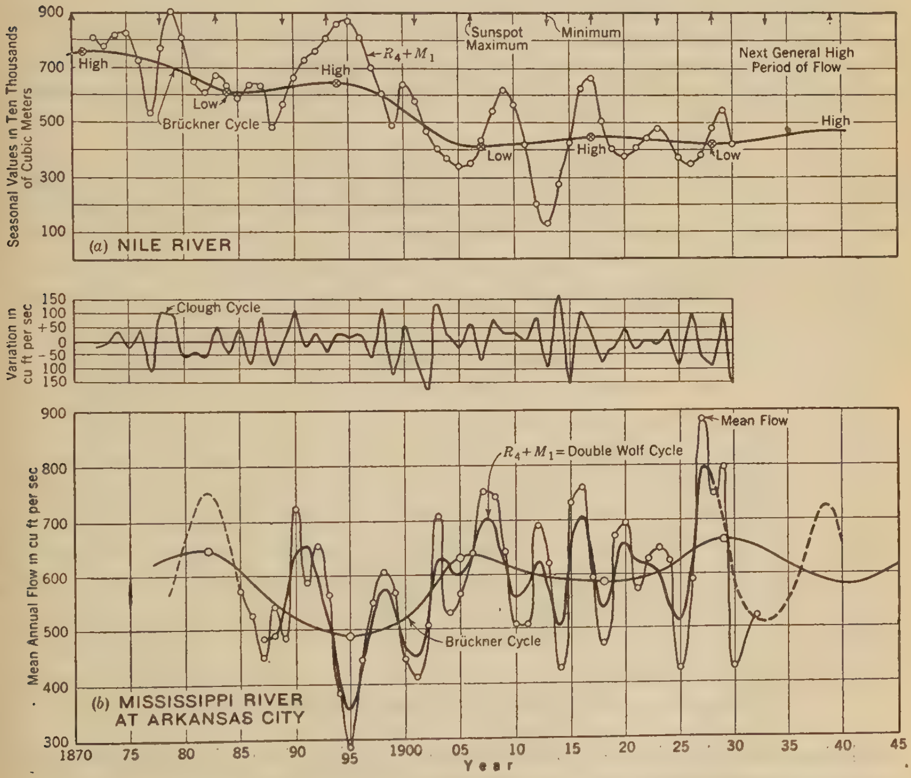


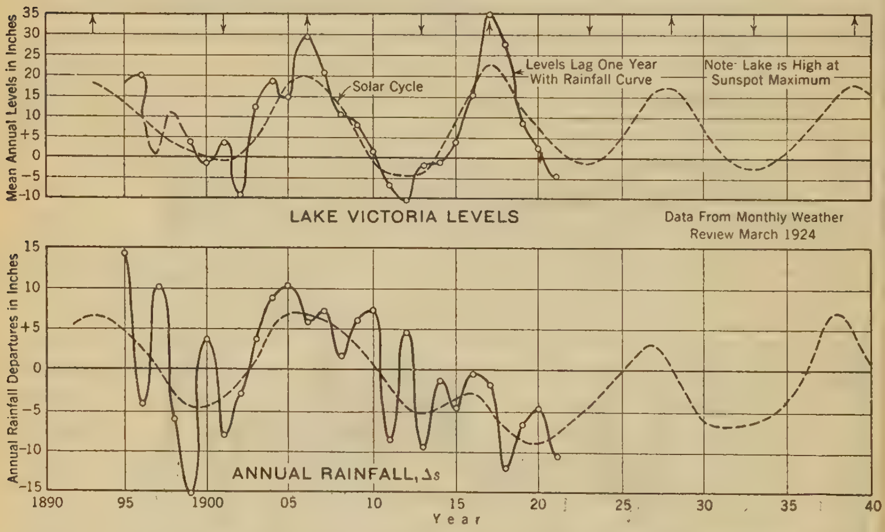
FIG. 14.—COMPARISON BETWEEN THE NILE RIVER AND THE MISSISSIPPI RIVER.

one year with the same cycle in sunspot numbers, or is nearly if not in phase (although the writer has placed small circles at each turning point of the Bruckner cycles shown in all the graphs in this discussion, it must be understood that it is probably impossible to define the actual turning point so accurately, at least with the type of mathematical procedure used herein. The circles are placed simply to give accent to the change of the trend in that region.)

Although the Secular cycle has not been drawn in, an inspection readily indicates that it is present in the data. For the Nile River, the Clough cycle is shown, secured by subtracting M_1 from S_1 (not shown herein). The algebraic sum of $R_4 + M_1$ and the Clough cycle equals the yearly discharge value.

The same cycles for the Mississippi River are shown in Fig. 14(*b*) as well as the annual values connected by a lightly drawn line. In this case, as in the case of Fig. 11, the same opposition in phase of the Bruckner cycles is in evidence. The Bruckner cycle is trending downward to a low point about 1940, although a peak in value of $R_4 + M_1$ may be expected before that time, of a magnitude considerably less than that in 1927.

Graphs[56] illustrating the varying level of Victoria Nyanza, the source of the White Nile, and the variation in the rainfall of that region are shown in Fig. 15. The sunspot maxima and minima are marked on the time scale, and the dotted line drawn freehand through the lake levels is the Solar cycle, which is beautifully clear. Note that the lake levels are at peak always at the sunspot maximum.



LAKE VICTORIA LEVELS

Data From Monthly Weather Review March 1924

ANNUAL RAINFALL, Δs

FIG. 15.—DIAGRAMS SHOWING VARYING LEVELS OF VICTORIA NYANZA, THE SOURCE OF THE WHITE NILE, AND VARIATION IN RAINFALL IN THAT REGION.

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[56] Original data pub. by Mr. C. E. P. Brooks, and later reviewed by Alfred J. Henry in *Monthly Weather Review*, U. S. Weather Bureau, Washington, D. C., March, 1924.

The outflow of this lake forms the base flow of the Nile, but there appears to be but little correlation, as would be expected, between the graphs of the river and the lake. If, however, comprehensive rainfall records were available, distributed over the drainage area of the Nile River, their statistical treatment taken in conjunction with that of the river, would yield immensely important information. This importance would be further enhanced if, in addition, records for temperature were included, and similarly treated, because temperature is a primary factor, whereas rainfall is secondary. For this lake Brooks found[57] that the correlation coefficient between the lake levels and rainfall was + 0.92, and between lake levels and sunspots, + 0.80.

The annual flood stages of the Nile River are of great importance to Egypt, and in bringing together the data described in his paper, the author has placed at the disposal of the profession valuable information, gleaned at great pains, which should ultimately lead to new interpretations. If mean monthly flow records were available for a period of years, this type of analysis would enable one to forecast the probable mean flow for the ensuing future year with considerable accuracy. In this discussion the author's records have been treated only back to 1738, and it is believed that sufficient knowledge can be deduced from this range of values because, although the regimen of the River Nile may be affected by the various causes, as pointed out by the author, the writer wishes to suggest that any such change in behavior would be gradual; so that in comparing values of to-day with those of 4 000 yr ago, when the average annual flood stages were about 23 ft higher than at present, the differences encountered would be of historical interest only. One would not need to consider the old records in designing a dam or other controlling works, which will endure probably not more than 150 to 200 yr.

Cycle analysis presents a conception of the variation in weather as registered by the river's flow, and suggests that extreme values, high or low, are not suddenly brought about, but are approached in an oscillating, pyramiding manner, culminating at the extreme value.

It is not yet 450 yr since Columbus discovered America, so that no structure in this country is that old. Prime movers are likely to become obsolescent in about 25 to 30 yr, and the life of a concrete dam is 50 to 200 yr. If any confidence can be placed in cycle analysis of a river's flow, that attempts to relate it with the changing weather, it follows that only values embraced in, say, 200 to 300 yr, are necessary for any type of analysis. Great catastrophies such as those in the past, probably did not strike from "a clear sky", but were the culminating point in a stormy period of greater intensity than normal, and which developed gradually enough so, that with proper knowledge, due warning of their approach was given.

The Ministry of Public Works of Egypt, in discussing the hydrology of the Nile Basin, states[58] that these records of the flow of the river have been analyzed for periodicities, and periods of small amplitude have been found. Quoting:

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[57] *Geographical Memoirs No. 20*, Air Ministry, Met. Office, London, England, 1923.

[58] "The Nile Basin", Vol. I, *Physical Dept. Paper No. 26*, Govt. Press., 1931, Cairo, Egypt.

> "They are, however, so masked in irregularities as to be useless for forecasting purposes. A connection exists between conditions in the South Atlantic and the Nile, but up to the present it has not been possible to make a forecast of the flood which is sufficiently reliable for practical purposes. However, as the science of meteorology develops and the nature of the mechanism producing the flood becomes known in detail, it may be possible some day to produce an accurate forecast of the nature of the flood some months in advance * * *."

The writer has already called attention to the connection that must exist in weather at different places, the reflection of which is given in river performance (for one index). It appears from the foregoing quotation that the authorities in charge of the controlling works of the Nile are cognizant of this fact and are working with it. The connection in the long swings, as defined by the Bruckner cycle, has been demonstrated in Fig. 11 between the Nile and the Mississippi Rivers. Doubtlessly, further search would bring to light other connections.

If the entire record of the Nile back to 620 A. D. were analyzed to secure the cycles described herein, interpretations of their probable effect upon the economic life of that region could be made by recourse to the vast storehouse of ancient literature that exists, which mentions here and there conditions that are ascribable to the Nile's behavior at the time. This river alone seems to have not only a very long record but also the written history of the times to accompany it, so that the field should be very fruitful.

Pharaoh Necho II (610 B. C.) was the first ruler[59] to attempt to create a connection between the two seas (Mediterranean Sea and Red Sea). Herodotus records this effort. The canal tapped the Nile at Bubastis, near Zagazig, and mention is made that the Red Sea extended farther inland then than at present. The length of this canal was computed by Pliny to be about 57 English miles. Herodotus states that its construction cost the lives of 120 000 men. Due to an adverse Oracle, warning that it might provide too easy means for approach to Egypt, the project was abandoned. Note that this was a period of ample water on the Nile. The canal was restored under Ptolemy Philadelphus (285 B. C.), and locks and sluices were used. This period also had ample water. Under Cleopatra (31 B. C.) the canal was impassible (a low-water period). Emperor Trajan (98–117 A. D.) found the Pelusiac Branch of the Nile very low, with practically no water. He tapped the Nile very much farther up stream.

Ptolemy wrote, in 150 A. D., the first clearly intelligent account of the origin of the Nile, something that had always been a great mystery. He writes of the two lakes, Victoria and Albert Nyanza, and the Mountains of the Moon. No less than 1 170 yr had to elapse before justice could be done this ancient geographer, and his account verified.

When one scans the history of regions not greatly remote from Egypt, and notes the vicissitudes of climate they have endured, it is significant that the records of the Nile disclose no such changes—merely changes of the weather. It was evidently with good reason that Egypt, with its droughts and famines notwithstanding, was a good place in which to live, by contrast with other not far distant regions, the flow of the Nile River being uniform enough so that even in its low periods, it carried sufficient water to keep the nation from migrating.

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[59] "History of Egypt", by Rappoport, The Grolier Soc., London, England, 1904.

KAMEL OSMAN GHALEB,[60] ESQ. (by letter).—Since 1926 the writer has been connected with the nilometer on Roda Island, on behalf of both the Irrigation Department and the Committee for the Preservation of Arabic Monuments.

In 1925, the column of the nilometer subsided about 7 cm and became detached from its upper support; it was propped up, to prevent it from falling. In December, 1926, pulsometers were used to unwater the well, so as to fix the column in position but, when the water level was lowered to about 2 m from the bottom of the well, dangerous cracks appeared in the walls and the unwatering had to be stopped lest the entire structure collapse. The Egyptian Parliament voted £20 000 (Egyptian) in the financial year beginning April, 1929, for the repair and restoration of the monument (see Fig. 16), as well as the expropriation of land surrounding it.

The column is an octagonal marble pillar with an enlarged base; the capital is an addition, probably of the Eighteenth Century. The engineer who erected the column in 861 A. D., has left a full description of its graduations. He states that it is divided into 19 equal cubits; in fact, at the upper end of the cubit just below the capital is inscribed in relief, in Arabic in Cufic characters, the words, "nine (&) ten cubits." As far as has been ascertained, the column is broken in three pieces: The top break, which is indicated on the drawing made during the Napoleonic Expedition, has been well repaired; the two ends of the lower break have been badly joined together, and the corresponding cubit now measures 31 cm only instead of the usual length of 54 cm. As in the case of the Suez Canal, the French savants attached to the Expedition, who have left such a wonderful detailed record of the monument, made an extraordinary mistake; in the Canal they had found a great difference in levels between the Mediterranean and the Red Seas; in the Roda nilometer they mistook the top cubit for the sixteenth; and one of them—the famous Arabic scholar, Marcel—when he returned to take service in Egypt years later, indicated that mistake and read the top cubit as the seventeenth. When the well is completely unwatered it will be possible to ascertain whether the French embedded the lowest part of the column, or whether they replaced it on the floor, as formerly, and simply neglected to take account of its base in counting the number of cubits to the top.

However that may be, the reason for the apparent unpredecented rise during the final century of record of another 6 cubits, mentioned in the paragraph following Table 1 of the paper, can now be easily explained; the 3 full cubits suppressed by the French savants, become 6 cubits in the readings for the maxima stages of the river, which are above the sixteenth cubit, where half cubits are read as full cubits. The French thought accordingly that lands watered at the beginning of the Nineteenth Century, when the Nile level was at 16 cubits, were previously irrigated at between 13 and 14 cubits. Another similar error seems to be current now; it is stated in the paragraph following Table 1 of the paper, that the flood-stages that indicated an assurance of plenty progressed gradually from 16 to 20 cubits on the Roda gauge. This statement must be due to a confusion in the translation of the mediaeval chronicles, the three stages of the rise of the river corresponding with the three categories into which the agricultural land of Egypt was divided being taken for one stage only.

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[60] Insp.-Gen. of Irrig., Lower Egypt, Cairo, Egypt.

The low lands were watered when the Nile reached 16 cubits; the middle lands were irrigated at 18 cubits; and the high ones at 20 cubits. On the occasion of each of these stages there was a special festival, and water was let into a canal by the cutting of a cross-bank. The first of these festivals was for the "wafa", mentioned by Mr. Jarvis, the second was for the "Neirouz" (the New Year's day of the agricultural year, which has now become the beginning of the Coptic year); and the third for the "Saleeb" which falls seventeen days after, and which is now also kept as a religious Coptic festival.

As far as historical times are concerned, the 16 cubits of the Nile have allowed a sufficient supply; that is, there was no chance of starvation when that level of the river was reached. The Arabs, on conquering the country, adopted this same criterion and called it the "wafa." Consequently, there has been no rise in the level of the water required to irrigate the land and, therefore, that land itself cannot have risen appreciably. The writer considers that deducing the secular rise of the Nile flood-plain by dividing the thickness of a local deposit by a number of centuries results in showing Egypt to be much younger than it really is and that (as mentioned by the author under the heading "The River Nile in Egypt") other methods of reckoning, etc., almost invariably indicate much longer periods of geo-morphic development and are more accurate.

Fig. 17 shows the niches in the western wall of the well by which the gauge-reader was guided at the beginning of the rising flood; the "wafa" stage was reached when the water attained the small bottom niche to the right, as the sixteenth cubit coincides with the bottom of the rounded cornice which touches the top of the arch of the main opening. At that stage the column could only indicate the thirteenth or fourteenth cubit due to the misreading of the graduations; that is, considering the highest cubit as the sixteenth or the seventeenth instead of the nineteenth. This led the gauge-reader to add the difference to the "supposed" correct reading of the column, and also to consider the length of the cubits from the sixteenth to the twenty-second as one-half that of the ordinary cubits (this method of reckoning is imaginary, as there are no such graduations as those mentioned in the text following Fig. 1 of the paper; it is amusing to note how a scientific meaning can be attributed to a popular error). The seventeenth cubit coincides with the bottom of the lower Arabic inscription and the highest niche.

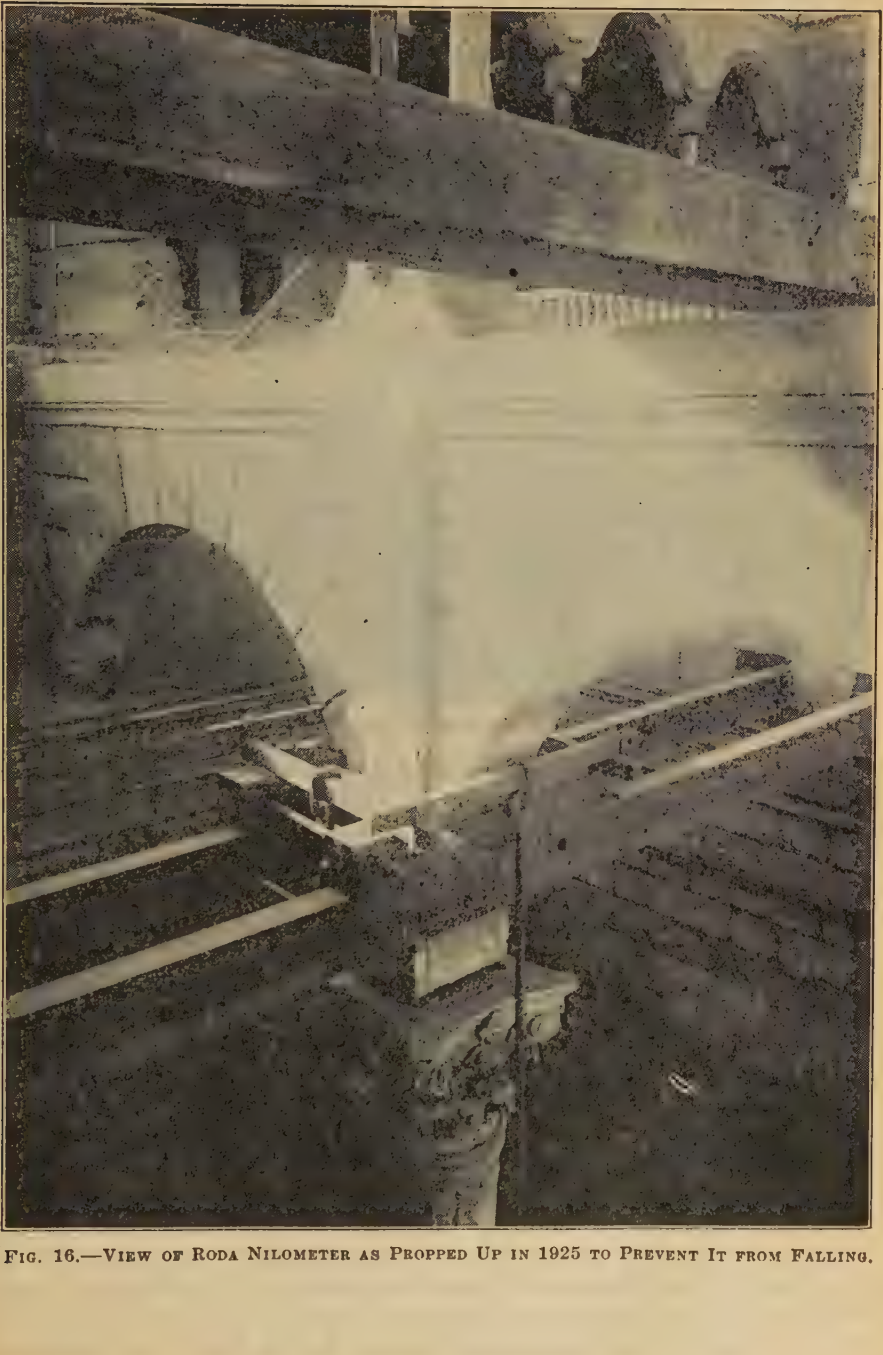

FIG. 16.—VIEW OF RODA NILOMETER AS PROPPED UP IN 1925 TO PREVENT IT FROM FALLING.

The Egyptian peasant may be illiterate, but he has known, from time immemorial, the level at which his land is watered; he will say, for example: "My land is irrigated at the fifteenth cubit" meaning that it will get its water when the Nile rises to that cubit. This is actually the case for the land still remaining under basin irrigation in Upper Egypt. It is the custom to design the level of the floor of a head-sluice of a flood canal to a given cubit, corresponding to the level of the land it has to water. It seems futile, therefore, to resort to the exaggeration of the Nile flood levels and create false records for increasing revenues; a "bad" government can force peasants to pay undue taxes without resorting to such a farce.

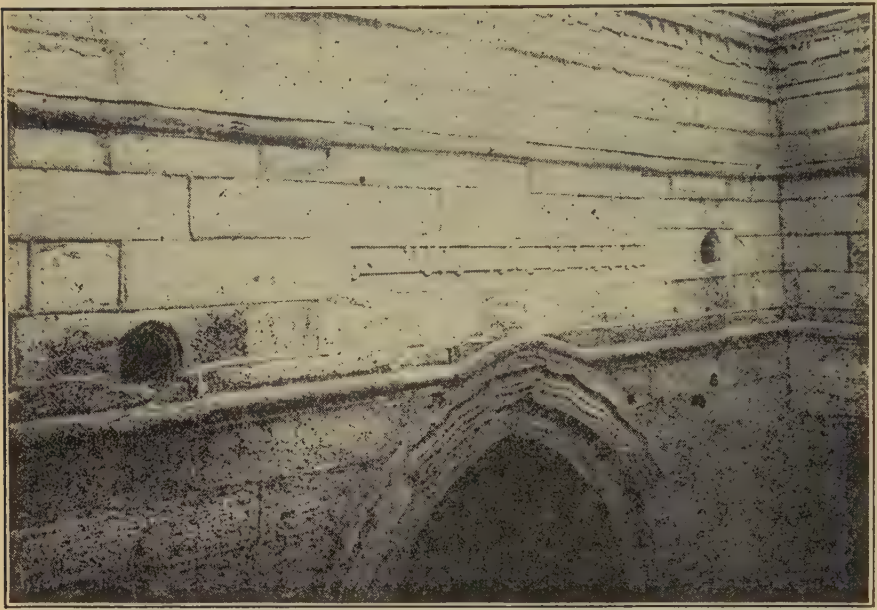

Fig. 17.—Western Wall of Well Showing Method of Graduating the Nile Gauge.

The Roda gauge records under review indicate the yearly highest and lowest readings of the level of the Nile at Cairo. The minimum readings that used to occur in May and June (it has even happened that the Roda branch of the river has been quite dry in certain years) have been affected by the regulation on the Delta Barrage during the latter part of the Nineteenth Century and, from the beginning of the present century, the stored water at Aswan has also had its effect. Consequently, the bottom underground-water passage to the well on the eastern side is now inaccessible and it is probable that the short top tunnel (which is not indicated on the drawing in the atlas prepared by the French Expedition) was added in the middle of the Nineteenth Century to replace it. The maximum levels might have been slightly affected in certain years by the emptying of basins, but the greatest error that must be corrected is that due to the misreading of the graduations of the column; it will be fortunate if it can be proved that the pillar was considered as originally designed (that is, 19 cubits), until the end of the Eighteenth Century.

From the Napoleonic invasion onward, the records are unreliable, and it is necessary to know the period during which the readings were made on a column the top cubit of which was considered to be the sixteenth; then it is necessary to know the date at which it became the seventeenth cubit. When was the column broken anew and one of its cubits considerably shortened? When were the sixteenth to the twenty-second cubits considered to be half cubits? In 1887, a metrical scale was fixed in the Roda branch of the Nile; the readings on that scale are trustworthy, but their conversion to cubits gives false results. The two following examples are to the point.

Example (*1*).—Sir W. Garstin the Under Secretary of State for the Irrigation Department, writes[61] when comparing the flood of 1892 with those of previous years: "The recorded maximum gauges of 1874 and 1878 are appallingly high, but it is more than probable that they are exaggerated, and incorrect." Referring to the year 1887, Sir Garstin continues: "Working on the above lines, it may then be assumed that the maximum height reached at Roda was 25 cubits 15 digits in 1874 (instead of 26 cubits 12 digits as recorded) and 25 cubits 14 digits, in 1878 (instead of 26 cubits 6 digits)."

Example (*2*).—When Sir William Willcocks was editing an Arabic technical review in the early Nineties, one of his party of engineers (engaged on precise leveling in connection with the fixing of the site for the Aswan Dam) contributed an article on land levels in Egypt. Among other items of information he states that, on October 15, 1893, the gauge reading of the Roda nilometer was 20 cubits 7 digits, which corresponds to a level of 18.04 m according to the method of reading indicated by Falaki (the well-known astronomer), whereas the reading on the 1887 metrical scale is 18.50 m. Where is the truth?

C. S. Jarvis,[62] M. Am. Soc. C. E. (by letter).—The generous response to this paper's presentation as indicated by the various discussions, each contributor working painstakingly along a separate thread and apparently disentangling some of the knots, has amply repaid the writer for the effort.

Mr. Gillette's views regarding cycles as affecting rainfall and related phenomena are the product of many years of study in this field, and deserve serious consideration. Mr. Davenport's studies have resulted in definite contributions to the science of hydrology, particularly as applied to the River Nile. His comparsion of the Nile with the Mississippi River and also with the Colorado River enables one to visualize the Nile floods more clearly and to account for their seeming regularity.

Both Mr. Hurst and Mr. Ghaleb are particularly equipped with information gained from actual experience, in contrast with that collected from

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[61] "Note on the High Flood of 1892", p. 9, Egyptian Govt. Press, 1893.

[62] Hydr. Engr., Soil Conservation Service, Washington, D. C.

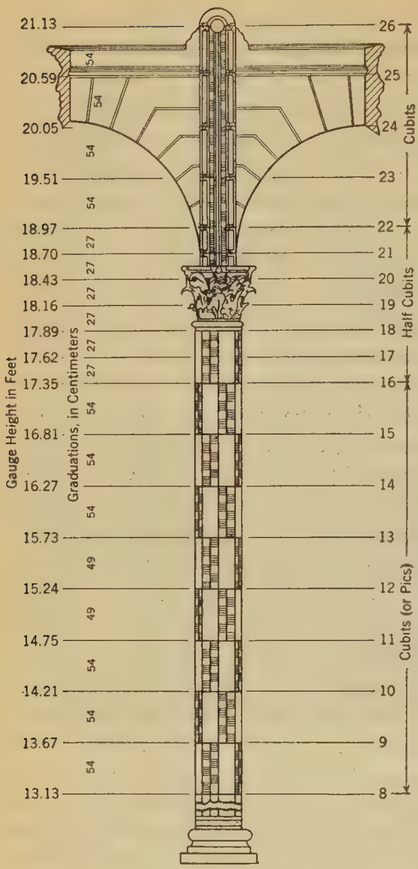
Gauge Height in Feet
Graduations, in Centimeters
Cubits
Half Cubits
Cubits (or Pics)
21.13
20.59
20.05
19.51
18.97
18.70
18.43
18.16
17.89
17.62
17.35
16.81
16.27
15.73
15.24
14.75
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FIG. 18.—THE NILE GAUGE AT RODA.

technical research and forming the basis of this presentation. Even the differences of viewpoint and opinion between two qualified observers, such as these, are interesting and instructive. Perhaps the graduations of the Roda nilometer will be better understood by reference to Fig. 18.[63]

Mr. Means was engaged in a study of the Nile River flood records about the same time that this paper was being prepared. His personal observations in Egypt during a period of duty and travel there some years ago probably fitted him for discussing the broad phases of the Nile River discharge so clearly and authoritatively. Mr. Beardsley's personal observations, including the photograph which he furnished (Fig. 7), showing the Roda nilometer, are valuable contributions; likewise, his drawing constituting Fig. 8.

The significance of Mr. Stevens' work in smoothing out the entire record by means of both 10-yr and 50-yr averages should increase with each reference to it. Mr. Shuman has made a strikingly clear exposition of his theories regarding the applicability of various cycles to the Nile flood records, and has added some significant historical notes.

During the preparation of the paper, the writer found the work of making complete statistical analyses of the records of the various centuries too extensive and time-consuming to complete before publication. He has since computed all the arrays, with plotting positions computed according to the modified California method.[64] It is remarkable how nearly coincident the first seven or eight centuries of record are thereby proved to be, with gradual displacement upward to keep pace approximately with the known sedimentation rate in the Lower Valley. Fig. 19 portrays the flood-frequency trends for both the earliest and the latest century of record, and also for the 1 173 items representing the 1 300-yr period.

By reference to the statistical arrays covering the thirteen centuries of record, it is found that, of the 51 annual flood heights registering 20 m, or more, above mean sea level as observed at the Roda gauge, 30 of these occurred during the Thirteenth (or final) Century, whereas 13 occurred during the Twelfth, 3 during the Eleventh, 4 during the Tenth, and 1 during the Eighth of these consecutive centuries. Of the total number of annual flood heights at or above 18.00 m, the tabulated totals for the respective centuries, listed in chronological order, are 24, 20, 22, 13, 24, 44, 33, 67, 79, 32, 37, 61, and 96. Except for the Tenth and Eleventh Centuries, all are complete, or within one or two items of complete century records. The Tenth and Eleventh Centuries have only 33 and 44 items, respectively, and should be combined for comparison with other century records. The number of annual flood events registering lower than 17.00 m for the consecutive centuries presents another series; thus 11, 20, 11, 13, 2, 4, 2, 0, 1, 1, 2, 0, and 0.

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[63] *Memoirs*, Inst. of Egypt, Vol. 9, 1925, Pl. 21.

[64] *Water Supply Paper 771*, U. S. Geological Survey.

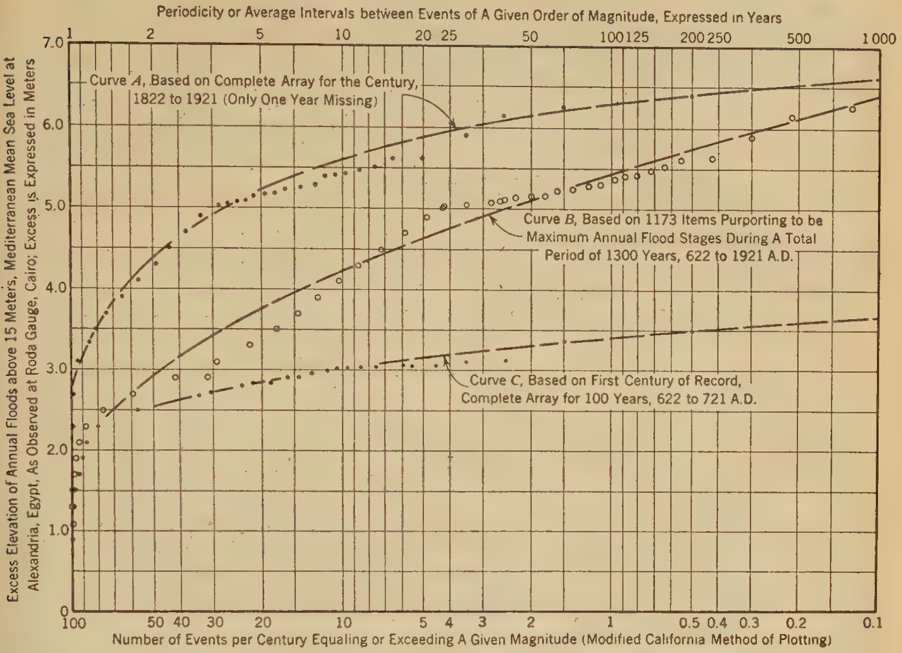

FIG. 19.—COMPARISON OF FLOOD FREQUENCY TRENDS OF THE NILE RIVER AT RODA FOR EARLIEST AND LATEST CENTURIES OF 1300 YEARS.

The foregoing references to the statistical arrays and plottings of the data heretofore presented in this paper and known to be affected by errors and elements opposed to comparability and homegeneity, are made for the purpose of presenting results of tedious calculations so that others may avoid repetition of the process.

Careful comparison of Roda gauge heights attained by annual floods since 1870, with flood volumes, or total annual discharge, as shown in Figs. 1 and 3 of the paper, should convince the most skeptical that Nile River flood peaks are usually fair indications of either the four-month flood discharge, or of the annual volume passing the respective gauging stations (see Figs. 20 and 21).

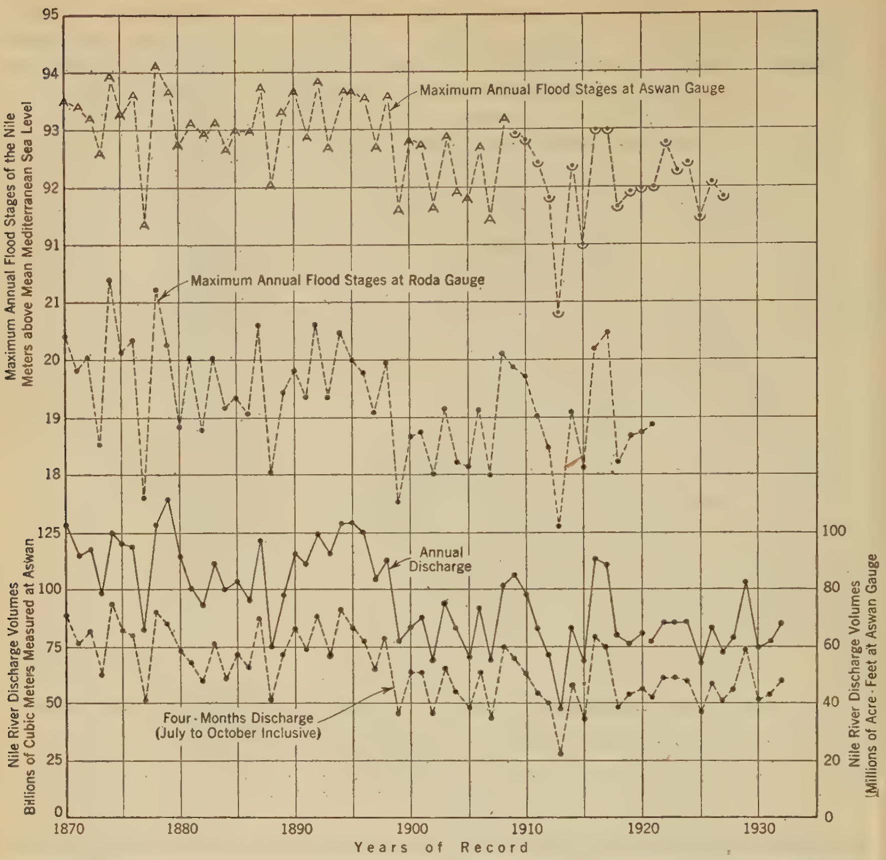

FIG. 20.—TRANSCRIPTS OF DATA FROM FIGS. 1 AND 3 FOR COMPARISON AND CORRELATION.

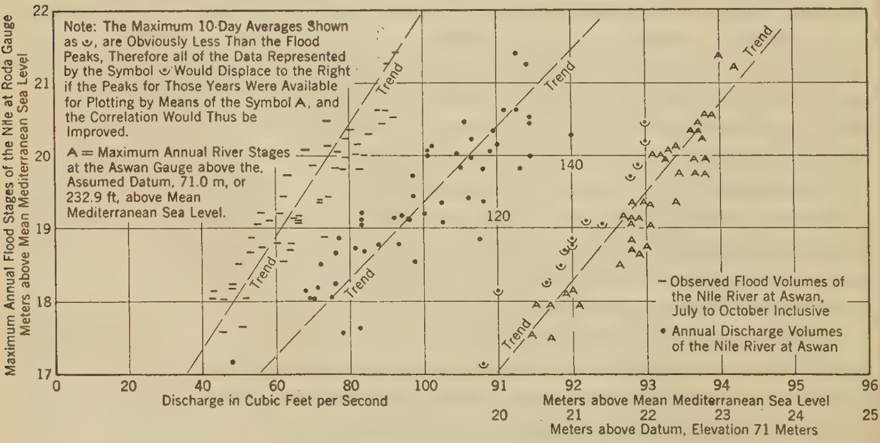

FIG. 21.—CORRELATION OF NILE RIVER DISCHARGES AT ASWAN WITH MAXIMUM FLOOD STAGES AT RODA AND ASWAN.

However, in the textual notes available in French[65], several instances are mentioned in which the peak attained "wafa", but receded so promptly as to cause famine and distress; or, having attained a moderate stage, it continued thus in spite of the use of a portion for basin irrigation, and brought bounteous crops. Generally, however, the high-flood peak meant plentiful water supply and full harvests, unless the stage was too high; and low-flood peaks meant famine and privation. Throughout the ages of history, the gifts of the Nile have been numerous, and reasonably dependable.

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[65] *Memoirs*, Inst. of Egypt, Vols. 4 and 9, pub. about 1925.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 1945

DISTRIBUTION OF STRESSES UNDER A FOUNDATION

By A. E. Cummings,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. Clement C. Williams, D. P. Krynine, L. C. Wilcoxen, Marshall G. Findley, M. A. Biot, Jacob Feld, George Paaswell, G. S. Salter, W. S. Housel, N. M. Newmark, A. A. Eremin, A. Casagrande, and A. E. Cummings.

Synopsis

The question of the distribution of stress in the ground under a foundation has engaged the attention of engineers for many years. The problem has been studied both theoretically and experimentally and it is the purpose of this paper to compare theory and experiment and to indicate several important factors which must be considered when problems of this kind are being analyzed. The symbols in this paper are introduced in the text as they occur and are summarized for reference in the Appendix. An effort has been made to conform essentially with "Symbols for Mechanics, Structural Engineering, and Testing Materials"[2] compiled by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932.

The Theory

The problem of determining the distribution of stresses and strains in a semi-infinite, elastic, isotropic solid, bounded by a plane and loaded at a point on that plane by a single concentrated force, was first solved by Lamé and Clapeyron[3] not long after the discovery by Navier of the differential equations of elastic equilibrium. Their solutions in the form of quadruple integrals are of little use for practical calculations. More recently the problem was

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Note.—Published in August, 1935, *Proceedings*.

[1] Dist. Mgr., Raymond Concrete Pile Co., Chicago, Ill.

[2] A. S. A.—*Z* 10 *a*—1932.

[3] "Sur l'équilibre intérieur des solides homogènes," Savants étrangers de l'Académie des Sciences de Paris, 1833, Tome IV, p. 541.

developed in considerable detail by Boussinesq.[4] His solutions are much more usable than those of the older elasticians and have become so well known that the entire problem is often referred to as the "problem of Boussinesq."

Fig. 1 shows the distribution of stresses across planes parallel to the surface as given by Boussinesq. The external force, P, applied perpendicularly to the plane boundary of the semi-infinite solid produces a stress in the direction of the radius vector, R, the magnitude of which per unit of area on any plane parallel to the surface is given by the expression,

$$p = \frac{3\,P\,z^2}{2\,\pi\,R^4} \quad \text{..........................} (1)$$

in which z is the depth from the surface to the plane for which the calculation is being made, and $R = \sqrt{x^2 + y^2 + z^2}$. At the point of intersection

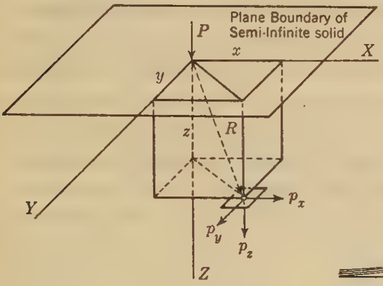

FIG. 1.

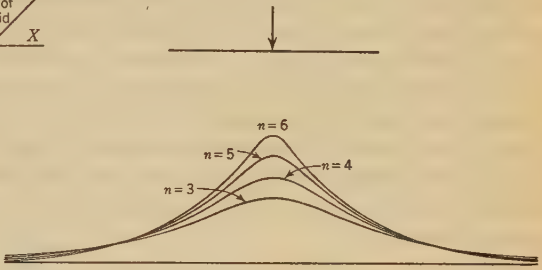

FIG. 2.—RELATIVE FORMS OF STRESS SURFACE FOR DIFFERENT VALUES OF n.

of R with any plane parallel to the surface, this stress may be resolved into three components as shown. The equations for these components as given by Boussinesq[5] are, as follows:

$$p_x = \frac{3\,P}{2\,\pi} \times \frac{z^2}{R^4} \times \frac{x}{R} \quad \text{......................} (2a)$$

$$p_y = \frac{3\,P}{2\,\pi} \times \frac{z^2}{R^4} \times \frac{y}{R} \quad \text{......................} (2b)$$

and,

$$p_z = \frac{3\,P}{2\,\pi} \times \frac{z^2}{R^4} \times \frac{z}{R} \quad \text{......................} (2c)$$

It is evident from Fig. 1, that the two horizontal components, p_x and p_y, are shears and that the vertical component, p_z, is a normal stress. It is also to be noted from Equations (2) that no elastic constants are involved. In connection with these two facts a little further explanation seems desirable.

The three stress components given by Equations (2) apply only to horizontal planes, that is, planes parallel to the boundary plane of the semi-infinite solid. For planes having other directions the stress distribution depends on the elastic properties of the material. In the general case the complete specification of stress at a point within the solid would require six components of stress, three being normal stresses and three being shears, and the elastic constants of the material appear in some of these equations. Because of the fact that Equations (2) do not include any elastic constants, the statement is sometimes made[6] that the distribution of stress is independent of the type of material. This is not exactly true and Boussinesq himself, after calling attention to the non-appearance of elastic constants in the equations, stated[7] that the distribution of stress across planes parallel to the surface is the same in all isotropic solids.

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[4] "Application des Potentiels a l'Étude de l'Équilibre et du Mouvement des Solides Élastiques," Paris, 1885.

[5] *Loc. cit.*, p. 104.

As long ago as 1897, August Föppl[8] arrived at the conclusion that the behavior of earths under load was not exactly in accordance with the theory of elasticity. He attributed this to the fact that the stresses and strains in earths are not proportional in the manner required by the assumptions on which that theory is based. In 1929, John H. Griffith,[9] M. Am. Soc. C. E., and, in 1932, Dr. Ing. O. K. Froehlich,[10] published stress equations involving a parameter which may be adjusted to suit materials other than the elastic isotropic solids. The two derivations differ in several important respects, but each makes use of the necessary condition of equilibrium between the summation of the vertical forces within the solid and the load applied at the surface. In the notation of Fig. 1, the equation given by these writers for the vertical normal stress is,

$$p_z = \frac{n\,P}{2\,\pi\,R^2} \times \frac{z^n}{R^n} \quad \text{........................} (3)$$

in which n is a parameter that may have different values for different soil structures and loading conditions. When $n = 3$ Equation (3) is exactly the same as Equation (2c). In other words, for elastic isotropic solids stressed within their elastic limits, the value of the parameter, n, is 3. It is also obvious from Equation (3) that the stress on the vertical center line of the load is directly proportional to n since the ratio, $\frac{z}{R}$, is 1 on this line. The general form of the bell-shaped stress surfaces computed from Equation (2c) is well known, and Fig. 2 shows the effect produced on these surfaces by varying the value of n. Because of this effect, n is sometimes called a concentration factor. It is of considerable importance in problems relating to earths and deserves much more attention than it has received.

It is generally conceded that the vertical normal stress, p_z, is the only one that needs to be considered as far as practical problems relating to the settlements of foundations are concerned. Its maximum values occur at points on the vertical center line of the loaded area ($x = 0$ and $y = 0$), where the shears vanish entirely. In the analysis which follows only the vertical normal stress on the vertical center line of the load will be considered.

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[6] Progress Report of the Special Committee on Earths and Foundations, *Proceedings*, Am. Soc. C. E., May, 1933, p. 780.

[7] "Application des Potentiels," p. 106.

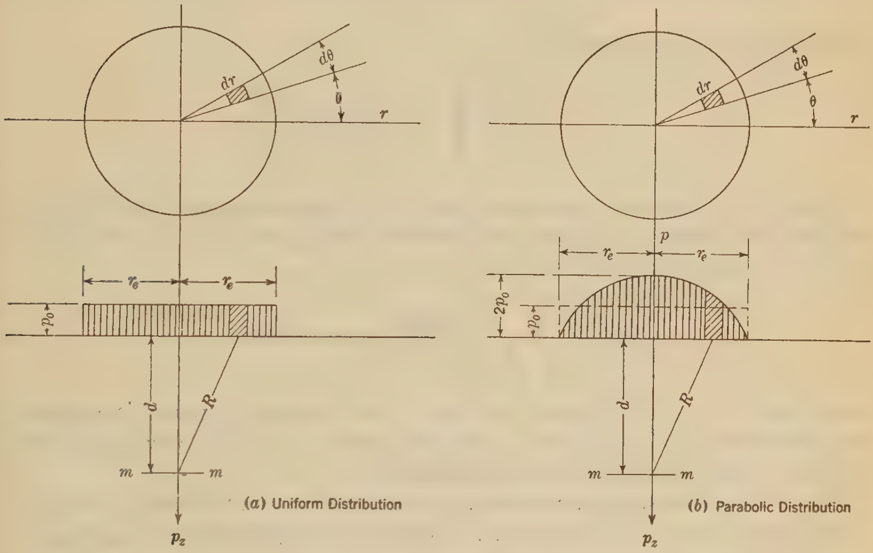
[8] "Versuche über die Elastizität des Erdbodens," *Zentralblatt der Bauverwaltung*, 1897.

[9] "Pressures under Substructures," *Engineering and Contracting*, March, 1929, pp. 113–119.

[10] "Drukverdeeling in Bouwgrond," *De Ingenieur*, April 15, 1932, p. B-52.

The factor, P, in Equations (1), (2), and (3), is a concentrated load applied at a single point on the plane boundary. In practical problems relating to foundations it is necessary to calculate with loads distributed over a certain part of the plane boundary. Boussinesq[11] discussed a number of cases of distributed loads and there have been additional contributions to the subject by later writers. The simplest case is that of a load uniformly distributed over a circular area. For bearing areas other than circles and for loads non-uniformly distributed the integrations become increasingly difficult. Inasmuch as the experiments discussed subsequently herein were made with circular plates, the theoretical analysis will be made for the circular bearing area. Two types of loading—uniform and parabolic— will be considered.

Fig. 3(a) is a representation in plan and section of a load uniformly distributed over a circular bearing area; and Fig. 3(b) is a similar representation of a parabolic distribution that is maximum at the center and zero at the edges. Four different cases will be analyzed and in each case a formula will be derived for the vertical normal stress on the vertical center line of the load on a small element of area (Plane m-m, Fig. 3), at a depth, d, below the plane of the load.



(a) Uniform Distribution

(b) Parabolic Distribution

FIG. 3.—LOAD DISTRIBUTION ON A CIRCULAR BEARING AREA.

Case I.—Uniform Distribution ($n = 3$).—This case has been developed by Föppl.[12] The formula for the vertical normal stress produced by a concentrated load is expressed by Equation (3) when $n = 3$ and $z = d$.

---

[11] "Application des Potentiels," pp. 139–179.

[12] "Drang und Zwang," Vol. 2, p. 205.

Using the polar co-ordinates, r and θ, as shown in Fig. 3, the total load distributed over the circular area will be given by,

$$P = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} p_o \, r \, dr \, d\theta \quad \ldots\ldots\ldots\ldots (4)$$

in which p_o is the density of the uniformly distributed load. It should be noted that Equation (4) gives the volume of the circular disk of height, p_o. From Fig. 3 there is also the obvious relation,

$$R^2 = d^2 + r^2 \quad \ldots\ldots\ldots\ldots (5)$$

Substituting Equations (4) and (5) in Equation (3), with $n = 3$ and $z = d$;

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{3\,p_o}{2\,\pi} \, \frac{d^3}{(d^2 + r^2)^{\frac{5}{2}}} \, r \, dr \, d\theta \quad \ldots\ldots\ldots\ldots (6)$$

Integrating and simplifying:

$$p_z = p_o \left[1 - \left(\frac{1}{1 + \left(\frac{r_e}{d}\right)^2} \right)^{\frac{3}{2}} \right] \quad \ldots\ldots\ldots\ldots (7)$$

Case II.—Uniform Distribution (n = 6).—Equations (4) and (5) also apply in this case and they may be substituted in Equation (3) when $n = 6$ and $z = d$; thus:

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{3\,p_o\,d^6}{\pi\,(d^2 + r^2)^4} \, r \, dr \, d\theta \quad \ldots\ldots\ldots\ldots (8)$$

Integrating and simplifying as in Case I, the resulting formula is identical with Equation (7), except that the three-halves power becomes the third power; that is, the quantity is simply cubed.

Case III.—Parabolic Distribution (n = 3).—The equation for the vertical normal stress due to a concentrated load is again expressed by Equation (3) when $n = 3$ and $z = d$. In order that Case III may be comparable to Case I, it is necessary to find a paraboloid of revolution with a volume equal to that of the circular disk of Case I. These volumes represent the total load on the bearing area. The equation of the parabola (Fig. 3(*b*)) will be taken as:

$$p = 2\,p_o \left(1 - \frac{r^2}{r_e^2} \right) \quad \ldots\ldots\ldots\ldots (9)$$

The volume of the paraboloid is then:

$$V = \pi \int_{p=0}^{p=2p_o} \left(r_e^2 - \frac{r_e^2\,p}{2\,p_o} \right) dp = p_o\,\pi\,r_e^2 \quad \ldots\ldots\ldots\ldots (10)$$

which is also the volume of the disk. Using the polar co-ordinates, r and θ, the total load is now given by,

$$P = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} 2\, p_o \left(1 - \frac{r^2}{r_e^2}\right) r\, dr\, d\theta \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Equation (5) applies also to this case and, when Equations (5) and (11) are substituted in Equation (3) (with $n = 3$ and $z = d$), the formula for the vertical normal stress becomes,

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{3}{2\,\pi}\, \frac{d^3}{[d^2 + r^2]^{\frac{5}{2}}}\, 2\, p_o \left(1 - \frac{r^2}{r_e^2}\right) r\, dr\, d\theta \quad \ldots\ldots\ldots (12)$$

Equation (12) may be integrated by expanding the quantity in the square brackets as a binomial series, then multiplying, and, finally, integrating term by term. This process leads to the equation,

$$p_z = p_o \left[\frac{3}{2} \left(\frac{r_e}{d}\right)^2 - \frac{5}{4} \left(\frac{r_e}{d}\right)^4 + \frac{5 \times 7}{4 \times 8} \left(\frac{r_e}{d}\right)^6 - \frac{5 \times 7 \times 9}{4 \times 8 \times 10} \left(\frac{r_e}{d}\right)^8 + \frac{5 \times 7 \times 9 \times 11}{4 \times 8 \times 10 \times 12} \left(\frac{r_e}{d}\right)^{10} - \ldots + \right] \quad \ldots\ldots\ldots\ldots (13)$$

which infinite series is convergent only for $d > r_e$ and, therefore, can be used only at depths greater than the radius of the loaded area.

Case IV.—Parabolic Distribution ($n = 6$).—Equations (5) and (11) apply to this case, and they may be substituted in Equation (3) (with $n = 6$ and $z = d$), giving,

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{3}{\pi}\, \frac{d^6}{[d^2 + r^2]^4}\, 2 p_o \left(1 - \frac{r^2}{r_e^2}\right) r\, dr\, d\theta \quad \ldots\ldots\ldots (14)$$

and simplifying as in Case III, the resulting formula is identical with Equation (13), except that the numerical coefficients of corresponding quantities become 3, 4, 5, 6, 7, etc. This series is also convergent only for $d > r_e$. By a transformation to trigonometric functions, Froehlich[13] has derived expressions similar to Equations (12) and (14) in a closed form instead of the infinite series of Equation (13) and its parallel in Case IV.

Equations (7) and (13) and their respective parallels in Cases II and IV may be used to make theoretical calculations for comparison with the stresses measured in the experiments.

THE EXPERIMENTS

Among the earliest experiments made to determine the distribution of vertical normal stresses on horizontal planes at various depths beneath a loaded disk, were those of Steiner-Kick[14] at Prague, in 1879 (see Table 1(*a*)).

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[13] "Druckverteilung im Baugrunde," Julius Springer, Vienna, 1934.

[14] Handbuch der Ingenieurwissenschaften, 1 Aufl., II. Bd.; *Der Brückenbau*, 2 Abt., S. 195, Leipzig, 1882.

These experiments were made with sand and the bearing area was a circular disk 10 cm in diameter. In 1909, at Graz, Austria, Strohschneider[15] experimented with small circular disks 1.5 cm in diameter, subjecting them to loads of a few kilograms (Table 1(*b*)). He made his experiments with sand and measured the vertical normal stress on horizontal planes at depths to about 4 cm below the plane of the disk. A. T. Goldbeck, M. Am. Soc. C. E., made similar experiments,[16] in 1917, at the United States Bureau of Public Roads (Tables 1(*f*) and 1(*g*)). For bearing areas Goldbeck used circular disks of different diameters (8 in. and 13.5 in.) and measured the stresses transmitted through sand at depths to 60 in. below the plane of the load. At the University of Illinois, M. L. Enger, M. Am. Soc. C. E., conducted further experiments[17]

TABLE 1.—COMPARISON OF RESULTS; DISTRIBUTION OF VERTICAL NORMAL STRESSES BENEATH FOUNDATIONS

DEPTH, d, MEASURED IN:		DIAMETER, $D = 2\,r_e$, OF THE LOADED PLATE, MEASURED IN:		Ratio, $\frac{d}{r_e}$	Values of p_z expressed as percentages of p_o
Centimeters (1 (*a*))	Inches (1(*b*))	Centimeters (2 (*a*))	Inches (2(*b*))	(3)	(4)
(*a*) STEINER–KICK					
8.0	3.1	10.0	3.9	1.60	107.2
13.2	5.2	10.0	3.9	2.64	43.0
(*b*) STROHSCHNEIDER					
2.0	0.8	1.5	0.6	2.67	38.3
3.0	1.2	1.5	0.6	4.00	17.8
4.0	1.6	1.5	0.6	5.33	9.9
(*c*) KÖGLER AND SCHEIDIG; SERIES I					
20.0	7.9	33.9	13.3	1.18	128
30.0	11.8	33.9	13.3	1.77	76
40.0	15.7	33.9	13.3	2.36	41
50.0	19.7	33.9	13.3	2.95	30
60.0	23.6	33.9	13.3	3.54	21
(*d*) KÖGLER AND SCHEIDIG; SERIES II					
10.0	3.9	45.0	17.7	0.45	220
20.0	7.9	45.0	17.7	0.89	100
30.0	11.8	45.0	17.7	1.33	80
40.0	15.7	45.0	17.7	1.78	65
50.0	19.7	45.0	17.7	2.22	48
60.0	23.6	45.0	17.7	2.67	31

Depth, d, measured in inches (1)	Diameter, $D = 2\,r_e$, of the loaded plate (2)	Ratio, $\frac{d}{r_e}$ (3)	Values of p_z expressed as percentages of p_o (4)
(*e*) M. L. ENGER			
38	36	2.11	50
30	36	1.67	74
24	36	1.34	111
18	36	1.00	160
12	36	0.67	184
30	30	2.00	64
24	30	1.60	101
18	30	1.20	136
12	30	0.80	199
38	21	3.62	18
30	21	2.86	38
24	21	2.28	72
18	21	1.72	108
12	21	1.14	139
(*f*) A. T. GOLDBECK; SERIES I			
12	13.5	1.78	79.5
12	13.5	1.78	75.7
12	13.5	1.78	81.5
12	13.5	1.78	89.1
Average	....	....	81.5
24	13.5	3.56	19.8
24	13.5	3.56	26.7
24	13.5	3.56	25.7
Average	....	....	24.1
36	13.5	5.33	8.7
36	13.5	5.33	9.1
36	13.5	5.33	10.7
Average	....	....	9.5

Depth, d, measured in inches (1)	Ratio, $\frac{d}{r_e}$ (3)	Values of p_z expressed as percentages of p_o (4)
(*f*) A. T. GOLDBECK; SERIES I (*Continued*) (D = 13.5 INCHES)		
48	7.11	3.96
48	7.11	4.28
48	7.11	4.28
Average	....	4.17
60	8.89	2.38
60	8.89	2.86
60	8.89	2.29
Average	....	2.51
(*g*) A. T. GOLDBECK; SERIES II (D = 8 .0 INCHES)		
6	1.5	116.5
6	1.5	98.3
6	1.5	93.9
6	1.5	94.5
6	1.5	94.3
Average	....	99.5
12	3.0	36.7
12	3.0	49.1
12	3.0	45.5
12	3.0	45.0
Average	....	44.1
24	6.0	8.33
24	6.0	8.33
24	6.0	7.22
24	6.0	8.75
24	6.0	8.56
Average	....	8.24
36	9.0	3.33
36	9.0	3.33
36	9.0	3.93
Average	....	3.53

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[15] "Elastische Druckverteilung," *Sitzungsberichte* d. Kais. Akad. d. Wissenschaften in Wien, Vol. 71, Abt. II*a*, February, 1912, p. 299.

[16] "Distribution of Pressures through Earth Fills," *Proceedings*, Am. Soc. for Testing Materials, Vol. 17, 1917.

[17] Second Progress Report of the Special Committee on Stresses in Railroad Track, *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–1920), p. 1409, and Vol. 93 (1929), p. 372.

of the same kind, using circular plates of various sizes to 36 in. in diameter (Table 1(*e*)). At Freiburg, Germany, Kögler and Scheidig have concluded an extensive experimental program[18] which included the measurement of stresses produced in sand fills by loaded plates on top of the sand (Tables 1(*c*) and 1(*d*)).

Values in Column (4), Table 1, are the measured vertical normal stresses on the vertical axis of the loaded area, expressed as a percentage of the average load on the bearing area. The average load is simply the total load applied to the plate divided by the area of the plate.

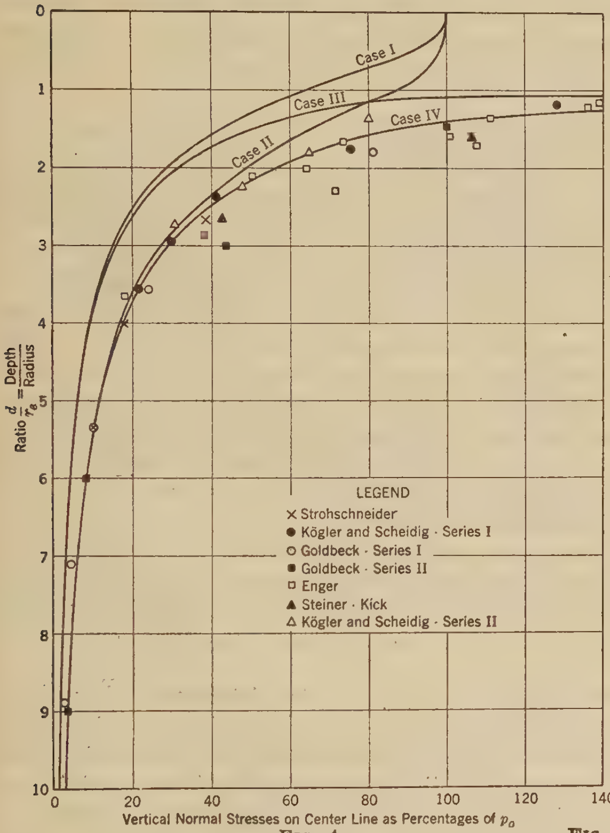
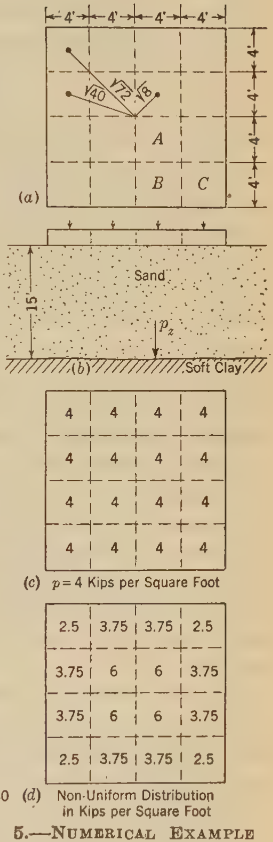


FIG. 4.



4	4	4	4
4	4	4	4
4	4	4	4
4	4	4	4

(*c*) $p = 4$ Kips per Square Foot

2.5	3.75	3.75	2.5
3.75	6	6	3.75
3.75	6	6	3.75
2.5	3.75	3.75	2.5

(*d*) Non-Uniform Distribution in Kips per Square Foot

FIG. 5.—NUMERICAL EXAMPLE (1 KIP = 1 000 POUNDS)

Fig. 4 is a graphical representation of these experimental results, together with the four theoretical curves represented by Equations (7), and (13) for Cases I and III, and their parallels for Cases II and IV. Before considering the relationship between the curves and the experimental points, it is desirable to notice several remarkable facts about the experiments themselves. In time, they cover a period of about fifty years. Geographically, they represent several places on two continents. The sands that were used came from various sources and the stress-measuring devices were of different kinds. The apparatus ranged from the miniatures used by Strohschneider to the large plates used by Enger. Notwithstanding these many possible variables, there is a most noticeable uniformity in the results of the experiments.

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[18] "Druckverteilung im Baugrund," *Die Bautechnik*, 1927, Heft 29 und 31; 1928 Heft 15 und 17; 1929, Heft 18 und 22.

The theoretical curves show several interesting facts. For a uniform distribution of surface pressure (Cases I and II) the curves show a stress under the center of the plate at the surface equal to 100% of this pressure. This is in accordance with the assumptions made in the analysis. For a parabolic distribution (Cases III and IV) the curves continue out toward the right and these particular curves plotted from the infinite series of Equation (13) and its adaptation to Case IV become asymptotic to the line, $\frac{d}{r_e} = 1$. With the finite integrals obtained by Froehlich[19] for the parabolic distribution, the curves turn upward and give a stress under the center of the plate at the surface which is 200% of the average surface pressure. In the region represented by Fig. 4, Froehlich's integrals and the infinite series give the same results. The importance of the parameter, n, of Equation (3) is clearly indicated by the curves. Stress calculations based on the theory of elasticity in which n is 3 (Cases I and III), give results at any depth on the vertical center line which are much smaller than those calculated with $n = 6$ (Cases II and IV). The character of the pressure distribution at the surface (uniform or parabolic) is especially important near the surface but loses its importance as the depth increases. This is evident from the fact that the curve of Case I (uniform) is the same as the curve of Case III (parabolic) at depths greater than about twice the diameter of the loaded area. The same phenomenon applies to the other pair of curves.

It is easy to note which of the four theoretical curves gives the best interpretation of the experiments. The distribution of pressure at the under side of these test plates was approximately parabolic and the parameter, n, was approximately 6. Both these factors are required to account for the measured stresses.

It seems desirable to point out the fact that the distribution of pressure on the surface is not necessarily parabolic in the manner shown in Fig. 3(*b*). The surface pressure might be a maximum at the center and zero at the edge, but it can vary from this range to a distribution that would be a minimum at the center and infinite at the edge. Boussinesq[20] has discussed several of these possible surface distributions and, in general, they depend on the relative elastic properties of the two bodies in contact. For a square or a rectangular area the distribution of pressure at the ground surface can be fairly complicated as has been shown experimentally by Frederick J. Converse,[21] Assoc. M. Am. Soc. C. E. A. E. H. Love[22] has given a theoretical analysis of the problem for both uniform and variable pressures over rectangular bearing areas. In most practical problems involving large bearing areas, it is customary to divide the large area into a number of small areas and then to make the stress calculation for each of the small areas and add the results. A non-uniform distribution can be treated in this manner by varying the load on the several small areas in accordance with the non-uniformity of the surface distribution.

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[19] "Druckverteilung im Baugrunde," p. 52.

[20] "Application des Potentiels," pp. 149–166.

[21] "Distribution of Pressure Under a Footing," *Civil Engineering*, April, 1933, Vol. 3, No. 4, p. 207.

[22] *Philosophical Transactions*, Royal Soc. of London, Vol. 228, Series A, 1929, p. 377.

Very little experimental work has been done for the purpose of determining the distribution of pressure in the contact area between the plate and the sand-fill. However, it is very probable that this pressure distribution varies to some extent with the load itself. For very light loads or for very large bearing areas the pressure is approximately uniform and the parameter, n, is approximately 3. As the load is increased or as the size of the bearing area is decreased the pressure at the contact plane becomes non-uniform and, at the same time, the value of n increases to 6 if not higher. The value of n also depends to some extent on the elastic properties of the soil. A value of 3 can be used only for elastic isotropic solids stressed within their elastic limits. A very hard dry clay, not too heavily loaded, would behave approximately as an elastic solid, and $n = 3$ would be nearly correct for this kind of material. Granular soils offering almost no resistance to tension cannot act as elastic solids and, in such materials, n has values which may be as great as 6.

NUMERICAL EXAMPLE

Figs. 5(*a*) and 5(*b*) show a footing 16 ft square which is to be built on a 15-ft stratum of sand, overlying soft clay. It is desired to know the vertical normal stress that will be exerted on the top surface of the clay. The footing is subdivided into small squares each 4 ft on a side and because of symmetry there will be only three types of squares, designated *A*, *B*, and *C*. The load will be assumed as concentrated at the center of gravity of each square, and the horizontal distances from the center of the footing to these centers of gravity are shown in Fig. 5(*a*). Fig. 5(*c*) shows this footing with a uniform distribution of surface pressure of 4 000 lb per sq ft. The total load is 1 024 000 lb. Fig. 5(*d*) shows the footing with a non-uniform distribution of

TABLE 2.—STRESS COMPUTATION; NUMERICAL EXAMPLE
($z = d = 15.0$ feet)

Type (see Fig. 5 (*a*)) (1)	Number of areas (2)	Radius vector, *R*, in feet (3)	(*a*) UNIFORM LOAD			(*b*) NON-UNIFORM LOAD		
			Concentrated load, *P*, in pounds (4)	Vertical Normal Stress, in Pounds per Square Foot		Concentrated load, *P*, in pounds (7)	Vertical Normal Stress, in Pounds per Square Foot	
				By Equation (3)* (5)	Total (Column (2)) × (Column (5)) (6)		By Equation (3)† (8)	Total (Column (2)) × (Column (8)) (9)
A......	4	15.3	64 000	123	492	96 000	349	1 396
B......	8	16.3	64 000	90	720	60 000	131	1 048
C......	4	17.2	64 000	68	272	40 000	57	228
Total...	....	......	....	....	1 484	....	....	2 672

* $n = 3$ † $n = 6$

pressure similar to the experimental determination of Converse.[21] The pressure is more than the average in the center squares and less than the average in the middle of the sides and in the corners. The total load for Fig. 5(*d*) is also 1 024 000 lb.

Table 2 gives the stress calculations in condensed form. Table 2(*a*) refers to Fig. 5(*c*); and, in addition to the uniform distribution of pressure at the surface, it is assumed that the distribution of stress through the sand is in accordance with the theory of elasticity; that is, $n = 3$. The total stress exerted on the top surface of the clay on the vertical center line of the footing is 1 484 lb per sq ft. Table 2(*b*) refers to Fig. 5(*d*); and, in addition to the non-uniform pressure distribution, it is assumed that $n = 6$. The calculated stress is 2 672 lb per sq ft, which is almost twice that of the other assumption.

Summary and Conclusions

In most problems of soil mechanics dealing with the probable behavior of foundations, the first requisite is a calculation of the stresses produced in the underlying strata by the foundation loads. The accuracy of the subsequent calculation of the effects produced by these stresses will depend on the degree of approximation obtained in the calculation of the stresses themselves. For the problem of the calculation of the stresses, it is believed that the foregoing analysis leads to two important conclusions: (1) The manner in which the pressure is distributed over the contact surface must be considered; and (2) the equations of the theory of elasticity must be modified before they can be applied to soils.

Conclusion (1) is particularly true when the depth for which the computation is being made is equal to, or less than, the diameter of the loaded area. It is to be understood that Equations (7) and (13) for Cases I and III, and their adaptation to Cases II and IV, are not offered for immediate use in the solution of practical problems. These formulas can be used only when the practical problem fulfills all the conditions under which they were derived. Most practical problems involve bearing areas other than circles and the distribution of pressure under the foundation is neither perfectly uniform nor parabolic. However, a reasonably accurate determination of the stresses can be expected only after a careful consideration of the probable pressure distribution at the contact surface and a proper choice of the value of the parameter, n. As a general rule, it is not satisfactory to assume that the pressure at the surface is uniformly distributed and that the equations of the theory of elasticity can be used without modification.

APPENDIX

Notation

d = depth from ground surface to a given plane.
e = a subscript denoting "edge."
n = a parameter with values that vary with different conditions; a concentration factor.

o = a subscript denoting uniform distribution.
p = pressure per unit area; p_o = intensity of pressure under a uniformly distributed load; p_x, p_y, and p_z = components of unit stress at a given horizontal plane, distant $z = d$ below the ground surface.
r = variable radius; r_e = radius to the edge of a circular foundation.
z = a variable distance measured parallel to the Z-axis; the depth, d, from the ground surface to the plane for which a given computation is made.
D = diameter of circular foundation $= 2r_e$.
P = externally applied, concentrated load.
R = a radius vector.
V = volume.
θ = angular distance.

DISCUSSION

Clement C. Williams,[23] M. Am. Soc. C. E. (by letter).—In his excellent review of theory and experiments relative to pressures under a foundation, the author has performed a helpful service in that his presentation is pointed toward a practical evaluation of their significance. The rather amorphous state of foundation literature represents a stage preliminary to its crystallization into usable principles involving known soil factors. The paper seeks to give point to the theories and experiments that cluster about the geometry of stress distribution in an isotropic elastic solid.

In most experiments relating to pressure distribution under a foundation slab sand has been utilized as the soil. Rankine's theories contemplate a material like dry sand. Within the shear limits of internal friction, the behavior of dry sand in transmitting pressure is similar to that of a granular sandstone, since the cementitious binder of the latter performs no essential function until friction is overcome, except in so far as the solid stone has a weak tensile strength. That there is a close correlation between the theories that are based on either an hypothetical elastic solid, or on a sand bed (which, within limits, behaves similarly to an elastic solid), should not occasion surprise; nor, on the other hand, should it lead to a generalized conclusion applicable to all foundation soils.

Not enough experimental observations have been made with loads resting on elastic soils. The limited observations made by the writer, by William S. Housel, Assoc. M. Am. Soc. C. E., and by others, indicate a behavior different in essential respects to that of granular material under a bearing plate.

As an extreme departure from a load resting on an elastic solid, the case of a box containing a load floated in a lake of water may be cited. The box is supported at a certain flotation depth, but the pressure on the bottom of the lake is not increased (except in so far as the water surface may be raised slightly) and the pressure is evenly distributed over the bottom of the box. In a fluid, the pressures so increase with the depth that the upward reactions on the bottom supports the load.

In a viscous material, such as heavy oil or asphalt, a similar phenomenon occurs, except that flow occurs slowly and only after a long period of time is the theoretical flotation sinking attained. In heavy asphalt, the sinking stops short of the fluid flotation depth, showing that the shear stresses developed have a certain permanent sustaining capacity for holding the load in equilibrium.

The behavior of plastic earths under load has certain points of similarity to viscous fluids. Whereas the pressures assume the typical bell-shaped distribution pattern in a sand bed under load, by virtue of the internal friction, in a plastic soil that is essentially devoid of internal friction, this pressure pattern is fundamentally modified. The spread of the load is probably much wider by virtue of shear and tension than observations seem to indicate, because of the experimental difficulty of measuring small increments of pressure. It is probable also that the depth to which the pressure is perceptibly transmitted in plastic soils is much less than is commonly assumed, owing to the wider spread of the load and the accompanying effect of immeasurably small pressures distributed over a large area, which is much greater than has been generally recognized. An artificial foundation bed for experimentation that would correspond more closely than sand to the prototype would be a thick layer composed of small rubber bands. Instead of using such an artificial bed for laboratory investigations, however, typical earths themselves should be used, because the present plethora of discordant foundation discussion has resulted largely from not basing theories and experiments on soils as they actually occur in Nature.

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[23] Pres., Lehigh Univ., Bethlehem, Pa.

A relatively small clay content will convert sand to a plastic material in so far as friction and cohesion are concerned. Limited observations made by the writer indicate that, with fine sand of fairly uniform grain, 20% clay will make the mass essentially plastic. Probably with graded sand having a lower percentage of void space, even a smaller proportion of clay would be required. Practical foundations should be considered as saturated, for such will almost certainly be their condition at times, and experiments on dry soils are likely to be misleading.

The bell-shaped pattern of pressure distribution has relatively little significance, therefore, in the practical design of a particular footing, partly because only in clean sand or gravel, or on solid rock, is such a distribution likely to occur. In the second place, even if it should occur when the load is first applied, adjustment in time would tend to make the pressures fairly uniform. Moreover, shifting eccentricities of practical loadings tend to render inapplicable conclusions that may be predicated on an assumption of a fixed point of application of a load. Since the higher pressures at the center of a footing result only from greater rigidity of the soil in that region, the unequal distribution of the pressures will not result in settlement of a footing that is proportioned for a uniform distribution of soil pressure. Any giving way at the center would immediately distribute pressures toward the periphery. Therefore, even if any flexure of the footing slabs that may occur would tend to accentuate the non-uniformity of soil-pressure distribution, a footing designed for uniform soil pressure would generally not result in failure.

In expressing appreciation of Mr. Cumming's excellent paper, which so succinctly sets forth the conception of a foundation as an hypothetical elastic structure, the writer wishes to venture the hope that more experimental and theoretical attention may be devoted to a consideration of actual soils with a view to determining their structural behavior under load. At any rate, more experimentation on plastic soils as a typical foundation material is needed, in order to establish values at the other extreme of soil types.

D. P. KRYNINE,[24] M. AM. SOC. C. E. (by letter).—The most comprehensive discussion of the Boussinesq formula and related problems that has ever appeared in the English language is contained in this paper. For advanced students of soil mechanics the completeness of references and the interesting historical data may be of great value; and readers less advanced are given a concise and clear presentation of the material which furnishes an excellent view of the subject as a whole. In reading this paper, however, the writer faced some difficulties which he wishes to clarify.

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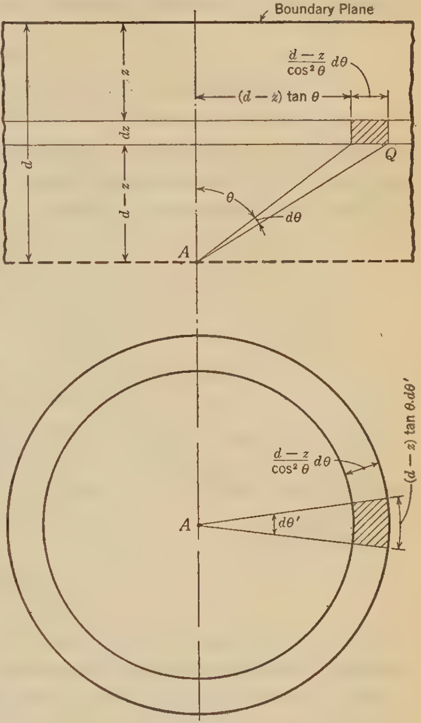
[24] Research Associate in Soil Mechanics, Dept. of Civ. Eng., Yale Univ., New Haven, Conn.

Conception of Isotropy.—Bodies termed "isotropic" in reality may possess quite different properties, depending on their degree of isotropy. If the material in a body possesses exactly the same properties throughout, including the stress-transmitting capacity, any point of any line drawn within that body will be a part of an isotropic "continuum," and such a state of isotropy may be termed "monotonous." On the other hand, if the material consists of units or particles interspersed with voids or some other material, with different stress-transmitting capacities, the body may possess "statistical" isotropy. Owing to the presence of a great number of particles pressed against each other, stresses within such a body may be generated so as to simulate a "continuum," and this similarity is the greater, the greater the degree of isotropy approaches the "monotony" referred to; for instance, if two straight lines, 1 in. long, drawn within a sand and a clay mass, pierce 100 and 100 000 soil particles, respectively, it is evident that the clay mass is much more properly termed "monotonously" isotropic than the sand mass. Furthermore, attracted moisture, which is almost solid in the case of a moist clay mass, approaches more the stress-transmitting capacity of the soil particles than water in the voids of a sand mass. Hence, formulas of the theory of elasticity, the Boussinesq formula among them, developed for the case of an isotropic elastic "continuum" should be applied to clays rather than to sands. It does not follow, however, that clays are elastic, and sands inelastic. Each sand particle is an elastic body, as demonstrated by the rebound after releasing a load from a sand mass; but the reason why the Boussinesq formula cannot be applied unconditionally to sands, lies in their lack of monotonous isotropy. Since statistical isotropy requires the presence of a great number of particles, it does not exist at all where this condition is not fulfilled. Hence, an earth mass, especially a natural sand mass, is not isotropic close to its boundary.

The explanations advanced are required in order to understand the following apparent inconsistency in Mr. Cummings' paper. First, he quotes Boussinesq himself who stated that the distribution of stress across planes parallel to the surface is the same in all isotropic bodies, and a few pages following a numerical example dealing with stress distribution in sand shows that an actual vertical pressure is almost twice that computed according to the Boussinesq formula, and so do the experiments described. The reader is thus induced to believe that sands are not isotropic and perhaps not elastic; but the situation clears up if one realizes that under the term, "isotropic," Boussinesq could mean monotonously isotropic bodies only, such as those studied in the theory of elasticity.

Interrelationship Between the Degree of Isotropy and the Value of the Concentration Factor.—Commenting on the opinion of Mr. Cummings, as to the values of the concentration factor ($n = 3$ for clays and $n = 6$ for sands), the writer agrees with it in so far as shallow depths are concerned, but wishes to offer an interpretation. At each point of the earth mass, loaded with its own weight, there is a vertical pressure, p_z, and a horizontal pressure, $p_x = p_y$. The ratio, $\frac{p_x}{p_z} = \frac{p_y}{p_z} = K$, is the coefficient of pressure at rest, according to the terminology of Professor Terzaghi. The value of K in incompressible isotropic elastic bodies (confined water, some metals, etc.), is equal to 1. This value decreases as isotropy ceases to be monotonous; for instance, it equals about one-fourth in the case of coarse sands. Simultaneously, the concentration factor, n, increases; and this can be shown mathematically. In Fig. 6 the vertical and the horizontal angular distances are designated by θ and θ', respectively; d is the depth of a point, A, at which the vertical and horizontal pressures, p_z and p_x, respectively, are to be determined; and z is the variable depth of a point, Q. Equation (3) represents the vertical normal stress, p_z, which may be conceived as the vertical projection of the full stress, p; thus:

$$p = \frac{nP}{2\pi R^2}\left(\frac{z}{R}\right)^{n-2} = \frac{nP}{2\pi R^2}\cos^{n-2}\theta \ldots (15)$$



Boundary Plane
z
dz
d
$d-z$
$\frac{d-z}{\cos^2\theta}\,d\theta$
$(d-z)\tan\theta$
Q
θ
$d\theta$
A
$\frac{d-z}{\cos^2\theta}\,d\theta$
A
$d\theta'$
$(d-z)\tan\theta.d\theta'$

FIG. 6

It will be assumed that a projection of the stress, p, on the horizontal plane does not depend on the value of the Poisson ratio. At the point, Q, there is a small earth element (shaded area in Fig. 6). If w is the unit weight of the earth material in question, the weight of this small element is:

$$dP = w\left[\frac{d-z}{\cos^2\theta}d\theta\right]\left[(d-z)\tan\theta\,d\theta'\right]\left[dz\right] = \frac{w\,(d-z)^2\sin\theta}{\cos^3\theta}\,dz\,d\theta\,d\theta' \ldots (16)$$

In Equation (16) the three dimensions of the earth element at the point, Q, (its length, width, and height) are written separately in brackets. According to Equation (15) the value of the elementary stress acting along the direction, QA, is:

$$dp = \frac{n\,dP}{2\pi(d-z)^2}\cos^n\theta \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

and the horizontal projection of this elementary stress:

$$dp \sin^2 \theta = \frac{n\, dP}{2\, \pi\, (d - z)^2} \cos^n \theta \sin^2 \theta \ldots\ldots\ldots (18)$$

Taking the sum of the horizontal projections (Equation (18)) of the elementary stresses reaching Point A from all parts of the earth mass lying above it, the horizontal pressure at Point A would be obtained. Hence:

$$p_x + p_y = 2\, p_x = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{z=0}^{z=d} \int_{\theta'=0}^{\theta'=2\pi} \sin^2 \theta\, dp \ldots\ldots\ldots (19)$$

Introducing into Equation (19) the values of dP and dp from Equations (16) and (17), the first step is to integrate with respect to both z and θ', and then with respect to θ; thus:

$$p_x = \frac{1}{2}\, n\, w\, d \int_0^{\frac{\pi}{2}} \cos^{n-3} \theta \sin^3 \theta\, d\theta$$

$$= \frac{1}{2}\, n\, w\, d\, \frac{2}{n} \left[- \frac{\cos^{n-2} \theta}{n - 2} \right]_0^{\frac{\pi}{2}} = \frac{1}{n - 2}\, w\, d \ldots\ldots\ldots\ldots (20)$$

Since the value of the vertical pressure at Point A is $p_z = w\, d$, the value of the coefficient of pressure at rest, K, would be: $K = \frac{p_x}{p_z} = \frac{1}{n - 2}$, from which:

$$n = 2 + \frac{1}{K} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Equation (21) expresses the relation between the concentration factor, n, and the coefficient of pressure at rest, K. It is an approach to the solution of this problem since further research may prove that some other causes also influence this relation. For an incompressible isotropic elastic body, $\frac{p_x}{p_z} = 1$ and, hence, $n = 3$. If the horizontal pressure exerted by a clay is three-fourths of the vertical, the value of n in that case is about 3.3. For a sand possessing an angle of friction of $\phi = 35°$ under given circumstances,

$$K = \frac{p_x}{p_z} = \tan^2 \left(45° - \frac{\phi}{2}\right) = \tan^2 27° 30' = \frac{1}{4}$$

Hence, according to Equation (21), the value of its concentration factor is $n = 6$. It is also obvious that there is a certain relation between the value of the concentration factor, n, and the angle of friction, ϕ, of a given sand.

The writer wishes to emphasize, however, that the few known values of K have been determined from experiments practically all of which were in rather shallow test containers. Hence, the corresponding values of the concentration factor, n, cannot be applied unconditionally for computations of stresses at greater depths. The writer's belief is that the concentration factor, n, is a function of depth also; apparently, it decreases with the depth, tending to approach the asymptote at $n = 3$. Evidently, the latter value is a minimum, since one cannot visualize any material more free from voids than a "continuum." Probably, at a certain depth within the earth all matter obeys the Boussinesq law with a concentration factor, $n = 3$, and the difference between the stress distribution in sand and clay is merely a surface phenomenon.

The writer bases his opinion on the experiments of Messrs. Terzaghi[25] and Hatch,[26] who proved that the coefficient of friction of sand decreases as the normal pressure increases; and this means that the coefficient of pressure at rest, K, is greater and the factor of concentration, n, smaller, at a relatively greater depth than close to the ground surface. The statement to which Mr. Cummings objects, namely, that the distribution of stress is independent of the type of material, should be corrected by adding the phrase, "at a certain depth." It is true that this unknown depth may be very considerable.

Stresses at the Plane of Contact.—Mr. Cummings' remark that the distribution of pressure on the under side of the experimental test plates in the experiments described is responsible for the stress distribution in the earth mass, cannot be over-emphasized. An interpretation of the fact should be given, however. According to the third Newton law, action and reaction are equal and opposite, and it appears at first glance that if the load on a foundation or experimental plate is uniform, the reaction of the earth mass under it should also be uniform. Such a manner of thinking has proved disastrous in the history of civil engineering and has retarded the development of soil mechanics for many years. What actually matters is not the distribution of load around the foundation, but the distribution of stresses within the footing close to its plane of contact with the earth. The structure and the earth mass together form a statically indeterminate system, and the stress distribution depends on the elastic properties of both bodies in contact. In all cases stresses on both sides of the plane of contact are equal, thus satisfying Newton's third law (Fig. 7(a)). As a rule, the stress distribution in question is not uniform, although it may become so in a particular case.

In reference to the interesting statement by Mr. Cummings that for very large areas the pressure is approximately uniform, the writer wishes to state that in some cases there is only an appearance of uniformity. In Fig. 7(b) a narrow structure, and in Fig. 7(c), a wide structure are represented; both are subjected to the same uniform unit load and both are rigid and constructed on rigid clay, so that the stress would be a minimum at the center. The ordinates, p_1, p_2, p_3, etc., of the stress distribution curve are exactly equal at corresponding points of both structures; but the rate at which the stress increases per foot of width of the foundation is less in the case of the wide foundation.

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[25] "Handbuch der physikalischen und technischen Mechanik," edited by F. Auerbach and W. Hort, Vol. IV, Pt. 2, p. 523, Leipzig, 1931.

[26] "Tests for Hydraulic-Fill Dams," by Harry H. Hatch, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 99 (1934), pp. 228–229.

This may create the impression of a rather uniform stress distribution close to the middle of a wide structure.

Mathematics of the Stress Distribution Formulas.—If trigonometric functions were used in developing Equation (7), it would be in a more convenient form for application. In Fig. 8, θ_o, designates a half angle at the vertex of the cone formed by radii vectors drawn from a given point, A, within the earth mass to the perimeter of a round foundation. Then, Equation (7) becomes,

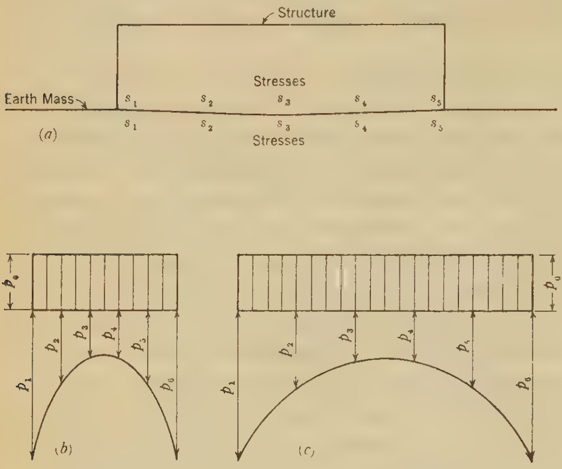

FIG. 7

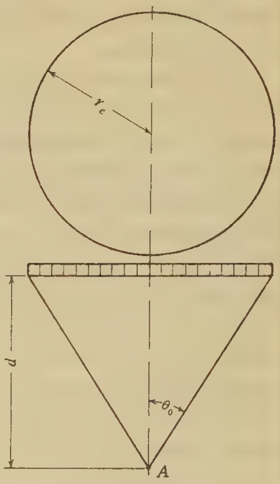

FIG. 8

$$p_z = p_o\,(1 - \cos^3 \theta_o) \ldots\ldots\ldots\ldots (22)$$

Equation (22) is a particular case of a general formula. Actually, integrating Equation (6) for a general case (that is, for a concentration factor, n):

$$p_z = p_o\,(1 - \cos^n \theta_o) \ldots\ldots\ldots\ldots (23)$$

from which, by making $n = 6$, the stress distribution for Case II may also be obtained. Equation (23) was first developed by Dr. O. K. Fröhlich, of Holland.[27]

To integrate Equation (12), Mr. Cummings uses an infinite series; the problem was also solved by Dr. Fröhlich by a transformation to trigonometric functions.[19] The formula, however, may be integrated in a closed form by substituting t for $\frac{r^2}{r^2_e}$, thus:

$$r\,dr = \frac{1}{2}\,r^2_e\,dt \ldots\ldots\ldots\ldots (24)$$

For convenience, let $d = a\,r_e$ in which a is merely a coefficient. Remembering that the limits of integration of the variable, t, are 0 and 1, which corresponds to $r = 0$ and $r = r_e$, respectively, and integrating with respect to θ:

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[27] "Druckverteilung im Baugrunde," p. 50, Julius Springer, Vienna, 1934.

$$p_z = 3\,p_0\,a^3 \int_0^1 \frac{1 - t}{(a^2 + t)^{\frac{5}{2}}}\,dt$$

$$= 6\,p_0\,a^3 \left[\frac{1}{\sqrt{a^2 + t}} \left(1 - \frac{1}{3}\,\frac{a^2 + 1}{a^2 + t} \right) \right]_0^1$$

$$= 2\,p_0 \left[1 - 2\,a^2 + \frac{2\,a^3}{\sqrt{a^2 + 1}} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

By making $a = 1$ in Equation (25), the true value of the vertical pressure, p_z, at a depth, $d = r_e$, may be obtained. It is equal to 0.83 p_0, and not infinity, as indicated by Fig. 4, Case III. A mathematical tool, like any other, can give adequate results only within the limits of its applicability. This is why in this case an infinite series proved inadequate to reflect the actual physical condition between $a = 1$ and $a = 0$. By making $a = 0$, in Equation (25), p_z becomes equal to 2 p_0 or 200% of the average surface pressure as obtained from Dr. Fröhlich's integrals. Case IV may be dealt with in a similar manner.

Limits of Application of the Formulas Discussed.—So far as may be concluded from this paper, the following methods are available for computing stresses within an earth mass: (*a*) By Equations (1) or (3) for the case of a concentrated load; (*b*) by Equation (7) for uniform load distribution; and (*c*), by Equations (12) and (14) for parabolic load distribution. A case is also mentioned by Mr. Cummings in which a distribution would be a minimum at the center and infinite at the edge; but no formula is given for this case.

In comparing the curves shown in Fig. 4, the author reaches the conclusion that "the character of the pressure distribution at the surface (uniform or parabolic) is especially important near the surface but loses its importance as the depth increases." This statement, although quite true, should be generalized and strengthened as follows: "At a certain depth, about twice the diameter of the loaded area, ($d \geq 4\,r_e$), any structure acts practically as a concentrated load." In other words, if stresses are to be determined at a depth greater than twice the diameter of the loaded area, the designer need not worry about how the loads are distributed along the structure, nor need he worry as to whether or not the structure is rigid. The Washington Monument, in Washington, D. C., has a base of about 125 by 125 ft; hence, it acts as a concentrated load at a depth of 250 ft beneath the base. On the other hand, the water reservoir behind Boulder Dam, with a length of, say, 100 miles, acts as a distributed load to a depth of 200 miles beneath the site.

The situation changes in the upper layers ($d < 4\,r_e$). If a non-rigid, round structure, such as a water or oil tank, non-rigid masonry structures, etc., is to be constructed on clay or similar soil, Equations (7) and (8) for uniform load distribution should be applied; and if a rigid, round structure is constructed on sand or similar soil, Equations (12) and (14) should be used.

The most interesting problem—that of determining accurately, the value of the bending moment in a footing or in a spread foundation—still remains open. A definite answer can be given only when the behavior of the "disturbed zone" under the structure itself can be studied properly. Although the integral Equation (25) gives a pressure of 200% of the average unit load close to the base of the structure, M. L. Enger, M. Am. Soc. C. E., found about 300% by experiment.[17] The writer believes that this value was not excessive and that stresses in the "disturbed zone" are exceedingly high.

Conclusion.—Mr. Cummings has brought to the attention of the profession one of the most serious problems of soil mechanics and has displayed, in this connection, an unusually broad knowledge of the technical literature. No one can be expected to solve such problems alone; hence, if in discussing this interesting and important paper, corrections and additions are made, this cannot diminish its value.

L. C. WILCOXEN,[28] ASSOC. M. AM. SOC. C. E. (by letter).—The distribution of stresses in an elastic soil under a foundation, based on certain specific assumptions, has been developed mathematically by Mr. Cummings. His conclusions are logical, and he has done well to stress the limitations of his formulas to the conditions assumed.

Examination of the general conditions affecting the distributions of stresses under a foundation more clearly sets forth the significance of the limitations prescribed by the author. The distribution of stresses is modified by: (*a*) The load distribution on the footing; and (*b*) the degree of flexibility of the footing. In practice, there are three distinct types of simple foundations, each having its characteristic stress distribution: (1) The uniformly loaded, perfectly flexible foundation; (2) the concentrated load on a semi-flexible foundation; and (3) the rigid (non-flexible) foundation, in which the character of the load is not a factor. Naturally, there are modifications and hybrids of these three types. Their recognition, however, explains much of the conflicting evidence regarding the behavior of foundations.

The uniformly loaded, perfectly flexible foundation produces a uniform surface load on the soil, as Mr. Cummings has assumed in Cases I and II. The concentrated load on a semi-flexible foundation (and with the proper degree of flexibility) produces a surface load that is parabolic in sections. This covers Cases III and IV. It may also yield a uniform surface load as in Cases I and II, and as will be explained.

The rigid foundation, regardless of the type of loading, produces a maximum surface loading at its perimeter, decreasing toward the center. This type of distribution has been demonstrated by the writer,[29] using the photo-elastic method. The soil loading produces an inverted arch, or dome of maximum stress, the bottom of which (in the case observed) was at a depth, 0.7 *D*, below the foundation. This dome of stress is probably somewhat akin to the cone developed under a small "foundation" when the latter is driven through a soil.

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[28] Asst. Civ. Engr., City Engr.'s Office, Detroit, Mich.

[29] Progress Rept. of the Special Committee on Earths and Foundations, *Proceedings*, Am. Soc. C. E., August, 1933, p. 1054.

In all these different cases, the foundation flexibility is the key to the resulting type of surface stress distribution. Deformation is probably as important in foundation problems as retaining wall movements (however small) are, in determining soil reactions. The latter has recently been demonstrated in a striking manner by Charles Terzaghi, M. Am. Soc. C. E.[30] The former remains to be demonstrated.

Given a concentrated load on a rigid footing, it has been stated that the maximum surface stress is at the perimeter, decreasing toward the center. If flexibility of the footing could be introduced progressively, the pattern of the surface reaction on the footing would change progressively. It would first become uniform; later, it would assume the parabolic distribution, and, ultimately, it would become a concentrated reaction under the load itself. It is one of these intermediate cases which was reported by Frederick Converse, Assoc. M. Am. Soc. C. E., and referred to by the author.[21] Failure by Mr. Converse to report the nature and extent of the footing deformation made the experiment incomplete.

The classical experiment[31] by M. L. Enger, M. Am. Soc. C. E., has been assumed by the author to have been a parabolic surface loading, as in Cases III and IV. This is probably an error, although in this case, it is not important because the data which he borrows are at a depth of at least one footing diameter below the footing. At this level, Dean Enger has shown such a distribution to be correct.

In the original report of this experiment the lines of equal stress were extrapolated from the uppermost plane of observed values (6 in. below the footing) upward to the foundation. Although not supported by tests, and although no conclusions were drawn from it, the diagram showing the greatest surface load at the center has gained wide credence. The footing was relatively rigid, at least in one case (a heavy plate, 13½ in. in diameter, with a load as great as 18 000 lb applied by a jack with a large base). Under this footing, a dome of maximum stress probably occurred with the distribution of the soil reaction, being a maximum at the perimeter and decreasing toward the center. Fig. 9 shows the original stress diagram (*a*) modified, and (*b*) in accordance with the writer's theory of rigid foundation surface stress distribution.

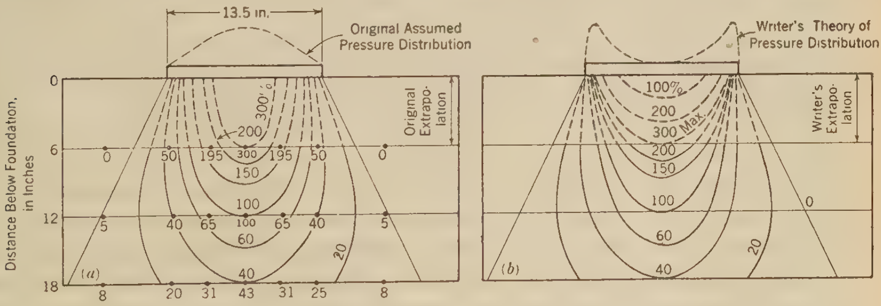
13.5 in.
Original Assumed Pressure Distribution
Distance Below Foundation, in Inches
0
6
12
18
300%
200
0
50
195
300
195
50
0
150
100
5
40
65
100
65
40
5
60
20
40
(a)
8
20
31
43
31
25
8
Original Extrapolation
Writer's Theory of Pressure Distribution
100%
200
300 Max.
200
150
100
0
60
20
40
(b)
Writer's Extrapolation

FIG. 9

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[30] *Engineering News-Record*, February 1 and 22, March 8 and 29, and April 19, 1934, Vol. 112, pp. 136, 259, 315, 403, and 503.

[31] *Engineering Record*, January 22, 1916, Vol. 73, p. 106; and *Transactions*, Am. Soc. C. E., Vol. 93 (1929), p. 372.

That the pressure under a rigid foundation is greatest at the center is a theory that is widely held. The writer is aware that in many cases the opinion is authoritative, especially when the soil is a granular one. Nevertheless, he offers the following facts and deductions in support of his own view.

It has been demonstrated[29] that, with a uniform surface loading, in both plastic and granular soils, settlement was greatest at the center. Fig. 10 (*a*) is a diagram of such a bulb of displacement and Fig. 10 (*b*) is a magnified incremental cube. With an increased footing load this cube would be distorted as indicated by the dotted diagram. The tests indicate that this deformation, and, hence, its stress distribution, would be the same regardless of whether the soil is cohesive or granular.

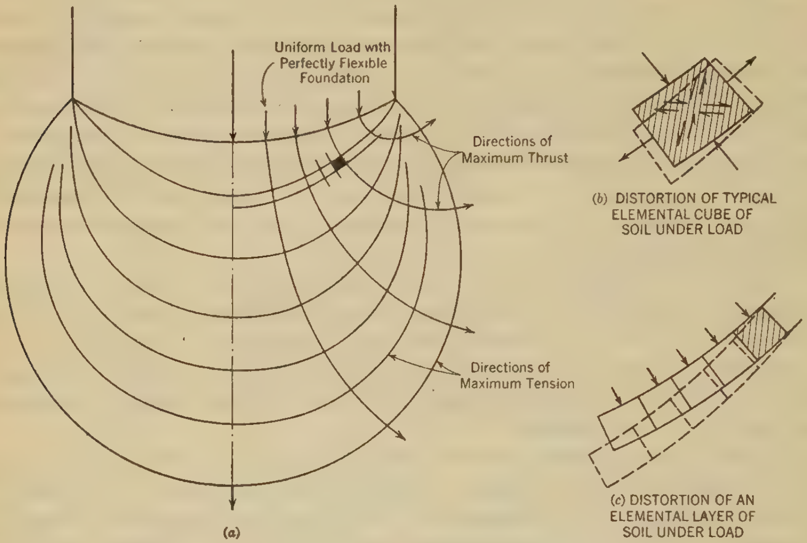
Uniform Load with Perfectly Flexible Foundation
Directions of Maximum Thrust
Directions of Maximum Tension
(a)
(b) DISTORTION OF TYPICAL ELEMENTAL CUBE OF SOIL UNDER LOAD
(c) DISTORTION OF AN ELEMENTAL LAYER OF SOIL UNDER LOAD

FIG. 10.—CHARACTERISTICS OF A BULB OF STRESS UNDER A FOUNDATION WITH A UNIFORM SURFACE LOAD

Fig. 10 (*c*) is a diagram of an incremental layer of soil as distorted by an increased load. It is apparent that each successive cube toward the center is displaced downward not only by its own direct load but also by the loads on all the adjacent cubes toward the perimeter. Thus, it seems that, regardless of the character of the soil, the stress distributions under uniform surface loads are the same and the area of greatest resistance is toward the perimeter. If the footing were a rigid one the greatest pressure would necessarily be in the region of greatest soil resistance; that is, toward the perimeter.

That this holds true in sand has been demonstrated conclusively by the writer in an elemental test. A lightly tamped, damp mortar sand was struck level with a straight-edge. First, a piece of typewriter carbon paper (with the carbon side up) was laid on the surface; then a piece of white paper; and, finally, a steel plate, 4 in. square. Pressure was then applied to the plate, by twice plunging a round-ended fence post down upon it. The results (see Fig. 11) showed that the region of greatest carbon transfer (and, hence, the greatest pressure) occurred under the part of the foundation near the perimeter.

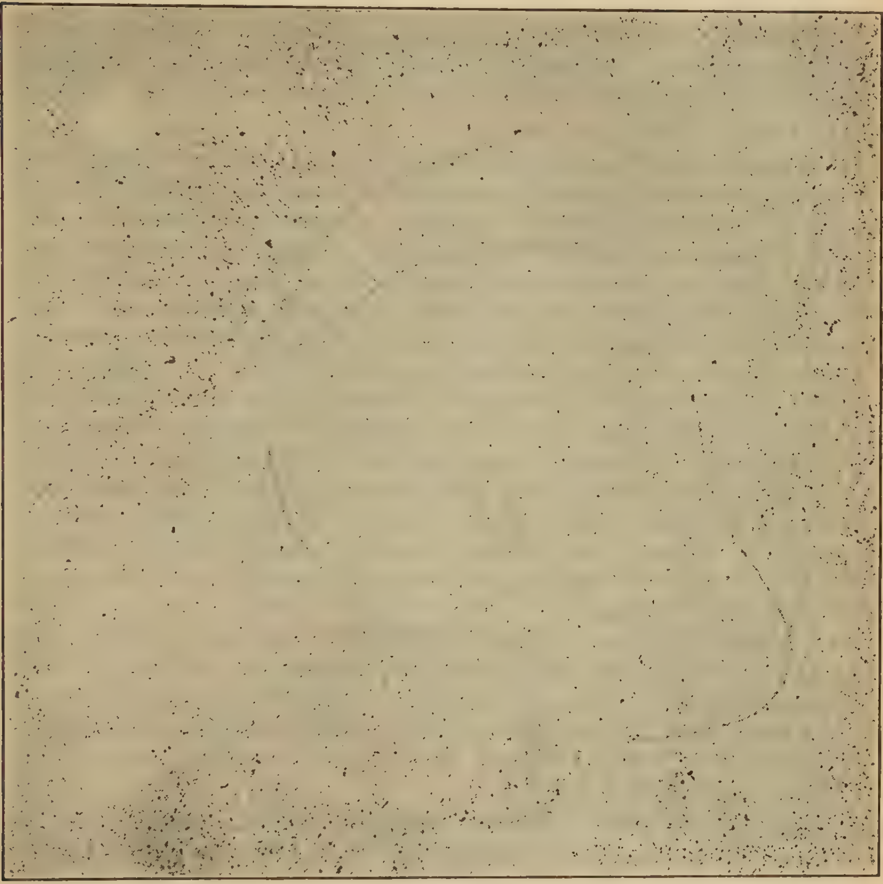

FIG. 11

The problem of analyzing the distribution of pressure under a foundation is a comprehensive one. The author has avoided the complications of footing characteristics by assuming surface, rather than footing, loadings, and has skilfully analyzed two of the three types of surface loadings that occur under foundations.

MARSHALL G. FINDLEY,[32] ASSOC. M. AM. SOC. C. E. (by letter).—The problem discussed by Mr. Cummings is one of a group, which may be suggestively subdivided as follows:

1.—Footings in which one plane section only may be analyzed: (*a*) Wall footings, centered (the simplest case); (*b*) flume footings (rectangular slabs with parallel walls along two opposite edges); and (*c*) continuous footings under a series of parallel walls.

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[32] Structural Designer, Water Purification Plant, Milwaukee, Wis.

2.—Simple footings symmetrical around a central point: (*a*) Square footings with central load; (*b*) circular shaft footings (chimneys, etc.); and (*c*) square shaft footings.

3.—Eccentric footings: (*a*) Square footings with eccentric load; (*b*) building column footings near property lines; (*c*) retaining wall footings; (*d*) footings for crane columns and movable bridges; and (*e*) base slabs of concrete reservoirs.

In all these types of footings, in the usual varieties of soil, actual pressures are unquestionably small at the edges of the footings, and greatest under or near heavy load concentrations. The usual assumption of design is that under a complex footing, earth stresses vary uniformly from one edge to the other. This usual assumption is clearly just as inaccurate in the case of complex footings as it is in simple cases, such as Footings 1(*a*) and 2(*a*). Furthermore, this inaccuracy is greater in some types of soil than in others.

In consideration of footing problems, two points of view have arisen. One is that of soil mechanics, or the analysis of conditions in the soil under the structure. From this point of view the assumption of uniform or uniformly varying pressure at the plane of contact is often actually unsafe. In certain types of clay, for instance, it is now known that a large footing sometimes cannot carry proportionately as large a load as a small footing; the greater building up of pressure in the center may start a crushing action of particles of clay holding water in integral union, which in time will lead to the settlement of the larger footing.

The second point of view is that of the designer of the footing slab. From his point of view, the assumption of uniform or uniformly varying pressure is almost always somewhat on the safe side. Usually the loss of economy is not very serious. In the unusual conditions that occur occasionally in which this assumption is very extravagant there is little published precedent as to the proper method of slab design.

The author's review of conditions beneath a simple footing should be studied carefully not only by students of soil mechanics, but also by designers of foundation slabs. Strangely enough, these two points of view are not actually co-ordinated as well as they might be.

In the design of footing slabs the exact theoretical formula is often not of great importance. For instance, in the case of a simple wall footing, a sine formula seems to be theoretically correct in stiff homogeneous material for distribution of pressure at the plane of contact; but considering the resulting design of footing, it may not make any appreciable difference whether a sinoid, parabolic, or an elliptic distribution curve of footing pressure is assumed. The important point is to take account of the deflection of the cantilever edges; this may be done even by means of a very rough, straight-line diagram, without serious error in shear or moment diagrams.

Such rough straight-line diagrams may be used with profit in the design of certain more complex types of footings. It is not the intention to cast any doubt on the advisability of theoretical and experimental work such as that cited by Mr. Cummings; but there are certain types of footings, some of which are among those outlined herein, in which even with rough straight-line assumptions as to footing pressures, design becomes very complex mathematically. The exact conditions are not known through experiment; nor are they accessible to exact scientific imagination; the soil is not perfectly homogeneous, although on the whole it is substantial and dependable; and the final conditions as to ground-water are not determined or are not constant. Under such conditions, it seems more in line with exact scientific procedure to make a rough but probable assumption and then to compute, exactly, certain of its consequences, than to use the very improbable assumption of uniformly varying load. Furthermore, the less probable assumption occasionally results in design that is economically impossible, and obviously unnecessary. Painstaking detailed investigation has clearly proved certain general truths about foundation pressure distribution. It is possible to use these general truths in a rough approximate way, under conditions where detailed application is impossible.

M. A. BIOT,[33] ESQ. (by letter).—The classical theory of elasticity is based on a linear relation between stress and strain. This fact introduces an important simplification in mathematical investigations in that the superposition principle can be used and the summation of elementary solutions leads to other solutions.

It is important to note that for a material that does not satisfy Hooke's law, this superposition principle does not hold and, in general, it will not be permissible to add up solutions due to loads acting independently in order to find the stresses due to their combined action.

In the case of certain materials like sand, however, the pressure distribution is assumed to be given by a semi-empirical law more general than that of Boussinesq formulated by Equation (3) of the paper. In case the pressure distribution does not coincide with that of Boussinesq, this must be due to the fact that the material is either not isotropic or does not satisfy Hooke's law. The latter is true, for instance, if some kind of gliding occurs inside the material. Obviously, then, it is not permissible to superpose solutions of the form of Equation (3) in order to find the effect of distributed loads.

JACOB FELD,[34] M. AM. SOC. C. E. (by letter).—The subject touched by this paper really involves two major classes of problems in foundation engineering, namely:

(1) For a foundation unit of a given size, both in area and in shape, carrying a known load, either as a point concentration or as a distribution of known variation: (*a*) What is the distribution of stress at the base of the foundation unit or footing? (*b*) what is the variation in the distribution with changes in the assumed relative and also absolute (for end conditions or cases) stiffness of the footing and of the supporting soil? and (*c*) what is

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[33] Univ. of Louvain, Louvain, Belgium.

[34] Cons. Engr., New York, N. Y.

the variation in the distribution with increases in the loading, for a given ratio of stiffness of footing and soil, including the end condition where the loading causes a change of the soil, or of the footing, from elastic to plastic material?

(2) For a given load on a footing of given size, the distribution of stress at the base having been determined: (*a*) What is the distribution of pressures through the soil? and, (*b*) what changes in distribution occur with increased loadings?

In all the foregoing problems, it must be remembered that the summation of stresses from various loads can only be made as long as the total does not exceed the elastic limit. As soon as plastic flow (in clays), or failure in shear (lack of tension in sands), occurs, the problem changes.

The author has noted clearly that his paper covers only one case of the general problem, namely, what is the distribution of stresses along the vertical axis of the soil supporting a circular load for: (*a*) Uniform; and (*b*) parabolic distributions of stress at the base of the circular footing? He shows definitely that the assumption of parabolic distribution gives theoretical results which agree with the experimental work with sand, in which all the conditions of the problem are met.

As an opening wedge, the paper is valuable. It is hoped that the author's warning (see "Summary and Conclusions") will be heeded: "These formulas can be used only when the practical problem fulfills all the conditions under which they were derived."

The theoretical formulas are based on the assumption of action in isotropic bodies. The definition of the term, "isotropic", is "having the same physical properties in all directions." This condition eliminates all soils except for very small loadings. In general, the approximate maximum load that can be considered, is the largest load which will not cause a permanent set or deflection after it is removed.

All the tests enumerated in Table 1 are for sand and circular disks. No attempt has been made to correlate the effect of the amount of unit loading with the variation or law of distribution in the soil.

In the tests by Messrs. R. B. Fehr and C. R. Thomas[35] on square and rectangular test areas, for sand, clay, and loam, a definite change in distribution was noted with increased loadings.

The very high unit stresses found in the area at the center of, and just below, the circular disk footing is not serious because in the actual condition there will be a release of support and, therefore, a redistribution of loading in this area. The theoretical solution of the problem is equivalent to a first approximation of a statically indeterminate structure in which deflections are neglected. In addition, the center core of soil under a footing is better retained against lateral movement and, consequently, against vertical movement than the remainder of the soil body. The effect is to permit greater loading without large settlement or flow.

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[35] "Experiments on the Distribution of Vertical Pressure in Sand", by R. B. Fehr and C. R. Thomas, *Bulletin No. 8*, Pennsylvania State Coll., 1913; and "Further Experiment on the Distribution of Vertical Pressure", by R. B. Fehr, *Bulletin No. 10*, Pennsylvania State Coll., 1913.

Where long loadings, such as quay walls, on soft ground, give high stress concentrations on a long strip of the supporting soil, with consequent settlement and flow of the mud or silt and often failure of the wall, it has been found most economical to eliminate the soft soil directly beneath the wall and replace it with a layer of sand. The entire picture of stress distribution in the soil is then changed not only because the high localized stress at the base of the wall is taken by the sand, but also by the formation of a new soil at the contact zone. A very fine description of the methods and large scale tests on this method of solving the problem is given by Gen. Carlo Barberis.[36]

Under "Numerical Example", the author points out that although greater areas of soil are affected at increasing depths below a loaded footing, the distribution at any depth is not uniform. The assumption of parabolic distribution, made in this example, may apply to sand, but certainly should be modified if clay or any cohesive material is used, chiefly because of the edge effect on the perimeter of the footing. For all soils, the assumed distribution will change as soon as a deflection occurs.

The true distribution at the base of a footing is such that the sum of the works performed in bending, in shear, and in compression of the footing is a minimum, subject always to the condition that no stress exceeds the elastic limit. It would be a very unusual design problem which would warrant such a computation. Probably, photo-elastic methods to show the qualitative conditions would be advisable to reduce the number of possible distributions that should be studied.

George Paaswell,[37] M. Am. Soc. C. E. (by letter).—It is gratifying to note the interest in the development of accurate methods of obtaining pressure distributions under foundations. The mechanics of foundations is predicated upon a correct determination of stress distribution in the underlying strata. However, correctness may be defined herein not as a rigorous mathematical solution consistent with all the data, but rather as a general correctness of stress distribution over the broad extent of the strata, subject to distortion by pressure of the foundation loading. In this sense of the word, the "local perturbation effect" immediately adjacent to the point of loading is of no significance in the general stress distribution. The problem may be viewed as analogous to the stress determination in a girder subject to concentrated loads, where the effect on the point of application of the load on the local part of the girder is considered of secondary importance, the general distribution of moment and stresses being the primary problem involved. The manner of loading a foundation and the soil pressures near its loading can generally be ignored in the determination of the stress distribution in the deep strata the distortions of which will usually determine the final movement of the supported structure. The analysis as presented by the author—although of interest to the general theory of stress distribution—

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[36] "Recent Examples of Foundations of Works, Such as Quay-Walls", etc., by Carlo Barberis, XVI International Congress of Navigation, Second Section, Brussels, 1935. "Ocean Navigation", Third Communication by Carlo Barberis, Rome, Italy.

[37] Mgr., Spencer, White & Prentis, Red Wing, Minn.

should not leave a serious doubt in the reader's mind as to the general validity of the Boussinesq equations. These equations, as given in the report[6] of the Society's Special Committee on Earths and Foundations, are sufficiently rigorous and consistent with experiment to permit a scientific prediction of foundation behavior. Their use in foundation design involves the same degree of accuracy as the use of the ordinary Bernoulli formulas for flexure.

It is unfortunate that the development of the Boussinesq equations is buried in the intricate treatises on the mathematical theory of elasticity. The fundamental assumptions of the theory of elasticity are that: (1) All displacements are infinitesimal; and (2), Hooke's law is correct.

Boussinesq was fortunate in discovering an ingenious device for the solution of the differential equations for displacement of elastic solids under a surface load. A function is found, such that its derivatives express the displacement of a given point under a certain type of surface loading. By combining two forms so as to induce the surface loading to a single force at a point, the displacements are evaluated, and from these the forces are found. The force system in horizontal planes (if the applied load is vertical and the boundary plane horizontal) involves no elastic constants. It so happens that the force system on vertical planes contains elastic constants, which, for the usual soils, permits values that cause the terms containing these constants to vanish, so that it may be stated, in general, that the force systems on both vertical and horizontal planes do not involve any elastic constants so far as foundation problems are concerned.

G. S. SALTER,[38] ASSOC. M. AM. SOC. C. E. (by letter).—The problem of the distribution of stresses in a foundation is quite complicated, and much good work has been done on the subject in the last few years, to which this paper is a commendable addition.

The writer has found that most of the investigations on this subject, both theoretical and experimental, have dealt with uniform or parabolic load distributions on the footings (either actual or assumed), whereas in so much of the foundation design encountered in his work the loading is line distribution on semi-flexible footings; also, while most of the literature deals with the determination of stresses at some distance below the contact surface, in many cases (especially on a clay varying from medium stiff through the various gradations to hardpan), the problem is more to determine the pressure distribution directly under the footing than the pressure intensities at some lower depth where the clay is more compact.

It has long been realized that variations in conditions lead to entirely different pressure distributions and that those resulting from line distribution on a semi-flexible footing, such as a heavy wall on a relatively thin slab, give high pressures directly under the wall.

The writer has made some investigations in this field and has come to the conclusion that the following procedure is a logical method of dealing with this particular problem. Make soil-bearing tests to determine settlement for different loading values and then determine, by trial if necessary, such pressure distribution values on the contact area between the footing and the soil so that the differential settlement resulting from the varying soil pressure equals the flexure in the footing.

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[38] Structural Designer, The San. Dist. of Chicago, Chicago, Ill.

Such line loads give a contact loading distribution somewhat parabolic in character, the variation depending upon the load-carrying capacity of the soil and the relative weights of the wall and the slab. Sometimes, it has been desirable to make various assumptions (within reason) on the bearing capacity of the soil and to design for the worst probable condition. Pressure distributions thus found are far from the uniform pressures commonly assumed, and where footings are somewhat flexible, or when the soil has high load-carrying values, a line load may be distributed over only a small portion of the footing directly under the load.

The foregoing is only one of the many problems on the behavior of foundations and is presented in an endeavor to show, as the author has stated, that, as a general rule, it is not satisfactory to assume that the pressure at the surface is uniformly distributed.

For the general problem of stress distribution which exists in the ground under a foundation to which the theory of Boussinesq applies, the University of Illinois has recently published[39] a *Circular* on simplified computations of vertical pressures.

W. S. Housel,[40] Assoc. M. Am. Soc. C. E. (by letter).—A contribution to the subject of foundation pressures is embodied in this paper, which has immediate value to the practicing engineer and student of soil mechanics. The concise presentation, with complete references, should make it easy for any one interested in the subject to be well informed on the mathematical treatment of a problem which heretofore has been exceedingly difficult to review completely. The solutions of several cases of pressure distribution, although still dealing with conditions more ideal than real, are a closer approximation to actual conditions encountered in practice and should form the basis of enlightened discussion and further development.

The author's analysis of various experimental results is most valuable in calling attention to the fact that practically all previous investigations introduce a variable pressure distribution on the bearing area which controls the maximum pressure ordinate at depths beneath that area. Scarcely less important is the emphasis on the concentration factor which is discussed in connection with solutions. The writer is inclined to feel that the equations presented are not as convenient as those which have been developed in terms of trigonometric functions, but that does not detract from the value of the analysis and discussion. There is little to be added to the author's use of the tools supplied by the mathematical theory of elasticity and his conclusions mark the point from which discussion may begin. These conclusions are: "(1) The manner in which pressure is distributed over the contact surface must be considered; and (2) the equations of the theory of elasticity must be modified before they can be applied to soils."

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[39] "Simplified Computations of Vertical Pressures in Elastic Foundations", by Nathan M. Newmark, Jun. Am. Soc. C. E., *Circular No. 24*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

[40] Asst. Prof., Civ. Eng., Univ. of Michigan, Ann Arbor, Mich.

In the writer's opinion the only fault to be found with the first conclusion is that it might well be made more emphatic. So much confusion and false deduction has arisen from a failure to appreciate the significance of pressure distribution on the bearing area itself that it certainly is gratifying to find this fact recognized as being one of two major conclusions supported by the author's study.

The second conclusion also impresses the writer and, when generally accepted, opens the door to definite progress toward the real problem of how the elastic equations may be altered so that they may be applied to soils. The author suggests the concentration factor, n, as the means of altering the equations and indicates that n must be varied in some manner, depending on the type of material, the load applied whether high or low, and the size of the bearing area whether large or small. With all this there seems to be no room for anything but enthusiastic and unqualified agreement. Thus, the paper establishes the fact that research in foundation pressures in its broadest sense implies much more than a mathematical study of the distribution of load concentrations through bodies of material indefinite in extent. It is also recognized that such mathematical studies have not yet provided foundation engineering with a practical working theory which completely describes the effect of foundation pressures on the soil masses upon which structures must be built.

The writer proposes to discuss the correlation of pressure distribution studies with a practical working theory of soil resistance which may describe soil behavior more completely and which may provide definite methods of measurement by which variable factors not included in the elastic theory may be evaluated. In its fundamental aspects the analysis of soil behavior does not differ from that of any other problem in mechanics of materials. There are two major steps involved in such a problem: First, it is necessary to determine the stresses imposed on the material by application of load; and, second, it becomes necessary to evaluate the resistance of that material to those stresses in terms of deformation and ultimate capacity to carry load. It makes little difference whether the material is in an indefinite mass or an articulated structure, a complete solution of the problems presented cannot be found until both aspects are subjected to analysis.

Pressure distribution based on elastic theory bears only on the first aspect of the major problem and, being limited to materials of assumed ideal properties, has not entirely fulfilled its purposes even there. In the writer's opinion elastic theory has almost, if not entirely, been exhausted in producing results of practical value, and valuable as these results have been they leave a considerable gap to be filled in before foundation problems can be treated on a rational basis. They have given foundation engineers, however, a number of sound conceptions as to how load may be transmitted through soil masses and the type of stresses that would result. The task of securing recognition of the limitations of the elastic theory and the building of new conceptions which may be free from these limitations are fields for further progress.

Before attempting further development along this line it seems pertinent to define the attitude with which the writer proposes to approach these problems and to set up certain criteria which may have value in testing the validity of theories which may be proposed. These criteria which represent the attitude of modern physics toward natural phenomena are nowhere more aptly presented than in the definition of a theory given by Dushman[41]. According to this eminent physicist, a theory is nothing more than a system of conceptions which will enable one to describe the results of experience and observation, a system which shall be as complete as possible but, at the same time, simple and comprehensive. Born[42] also serves warning that "the true laws of Nature are relations between magnitudes which must be fundamentally observable. If magnitudes lacking this property occur in our theories it is the symptom of something defective."

Accepting this creed, the development of theories concerned with foundation pressures consists of an orderly process of collecting data by careful observation; the analysis and correlation of these data; and, finally, the formulation of conceptions describing the results of observations only in terms of magnitudes which may be subject to direct observation.

It is not necessary to attempt complete citation of the various experimental and mathematical investigations in the light of the author's excellent review of both. The writer will attempt only to discuss conclusions which may be drawn from these investigations and limitations which must be recognized in applying the conclusions to practical problems.

Distribution of Pressure Beneath the Surface.—From a survey of the results of the various experimental investigations it appears that there are two types of pressure distribution curves which show a significant difference. This difference is illustrated in Fig. 12, in which Curve 1 denotes the kind of distribution found on horizontal planes relatively close to the bearing area. It yields definite evidence of a concentration of pressure in the region beneath the bearing area, which may be referred to as the central column of the compression cone. This concentration of pressure is a phenomenon that has been noted in practically all experimental studies. It has generally been described in two ways: First, by increased values of the vertical pressure, p_z, in the central column; and, second, by decreased values of the

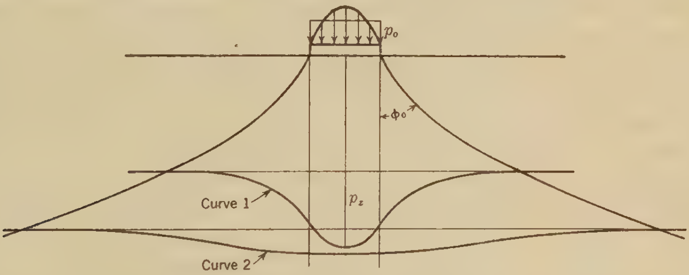

FIG. 12.

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[41] "Modern Physics—A Survey", by Dr. Saul Dushman, *General Electric Review*, June and July, 1930.

[42] *General Electric Review*, June 30, 1930, p. 332.

angle of lateral distribution, ϕ_0, for planes close to the bearing area. Curve 2, which is typical of pressure distribution at greater depths where intensities of pressure are not excessive, shows no evidence of the foregoing concentration of pressure. Intensities beneath the bearing area appear to be more consistent with those in the compression cone outside the central column and the angle of lateral distribution appears to approach a constant maximum value which, by experiment, has been found to be approximately 60 degrees.

The concentration of pressure in the central column is accompanied by a non-uniform distribution of pressure on the bearing area which the author has demonstrated in his analysis of experimental results. Rather than to state that the non-uniform pressure distribution on the contact surface is the cause of the concentration of pressure in the central column, it seems more logical to conclude that both result from some other phenomenon of primary importance. It is at this point that elastic theory becomes quite inadequate and must be displaced by more basic conceptions not limited by the assumptions of elasticity.

In the first place soils do not possess the properties of the perfectly elastic material necessary to satisfy the very important condition of continuity of stress-strain relationship. The fact that the equations of elasticity yield pressure distribution curves which are at least qualitatively adequate does not alter the fact that the quantitative results are of limited value. This situation arises from the fact that distribution of pressure through soil masses follows laws depending upon statistical regularity which may obtain even in an essentially discontinuous medium.

Failure of the elastic theory appears to originate from two distinctly different sources. First, there are discontinuities arising from boundary conditions which are introduced by the problem of treating load applications over finite areas resting on an indefinite body of material. These considerations are entirely a function of dimensions not peculiar to any particular material and, to that extent, are independent of the physical properties of the material. Second, there are discontinuities arising from the manner in which the material develops resistance to stress or rather fails to develop resistance to stress. The latter phenomenon is affected by dimensions of the loaded area, but is primarily related to properties of the material.

Strohschneider[43] was the first investigator to recognize the significance of this particular characteristic of pressure distribution in soils, and he described pressure distribution as being confined to a cone-shaped region of stress, outside of which there was no apparent effect due to the applied load. He also attempted to alter the elastic equations to produce the concentration of pressure in the central part of the compression cone. His attempt to extend these equations was satisfactory only in special cases, as no single continuous function appeared to be flexible enough to satisfy the wide variation in soil behavior.

In the writer's opinion the concentration factor introduced by later investigators and discussed by the author will not prove to be any more satisfactory in the final analysis. The imperfections in the elastic theory as applied to soils are more deeply seated than that. The discrepancies arise from actual discontinuities in behavior, and it appears fairly obvious that no amount of adjustment will fit these phenomena into a framework based fundamentally on continuity.

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[43] See "Pressure Distribution in Building Soil", by Dr. F. Kögler and Dr. A. Scheidig, *Die Bautechnik*, No. 29, 1927.

The lateral distribution of vertical pressure from a finite area is primarily a boundary phenomenon. This may be illustrated by some results obtained experimentally by Kögler and Scheidig[44]. These results, shown in Fig. 13, represent the pressure distribution obtained at some depth beneath a large rigid plate resting on the surface of a sand fill. The dimensions and rigidity of the plate are such that the load distribution at the contact surface is essentially uniform except for some secondary boundary effects which may be neglected in the present illustration. The important point is that at a depth slightly less than one-fourth the diameter of the loaded area the initial pressure, p_0, was transmitted downward without change in the central region beneath the bearing area. The original pressure distribution is shown assuming a uniform pressure, which is approximately true in this case. The manner in which lateral distribution occurs may be described in simple terms and demonstrates that it is essentially a boundary effect.

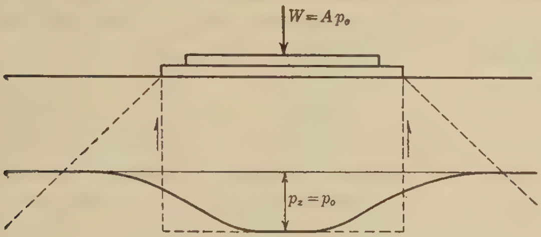

FIG. 13.

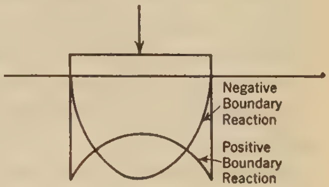

FIG. 14.

If the central column is to be relieved of any of the applied pressure as the load is carried downward this must be accomplished by means of shearing resistance developed on the vertical planes through the edge of the bearing area. The increase in pressure ordinates outside the central column and the decrease in ordinates inside the central column both originate at the boundary and this combined effect extends horizontally in both directions from that plane as the depth increases. As long as the horizontal plane considered is sufficiently close to the bearing surface, the effect from opposite edges does not overlap and the maximum pressure, p_0, may be transmitted downward without change over some width dependent on the limiting angles of influence.

It is also a necessary condition that the areas of the pressure diagram at each depth be equal. The areas taken from the original pressure diagram should be equal to the areas transferred to the region outside the central column. This condition is not entirely fulfilled in the present illustration due to inaccuracy in assuming a uniform distribution of pressure in the absence of more complete information regarding distribution on the contact surface.

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[44] "Pressure Distribution in the Soil", by Dr. F. Kögler and Dr. A. Scheidig, *Die Bautechnik*, 1927, No. 31, pp. 1–3.

One further point to be brought out in connection with Fig. 13 introduces the resistance of the soil itself as influencing the transmission of pressure. If the loading is continued beyond the point that produces shearing stress on the perimeter planes in excess of the shearing resistance of the soil no more pressure can be transmitted to the region outside the central column. It is obvious then that such additional increments of load must be carried by the central column as a concentration of pressure, a condition which may obtain until the central column is loaded beyond its capacity. Thus several distinct stages of behavior are introduced between which the relations involved are quite dissimilar and, therefore, may not be expressed by a continuous mathematical function.

Distribution of Pressure on the Contact Surface.—The importance of the pressure distribution at the surface of contact has been emphasized by the author and in this discussion. The type of pressure distribution depends primarily on boundary conditions and resistance developed by the soil mass; and it varies depending on the relative rigidity of the bearing plate, the dimensions of the loaded area, and the relative intensity of the load. With all these factors involved, the most fruitful line of attack appears to be to consider types of pressure distribution and their relation to the manner in which resistance to load is developed. It may then be possible to formulate methods of evaluating resistance to load which are dependent on direct measurements and independent of quantitative determination of the actual pressure distribution. This seems to be the only feasible method of approach as it is evidently impracticable in general practice, if not impossible, to measure actual pressure distribution on all foundation substructures.

There are two major types of pressure distribution curves which are illustrated in Fig. 14, and which depend upon the type of boundary reaction developed. In general, materials that are cohesive develop stress concentration at the edge of the bearing area which may be described as a positive boundary reaction. As a result the pressure on the bearing area is maximum at the edge and minimum at the center. In this case the boundary is a source of strength and the reaction developed at that point contributes to the bearing capacity in proportion to the perimeter-area ratio. There are also cases in which the boundary is a source of weakness due to inability to develop resistance to vertical pressure which may be expressed as a negative boundary reaction. The resulting pressure distribution gives maximum ordinates at the center of the bearing area and minimum ordinates at the edge. Such a condition obtains at the surface of bodies of granular material and is typical of most of the experimental investigations of pressure distribution that have been made.

The loss of bearing capacity at the boundary of the loaded area is not difficult to explain by the manner in which load is transmitted through a granular mass. Particles in contact with the bearing plate transmit vertical load to other particles below the surface through points of contact. There being no cohesion the only means of transmission is by arching action, with the reaction to horizontal thrust in the elementary arches being furnished by adjacent particles. At the edge of the area and at the free surface the resistance to horizontal thrust is zero, and the mass is incapable of furnishing resistance to vertical load. In the center of the bearing area horizontal thrusts are partly balanced and, consequently, the elementary soil arches have some stability and are capable of carrying higher intensities of vertical load.

The situation described is not generally encountered in practice although it is representative of the conditions under which most experimental investigations have been performed. Substructures and footings are seldom located on an exposed surface, but are located at some depth with sufficient static head, due to overburden, available to furnish a considerable resistance to lateral displacement. Bearing-capacity tests on granular material conducted under the writer's supervision have shown a positive boundary reaction for the range during which there is sufficient resistance to prevent lateral displacement of the granular material at the edge of the bearing plate[45]. The behavior of foundation mats in general indicates that boundary reactions are more often positive than negative, inasmuch as such mats almost invariably become dish-shaped with the concave surface upward. The smaller settlements at the edge of such mats, over which the loads are fairly well distributed, can only be accounted for when the reaction of the soil mass is greater at that point. The writer has encountered very few cases in which foundation mats settled more at the edge than at the center, and these settlements could be traced definitely to heavy load concentration at the edge of the mat.

Somewhat apart from dimensional effects and soil reactions which affect pressure distribution on the contact surface, consideration must be given to rigidity of the bearing plate. All other things being equal, the pressure distribution may vary from one extreme type to the other, depending upon this rigidity. Flexible bearing plates carrying a distributed load will conform in general to the deformation of the soil mass sufficiently close to develop a uniform pressure on the plane of contact. Concentrated loads on flexible plates are generally transmitted through the plates without fully developing the contact area, resulting in high concentrations of pressures directly under the load.

Pressure distributions on rigid bearing areas are largely controlled by reactions in the soil mass regardless of whether the load applied to the plate is concentrated or distributed. Some of the most effective studies of this type of problem have been made by photo-elastic methods[46], an example of which is shown in Fig. 15. A bakelite sheet, pressed on by a rigid steel die, yields a stress pattern with high concentrations at the edges of the die and a point of minimum stress in the center. Similar experiments with flexible plates show that the concentrations of pressure at the edge disappear, and that the distribution may become practically uniform.

In the preceding discussion the writer has attempted to call attention with some degree of completeness to those factors which enter into a determination of stresses in a foundation. Most, if not all, of these factors are fairly well recognized although they may be described in different terms. The attempt has been made in this discussion to place emphasis on certain aspects of pressure distribution which the writer believes may be correlated and built into a system of conceptions that will serve as a practical theory of soil resistance.

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[45] "Subgrade Soil Testing Methods", *Proceedings*, Am. Soc. for Testing Materials, Vol. 34, Pt. II, pp. 730–737.

[46] "Kinematography in Photoelasticity", by M. M. Frocht, *Transactions*, Am. Soc. M. E., Vol. 54, No. 11, p. 83.

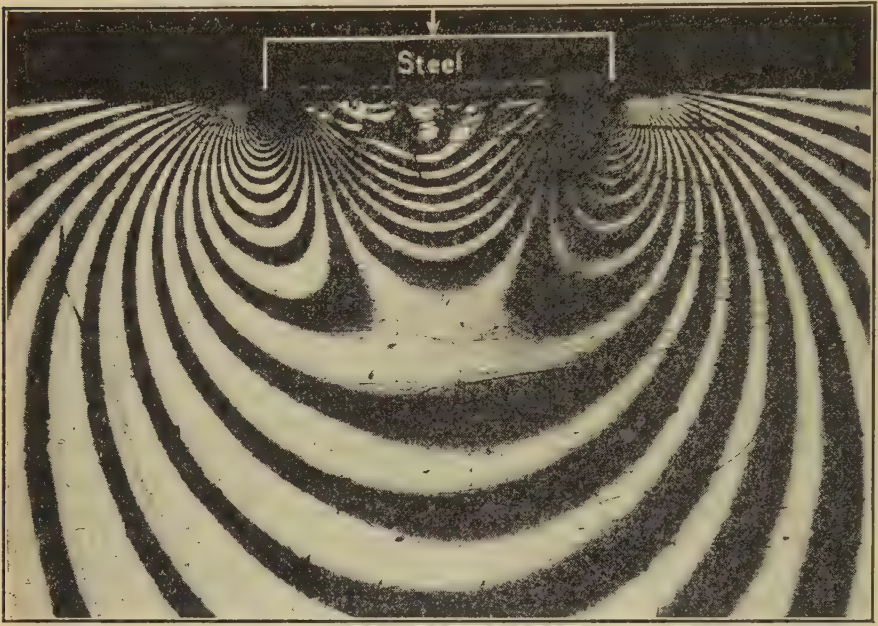

Fig 15.—Stress Distribution by Photo-Elasticity.

Because of the fact that some of the assumptions of elastic theory cannot be applied to soils, the distribution of stress predicated on elasticity cannot be considered satisfactory except in a qualitative sense. These limitations of elastic theory lead one to the conclusion that it is essential to simplify the methods of dealing with the problem, thus making it possible to include some important factors that otherwise may be ignored. It seems entirely logical in a qualitative treatment of pressure distribution to accept such simplification as long as the results are only used to establish other relations which may be determined on a proper quantitative basis. This procedure is particularly necessary in order to describe pressure distribution in regions close to the bearing area where dimensional effects and discontinuities due to several different stages of soil resistance are most critical.

A Simplified Method of Pressure Distribution.—Such a simplified method of pressure distribution is shown in Fig. 16, in which it is proposed to deal with pressure distribution by linear relations combining triangular and rectangular pressure distribution. All the examples in the succeeding discussion will be treated in terms of the two-dimensional problem for a bearing area of finite width, b, and indefinite length. The solution of this problem based on the elastic theory is credited to Michell and has been reviewed[47] by George Paaswell, M. Am. Soc. C. E. The problem will be treated in terms of a uniform pressure, p_o, at the contact surface and the effect of a non-uniform distribution discussed as a separate consideration. The transition from the pressure diagram for the bearing area to the pressure diagram on any horizontal plane is accompanied by satisfying the condition that the area of the respective diagrams is equal, as shown in Fig. 16(b).

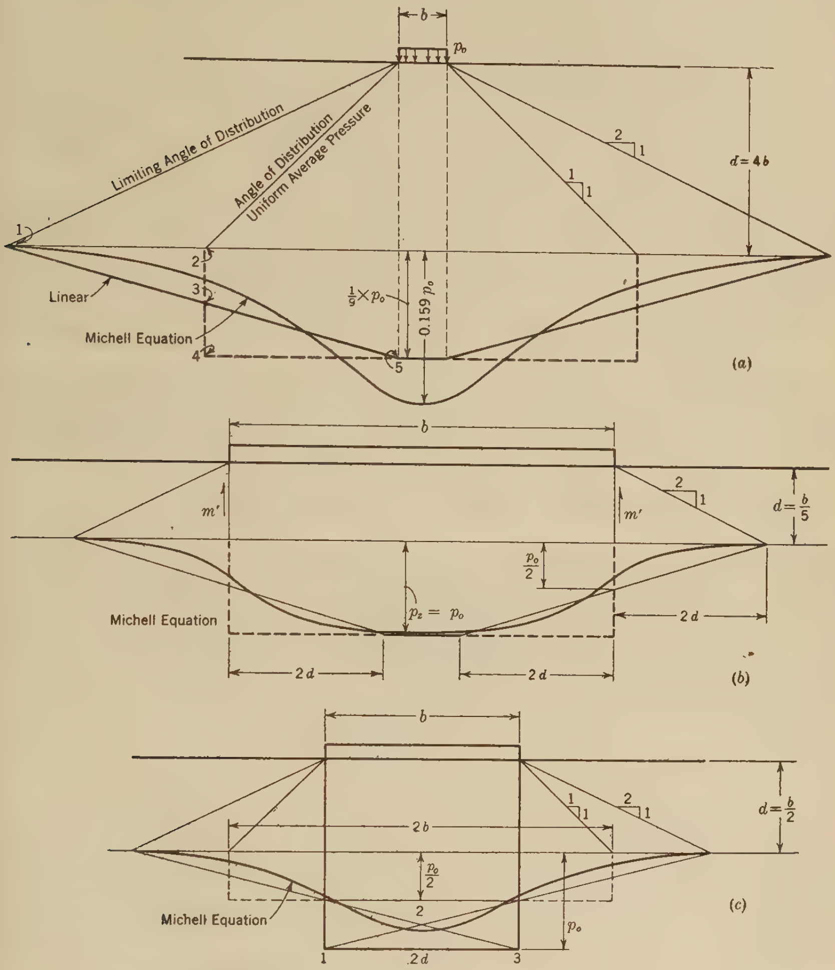
b
p_o
Limiting Angle of Distribution
Angle of Distribution
Uniform Average Pressure
2
1
1
1
d = 4b
1
2
3
4
5
Linear
Michell Equation
1/9 × p_o
0.159 p_o
(a)
b
m'
m'
2
1
d = b/5
p_o/2
p_z = p_o
Michell Equation
2d
2d
2d
(b)
b
1
1
2
1
d = b/2
2b
p_o/2
2
p_o
Michell Equation
1
2d
3
(c)

FIG. 16.

In Fig. 16(a) the distribution curve as given by the Michell equation is replaced by a straight line which varies from zero at the limiting angle of distribution with a slope of 1 on 2 to a maximum ordinate at the vertical plane through the edge of the bearing area. A uniform pressure or rectangular distribution is used under the central column of the compression cone. An average or uniform pressure commonly assumed in practical problems is obtained by a redistribution or juxtaposition of equal triangles, 1-2-3 and 3-4-5. In order that the areas of the pressure diagram shall be the same, the angle of average pressure distribution must be 45° or a 1 on 1 slope which is a value commonly assumed. The proper value of the limiting angle of distribution is an important factor and is taken as a slope of 1 on 2 or 63° 26′ for several reasons. In the first place the experimental investigations have indicated an angle of about 60 degrees. It is improbable that the very small pressures near the limiting zone of influence would be recorded by the measuring devices thus far used; therefore, the difference of 3.5° is not significant. As will be shown subsequently the relation between the angle of average pressure distribution and the limiting angle of influence verifies the slope of 1 on 2 since the equilibrium of elements within the compression cone requires that the average angle of distribution be 45 degrees.

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[47] "Transmission of Pressure Through Solids and Soils and the Related Engineering Phenomena", by George Paaswell, *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 1563.

For the purpose of discussion, accepting the assumption that the distribution angles are correct, three cases of pressure distribution arise, defined by the ratio existing between the width of the bearing area and the depth to the horizontal plane considered. First, if the depth is less than $\frac{b}{4}$ as in Fig. 16(*b*) the full intensity of the applied pressure reaches the horizontal plane over the middle of the central column. Such a distribution takes into account the fact that the lateral distribution is a boundary effect which depends upon the shearing resistance, m', acting on the vertical planes through the edge of the bearing area. The result is also in conformity with results of Kögler and Scheidig shown in Fig. 13. Second, a special case exists in which the depth, d, is equal to $\frac{b}{2}$. As illustrated in Fig. 16(*c*) the same principles are applied in obtaining the pressure distribution as were used in the previous example. Pressure triangles are taken out of the original pressure diagram equal to the pressure distributed on either side of the central column. The triangle, 1-2-3, is subtracted twice, being included in the distribution to both sides and reducing the ordinate of the rectangular part of the diagram to $\frac{P_o}{2}$. The average pressure distribution may be used to obtain the same result, considering the average pressure within a 45° angle of distribution, and changing from the rectangular to the triangular distribution outside the central column in the same manner as indicated in Fig. 16(*a*). Third, in all cases when the ratio is greater than $\frac{b}{2}$ it is necessary first to obtain the average pressure to determine the center ordinate as shown in the diagram, as it is impossible to work directly from the original pressure diagram as illustrated in Fig. 16(*b*).

The comparison between the simplified pressure distribution and the elastic solution by Michell has been made for the purpose of correlating the proposed method with previous investigations. In the light of the limitations of the elastic theory, the writer does not feel, however, that the mathematical solution is entitled to any pre-eminence as a criterion of accuracy as far as quantitative results are concerned. Nevertheless, it is interesting to note wherein the two methods are at variance and, in the light of all factors that should be considered, to discuss the probable accuracy of the results obtained.

For the case in which the depth considered is less than $\frac{b}{4}$, the agreement is quite close, but the Michell solution fails to give a center ordinate equal to p_0 which should be the case if the boundary effects are properly treated. As soon as the boundary effect from either edge overlaps, as for depths greater than $\frac{b}{4}$, the elastic solution gives a center ordinate greater than that obtained from the simplified method. In view of the factors involved, the writer can see little choice in either method in judging the accuracy of the result. The simplified method appears to include the more rational treatment of boundary effects; but at best it is only an approximation. In all cases the elastic solution gives smaller ordinates in the regions more remote from the central column.

When non-uniform distribution on the contact surface is considered in relation to the phenomenon of lateral distribution of pressure the results of the simplified method appear more favorably. In that range of load where shearing resistance on the boundary planes has not been exceeded, distribution on the contact surface would tend toward maximum pressures at the edge and minimum pressures at the center. Translating these conditions into pressure distribution on planes below the bearing area, it is apparent that center ordinates would be decreased and ordinates outside the central column increased in closer conformity with the simplified pressure diagrams. It is felt that this is the case more commonly encountered in practice.

If the pressure distribution on the contact surface is maximum at the center and minimum at the edge as in most experimental results available, this behavior may be traced to a failure to affect lateral distribution at the boundary, and there is a consequent concentration of pressure in the central column. In this case the simplified method does have a real advantage because it affords a simple and logical method of treating concentration of pressure in the central column. The importance of this advantage lies not so much in the determination of pressure ordinates in the central column as it does in the correlation of stress distribution beneath a loaded area with ability to carry load. After all, the determination of stresses in the soil mass is merely the first step in developing methods of evaluating bearing capacity. Thus, from a study of pressure distribution, it has been found that one of the two major types of stress is shown to be concentration of pressure in the central column, and the other, to be resistance developed on the boundary planes of the central column which affects a lateral distribution of vertical pressure.

The problem now becomes one of evaluating the resistance of the soil mass to these two types of stress in terms that may be subject to direct measurement. In Fig. 17 are shown the stress conditions in the compression cone in terms of which it is possible to study the equilibrium of the various elements and establish their ultimate resistance to load.

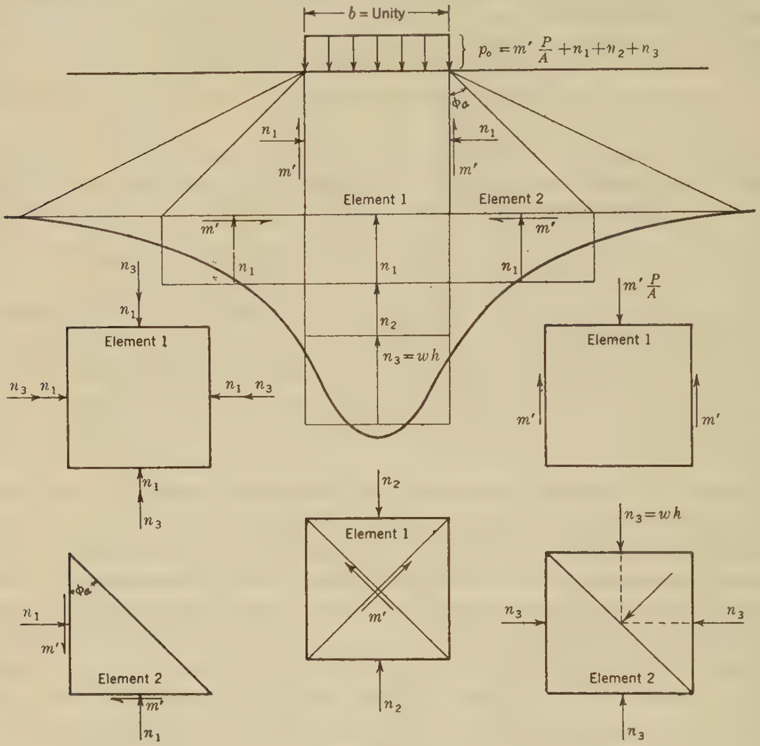

FIG. 17.

Element 1, taken from the central column, offers resistance to deformation by two distinct stress reactions. The first is the summation of three similar pressure components: n_1, the vertical pressure representing cubical compression with lateral components furnished by the confining influence of the surrounding body of material and involving no shearing stresses within the element; n_2, the vertical pressure transmitted downward without increase in lateral components up to the full capacity of the soil to develop shearing resistance on 45° planes of maximum shear; and n_3, a third increment of vertical pressure arising from the static pressure due to surcharge, which in some cases may be available as additional confining influence. All three of these factors are included in a stress reaction which may be defined as developed pressure and represents the capacity of the central column to transmit vertical load directly downward without failure due to lateral displacement.

The second stress reaction must be considered because of the lateral distribution of pressure through the shearing resistance, m', acting on the boundary planes. The vertical load carried in this fashion is a function of the perimeter of the area and may be expressed in terms of the bearing pressure on the area by dividing the total boundary force, $m'P$, by the area, A, as shown in the equation for total bearing capacity, p_0. The equilibrium of Element 2 within the compression cone and outside the central column establishes certain conditions which must be satisfied and leads to some significant relations. As long as the shearing resistance, m', is not exceeded, no concentration of pressure will develop within the central column and the pressure distribution across the compression cone may be represented as uniform pressure, n_1. Element 2 must also furnish the lateral component, n_1, resulting in a shearing stress, m', on the horizontal plane bounding the element. Under the condition that the normal pressures, n_1, acting on both the horizontal and vertical planes of Element 2 are equal, the shearing resistance, m', on both planes must also be equal to n_1. In the first place this condition requires that the angle of average distribution be 45° as no other value of the angle will satisfy all three equations of equilibrium, which require that the summation of horizontal and vertical forces and the summation of moments be zero. The quality of shearing stress on planes at right angles to each other, as stated in Cauchy's theorem[48], also may be used to define the angle, ϕ_a, as having a specific value of 45 degrees. Thus, all the requirements for equilibrium of elements in a body of material are satisfied except the commonly accepted condition of continuity of stress. In the problem under consideration the discontinuity of stress introduced at the boundary of the compression cone is an agreement with experimental observations, particularly the cone-shaped region of stress noted by Strohschneider.[43]

The increments of pressure transmitted as concentrations in the central column do not affect the lateral distribution, as this path of force transmission is cut off when the ultimate shearing resistance of the soil is exceeded on the perimeter planes. The supporting power of the soil is not exhausted, however, until the capacity of the mass to develop pressure is exceeded. Thus, the concentration of pressure in the central column is possible, resulting in a pressure distribution that cannot be represented by an average pressure for the full width of the compression cone. Any attempt to use such an average pressure distribution results in an erroneous value of the angle of distribution. It is important to note that apparent values of this angle of less than 45° are due to a changing proportion of two separate and dissimilar stress reactions rather than to a change in the angle of lateral distribution.

Correlation of Pressure Distribution and Bearing Capacity.—With a complete understanding of the nature of the stresses imposed on a body of soil by foundation loads, it is possible to correlate these stresses with the resistance developed by the soil and to isolate those factors which must be measured in order to evaluate the bearing value of the soil. It is apparent from the review of pressure distribution that two separate stress reactions are involved in the problem and these may be defined as developed pressure and perimeter shear. The bearing capacity, which is the combined effect of these two types of resistance, may be expressed by the following straight-line equation developed in connection with Fig. 17 which is valid for constant value of a third variable, settlement:

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[48] "The Mathematical Theory of Elasticity", by A. E. H. Love, pp. 74 and 79.

$$p = \frac{mP}{A} + n \qquad (26)$$

in which p = bearing pressure, in pounds per square foot; m = perimeter shear, in pounds per linear foot; P = perimeter, in feet; A = area, in square feet; and, n = developed pressure, in pounds per square foot.

The developed pressure, n, is the summation of all the increments of pressure developed by the soil mass and designated as n_1, n_2, and n_3 in the previous discussion. For the purpose of interpreting load tests on soil, it is not necessary to measure the separate effects although the subdivision is useful in some other respects. The developed pressure is the source of the major part of the bearing capacity in practical design, the effect of perimeter shear being small for large bearing areas with small perimeter-area ratios.

The perimeter shear is an important factor, however, in the usual sizes of bearing areas for testing purposes and, unless it is properly evaluated, load tests cannot be used to determine the supporting value of large foundations. The perimeter shear, m, is expressed in terms of a concentrated force per unit length of the perimeter because the boundary effects, either positive or negative, involve irregularities in the distribution of shearing stress on the boundary planes which have not been subjected to analysis.

Evidently, there are two types of behavior involved due to boundary stresses. The primary effect of the shearing stress on the perimeter planes is in the lateral distribution of the pressure which controls the depth affected by the surface load. The larger the area the greater the depth required for distribution and, consequently, the greater the settlement for any given intensity of bearing pressure. This leads to the familiar statement that the settlement for a given intensity of pressure is proportional to the diameter of the bearing area. Conversely stated, it may be said that for a constant settlement the bearing pressure is inversely proportional to the diameter, or that it is proportional to the perimeter-area ratio. This statement is correct in so far as one type of reaction is concerned, but it ignores the fact that developed pressure is an important factor in the bearing capacity and it is to some degree independent of the stress reaction at the boundary.

In addition to the primary boundary stresses, there are secondary stress reactions which lead to the non-uniform pressure on the bearing area. These secondary stress concentrations appear to be independent of the width of the area and are rather a function of rigidity of the bearing area, or as in granular materials, are controlled by a lack of resistance to lateral displacement. It appears impossible at present to measure these boundary stresses except in terms of a concentrated force acting at the perimeter and being the integrated result of the complex reactions which have been recognized.

The analysis of a series of bearing-capacity tests has shown that the straight-line equation does reproduce the load-settlement relations for different sizes of bearing area to a satisfactory degree of accuracy within a range of size for which the secondary boundary effects are similar. However, due to a change in boundary conditions, load tests of bearing areas of somewhat less than 1 sq ft do not conform to the linear equation for larger plates. Although the reasons for this discontinuity are not completely exposed by present data it appears to be due to the secondary boundary reaction which is independent of the width of the bearing area for the larger test sizes. In the smaller sizes the secondary boundary effects from opposite sides of the area overlap, and failure takes place due to lateral displacement before the full supporting value of the soil can be developed. In both cases, however, the only factor affected is the boundary stress or perimeter shear, and the analysis of tests indicates that the developed pressure remains the same for all sizes of bearing areas. Until further investigation has been made, it appears desirable to avoid the discontinuities referred to and this may be done by the proper selection of test sizes. When this procedure is followed, the straight-line equation is valid and furnishes a reliable determination of soil resistance.

Conclusion.—In this discussion limited space does not permit of a more complete elaboration of the conceptions of soil resistance herein presented, and particularly a general relation which correlates bearing capacity with settlement of foundations. Such a general equation has been developed with examples in which the relations were applied to the actual design of substructures.[49] Examples of a number of series of load tests are available in which the factors of soil resistance have been evaluated successfully[50]. Some progress has been made in developing a method of measuring the resistance of cohesive soils and correlating the cohesion or shearing resistance with the concentration of pressure beneath a loaded area, which has been emphasized in the paper[51].

Progress in foundation engineering or, more exactly, soil mechanics has been rapid in the last few years and important contributions to the science have been made by a number of investigators working in widely separated localities. The most difficult problem to-day, in all this development, is for the investigators themselves, to say nothing of others who are interested, to find an opportunity to engage in a free exchange of ideas which is essential for real progress. It is seldom that a paper as comprehensive and thorough as that presented by the author is available for discussion, and in providing this opportunity Mr. Cummings has made a contribution of double value to the profession.

N. M. NEWMARK,[52] JUN. AM. SOC. C. E. (by letter).—The close agreement between the wide range of experimental data gathered by Mr. Cummings and the hypothesis of stress distribution in sand attributed to Professors Griffith and Fröhlich is quite remarkable. It seems that the experimental data are limited to measurements at depths of less than 5 ft, and in all but a few cases at depths of 3 ft or less. It appears reasonable that, at great depths, sand should act more nearly in the manner of an elastic and homogeneous material than at shallow depths. The experimental data cited, for example, give, for a ratio of depth to radius of plate of approximately 9, and for depths of 60 in. and 36 in., the respective values of pressure as 2.5% and 3.5% of the intensity of load applied. Thus, it seems that n possibly decreases with the depth.

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[49] "Bearing Value of Clay is Determinable", *Engineering News-Record*, February 23, 1933, p. 244.

[50] "Bearing Capacity of Clay", *Engineering News-Record*, September 19, 1935, p. 408.

[51] "A Penetration Method of Measuring Soil Resistance", *Proceedings*, Am. Soc. for Testing Materials, Vol. 35, 1935.

[52] Research Asst. in Civ. Eng., Coll. of Eng., Univ. of Illinois, Urbana, Ill.

It does not appear to be convenient to consider a varying value of n in the equations given by the author. However, other empirical relations may be found by which it is possible to take into account the change in properties of the sand with depth and also to account for the proper effect near the surface.

A simple method which the writer has devised seems to offer certain advantages. It consists, briefly, in transforming the depth of the given material in an arbitrary manner, the transformation depending only on the depth, and then applying Boussinesq's equations to find the vertical stress on horizontal planes of the transformed material. The vertical stresses thus obtained are taken as acting at the corresponding untransformed depth and will be in statical equilibrium. Statically possible stresses in other directions may be determined as well, but it is not necessary to discuss this matter herein.

Let the depth to a given horizontal plane in the given material be denoted by z, and the depth after transformation by h. Assume that each element of depth, dz, is to be transformed by the relation,

$$dh = \psi\, dz \qquad (27)$$

in which ψ is a function of the depth, z, alone. Then the transformed depth, h, is given by the equation, $h = \int_0^z \psi\, dz$. When ψ is a constant, the transformed depth is, $h = \psi\, z$.

The value of ψ may be chosen in such a way that the stress at various depths along the line of action of a single concentrated force will be the same as the stress computed for a particular value of n. This value of ψ is found to be,

$$\psi = \sqrt{\frac{3}{n}} \qquad (28)$$

In general, the stress surface for a particular value of n and that for a value of ψ given by Equation (28) will not be the same. However, the difference will be very small, as it apparently must be, since the volumes under each of the stress surfaces are equal, the center heights are equal, and the shapes are somewhat similar. For example, the curves in Fig. 18 show vertical sections through the stress surfaces for $n = 3$, $n = 6$, and for a constant value of ψ equal to $\sqrt{0.5} = 0.707$.

Computations for varying values of ψ are no more difficult than for constant values. Furthermore, all existing tables, coefficients, and formulas for vertical stress due to various given loadings may be used directly merely by taking the value of the transformed depth, h, instead of the actual depth, z. The entire process is essentially the same as a change in the scale of depth, other conditions remaining the same.

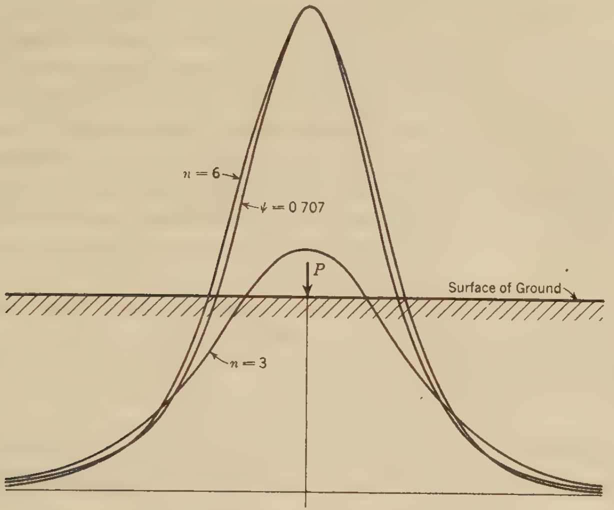

FIG. 18.—RELATIVE FORMS OF STRESS SURFACE.

Close agreement with the experiments cited by the author will also be obtained by use of a transformed depth. For example, in Fig. 4, curves corresponding to Cases II and IV may be obtained from Cases I and III by transforming the vertical scale of these curves by multiplying by the quantity, $\frac{1}{\psi} = \sqrt{2}$. These curves will lie very close to Curves II and IV, and, consequently, will fit the experimental data as well.

Approximate solutions of problems similar to those discussed by the author may also be obtained by various depth transformations. Pressure distribution through sand to clay, or through alternate layers of sand and clay, might possibly be treated. Of course, the proper method of depth transformation must be chosen, and, apparently, there is very little experimental data upon which to base a choice.

A. A. EREMIN,[53] ASSOC. M. AM. SOC. C. E. (by letter).—Basing his computations on the Boussinesq equations, Mr. Cummings has developed some empirical formulas for computing stresses under a foundation resting on sand fill. These formulas yield stresses that are smaller than those obtained from the experiments shown in Fig. 4. The stresses at the center of the boundary area in Cases III and IV are equal to 200% of the stresses produced by uniformly distributed loading.

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In Fig. 4 they are shown, erroneously, as being infinite. Evidently, stresses under a foundation computed on the assumption that the load distribution is in the form of a cone, will check experimental observations closer than those computed for Cases I to IV, inclusive.

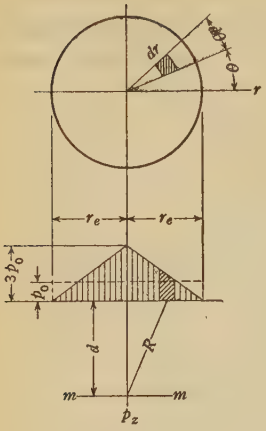

FIG. 19.—CONIC DISTRIBUTION OF LOADING BENEATH A CIRCULAR BEARING AREA.

Case V.—Conic Distribution (n = 3).—In the case of a conic distribution of stress beneath a foundation, the volume of the cone equals the total load and the reaction at the center is equal to $3\,p_0$ (see Fig. 19). The equation of the cone is expressed by:

$$p = \frac{3\,p_0}{r_e}\,(r_e - r)\ldots\ldots\ldots\ldots(29)$$

Substituting Equation (29) and Equation (5) in Equation (3) (with $n = 3$ and $z = d$), the formula for the vertical normal stress becomes,

$$p_z = \int_{r=\theta}^{r=r_e}\int_{\theta=0}^{\theta=2\pi} \frac{9}{2\,\pi}\,\frac{d^3}{(d^2+r^2)^{\frac{5}{2}}}\,\frac{p_0}{r_e}\,(r_e - r)\,r\,dr\,d\theta\ldots\ldots\ldots\ldots(30)$$

Integrating and simplifying:

$$p_z = 3\,p_0\left(1 - \frac{d}{\sqrt{d^2 + r_e^2}}\right)\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

In Equation (31), by making $d = 0$, the pressure at the boundary plane, p_z, becomes equal to $3\,p_0$, or 300% of the pressure for uniformly distributed loading.

Case VI.—Conic Distribution (n = 6).—Substituting Equation (29) and Equation (5) in Equation (3) (with $n = 6$ and $z = d$), the formula for the vertical normal stress is,

$$p_z = \int_{r=0}^{r=r_e}\int_{\theta=0}^{\theta=2\pi} \frac{18}{2\,\pi}\,\frac{d^6}{(d^2+r^2)^4}\,\frac{p_0}{r_e}\,(r_e - r)\,r\,dr\,d\theta\ldots\ldots\ldots\ldots(32)$$

Intergrating and simplifying,

$$p_z = 3\,p_0\left(1 - \frac{d^4}{4\,(d^2+r_e^2)^2} - \frac{3\,d^2}{8\,(d^2+r_e^2)} - \frac{3\,d}{8\,r_e}\tan^{-1}\frac{r_e}{d}\right)\ldots(33)$$

By making $d = 0$ in Equation (33) the pressure at the boundary plane is $p_z = 3\,p_0$, or 300% of the pressure computed for uniformly distributed loading. At the deeper planes the pressures computed for conic distribution of loading, with $n = 6$, are close to the average stresses shown in Fig. 4, which were determined by experiments.

The Boussinesq equations have considerable theoretical value. Various simplified theories of computing the stresses under a foundation are based on them. In computing the stresses under a large foundation, however, a theory based on an assumed pyramid distribution of the stresses has considerable advantage. Paul Müller[54] has computed the stresses and deformations under a foundation in this manner and has found that such stresses and deformations are closely in agreement with those determined by the Boussinesq equations. He assumed a pyramid the sides of which were inclined at an angle of 35° to the vertical. In plastic soils the side angles vary from 35 to 90 degrees.

The problem of computing stresses under a foundation is very complicated and requires extensive experimental and theoretical study. Mr. Cummings has contributed a valuable experimental analysis.

A. CASAGRANDE,[55] ESQ. (by letter).—The question of stress distribution in a semi-infinite elastic body, which was treated so successfully by Boussinesq fifty years ago, has since been extensively elaborated. Unfortunately, most publications on this subject are little known to foundation engineers, in spite of the fact that these solutions could be applied to many problems in the field of earth and foundation engineering. This situation is probably due to the intricate mathematics involved. It is desirable, therefore, that the results of such theoretical findings should be presented to the Engineering Profession in a form easily understood and adapted to ready application.

The author has been successful in presenting, in a concise and clear manner, those results of Boussinesq's studies which are most important in foundation engineering. He has also presented a recent modification of Boussinesq's equations which makes possible a semi-empirical approach to the problem of stress distribution in soils that do not follow Hooke's law; and, finally, he has compared the most outstanding experimental investigations with theory. This comparative analysis of experimental results and theoretical solutions is very enlightening, and its careful study is heartily recommended to those who desire information on the degree of deviation which may be expected between computed and actual stress distribution in a homogeneous mass of soil.

Mr. Cummings has confined himself to considerations of stress distribution within a semi-infinite, elastically isotropic body. Frequently the boundary conditions or the lack of homogeneity of the soil mass is such that the assumption of an isotropic, semi-infinite body represents only a rough approximation. Fortunately, mathematical solutions are available for a number of cases with more complicated boundary conditions, and even for elastically anisotropic materials, restricted only by the assumption of the validity of Hooke's law.

A condition frequently encountered is the presence of a practically incompressible stratum (rock) underlying a compressible soil stratum. Usually, it will be correct to assume the surface of the incompressible stratum to be so rough that no slippage between the two strata can occur. The other extreme would be represented by a frictionless rigid bed. In either case the vertical displacement in the elevation of the rigid surface is reduced to zero by forces acting in an upward direction, thereby increasing the concentration of stresses.

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[54] *Die Bautechnik*, 1934, p. 377.

[55] Prof., Graduate School of Eng., Harvard Univ., Cambridge, Mass.

The presence of thin sand layers within an otherwise isotropic mass of clay has a restraining influence which can be idealized by the assumption of an infinitesimally thin, inextensible, flexible layer.

Following a suggestion by the writer, M. A. Biot[56] made a comparative study of the effects of such discontinuities on the stress distribution for point and line loading. For this analysis he used an original approach, although some of the solutions were found before by Michell, Melan, and Marguerre.

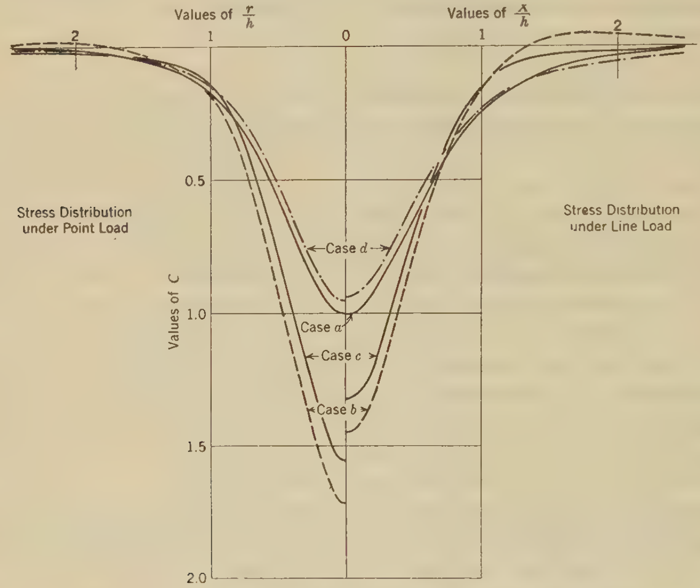

FIG. 20.

The results are shown in Fig. 20, in which the stresses are plotted to such a scale that the maximum stress for Boussinesq's solution, is taken as unity. For the case of a vertical point load, P, the maximum normal stress on a horizontal plane at a depth, z, is expressed by:

$$\sigma = C \frac{3\,P}{2\,\pi\,z^2} \qquad (34)$$

in which, in addition to the notation of the paper, σ = the maximum normal stress on a horizontal plane; and C = a constant with values as shown in Table 3. For the case of a line load, $\bar{p}$, per unit length, the maximum normal stress on a horizontal plane at depth, z, is expressed by:

$$\sigma = C \frac{2\,\bar{p}}{\pi\,z} \qquad (35)$$

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[56] "Effect of Certain Discontinuities on the Pressure Distribution in a Loaded Soil", by M. A. Biot, *Physics*, December, 1935.

It will be noted that Equations (34) and (35) are built up so that they consist of the product of a numerical constant and another term which is identical with Boussinesq's solution. Thus, these constants immediately indicate the degree of deviation which the presence of the boundary produces if compared with the stresses in the semi-infinite homogeneous body.

TABLE 3.—VALUES OF C IN EQUATIONS (34) AND (35)

Case (see Fig. 20)	Description	Equation (34)	Equation (35)
a	Boussinesq's distribution	1.000	1.000
b	Frictionless rigid base	1.711	1.441
c	Rough rigid base	1.557	1.291
d	Inextensible flexible layer	0.942	0.935

The relative amount of concentration caused by the presence of an incompressible sub-stratum decreases with an increasing ratio of the width of the loaded area to the depth of the rigid bed. When this ratio is greater than unity, the stresses beneath the interior of the loaded area approach those computed by the integration of Boussinesq's solution.

Although the presence of a single, inextensible layer causes only a small reduction in the maximum stresses, it is likely that the presence of many such layers (as is the case, for example, in certain varved clays) would cause an appreciable spreading of the load, if compared with Boussinesq's distribution. According to Professor Biot it would be possible to solve such cases, although it would require a large number of computations. In view of the fact that such computations need to be made only once, and if available in published form, could readily be applied, they would represent a worth-while contribution to soil mechanics.

Another way to approach the problem of determining the stress distribution in stratified soils is by assuming an anisotropic material possessing a greater modulus of elasticity in the direction parallel to the planes of stratification. Solutions for anisotropic, elastic materials, corresponding to those of Boussinesq, have been found by Michel[57] and Wolf[58].

If, for example, a soil stratum consists of alternating layers of sandy silt and clay, the moduli of elasticity, E_1 and E_2, of each of these materials can be determined from tests on undisturbed samples. Assuming, for the sake of simplicity, these layers to be of equal thickness, then the average moduli of elasticity in a horizontal and a vertical direction for the entire stratum can easily be derived mathematically, and are found to be:

$$E_h = \frac{E_1 + E_2}{2} \quad \text{.........................(36)}$$

and,

$$E_v = \frac{2\,E_1\,E_2}{E_1 + E_2} \quad \text{.........................(37)}$$

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[57] *Proceedings*, London Math. Soc., Vol. 32, 1901.

[58] *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 15, No. 5.

Equations (36) and (37) become somewhat more complicated if the thicknesses of the alternating layers are not assumed to be equal. For example, if $E_1 = 1\,000$ kg per sq cm, and $E_2 = 100$ kg per sq cm, Equations (36) and (37) yield $E_h = 550$ kg per sq cm, and $E_v = 182$ kg per sq cm, respectively. When substituting these values in the formulas derived by Wolf[58] for a line load, one arrives at a maximum stress along the center line which is only about one-half the value for such a stress in an isotropic material. It is true that integration over a finite area and relatively shallow depth results in a smaller reduction of the stresses. However, even an average reduction of 20% or 30% in the stresses in that part of the soil which contributes most to the settlement of a structure, is of great importance in a settlement analysis, particularly when the soil is preconsolidated under loads greater than the present over-burden.

Such cases are much more difficult to evaluate analytically where a thick stratum of sand or gravel overlies a stratum of compressible soil, because the assumption of an average modulus of elasticity for the sand layer deviates materially from the actual conditions. The modulus of elasticity within the sand layer increases in proportion to the depth, and there is an additional increase in its magnitude beneath the loaded area, due to the stresses produced by the load. Taking all factors into account, the modulus of elasticity in the sand layer is largest below the center of the loaded area.

For such conditions, computations based on Boussinesq's stress distribution may lead to estimates of amount and distribution of settlements far in excess of those actually observed. The writer has had opportunity to study a number of such cases and he has noticed invariably that such sand and gravel layers, overlying compressible strata, reduce the magnitude of the stresses in the compressible layer by spreading the load, and are very effective in smoothing out those differential settlements which would take place if Boussinesq's equations were applicable.

An analytical approach to the problem of stress distribution in a stratum of clay that is overlaid by a stratum of granular soil is very difficult. For an approximate solution one may replace the actual thickness of the harder, upper stratum by a thicker layer of the softer underlying soil, whereby the assumed increase in thickness is a function of the ratio of the moduli of elasticity of both soils, a method suggested by Terzaghi[59].

In connection with an investigation of the probable settlements of an extension to an existing utility structure, of which accurate settlement records were available, the writer has applied the foregoing method, as well as Equation (3) in the paper, with n-values less than three, to obtain computed settlement curves which would correspond in shape to the observed settlements. These buildings are resting on a layer of sand 30 ft thick, that is underlaid by 100 ft of clay. The results obtained with both methods are practically identical. One might argue that, from a purely theoretical standpoint, the method of increasing the thickness of the upper stratum is preferable because Boussinesq's formula permits the application of the law of superposition.

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[59] "Bodenpressung und Bettungsziffer", von Charles Terzaghi, M. Am. Soc. C. E., *Oesterreichische Bauzeitung*, No. 25, 1932.

However, when one considers the large variations of the modulus of elasticity within the sand stratum, both methods are equally objectionable from a theoretical point of view.

The actual settlements of the edge of the loaded area are larger in relation to the settlement of the central part, than according to either of these approximate methods. Due to the larger stresses beneath the central part, the modulus of elasticity in that region of the sand is large and hence individual loads are distributed over a wider area, than the same loads near the edge, where the smaller stresses mobilize a smaller modulus of elasticity. Furthermore, the distribution of the loads near the edge is unsymmetrical with the greater concentration on the outside of the loaded area. Although it is not possible to consider these variations by either of the aforementioned approximate methods, the writer has found that one can arrive at satisfactory solutions by assuming the loads in the central part acting at a higher elevation than the loads along the edge, with a gradual transition for loads intermediately situated. This procedure is similar to the method in which the thickness of the harder stratum is increased, except that the increase is not constant, but a minimum at the edge. The writer found that, for a layer of sand overlying a medium stiff clay, the loads may be assumed distributed over a thickened layer, the upper border forming a semi-ellipse with the horizontal axis equal to the width of the loaded area, and the vertical axis equal to one-half the thickness of the sand stratum. This simple rule is, of course, subject to modification as more observation data become available.

The objection may be raised against this solution that the surface on which the loads are applied is not plane, and that, therefore, Boussinesq's formulas do not apply. However, it must be remembered that as soon as one deals with sand, none of the suggested methods has any scientific value, but that these methods only represent rules-of-thumb for arriving at stress distributions which are similar in character to those actually observed. There would be neither more nor less reason to assume a probability function for such a distribution, rather than Boussinesq's equation, or the modification by Griffith-Froehlich, if it can be shown that such a function also approaches the observed distributions. If one tried to retain Boussinesq's formulas, it would be merely for the sake of convenience, since very useful numerical and graphical solutions are available for such formulas.[60]

Stress computations of the foregoing type are used, in combination with the necessary data on the physical properties of the soils, for analyzing settlements and the stability of foundations.

For a settlement analysis one is often satisfied to compute the volume decrease due to the normal stresses on horizontal planes. This approach is simple but involves two approximations: First, that the rate of volume decrease is equal to the volume decrease observed in a compression test with lateral confinement (consolidation test), and is independent of the distortion of the mass; and, second, that the effect of the normal stresses on vertical planes, and of the shearing stresses on volume decrease, can be neglected.

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[60] Influence table by Glennon Gilboy, Assoc. M. Am. Soc. C. E., pub. in the Progress Report of the Committee on Earths and Foundations, *Proceedings*, Am. Soc. C. E., May, 1933, p. 781; tables for normal and shearing stresses, for various load conditions, by L. Jürgenson in "The Application of Theories of Elasticity and Plasticity to Foundation Problems", *Journal*, Boston Soc. of Civ. Engrs., July, 1934; and graphical solutions for distribution of normal stresses and deformations under rectangular loaded areas, by W. Steinbrenner, in "Tafeln zur Setzungsberechnung", *Die Strasse*, No. 4, 1934; and influence tables for the same case by N. M. Newmark, Jun. Am. Soc. C. E., in "Simplified Computation of Vertical Pressures in Elastic Foundations", *Bulletin*, Univ. of Illinois, Vol. XXXIII, No. 4, 1935.

Regarding the first assumption, in 1931, the writer[61] succeeded in demonstrating with a new type of shearing apparatus that the compressibility of clays is increased by simultaneous deformation. Unfortunately, no data are as yet available as to the amount of this increase for undisturbed clay, and for the magnitude of distortion as normally encountered in the soil underlying structures. The writer believes that for such conditions this influence is negligible.

The second assumption leads to errors which, for the majority of foundation problems, are tolerably small. Besides, in many cases, where the loading is simple, it is not necessary to introduce such an approximation, because it is possible with equal, or even with less, work to apply directly the formulas for strain which Boussinesq and others have derived. This procedure will be illustrated by the following example which the writer has used for instruction purposes since 1932.

A circular area with the diameter, D, resting on the surface of a semi-infinite body with a constant modulus of elasticity, is loaded with p per unit of area. The theory of elasticity gives for the total displacement, Δ, of the center of the area (settlement) the following formula:

$$\Delta = \frac{p}{E} \times \frac{m^2 - 1}{m^2} \times D \qquad (38)$$

in which m represents Poisson's number.

For the extreme cases of a perfectly incompressible material, with $m = 2$, and a perfectly compressible material, with $m = \infty$, the displacements are as follows:

For $m = 2$,

$$\Delta = \frac{3\,p\,D}{4\,E} \qquad (39)$$

and for $m = \infty$,

$$\Delta = \frac{p\,D}{E} \qquad (40)$$

The settlement in the center, therefore, corresponds to the decrease in length of a cylinder of the same material, loaded with the same unit load, and of a height equal to 0.75; or, 1.00 times the diameter of the area.

Although in most materials, deformation and volume change cannot be separated, their relation being expressed by Poisson's number, such a separation is possible for clays. If stresses are introduced into a mass of plastic soil it will first deform without volume change, and, subsequently, a gradual volume decrease will take place at a rate which is dependent on the dimensions, drainage, and consolidation characteristics of the mass of soil.

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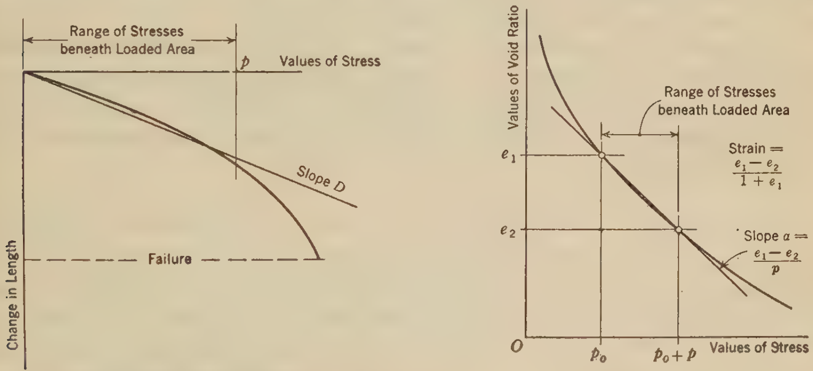
[61] "Research on the Shearing Resistance of Soils", by A. Casagrande and S. G. Albert, Mass. Inst. Tech., 1932; see, also, "The Shearing Resistance of Soils", by L. Jürgenson, *Journal*, Boston Soc. of Civ. Engrs., July, 1934; "New Facts in Soil Mechanics from the Research Laboratories", by A. Casagrande, *Engineering News-Record*, September 5, 1935; and "Characteristics of Cohesionless Soils Affecting the Stability of Slopes and Earth Fills", by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., January, 1936.

The settlement due to immediate deformation, without volume change, can be found by utilizing the stress-strain diagram from an unconfined compression test on a undisturbed sample of the clay. A typical result of such a test is shown in Fig. 21(*a*). In the majority of cases the stress-strain curve for the range of the stresses which are introduced into the soil by the load, can be very well replaced by a straight line. The slope of this line is designated as the modulus of deformation, M. In its effect this modulus corresponds to the modulus of elasticity used in the theory of elasticity. With the assumption of a straight-line relationship with the same slope in the entire mass of soil one fulfills all requirements to permit the application of the theory of elasticity. Since for the deformation without volume change, $m = 2$, one arrives at the settlement in the center of the circular area, due to deformation only, by replacing, in Equation (39), the modulus of elasticity, E, by the modulus of deformation, M.

The subsequent settlement, due to gradual volume decrease, can be determined with the help of the pressure-void ratio relationship shown in Fig. 21(*b*), which is obtained from a consolidation test (confined compression test) on an undisturbed sample. Since a homogeneous mass of soil is assumed, the entire mass is preconsolidated under the same load, p_0. The loading raises this intrinsic stress in the soil in varying amounts, to a maximum of $(p_0 + p)$. For this range of stresses the change in volume may be represented with sufficient degree of accuracy by a straight line with the slope equal to the coefficient of compressibility[62], $a = \dfrac{e_1 - e_2}{p}$. From this value the corresponding ratio of stress over strain is derived, thus:



(*a*) UNCONFINED COMPRESSION TEST (*b*) CONSOLIDATION TEST

FIG. 21.

$$A = \frac{p}{\dfrac{e_1 - e_2}{1 + e_1}} = \frac{1 + e_1}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

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[62] "Erdbaumechanik", von Charles Terzaghi, M. Am. Soc. C. E., Vienna, 1925.

which is defined as the modulus of volume change. This modulus corresponds to the modulus of elasticity of a material following Hooke's law, and with $m = \infty$. The only assumption made in this analysis is the same as that made previously, namely, that the rate of volume change is independent of deformations.

Thus, the consolidation of the mass of clay has been reduced to the behavior of a material following Hooke's law, and with $m = \infty$. Therefore, the settlement of the circular area, due to consolidation only can be found by replacing in Equation (40) the modulus of elasticity, E, by the modulus of volume change, A. Hence, the total settlement of the center of this area will be:

$$\Delta = \left(\frac{3}{2}\frac{1}{M} + \frac{2}{A}\right) p\, D \quad \ldots\ldots(42)$$

In a similar manner one can derive formulas for the settlement of other points on the surface, and for other areas.

In closing, the writer wishes to note that, in his experience, settlement due to consolidation in genuine clays can be analyzed and predicted, on the basis of tests on undisturbed samples, with a satisfactory degree of accuracy. However, for slightly plastic silt clays, non-plastic silts, and certain organic silts, no amount of care can prevent sufficient disturbance to the samples to cause an appreciable increase in compressibility. For such materials, predicted settlements have frequently been too large.

The immediate settlement due to deformation of clay under large loaded areas was found by the writer to be always less than the values computed on the basis of unconfined compression tests on undisturbed samples. Sometimes, this discrepancy has been very large. The cause of this behavior is, at present, unexplained.

A. E. CUMMINGS,[63] ASSOC. M. AM. SOC. C. E. (by letter).—The discussion has served to bring out at least three important phases of the problem of the distribution of stresses through soils. These appear to be: (1) Questions of isotropy and Hooke's law; (2) the distribution of pressure on the contact surface; and (3) the effect of depth on the stress distribution. The writer proposes to devote this closing discussion to a consideration of these three topics.

Before considering the various topics mentioned during the discussion, however, the writer wishes to demonstrate the derivation of the stress equations for another type of surface-load distribution as shown in Fig. 22. To distinguish it from the parabolic distribution of Fig. 3(*b*), this will be referred to as an inverse parabolic distribution, varying according to the equation:

$$p = 2\, p_o \left(\frac{r^2}{r^2_e}\right) \quad \ldots\ldots(43)$$

so that the volume generated by rotating the shaded area around the vertical axis is equal to the volume of the circular disk of Fig. 3(*a*).

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[63] Dist. Mgr., Raymond Concrete Pile Co., Chicago, Ill.

Case VII.—Inverse Parabolic Distribution ($n = 3$).—Substituting Equation (43) and Equation (5) into Equation (3), with $n = 3$ and $z = d$, gives:

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{3}{2\pi} \, 2 \, p_0 \left(\frac{r^2}{r_e^2}\right) \frac{d^3}{(d^2 + r^2)^{\frac{5}{2}}} \, r \, dr \, d\theta \qquad (44)$$

With the method used by Professor Krynine in the derivation of Equation (25), the integration and simplification of Equation (44) leads to:

$$p_z = p_0 \left[4\,a^2 - \frac{4\,a^3}{(a^2 + 1)^{\frac{1}{2}}} - \frac{2\,a^3}{(a^2 + 1)^{\frac{3}{2}}} \right] \qquad (45)$$

Case VIII.—Inverse Parabolic Distribution ($n = 6$).—In a manner similar to Case VII, the substitution of Equation (43) and Equation (5) into Equation (3), with $n = 6$ and $z = d$, can be shown to give:

$$p_z = p_0 \left[\frac{a^2 + 3\,a^4}{(a^2 + 1)^3} \right] \qquad (46)$$

It is easily seen by substitution of $a = 0$ into Equations (45) and (46) that the vertical normal stress on the vertical center line at the ground surface is zero. This is in accordance with the load condition of Fig. 22. The manner in which this stress varies with the depth is of considerable interest and the graphs of Equations (45) and (46) are shown in Fig. 23.

Isotropy and Hooke's Law.—This subject is mentioned by Messrs. Williams, Krynine, Biot, Feld, Paaswell, Housel, and Casagrande. As is well known, Hooke's law is a very simple statement that "stress is proportional to strain." Isotropy is not quite so easily defined. Professor Krynine uses the terms, "monotonous" and "statistical", to describe different degrees of isotropy. His conception of a "continuum" and the number of particles in a given volume connotes the idea that in some way isotropy is related to density. Mr. Feld describes an isotropic body as one "having the same physical properties in all directions." The writer prefers Mr. Feld's statement, and believes that the chances of misunderstanding will be lessened materially by a strict adherence to the definition of isotropy commonly used in the mathematical theory of elasticity. Max Planck[64] states that,

> "The necessary and sufficient condition for a body to be elastically isotropic—that is, that it should have no favored directions at all—is that its elastic constants should all be invariant with respect to any change of the coordinate system, or, what amounts to the same thing, that it can be made to coincide with itself by means of any arbitrary rotation. * * * Hence, it follows that the elastic potential of an isotropic body has only two constants which are independent of each other * * *."

It is the writer's belief that this definition precludes the idea of different "degrees of isotropy." A body is either isotropic, or it is not, and if it is not isotropic it is anisotropic or aeolotropic.

With the exception of Mr. Paaswell, all who contributed to the discussion appear to be agreed that soils do not always follow Hooke's law and that they are not isotropic in the sense in which this word is defined in the mathematical theory of elasticity. Mr. Paaswell states that the use of the Boussinesq equations "in foundation design involves the same degree of accuracy as the use of the ordinary Bernoulli formulas for flexure." These flexure formulas, of course, are based on the assumption that the displacements are infinitesimal and that the material in question follows Hooke's law. However, in the closing sentence of his discussion, Mr. Paaswell mentions the horizontal normal stresses as well as the vertical normal stress which is represented by Equation 2(c). The equations for these horizontal normal stresses include the elastic constants of the material and the sum of the three normal stresses is one of the well-known invariants of the mathematical theory of elasticity. In other words, in an elastic isotropic solid obeying Hooke's law and subjected to infinitesimal displacements which are within the elastic

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[64] "Introduction to Theoretical Physics", Vol. II, p. 66.

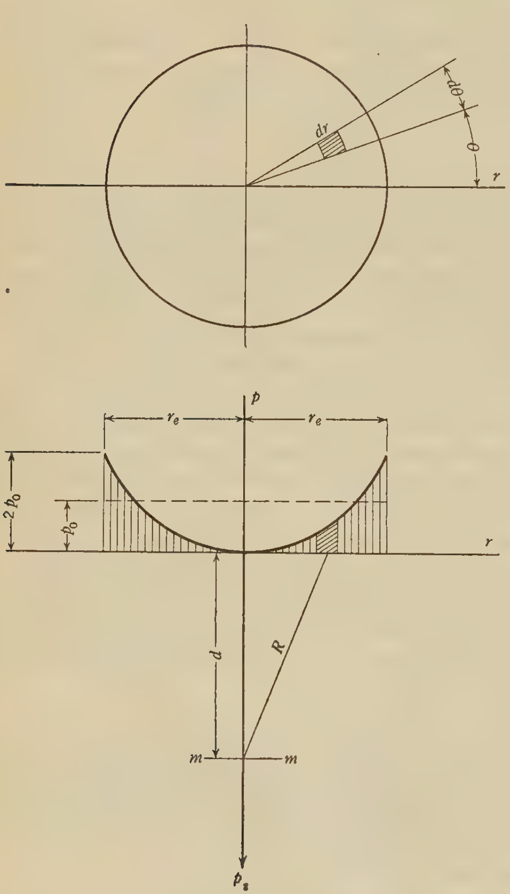

FIG. 22.—INVERSE PARABOLIC DISTRIBUTION.

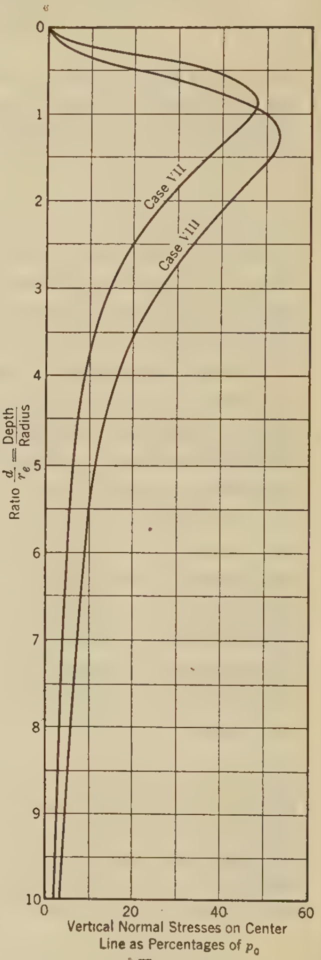

FIG. 23.

limits of the material, the stress system at a point within the solid includes three normal stresses and two of these stresses contain elastic constants.

For ordinary soils, Mr. Paaswell states that the terms containing these elastic constants will vanish. However, the writer is unable to understand how this problem can be expected to work both ways. According to Boussinesq, two of the three normal stress equations contain elastic constants. With the terms containing the elastic constants eliminated from the stress equations, it is evident that they are no longer those given by Boussinesq. It appears, therefore, that Mr. Paaswell agrees with the writer's Conclusion (2), namely, that the equations of the theory of elasticity must be modified before they can be applied to soils.

If it is taken for granted that soils do not obey Hooke's law and that they are not elastically isotropic, there arises the question as to what is to be done about calculating stress distributions in soils. In the writer's opinion there are several possible methods of approach: (1) By assuming that soils are anisotropic; (2) by modifying the equations for elastic isotropic solids; and (3) by eliminating the theory of elasticity in favor of a "rational method".

(1) The problem can be attacked on the assumption that soils are anisotropic and that it is necessary to use additional elastic constants to represent the variations of the elastic properties of the soil in different directions. This method has been demonstrated by Professor K. Wolf.[65] It is also the method used in the study of the elastic behavior of crystals.[66]

(2) The equations for elastic isotropic solids may be modified with parameters in the manner demonstrated by Griffith[9] and Froehlich.[10] The numerical values of the parameters are to be determined by comparison with experiments. This is the method used by the writer.

(3) The theory of elasticity may be eliminated entirely and the problem attacked on what may be termed a rational basis. This method is represented by the procedure outlined by Professor Housel who attributes considerable importance to surface phenomena, particularly the edge effect at the perimeter of a footing. Professor Housel's method is similar to that developed by Froehlich[67] and referred to by him as "die kritische Randbelastung." In this method the coefficient of internal friction, ϕ, plays an important part.

The interesting fact about these several methods of attacking the problem of stress distributions in soils is that, to some extent, they appear to be related in various ways. By his Equation (21), Professor Krynine shows a relationship between the Griffith-Froehlich n and Terzaghi's coefficient of pressure at rest, K. The coefficient K, in turn, is related to the coefficient of internal friction, ϕ. Mr. Newmark has developed a function, ψ, which he expresses in terms of n by Equation (28). With the aid of the Rankine stress circle, Froehlich[68] has shown a relationship between the concentration factor, n, and the coefficient of internal friction, ϕ. The same writer[69] has demonstrated mathematically that the concentration factor, n, is more or less than 3, depending on whether the elastic modulus of the soil increases or decreases with the depth. By means of Fig. 16, Professor Housel compares his solution with that of Michell which is based on the elastic theory.

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[65] "Ausbreitung der Kraft in der Halbebene und im Halbraum bei Anisotropen Material", *Zeitschrift für ange, Math. u. Mech.*, 1935, Vol. 15, No. 4, p. 249.

[66] "Mathematical Theory of Elasticity", by A. E. H. Love, Fourth Edition, p. 159.

[67] "Druckverteilung im Baugrunde", p. 83.

[68] "Drukverdeeling in Bouwgrond", *De Ingenieur*, April 15, 1932, p. B–60.

This problem presents an interesting field for further study particularly by means of large-scale experiments on actual structures. Information available on the subject at present is insufficient to enable one to state that any particular method of calculating stress distributions is to be preferred to all others.

Distribution of Pressure on the Contact Surface.—This problem has been discussed by Messrs. Williams, Krynine, Wilcoxen, Findley, Feld, Salter, Housel, and Eremin. There is no question but that it is an important factor in the behavior of foundations. The subject is not fully understood, and yet, of all the unsolved problems of foundation engineering, it is probably the least difficult to investigate experimentally on a large scale. It is neither difficult nor expensive to bury pressure cells under full-sized structures in order to determine actual pressures in the plane of contact between foundation and soil. Pressure readings continued over a period of time would show the nature of any changes that might occur in these surface-pressure distributions.

In his discussion of the problem, Mr. Wilcoxen presents Fig. 9(b), showing a surface distribution which is a maximum at the edges and zero in the center of the plate. Within the solid he shows a stress distribution in which the vertical stress on the vertical center line reaches a maximum of 300% of the average surface load. This maximum is shown at a depth of about one-half the diameter of the plate. Mr. Wilcoxen implies that an error has been made in the interpretation of this experiment, and that Fig. 9(b) represents the true explanation.

The distribution of surface pressure in Fig. 9(b) is the same as that of Fig. 22. The vertical normal stress on the vertical center line, for the load condition of Fig. 22, is shown in Fig. 23. It is easily seen from Fig. 23 that at no point on the vertical center line is this stress anywhere near 300% of the average surface load either in the elastic isotropic solid ($n = 3$) or in sand ($n = 6$). The distribution of surface pressure and the stress distribution within the solid, as shown in Fig. 9(b), are not compatible. The experiment performed by Mr. Wilcoxen is difficult to interpret, but it appears to have been an impact experiment rather than one of static pressure. The thickness of the steel test plate is not given so that it is impossible to state whether or not the plate would act as a rigid body under the blows of the fence-post.

However, some very careful experiments have been performed by Kögler and Scheidig[70] for the purpose of determining the distribution of pressure in the contact plane between a rigid body and a bed of sand. For the rigid body, they used a heavy circular block of concrete, 63 cm (24.8 in.) in diameter. Pressure cells were embedded in the lower face of the block so that the variation of pressure over the contact plane could be determined. At the same time, pressure cells were placed in the sand bed at a depth f 40 cm (15.7 in.) The block was loaded and simultaneous readings were made of the pressure distribution at the surface and of the stress distribution in the sand. The results obtained in these experiments agreed with Mr. Wilcoxen's Fig. 9(*a*). In the contact plane the pressure was a maximum at the center. Mr. Wilcoxen states that "foundation flexibility is the key to the resulting type of surface stress distribution." In the writer's opinion this is only half the problem. The other half depends on the elastic properties of the material on which the foundation is built.

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[69] "Druckverteilung im Baugrunde", p. 90.

[70] "Druckverteilung im Baugrunde", *Die Bautechnik*, November 29, 1929, Heft 52, p. 828.

A factor in this problem that is sometimes overlooked is the horizontal friction force that may be developed in the plane of contact. The point load, *P,* in Fig. 1, is normal to the surface of the solid. When Equation (3) is integrated over a finite area there is an implied assumption that the pressure under the finite area is normal to the surface of the solid and that no horizontal forces are acting in the contact plane. In the case of an actual footing, it is almost certain that, under some conditions, appreciable horizontal forces are generated in the contact plane. These forces affect the stress distribution within the solid. Almost no information is available as to the nature and magnitude of these horizontal forces under a foundation or even under a test plate. However, Boussinesq[71] has discussed several theoretical solutions involving horizontal forces applied at the surface of the solid in various ways.

Messrs. Findley and Salter have discussed this question from the point of view of the practical designing engineer. Serious complications are involved, of course, in dealing with the general problem of a given foundation on a given soil. There is also the question of how far a practical designer should go in his efforts to determine the probable distribution of pressure on the contact plane. At present, the practical designer is not able to go very far in this direction due to the lack of field data as to the actual pressure distributions that exist under full-sized structures. Without these data with which to check the theoretical analysis, the problem is still more or less in the field of speculation as far as actual foundations and actual soils are concerned.

In this connection it is to be noted that Boussinesq's methods do not lead to a solution of this general problem. The surface load distributions shown in Figs. 3(*a*), 3(*b*), 19, and 22, are not to be confused with actual plates or footings. They are simply normal loads distributed over a part of the surface of the solid. If these surface distributions are known, Boussinesq's methods of applying potential functions can be used to determine the distribution of stresses within the solid. If, instead of these surface pressures, the displacements at the surface are given, Boussinesq's analysis can also be applied, and the stress distribution within the solid can be found from the surface displacements.

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[71] "Application des Potentiels", p. 72.

However, in the general problem, neither the pressure distribution nor the surface displacements is given. A footing with certain elastic properties is to be placed on a soil with other elastic properties. The footing is to be loaded by means of a concentrated column load, or otherwise. The available methods for attacking this problem and the solution of a practical problem of this type, were outlined in 1935 by Froehlich.[72] The solutions for a number of cases involving circular plates carrying various types of loads have been published by Dr. Ing. Ferdinand Schleicher.[73] The equations for stresses and displacements include the elastic properties of the plate as well as the elastic modulus of the foundation material.

The Effect of Depth on Stress Distribution.—This subject was mentioned several times during the discussion. Professor Krynine states that, "probably at a certain depth within the earth all matter obeys the Boussinesq law * * *." Mr. Newmark calls attention to the fact that "* * * the experimental data are limited to measurements at depths of less than 5 ft * * *", and that, "* * * at great depths, sand should act more nearly in the manner of an elastic and homogeneous material * * *".

It is true, of course, that the physical properties of soils depend to some extent on the pressures under which they exist. In sand, it is generally agreed that the elastic modulus increases directly with the depth. For a homogeneous sand bed of great depth, the elastic properties at a depth of 50 ft would certainly differ from those at a depth of 5 ft. It is a question, therefore, as to how far the experiments may be extrapolated for application to full-sized structures. As has been shown, the experiments indicated a value of about 6 for the concentration factor, n. Whether or not a stress concentration factor as high as 6 would apply to a full-sized structure on a deep bed of sand is largely a matter of speculation. It seems probable that, in a bed of compact sand of indefinite depth, the average stress concentration factor might not reach a value of 6, although it would be more than 3.

In discussing this question of depth, Mr. Paaswell mentions Boussinesq's theory of "local perturbations" and states that, "the manner of loading a foundation and the soil pressures near its loading can generally be ignored in the determination of the stress distribution in the deep strata * * *". This is correct if the strata under consideration are deep enough. However, when an analogy is drawn between the foundation problem and the girder problem, as is done by Mr. Paaswell, there is one important factor in the foundation problem that must not be overlooked.

"Depth" in the foundation problem is a relative term and the vertical co-ordinates of Figs. 4 and 23 are not absolute depths; they are ratios of depth to radius of loaded area. The concentrated load in the girder problem is not exactly analogous to the distributed loads in the foundation problem. In general, the stress distribution calculations are made for structures that cover ground areas on the order of 100 or 200 ft in diameter. Even if the structure were supported by isolated footings, it is the entire loaded area that would have to be considered. At the same time, the depth from ground level to rock or hardpan, over the greater part of the surface of the earth, is also on the order of 100 or 200 ft. There are well-known exceptions, of course, such as the City of Mexico and the mouths of the Mississippi and the Yangtse Rivers. However, the most common condition is one in which the depth of soil and the diameter of the loaded area are of the same order of magnitude. In other words, in a practical foundation problem, it is the region represented by the upper parts of Figs. 4 and 23 that is of particular interest. In this region the distribution of surface pressure is especially important and, regardless of the probable value of the concentration factor, there are large variations in the magnitude of the soil stresses due entirely to the manner in which the ground surface is loaded. The foundation problem, therefore, appears to be largely a question of surface phenomena and local perturbations rather than one of stress distributions in a semi-infinite solid at depths so great that the surface load conditions can be ignored.

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[72] "Die Bemessung von Flachgründungen aus Eisenbeton und die neuere Baugrundforschung", *Beton und Eisen*, 1935, Heft 12.

[73] "Kreisplatten auf elastischer Unterlage", Berlin, 1926.

This problem of stress distributions is also influenced by other important factors, such as those mentioned by Professor Casagrande. Rigid underlying strata; planes of discontinuity due to stratification in the soil mass; soft clay beds under beds of dense sand—all affect the stress distributions. These problems have been attacked theoretically and for some of them a theoretical solution is available. Proof of the accuracy of the theoretical analysis must await the collection of data in the field. It is interesting to note that in the case of a rigid underlying stratum, Dr. Biot's equations (Fig. 20) indicate stress concentrations in excess of those given by the Boussinesq equation. In most of the experiments shown in Table 1 and Fig. 4, the pressure cells were placed on the more or less rigid concrete floor of the laboratory. The experiments produced stress concentrations similar to those required by Dr. Biot's analysis.

Summary and Conclusions.—The number and variety of the discussions on this paper are indicative of the great interest that has developed in foundation problems and soil mechanics in recent years. Nevertheless, there can be no question as to the truth of Professor Williams' remark concerning the "rather amorphous state of foundation literature" at the present time. However, Professor Williams sees no reason for discouragement in this situation; nor can the writer. It is believed that a parallel can be drawn between the development of soil mechanics and the development of the theory of elasticity which is the basis of modern structural analysis.

Galileo, in 1638, was the first mathematician to endeavor to determine the nature of the resistance of a beam to rupture. During the next 200 yr the development of the elastic theory was in the hands of such men as Galileo, Hooke, Mariotte, Young, Euler, Daniel Bernoulli, James Bernoulli, La Grange, Coulomb, and others. These are all well-known names and yet, in the "Historical Introduction" to his "Mathematical Theory of Elasticity," Love writes, as follows:

> "At the end of the year 1820 the fruit of all the ingenuity expended on elastic problems might be summed up as * * * an inadequate theory of flexure, an erroneous theory of torsion, an unproved theory of the vibrations of bars and plates, and the definition of Young's modulus."

However, Love goes on to state that "* * * such an estimate would give a very wrong impression of the value of the older researches". The year 1821 marks the discovery by Navier of the general differential equations of elastic equilibrium. Since then another hundred years have elapsed during which elastic theory and structural analysis have made great progress in many directions.

In comparison with this record, soil mechanics and foundation engineering are very young sciences. Rankine's work is about eighty years old and his theories are now termed "classical". Boussinesq's great works on pulverulent masses and on the theory of stress distributions are only fifty years old. Terzaghi published his theory of the consolidation of clay about fifteen years ago. In the past ten years much progress has been made, although many important problems remain unsolved. The most serious obstacle in the way of further progress at the present time is the lack of accurate information as to the behavior of existing structures.

In conclusion, the writer wishes to express his sincere thanks to those who contributed to the discussion. He feels that the value of his paper was materially increased by reason of the discussion, and he believes that the discussion has demonstrated the validity of his original conclusions.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1946

SOME LOW-TEMPERATURE CHARACTERISTICS OF BITUMINOUS PAVING COMPOSITIONS

BY HUGH W. SKIDMORE,[1] ASSOC. M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. PHILIP W. HENRY, J. T. L. MCNEW, ROY M. GREEN, W. W. CROSBY, JOSEPH ZAPATA, M. HIRSCHTHAL, AND HUGH W. SKIDMORE.

SYNOPSIS

Stability at temperatures approximating hot-weather temperatures in the pavement has become thoroughly established as a basic requirement of bituminous compositions. Means of measuring and rating this important property have been perfected as a result of laboratory experiments and have been proved sound through several years of practical application. Two contemporary tests have evolved with resistance to shear the basic principle of both: In one (the extrusion test) the shear is measured indirectly and in the other (the direct test), simple shear is measured directly, and values are expressed in pounds per square inch. The latter method is distinctly convenient in comparing a variety of compositions which may have been tested in cylinders of from 1 in. to 6 in., or more, in diameter, since values by the extrusion test are not reducible to unit loads.

Although much work has been done in connection with summer temperature characteristics, with the result that the elements of mixture design involved have become thoroughly understood, little, if any, study has been made concerning low-temperature performance in paving compositions. With the thought that such an investigation should throw much light upon several questions that have long been unanswered and have given engineers much concern, the results of a systematic course of study, begun in 1931, are reported in this paper.

INTRODUCTION

It was recognized that some engineers have avoided the use of bituminous pavements because they were not always certain of the capacity of these types

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NOTE.—Published in August, 1935, *Proceedings*.

[1] Pres., Chicago Paving Laboratory, Chicago, Ill.

to withstand winters, although it is well known that many bituminous compositions have withstood severe low temperatures for long periods. Just why some pavements have failed to do so has never been clearly explained. Aside from failures definitely attributable to base or to sub-grade, bituminous compositions may crack too profusely, or they may become brittle, and ravel at low temperatures. Definite means of avoiding these faults have not been at hand because scientific data have not been available with reference to the characteristics of bituminous paving compositions over a wide range of temperatures, particularly low temperatures.

One of the early disclosures in experimental work with stability at temperatures above normal air temperature was that mixture resistance to shearing forces mainly resided in the mineral structure (especially the finely divided mineral filler) and that the bituminous cement was of decidedly secondary importance. Although there were noticeable differences between various cements in a given mineral structure, and although differences in consistency of a cement accounted for some change in stability, these differences were of minor importance as compared with those in stability that were due specifically to changes in mineral composition.

One of the first items of significance encountered in studying mixtures at low temperatures was the important rôle of the bitumen in such mixtures, both with respect to its normal consistency and to its inherent qualities, such as may result from refining methods or from the parent crude oil. As the temperature is reduced, the bituminous cement becomes more and more the outstanding factor in the strength of the composition. For example, there are obvious characteristic differences between straight-run materials and cracked or oxidized cements; and there are equally as distinctive differences between tar and some asphalts and between asphalts from mid-Continent, California, and South American sources. Such facts were suspected, but certainly they were not actually known until the studies were undertaken.

Although the data reported herein pertain to sheet asphalt mixtures of the hot-mix type, sufficient work has been done with other compositions to demonstrate that typical characteristics as disclosed by these studies are common to bituminous paving compositions in general. Important avenues of further investigation with reference to specific differences between fine and coarse aggregate mixtures, hot and cold-mix types, and other such comparisons, are clearly indicated.

Composition of Mixtures Studied

In order to develop a general comparison of the twenty or more cements included in this study, a normal sheet asphalt composition was chosen, with the bitumen the only variable. The commonly used consistency of 50 to 60 penetration was selected, with a few examples of a given cement over a complete range of consistencies from low to high penetration. Normal mixtures were designed with bitumen filling the voids in the dry, compacted, mineral aggregate. Variations from normal design were also studied with the bitumen content both less and more than mineral voids. A moderate filler content of 20% by weight of total mineral aggregate (about 14% of mineral passing a 200-mesh sieve in the finished mixture) was used in the main group, with more, and less, filler used in one group to determine the general effect of variations in filler content. Thus, a fairly wide scope of possible variables applicable to fine aggregate compositions was covered by these studies.

MINERAL AGGREGATES

Sand.—In order to eliminate possible sources of uncontrolled variables, a sand of controlled grading, obtained from a single source, was selected. It is recognized that, in all likelihood, similar mixtures with sands from other sources and gradations would produce different values, but extensive investigations in the past have indicated quite conclusively that there is a characteristic order of values for any given aggregate material and that, in so far, as practical sheet asphalt gradations are concerned, the order of stability is rather well defined, depending mainly upon filler content for marked increases or decreases. Hence, it was felt that although the studies of a variety of sands and fillers would be interesting academically, the chances were slight that any really vital information would be developed by such studies.

The Lake Michigan sand that was selected is common to the territory in the Great Lakes region, and has been used for many years in asphalt paving construction. Although it is not the best sand encountered, it has the advantage of extreme cleanliness, uniformity of composition, grain shape, and size within the range of material passing any given sieve opening. The sand was separated on the 10, 40, and 80-mesh sieves and recombined to a definite grading as follows:

Percentage passing the 80-mesh sieve, and retained on the 200-mesh sieve..........	21.6
Percentage passing the 40-mesh sieve and retained on the 80-mesh sieve...........	47.3
Percentage passing the 10-mesh sieve and retained on the 40-mesh sieve...........	31.1
Total (percentage)..................	100.0

The specific gravity of this sand was 2.65.

Mineral Filler.—The limestone dust filler has also been used extensively in the Chicago area for many years. It is made by pulverizing a hard, durable white limestone and, in the tests, had the following characteristics:

Percentage passing a 325-mesh sieve........	61.3
Percentage passing a 200-mesh sieve and retained on a 325-mesh sieve...........	15.0
Percentage passing a 80-mesh sieve and retained on a 200-mesh sieve...........	23.7

The specific gravity in this case was 2.71.

Sand-Filler Aggregate.—Voids were determined[2] for several sand-filler proportions such as 0, 12, 20, 25, 30, and 100% of filler. Fig. 1, which is the voidage curve for these two minerals, is the basis for calculating the bitumen required in the mixtures. Because this curve was so typical for the materials, points intermediate between 30 and 100% of filler are omitted, since more than 30% of filler is rarely used in practice.

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[2] Symposium on Mineral Aggregates: "Fine Aggregate in Bituminous Mixtures," by Hugh W. Skidmore, *Proceedings*, Am. Soc. for Testing Materials, 1929, p. 788 (also reprinted in *Roads and Streets*, October, 1929); "Sheet Asphalt Mixture Research," by Hugh W. Skidmore, *Proceedings*, Wisconsin Eng. Soc., 1925 (also reprinted in *Roads and Streets*, April, 1925).

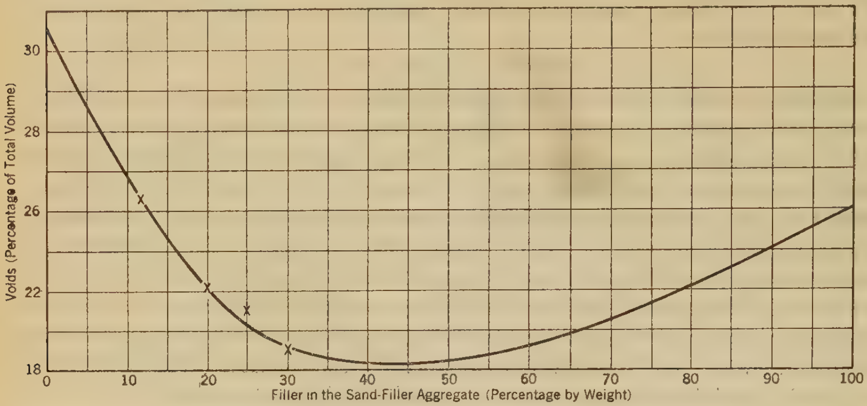

FIG. 1.—VOIDAGE CURVE; MINERAL COMPACTED DRY, TO REFUSAL.

Bituminous Materials.—The characteristics of the twenty-four cements in the main group of mixtures are detailed in Table 1, together with such additional

TABLE 1.—CHARACTERISTICS OF BITUMINOUS CEMENTS

Specimen	Source	Method of refining	Penetration			Ductility			Cementation Values ‡		Specific gravity	Bitumen; solubility in carbon disulphide
			At 77° F; 100 g, 5 sec	At 32° F; 200 g, 60 sec	At 115° F; 50 g, 5 sec	At 77° F; 5 cm, 60 sec	At 32° F; 0.25 cm, 60 sec	At 41° F; 5 cm, 60 sec	Kilogrammeters at 41° F	Elongation, in centimeters		
1	California	Steam	54	11	400†	150†	14.5	0	0.377	15	1.014	99.9†
2	California	Vacuum and steam	53	8	400†	150†	....	0.5	.....	....	1.020	99.9†
3	California	Steam	52	7	400†	150†	11.0	0	0.430	17†	1.019	99.9†
4	Coal tar	Steam	56	1	400†	121	0	0	0.045	0.75	1.181	93.2
5	Pennsylvania*	Cracking coil	53	10	380	150†	0	0	0.120	3.5	1.113	....
6	Mexico	Steam	50	16	225	150†	10.5	7.5	0.180	11.5	1.040	99.9†
7	Mexico	Steam	54	17	248	150†	....	7.25	0.160	9.0	1.031	99.9†
8	Mexico	Steam	53	14	215	150†	....	6.75	.....	....	1.040	99.9†
9	Mid-Continent	....	55	21	261	126	5.5	5.0	0.074	5.5	1.014	99†
10	Mid-Continent	....	52	19	221	150†	5.25	4.5	0.111	5.0	1.005	99†
11	Mid-Continent	....	48	14	215	100.0	....	4.75	0.193	5.75	1.018	99†
12	Mid-Continent	....	60	23	226	150†	....	6.25	0.133	6.5	1.010	99†
13	Wyoming	....	52	16	225	133	....	5.75	0.1572	5.75	1.029	99†
14	Arkansas	Vacuum and steam	49	18	234	150†	5.5	4.75	0.162	5.0	1.027	99†
15	Arkansas	Vacuum and steam; oxidized	56	28	185	62	5.75	5.5	0.053	6.0	1.016	99†
16	Mexico	Oxidized	56	31	123	4	3.5	0	0.070	3.5	1.017	99†
17	Texas	Steam	52	14	235	150†	....	5.0	0.1975	5.0	1.009	99†
18	Texas blend	Partly oxidized	51	22	202	92	6.5	....	0.096	6.0	1.011	99†
19	Colombian	Flash-vacuum oxidized	52	15	219	150	5.75	5.0	0.137	5.0	1.005	99†
20	Venezuela	Steam	54	16	243	150†	....	6.75	0.186	9.0	1.030	99†
21	Bermudez	South American flux	53	17	290	58	....	7.5	0.240	10.0	1.061	97.7
22	Blend	Steam and cracking	52	16	204	150†	7.75	6.0	0.127	6.5	1.056	99†
23	Trinidad	South American flux	54	18	251	62	....	5.25	0.2235	8.0	1.257	69.2
24	Trinidad	11° API† Mexican flux	53	..	...	....	....	....	.....	....	.....	71.9

* Cracked kerosene residuum. †American Petroleum Institute. ‡ Test described in *Journal of Industrial and Engineering Chemistry*, Vol. 6, No. 12, p. 976.

tional information as source and methods of refining, in as complete form as such data could be obtained from the producers.

In procuring samples of representative products, asphalt of three ranges of penetration were requested: 40–50, 50–60, and 60–70. Although these penetrations represent the grades commonly used in paving construction, it was considered desirable, after much work had been done, to include some softer materials, and these were secured in some cases.

Specimens 4, 16, and 24, Table 1, were three special cements prepared in the laboratory. For Specimen 4, a vertical-oven gas-house tar was reduced to grade by steam distillation. The crude tar showed the following characteristics:

Water content (percentage by weight)........	1.3
Specific gravity (at 25° C.)...................	1.098
Viscosity (in seconds, Saybolt Furol) at 50° C.	21
Viscosity (in seconds, Saybolt Furol) at 40° C.	37
Viscosity (in degrees, Engler, specific) at 40° C.	8.7
Percentage soluble in carbon disulphide.......	98.2
Distillation (percentages by weight) to:	
170° C................................	0.72
235° C................................	14.12
270° C................................	23.34
300° C................................	32.61
Softening point of residue..................	106° F

For Specimen 16, a regular Mexican flux (gravity, 11°, American Petroleum Institute) was blown to grade in the laboratory blowing-still. This material had a penetration of about 200 before blowing. For Specimen 24, a refined lake asphalt from the Island of Trinidad was fluxed to grade with the same Mexican flux used in manufacturing the oxidized Mexican cement. Tests were made for cementing value, with contained mineral present.

Supplementary comments referring to Table 1 should also be made regarding Specimens 5, 15, 19, 21, 22, and 23, as follows: Specimen 5 was a synthetic asphalt derived from residual, cracked kerosene; Specimen 15 was run to a penetration of 150 to 200 and then blown to grade; Specimen 19 was a flash-vacuum residual of 100 penetration, blown to grade; in the case of Specimens 21 and 23, tests for cementing value were made with contained mineral present; and Specimen 22, identified in Table 1 as a "blend," was probably a blend of steam-refined Venezuelan, and cracked domestic asphalt.

PREPARATION OF MIXTURES

Sand-filler aggregate was thoroughly mixed dry and heated to 350° F; then it was mixed thoroughly with hot asphalt cement at the same temperature. In the case of the more fluid binders, such as coal tar and the cracked asphalts, lower temperatures were used for both aggregates and binders, depending upon the viscosity of the bitumen. Some were heated to 325°F, some to 300° F, and the tar, to 250° F. In this particular the best field practice was simulated.

During the preliminary experiments, mixtures were prepared in a small, laboratory, power mixer, but the practice was abandoned in favor of careful hand-mixing for the study proper, due to the fact that too large a batch was required for the mechanical mixing. It was found that, in the case of an open pug-mill and hand-mixing, oxidation was identical. Therefore, accuracy was in no manner affected as compared with actual practice, since open mixers of the pug-mill type are still predominantly characteristic of asphalt paving manufacture despite important advantages inherent in sealed rotary mixers.

Each mixed batch was then weighed into the small batches to be moulded into the set of test cylinders. The small batches were stored in an oven at mixing temperatures during the few moments required for moulding.

Preparation of Test Cylinders

The normal size of compressed cylinders for sheet mixtures has commonly been 2 in. in diameter by approximately 1.5 in. in depth. For ordinary temperatures used in testing for stability, this is quite satisfactory for the capacity of the testing machine, but it was soon discovered that low temperatures would produce shearing strengths in excess of the capacity of the machine when using that size of cylinder. It was also found that results on cylinders 1 in. and 2 in. in diameter agreed for sheet mixtures. Therefore, the smaller diameter was used in these studies so as to provide for shearing strength possibilities in excess of 3 000 lb per sq in., if required, without building a new testing machine.

Cylinders were compressed at mixing temperatures in seamless steel tube moulds, using a double plunger, on a hydraulic press under a vertical load of 5 000 lb per sq in. Pressure was released as soon as the dial registered the required load. The compressed mixture was then cooled before being extruded from the moulds on the press, by means of a plunger and over-sized tube or split mould.

As has previously been described elsewhere,[3] this procedure has become standard laboratory practice after several years of experimenting and checking against field compression of paving mixtures. It has been found, repeatedly, that densities thus obtained in the laboratory agree well with those obtained in the field by adequate compacting with 10-ton rollers. The use of a double plunger assures uniform density throughout the specimen.

Testing Procedure

Density determinations (specific gravity) were made upon compressed cylinders a few hours after their preparation. Customary laboratory routine required that the mixtures be tested the day following their preparation, but there is no reason why they may not be tested as soon as they have been properly cooled and their densities determined.

The testing machine stands in a bath sufficiently large to provide storage for a considerable number of cylinders. Small cylinders, 1 in. to 2 in. in diameter, are maintained in the bath at testing temperature for 1 hr before they are tested, whereas larger ones are held for 2 hr or more, depending upon their size. It is not necessary to remove them from the bath to place them in the machine.

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[3] "Practical Application of the Shear Test to Bituminous Mixtures," by H. W. Skidmore, *Proceedings,* 7th Annual Asphalt Paving Conference; also, *Roads and Streets,* January, 1929.

For all temperatures above 32° F, water was used for the bath, adding sufficient ice to secure a temperature of 41° F. Alcohol and dry ice were used for 32° F, 0° F, and −20° F, and proved to be an excellent bath medium for the low temperatures; accurate temperature control was readily maintained.

When the shearing strength test was developed several years ago, it was found by experiment that, by applying the shearing load on cylinders 2 in. in diameter at the rate of 2 lb per sq in. per sec, the most concordant results were obtained for a wide range of mixture composition and temperatures. For this reason it was adopted as the standard rate, and in order to maintain consistent results for various diameters of test cylinders, the rate must be kept proportional to the diameter, since the load is applied circumferentially; for example, one-half the 2-in. rate was required for cylinders, 1 in. in diameter, and double the 2-in. rate was necessary for 4-in. cylinders. These were the rates of load application used in the studies reported herein.

The dynamometer of the testing machine is equipped with a maximum load indicator. In the present studies, values are the average of three test cylinders of each mixture at each temperature. In the application of the stability test at normal air temperature and above, maximum variation in well designed mixtures has been found to be within 5% of the average of the group of individual test cylinders. Poorly designed mixtures usually show greater variation, and if badly out of balance, wide differences between minimum and maximum results are common. In studying mixtures at low temperatures the same general rule seems to apply, except that with highly susceptible bitumens, such as tars and cracked asphalts, even well designed mixtures show erratic results at 0° F and below. In fact, if the curve for a given mixture tends to reverse its direction as the temperature drops, the spread between minimum and maximum becomes much wider as the temperature becomes progressively lower beyond the approximate point of reversal of direction.

TEST RESULTS

Results obtained in the use of a common mineral composition are plotted in Fig. 2(*a*), with Fig. 2(*b*) an enlargement of horizontal scales for the lower temperatures. The principal value of this general curve is to demonstrate graphically the spread from maximum to minimum values for the several binders (of common consistency range) studied, and it reveals nicely the characteristic trends of the widely divergent types of cement, with respect to source and refining methods.

The common mixture was composed approximately as follows: Bitumen, 10%; filler, 18%; and sand, 72 per cent. Pro rata variations were made to compensate for differences in the specific gravity and bitumen content of the binder, in order to maintain a constant mixture with reference to pure bitumen, filler, and sand.

As a sidelight upon the accuracy of the shear test in detecting characteristic differences between bitumens at low temperatures, attention is called to the fact that points on the curves for two Mexican cements of practically the same consistency and the same refining process (Specimens 7 and 8, Fig. 2), agreed so closely that one curve serves for both, and the third Mexican cement, of slightly lower penetration (Specimen 6), falls close to the other two; likewise, the California cements from three different refineries (Specimens 1, 2, and 3) fall upon a single curve.

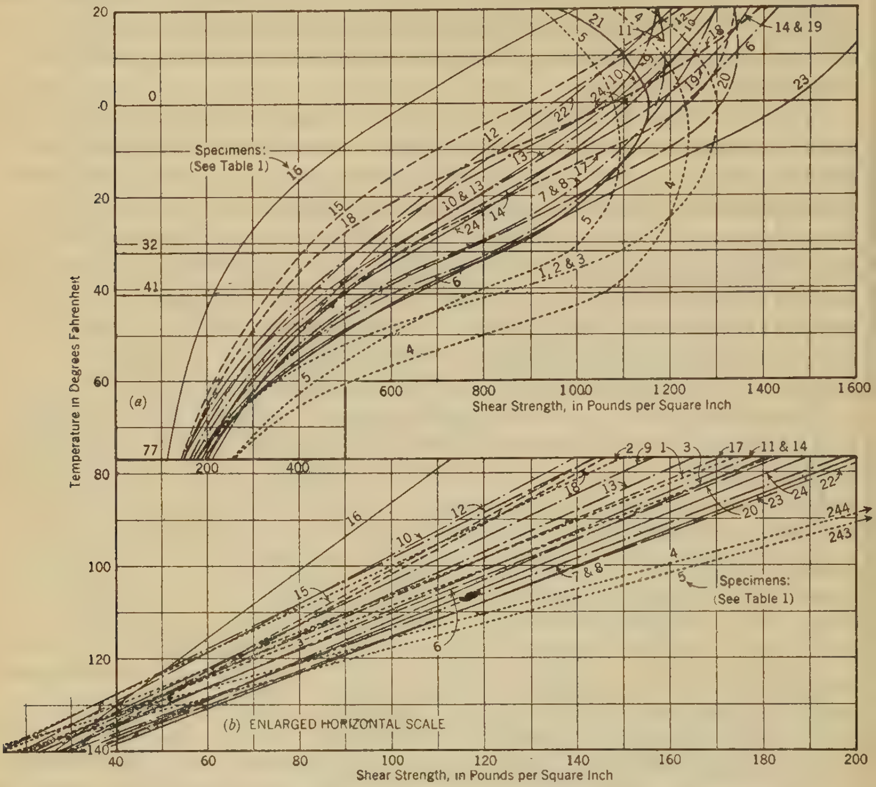

FIG. 2.—SPREAD OF TEST RESULTS FROM MAXIMUM TO MINIMUM VALUES.

Bituminous binders most sensitive to great extremes of temperature are strongest at moderate temperatures, of the order of 20° F to 90° F, while being characteristically weaker at summer sun heat and severe winter cold, of zero, or below. This is most certainly borne out in practice.

Oxidation tends to smooth out the curve for a given cement and, in doing this, stability (or strength) is reduced at all temperatures, especially at low and moderate temperatures. Past experience has demonstrated repeatedly that highly oxidized asphalts are poor paving cements, particularly in climates in the temperate and frigid zones. Such binders have caused severe cracking of pavements in these localities. Likewise, asphalts which were not oxidized during the process of manufacture, but which underwent severe oxidation while in thin films on hot mineral aggregate (in the mixer), have cracked badly in pavements which have undergone very low temperatures. Furthermore, the addition of certain salts to the paving mixture for the purpose of oxidizing or hardening the binder, has caused excessive cracking of such compositions. Every student of bituminous pavements has seen these facts demonstrated repeatedly.

This delayed effect was especially noticeable some five or six years after the introduction of the first so-called high filler mixtures (about 1922). At that time, the necessity had not been generally recognized for providing ample dry-mixing periods before introducing the asphalt and thus reducing the temperature of the hot sand (which had to be hotter because of increased filler content), by contact with cold mineral filler, to a degree not dangerous to the asphalt. Most pug-mills revolve at about 75 rpm. Consequently, freshly coated mineral is agitated severely and the asphalt (in thin films) is exposed to air rapidly, during a period of 15 to 60 sec, depending upon the enlightment of the specification writer and enforcer. If one is inclined to doubt the effect of rapid agitation of asphalt under these conditions, let him perform the following simple experiment: Weigh out 50 grams of asphalt into a tin and place it upon a hot-plate, heating it to, say, 400° F and maintaining it at that temperature for about 2 hr. Then determine its penetration and ductility. Repeat the process with a fresh sample of the same asphalt, but this time stir it for 15 min; then determine its penetration and ductility. In the first instance, it shows little if any alteration from its original characteristics, whereas in the second case it has hardened considerably and its ductility has been drastically curtailed.

Many supposedly experienced technologists still fail to take cognizance of these extremely important facts; poorly informed or ill-advised engineers continue to write specifications requiring the introduction of the bituminous binder without adequate (if any) dry-mixing of aggregates or with long wet mixing periods of 1 min, or more; and few plant operators and their employers recognize this important and inescapable result of exposing thin films of bitumen to temperatures in excess of 300° F to 400° F (depending upon the cement) in the presence of free air. If any engineer feels that he must require long mixing time to be perfectly safe, let him apply such precaution to the dry-mixing of aggregates and, to be really safe, require only 15 to 30 sec of wet mixing in any type of mixer, such as a pug-mill, to which free air has access.

Even a hasty inspection of Fig. 2 will disclose the unusual performance of fluxed Trinidad (with South American flux) cement through the low temperature range. This disclosure quite naturally indicated further study to the investigators. Steam-reduced Mexican cement (Specimens 6, 7, and 8, Fig. 2) seemed to come nearest to duplicating the performance of fluxed Trinidad cement, taking into consideration the shape of curve and values throughout the temperature range. Fig. 3(*a*) is introduced to demonstrate the effect of increasing the filler content in the case of Mexican cement, as well as the effect of retaining the same filler content but using a slightly softer cement. The composition of the mixture is shown in Table 2.

The filler was increased under the assumption that the self-contained mineral in the Trinidad cement might possess some special value as a mineral filler, and softer asphalt was tried because the Trinidad product does not oxidize as readily as the Mexican cement. The slightly softer Mexican cement (Curve *B*, Fig. 3(*a*)) produced an excellent and very smooth curve only slightly above the Trinidad curve (Curve *A*) throughout the temperature range; whereas the higher filler with the same consistency of cement (Curve *C*, Fig. 3(*a*)), practically duplicated the results of the Trinidad product. This is simply an illustration of intelligent design of composition in the light of the materials to be used.

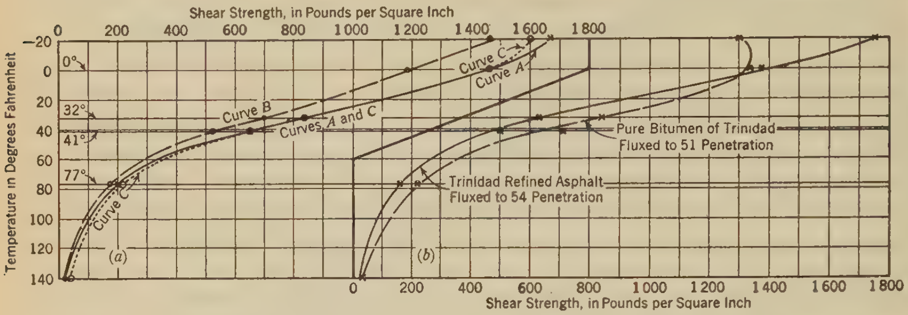

FIG. 3.—COMPARISON OF DESIGN VALUES, MEXICAN AND TRINIDAD CEMENTS.

TABLE 2.—MIXTURE COMPOSITION; CURVES *A*, *B*, AND *C*, FIG. 3(*a*)

Material	Curve *A*	Curve *B*	Curve *C*
54-Penetration, Trinidad asphalt cement; South American flux	14.5	....	....
Pure bitumen	10.0	....	....
51-penetration, Mexican asphalt cement; Panuco	....	....	9.6
64-penetration, Mexican asphalt cement; Panuco	....	9.9	....
Filler; limestone dust	13.5	18.0	22.6
Filler; Trinidad mineral	4.5	....	....
Sand; identical grading	72.0	72.1	67.8

From these studies it seems clear that the mineral in Trinidad asphalt is an efficient filler, but that this value can be duplicated readily by proper design when using another cement. Additional indication of the filler value of Trinidad mineral is apparent from the examination of Fig. 3(*b*), in which a South American flux was used in both cements plotted. The relatively pure bitumen of the Trinidad refined asphalt was recovered by an excellent method described[4] in 1933 by Mr. Gene Abson. This asphalt, in its almost mineral-free condition, was fluxed to the consistency shown by the addition of South American (Trinidad) flux; it showed 1.041 specific gravity at 77° F, and was 95.72% soluble in carbon disulphide.

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[4] "Methods and Apparatus for the Recovery of Asphalt," by Gene Abson, ***Proceedings***, Am. Soc. for Testing Materials, 1933, Pt. II, p. 704.

This investigation gives proof that the presence of the mineral in the Trinidad cement is of value in the pavement to the extent indicated herein. It is important to agitate the fluxed Trinidad cement adequately in the melting kettles, or in the tank car, as it is drawn for use in the mixture. If it is necessary (as has often been the case) to remove several inches of sludge from the kettle at more or less regular intervals, the filler value of the product has been largely dissipated.

In recent years, practical experience has shown, without question, the desirability of using softer bitumen in hot-mix compositions. Stability against displacement at summer temperatures can be secured so easily by suitable mineral content, that the very slight increased stability afforded by ten points of lower penetration becomes insignificant, especially in view of the danger attending the use of harder asphalts when they are likely to be subjected to more or less severe oxidation in the pug-mill mixer commonly used. This study shows conclusively that in climates where comparatively low winter temperatures prevail, bitumens of relatively soft consistency should be used. Asphalts of 70 to 80 penetration have been used successfully in hot-mix pavements in the North Central States to withstand trunk-line traffic during the summer and the sub-zero temperatures of severe winters. (For many years 90-penetration asphalt has been used in hot-mix pavements at Calgary, Alberta, Canada, with remarkable freedom from cracking.) When they are well designed, such pavements have been practically free from contraction cracks and have not been displaced during hot weather.

Further proof of the effectiveness of soft asphalts at low temperatures is shown in Fig. 4, the mixture composition being: Asphalt cement, 9.9% (9.8% for 91-penetration and 175-penetration); limestone dust, 18.0%; and, sand, 72 per cent. The only variable was the consistency of the asphalt, which was the product of one refiner in Fig. 4 (*a*) and of another in Fig. 4(*b*), using the same crude (Mexican) and the same refining process supposedly for all grades. The reversal of the curve for the 44-penetration cement (supplied by the refiner) is a rather conclusive indication that in the case of this harder grade the refiner varied the manufacture, either by altered processing of the residuum or by blending it with another material. The striking result of

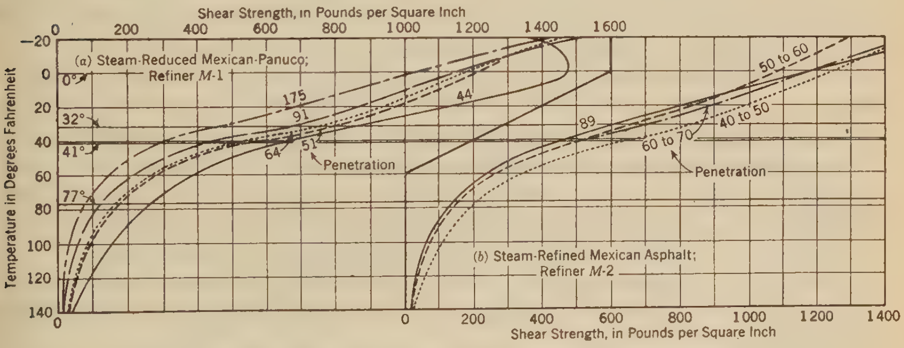

FIG. 4.—EFFECTIVENESS OF SOFT ASPHALTS AT LOW TEMPERATURES.

the 175-penetration material undoubtedly will be a surprise to many who have always thought in terms of asphalts with 40 to 60 penetration; but when one takes into consideration the more or less satisfactory performance of some cold-lay mixtures and some rock asphalts, it seems more logical. There is no doubt that, with proper design of mixture, 100-penetration cement may be used successfully in sheet asphalt mixtures in almost any climate, and although there is no need for such a soft material in tropical or semi-tropical regions, it would be highly desirable in all sections subject to temperatures of 0° F, or below. The use of soft bitumens in such sections would probably be found the most important forward step in recent years.

Effect of Varying Mixture Composition

This study has disclosed two important facts not heretofore thoroughly appreciated: First, that there is an optimum proportioning of ingredients that will produce the greatest strength of mixture throughout the entire temperature range from sub-zero to summer sun temperature; and, second, that for best resistance to low temperatures, mixtures should contain slightly more bitumen than will just fill mineral voids, as determined in compacted, dry aggregate. To the present time, technologists have been of the opinion that mixtures should carry slightly less bitumen than will fill the voids, as determined in the dry mineral.

In Fig. 5(*a*) will be seen the results obtained in four designs of mixture using the same materials in all, and ranging from low filler content to high, with bitumen filling the voids in dry aggregate. The mixture composition of materials in Fig. 5 is listed in Table 3. It seems quite certain that too much filler can be used with a given sand or coarse aggregate and a definite guide in this respect is indicated in Fig. 5(*a*). It will be seen that Mixture *D* is inferior to Mixture *C* throughout the temperature range, so that low temperature results, in this case, are not necessary for intelligent selection. Higher bitumen, a greater compaction of mixture, might tend to prevent Curve *D* from reversing itself, and might also increase its strength at all temperatures. There is positive danger of bitumen films being too thin. Only so much compaction is available with practicable construction equipment and safe working temperatures, and although increased filler may continue to lower voids in the mineral, a certain point is reached at which the mixture becomes less compressible, and, consequently, instead of increasing the strength and density of the finished pavement, additional filler actually decreases these important properties.

Increasing the bitumen beyond that required to fill voids in the dry mineral is certainly beneficial at low temperatures, whereas decreasing it has the opposite effect. In Fig. 5(*b*), Mixture *B* of Fig. 5(*a*) was made with both plus and minus 1% bitumen content (equivalent to 10% of the normal bitumen content) with the results shown. Decreased bitumen tended to decrease the stability very slightly at high temperatures and drastically reduced the strength at temperatures of from +20° F to −20° F. This tendency is reliable proof of the danger of using mixtures with less bitumen than will fill the voids; and it offers an excellent explanation of cold-weather cracking in dry mixtures. Reasonable excess of bitumen, on the other hand, only slightly reduced the stability at maximum pavement temperature and substantially increased the strength at low temperatures. Obviously, in warm climates, this particular phenomenon has no significance.

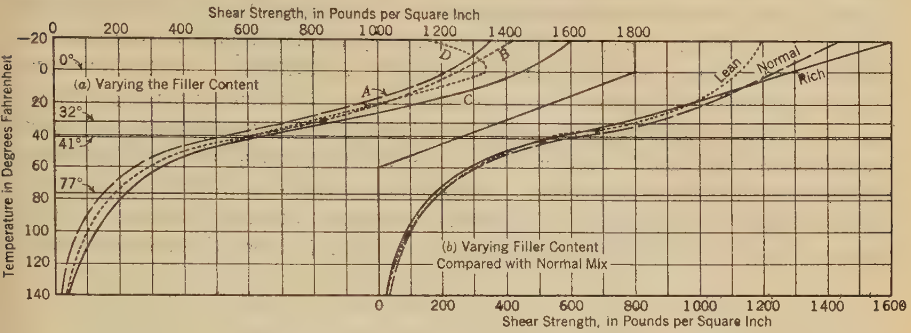

FIG. 5.—DESIGN OF ASPHALT MIXTURES.

TABLE 3.—MIXTURE COMPOSITION (ALL PERCENTAGES), DESIGNS IN FIG. 5

Description	(a) VARYING THE FILLER CONTENT; FIG. 5(a)				(b) VARYING THE FILLER CONTENT COMPARED WITH NORMAL MIX; FIG. 5 (b).		
(1)	Mixture *A* (2)	Mixture *B* (3)	Mixture *C* (4)	Mixture *D* (5)	Lean mix (6)	Normal mix (7)	Rich mix (8)
Mexican asphalt cement *...	12.0	9.9	9.6	8.4	8.9	9.9	10.9
Limestone dust............	10.6	18.0	22.6	27.5	18.2	18.0	17.8
Graded sand...............	77.4	72.1	67.8	64.1	72.9	72.1	71.3
Total................	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Voids in the compressed mix.	3.1	4.0	2.6	4.4	0†	0†	0†

* 51-penetration. † Normal bitumen fills the voids in the mineral.

In Fig. 6, eight distinctly different asphalt cements of the same grade, as determined by the penetration test at 77° F, are compared as to penetration

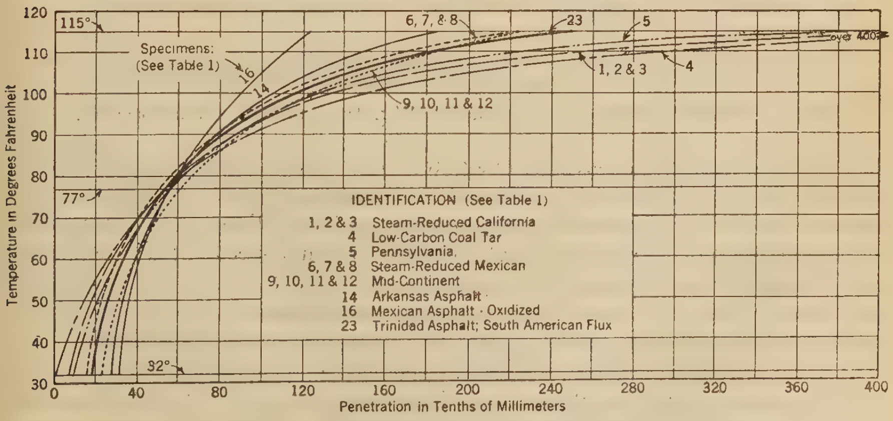

FIG. 6.—CONSISTENCY-TEMPERATURE RELATION.

at the other two standard temperatures, in order to show the wide differences in susceptibility as measured by this test. Such a test, as usually interpreted, can be very misleading. The ratio of penetrations at the several temperatures, when specified with a maximum limit only (as is usually the case) is no guaranty of performance in the pavement. In stipulating requirements for a high-grade paving cement, penetration susceptibility should be omitted in favor of a minimum low-temperature ductility requirement (in the absence of a better test at this time) in all regions where low temperature performance is critical. Specifications containing maximum penetration ratios may invite over-oxidation of what may be a reasonably good asphalt, as a measure of safety on the part of the refiner. Rigid requirements as to low temperature ductility have the opposite effect, since in nearly all cases oxidation will decrease ductility at the lower temperatures. Fig. 7 shows the relative characteristics of typical cements common to the market in the Central, West-Central, North and South-Central, and Southwestern States, with respect to ductility at 5° C. These studies are convincing in the light of the fact that, in general, cements of relatively high ductility (normal speed) at low temperatures likewise impart superior strength to mixtures at the lower temperatures. In climates where low temperature performance is important, such a requirement should be specified until a better test upon the asphalt is available.

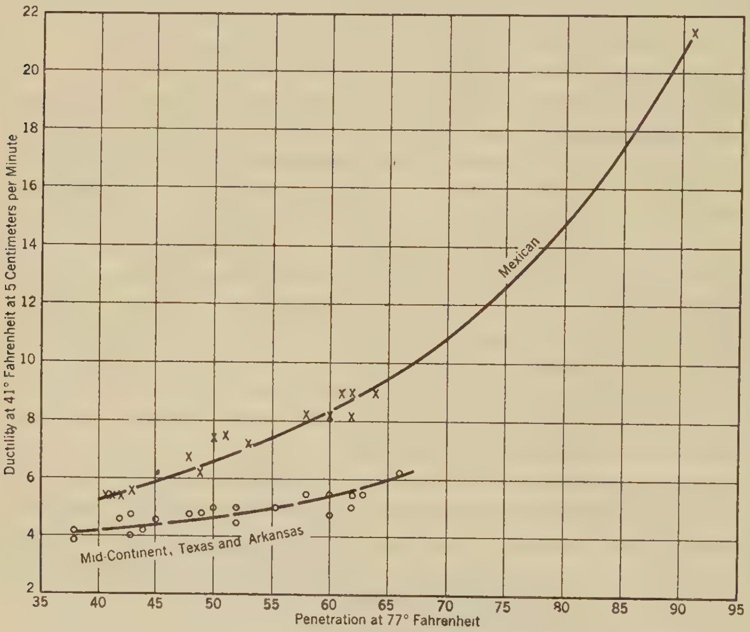

FIG. 7.—RELATION BETWEEN LOW-TEMPERATURE DUCTILITY AND PENETRATION.

Ductility as a specified requirement, when the test is conducted at 77° F, or above, means nothing for at least two important reasons: (1) This is not a critical temperature in any bituminous pavement; and (2) all binders possess ample ductility at such temperatures. Although this test may have some value in comparing a variety of cements, it is practically worthless in a pavement specification.

In an effort to translate shear strengths of bituminous pavement mixtures into comparative values familiar to engineers in general,[6] standard Portland cement mortar cylinders were subjected to the same test over the same range in temperatures. In offering these data, it is recognized that standard mortar is not concrete of the composition used in pavements, but such mortar is generally used as an indicator of concrete strength, and its values are well known from the vast amount of published data available. On the other hand, almost no data are available with respect to bituminous concrete mixtures, and, therefore, the average engineer has a limited understanding of the strength of such compositions, especially at low temperatures. Consequently, a comparison, limited as this one is, seems sufficiently valuable from the standpoint of engineering knowledge in general, to be included in this record as a means of evaluating previously unknown values in terms of well known ones.

Even a brief analysis of Fig. 8 will convince any engineer that bituminous and Portland cement concrete structures are not directly comparable. Portland cement concrete has a non-ductile binder as a cementing medium in contrast with a ductile binder in a bituminous mixture, of otherwise quite similar composition. Plastic flow is much more pronounced in bituminous compositions than in Portland cement concrete mixtures. Furthermore, the strength of a bituminous mixture is much more a function of temperature than is the case of Portland cement concrete, except at temperatures below freezing. Plastic flow is definitely related to ductility. Contraction forces in bituminous concrete mixtures are obviously absorbed by the ductility of the binder in addition to being resisted by the tensile strength of the structure, whereas in Portland cement concrete these forces are resisted by tensile strength. Patently, the modulus of elasticity will be much higher in the case of Portland cement concrete than in that of bituminous concrete.

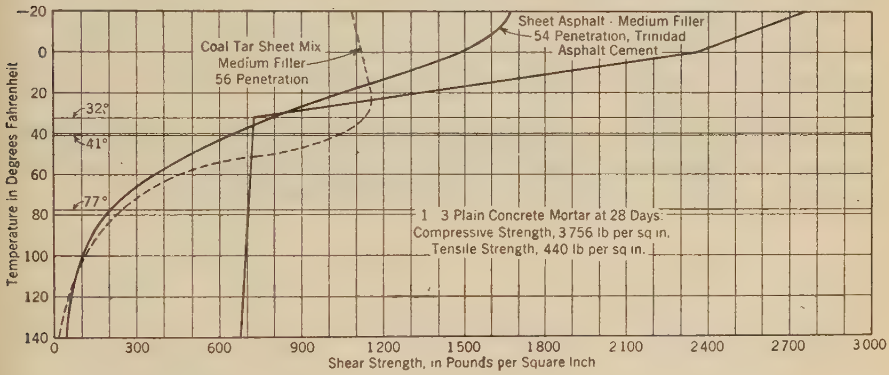

FIG. 8.

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[6] A. S. T. M. standard mortar cylinders using standard Ottawa sand, with strengths as shown on Fig. 8.

SUMMARY

The studies described herein have revealed some facts that were previously unknown and have confirmed others that were suspected as a result of experience and observation, as follows:

(1) The inherent characteristics and quantity of the bitumen in the mixture are much more important at low temperatures than at normal and higher temperatures in the pavement;

(2) There is an optimum proportioning of the several ingredients of the paving mixture that will insure the best performance over the entire range of temperatures;

(3) In regions where low temperatures obtain, the adoption of considerably softer bitumens than have been used commonly, will constitute one of the most important advances in the direction of better pavements;

(4) Ample bitumen content is essential; relatively rich mixtures are much safer than lean ones (bitumen slightly in excess of the void content of the mineral aggregate improves low temperature performance without impairing stability at high temperatures);

(5) Mineral filler in excess of the optimum for a given binder and coarse aggregate should not be used; and,

(6) The ductility of the binder at the lower temperatures appears to be an important characteristic. The test for ductility probably should be made at the standard rate of 5 cm per min, and a temperature of 4° or 5° C is about as low as may be used.

(7) At least in so far as resistance to shear is concerned, Portland cement concrete and bituminous concrete are not similar structures due to the fact that their cementing mediums are of decidedly different character.

DISCUSSION

Philip W. Henry,[6] M. Am. Soc. C. E. (by letter).—Although the author may be justified in stating (see "Synopsis") that "little, if any, study has been made concerning low-temperature performance in paving compositions" and (see "Introduction") that "scientific data have not been available with reference to the characteristics of bituminous paving compositions over a wide range of temperatures, particularly low temperatures," nevertheless, as early as 1887, it was the practice to use a softer asphalt cement in such cities as Buffalo, N. Y., and Omaha, Nebr., where asphalt pavements laid a few years earlier had cracked badly. In 1887 the Company with which the writer was connected used a softer asphalt cement without any change in the mineral aggregate or percentage of bitumen to meet this situation. When the writer became Superintendent of the Company in Omaha, in 1889, he continued the practice, with the result that although the asphalt pavements laid a few years previously showed the worst cracking that he ever beheld before or since, all the pavements laid with the softer asphalt cement came through the winters without cracking. It was also the practice to use a still softer asphalt cement on streets of light traffic.

In those days no attention was paid to the mineral aggregate, and little to the percentage of bitumen. In all cities practically the same number of pounds of sand, stone dust, and asphalt cement were used in the batch, regardless of the grading of the sand. Fortunately, in most cities, suitable sand was at hand, although in a few cities, St. Louis, Mo., particularly, the sand then used was deficient in fine material (as was discovered later), and the pavements lacked the durability which pertained to most of the asphalt pavements of those days.

Until 1888 there was no method of recording the consistency of a given asphalt cement. Previous to that time the consistency was determined by chewing the cement, and the foremen of the asphalt plants became so expert that they could determine in this way the penetration of a given asphalt cement within four or five points.

Mr. Clifford Richardson, the author of the first authoritative publication concerning all phases of an asphalt pavement, was well aware of the importance of meeting low temperatures with a softer asphalt cement. In discussing certain pavements laid in 1897 he wrote[7]:

> "Experience has shown, however, that in these surfaces, although the average percentage of bitumen reached 10.5, it was in most cases too hard, averaging 58, which has resulted in some cracking. In subsequent years, therefore, it has been the practice to use softer asphalt cement. The results have been very satisfactory."

In another place he states[8]:

> "Cracks of the second description, due to the use of asphalt cement which is too hard or which has become hardened by being mixed with too hot sand,

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[6] Cons. Engr., New York, N. Y.

[7] "The Modern Asphalt Pavement", Second Edition, p. 321.

[8] *Loc. cit.*, pp. 480–483.

or to this cause combined with others, are the form which is most commonly met with * * *. Such cracks are due to the fact that the hard asphalt is too brittle at low temperatures to yield to the contraction of the surface. It fractures under the tensile stress imposed upon it. * * * Cracks which are due to the fact that the mixture is deficient in bitumen, in consequence of which the surface does not possess sufficient tensile strength, regardless of ductility, at low winter temperatures, are not as frequent as those due to a hard bitumen, since in such a mixture, disintegration with the formation of holes takes place, as a rule, before cracking."

Notwithstanding this early general knowledge, arrived at through costly experience (in those days, asphalt pavements were guaranteed from 5 to 20 yr), the data furnished by the exhaustive study made by Mr. Skidmore would seem to establish a scientific basis for designing asphalt pavements to meet the low winter temperatures.

Incidentally, it is curious to note that there has been practically no change in the method of laying asphalt pavements on city streets in the past fifty years and more. In the Eighties the same types of tampers, smoothers, fire-wagons, and rollers were used as are used to-day. This is in contrast to the laying of the concrete base, which was then mixed entirely by hand.

J. T. L. McNew,[9] M. Am. Soc. C. E. (by letter).—It is indeed gratifying that Mr. Skidmore has presented the results of this particular investigation. Undoubtedly, he has treated a phase of the subject which is very important, and one, perhaps, which has been neglected too long.

Notwithstanding the fact that technologists have always admitted that asphalt mixtures have no appreciable strength in flexure, the tendency of many engineers in practice has been to build as much beam strength in the mixes as possible, with the result that flexibility, self-healing, and shock-resisting properties were sacrificed. This tendency is indicated by specifications which provide for extremely high filler content, low bitumen content, and harder asphalt. Part of this tendency toward so-called highly stable mixtures has been due to a misconception as to what construction methods are feasible for field use. Mixtures that are compressible under laboratory conditions may be entirely unworkable on the road or on the street. When such conditions are encountered the novice resorts to hotter mixtures and still less bitumen, all of which tends toward making a bad mixture worse.

The effect of the mixing temperature on the bituminous cement is of considerable importance, especially since many of the mixtures classified as highly stable are also low in aggregate voids and, in turn, low in total bitumen-carrying capacity. Reference to analyses of samples from existing pavements show conclusively that plant heat and agitation during mixing may lower the penetration of the asphalt as well as the percentage of bitumen so much as to destroy the self-healing properties entirely. As early as 1919, Roy M. Green, M. Am. Soc. C. E.,[10] reported numerous examples of pavements in which the aggregate was bonded with asphalt from 10 to 20 penetration and yet it was known that the material used was not as hard as the analyses indicated. These extremely low penetrations could not be chargeable entirely to a natural loss of the more volatile fractions by weathering.

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[9] Prof. of Highway Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

[10] Bituminous Pavement Investigations, Pt. I and Pt. II, Texas Eng. Experiment Station, 1919 and 1921.

The writer has always been unable to find much good in the theory requiring the use of harder asphalts for increasing stability as measured by shearing tests. Shearing displacements are most likely to occur during periods of high temperature; and yet when the air temperature is 100° F the pavement temperature will often be 145° F, or so hot that any usable steam-refined asphalt cement will be in a liquid, or a very soft, condition. Obviously, the cementitiousness of two similar residues having slightly different degrees of fluidity cannot be measured as a large quantity. If most asphalt binders are in reality liquids or near liquids at summer pavement temperatures, then perhaps the real question to be answered is: What mixture design will serve both the summer conditions as well as the winter extremes? The author's investigations are interesting in that the results give some information on that question and cast suspicion on the theory sometimes advanced that harder asphalt cements produce better pavements.

If it is assumed that an asphalt surface is on an adequate foundation, low temperature failures of the surface are likely to occur as a result of mixtures that are too lean in asphalt cement, as a result of brittleness of the binder, or by raveling of the surface. Winter raveling may be due to the brittleness of the binder or it may be caused by freezing of impregnating water in the open mixtures. Brittleness in a pavement probably is more easily measured by rapidly applied loads than by slowly applied increments, and such a thought gives rise to the question: Why should not all stability tests on asphalt mixtures be made either with rapidly increasing loads or with loads applied suddenly? Regardless of what stability test is used on a bituminous paving mixture, the strength increases as the percentage of bituminous binder increases from a decided deficiency to some percentage that provides a maximum strength. After the maximum strength is reached, if the bituminous cement is further increased, the strength of the mixture decreases and continues to decrease upon the addition of bitumen until the samples are made of a preponderance of bituminous cement and little or no aggregate. The difficulty encountered in correlating the results of the different types of tests lies in the fact that the maximum strength does not occur for the same percentage of bitumen; for example, one testing procedure may indicate a maximum strength at about 8% bitumen, whereas another type of test will probably show a maximum strength for as high as 9½ per cent.

For a number of years the writer has been using a modified shear test made by forcing a frustum of a cone out of a cylindrical specimen of pavement under a gradually applied load, and has been able to verify conclusions of other investigators who used somewhat different tests to study the effects of different variables on the strength of mixtures. He has never been completely satisfied with the test as performed, however, because of the conviction that the manner of applying the load does not simulate the load application made by traffic; nor has he been able to propose such a loading device that is entirely satisfactory. Even the most rigid or stable asphalt pavements, as measured by the shearing strength tests, are in reality plastic mixtures that undergo movement on the street; and such movement does, to an appreciable extent, cause an internal re-arrangement of particles. Plasticity, then, is a property of a bituminous mixture that must be admitted and the most made of it as a quality of the mixture, because it cannot be eliminated entirely. A bituminous mixture may be plastic enough to flow gradually when subjected to a load of any kind for a long time and yet if the same mixture were loaded suddenly either in shear, compression, tension, or flexure, the plastic properties would not prevent failure.

As stated by the author, the ductility of the bituminous cement at low temperatures is of great importance. For that matter, any test for consistency at low temperatures is of value because plasticity is the desirable quality rather than brittleness in the lower ranges of temperature. It appears that the mixture which may be judged the strongest in shear at 140° F may be decidedly too brittle at 4° or 5° C.

In 1929, the writer undertook the direction of some research work on bituminous paving mixtures in the hope of developing certain modifications of test procedure that would detect properties of brittleness. Specimens were tested first by determining the intensity of load, applied gradually, that would be required to push a frustum of a cone out of a specimen 5 in. in diameter and 2 in. thick. From this test a strength curve could be plotted to show the effect of varying the bitumen. Later, another series of samples like the first was tested by knocking out the frustum of a cone of the same size by means of a hammer falling from a constant height. In this latter case the number of blows to cause failure was plotted as the ordinate against the percentage of bitumen as the abscissa. As was to be expected, the general shape of the strength bitumen curve was the same for the two tests, but the percentage of bitumen required for maximum strength was more for the impact test than for the shear test. In other words, the samples that were strongest under shear were not the strongest under impact. Of course, no one has proposed that the mixture which is strongest in shear should necessarily be the one chosen for the street; however, it has been a debatable question as to what variation from maximum results should be specified for the paving job. The writer is convinced that the foundation of a road must be able to withstand the traffic load before any surface will be capable of fulfilling its mission. If such a foundation is adequate, the function of the surface is to prevent wear, absorb impact, distribute load, and, in other ways, preserve the life of the load-carrying medium, regardless of weather. To fulfill that function the surface must be plastic enough to be either flexible or self-healing, stable enough to prevent waving, and designed so that it may be controlled as to uniformity in all the construction operations.

Since ground movement is also prevalent in winter months there appears to be every evidence to indicate that flexibility must be a quality of bituminous surfaces during winter extremes of temperature. It appears also that the mixture design, as well as the choice of all materials, must be predicated upon a proper evaluation of all demands to be made upon a surface, and not on the extremes of summer temperatures alone.

Roy M. Green,[11] M. Am. Soc. C. E. (by letter).—Engineers and asphalt technologists are indebted to the author for this contribution toward placing the design and performance of asphaltic mixtures upon a more sound and rational basis. He is to be particularly commended for his demonstration of the following facts: (1) A desirable design of an asphaltic mixture for cold climates is one in which the asphalt cement slightly over-fills the voids in the mineral; (2) in attempting to obtain high stability in a mixture there is grave danger of using a filler content which is too high; and (3) asphalt cements of higher penetration are desirable in the colder climates.

A reading of this paper and a study of the curves in Fig. 2 leaves the impression that the California cements (Specimens 1, 2, and 3) are undesirable in cooler climates. This is not a stated conclusion of the author, however. In the Cities of Omaha and Lincoln, Nebr., there are a number of sheet asphalt pavements which have been in service since 1905 and before. These pavements were laid with California cements, and have withstood the winter weather of this section of the country without the least distress and are in excellent condition to-day. However, the mixtures are rich and contain asphalt cement in excess of that necessary to fill the voids in the mineral.

It would have been very enlightening had the author made further mixtures with these cements, carrying the asphalt cement to 10% above normal, as he did the mixtures given in Fig. 5. Would this added asphalt cement produce a mixture such that the curve would not reverse itself at the lower temperatures? Is it possible, to a certain extent, to improve upon the performance of a given asphalt cement by altering the composition of the mixture from the standard conditions set up by the author's experiments?

W. W. Crosby,[12] M. Am. Soc. C. E. (by letter).—In the years since the writer first proposed a scientific research in the field of bituminous paving compositions[13], these analyses have been greatly developed and improved in many respects. Mr. Skidmore is to be complimented on having pursued a line of great importance, but one that has been too much neglected in the past.

In the beginning of the introduction of bituminous highways it was known that a higher percentage of "free carbon" was desirable in tars for roofing purposes or sidewalks in New Orleans, La., than would be the case in Bangor, Me., because of the lower temperatures in the latter place and their "brittling" or cracking effects. Furthermore, the higher temperatures of New Orleans seemed to render a higher percentage of inert material desirable in order to prevent "running" at those temperatures.

Similarly, when asphaltic oils were offered for road use, too much "middle oil" was recognized as undesirable because of the instability of these materials —which lack "free carbon"—under high temperatures. As the latter phase was of much wider observance for a long time, the researches have been mainly along the lines of perfecting bituminous materials for the warmer climates.

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[11] Pres. and Mgr., Western Laboratories, Inc., Lincoln, Nebr.

[12] Cons. Engr., Coronado, Calif.

[13] "Sampittic Surfacing", by W. W. Crosby, *Transactions*, Am. Soc. C. E., Vol. LXIV (1909), p. 352.

Now that the roads of the colder climates are being regularly cleared of snow, and their surfaces exposed to traffic, the importance of researches along the lines suggested by Mr. Skidmore should be appreciated.

The writer has previously expressed[14] his opinion on the value of ductility at 77° F. He agrees in the main with the summary of this paper, although he feels that some of the "facts" stated therein may be expressed somewhat loosely and thereby possibly may permit of dangerous interpretations; for instance, under Items (1) and (4) material might be used that, from its characteristics or quantities, would result in 'bleeding" or "shoving" under traffic, in the hot suns of many localities.

Joseph Zapata,[15] Esq. (by letter).—The author states that although the data reported in his paper pertain to sheet asphalt mixtures of the hot-mix type, sufficient work has been done with other compositions to demonstrate that typical characteristics disclosed by his studies are common to bituminous paving compositions in general. This conclusion does not agree with the results obtained by the writer in his experience and laboratory work.

In the first place, Mr. Skidmore gives no data on the stability characteristics of the aggregate mixture (sand-filler aggregate). Doubtless, this is due to the type of test that was used to evaluate the mixes discussed. During the last few years the writer has been working on the development of a stability test of wider range of application than either of the tests mentioned by Mr. Skidmore.[16] Under the conditions governing the test it was found that the addition of bitumen definitely lowers the resistance to displacement shown by the bare aggregate.

One of the conclusions reached by Mr. Skidmore is that the inherent characteristics and quantity of bitumen are much more important at low temperatures than at normal and higher temperatures in the pavement. The only characteristics given for the bitumens are penetration, ductility, cementation value, specific gravity, and the percentage soluble in carbon disulphide. The penetration has been limited to a range of 50 to 60; the available data are not sufficient to provide a clear understanding of the meaning of cementation values, (the test does not appear in standard reference texts); and the specific gravity and solubility values are not sufficient in themselves to establish inherent characteristics; therefore, the only characteristic available for making comparisons is the ductility. On the basis of the data shown in Table 1 the ductility seems to depend upon the origin of the petroleum. In view of the shear strengths shown on Fig. 2 is it possible, then, to assume that the choice of asphalts should be made on the basis of source?

Furthermore, Mr. Skidmore states that the ductility of the binders at the lower temperatures appears to be an important characteristic. His data do not confirm his statement inasmuch as the binder giving the highest shear strength is one that has no ductility at 41° F, 5 cm, 60 sec; also, in comparing the various binders it is noted that some show a difference of about 61% in ductility (41° F, 5 cm, 60 sec), with a difference of only 16% in shear strength. In making these comparisons, only data for temperatures of 41° F, 5 cm, 60 sec were considered for the following reasons: (1) A temperature of 41° F is practically that of the mean annual temperature for Wisconsin; (2) other ductility data available to the writer were obtained at a temperature of 39.2° F, 5 cm, 60 sec; (3) no data are given for ductilities at temperatures lower than 41° F, with a speed of 5 cm per min (although ductilities at 32° F are given, the conditions of test were changed, and thus an entirely new type of data are presented. What happens to the ductility, regardless of the method of test, when the temperature drops below 32° F?); and (4) in considering low temperatures, weather records were kept in mind; for example, in Wisconsin one finds the following temperature conditions: During approximately 5.5% of the year the temperature is 0° F, or below; for about 39% of the year it is 32° F, or below; and during the remainder of the year it is above 32° F.

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[14] *Proceedings*, Am. Soc. for Testing Materials, Vol. XI (1911), p. 685.

[15] Asst. Materials Engr., State Highway Comm. of Wisconsin, Madison, Wis.

[16] *Journal of Industrial and Engineering Chemistry*, Vol. 6, No. 12, p. 976.

No proof has been furnished that the most destructive changes in pavements occur when the temperature is below 0° F, or between 0° F and 32° F. In an investigation conducted by the writer to determine the behavior of asphalts used as joint fillers, it was found that more damage was shown by the material during periods of alternate freezing and thawing than during periods of nearly zero weather.

Mr. Skidmore concludes that the ductility of the binder at the lower temperatures appears to be an important characteristic, and he recommends that the test for ductility be made at a standard rate of 5 cm per min and at a temperature of 4° C or 5° C. Making the safe assumption that by binder is meant a tar as well as an asphalt, Mr. Skidmore's data do not confirm his statement, especially if evaluations are to be made at 4° C or 5° C. The three binders showing high shear strengths at 4° C or 5° C have no ductility at those temperatures.

TABLE 4.—COMPARISON OF THE RATIONAL PERCENTAGE OF BITUMEN IN TWO BITUMINOUS MIXES

Material	Percentage by weight	Specific gravity	Rational proportion	Rational percentage
(a) MIX CONTAINING SPECIMEN 10				
Aggregate	90	2.662*	33.80	77.25
Bitumen	10	1.005	9.95	22.75
Total			43.75	100.00
(b) MIX CONTAINING SPECIMEN 5				
Aggregate	90	2.662	33.80	79.01
Bitumen	10	1.113	8.98	20.99
Total			42.78	100.00

* Computed from the specific gravity of the sand and the specific gravity of the mineral filler.

The design of the mixes was made on the basis of the voids theory; Mr. Skidmore does not indicate clearly, however, what steps were taken to balance differences in specific gravities. To illustrate the point, Table 4 offers a comparison. The question is raised in view of the statement that the common mixture was composed approximately of 10% bitumen, 18% filler, and 72% sand.

Mr. Skidmore fails to give data on the most important factor that should be considered in designing bituminous mixes, namely, the durability of asphalt cements. Without these data it is not possible to arrive at a complete understanding, or even to anticipate to a reasonable degree the possible behavior of mixes containing asphalts of various characteristics.

M. Hirschthal,[17] M. Am. Soc. C. E. (by letter).—The question raised by Mr. Skidmore in his analysis of bituminous mixtures as used for paving materials is also of great importance in the use of bitumens for water-proofing purposes where exposed to weather.

Unfortunately, it is a fact that although bitumens have been refined to develop qualities that will prevent "running" in high temperatures, the attempts at prevention of brittleness at low temperatures (at and below freezing) have been far from successful.

If a water-proofing material or a paving material is to be used in a location where it will be subjected to temperatures below freezing, the material should have some ductility at those temperatures to meet the contraction which is occurring and the expansion that is to follow with a future rise in temperature. It has been claimed that ductility tests on bitumens at 40° F are not reliable, as slight jars tend to crack the bitumen at that temperature before the test is completed. The claim itself indicates the necessity of ductility that is measurable. At the same time, it indicates the requirement of improved technique in making the test. Another claim frequently made is that a ductility test at 77° F, together with a melting point and penetration of the bitumen, will indicate the ductility at 40° F without the test at that temperature.

In 1921 and 1922 the writer required tests of quite a number of bitumens by the laboratory of the Delaware, Lackawanna and Western Railroad Company, at Scranton, Pa., for the purpose of obtaining results as a basis for writing a specification for water-proofing for the improvements on that road at East Orange, N. J. (at that time there was no water-proofing specification for railroad structures in existence). Some of the bitumens then tested and others submitted later to the laboratory showed melting points, penetrations, and ductility at 77° F, within the range of those found acceptable, but had no ductility at 40° F—a confutation of the claim that the other qualities of the bitumen will be sufficient to indicate a ductility of the material at 40° F. The writer feels strongly the necessity of knowing the quality of ductility at low temperatures, in a material that is known will be subjected to such temperatures on structures which are subject to movement due to such changes in temperature.

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[17] Concrete Engr., D. L. & W. R. R., Hoboken, N. J.

Hugh W. Skidmore,[18] Assoc. M. Am. Soc. C. E. (by letter).—The reasoning of Professor McNew in connection with the fallacy of using hard asphalt in pavements, is very sound. It coincides perfectly with the ideas of the writer and his associates.

The displacement of paving mixtures under traffic during hot weather is the direct result of shearing action in the structure, induced by an applied force which is mainly compressive in nature. This compressive force is applied by means of the wheels of vehicles passing over the pavement. Fast-moving traffic ruts the pavement longitudinally in the direction of travel by displacing the composition laterally; the tendency, if any, toward forward displacement is converted apparently into lateral displacement due to the speed of the moving wheels. Vehicles coming to a stop on the pavement tend to displace the composition in a forward direction, creating humps across the line of travel. Very heavy, slow-moving traffic causes both types of failure.

In so far as hot-mix, filled, or dense, graded types are concerned, the problem of designing mixtures which will resist displacement successfully, has been readily solved, following the development of stability tests in which shear is involved. In the case of oil aggregate and some of the cold-lay mixtures, shear tests cannot be used at temperatures consistent with service conditions because these compositions possess little resistance to direct shear, except at very low temperatures; and yet many of the very soft and open-graded types of mixtures have shown satisfactory stability against displacement under traffic. This has led to the development of a new type of test as reported by Thomas E. Stanton, Jr., M. Am. Soc. C. E. and Mr. F. N. Hveem.[19]

Research in soil science is throwing much light upon the matter of the stability of cohesive masses subject to plastic flow, as well as upon the mechanics of stability of masses of non-cohesive, granular materials. Bituminous paving mixtures obviously are cohesive and subject to plastic flow. The stabilometer developed by Messrs. Stanton and Hveem appears to show much promise in the determination of the stability of all classes of bituminous paving mixtures as well as soils.

The low-temperature performance of bituminous paving mixtures, however, does not involve stability of the compositions. It has been shown clearly that the characteristics of bituminous cement are highly important with respect to low-temperature performance of the mixture. Cements that remain ductile at low temperatures unquestionably resist cracking. In this connection, Mr. Green points out that California asphalts have generally given good service in cold weather. The writer is entirely familiar with this fact. Although California cements of 50/60 penetration show no ductility at 0° C or 5° C, when elongated at a speed of 5 cm per min, they do show excellent ductility when a rate of $\frac{1}{4}$ cm per min is used; also, they show excellent elongation in the cementing value test, in which case the rate is 1 cm per min, whereas the tar and the highly cracked synthetic asphalt show no ductility at these low temperatures at either slow speed or fast speed.

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[18] Pres., Chicago Paving Laboratory, Chicago, Ill.

[19] *Proceedings*, 14th Annual Meeting, Highway Research Board, Washington, D. C., December 6 and 7, 1934.

Mr. Zapata raises the point of the stability of the bitumen-free aggregate, although he does not describe the nature of the test which he uses for this purpose. Mr. Zapata's test measures the horizontal shear of a confined specimen of either coated or uncoated mineral aggregate, and he finds that the uncoated mineral always shows a higher value than the coated aggregate. Obviously, such a test does not measure stability as stability functions in the pavement or in the sub-grade; neither does it measure direct shear, because the specimen is confined laterally. Possibly, it is a measure of the mechanical interlocking of mineral grains, although it does not simulate service conditions. According to the Zapata test, uncoated, dry sand is more stable than a sand-bitumen or a sand-water mixture, and yet it is well known that dry sand will not support vehicular traffic, whereas wet sand will, as will also sand-asphalt mixtures.

With reference to Mr. Zapata's criticism of the lack of sufficient data concerning the characteristics of the bituminous cements, suffice it to state that the cements used in the studies were subjected to several tests not included in the general run of specifications for paving binder. No doubt others also could have been included as an after-thought. Most certainly the ductility test is one for quality and not for identification of source, as Mr. Zapata infers. The cementing-value test is well known to technologists and is described in detail in the reference given.[16]

Reference to Fig. 4 will show that binders of 44 to 175 penetration were studied. Surely this pretty well covers the range of cements suitable for hot-mix construction. Also, contrary, to Mr. Zapata's statement, the cement showing the highest shear strength at the low temperatures has good ductility at 41° F, even with its high mineral content present. Mineral-free Trinidad cement is unusually ductile at low temperatures unless an inferior flux is used.

There can be little question but that alternate freezing and thawing is destructive to any structure sufficiently porous to admit appreciable quantities of moisture. It is to avoid this difficulty, as well as to increase stability, that highway engineers generally are recommending closer grading and better filling of the aggregates used in low-cost road mixtures. Thermal contraction cracking unquestionably will be more severe at $-20°$ F than at 32° or 41° F. Bituminous pavements must not be confused with Portland cement concrete in this respect, since concrete slabs are generally high in moisture when cold weather comes, and are probably most susceptible at or near 32° F. The moisture content of a well designed and well built hot-mix asphalt paving mixture is very low, being usually less than 0.5% after total immersion of 3 to 5 days. Bituminous surfaces on concrete bases are compelled to follow the movement of the base irrespective of their own ability to resist contraction forces.

The writer fails to appreciate Mr. Zapata's analogy to the effect that since the tar mixture showed the highest shear strength at 41° F, the tar binder should have shown the highest ductility at that temperature. There is nothing in the data to indicate remotely that such a conclusion could be drawn. The fact remains demonstrated by these studies that those bitumens which showed the best ductilities at 41° F also produced the best shear strengths throughout the low temperature range, except the Bermudez cement, which reversed its curve in the sub-zero range.

The exact composition of the two mixtures which Mr. Zapata selected was:

	Specimen 5	Specimen 10
Bitumen (pure), by weight........	10.6%	9.7%
Filler, by weight................	17.9%	18.1%
Sand, by weight..................	71.5%	72.2%

In every case adjustment was made to accommodate differences in specific gravity and purity of cement. Actual proportioning was by volume although the formula is based on weight.

It is not clear what Mr. Zapata means by "the durability of the asphalt cements." It was the writer's purpose in presenting the results of the study to try and throw additional light upon the matter of durability of bituminous pavements especially with respect to low temperatures.

It is quite possible that the ductility test will be supplanted by a better tool. Considerable can be said for the cementing value test. Neither of these tests seems to give all the information desired, but it must be remembered that several tests are required to establish the quality of a bituminous material and that some of the tests now being made are worse than useless since they may give fictitious information. A particularly culpable test which falls in the latter category is the reduction test at 500° F, made upon liquid binders to provide a so-called asphaltic residue of 100 penetration at 77° F.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1947

FAILURE THEORIES OF MATERIALS SUBJECTED TO COMBINED STRESSES

BY JOSEPH MARIN,[1] JUN. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. J. J. SLADE, JR., T. MCLEAN JASPER, I. K. SILVERMAN, W. P. ROOP, H. F. MOORE, A. A. EREMIN, A. FLORIS, AND JOSEPH MARIN.

SYNOPSIS

Several theories have been devised and tests made to determine the laws of failure of materials subjected to combined stresses. The main purpose of this paper is to extend the correlation of such theories and test results, representing them by a common set of co-ordinate axes. Without such a correlation, some test results on combined stresses have been interpreted incorrectly. In addition, some new theories are given. The graphical representation used is that of Mr. B. P. Haigh[2] and H. M. Westergaard,[3] M. Am. Soc. C. E.

Notation.—The symbols in this paper are summarized in the Appendix. An effort has been made to conform essentially with "Symbols for Mechanics, Structural Engineering, and Testing Materials,"[4] compiled by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932.

THEORIES OF FAILURE

The theories represented will be limited, for the present, to bi-axial stresses, isotropic materials, and gradually applied loads. The assumptions, limitations, and inconsistencies in each particular theory have been treated. The term, "failure," is subject to a variety of definitions. It implies a limiting value of change in shape, stress, sliding, energy, or a combination of these factors,

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NOTE.—Published in August, 1935, *Proceedings*.

[1] Asst. Prof. of Eng. Materials, Coll. of Eng., Rutgers Univ., New Brunswick, N. J.

[2] "The Strain-Energy Function and the Elastic Limit," by B. P. Haigh, British Assoc. for the Advancement of Science, 1919.

[3] "The Resistance of Ductile Materials to Combined Stresses in Two or Three Directions Perpendicular to One Another", by H. M. Westergaard, *Journal*, Franklin Inst., May, 1920.

[4] A.S.A.—Z10*a*—1932.

a point of failure being determined experimentally by a stress-strain curve.

Classification.—An element in a stressed member can be conceived as failing when a stress, deformation, sliding, or a combination of these three factors, reaches a limiting value. Based on these physical concepts, theories of failure are classified, as follows:

(*a*) General Stress Theories.—
  (1) Maximum Stress Theory (Rankine).
  (2) Maximum Normal-Stress Theory.
  (3) Maximum Stress-Normal Stress Theory.

(*b*) Deformation Theories.—
  (4) Maximum Strain Theory (St. Venant).
  (5) Maximum Distortion Theory.
  (6) Maximum Strain-Distortion Theory.

(*c*) Shear Stress Theories.—
  (7) Maximum Shear Theory (Coulomb and Guest).
  (8) Internal Friction Theory (Special Cases of Coulomb's Theory).
  (9) General Shear Theory (Special Cases of Mohr's Theory).

(*d*) Energy Theories.—
  (10) Maximum Strain-Energy Theory (Beltrami and Haigh).
  (11) Maximum Shear Strain-Energy Theory (von Mises, Hencky, and Huber).
  (12) Maximum Strain-Shear Strain-Energy Theory.
  (13) Maximum Volume-Energy Theory.

(*e*) Miscellaneous Theories.—
  (14) Wehage's Theory.
  (15) Maximum Change-in-Volume Theory.
  (16) Maximum Shear-Strain Theory (Becker).

It is possible that combinations of the aforementioned theories, depending on the signs of the principal stress, may be the true ones. Another approach to the problem is to consider the material as non-isotropic. Such a procedure has been followed by Brandtzaeg.[5]

For simplicity, the theories will be represented for the case of bi-axial stresses. In most cases they are easily extended to the case of tri-axial stress. In such cases, instead of a graphical representation in one plane, the theory is represented by a surface symmetrical about three planes at right angles. For brevity, the assumptions and limitations involved will be omitted.

Let s_1 and s_2 be the principal stresses at failure in a material subjected to bi-axial stress; s_t, the stress at failure in simple tension; and s_c, the stress at failure in simple compression. The theories are represented graphically by diagrams with co-ordinates, x and y, in which $+x = \frac{s_1}{s_t}$ and $+y = \frac{s_2}{s_t}$. The remaining quadrants are explained by the illustrations.

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[5] "Failure of Materials Composed of Non-Isotropic Elements," by A. Brandtzaeg, Assoc. M. Am. Soc. C. E., Det Klg Norske Videnskabers, *Selskake Skrifter*, 1927, No. 2, Trondhjem, Norway.

General Stress Theories

(*1*) *Maximum Stress Theory* (*Rankine*).—In Rankine's maximum stress theory, failure is defined as the condition of a material when one of the principal stresses equals the stress, at failure, in simple tension or compression. In other words, failure has occurred when $s_1 = s_t$; $s_2 = s_t$; $s_1 = s_c$; or, $s_2 = s_c$. Referring to Fig. 1, for the case, $s_c = s_t$, a material is defined as having failed when: $x = +1$; $y = +1$; $x = -\frac{s_c}{s_t}$; or $y = -\frac{s_c}{s_t}$. A more complete treatment of Theory (1) (and of Theories (4), (7), and (10), to follow), has been written by Professor S. Timoshenko.[6]

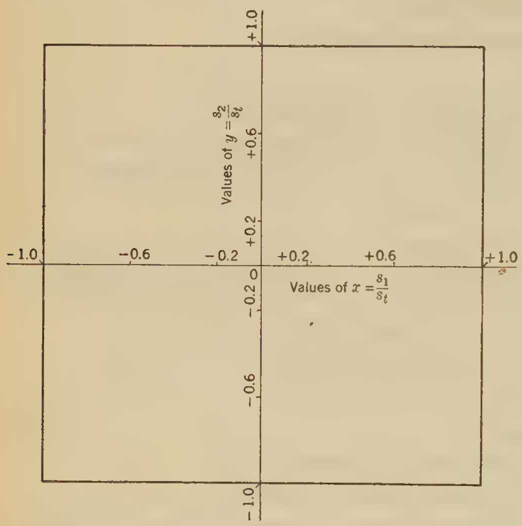

FIG. 1.—RANKINE'S MAXIMUM STRESS THEORY.

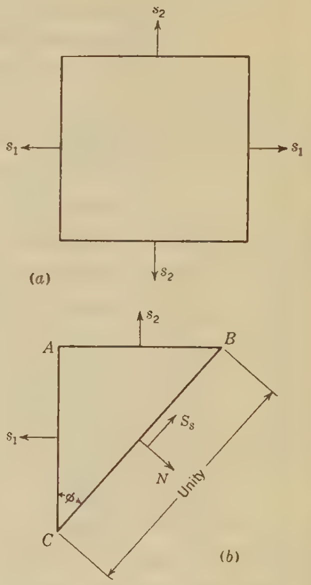

FIG. 2.

(*2*) *Maximum Normal-Stress Theory.*—Experimental evidence shows that materials do not fracture in a direction perpendicular to the greatest principal stress as required by the maximum stress theory. Consider the fracture to occur at right angles to some intermediate direction, and that the material in question has failed because the normal stress on this plane has reached a limiting value. Furthermore, let this limiting stress be composed of two parts: A normal component, *N*, of the principal stresses; and a normal component due to internal friction. Assuming the normal component due to friction to be proportional to the shearing stress, s_s, its value is $s_s f$, in which f equals a coefficient of internal friction. Fig. 2 represents the forces, s_s and *N*, acting on the plane of fracture, *BC*. Hence, the limiting normal stress, *s*, is:

$$s = N + s_s f \quad \ldots\ldots(1)$$

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[6] "Strength of Materials," by S. Timoshenko, Pt. II.

Substituting values of N and s_s in terms of the stresses, s_1 and s_2, and differentiating s with respect to ϕ, $\tan 2\phi = -f$ gives the condition for the maximum value of s. With $\tan 2\phi = -f$, the maximum or limiting value of s is expressed by:

$$s = \frac{s_1 + s_2}{2} + \frac{s_1 - s_2}{2}\sqrt{1 + f^2} \;\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

For the case of simple tension: $s_2 = 0$; $s_1 = s_t$; and,

$$s = \frac{s_t}{2}\left(1 + \sqrt{1 + f^2}\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Combining Equations (2) and (3):

$$s_1 + \frac{s_2\left(1 - \sqrt{1 + f^2}\right)}{1 + \sqrt{1 + f^2}} = s_t \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

or, when s_1 and s_2 are positive, and $s_1 > s_2$:

$$x + \frac{y\left(1 - \sqrt{1 + f^2}\right)}{1 + \sqrt{1 + f^2}} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Fig. 3 represents Equation (5) for values of f = 0.0, 0.3, and 1.0, in the four quadrants.

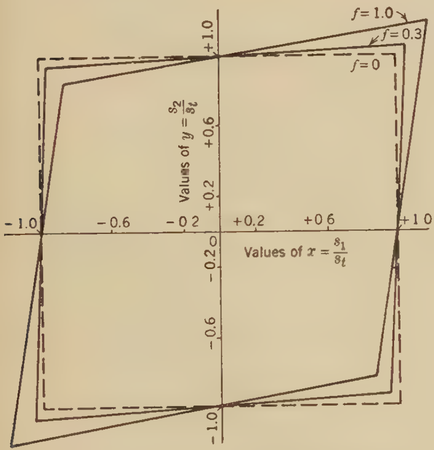

FIG. 3.—MAXIMUM NORMAL STRESS THEORY.

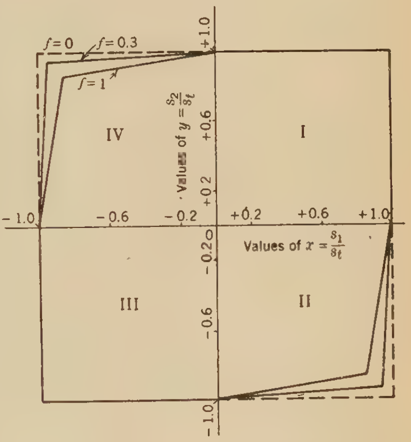

FIG. 4.—MAXIMUM STRESS-NORMAL STRESS THEORY.

(3) *Maximum Stress-Normal Stress Theory.*—Consider that the limiting stress is governed by either general stress or normal stress theories, depending on the signs of the principal stresses. Then, the limiting equations are:

$$x = \pm 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6a)$$

$$x + \frac{y\left(1 - \sqrt{1 + f^2}\right)}{1 + \sqrt{1 + f^2}} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots(6b)$$

$$y = \pm 1 \qquad (6c)$$

and,

$$y + \frac{x\left(1 - \sqrt{1 + f^2}\right)}{1 + \sqrt{1 + f^2}} = 1 \qquad (6d)$$

Fig. 4 represents Equations (6) for values of $f = 0$, 0.3, and 1.0, and for $s_o = s_t$, in Quadrants II and IV.

DEFORMATION THEORIES

(*4*) *Maximum Strain Theory* (*St. Venant*).—In the maximum strain theory failure is assumed to occur when the elongation or contraction in the direction of one principal stress reaches a limiting value equal to the deformation at failure in simple tension. Expressed algebraically, this is equivalent to stating that failure occurs when:

$$x - \sigma y = \pm 1 \qquad (7a)$$

or,

$$y - \sigma x = \pm 1 \qquad (7b)$$

in which σ = Poisson's ratio. Fig. 5 represents Equations (7) for $s_c = s_t$ and $\sigma = 0.25$ and 0.35.

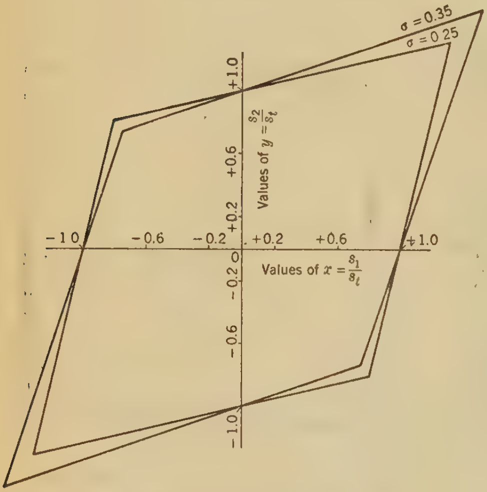

FIG. 5.—ST. VENANT'S MAXIMUM STRAIN THEORY.

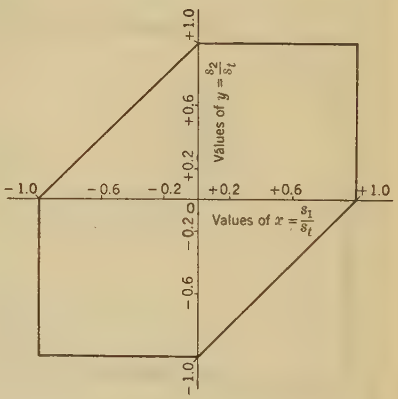

FIG. 6.—MAXIMUM DISTORTION THEORY.

(*5*) *Maximum Distortion Theory.*—A distortion theory is based on the assumption that a limiting angular deformation or shear distortion is the factor that produces failure. By equating the angular distortion in the case of bi-axial stresses to that for the case of simple tension, the following limiting equations are obtained: $x = \pm 1$; $x - y = \pm 1$; $y = \pm 1$; or $y - x = \pm 1$ (see Fig. 6). As might be expected, this theory reduces to the maximum shear theory (Theory (7)), the discussion of which follows.

(6) *Maximum Strain-Distortion Theory.*—Consider failure as being governed by a limiting elongation, contraction, or angular distortion, equal to the values in simple tension, but depending on the signs of s_1 and s_2. Then the equations defining failure are (see Fig. 7): $x - \sigma y = \pm 1$; $x - y = \pm 1$; $y - \sigma x = \pm 1$; or, $y - x = \pm 1$.

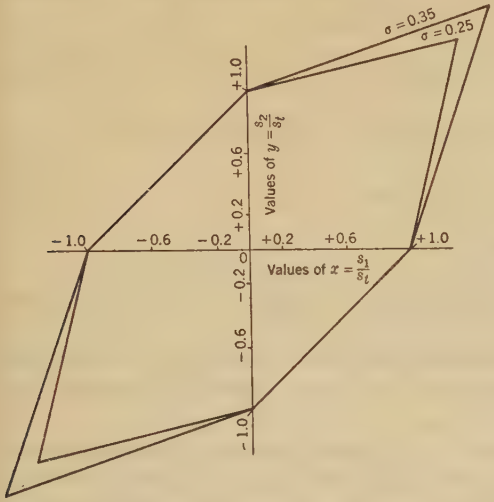

FIG. 7.—MAXIMUM STRAIN-DISTORTION THEORY.

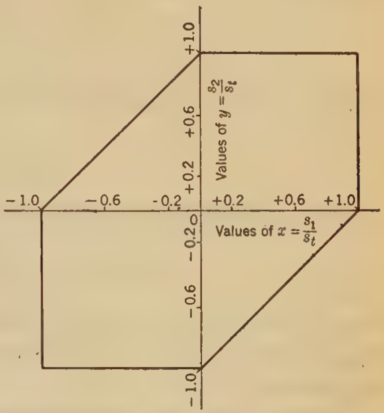
+1.0
+0.6
+0.2
Values of $y = \frac{s_2}{s_t}$
−1.0
−0.6
−0.2
+0.2
+0.6
+1.0
0
Values of $x = \frac{s_1}{s_t}$
−0.2
−0.6
−1.0

FIG. 8.—COULOMB AND GUEST'S MAXIMUM SHEAR THEORY.

SHEAR STRESS THEORIES

Shear theories are based on the assumption that failure results along some plane as a result of shear stress on the plane having reached a value equal to the limiting value of the shear stress in simple tension. Several theories are rooted in this concept, by defining the direction of the plane of the limiting shear or the value of the shear stress.

(7) *Maximum Shear Theory (Coulomb and Guest).*—In this theory, the plane of sliding or limiting shear is considered to be the plane of theoretical maximum shear, and may be expressed algebraically as follows (see Fig. 8): $x = \pm 1$; $y = \pm 1$; $x - y = \pm 1$; or, $y - x = \pm 1$.

(8) *Internal Friction Theory (Special Cases of Coulomb's Theory).*—Assume the plane of failure or sliding to be at an angle different from that of the theoretical maximum shear. For example, in Fig. 2, consider that the resistance to sliding, or the limiting stress, s_L, on the limiting plane, *BC*, is equal to the sum of a shearing stress, s_S, which is due to s_1 and s_2, and a frictional force proportional to the normal stress, N, equal to fN. Then:

$$s_L = s_S + fN \qquad \text{(8)}$$

Coulomb showed that, by substituting values of s_1 and s_2 for s_L and N:

For $s_1 \neq s_2$:

$$s_s = \frac{1}{2 \cos \alpha} [s_1 - s_2 + (s_1 + s_2) \sin \alpha] \qquad \text{(9)}$$

and, for $s_1 = s_2 = s_T$,

$$s_s = \frac{s_T}{\cos\,\alpha} \quad \text{.........................(10)}$$

in which $\alpha = \tan^{-1} f$; and s_T = shear stress in torsion.

Special cases of this theory can be obtained and represented by the co-ordinates used for Theories (1) to (7). Thus, consider,

$$s_T = 0.6\,s_t \quad \text{.........................(11)}$$

Such a value is representative for several steels. From Equation (9):

$$s_1 - s_2 + (s_1 + s_2)\sin\alpha = 2\,s_T \quad \text{...............(12)}$$

From Equations (11) and (12):

$$s_1 - s_2 + (s_1 + s_2)\sin\alpha = 1.2\,s_t \quad \text{...............(13)}$$

When $s_2 = 0$, $s_1 = s_t$ and from Equation (13) $\sin\alpha = 0.2$, or $\alpha = 11°\ 30'$. Substituting this value of α in Equation (13) (and, when s_1 is positive, s_2 is negative, and when $s_1 > s_2$), $s_1 - 0.67\,s_2 = s_t$, or $x - 0.67\,y = 1$.

When s_1 and s_2 are both positive, $x = +1$ and $y = +1$. Considering all quadrants the theory requires that, for failure: $x = +1$; $y = +1$; $x - 0.67y = 1$; or, $y - 0.67x = 1$. Fig. 9 represents this special case of Coulomb's theory for $s_T = 0.6\,s_t$.

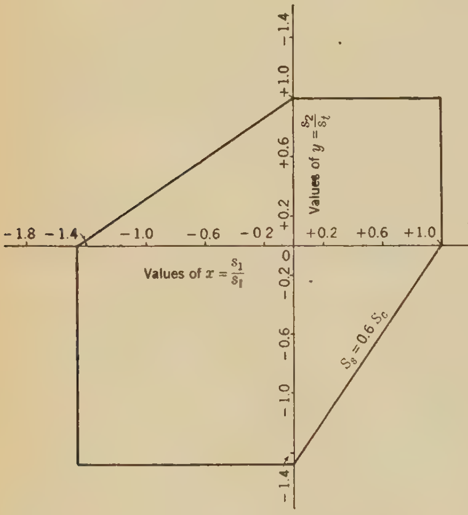

FIG. 9.—INTERNAL FRICTION THEORY, SPECIAL CASES OF COULOMB'S THEORY.

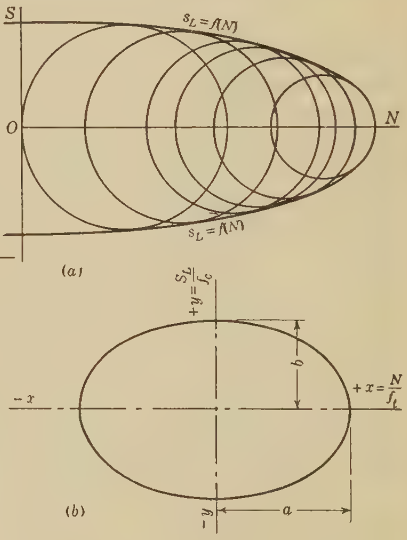

FIG. 10.

(9) *General Shear Theory (Special Cases of Mohr's Theory).*—Assuming that the plane of limiting shear or sliding is dependent in direction on the

relative values of the combined stresses, Mohr stated a general shear theory. It is assumed that the limiting stress, s_L, is equal to a constant stress plus a frictional stress produced by the normal stress, N, on the plane of sliding. The value of s_L is thus a function of N, or,

$$s_L = f(N) \dots\dots\dots\dots (14)$$

Referring to Fig. 2, in which ϕ defines the direction of sliding, by statics:

$$N = \frac{s_1 + s_2}{2} \times \frac{s_1 - s_2}{2} \cos 2\phi \dots\dots\dots\dots (15)$$

and,

$$s_L = \frac{s_1 - s_2}{2} \sin 2\phi \dots\dots\dots\dots (16)$$

Mohr represents the values of s_1 and s_2 graphically as shown in Fig. 10(a) for different values of s_1 and s_2 as represented by a series of circles. The lines drawn tangent to the circles represent Equation (14) and the co-ordinates of the points of tangency are given by Equations (15) and (16). The envelopes to the circle, therefore, represent the limiting values of N and s_L that define failure.

An empirical correlation between Mohr's theory and other theories is desirable. For this purpose, test results of several experimenters were plotted on the co-ordinate axes as indicated in Fig. 10(a), and the average envelopes to the circles were drawn. It was found that the experimental data could best be expressed by the equation of an ellipse. Actual values are given subsequently in Equations (21) and (22). Fig. 10(b) represents the relation between N and s_L—the envelope to the circles shown in Fig. 10(a). Then $N^2 + \frac{a^2}{b^2} s^2_L = a^2 s^2_t$; or,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots\dots\dots\dots (17)$$

in which, $x = \frac{N}{s_t}$; and $y = \frac{s_L}{s_t}$. Substituting Equations (15) and (16) in Equation (17):

$$(y + x)^2 + (y - x)^2 \cos^2 2\phi - 2(y^2 - x^2)\cos 2\phi + \frac{a^2}{b^2}(y - x)^2 \sin^2 2\phi = 4a^2 \dots\dots\dots\dots (18)$$

For Equation (18) to be consistent for simple tension, let $x = 1$ and $y = 0$. Therefore,

$$1 + \cos^2 2\phi + 2\cos 2\phi + \frac{a^2}{b^2}\sin^2 2\phi - 4a^2 = 0 \dots\dots\dots\dots (19)$$

Assuming $\phi = 45°$ for the case of simple tension, Equation (19) becomes:

$$b^2 = \frac{a^2}{4a^2 - 1} \dots\dots\dots\dots (20)$$

From the plotted experimental data and for average limiting values of the envelopes, the values of a and b, consistent with Equation (20) and the test data, are: For $a = 1.30$, $b = 0.54$; and, for $a = 1.00$, $b = 0.58$. Substituting these values in Equation (17):

For $a = 1.30$,

$$N^2 + 5.80\, s_L^2 = 1.69\, s_t^2 \qquad (21)$$

and for $a = 1.00$,

$$N^2 + 3.02\, s_L^2 = s_s^2 \qquad (22)$$

Substituting the values of N and s_L from Equations (15) and (16) in Equation (12) and assuming that $\phi = 45°$:

For $a = 1.0$,

$$y^2 - x\,y + x^2 - 1 = 0 \qquad (23)$$

and, for $a = 1.3$,

$$y^2 - 1.41\, x\,y + x^2 - 1 = 0 \qquad (24)$$

Fig. 11 represents Equations (23) and (24) for values of $a = 1.00$ and $a = 1.30$.

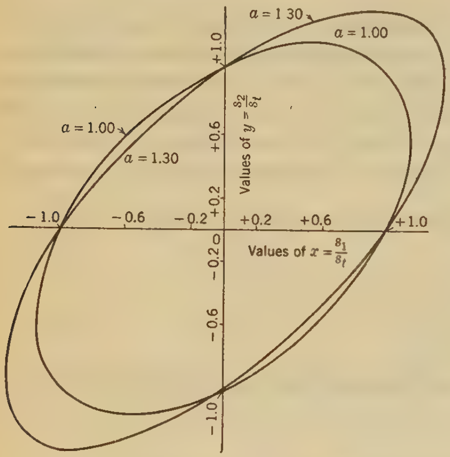

Fig. 11.—General Shear Theory: Special Cases of Mohr's Theory.

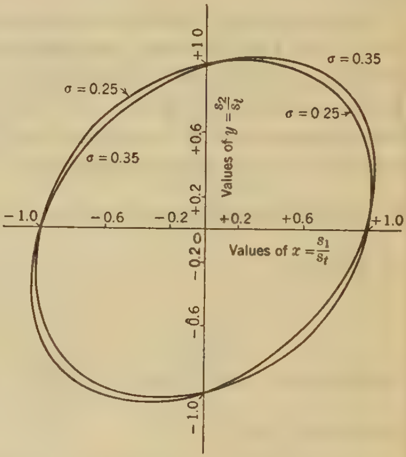

Fig. 12.—Maximum Strain-Energy Theory: Beltrami and Haigh.

Energy Theories

(*10*) *Maximum Strain-Energy Theory* (*Beltrami and Haigh*).—This theory is based on the assumption that the strain energy at failure for combined stresses equals the value of the energy at failure in the case of simple tension. In other words, failure occurs when,

$$x^2 + y^2 - 2\,\sigma\, x\, y = 1 \qquad (25)$$

Fig. 12 shows Equation (25) for $\sigma = 0.25$ and 0.35.

(*11*) *Maximum Shear Strain-Energy Theory* (*von Mises, Hencky, and Huber*).—Assuming that the energy due to shear distortion at failure for combined stress equals the value of shear energy of distortion for simple tension, this theory reduces to:

$$x^2 - x\,y + y^2 = 1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

This was obtained by subtracting from the total strain energy the energy due to change in volume, which yields the resulting energy due to change in shape.

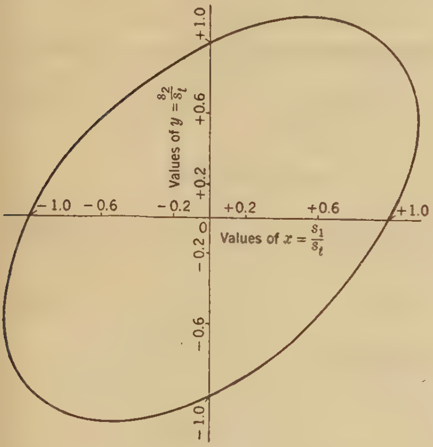

FIG. 13.—MAXIMUM SHEAR-STRAIN THEORY: VON MISES, HENCKY, AND HUBER.

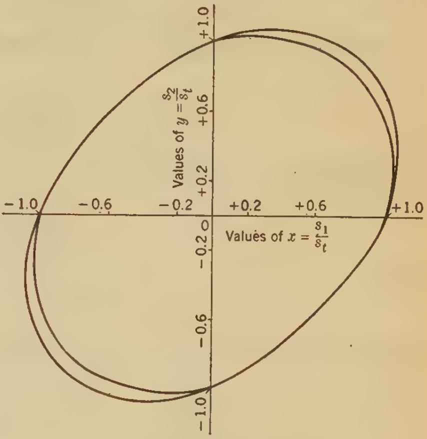

FIG. 14.—MAXIMUM STRAIN-SHEAR STRAIN-ENERGY THEORY.

Equation (26) (see Fig. 13) is identical with the special case of Mohr's theory for $a = 1.0$. Theory (11) and other theories have been written by A. Nadai.[7]

(*12*) *Maximum Strain-Shear Strain-Energy Theory.*—Consider that the energy due to either the normal strains or the angular distortions are the limiting factors. The equations defining failure are then expressed as:

$$x^2 + y^2 - 2\,\sigma\,x\,y = 1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

or,

$$x^2 - x\,y + y^2 = 1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

The equation which defines failure in Equations (27) and (28) depends on the value of σ. Equations (27) and (28) are represented in Fig. 14 for $\sigma = 0.25$ and $\sigma = 0.35$.

(*13*) *Maximum Volume-Energy Theory.*—Let the energy to produce change in volume be the limiting factor. In addition, let the volume energy at failure for simple tension equal the volume energy at failure for combined stress.

---

[7] "Plasticity," by A. Nadai, 1931.

The energy required to change the volume of an element subjected to the principal stresses, s_1 and s_2, is:

$$\xi = \frac{1}{2}\left(\frac{s_1 + s_2}{3}\right)(\epsilon_1 + \epsilon_2 + \epsilon_3) \quad \ldots\ldots\ldots\ldots\ldots (29)$$

in which, ϵ_1, ϵ_2, and ϵ_3 are the principal strains and $\epsilon_1 + \epsilon_2 + \epsilon_3$ = change in volume. Substituting for stresses in terms of strains in Equation (29):

$$\xi = \frac{1}{6\,E}(s_1 + s_2)^2 (1 - 2\sigma) \quad \ldots\ldots\ldots\ldots\ldots (30)$$

in which E = Young's modulus. For simple tension: $s_1 = s_t$; $s_2 = 0$; and,

$$\xi = \frac{1}{6\,E}(s^2_t)\ (1 - 2\sigma) \quad \ldots\ldots\ldots\ldots\ldots (31)$$

From Equations (30) and (31), $s_1 + s_2 = s_t$; and, therefore;

$$x + y = \pm\ 1 \quad \ldots\ldots\ldots\ldots\ldots (32a)$$

and,

$$x - y = \pm\ 1 \quad \ldots\ldots\ldots\ldots\ldots (32b)$$

Equations (32), for $s_c = s_t$, are plotted in Fig. 15.

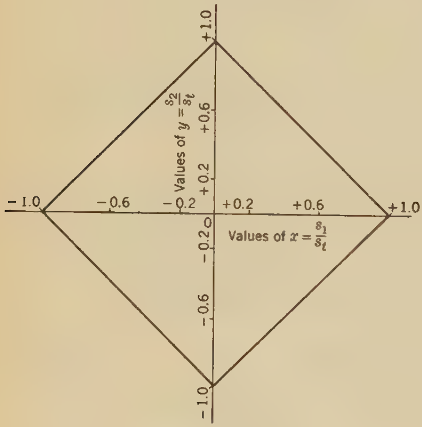

FIG. 15.—MAXIMUM VOLUME ENERGY THEORY.

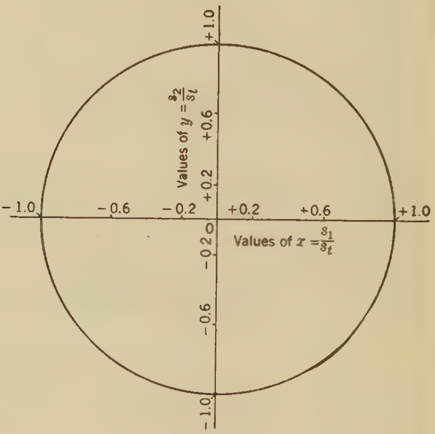

FIG. 16.—WEHAGE'S THEORY.

MISCELLANEOUS THEORIES

(*14*) *Wehage's Theory.*—This theory is empirical and is based on experiments made with pieces of paper submitted to tension in two directions at right angles. Algebraically, the theory reduces to:

$$x^2 + \frac{a^2\,y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots (33)$$

in which a and b are the "yield" point stresses in the two directions. For $a = b$, the theory becomes:

$$x^2 + y^2 = 1 \quad \text{.......................} \quad (34)$$

Equation (34) is represented in Fig. 16 for $a = b$.

(15) Maximum Change in Volume Theory.—Consider that the volume change is the limiting factor and that the change in volume to the point of failure in simple tension is equal to the change in volume for combined stresses. Let the change in volume $= \Delta V$. For the case of an element subjected to the principal stresses, s_1 and s_2:

When s_1 and s_2 are both positive:

$$\Delta V = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1}{E}\,(1 - 2\,\sigma)\,(s_1 + s_2) \quad \text{.........} \quad (35)$$

When s_1 and s_2 are both negative:

$$\Delta V = -\frac{1}{E}\,(1 - 2\,\sigma)\,(s_1 + s_2) \quad \text{............} \quad (36)$$

When s_1 is positive, s_2 is negative, and $s_1 > s_2$:

$$\Delta V = \frac{1}{E}\,(1 - 2\,\sigma)\,(s_1 - s_2) \quad \text{................} \quad (37)$$

When s_1 is negative, s_2 is positive, and $s_2 > s_1$:

$$\Delta V = \frac{1}{E}\,(1 - 2\,\sigma)\,(s_2 - s_1) \quad \text{..............} \quad (38)$$

When (in the case of simple tension), $s_1 = s_t$, and $s_2 = 0$:

$$\Delta V = \frac{1}{E}\,(1 - 2\,\sigma)\,s_t \quad \text{....................} \quad (39)$$

and, when (in the case of simple compression), $s_1 = s_c$, and $s_2 = 0$:

$$\Delta V = \frac{1}{E}\,(1 - 2\,\sigma)\,s_c \quad \text{....................} \quad (40)$$

Equating the formulas, (35) to (38), inclusive, to Equation (39), and letting $s_c = s_t$, the following equations are obtained, defining failure as a limiting volume change; thus:

For $s_c = s_t$,

$$x + y = \pm 1 \quad \text{........................} \quad (41a)$$

and,

$$x - y = \pm 1 \qquad (41b)$$

and, for $s_c \neq s_t$,

$$x + y = \frac{s_c}{s_t} \qquad (42a)$$

and,

$$x - y = \frac{s_c}{s_t} \qquad (42b)$$

Equations (41) and (42) are represented in Fig. 17.

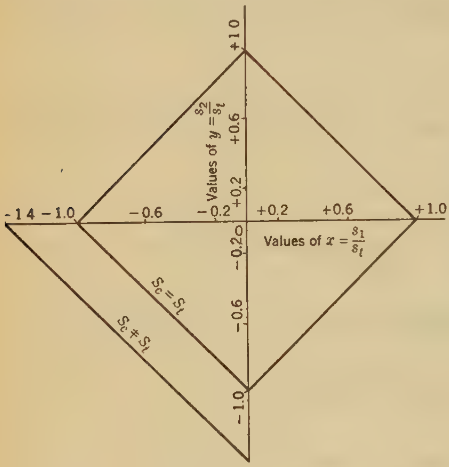
+1.0
+0.6
+0.2
−0.2
−0.6
−1.0
−1.4
−1.0
−0.6
−0.2
+0.2
+0.6
+1.0
Values of $y = \frac{s_2}{s_t}$
Values of $x = \frac{s_1}{s_t}$
$S_c = S_t$
$S_c \neq S_t$

FIG. 17.—MAXIMUM CHANGE IN VOLUME THEORY.

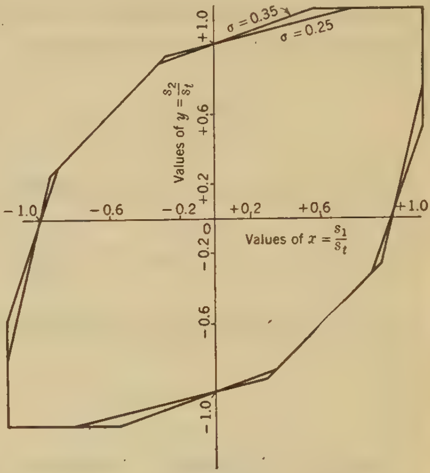
$\sigma = 0.35$
$\sigma = 0.25$
+1.0
+0.6
+0.2
−0.2
−0.6
−1.0
−1.0
−0.6
−0.2
+0.2
+0.6
+1.0
Values of $y = \frac{s_2}{s_t}$
Values of $x = \frac{s_1}{s_t}$

FIG. 18.—BECKER'S MAXIMUM SHEAR-STRAIN THEORY.

(*16*) *Maximum Shear-Strain Theory* (*Becker's Theory*).—Consider that the limiting condition for failure is either a maximum strain or a maximum shear. Then from the formulas of Theory (4) Equations (7), and Theory (7), the conditions defining failure are: $x - \sigma y = \pm 1$; $x - y = 1$; $x = \pm 1$; $y - \sigma x = \pm 1$; $y - x = 1$; and $y = \pm 1$.

Becker introduces an experimental constant in the foregoing equations by assuming that $\frac{s_s}{s_t} = 0.6$ instead of 0.5. Using this value the corresponding formulas become: $x - \sigma y = \pm 1$; $x = \pm 1.2$; $x - y = + 1.2$; $y - \sigma x = \pm 1$; $y = \pm 1.2$; and $y - x = 1.2$. These formulas, for values of $\sigma = 0.25$ and 0.35, are shown graphically in Fig. 18. A number of theories of this type can obviously be obtained by the superposition of others, as seen from Fig. 19.

COMPARISON OF THEORIES

The differences in the dimensions of machine and structural members as a result of designing by the various theories are of interest, if not of importance. The importance is evidently reduced because of the indeterminate factors involved in the selection of a working stress. The theories are compared graphically in Fig. 19 for the values of the mechanical properties given. From this graphical representation, a numerical comparison can be obtained between the limiting stresses as required by the various theories and the resulting differences in design.

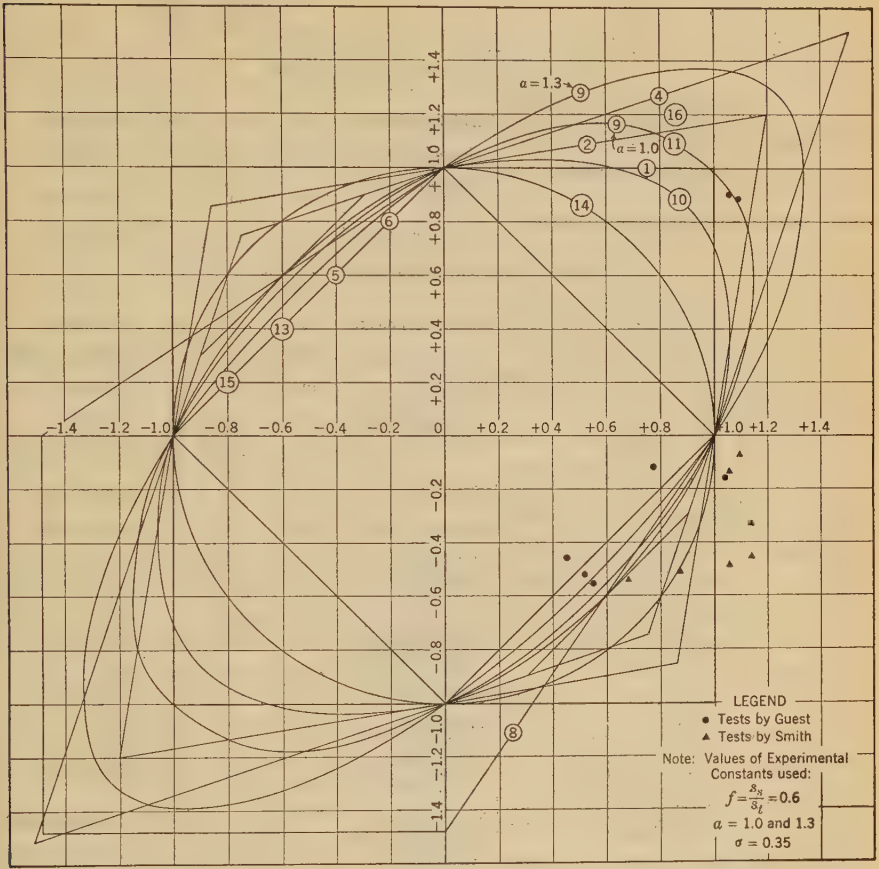

Fig. 19.—Comparison of Measured and Theoretical Strength for Brass Tubes.

Comparison of Theories and Tests

Numerous tests were conducted on specimens of cast iron, steel, copper, and brass, to study the relation between the measured and the theoretical strengths in each case. The specimens were subjected to combined stresses and the results plotted with the corresponding theoretical curve in the manner demonstrated in Fig. 19 for brass tubes. The numerical comparison given in Table 1 is obtained from these plotted results. The percentage values for the discrepancies between the test results and all the theories are approximate. These values for the percentage difference between theoretical and actual

TABLE 1.—TESTS TO STUDY COMBINED STRESS

Stress ratio, $\frac{x}{y}$ (1)	Type of specimen (2)	Method of loading (3)	Criterion of failure (4)	Number of tests (5)	Average Percentage Difference Between Tests and Theories		Theories that were most consistent with tests (8)
					Maximum (6)	Minimum (7)	
		(a) COMBINED TENSION AND TENSION IN STEEL					
>1.0	Tube	Axial tension with torsion and axial tension with pressure	Initial yield point	25	37	4	(1) and (10)
		Axial tension with internal pressure	Elastic limit	3	20	6	(11) and (1)
		Axial tension with internal pressure	Johnson's yield point	7	30	0	(4) and (16)
<1.0	Tube	Tension and torsion and tension and pressure	Initial yield point	12	24	0	(1) and (10)
		(b) COMBINED TENSION AND COMPRESSION IN STEEL					
>1.0	Tube	Tension and torsion and tension and pressure	Initial yield point	22	11	3	(11) and (8)
		Tension with torsion and compression with torsion	Proportional limit	14	27	2	(8), (4), and (9)
		Tension with torsion and compression with torsion	Yield point	19	26	1	(7) and (9)
		Bending and torsion	Intersection of elasticity and flow	7	40	9	(7) and (9)
		Tension with internal pressure	Elastic limit	4	37	4	(7)
		Internal pressure	Yield point	21	24	2	(10), (2), and (9)
		Axial tension with internal pressure	Johnson's yield point	4	16	0	(16) and (10)
<1.0	Tube	Tension with torsion and tension with pressure	Initial yield point	1	26	4	(2) and (10)
		Tension with torsion and compression with torsion	Yield point	2	44	1	(4) and (9)
		Tension with internal pressure	Elastic limit	5	29	2	(2) and (9)
=1.0	Tube	Tension with torsion and tension with pressure	Initial yield point	11	55	1	(9)
		Compression with torsion	Proportional limit	3	44	0	(2) and (16)
		Bending with torsion	Intersection of elasticity and flow	2	63	9	(8)
		Tension with internal pressure	Elastic limit	10	44	0	(16) and (8)
		Axial tension with internal pressure	Elastic limit	12	46	0	(9)
		Torsion	"Primitive" elastic limit	3	37	2	(10)
>1.0	Solid rods	Tension with torsion	Initial yield point	2	21	1	(2) and (9)
		Tension with torsion	Proportional limit	14	25	4	(1)
		Tension with torsion and compression with torsion	Yield point	9	26	1	(2), (8), and (10)
		Bending with torsion	Intersection of elasticity and flow	16	20	2	(4), (9), and (2)
<1.0	Solid rods	Tension with torsion and compression and torsion	Yield point	3	32	2	(9), (2) and (8)
		(c) COMBINED TENSION AND COMPRESSION IN CAST IRON					
>1.0	Tube	Internal pressure	Yield point	8	38	2	(1)
<1.0	Solid rods	Bending with torsion	Rupture point	12	12	1	(4)
		Bending with torsion	Rupture point	16	13	1	(4)
		Bending with torsion	Rupture point	21	12	3	(4)
		Bending with torsion	Rupture point	23	14	1	(4)
		(d) COMBINED TENSION AND TENSION IN COPPER					
>1.0	Tube	Tension with torsion, tension with pressure, and torsion with pressure	Initial yield point	3	14	1	(9) and (16)
<1.0	Tube	Tension with torsion, tension with pressure, and torsion with pressure	Initial yield point	2	21	5	(9) and (1)

TABLE 1.—(*Continued*).

Stress ratio, $\frac{x}{y}$ (1)	Type of specimen (2)	Method of loading (3)	Criterion of failure (4)	Number of tests (5)	AVERAGE PERCENTAGE DIFFERENCE BETWEEN TESTS AND THEORIES		Theories that were most consistent with tests (8)
					Maximum (6)	Minimum (7)	
(*e*) COMBINED TENSION AND COMPRESSION IN COPPER							
>1.0	Tube.....	Bending with torsion..........	Intersection of elasticity and flow..	8	27	8	(4) and (9)
		Tension with torsion and tension with pressure..............	Initial yield point.	3	21	0	(9) and (4)
(*f*) COMBINED TENSION AND TENSION IN BRASS							
>1.0	Tube.....	Tension with torsion, tension with pressure, and torsion with pressure......................	Initial yield point..	2	22	2	(9)
(*g*) COMBINED TENSION AND COMPRESSION IN BRASS							
>1.0	Tube.....	Tension with torsion..........	Yield point.......	7	24	2	(1) and (2)
		Tension with torsion, and tension with pressure..........	Initial yield point..	5	37	0	(9) and (4)

stresses were obtained by arithmetical averages. The values in Column (8), Table 1, are based on these results.

A study of the test data and test procedure shows that in some cases the discrepancies between theories and tests may be due to inaccurate test data. Computations were made showing that the errors in the stresses, due to errors in measurement or errors in the values of experimental constants used, may be appreciable. Another source of error is in the computation of the stresses, s_1 and s_2, from the applied loads and the dimensions of the member.

CONCLUSION

The foregoing comparison shows that no one theory now in use agrees exactly with test results. The precisely correct theory of failure is probably a combination of a number of these theories, depending upon the ratio, $\frac{s_1}{s_2}$. Fig. 19 indicates a number of such cases. Furthermore, there may be several theories equally well supported for one material and the correct theories may vary with the material. Obviously, an ultimate solution will depend on the results of further study in this field.

ACKNOWLEDGMENT

This paper is based in part on a research[8] sponsored by the Engineering Experiment Station, University of Illinois, and the Chicago Bridge and Iron Works. The study was made under the direction of Wilbur M. Wilson, M. Am. Soc. C. E.

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[8] Presented as a thesis to the University of Illinois in 1930 in partial fulfillment of the requirements for the degree of Master of Science in Enginering.

APPENDIX

NOTATION

The following symbols have been adopted for use in this paper:

a = major axis of an ellipse; also, a = major yield point stress.

b = breadth; minor axis of an ellipse; also, b = minor yield point stress.

c = a subscript denoting "compression."

f = coefficient of internal friction.

s = unit stress in general; s_L = limiting normal stress; s_1 and s_2 = principal stresses at failure; s_t = stress at failure in simple tension; s_c = stress at failure in simple compression; s_s = unit shearing stress; and s_T = shear stress in torsion.

t = a subscript denoting "simple tension."

x = variable distances measured parallel to the X-axis.

y = deflection; variable distances measured parallel to the Y-axis.

E = modulus of elasticity; Young's modulus; E_s = modulus of elasticity in shear.

L = length; as a subscript, L, denotes "limiting."

M = moment of force.

N = normal component of principal unit stress.

T = a subscript denoting "shearing torsion."

V = volume; ΔV = change in volume.

α = $\tan^{-1} f$.

Δ = "change in."

ξ = energy of distortion on an element subjected to the principal stresses, s_1 and s_2.

σ = Poisson's ratio.

ϕ = angle between planes of stress.

DISCUSSION

J. J. SLADE,[9] JR., ESQ. (by letter).—The interpretation of the results presented by Professor Marin offers some serious difficulties, the principal of which is due to the large radial scatter of the representative points. It is apparent, however, that none of the theories outlined by the author is adequate to represent the phenomenon throughout the range of values of $\frac{s_2}{s_1}$. In noting this the author states (see "Conclusion") that "the precisely correct theory of failure is probably a combination of a number of these theories, depending upon the ratio, $\frac{s_1}{s_2}$."

Assuming a material essentially homogeneous, such as steel, it is not likely that the failure values, s_1 and s_2, of the principal stresses would jump, at some ratio, $\frac{s^2}{s_1}$, from those given by one theory to those given by another; that is, it is more conformable with experience that the transition should be gradual. Such transitions, then, would most likely occur close to the points of intersection of the representative curves. Tracing the various possibilities it is seen that, although one may obtain a locus much closer to the average of the test data than any of those given by the simple theories, still there are serious discrepancies. For instance, near the ratio, $\frac{s^2}{s_1} = 1$, the average of the tests falls between loci of the general shear and the maximum strain-energy theories. These observations do not rule out any of the simple theories presented, however, but only their mathematical form; most of these expressions have been obtained on the assumption that Hooke's stress-strain relation holds, and failure, by any reasonable definition, occurs beyond the linear relation.

In view of this the writer proposes a different approach to the problem. Considering sufficient homogeneity of the material and accuracy of tests to determine the failure point, it will be assumed: (*a*) That there is a law of failure; (*b*) that the curve representing this law is continuous and continuously changing; and (*c*) that, for a given value of the ratio, $\frac{s_2}{s_1}$, the resultant of the principal stresses is single valued. As was stated, Assumption (*b*) is a requirement that conforms with experience. Requirement (*c*) states, in other words, that, if two identical samples are subjected to identical conditions, the values of the principal stresses at failure will be identical for the two. It is important to notice that the test results will not generally be identical, but these results are subject to errors of interpretation and mechanical errors which cannot be avoided. The scatter of the points representing the tests presented by Professor Marin will be considered later.

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In order to develop the present analysis it will be convenient to adopt the polar co-ordinates, ρ and θ, which are connected with the co-ordinates, x and y, by the relations $x = \rho \cos \theta$; and $y = \rho \sin \theta$; or,

$$\rho = \sqrt{x^2 + y^2} \qquad (43)$$

and,

$$\theta = \text{arc tan} \left(\frac{y}{x}\right) \qquad (44)$$

It is thus seen that ρ is the resultant of the principal stresses divided by s_t. For brevity, ρ will be called the resultant and x and y, the principal stresses. In terms of these co-ordinates the law of failure embodying the foregoing three requirements will be given by a functional relation,

$$\rho = F(\theta) \qquad (45)$$

Besides the aforementioned requirements, it is seen that $F(\theta)$ is periodic in θ (this is so by the very nature of the representation of the law; for instance, the value of ρ cannot change by adding 360° to θ). Therefore, $F(\theta)$ is ideally suited to be expanded in a Fourier series:

$$F(\theta) = b_0 + \sum_{k=1}^{\infty} a_k \sin k\,\theta + \sum_{k=1}^{\infty} b_k \cos k\,\theta \qquad (46)$$

With very extensive data the coefficients of this expansion may be determined readily by the method of least squares. A number of the terms of the series may be discarded *a priori* by Requirement (*d*) that the curve must pass through points, ±1, on the axes, and Requirement (*e*) that the curve must be symmetrical about the two lines, $\theta = \pm 45$ degrees. The number of the remaining terms of the expansion, Equation (46), that may be retained, will depend on the completeness of the available data. In another connection, the writer has discussed the significant flexibility of curves,[10] particularly of curves represented by a series in which the coefficients are adjusted by the method of least squares or related procedures. It is sufficient to state that, depending on the data, there is a point beyond which added terms of the series will give a poorer representation of the function than will be given by the series with these terms omitted. The first two terms of the series, Equation (46), that satisfy Requirements (*d*) and (*e*) are:

$$F(\theta) = 1 + a \sin 2\,\theta \qquad (47)$$

Equation (47) will be tried as a first approximation to the law of failure. There is only the constant, a, to be determined, so that the method of least squares leads to the simple formula:

$$a = \frac{1}{n} \sum_{k=1}^{n} \frac{\rho_k - 1}{\sin 2\,\theta_k} \qquad (48)$$

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[10] See p. 94.

for the best value of a. In Equation (48) ρ_k is the resultant of the recorded tests; θ_k, the corresponding angle in the polar representation; and n, the number of items. Equation (45) has been used to substitute ρ for F. To every experimental result there corresponds a definite set of co-ordinates, ρ_k and θ_k, which may be readily obtained from the data made available to the writer by Professor Marin; but instead of computing each term of the series, Equation (48), separately, the data will be grouped as follows: The right half of the stress plane (where all the experimental data fall) will be subdivided into eighteen 10° sectors numbered as shown in Fig. 20, and to the points falling within each sector will be assigned the angle of the sector mid-line, as shown in the same diagram. This subdivision is not only convenient but necessary for the statistical analysis of the problem. The magnitude of the angular opening of the class sectors adopted, has been dictated by personal judgment; it is desirable to make this magnitude small, but if it is made too small, the classes into which the data are thereby grouped will not contain a sufficient number of items with which to compute their statistical elements.

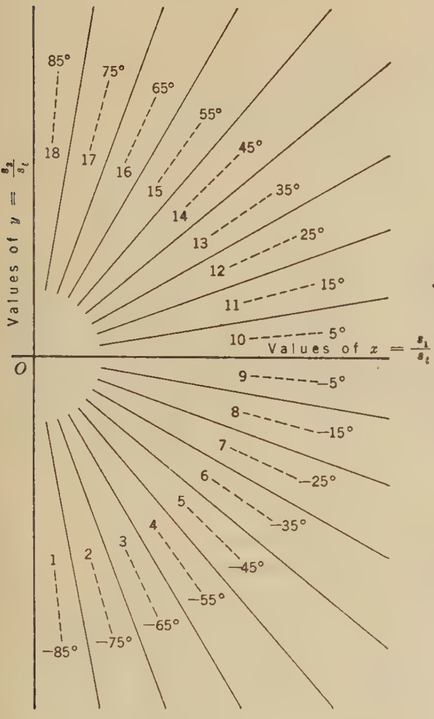

FIG. 20.— CLASS SECTOR DIVISION OF THE STRESS PLANE

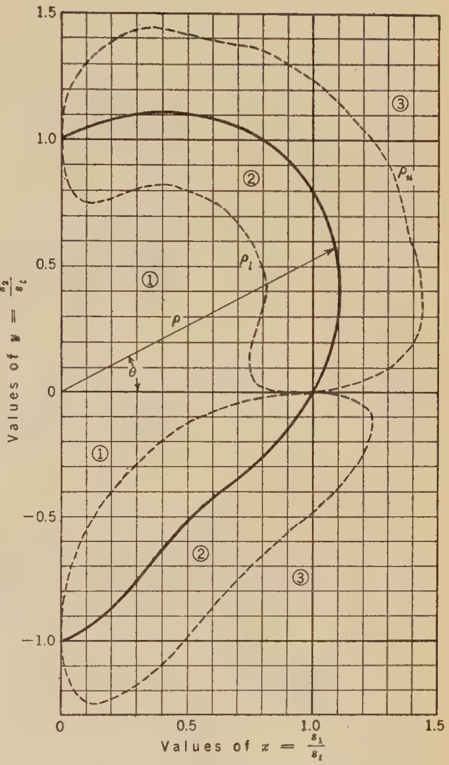

FIG. 21.— FAILURE CURVE WITH REGIONS OF ACCIDENTAL VARIATION

Adopting this class subdivision and inserting the values in Equation (48), obtained from the data on steel tubes presented by Professor Marin (which is the most extensive), the magnitude, 0.278, is obtained for the best value of a. Thus, the best two-term Fourier approximation to the failure law for steel tubes is:

$$\rho = 1 + 0.278 \sin 2\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

TABLE 2.—Comparison of Failure Curve with Test Results on Steel Tubes

Sector (1)	No. of items (2)	θ, in degrees (3)	Sector means (4)	Dispersion about means (5)	ρ (6)	Deviations of means (7)	Standard errors of means (8)	Variation limits (9)	ρ_l (10)	ρ_u (11)
1	0	—85	....	....	0.951	....	....	0.285	0.666	1.246
2	1	—75	(0.806)	....	0.862	—0.056	....	0.343	0.519	1.205
3	3	—65	0.778	0.026	0.789	—0.011	±0.014	0.335	0.454	1.124
4	1	—55	(0.940)	....	0.739	+0.201	....	0.315	0.424	1.054
5	60	—45	0.759	0.107	0.722	+0.027	±0.014	0.308	0.414	1.030
6	14	—35	0.877	0.057	0.739	+0.138	±0.015	0.315	0.424	1.054
7	17	—25	0.800	0.125	0.789	+0.011	±0.030	0.335	0.454	1.124
8	20	—15	0.856	0.084	0.862	—0.006	±0.019	0.343	0.519	1.205
9	25	— 5	0.911	0.098	0.951	—0.040	±0.019	0.285	0.666	1.246
10	4	5	0.981	0.076	1.049	—0.068	±0.038	0.285	0.764	1.334
11	4	15	1.091	0.166	1.139	—0.048	±0.083	0.343	0.796	1.482
12	6	25	1.253	0.067	1.212	+0.041	±0.028	0.335	0 877	1.547
13	11	35	1.247	0.109	1.261	—0.014	±0.033	0.315	0.946	1.576
14	13	45	1.357	0.186	1.279	+0.078	±0.052	0.308	0.971	1.587
15	7	55	1.186	0.043	1.261	—0.075	±0.016	0.315	0.946	1.576
16	2	65	1.092	0.039	1.212	—0.120	±0.027	0.335	0.877	1.547
17	0	75	....	....	1.139	....	....	0.343	0.796	1.482
18	0	85	....	....	1.049	....	....	0.285	0.764	1.334

Before discussing Equation (49) to determine its meaning as a mathematical expression for the theory of failure under bi-axial stresses, it is well to test its adequacy to represent the given data. For this purpose Table 2 has been constructed. The first column of this table lists the class numbers given in Fig. 20. The second column gives the number of experiments with representative points falling within the corresponding sectors. The third lists the corresponding sector mid-angles, also shown in Fig. 20. The mean resultant of the observed points for each class is given in Column (4). The standard radial deviation was computed for each class about the class mean and the results listed in Column (5), Table 2. Column (6) gives the values of ρ computed for the values of θ listed in Column (3). In Column (7) are listed the differences, Column (4)—Column (6); these, then, are the deviations of the sector means from the failure curve, Equation (49). For every sector, from the number of items in the class and the corresponding standard deviation, the standard error of the mean was computed and the results are listed in Column (8).

In the first place, it should be noted that the deviations from the failure curve of the sector means do not show any marked trend; their distribution is not incompatible with one of a random nature. Now, of the thirteen classes for which the statistical elements can be computed, the deviations from the curve of the means of five of them are less than the standard errors in the mean, and five others come well within three times the standard errors. Of the remaining three, two may be disposed of by the following arguments: In Sector 6, ten of the fourteen items represent experiments by Cook and Robertson, which, it will be noticed, show consistently high values for the resultant—as compared with tests by Scoble, for instance, which show consistently low results. This sector, therefore, is not a fair sample, and the high value of the deviation cannot be given full weight. Sector 16, with a high negative deviation, has only two items, from which a standard deviation of 0.039 was computed. Two items, however, cannot give a fair estimate of the dispersion, which is consistently high for the entire range of tests, averaging 0.091 for the thirteen classes and being 0.107 for Sector 5 which has sixty items. With such high dispersion it is not beyond the bounds of reasonable probability that the average of two experiments, particularly when performed by the same person (Guest), should deviate − 0.120 from the curve. The remaining class, Sector 15 with seven items, might be considered in like manner, but it is quite probable that one in thirteen classes should deviate from the norm by more than three times the standard error. Of the two classes with a single item each (Sectors 2 and 4) one deviates less than the average standard deviation, the other less than three times this deviation—a reasonable variation for single items.

Altogether, this analysis seems to show a high probability that Equation (49) is a close first approximation to the law of failure under bi-axial stresses for steel tubes.

The writer believes that it is important to make explicit recognition of the fact that the scatter of the experimental points is doubtless due more to errors of observation and faults of technique than to variations in the strength of the material, provided, of course, that the strength is kept within certain specification bounds. For the purpose of design it is desirable to set limits to the actual variation in the strength of the material, but there are no data available to the writer from which to make such an estimate. However, a rough estimate of the observational variation limits may be made from the data presented by Professor Marin. In the octant included between Line $\theta = 0°$ and Line $\theta = -45°$, where most of the experimental points fall, the circle of radius = 1 and the center at the point (1, − 1) follows quite well the trend of the lower limit of variation. The equation of this circle referred to the origin is:

$$r = \sin \theta' + \cos \theta' - \sqrt{\sin 2\theta'} \quad \ldots\ldots\ldots\ldots\ldots (50)$$

in which r is the radius vector and θ' equals θ in the fourth quadrant taken with a positive sign. The lower limits of experimental variation from Equation (49) will thus be given, in the fourth quadrant, by,

$$l = \rho - r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

Assuming an essential symmetry in the accidental variations two curves may be constructed in the four quadrants:

$$\rho_1 = \rho - l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

and,

$$\rho_u = \rho + l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

where, for all quadrants, l is computed from the fourth quadrant by Equations (50) and (51). The values for the limits of variation thus determined are given in Column (9) of Table 2. Columns (10) and (11) give ordinates of ρ_1 and ρ_u. In Fig. 21 are given the graphs of Equations (49), (52), and (53) in the right half of the stress plane. These loci divide the plane into three regions. Assuming material essentially homogeneous, and reasonably accurate laboratory technique, then one can state that in Region 1 no sample will be found to fail, all samples will be observed to fail somewhere in Region 2, and no samples will be found to persist in Region 3. These limits, no doubt, can be greatly narrowed when more is known regarding the differences between observed results and actual failures. It may be noted that three results of experiments by Becker (Sector 14) fall beyond the tentatively assigned upper limit.

From the available data little can be said about the distribution of failures throughout Region 2; but taking as typical the limits, ±0.315, and a standard deviation of 0.105 (this value, which is close to the average of the standard deviations, is taken so as to make the ratio, $\frac{0.315}{0.105} = 3$, an even number), one may construct a distribution curve from Table 22 of the closing discussion of the writer's paper.[11] From the first section of Table 22, for the symmetric distribution function, for the value, $\lambda = 3$, the quantities listed in Table 3 are obtained.

TABLE 3.—DISTRIBUTION OF OBSERVATIONAL VARIATIONS

Percentage of samples...	1	5	10	15	20	30	40	50	60	70	80	85	90	95	99
Deviation...	−0.233	−0.184	−0.151	−0.126	−0.104	−0.066	−0.031	0.0	0.031	0.066	0.104	0.126	0.151	0.184	0.233

The meaning of Table 3 is that, in testing a large number of samples, 5% of them, for instance, will be observed as weak or weaker than the value given by Equation (49), less 0.184; or, that 70% of samples will be observed as weak or weaker than the value given by Equation (49), plus 0.066.

Fig. 22 shows this distribution function. For comparison the actual distribution is given of the tests in Sector 5 about the class mean. The two are not strictly comparable because neither the means nor the standard deviations coincide; but the comparison is made, nevertheless, to show that, in general, the distribution of observed failures in Region 2 follows that required by theory.

The present statistical analysis of the problem has led to Equation (49) as a first approximation to the law of failure for steel tubes under bi-axial stresses. The available statistics do not warrant a closer approximation, but they seem to give to this equation a high probability of approximating closely the true law. It remains to discuss this law from the point of view of a possible theory of failure that might give rise to it.

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[11] "An Asymmetric Probability Function", see p. 95 *et seq.*

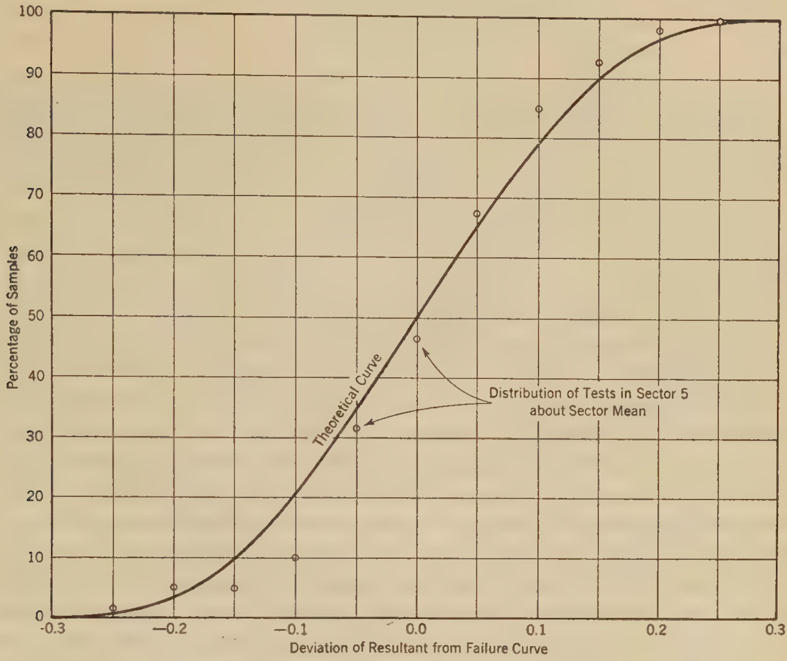

FIG. 22.— DISTRIBUTION OF FAILURES ABOUT FAILURE CURVE

The value, $a = 0.278$, found for the constant of Equation (47) suggests strongly that this constant might be Poisson's ratio; or, perhaps (as will appear subsequently in the discussion of the failure of brass tubes), some function of this ratio. One might indeed expect, *a priori*, to find this important constant included in the law of combined stress failure. Assuming $a = \sigma$ for the moment, and using the relation, $\sin 2\theta = 2 \sin \theta \cos \theta$, and the co-ordinate transformations, Equations (43) and (44), then the law of failure, Equation (49), becomes,

$$\sqrt{x^2 + y^2} = 1 + \frac{2\,\sigma\,x\,y}{x^2 + y^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

or, more compactly (using the value for ρ given by Equation (43)),

$$x^2 + y^2 - \left(\frac{2\,\sigma}{\rho}\right) x\,y = \rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

In this form the law is seen to bear a striking resemblance to the expressions for several of the theories presented by the author. Equation (55), in fact, may be considered as the equation of a continuously varying ellipse.

In Sectors 9 and 10 this curve coincides closely with the maximum strain-energy theory, but elsewhere it intersects or parallels the curves representing the other theories and thus does not seem to bear out the author's conjecture that the correct theory may be a combination of several of these theories depending on the ratio, $\frac{y}{x}$. In the writer's opinion the correct theory of failure, an approximation to the mathematical expression for which is given by Equation (49) or Equation (55), can be formulated only after taking into account: (*a*) The non-linear character of the stress-strain relation in the neighborhood of failure; (*b*) the modification of Poisson's ratio in that region; and (*c*) the relation of the direction of the plane of fracture to θ in the polar representation.

The experiments on brass tubes given in the author's paper would seem to indicate that the constant, a, of Equation (49) is not simply equal to Poisson's ratio, for the substitution of $\sigma = 0.35$ in Equation (55) gives a poorer representation of the results than a much smaller quantity would give. There are not enough tests given on brass tubes, however, to make a significant statement regarding the form of this constant. The few results given on copper tubes, on the other hand, more nearly confirm the conjecture that $a = \sigma$. The experiments on steel rods, on an average, give higher values than are obtained by Equation (49) but this is, perhaps, what should be expected.

In a tube the condition of failure can be attained more uniformly than in a solid rod. In the latter failure may occur at a point before failure can be detected experimentally. Where failure does not occur uniformly there is a chance for the phenomenon of "wire-drawing" to add its reinforcing effect to the test rod. In this case Equation (49) would give the first failure, or "point" failure, and whether this can be detected depends on the achievement of very exact laboratory technique. Cast iron presents characteristics of failure that seem to be definitely different from those of the more elastic and homogeneous materials. Experiments on this material show a scatter of the representative points which is smaller than that for tests on steel, so that a few careful experiments that plot in the first quadrant of the stress plane would no doubt give a good indication of the failure characteristics of this material. A negative value of the constant, a, of Equation (47) is necessary to represent the cast-iron data properly, but the law thus determined might not follow through into other quadrants. Another term in the Fourier series may be necessary to approximate closely enough the law of failure for this material, a term which becomes insignificant in the case of the failure of steel tubes. This case illustrates the importance of formulating carefully the correct theory of failure.

In conclusion, the writer wishes to express his appreciation of Professor Marin's work of unifying and presenting this valuable information collected from such wide sources; his paper is a great step toward the correlation of theory with experiment in this interesting branch of mechanics, which marks the way to greater efficiency in design.

T. McLean Jasper,[12] M. Am. Soc. C. E. (by letter).—In discussing the paper by Professor Marin it is well to point out two fundamental differences which might cause confusion in the discussion of theories of failures. It is believed, that the general testing methods adopted to discover the theories of material failure have not been given proper differentiation as between the failure of a structure because of its shape and the failure of a material based on its fundamental strength qualities.

When buckling is introduced into the test sample as a possible method of failure of a structure, it is obvious that failure can occur at values of stress that are much below the elastic limit or yield point of the material. Such a failure depends on the shape and on the elastic properties of the test pieces, rather than on the yield or general break-down of the material.

Examples of structure failure, in which buckling occurs at varying values of stress depending on the relative slenderness of the test samples, are contained in the column problem, the tube collapse problem, and the thin tube torsion problem. Many of the theories of material failure are based on test samples in which buckling, as a method of failure, enters, and the confusion which has associated itself with various theories, as shown in the author's Fig. 19, is largely attributable to lack of appreciation of this fact. The writer's experience with testing has brought to light failure of slender steel columns at less than one-third the yield strength of the material. The slender tube has collapsed at less than one-third the yield strength of the steel, and the slender torsion tube has collapsed at less than two-thirds of the torsional yield strength of the steel. In each case as the slenderness of the test sample has been reduced, the failure point of the various aforementioned structures has increased progressively.

Many of the experimenters have applied as experimental test samples one or more of the aforementioned methods which are associated with this buckling and as different investigators have varied the relative slenderness of test shapes, they do not check even within large percentages with each other.

A good example of what the writer wishes to convey is contained in the experimental work of Guest,[13] who is responsible for the maximum shear theory of failure. The record of his work shows that great care was taken in his experimentation, and since it was recorded in such detail, it allows a careful study of the test procedure. In his experimental work, Guest used thin cylinders and performed as many as twenty tests on a single test sample. He used a combination of torsion, internal pressure, and tension or compression, or various combinations of these testing methods. In certain cases, he repeated the combinations of stress application more than once on a single test sample. For instance, the first test might be that of simple shear, and after various other tests were made on the same sample, he again resorted to simple shear. Invariably, when he repeated identical tests on the same sample he obtained the same results.

The point the writer wishes to make is this; Since his standard of measurement depended on the stress-strain diagram and the deviation from proportionality, it is obvious, that if this point is exceeded in terms of yielding of the metal, then with each test a new yield point is established. If, on the other hand, elastic buckling only is responsible for the phenomena, the limit of proportionality of the structure can be exceeded without interfering with the yield of the metal and such elastic buckling will repeat itself indefinitely. If Guest had changed the slenderness of his test sample appreciably, he would have obtained quite different results.

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[12] Research Dept., A. O. Smith Corporation, Milwaukee, Wis.

[13] "On the Strength of Ductile Materials under Combined Stress", by J. J. Guest, *Philosophical Magazine*, July, 1900, p. 69.

There is exceeding good evidence that elastic structures fail very nearly according to the mathematical theory of elasticity, especially in the zone in which the elastic properties of the materials impose the controlling factor. The failure of materials, such as separation into two parts can occur only in tension or in shear according to the strict interpretation of these terms. The writer has obtained fatigue fractures from different samples of the same bar of steel which occurred separately as tension and as shear.

It is very probable, that neither the maximum shear theory nor the maximum stress theory can explain the break-down of the steel or other materials in bi-axial or tri-axial loading. The writer leans toward the idea that some form of maximum energy will yield the most reliable answer to the problem of the theory of failure of a material.

This discussion is not presented with the idea that the answer to the problem of material failure is known definitely. The method of determining the answer must be disassociated from buckling. It is possible that different materials may yield to different solutions; for instance, glass probably will produce a different result than steel. Steel will possibly have a different solution than aluminum. On the other hand, the failure of structures has yielded to consistent ideas, so that the service problem of the application of materials has not suffered because the various theories of material failure do not coincide.

I. K. Silverman,[14] Jun. Am. Soc. C. E. (by letter).—The conditions causing plasticity have been of great interest not only to physicists but also to engineers, since the limiting states of stress are intimately bound up with working stresses and factors of safety.[15] Civil engineers use the Rankine or maximum stress theory almost exclusively, whereas mechanical engineers favor the maximum shear theory, especially for ductile materials.

The author lists sixteen theories. The best known of these are Theories (1), (4), (7) to (11), (14), and (16). Theory (5) reduces to that of Theory (7); Theory (12) reduces to Theories (10) and (11); and, Theories (6), (13), (14), and (15) seem to have very little experimental support. By comparing the hypotheses of Theories (11) and (13) it is seen that the fundamental ideas thus expressed are diametrically opposed. The statement of Theory (11) is based upon the experimental fact that materials can withstand large hydrostatic pressures without failing. What is the basis for the statement expressed in Theory (13)?

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[14] With U. S. Bureau of Reclamation, Denver, Colo.

[15] "Factor of Safety and Working Stress," by C. R. Soderberg, *Transactions*, Am. Soc. Mech. Engrs., APM 52-2.

These various theories can be compared by examining the case of pure shear, in which $s_1 = -s_2$, and from which a relation between the yield stress in shear and that in uni-axial stress may be obtained. Writing this relation as,

$$s_T = k\, s_t \quad \text{.........................} (56)$$

the values of k for the various theories mentioned by the author, are given in Table 4.

TABLE 4.—COMPARISON OF k-VALUES

Theory No. (1)	Special condition (2)	Value of constant, k (3)	Theory No. (1)	Special condition (2)	Value of constant, k (3)	Theory No. (1)	Special condition (2)	Value of constant, k (3)
1	....	1	8	....	0.6†		By Equation	
2	....	0.83	9	$a = 1$	0.577		(27):	
3	....	0.83	9	$a = 1.3$	0.54	12	$\sigma = 0.3$	0.62
4	$\sigma = 0.25$	0.8	10	$\sigma = 0.25$	0.63	12	$\sigma = 0.35$	0.61
4	$\sigma = 0.30$	0.77	10	$\sigma = 0.3$	0.62	12	By Equation	
4	$\sigma = 0.35$	0.74	10	$\sigma = 0.35$	0.61		(28)‡	0.577
5	....	0.5	11	....	0.577	13	....	0.5
6	....	0.5	12	By Equation		14	....	0.71
7	....	0.5		(27): * $\sigma = 0.25$	0.63	15	....	0.5
						16	....	0.6†

* Same as Theory No. 10. † Assumed. ‡ Same as Theory No. 11.

Experiments by Becker[16] have shown that k is approximately 0.6, and that, apparently, for the state of stress expressed by $s_1 = s_2$ failure occurs when $s_1 = s_2 = 1.2\, s_t$. This value may have been influenced by the fact that the tests were not purely bi-axial. More recent experiments[17] seem to support the theory of Huber-Hencky-von Mises.

W. P. ROOP,[18] M. AM. SOC. C. E. (by letter).—The need for a "failure theory" is not immediately apparent; it is a rather naive but widespread idea that a safety factor, taken as the ratio of ultimate to working stress, represents the margin provided to prevent failure in spite of the designer's mistakes. In point of fact "there is a considerable number of variables outside of magnitude of stress, involved in the problem of strength"[19]; the burden thrown on the safety factor by uncertainty as to what determines failure under combined stresses is not responsible for a large share of the margins of strength found necessary.

Nevertheless, some sort of criterion by which to allow for combination of stress is necessary, and as the other variables in the problem of strength are subjected to more searching analysis, it is right that this one should also be re-examined. For this purpose sixteen different theories are reduced to forms suitable for comparison, both analytical and graphical. It is understood that the curves shown represent boundaries beyond which plotted points indicate failure. Although it is not clearly so stated, it is understood furthermore, that the co-ordinate of these points are the relative values of the principal stresses. Each such point thus represents a certain proportion between the two principal stresses, the scales being such that the ultimate strengths in simple tension and compression are correctly indicated, so that all curves (except that for Theory (8)) pass through all the four points $x = \pm 1$, $y = \pm 1$.

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[16] *Bulletin No. 32*, Univ. of Illinois, Vol. 13.

[17] W. Lode, *Zeitschrift für Physik*, Vol. 36, 1926; also, Ros and Eichinger, *Proceedings*, Second International Cong. for Applied Mechanics, Zurich, 1926.

[18] Lt. Comdr. (C.C), U. S. Navy, Office of Superintending Constructor, New York Shipbuilding Co., Camden, N. J.

[19] *Journal of Applied Mechanics*, 1935, p. 106.

It appears in Table 1 that "failure" may mean anything from rupture to exceeding the proportional limit. In the graphic study, these differences are ignored and only the effect of combining stresses is considered, with no reference to characteristics of the material.

The stress at a point in a homogeneous elastic medium is a condition that can be specified exactly, regardless of the combination of loads, each with its contribution to stress, by which the actual result is produced. The real questions thus appear to be: (*a*) Considering the stress at the point at which "failure" begins, what feature of the stress is it that determines the failure; or, in a word, what constitutes equivalence of stresses in causing failure? and (*b*) in strength calculations involving combined loads, how shall the effective feature of the resultant stress be estimated?

Fig. 19 indicates that the graphs generally agree in showing an elongation along the axis of half past one. Does this mean that a combination of principal stresses of the same sign is more easily withstood than a combination of opposite signs? If so, that would confirm the idea that somehow shear plays a special part in causing failure.

It appears that no more definite conclusion than this is possible from the data presented, since the spots in Fig. 19 and the data in Columns (6) and (7), of Table 1, show that the spread of observations exceeds the differences between the theories.

H. F. MOORE,[20] ESQ. (by letter).—An interesting summary of the present-day knowledge of theories of failure of materials subjected to combined stress, is presented in this paper. Table 1 is a very convenient statistical summary of the evidence. It is to be noted that, with the exception of the tests with solid cast-iron rods under combined tension and compression (at right angles), Professor Marin has used the criterion of elastic strength in all cases. He quotes the strength of a tube of cast iron as judged by "yield point". This reporting of a yield point in cast iron would seem to call for further explanation, as cast iron ordinarily does not have a yield point. At first sight, the various criteria used for elastic failure would rather confuse the evaluation of results, but a closer examination shows that for each comparison of theory with experimental results the same criterion is used, so that tests may be compared fairly satisfactorily. It would be of interest if Professor Marin had given references for the particular experiments in each case listed in Table 1; in fact his paper would be distinctly more valuable if references to original sources of data were given more fully.

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[20] Research Prof. of Eng. Materials, Univ. of Illinois, Urbana, Ill.

Two points are not covered by the paper. The first is the failure of materials by other than elastic failure. Professor Marin has, indeed, quoted tests on cast iron in which theory was judged by the strength at fracture. These tests seem to indicate that the maximum strain theory fitted test results more closely than the other theories. This result is checked by tests obtained by Matsumura and Hamabe[21] on tests of cast iron under combined bending and torsion. In the 1933 Marburg Lecture before the American Society for Testing Materials, Dr. H. J. Gough, of the British National Physical Laboratory, quoted fatigue tests of mild steel under combined stresses[22] which indicated that, using fatigue fracture as a criterion of strength, the maximum shear strain-energy theory (Theory (11) in the paper) fitted results best; and the maximum shear theory (Theory (7), in the paper) showed fairly close agreement with test results and was slightly on the "safe" side. It is quite possible that different theories hold for ductile and for brittle materials, and also for failure by fracture as opposed to failure by plastic action. This raises the question of failure by continued flow or creep, which has become important at high temperatures. No data on creep under combined stresses are available, so far as the writer knows.

Another point not covered by Professor Marin is the question of failure by "tri-axial" stress, especially a failure under equal tensile stresses along three axes of reference. So far, no effective experimental study of this phenomenon has been made. Under equal tensile stresses in three directions the shearing stress in a metal would be zero, and it is a matter of dispute whether this would strengthen the metal by preventing shearing failure, or weaken it by inhibiting the occurrence of slight localized plastic action which usually is found in metals even under light stresses, and which tends to reduce the "peaks" of localized stress which, without this plastic action, would be developed on minute areas. The writer does not know any more intriguing problem than the study of this "triple tension", which has been discussed so vigorously by Professor B. P. Haigh, of the Greenwich Naval Academy Laboratories, at Greenwich, England.

A. A. Eremin,[23] Assoc. M. Am. Soc. C. E. (by letter).—Tests of materials are not sufficient on which to base conclusions as to the validity of the theories of failure treated in this paper. Under "Conclusions", Professor Marin states that, "the precisely correct theory of failure is probably a combination of a number of these theories, depending upon the ratio, $\frac{s_1}{s_2}$".

Each theory with its basic assumptions and limitations has its individual historical and practical value. The theory of the strength of materials has always been a favorite subject of scientific research among mathematicians. In the first part of the Seventeenth Century the theories of elasticity were based on empirical values of either twenty-one or fifteen unknown constants.

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[21] *Memoirs*, Coll. of Eng., Kyoto Imperial Univ., Kyoto, Japan, February, 1915.

[22] *Proceedings*, Am. Soc. for Testing Materials, Vol. 33, Pt. II, pp. 103–104.

[23] Assoc. Bridge Designing Engr., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

Discussion of some of these theories continues to the present.[24] Theorists and mathematicians have contributed numerous equations for computing ultimate forces sustained by various elastic members. It is an imposing task to apply some of these equations and theories in practice. The Euler formula for a column sustaining direct force was considered erroneous for a number of years. Only recently have the practical limitations of Euler's equations been understood. His equations have found wide application in the theory of the strength of materials.[25]

An interesting practical application of the theory of maximum shear stress failure has been made by Professor J. Fritsche.[26] He has developed the equations for computing the relation between direct stresses in a concrete test prism and ultimate stresses in concrete under combined stresses. Furthermore, in the equations for computing the limiting stresses in concrete, Professor Fritsche introduced an exponential expression for variation of strain and stresses in concrete. However, as stated[27] correctly by the late George F. Swain, Past-President and Hon. M. Am. Soc. C. E., no one has yet proposed a reliable instrument for measuring the variation in stresses in material sustaining combined forces in space in such a manner as to verify the theories of failure.

A. FLORIS,[28] ESQ. (by letter).—The critical examination of the existing theories of failure advanced by various investigators, and their correlation by means of a common set of co-ordinate axes, is the subject of this interesting paper. To the theories of failure treated, the writer wishes to add one more, proposed by Professor G. D. Sandel.[29]

The change in shape of a body under the influence of external forces, caused by the change in position of its particles relative to each other, can be considered as the measure of ultimate strength of the material. Consequently, the deformation of a body is determined by the slip of its particles along principal shearing planes and also by its volume change. The theory proposed by Sandel is based upon this reasoning. It is identical with the maximum shear theory when the influence of the intermediate principal normal stress is equal to zero. In general, however, this influence can be explained only if it is assumed that the volume change contributes also to the rupture of the material. At the limiting state of stress, therefore, the greatest sliding reaches a limiting value which afterward decreases linearly with the positive volume change (increase) of the body. Evidently, Sandel's theory of failure is a generalized maximum shear theory.

In the case of a bi-axial state of stress and with the author's notation, Sandel's theory of failure is expressed by,

$$(n + 1)\, s_1 + (n - 1)\, s_2 = 2\, s_s \qquad \text{(57)}$$

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[24] "A Treatise on the Mathematical Theory of Elasticity", by A. E. H. Love, 1927, p. 13.

[25] "Strength of Materials", by S. Timoshenko, 1916.

[26] *Beton und Eisen*, April 5, 1935.

[27] "Strength of Materials", 1924, p. 539.

[28] Dipl.-Ing., Los Angeles, Calif.

[29] "Ueber die Festigkeitsbedingungen: Ein Beitrag zur Loesung der Frage der zulaessigen Anstrengung der Konstruktions-materialien", von G. D. Sandel, Leipzig, 1925.

in which, s_s is the shearing stress at failure; and $n = \frac{s_c - s_t}{s_c + s_t}$ denotes the degree of brittleness of the material. For structural steel, $n = 0.04$; for cast iron, $n = 0.60$; and for concrete, $n = 0.88$.

For $n = 0$, Equation (57) expresses the maximum shear theory and for $n = 1$, the maximum stress theory of failure.

The bi-axial state of stress is easily extended to that of tri-axial stress by adding the term, $n\ s_3$ of the intermediate principal normal stress to the left-hand side of Equation (57).

The author is to be commended for bringing this important subject to the attention of engineers who, in practice, are accustomed to deal with allowable stresses.

JOSEPH MARIN,[30] JUN. AM. SOC. C. E. (by letter).—The interpretation of the test results for materials subjected to combined stresses as given by Professor Slade offers an interesting method of attack. It is essentially the same as the method used in the modified Mohr theory (Theory (9)), since in both cases an average value of the test results is used. As might be expected, both these semi-empirical methods give approximately the same results. It is apparent that Professor Slade's interpretation is more flexible and basic in its application. However, the usefulness of his method in this particular problem, is reduced in view of the new data presented in this closure.

Mr. Jasper has revealed a significant factor in referring to the error introduced in some experiments due to a buckling type of failure. This danger, however, can be avoided by a preliminary analysis in which the size of the specimen can be determined in such a manner that failure by buckling is avoided.

The writer wishes to thank Mr. Silverman for referring to the work of Lode and that of Ros and Eichinger. Later work by these investigators has been reported.[31] In addition, the recent experimental work of Taylor and Quinney[32] should be mentioned. A further study by the writer, of the experimental results obtained by these investigators and others, shows that for ductile metals the maximum shear energy theory (Theory (11)) gives decidedly the best agreement with the experimental results, whereas, for a brittle material, such as cast iron, the few test results available show that the maximum stress theory agrees best with the experiment. The basis for the formulation of the maximum volume energy theory as requested by Mr. Silverman, is that failure in the case of an element subjected to combined stresses is assumed when these stresses have reached such values that the volume change produced as a result of deformation reaches the value of the volume change produced in the case of simple tension at failure.

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[30] Asst. Prof. of Eng. Materials, Coll. of Eng., Rutgers Univ., New Brunswick, N. J.

[31] "Der Einfluss der mittleren Haupspannung auf das Fliessen der Metalle", by W. Lode, V. D. I. *Forschungsarbeiten*, Heft 303, 1928; also, "Eigenössische Materialprüfungsanstalt an der E. T. H. in Zurich, Versuche zur Klarung der Frage der Bruchgefahr", by M. Ros and A. Eichinger, *Diskussionsbericht*, Nr. 28 (1928) and Nr. 34 (1929).

[32] "Plastic Distortion of Metals", by G. I. Taylor and H. Quinney, *Philosophical Transactions*, Royal Soc. of London, Series A, Vol. 23, 1931, p. 323.

The writer does not understand the statement made by Lieut. Comdr. Roop that "the need for a 'failure theory' is not immediately apparent." The ultimate strengths in simple stress are not always denoted by the points, $x = \pm 1$, $y = \pm 1$. It is the practice usually to define failure for a ductile material such as steel by the lower yield point (if such exists), or a proof stress, while for a brittle material such as cast iron it is the practice to use the ultimate stress. The more definite conclusion requested by Lieut. Comdr. Roop is given in the foregoing paragraphs.

Professor Moore mentions the tests on cast iron subjected to combined stresses made by Matsumura and Hamabe as supporting the maximum strain theory. However, tests on cast iron subjected to combined stresses made by Ros and Eichinger[31] and others made by Cook and Robertson[33] show that the maximum stress theory is in closer agreement with the test results. Regarding fluctuating stresses, tests by Gough[34] more recent than those mentioned by Professor Moore verify that, for the two kinds of steel tested, the maximum shear energy theory is a good approximation. The writer agrees with Professor Moore that there is sufficient evidence to show that different theories hold for ductile and brittle materials. Some recent work has been done on the subject of creep under combined stresses, both experimentally and analytically, principally by Bailey.[35]

The problem of the failure of materials subjected to tri-axial static stresses was intentionally omitted from the paper in order to present first the simplest case. In answer to Professor Moore's comment about tri-axial stresses, the researches of von Kármán[36] and Boker[37] on brittle materials, and more recently that of Cook[38] on steel, should be mentioned. There are apparently many problems yet to be investigated regarding the failure of materials subjected to static and fluctuating combined stresses at both normal and elevated temperatures.

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[33] "The Strength of Thick Hollow Cylinders Under Internal Pressure", by G. Cook and T. Robertson, *Engineering*, Vol. 92, p. 786.

[34] Rept. of National Physical Laboratory, England, by H. C. Gough, *Engineering*, July 12, 1935, p. 43.

[35] "Creep of Steel Under Simple and Compound Stresses", by R. W. Bailey, *Engineering*, January–June, 1930, Vol. 129, pp. 265, 327; also, "Utilization of Creep Test Data in Design", *Engineering*, December, 1935, and preprint for November meeting of the Inst. of Mech. Engrs., London, England.

[36] "Festigkeitsversuche unter allseitigem Druck", Th. von Kármán, M. Am. Soc. C. E.. *Mitteilungenüber Forschungsarbeiten*, V. D. I., Heft 118, 1912.

[37] "Die Mechanik der bleibenden: Formänderung in Kristallinischaufgebauten Körpern". by R. Boker, *Mitteilungenüber Forschungsarbeiten*, V. D. I., Heft 175–176, 1915.

[38] *Proceedings*, Royal Soc. of London, 1931, Vol. 37, p. 559.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 1948

ADAPTATION OF VENTURI FLUMES TO FLOW MEASUREMENTS IN CONDUITS

By Harold K. Palmer[1], M. Am. Soc. C. E., and Fred D. Bowlus[2], Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. N. F. Hopkins, Hunter Rouse, F. V. A. E. Engel, J. C. Stevens, Filippo Arredi, and Harold K. Palmer and Fred D. Bowlus.

Synopsis

A weir can be considered as a control section through which water flows at critical depth. The sharp angle at the face and the fact that the flow is convex where it passes the critical section introduce energy losses between the point of measurement and the control section, the amount of these losses depending upon the setting of the weir in the channel. The ordinary weir formula is an empirical equation that is accurate only as long as the fundamental conditions upon which it was developed can be duplicated. These conditions can rarely be complied with in a confined channel, such as a sewer or an irrigation canal.

This paper presents the theoretical hydraulic principles involved, and the results of special tests made, in the adaptation and construction of various Venturi flumes for measuring flow in conduits of uniform cross-section, where weirs have proved unsatisfactory.

Since the uncertainties or variations in weir coefficients are due to indeterminable energy losses, it is reasonable to suppose that if these losses can be eliminated or reduced to a negligible amount by the use of some other device, the uncertainties in the rating curve will be eliminated. Such conditions are found in the so-called Venturi flume which, in this paper, includes any streamlined device placed in an open channel, or a closed channel partly full, having a sufficient constriction to cause water to flow at critical depth with parallel filaments.

Note.—Published in September, 1935, *Proceedings*.

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Any shape of throat may be used, and the flow can be determined from a single depth measurement, using a rating curve drawn from rational formulas. Several of these Venturi flumes have been constructed, including one that was simply a flat slab on the bottom, and had no side contractions; one that was rectangular in cross-section; and several with trapezodial-shaped throats, all of which have given good results.

The writers show how the rating curve may be drawn, and give graphs for use with circular conduits.

INTRODUCTION

Weirs have long been considered the standard devices for measuring running water, but their use involves empirical formulas, and they must be built under restricted conditions, unattainable in many classes of conduits. The ponding effect of the water up stream from the weir causes deposits of suspended matter which often alter the hydraulic conditions so that the empirical formula fails to give the correct flow. In the case of sewers, sludge deposited in this manner will decompose in time and become an added source of trouble. Another objection to the use of weirs in a closed conduit is the relatively large loss of head.

Many factors affect the proper installation of weirs as shown by Schoder and Turner[3] in an able discussion of precise weir measurements. The United States Bureau of Reclamation has found from experiments[4] that the ordinary sharp-crested Cipolletti or rectangular type of weir must have certain very definite requirements for a fair degree of accuracy; for instance, the distances from the bottom and sides of the channel to the edges of the weir must not be less than twice the depth of water over the weir, and the channel above the weir for a distance of 20 or 30 ft must have a cross-sectional area at least six times that of the over-flowing sheet of water at the weir crest. When the depth of the water passing over a weir exceeds three-tenths of its width, the engineers of the Reclamation Bureau have found that the standard formulas indicate quantities that are too small. The error ranges from zero at three-tenths to 30% when the depth equals its width. These restrictions limit the capacity of the weir to about 40% of the capacity of a closed conduit.

Occasionally, the true Venturi flume has been used to advantage because it gives a minimum of ponding and the frictional losses, being small, result in little loss of head. However, it has the disadvantage of requiring the measurement of the area of cross-section as well as the velocity of the water flowing in the throat. This has only been possible in the past by making two simultaneous depth measurements.

Use of the Parshall meter,[5] which is one form of a Venturi flume, obviates some of the difficulties previously mentioned, but it is not readily installed in conduits already constructed because of the required 3-in. drop in the invert grade at the throat. It could scarcely be adapted to an existing, standard sewer manhole, say, 4 ft in diameter. In the design of a new system of conduits for which the desired points of flow determinations might be predetermined and a few inches of grade sacrificed, Parshall meters could be (and have been) installed to advantage. They have certain disadvantages in the continuous measurement of sewage, requiring daily inspection when a week or more of record is required, because the orifice between the channel and the wet-well often becomes clogged and the formation of sludge after a day or two in the wet-well tends to raise the float, thereby causing an error in the stage recorder reading.

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[3] "Precise Weir Measurements," by E. W. Schoder, M. Am. Soc. C. E., and the late Kenneth B. Turner, Esq., *Transactions*, Am. Soc. C. E., Vol. 93 (1929), p. 999.

[4] "Measurement of Irrigation Water", U. S. Dept. of Interior, Bureau of Reclamation. Third Edition, 1925.

[5] "The Improved Venturi Flume", by R. L. Parshall, Assoc. M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 841.

Ordinary Venturi tubes may also be used for liquid flow measurement, but larger structures are required for their installation. Such a tube must be entirely submerged and in sewers this is likely to prevent the proper handling of light floating material. Its efficiency is often impaired by clogging of the pressure recording pipes and, including the recorders, the initial installation is more costly than the weir and stage recorder units.

In 1920, Hinds[6] suggested placing a sufficient constriction in a channel to cause the water to flow at critical depth, on the assumption that in this case the energy head is fixed, whereas the depth is uncertain. He presented experiments made by the U. S. Bureau of Reclamation,[7] on a flume similar in many respects to one type considered by the writers and showed an error of less than 5% in measurement of flow, assuming no transition losses.

The following analysis of the adaptation of the Venturi flume to flow measurements in conduits of regular cross-section is applicable not only to clear water, but also to any liquid carrying suspended matter. Among the latter are included silt-laden irrigation water, storm water, and sewage. The investigations herein reported were made in connection with a sanitary sewerage system and as sewage presents most of the difficulties encountered in the other classes of water, the word, "sewage," will be used subsequently to denote water that carries settleable solids.

In passing, it may be stated that an accurate measurement of sewage flows in various parts of a sewerage system is quite important in order to be able to predict with reasonable accuracy the need of future works, relief sewers, additional pump capacity where pumping is required, etc. In joint sewerage works an accurate measurement of sewage is necessary to apportion costs to respective participants. Furthermore, as a check on pumping efficiency and power requirements, knowledge of the actual sewage flow is necessary. An accurate gauge of the flow is required for efficient and economical operation, where pre-chlorination[8] is practiced since the chlorine feed depends on both the strength and the quantity of the sewage. One difficulty encountered in an accurate measurement of sewage flow in new sewerage works is the fact that a sewer is usually constructed of a greater capacity than is required at the time it is first placed in operation. The flow in the line is then relatively shallow. Furthermore, in a new system of any large extent, the increase in sewage from month to month or year to year is relatively large and many measuring devices which might be satisfactory at one time would prove totally inadequate at a later date. Sewage flows range from a minimum in the early morning to a maximum of as much as two or three times the minimum amount later in the day; it will also vary from day to day, and any meter used for the purpose must be adaptable to such variations.

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[6] "Venturi Flume Data Throws Light Upon Control Weir", by Julian Hinds, M. Am. Soc. C. E., *Engineering News-Record* (1920), p. 1223.

[7] "The Improved Venturi Flume", by R. L. Parshall, Assoc. M. Am. Soc. C. E., discussion by Julian Hinds, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 864.

[8] "Control of Sewage Condition by Chlorination", by F. D. Bowlus, and A. P. Banta, Assoc. Members, Am. Soc. C. E., *Water Works and Sewerage*, November, 1932, Vol. 79, p. 369.

Difficulties encountered with ordinary weirs and other similar devices in the measurement of sewage indicate that the most satisfactory meter would incorporate the use of a stream-lined flume, with minimum head loss, preventing deposits of sludge above it; without sharp edges to catch rags and other floating refuse; and with sufficient capacity to measure the entire designed flow and yet be sensitive enough to record low flows in the first years of its use. The flume must be adaptable to the use of a simple float mechanism for water-stage recording, and must be designed so as to be readily installed in an existing conduit with minimum interference to the sewage flow. This paper deals with a measuring flume designed so that the flow may pass without objectionable ponding up stream; which offers no obstruction to floating or suspended solids; and which affords an opportunity to measure the flow from a continuous record of the water-surface elevation at a single point in the channel up stream from the flume.

The formulas developed for the flume are not empirical but are based entirely on theory and can be applied to conduits of any regular size or shape. Irregular channels would require regulation for a short distance up stream by lining of some kind. Uniformity of shape rather than factors affecting friction are important. The method has been developed especially for sewers, but is equally applicable to other channels, irrigation canals, storm drains, etc.

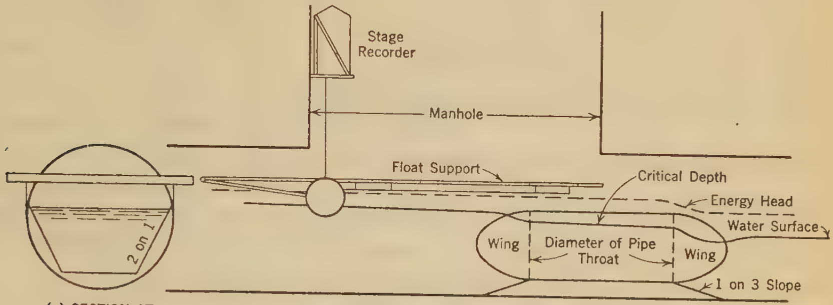
NOTATION

The symbols introduced in this paper are defined as follows:

b = a bottom width.
c = a subscript denoting "a critical depth."
d = depth of flow.
f = "function of."
g = acceleration due to gravity.
n = Kutter's coefficient of roughness.
v = a subscript denoting "velocity."
A = area.
B = surface width; width of water surface.
C = Chezy's coefficient.
D = diameter.
H = head; H_v = velocity head.
K = a constant in Kutter's formula = $A\,C\sqrt{R}$.
Q = rate of flow, or discharge.
R = hydraulic radius.
S = hydraulic slope; slope of a conduit, expressed as a percentage.
V = average velocity in a section.
Δ = "difference in;" ΔH_v = difference in velocity heads in throat and conduit.
ϵ = energy head; ϵ_c = energy head at critical depth.
ϕ = "function of."

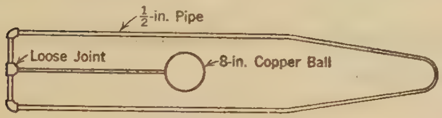
Theory of Design

According to Bernoulli's theorem, the energy head of each pound of water in a conduit is the height of the water surface above a given datum plane, plus the velocity head. In passing from one section of a conduit to another, frictional and other losses are measured by the decrease in the total energy head regardless of changes in depths and velocities at the two places. Therefore, measurement of the total energy head is the logical basis for the development of any theory in the design of a Venturi flume of the general type illustrated in Fig. 1.



(a) SECTION AT CENTER OF THROAT

(b) LONGITUDINAL SECTION OF METER IN 24-IN. PIPE



(c) PLAN OF FLOAT SUPPORT

Fig. 1.

Under certain conditions a calculation of the flow may be made by simply measuring the depth of the water flowing in a conduit of regular cross-section at any convenient point immediately above the throat of the measuring flume. This is only possible when the throat is designed to cause the water to flow through it at critical depth. For any given quantity of water, and size and shape of conduit, Fig. 2 shows that the energy head has a minimum value when the water is flowing at critical depth in the throat. For a given channel this critical depth is a function of the quantity of water, and attempts have been made by Woodburn[9] to use a broad-crested weir as a measuring device recording the critical depth. However, in this case, it was found impossible to locate the section of critical depth accurately as it changed position with different quantities of flow. Near the section of critical depth in the throat of the Venturi flume, the depth is uncertain, but the energy head is fixed; therefore, if the throat is designed so as to cause the water to flow at critical depth the depth in the conduit of the regular cross-section above will be the average energy head less the velocity head at that point, except for frictional losses, which will be discussed subsequently.

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[9] "Tests of Broad-Crested Weirs", by James G. Woodburn, Assoc. M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 387.

The proposed method of measurement is best illustrated by reference to Fig. 2. For simplicity, consider that a rectangular conduit is contracted on the sides only, forming a rectangular throat one-half the width of the conduit.

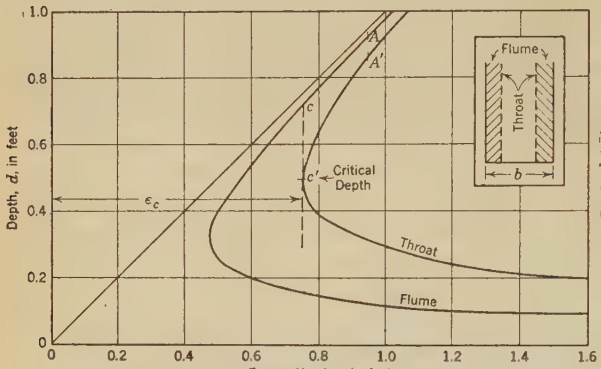

FIG. 2.—ENERGY HEAD CURVES IN A VENTURI FLUME.

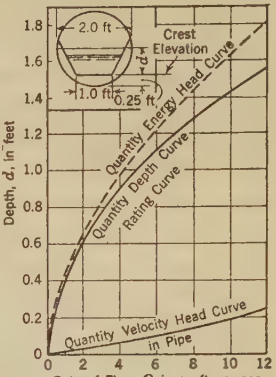

FIG. 3.—DERIVATION OF RATING CURVE, VENTURI FLUME.

For any given quantity of water, according to Bernoulli's theorem,

$$\epsilon = d + \frac{V^2}{2g} \qquad (1)$$

in which ϵ = energy head; d = depth of flow; V = average velocity in a section; and g = acceleration due to gravity.

Since $V = \frac{Q}{A}$,

$$\epsilon = d + \left(\frac{Q}{A}\right)^2 \frac{1}{2g} \qquad (2)$$

in which Q = rate of discharge; and A = area of cross-section. For a rectangular cross-section:

$$A = bd \qquad (3a)$$

$$V = \frac{Q}{bd} \qquad (3b)$$

and,

$$\epsilon = d + \left(\frac{Q}{bd}\right)^2 \frac{1}{2g} \qquad (4)$$

in which b = the bottom width of a channel; and ϵ and d are the variables. In Fig. 2, the curves represent Equation (4) for the conduit and throat, respectively, Q being the same in both cases. Neglecting frictional losses there would be no loss of energy in or through the Venturi flume, and if Q cu ft per sec flowed at a depth, A, in the conduit, it would flow at a depth, A', in the throat. Theoretically, Q could be determined by measuring these two depths, A and A'. The necessity in the past for the two measurements has been an inconvenience in the use of the Venturi flume. If Q, in cubic feet per second, flowed at the critical depth in the throat (Point C'), its depth in the conduit would be represented by Point C which would be its minimum depth above the throat, because for shallower depths the energy is insufficient to force Q through the throat.

By drawing similar pairs of curves for other values of Q, a rating curve showing the relationship between Q and d could be drawn, but in practice this requires an unnecessary amount of labor.

For a given size and shape of throat, the value of ϵ_c is a definite function of Q, or to write the inverse case:

$$Q = f(\epsilon_c) \qquad (5)$$

When ϵ and Q are known, H_v in the conduit can be found by trial, and subtracting this H_v from ϵ gives d (Fig. 3).

For rectangular throats, Equation (5) takes a simple form which will be evaluated from the general formula, but for other throat shapes, this becomes a transcendental function too complicated to be considered, and it is easier to compute a few points on the curve and prepare the rating curve graphically. It has been found from practice in computing these points that the best procedure is to assume various critical depths in the throat and compute Q and ϵ by the following formulas:

At critical depth, $\frac{d\epsilon}{dd} = 0$; differentiating Equation (2):

$$\frac{d\epsilon}{dd} = 1 - \frac{Q^2}{g} \times \frac{1}{A^3} + \frac{dA}{dd} \qquad (6)$$

Equating this formula to zero, and substituting B for its equivalent[10], $\frac{dA}{dd}$, Equation (6) becomes:

$$1 - \frac{Q^2}{g} \times \frac{B}{A^3} = 0 \qquad (7)$$

and, therefore,

$$Q = A\sqrt{\frac{A}{B}\,g} \qquad (8)$$

Equation (8) is the accepted formula for the quantity flowing at a critical depth in a channel of any shape. Since $Q = A\,V$, by substituting the value of Q in Equation (8):

$$\frac{V^2}{2g} = \frac{A}{2B} \qquad (9)$$

and Bernoulli's equation becomes,

$$\epsilon = d_c + \frac{A}{2B} \qquad (10)$$

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[10] "Hydraulics of Open Channels", by B. A. Bakhmeteff, M. Am. Soc. C. E., p. 31, Equation (19).

In a rectangular throat in which $A = B\,d_c$, Equation (8) becomes,

$$Q = B\sqrt{g}\,d_c^{\frac{3}{2}} \quad \text{(11)}$$

$$\frac{V^2}{2g} = \frac{d_c}{2} \quad \text{(12)}$$

and,

$$\epsilon = \frac{3}{2}\,d_c \quad \text{(13)}$$

Substituting the value of d_c obtained from Equation (13) in Equation (11):

$$Q = 3.09\,B\,\epsilon^{\frac{3}{2}} \quad \text{(14)}$$

which is the form taken by Equation (5) for a rectangular throat. For all other shapes it is necessary to use Equations (8) and (10).

Having drawn the quantity-energy head curve for a given throat (see Fig. 3), it is necessary to compute and draw the quantity-velocity head curve for the conduit section. If the conduit is circular this may be done by means of Fig. 4, which gives for various depths the velocity and velocity head for 1 cu ft per sec on a logarithmic scale, and, for other quantities, by merely adding log Q graphically. The intersection of the velocity abscissa with the "guide" line falls on the ordinate of the velocity head. The procedure is best

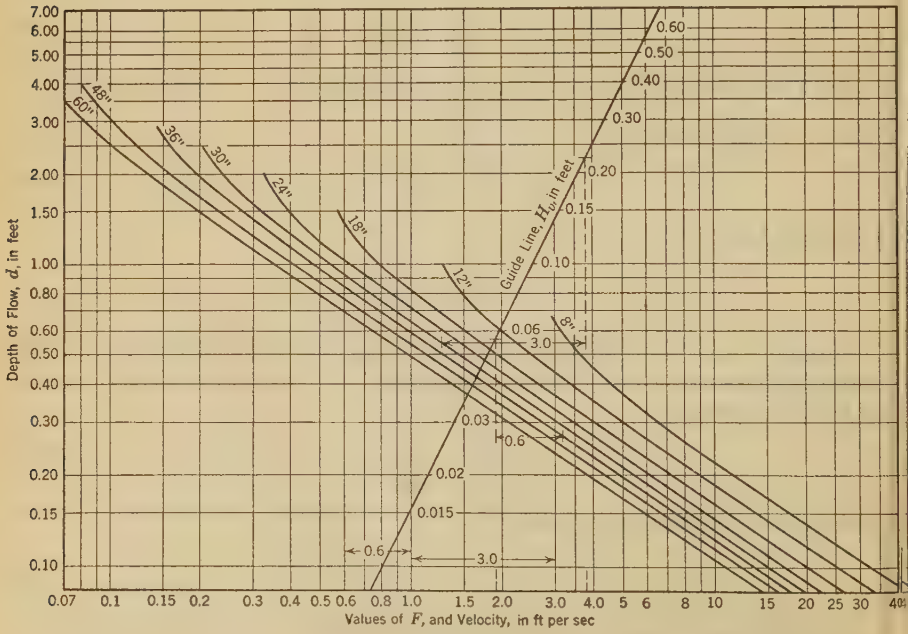

FIG. 4.—VELOCITY IN PIPES WITH WATER FLOWING AT VARIOUS DEPTHS.

explained by two examples: First, when $Q > 1$; and, second, when $Q < 1$. For example, let $Q = 3$ cu ft per sec with water flowing 0.55 ft deep in a 30-in. pipe. Set a pair of dividers to the distance from 1 to 3 and measure this distance to the right from the intersection of the 30-in. pipe with $d = 0.55$ ft, giving $v = 3.8$ ft per sec. The vertical line for $v = 3.8$ intersects the "guide line" at $H_v = 0.225$. In the second case let $Q = 0.6$ cu ft per sec, with water flowing 0.27 ft deep in a 36-in. pipe. Set the dividers for the distance 1.0 to the left to 0.6 and measure this distance to the left from the intersection of the 36-in. pipe with the $d = 0.27$ horizontal line, finding $v = 1.86$ ft per sec. The vertical line for $v = 1.86$ intersects the guide line at $H_v = 0.056$ ft.

To find the velocity head for any value of Q, first assume an approximate value of the velocity head, subtract it from the quantity-energy head, and note the assumed depth. Enter Fig. 4 with these values of Q and d and read the velocity head. If this differs materially from the assumed value, a second approximation should be made. To draw the velocity-head curve it is obviously better to begin with a small value of Q because the error in the assumed value of the velocity head is correspondingly small and one or two approximations are usually sufficient. It will be found that only a few points are required to draw in the quantity-velocity head curve. Ordinates on the quantity-depth curve are merely the differences between the ordinates of the quantity-energy head curve and the quantity-velocity head curve. This quantity-depth curve is the required rating curve.

As an illustration of the method of computing a rating curve, consider a Venturi flume with trapezodial throat cross-section as in Fig. 1(*a*), 12 in. wide at the base with side slopes of 2 on 1 and a flat slab base 3 in. above the invert of a 24-in. pipe. All measurements are in feet and quantities in cubic feet per second. As previously recommended, various critical depths will be assumed and Q and ϵ computed. In this case the formulas for the cross-sectional area and water-surface width in the throat are:

$$A = \left(1 + \frac{d_c}{2}\right) d_c \quad \ldots\ldots\ldots\ldots (15)$$

and,

$$B = 1 + d_c \quad \ldots\ldots\ldots\ldots (16)$$

The computations for energy heads are arranged in Table 1. In computing the velocity heads (see Table 2), it must be remembered that the crest of the throat is 0.25 ft above the invert; hence, the depth in the conduit must be increased by that amount. The velocity head in the conduit can be obtained only by trials preferably starting with the small values of Q. In Table 2 when $Q = 2$ cu ft per sec, the depth in the conduit will be somewhat less than $\epsilon + 0.25 = 0.908$. For 2 cu ft per sec, at a depth of 0.9 ft in a 24-in. pipe, Fig. 4 gives $H_v > 0.03$. Assuming $H_v = 0.03$, the depth becomes 0.88 with a resultant value of $H_v = 0.035$. This is not a sufficient change to effect the result so 0.035 can be assumed as the final value. Since the crest of the throat is taken as the datum, this 0.035 is to be subtracted from the value of ϵ. The velocity heads for other values of Q are found in the same way, except that after two or more have been computed an assumed value for the next one may be obtained by extending the curve drawn through the known points.

TABLE 1.—QUANTITY-ENERGY HEAD CURVE

d_c	0.2	0.4	0.6	0.8	1.0	1.2
$0.5\ d_c$	0.1	0.2	0.3	0.4	0.5	....
$b + 0.5\ d_c$	1.1	1.2	1.3	1.4	1.5	....
A	0.22	0.48	0.78	1.12	1.50	1.88*
B	1.2	1.4	1.6	1.8	1.93*	1.79*
$\frac{A}{B}$	0.183	0.343	0.487	0.622	0.777	1.050
Q	0.53	1.60	3.09	5.02	7.50	11.00
H_v	0.092	0.172	0.244	0.311	0.389	0.525
ϵ	0.292	0.572	0.844	1.111	1.389	1.725

* By scale and planimeter (see Fig. 1).

Table 1 is extended to a value, $Q = 11.0$. The energy head was so regular that it was extended by eye to $Q = 12$, and Table 2 is computed to this limit although it is subject to some suspicion for $Q > 11.0$. These curves are all shown in Fig. 3.

TABLE 2.—QUANTITY-VELOCITY HEAD CURVE.

Q =	2	4	6	8	10	12
ϵ =	0.658	0.978	1.225	1.438	1.630	1.812
$\epsilon + 0.25$ =	0.908	1.23	1.48	1.69	1.88	2.06
Assumed H_v =	0.03	0.07	0.09	0.14	0.19	0.25
$\epsilon + 0.25 - H_v$ =	0.88	1.16	1.39	1.55	1.69	1.81
H_v (final) =	0.035	0.064	0.100	0.145	0.198	0.256
$d = \epsilon - H_v$ =	0.623	0.914	1.125	1.293	1.432	1.556

An exponential formula may be written for some small sections of the final rating curve, but it is not advisable. The inaccuracies of such a formula can be appreciated by equating the value of ϵ in Equation (2) for both the throat and conduit sections; thus,

$$d_c + \left(\frac{Q}{A_c}\right)^2 \frac{1}{2\,g} = d + \left(\frac{Q}{A}\right)^2 \frac{1}{2\,g} \quad \ldots\ldots\ldots\ldots\ldots (17)$$

which reduces to,

$$Q = \sqrt{2\,g\,\frac{(d - d_c)\,A\,A_c}{A - A_c}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Writing $A = f(d)$ and $A_c = \phi\,(d_c)$:

$$Q = \sqrt{2\,g\,\frac{(d - d_c)\,f\,(d)\,\phi\,(d_c)}{f\,(d) - \phi\,(d_c)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Since $f(d)$ is not a simple function in a circular pipe, it is evident that the use of an exponential formula is generally inadvisable because of the many different combinations depending on the relative size of the pipe and throat, and the height of the crest of the throat above the invert.

ENERGY LOSSES

One of the chief disadvantages of water measurement with a weir is the relatively large and uncertain loss of energy. Schoder and Turner[3] show how the so-called constant of a weir is subject to wide changes depending on the size and setting. This change in the constant can be interpreted as a variation in the energy loss, and it follows that if such variations can be limited by the use of other devices, the change in so-called constants will likewise be restricted. In the case of the weir such control is difficult and at times impossible; with a Venturi flume it may be accomplished easily.

In the adaptation of the Venturi flume, to the writers' ideas, considerable attention was given to the design of the transitions to minimize loss of energy. In the hydraulic design of flume transitions Hinds[11] found, in making twenty-nine tests on ten flume inlets, that only three had losses amounting to more than $0.1 \Delta H_v$. In a number of cases no measurable loss occurred and the average was about $0.04 \Delta H_v$. The 4-ft diameter of a standard sewer manhole limits the length of the transition that can be built readily in an existing sewer. In the sewerage system of the Los Angeles County Sanitation Districts, a ratio of three longitudinally to one transversely has been adopted as a compromise. To determine the energy losses through Venturi flumes constructed by the Districts, a differential meter acting on the principle of a Pitot meter was devised. This is shown in Fig. 5.

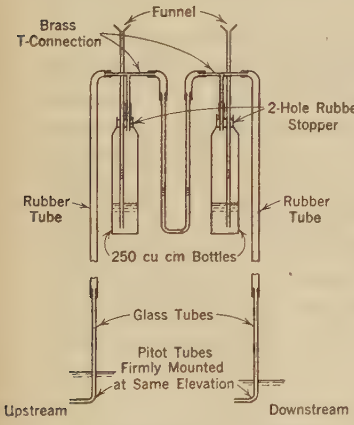

FIG. 5.—DIFFERENTIAL ENERGY METER.

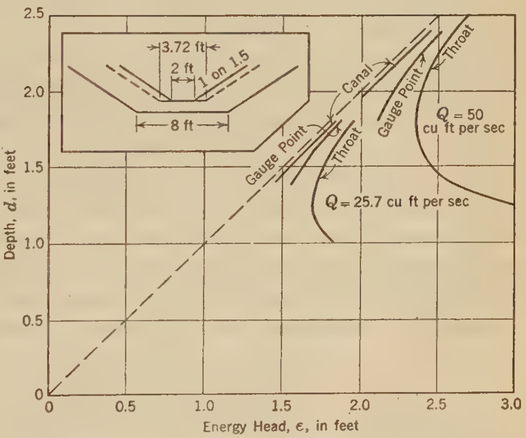

FIG. 6.—ENERGY CURVES, CONTROL SECTION WEIR.

Two bent tubes are set at two points in the stream with orifices at the same elevation and pointing up stream, one in the throat and one above the inlet transition. These two tubes are connected by means of air tubes to a manometer filled with colored water; the displacement of the water column in the manometer after all water is blown out of the bent tubes measures the difference in energy head at the two points. To allow for any compression of the air which might occur in the tubes it was found advisable to blow air into each from time to time during the test. This was accomplished by pouring water into the two bottles, forcing air out through the orifice tubes. In sewage, one or the other of the orifice tubes may become clogged by floating matter, throwing the manometer out of balance. The manometer resumes its balance when the obstruction has been cleared away and a little more air forced into the tubes. Any type of meter in which water is drawn into the tubes will be clogged by sewage. This simple, head-differential apparatus has been found to be accurate and sensitive.

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[11] "The Hydraulic Design of Flume and Siphon Transitions", by Julian Hinds, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 92 (1928), p. 1423.

Several tests with the differential-energy meter (Fig. 5) integrated over the entire cross-section have shown that no energy losses greater than 0.005 ft occur in any of the meter throats thus far installed, and as many stage recorders are not sensitive to such small changes this energy loss is practically negligible. Should an appreciable loss be found in a given installation for high velocities it can be added to the quantity-energy head curve.

In sewage measurement, the mean daily flow is usually required. This may be calculated with sufficient accuracy by using a planimeter on a water-stage record to get the mean depth. If it is suspected that there are energy losses through the Venturi flume at higher flows the differential energy meter can be used at various stages, including the maximum, to correct the curve; and if the flow is increasing from year to year, the corrections may be determined beforehand, as the maximum flow of to-day will be the mean flow some months or some years hence.

Another possible source of error seems to be that in an open conduit the surface velocity is about 25% greater than the mean, which should cause the velocity head at the surface (where the float is installed for measurements), to vary as the ratio of 1.00^2 to 1.25^2, or 56% greater than the mean. If this condition existed, 1.56 H_v should be subtracted from the quantity-energy head instead of 1.0 H_v, in order to obtain the proper depth in making a rating curve. This condition was tested at one meter formed by placing a flat slab, 6 in. thick (at the invert), on the bottom of a 54-in. sewer pipe without side contractions. In the regular pipe channel, at the upper end of the transition section and at mid-stream, a Pitot tube was turned up stream and then at right angles to the current. The difference between these two measurements was the true velocity head, which, added to the measured depth, gave the total-energy head. Applying this energy head to the previously computed quantity-energy head curve gave the quantity. Readings taken at various depths between the center and sides of the stream showed substantially

TABLE 3.—Relation Between the Measured and the Theoretical Velocity Heads at Mid-Stream.

Quantity, Q, in cubic feet per second	Calculated velocity, V, in feet per second	Theoretical head, H_v, in feet	Measured head, H_v, in feet
8.2	2.18	0.074	0.076
10.25	2.50	0.097	0.096
17.5	3.15	0.153	0.142

the same velocity head at all points. Table 3 shows the relation between the measured and the theoretical velocity heads at mid-stream. These results indicate that within the accuracy of the measurements the velocity head in the conduit may be assumed to be the mean velocity head.

An investigation of the velocity-head correction for hydraulic flow by O'Brien and Johnson[12] indicates that, in this case, such a correction might amount to from 4 to 10% of the velocity head. As this is less than the limit of accuracy for a stage recorder with a float set in the main stream it may be neglected. For measurements in clear water, using a float-well for more accurate measurements, it should be determined experimentally.

When the velocity above the meter is less than 2 ft per sec the water surface is usually smooth, but above this velocity waves begin to form, probably caused by the difference in velocity heads in the region adjacent to the sides of the conduit. The waves are too small to affect the ball-float, but would affect very precise measurements made with a hook-gauge. Where they are usable float wet-wells would obviate this trouble, but in sewers the floats are affected by the accumulation of decomposing sludge.

One question that may be raised is the effect on the energy head of different velocities at various points in the cross-section of the throat. It was found that the energy head was the same at all points in a transverse plane. As another check, small floats were dropped simultaneously into the upper end of the throat in pairs, one being near the side and the other nearer the center. Every time, both floats reached the lower end together. As another test a string was dropped into the throat transversely to the flow line and reached the lower end straight. The explanation of this phenomenon is that at, and near, the critical velocity in a short flume the side friction is not an appreciable factor.

Efforts were made to find a coefficient to be applied to H_v in the conduit. The flow was determined by measuring the energy head in the throat with a Pitot tube. Using this value of Q, the measured depth was subtracted from the energy head to give $C\,H_v$. Within the accuracy of measurements, flows of 10.3 cu ft per sec, at a velocity of 2.4 ft per sec, and 17.5 cu ft per sec, at a velocity of 3 ft per sec, in the conduit both gave $C = 1$. These were the largest flows and velocities available.

EXPERIMENTAL VERIFICATION

In the experiments made by the U. S. Bureau of Reclamation,[7] a wooden throat with a bottom width of 2 ft and side slopes of 1 on 1½ was installed in an earthen canal with a bottom width of 8 ft and similar side slopes. The bottom of the throat was 8 in. above the base of the canal and the downstream apron end of the throat sloped downward to a point 8 in. below the elevation of the canal base, similar to the Parshall meter. The gauge-point was placed in the upper transition where the bottom width was approximately

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[12] "Velocity Head Correction for Hydraulic Flow", by M. P. O'Brien, Assoc. M. Am. Soc. C. E., and J. W. Johnson, Jun. Am. Soc. C. E., *Engineering News-Record*, Vol. 113, 1934, p. 214.

3.72 ft. It was placed only 0.4 ft down stream from the upper end of the wooden flume section where it was certain no silt deposit would occur. The writers have drawn energy curves for the flume (Fig. 6) at the control, or throat section, the gauge-point, and in the main canal for 50 and 25.7 cu ft per sec, respectively.

Had the gauge-point been moved back into the main canal, the measured depth would have been greater by about 0.1 ft in the case when $Q = 50$ cu ft per sec, but the main argument in favor of moving up stream is that the drop in the water surface indicates convexity, especially as it increases rapidly with the tapering of the channel, and it is in the convex water surfaces that the relation of the water surface to energy head is not correctly given by the Bernoulli theorem. Had the gauging-point been placed in the regular channel above as suggested, and the sides and bottom lined for a few feet to maintain a uniform cross-section, the stream filaments would have been parallel and the calculated flow would have followed more closely the volumetric measurements. Friction due to the longer distance between the point of measurement and the throat could have been measured by means of the differential energy-head meter (Fig. 5) and the energy head increased by this amount.

With this change in the point of measurement the recorded tests would have shown closer agreement with the calculated rating curve. Since the quantity of water flowing through this Venturi flume tested by the Bureau of Reclamation was carefully measured volumetrically the data are considered most reliable. From the known dimensions a quantity-depth or rating curve was readily constructed by the writers using the method previously outlined. As stated by Hinds the difference between the quantities taken from this curve for given depths and the actual measurements in no case exceeded 5 per cent. Since the writers have pointed out that greater accuracy could have been attained by the selection of a better place for measuring the depths, and by measurement of frictional losses, this test is sufficient proof that all forms of the Venturi flume will give accurate measurements when it is possible to measure the depth in the conduit of a regular cross-section immediately above the throat and where the throat is properly designed to insure that the water reaches critical depth.

In the absence of an experimental laboratory the writers have had to make use of active sanitary sewers in the development and adaptation of the Venturi flume designs. Standard weirs were set in adjacent manholes, but comparative results were poor except where the flow was small in comparison with the capacity of the sewer, since the weir coefficients were uncertain. Therefore, these measurements were not considered sufficiently accurate to verify those obtained from the Venturi flumes.

The experiments made by the U. S. Bureau of Reclamation, previously referred to, were considered a positive check on the accuracy of the method outlined herein. Furthermore, a field test, comparing measurements of a temporarily installed Venturi flume with a permanent Parshall meter (made through the co-operation of the Cities of Los Angeles and Beverly Hills), supplied added proof of the accuracy of the adapted Venturi flume.

At the point selected for the test an elaborate underground structure was constructed on a 21-in. sewer line which included a standard 12-in. Parshall meter and an indicating and recording register, showing the depth of flow through the flume and the rate of flow, in cubic feet per second. Water continuously running into the float-well prevented the accumulation of sludge which otherwise would affect the float results. The wooden throat of the temporary Venturi flume was installed in an ordinary standard manhole about 1 000 ft above the permanent gauging station, where the grade of the 21-in. sewer was 0.70 per cent. The bottom of the throat with a width of 10 in. was placed 2 in. above the invert. Although the boards forming the side slopes were intended to be set at a slope of 2 on 1, the irregularities of the pipe caused them actually to be placed on a slope of 2.1 on 1.

Tin transitions for stream-lining the upper end were attached to the boards, but could not be set in exact position due to the roughness of the channel through the manholes. Some caulking had to be done with oakum. The entire apparatus was quickly installed in the sewer during the low early morning flow and readily withdrawn a few hours later when the flow was at its peak of about 6 cu ft per sec. The elevation of the bottom of the throat was referred to a straight-edge placed across the manhole rim in the street pavement and measurements to the water surface above the flume were made with a steel tape.

Readings were taken simultaneously at the Venturi flume and the Parshall meter, a correction in time being made between the two stations as determined by passing floats occasionally from one to the other. A correction was also required for the small flow entering the sewer from a side branch at a point between the two testing stations. Occasional depth measurements over a V-notch gave this correction with sufficient accuracy.

Table 4 gives the results of tests when the flow was within the capacity of the Venturi flume. In Table 4(a) the average discrepancy between the two methods of measurement was 4.2%, with a probable error of ± 0.4 per cent.

During the period of high flow (8:28 to 8:58) it was found that the critical depth in the throat exceeded the depth of the Venturi flume. These results are shown in Table 4(b). In this second group the discrepancy was + 6% ± 0.7%, showing that even when poorly adapted it was still comparatively correct. The 4% error in the first group could be corrected by allowing a head loss of 0.25 H_v, or by subtracting 0.75 H_v instead of H_v from the energy head. Doubtless the discrepancy could be reduced materially by carefully installing a permanent concrete Venturi flume instead of the temporary wooden one which was used. Any obstruction placed in this sewer laid on a slope greater than the critical causes the water to jump immediately to the conjugate depth on the energy-head curve (Fig. 2) and unless allowance is made for this effect, the capacity of the sewer would be curtailed seriously. Therefore, it was necessary to use a large throat with a consequent rather high velocity of approach. It has been found that the slower the velocity of approach the greater the accuracy. This is limited by the capacity of the conduit and the possibility of forming deposits in the approach channel which would affect the rating curve.

TABLE 4.—COMPARATIVE MEASUREMENTS WITH A VENTURI FLUME AND PARSHALL METER.

(Flow Q, in cubic feet per second.)

Time (1)	VENTURI FLUME		Parshall, Q'_p (4)	Side inflow, Q''_p (5)	$Q_p =$ Column (4) minus Column (5) (6)	$\triangle Q =$ Column (6) minus Column (3) (7)	Ratio: Column (6) / Column (3) (8)
	Depth, in feet (2)	Q_v (3)					
(a) FIRST GROUP							
5:57	0.545	1.45	1.57	0.14	1.43	—0.02	0.98
6:01	0.545	1.45	1.57	0.14	1.43	—0.02	0.98
6:03	.....	....	1.50	0.14	1.36	....	.....
6:05	.....	....	1.43	0.14	1.29	....	.....
6:06	0.495	1.22	....	....	1.23	+0.01	1.01
6:07	.....	....	1.30	0.13	1.17	......	.....
6:08	0.485	1.18	....	....	1.12	−0.06	0.95
6:09	.....	....	1.20	0.13	1.07	....	....
6:10	0.465	1.11	....	....	1.06	−0.05	0.95
6:13	0.443	1.04	....	....	1.02	−0.02	0.98
6:15	0.443	1.04	....	....	0.99	−0.05	0.95
6:17	.....	....	1.10	0.13	0.97	......	.....
6:18	0.443	1.04	....	....	0.96	−0.08	0.92
6:19	.....	....	1.07	0.12	0.95	......	....
6:22	0.433	0.98	1.07	0.12	0.95	−0.03	0.97
6:24	.....	....	1.07	0.12	0.95	......	....
6:27	0.437	1.00	1.05	0.12	0.93	−0.07	0.93
6:31	0.433	0.98	....	....	0.92	−0.06	0.94
6:32	.....	....	1.04	0.12	0.92	......	....
6:37	0.453	1.06	1.09	0.13	0.96	−0.10	0.91
6:42	0.516	1.31	1.39	0.13	1.26	−0.05	0.96
6:48	.....	....	1.48	0.14	1.34	......	....
6:50	0.558	1.50	....	....	1.41	−0.09	0.94
6:52	.....	....	1.62	0.14	1.48	......	....
7:23	0.730	2.43	2.47	0.16	2.31	−0.12	0.95
7:26	0.745	2.53	....	....	2.48	−0.05	0.98
7.28	0.790	2.78	....	....	2.58	−0.20	0.93
7:31	0.813	2.91	....	....	2.77	−0.14	0.95
7:33	.....	....	3.05	0.17	2.88	......	....
7:34	0.830	3.00	....	....	2.95	−0.05	0.98
7:35	0.850	3.13	....	....	3.02	−0.11	0.97
7:37	0.870	3.28	....	....	3.17	−0.11	0.97
7:38	.....	....	3.42	0.18	3.24	......	....
7:39	0.910	3.56	....	....	3.31	−0.25	0.93
7:41	0.953	3.86	....	....	3.54	−0.32	0.92
7:43	.....	....	3.80	0.19	3.61	......	....
7:48	.....	....	3.89	0.19	3.70	......	....
7:52	0.985	4.12	....	....	3.89	−0.23	0.94
7:53	.....	....	4.18	0.19	3.99	......	....
7.55	0.975	4.02	....	....	4.08	+0.06	1.02
7:58	1.015	4.37	4.40	0.19	4.21	−0.16	0.96
8.01	.....	....	4.47	0.20	4.27	......	....
8:03	1.015	4.37	....	....	4.31	−0.06	0.99
Mean...	.....	....	....	....	....	......	0.958
(b) SECOND GROUP							
8:28	1.057	4.75	5.33	0.20	5.13	+0.38	1.08
8:32	1.067	4.87	....	....	5.26	+0.39	1.08
8:33	1.077	4.97	5.50	0.21	5.29	+0.32	1.06
8:38	.....	....	5.65	0.21	5.44	......	....
8.40	1.083	5.00	....	....	5.51	+0.51	1.10
8:43	.....	....	5.79	0.21	5.58	......	....
8:44	1.120	5.35	....	....	5.59	+0.24	1.04
8:48	.....	....	5.83	0.21	5.62	......	....
8:50	1.120	5.35	....	....	5.61	+0.26	1.05
8:53	.....	....	5.81	0.21	5.60	......	....
8:56	1.135	5.50	....	....	5.65	+0.15	1.03
8:58	1.140	5.55	5.92	0.21	5.71	+0.16	1.03
Mean...	.....	....	....	....	....	......	1.06

Design of Venturi Flume

The proper design of a Venturi flume requires that parallel flow occur in the channel[13] above the flume and in the throat. The necessary critical velocity is obtained only when a drop in the energy head occurs just below the throat. A small jump in the water surface at this point (see Fig. 1) is positive evidence that critical velocity is occurring. In addition, the throat section must have sufficient length or the water will not be flowing in parallel filaments at the point of critical depth. The first meter stations installed by the Sanitation Districts of Los Angeles (Calif.) County had short throats which gave good results only on the small flows. Lengthening the throat to 3 ft insured parallel flow through more of the length, and tests with a Pitot tube showed no apparent change in energy head except at the ends. Data are lacking to formulate a rule as to the proper length, but experience indicates that the throat should be at least as long as the diameter of the pipe. A comparison of flow through two Venturi flumes in consecutive manholes, with 36-in. and 6-in. throat lengths, respectively, showed a ratio of 1.0 to 0.8, the longer throat indicating the larger flow.

The important features of the Venturi flume are its adaptation to stream-flow measurement in all shapes of conduits not flowing under pressure and its ready installation in lines to which access is much restricted. The device can always be installed in an existing line and often without serious interruption to the flow.

The size of the flume throat depends upon the size and grade of the conduit and the range of flow it is desired to measure. The ideal flume throat would have such a size and shape that the ratio of the cross-section of the water in the throat to that in the conduit would be the same for all quantities; but such a throat would be impossible to design and build because the ratio of the critical depth to the depth in the conduit is not a constant.

For a flow that is only a small percentage of the designed capacity of a conduit, a throat, V-shaped, or narrow at the bottom, with sloping sides, is satisfactory. However, such side walls are difficult to hold in place. A rectangular narrow throat may contract the flow so much as to give inadequate capacities for larger flows.

For circular pipes it has generally been found advisable to install a bottom slab in addition to the two sides, a feature which helps to support the side walls. The slab acts as a broad-crested weir when flows are only a small percentage of the designed capacity. Since a thin bottom slab properly stream-lined by approach transitions causes no deposition of solids except possibly at the very lowest velocities, such a slab is recommended, especially on the lighter grades, to offset any possible chance of back-water from an unforeseen obstruction below that might otherwise prevent critical velocity from being attained. In general, the flatter the slope, the more necessary is this precaution and the greater the slab thickness needed.

A throat of rectangular cross-section with proper transitions may be adaptable in some cases, as in the instance of a semi-elliptical section, or a rectangular-shaped conduit, where its use is recommended. Lack of accuracy in very low flows and inability always to pass the full design capacity, however, are restricting limitations to its general use.

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[13] "Hydraulics of Open Channels", by B. A. Bakhmeteff, M. Am. Soc. C. E., 1932, p. 28.

Where a large flow in either rectangular or circular-shaped conduits prevents the placing of forms for building the side walls, a simple device to create critical velocity consists of placing on the invert a pre-cast flat slab with transitions. Deposition can only occur above this type at extremely low velocities. In similar cases involving large flows critical velocity may be produced by simply inserting vertical side walls with the proper transitions.

The important factor in the construction and installation of any form of Venturi device is that a constriction of some sort be placed in the channel to produce critical velocity with the least loss of energy, and that the shape, size, and dimensions of the device are important only in so far as they meet the specific problem at hand in a practical manner.

The correct size and shape of throat for use in a given conduit is that one for which the energy head is greater than the normal energy head in the free flowing conduit at all values of Q, and with a bottom slab thin enough to prevent deposition of sludge at low flow. A practical method of making this selection is first to prepare quantity-energy head curves for several sizes of Venturi flume throats (Fig. 7) on tracing cloth. Table 5 can be used for constructing these curves for trapezoidal throats having side slopes of 2 on 1. Next, on the same scale, but on a separate piece of paper, a quantity-energy head curve is drawn for the conduit under consideration, assuming unobstructed flow. Any formula for flow in open channels may be used for first constructing a quantity-depth curve, but the following procedure is suggested. Bakhmeteff[14] expresses the Chezy formula, as follows,

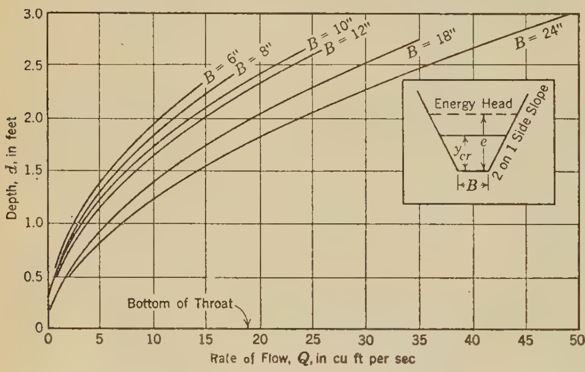

FIG. 7.—QUANTITY-ENERGY HEAD CURVE FOR VARIOUS THROAT SIZES WITH SIDE SLOPES OF 2 ON 1.

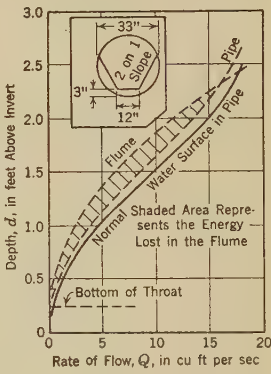

FIG. 8.—ENERGY HEAD CURVE, VARIOUS THROAT SIZES.

$$Q = K \sqrt{S} \quad \text{.......................} (20)$$

in which,

$$K = A C \sqrt{R} \quad \text{.......................} (21)$$

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[14] "Hydraulics of Open Channels", by B. A. Bakhmeteff, M. Am. Soc. C. E., 1932, p. 13.

Computing K as a function of the depth in a pipe is rather tedious. Values of K are shown for several pipe sizes in Fig. 9 which has been computed for $n = 0.013$ and $S = 0.01$ in Kutter's formula. Drawn to a logarithmic scale, it is merely necessary to add one-half the logarithm of the slope (expressed as a percentage) to obtain the quantity for any depth in the pipe. Velocity heads are obtained from Fig. 4 in the manner previously described. Adding these velocity heads to the depths shown on the quantity-depth curve gives the quantity-energy head curve. Fig. 8 shows a typical diagram.

TABLE 5.—ENERGY-HEAD TABLES FOR THROATS WTH SIDE SLOPES OF 2 ON 1

Rate of flow, Q, in cubic feet per second	Energy Head, in Feet, for Bottom Widths, in Inches					Rate of flow, Q, in cubic feet per second	Energy Head, in Feet, for Bottom Widths, in Inches				
	6	8	10	12	18		6	8	10	12	18
0.2	0.232	0.198	0.176	0.156	0.121	10.00	1.941	1.820	1.719	1.623	1.388
0.5	0.401	0.348	0.311	0.280	0.219	12.00	2.117	1.993	1.887	1.786	1.540
1.0	0.594	0.526	0.474	0.431	0.342	14.0	2.276	2.149	2.041	1.936	1.679
2.0	0.865	0.779	0.713	0.654	0.529	16.0	2.422	2.294	2.183	2.075	1.809
3.0	1.070	0.975	0.897	0.829	0.680	18.0	2.558	2.428	2.315	2.205	1.930
4.0	1.239	1.136	1.054	0.978	0.810	20.0	.....	.....	2.439	2.327	2.045
5.0	1.384	1.279	1.191	1.109	0.926	25.0	.....	.....	.....	2.604	2.308
6.0	1.515	1.406	1.314	1.228	1.032	30.0	.....	.....	.....	2.852	2.544
7.0	1.634	1.522	1.426	1.337	1.130	35.0	.....	.....	.....	3.078	2.759
8.0	1.743	1.628	1.530	1.438	1.221	40.0	.....	.....	.....	.....	2.958
9.0	1.845	1.727	1.627	1.532	1.306	....	.....	.....	.....	.....	.....

Referring to Fig. 9, the capacity, Q, of a pipe is given by Equation (20) when S = the slope, expressed as a percentage; $K = A \times \frac{C}{10} \sqrt{R}$ for any given depth; and C = Chezy's coefficient. The curves are drawn for a 1% slope. For values of S less than 1% the flow for any depth will be to the left of the curve and for more than 1% to the right. To find the value of $\sqrt{S}$ measure from the slope guide to the right 10-line for $S < 1\%$ and to the left for $S > 1$ per cent. For example: Let $S = 0.25\%$; D = 8-in. pipe; and, $Q = 0.1$ cu ft per sec. In this case $S < 1$ per cent. Set the dividers for the distance between the inclined slope line in percentage line and the right 1.0-line of the diagram at $S = 0.25$. Transfer this distance to the $Q = 0.1$-line, measuring to the right and intersect the 8-in. curve at $d = 0.20$ ft. If Q is given in million gallons daily, set the dividers for the distance between the inclined line and the million-gallon-daily guide line on the extreme right. For $S = 0.25\%$ and $Q = 0.1$ mgd, $d = 0.25$ ft in an 8-in. pipe.

On the other hand, let $S = 2.25$; D = 60-in. pipe; and $Q = 15$ cu ft per sec. In this case $S > 1$ per cent. Set the dividers for the distance from the inclined slope line to the 0.1-line to the left at $S = 2.25$. Transfer this distance to the Q = 15-line, measuring to the left and intersect the D = 60-in. line at $d = 0.66$ ft.

By superimposing the quantity-energy head curves for the throat upon that for the pipe and then moving the throat tracing up or down, it is possible to decide quickly which throat size should be used and how high it should be set above the invert. The difference in the two energy heads shows the

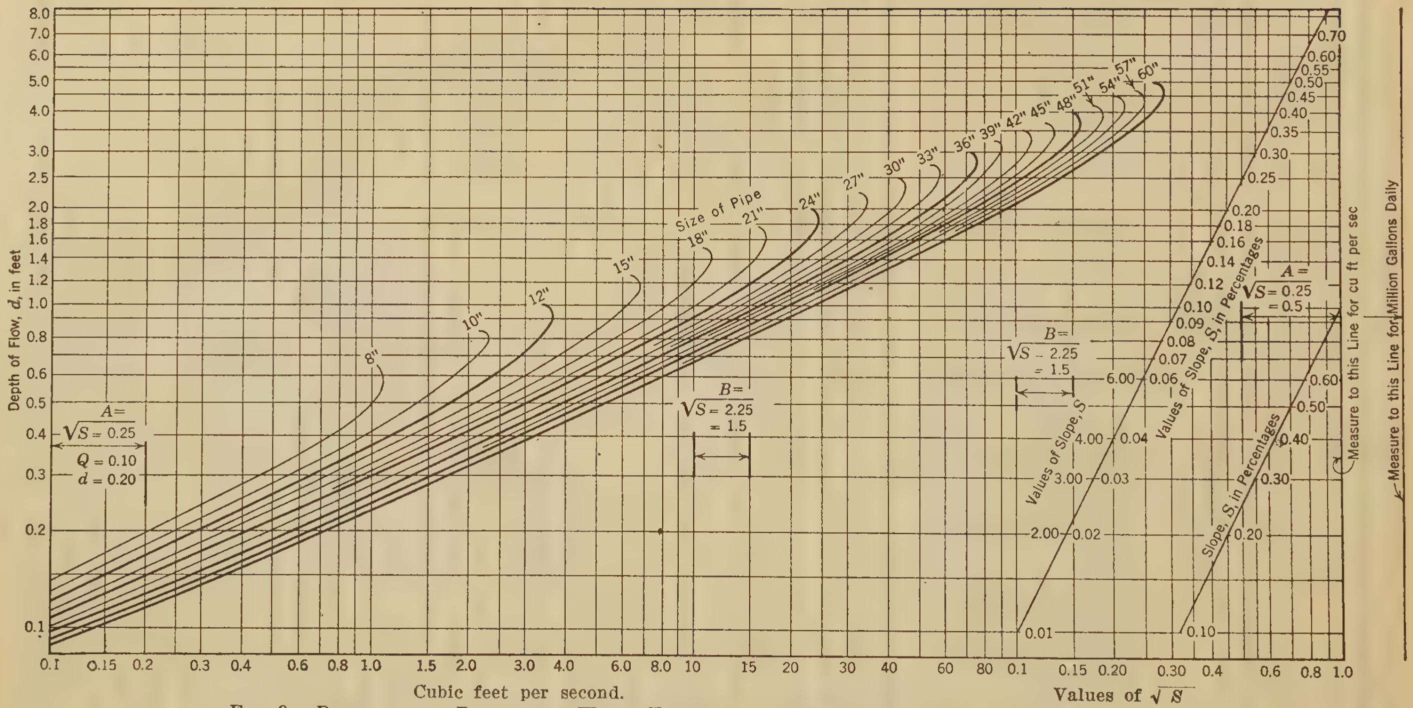

Fig. 9.—Discharge of Pipes with Water Flowing at Various Depths When Kutter's $n = 0.013$.

loss caused by the Venturi flume for all quantities. Fig. 10 shows typical quantity-energy head curves for both a trapezoidal and a rectangular throat superimposed upon a quantity-energy head curve for a 33-in. circular pipe,

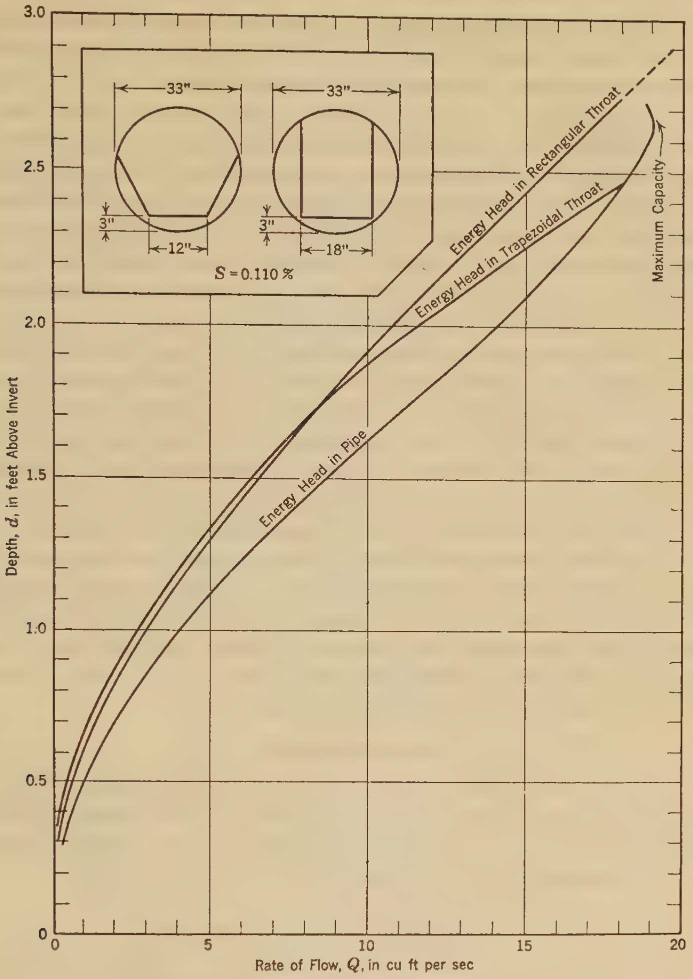

Fig. 10.—Comparative Energy Head in Trapezoidal and Rectangular Flumes.

and it is easily seen that the trapezoidal section can be used as a meter for flows as great as 18 cu ft per sec and that at 19 cu ft per sec, which is the maximum capacity of the pipe (there being no loss in energy through the flume), the carrying capacity of the pipe is not curtailed. The quantity-energy head curve of the rectangular flume (Fig. 10) being always above the energy head for the pipe, shows loss of energy through the flume at maximum designed capacity. This type of Venturi flume would interfere with the use of the sewer at full capacity. It is more accurate to compute the rating or quantity-depth curve from actual measurements of the flume as constructed than to base it upon the design curves. Figs 4 and 9 are for use with circular pipes, but similar curves may be computed for conduits of any shape.

For permanent installation in sewers of the Los Angeles County Sanitation Districts the Venturi flumes were readily constructed of concrete directly in the sewer manhole without interruption to sewage flow. A layout sketch of the meter was prepared for a single installation in a circular pipe and, using the plan as a pattern, forms for other sizes were obtained by proportion. When possible the entire form for the flume was assembled in the sewer manhole invert and concrete was poured into the forms from one side until the bottom slab was filled in order to assure complete displacement of the sewage. Use of a fast-setting cement had distinct advantages in such installations.

Placing the center of the throat at about the lower side of a manhole gave proper space for installing the stage-recorder float near the upper side at a point where the flow in the pipe was uniform. The float consists of an 8-in. copper ball fastened to a 2-ft length of $\frac{1}{2}$-in. pipe, hinged loosely up stream to a pipe framework (see Fig. 1). The latter is attached to the top of a plank placed cross-wise to the sewer and supported by the shelves of the manhole. Without some such apparatus to hold the ball, it tends to float down stream and disturb the record. The stage recorder is suspended on a specially built steel seat immediately under the manhole cover where it is readily accessible. Repeated checks of indicated stage records against actual measured depths show no appreciable error during the daily range of velocities. Braided, insulated, copper wire connects the float to the recorder. Considerable difficulty was experienced in keeping the stage recorder clocks running for a week at a time in the damp, gas-laden, sewer air, but the purchase of sealed clocks has practically solved this problem.

ACKNOWLEDGMENTS

The writers are particularly indebted to A. K. Warren, Assoc. M. Am. Soc. C. E., and A. M. Rawn, M. Am. Soc. C. E., for the opportunity to make the experiments and studies for this investigation, and for their constructive criticism in the preparation of the paper.

DISCUSSION

N. F. Hopkins,[15] M. Am. Soc. C. E. (by letter).—The experiments reported by the authors are interesting as confirming the formulas. Especially interesting are the experiments showing that the velocity is uniform throughout the cross-section of the flume at such comparatively low velocities.

It would seem as if the theory could have been set forth in a more simple manner. The velocity increases with the drop in the surface from the back-water to the flume, whereas the area of the cross-section decreases with the drop. The quantity is the product of the area and the velocity. If there is no drop, there would be no velocity and quantity; and, if the drop is a maximum, there would be no area and no quantity. At some point, the product of the velocity and the area is a maximum.

For example, in addition to the notation of the paper, let p = the pitch of the sides of the flume; Z = the elevation of the back-water above the bottom of the flume plus the velocity head at back-water, minus the friction loss in entering the flume; H_v = the velocity head in the flume; d = the depth of water in the flume = $Z - H_v$ (see Fig. 11); V = the velocity in the flume = $\sqrt{2\,g\,H_v}$; and A = the area of the water section in the flume. Furthermore, let $a = B\,d - A$, then;

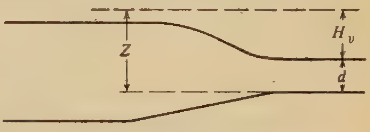

Fig. 11.

$$A = B\,d - a = B\,Z - B\,H_v - a \qquad (22)$$

$$Q = \sqrt{2\,g\,H_v}\,(B\,Z - B\,H_v - a) = \sqrt{2\,g}\,(B\,Z\,H_v^{0.5} - B\,H_v^{1.5} - a\,H_v^{0.5}) \qquad (23)$$

and for a maximum value of Q:

$$\frac{dQ}{dH_v} = \sqrt{2\,g}\left(\frac{B\,Z}{2\,H_v^{0.5}} - \frac{3\,B\,H_v^{0.5}}{2} - \frac{a}{2\,H_v^{0.5}}\right) = 0 \qquad (24)$$

In Equation (24), the quantity in parenthesis equals zero, and, consequently, $B\,Z - 3\,B\,H_v - a = 0$; that is:

$$H_v = \frac{Z}{3} - \frac{a}{3\,B} \qquad (25)$$

For a rectangular flume, $a = 0$ and Equation (25) becomes $H_v = \dfrac{Z}{3}$ and Equation (23) becomes:

$$Q = 3.09\ B\,Z^{1.5} \qquad (26)$$

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[15] Civ. and Min. Engr. (Harrop & Hopkins), Pittsburgh, Pa.

For a V-shaped flume, in which $a = \frac{B(Z - H_v)}{2}$, Equation (25) is:

$$H_v = 0.2\,Z \qquad (27)$$

and, Equation (23) is,

$$Q = 2.297\,p\,Z^{2.5} \qquad (28)$$

For a trapezoidal flume, in which $a = (Z - H_v)^2\,p$, Equation (25) becomes:

$$H_v = \frac{0.3\,b}{p} + 0.6\,Z - \sqrt{\left(\frac{0.3\,b}{p} + 0.6\,Z\right)^2 - \frac{Z}{5}\left(\frac{b}{p} + Z\right)} \qquad (29)$$

and Equation (22) becomes:

$$A = b\,(Z - H_v) + p\,(Z - H_v)^2 \qquad (30)$$

Substitute the value of H_v found by Equation (29) in Equation (30) to find A; and in Equation (23) to find Q. An approximate expression for the flow is:

$$Q = 3.06\,(b + 0.72\,p\,Z)\,Z^{1.5} \qquad (31)$$

For a parabolic flume, in which $a = \frac{B}{3}(Z - H_v)$, Equation (25) becomes:

$$H_v = 0.25\,Z \qquad (32)$$

Equation (22) becomes:

$$A = \frac{B\,Z}{2} \qquad (33)$$

and Equation (23) becomes:

$$Q = 2\,B\,Z^{1.5} \qquad (34)$$

Finally, it is necessary to find the value of B for $d = 0.75\,Z$.

For a circular flume (or, in fact, for a flume of any cross-section), it is sufficiently exact to assume that,

$$Q = 4\,A\,\sqrt{Z} \qquad (35)$$

in which A is the area of a section of the flume, the depth of which is 0.75 Z.

HUNTER ROUSE,[16] ESQ. (by letter).—In so far as the subject-matter specified in the title of this paper is concerned, the writer must commend the authors for their able solution of a practical hydraulic problem. The fact is not to be disputed that any appropriate measuring device may be expected to yield satisfactory results once its rating curve has been determined directly or indirectly by volumetric measurement. Hence, the authors were fully justified in choosing a measuring device which eliminated every possible objectionable feature, in adapting it to the needs of the situation, and then rating it in place through comparison with a device of known characteristics. In view of the accuracy required (an allowable error of perhaps 5%, or more), it was not amiss to apply certain empirical checks upon the characteristics of flow encountered, and to express the flow equations in simplified form through several common approximations.

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[16] California Inst. of Technology, Pasadena, Calif. Mr. Rouse was elected Assoc. M. Am. Soc. C. E., on January 13, 1936.

Approximate methods, however, are truly justifiable only if the investigator realizes the fact that the assumptions made do not fully describe the actual state of flow. When the authors claim that their paper "presents the theoretical hydraulic principles involved" in the performance of the Venturi flume, the writer believes that certain supplementary remarks have a definite place in this discussion. For although the authors state that "the formulas developed for the flume are not empirical but are based entirely on theory", the analysis of flow which they have presented is really only the first page of a long and complex story.

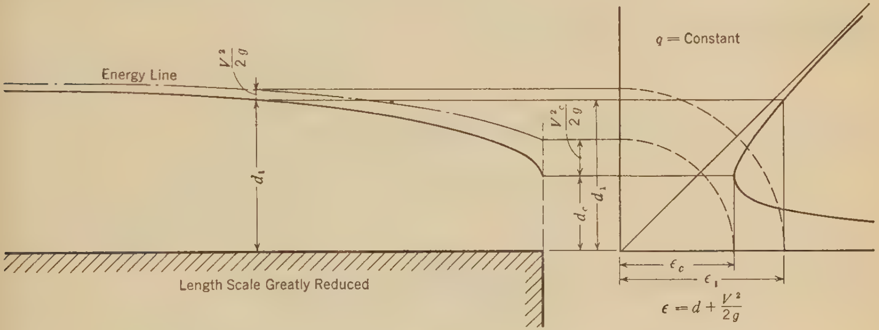


FIG. 12.

The specific energy diagram used by the authors applies correctly only to flow in long open channels in which the gradual change in surface profile is entirely the result of energy loss, and not of horizontal or vertical contraction of the flow section. Thus, in Fig. 12 is shown the surface curve in a long channel of mild slope ending in an abrupt fall, the specific energy of flow, and hence, also, the depth of flow, decreasing in the direction of motion; the specific energy must reach a minimum value, for the given rate of discharge, in the vicinity of the fall, in which region the depth is equal to the critical. It must be remembered that such an energy diagram is constructed on the assumption that the flow at every section is so nearly parallel that accelerative forces can be ignored; that is, the pressure distribution is considered static throughout, so that the sum of pressure head, $\frac{p}{\gamma}$, and elevation, z, must always equal the depth of flow. The vertical distance between the energy line and the free surface must then be equal to the velocity head.

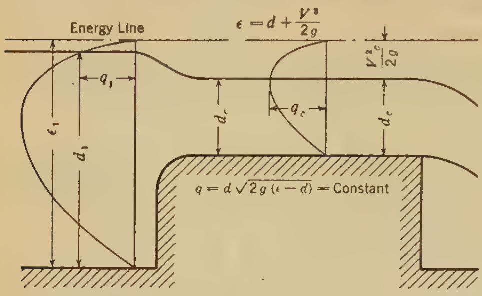

Energy Line
$\epsilon = d + \frac{V^2}{2g}$
$\frac{V_c^2}{2g}$
q_1
q_c
ϵ_1
d_1
d_c
d_c
$q = d\sqrt{2g(\epsilon - d)}$ = Constant

Fig. 13.

The authors' equations for flow through the Venturi flume, however, are practically identical with those used in the customary treatment of the broad-crested weir, in which the change in surface profile is entirely the result of contraction of the flow section (see Fig. 13). Not only is the flow assumed parallel throughout, but the further assumption is made that no energy loss occurs. Hence, instead of the specific energy diagram, recourse must be taken to the discharge curve, which considers the discharge per unit width as a function of the depth for a given energy line elevation. In the case of two-dimensional motion over the broad-crested weir, since $\epsilon = \frac{V^2}{2\,g} + d$, the unit discharge may then be expressed as:

$$q = \frac{Q}{b} = V d = d\sqrt{2\,g\,(\epsilon - d)} \;\ldots\ldots\ldots\ldots\ldots (36)$$

Taking the derivative of q with respect to d,

$$\frac{\delta q}{\delta d} = \sqrt{2\,g}\left(\sqrt{\epsilon - d} - \frac{d}{2\sqrt{\epsilon - d}}\right)$$

and setting this equal to zero $\left(\frac{2\,\epsilon - 3\,d}{2\sqrt{\epsilon - d}} = 0\right)$; and:

$$d_c = \frac{2}{3}\,\epsilon = \sqrt[3]{\frac{q^2}{g}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Obviously, the critical depth, d_c, derived by this method must be identical with that used by the authors, although the original premises are quite different.

From Equation (36) it is apparent that q must be zero when $d = 0$ and when $d = \epsilon$, and will attain its maximum value when $d = d_c$. Two such q-curves are plotted in Fig. 13, the water surface over the entire length of the weir coinciding with the point of maximum discharge on the smaller q-curve. For any other depth of flow the rate of discharge must be less than the maximum on the corresponding q-curve as shown in the region of approach.

As stated by the authors, a single measurement of depth in the throat should suffice for the determination of the discharge under these conditions, according to Equation (37). As also pointed out by the authors, a definite relationship must exist between the critical depth and the depth of approach, so that once this relationship is known a single measurement of the latter depth would also be sufficient.

A slightly different use may be made of the discharge curve in a procedure that is more directly applicable to the Venturi flume. As suggested to the writer by B. A. Bakhmeteff, M. Am. Soc. C. E., the surface curve for flow in either converging or diverging channels may be approximated in the following manner: Again assuming parallel flow at constant specific energy over a horizontal floor (the velocity then being normal at all points to horizontal concentric arcs of circles), the depth of flow at any section may be expressed in terms of the radius, or horizontal distance, r, from the fixed center, the total head, ϵ, and the rate of discharge per unit width, q. If the angle of convergence or divergence of the vertical walls is designated by θ, from the law of continuity the total discharge will be:

$$Q = \theta\, r\, V d = \theta\, r\, q = \text{constant} \quad \text{..............} (38)$$

Introducing the discharge per unit width at the critical section (where q must again be a maximum):

$$\theta\, r\, q = \theta\, r_c\, q_c \text{ and } \frac{q}{q_c} = \frac{r_c}{r} \quad \text{..................} (39)$$

Since,

$$\epsilon = \frac{V^2}{2\, g} + d = \frac{3}{2}\, d_c = \text{constant} \quad \text{..............} (40)$$

$$q = V d = d \sqrt{2\, g\, (\epsilon - d)} \quad \text{..................} (41)$$

and at the critical section,

$$q_c = \sqrt{g}\; d_c^{\frac{3}{2}} = \sqrt{g} \left(\frac{2}{3}\, \epsilon \right)^{\frac{3}{2}} \quad \text{..............} (42)$$

Substituting Equations (41) and (42) in Equation (39):

$$2.6\, \frac{d}{\epsilon} \sqrt{1 - \frac{d}{\epsilon}} = \frac{r_c}{r} = \frac{q}{q_c} \quad \text{..............} (43)$$

Equation (43) is thus a dimensionless relationship between the ratio of depth to total head and either the ratio of the critical radius to the radius at any section or the ratio of the unit discharge at any section to the critical unit discharge. This equation will yield the q-curve and the two surface curves shown in Fig. 14. It will be noted that in the region between $r = 0$ and $r = r_c$ no flow can occur under the assumed conditions, this region being known in hydrodynamics as a "singular" zone which does not belong to the region in which the function is continuous.

These relationships may be applied to the Venturi flume having vertical side contractions by assuming, as in the case of the broad-crested weir, that the critical depth occurs over the entire parallel section of the throat. Choosing arbitrarily the throat proportions shown in Fig. 15, this method would yield the profile given in the illustration. It is clear, as noted by the authors, that the hydraulic jump may form at any section beyond the throat, if the tail-water level is raised, for in this region the flow is in the rapid state. Thus, the Froude number, identical with Bakhmeteff's kinetic flow factor, λ, will be:

$$\lambda = 2\,\frac{\dfrac{V^2}{2\,g}}{d} = 2\,\frac{\dfrac{2\,g\,(\epsilon - d)\,d^2}{2\,g\,d^2}}{d} = 2\,\frac{1 - \dfrac{d}{\epsilon}}{\dfrac{d}{\epsilon}} \;\ldots\ldots\ldots\ldots(44)$$

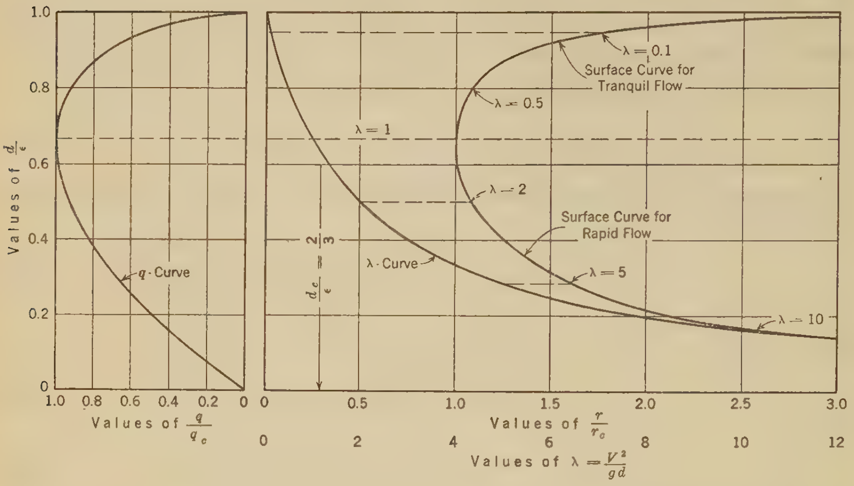
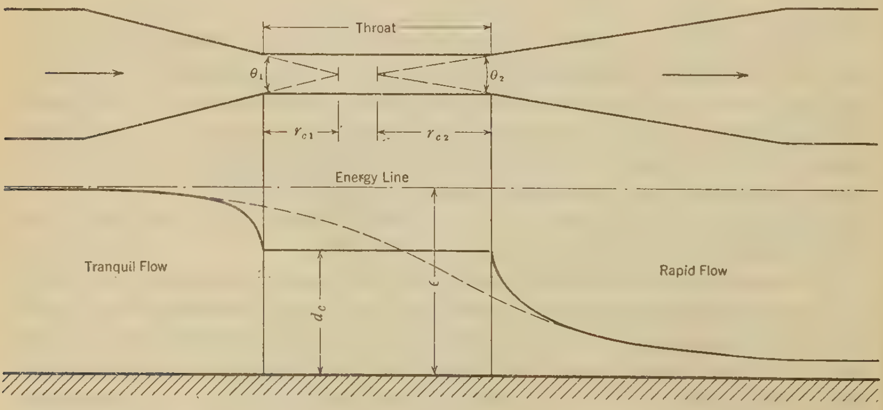


Fig. 14.

The characteristics of the jump at any section will then depend upon the magnitude of λ and of θ at that section; λ, of course, must have a value greater than unity before the jump can form. The curve of $\frac{d}{\epsilon}$ against λ is also shown in Fig. 14.



Throat
θ_1
θ_2
r_{c1}
r_{c2}
Energy Line
Tranquil Flow
Rapid Flow
d_c
ϵ

Fig. 15.

It should be apparent that a single measurement of the depth in the approach channel would suffice to determine the rate of discharge, the relationship between the depth at this point and the magnitude of q being given by Equation (43). Although this method has been applied only to straight contractions in the horizontal plane, a slight modification would permit simultaneous treatment of the vertical contraction of the lower boundary and even a study of any curvature of the converging and diverging boundaries. It is presumed, however, that the foregoing example will suffice to illustrate the point in question.

Although this procedure is fully as justifiable as the customary elementary treatment of flow over the broad-crested weir, the resulting picture of flow in the throat region is, to say the least, quite hypothetical. Actually, neither of the two basic assumptions made is completely fulfilled in the case of either weir or flume, and two major errors are thereby introduced by this elementary line of attack:

1.—The energy line is not horizontal, but slopes at a rate which varies with the discharge through a given flume or over a given weir. At any given discharge, the longer the meter the greater will be the discrepancy in total head between the channel of approach and the critical region. Since the energy line slopes, there can really be no critical "region" but only a critical section; and according to the energy diagram, which must now be used in addition to the q-curve, this critical section should be near the end of the parallel throat or horizontal crest.

2.—The flow is not even approximately parallel if the contracted section is short, and hence the calculations based upon assumed static pressure distribution cannot be correct. The shorter the contraction the greater will be the error introduced by this assumption. If parallel flow does not exist, surely the expression for critical depth based on this assumption cannot apply. Thus, the authors' statement that the sharp-crested weir is a "control section through which water flows at critical depth" is a fallacy, since the sharp-crested weir is the shortest possible contraction one can imagine and hence involves an extreme degree of departure from the basic assumption of parallel flow.

In order to clarify the error introduced by this assumption, consider the region of the fall shown in distorted scale in Fig. 12. If the length scale is given its correct proportion to the vertical scale, the profile of flow will be that shown in Fig. 16. As the flow approaches the abrupt drop, vertical acceleration will begin, and the nappe will be deflected in the downward direction. Since curvature of the filaments requires the action of centripetal forces, the pressure will no longer be statically distributed, and the pressure head will be less than the depth below the free surface by an amount depending upon the elevation and curvature of the filaments. Although the computed critical depth for parallel flow may still be found some distance up stream from the fall (a distance varying with the unit discharge, with the slope and roughness of the channel, and with the degree of ventilation of the space below the nappe), the fact that the energy line still slopes signifies that the true critical section—that is, that of minimum energy—exists at the fall itself, where the flow is highly curvilinear. Unfortunately, the treatment of critical flow under conditions of non-static pressure distribution has not yet been mastered by the hydraulician.

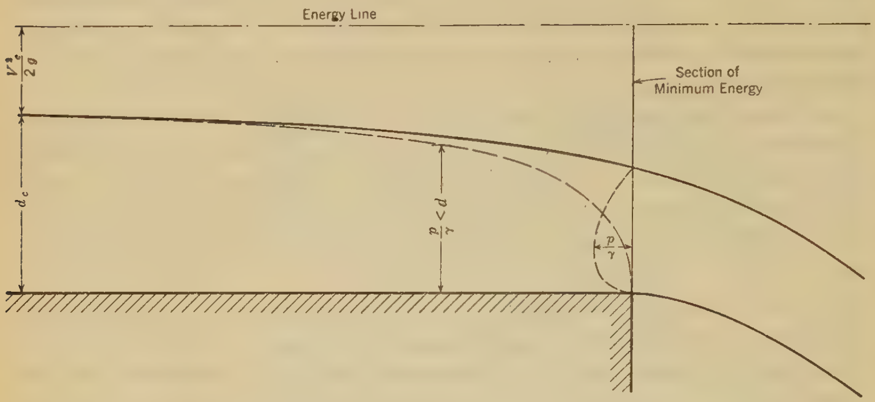


Fig. 16.

The situation at the free over-fall is closely parallel with that at the end of a broad-crested weir and is also an indication of what may be expected in the Venturi flume. Furthermore, the convergence of the boundaries between the channel of approach and the contracted section itself introduces another region of curvilinear flow which by no means can be neglected. If the meter is short these two transition regions partly overlap so that at no point throughout the contraction can parallel flow be assumed to exist. If the meter is long, reaction waves are likely to occur over the entire length of the contracted section, thereby introducing further curvature of the filaments.

These two basic fallacies in the customary elementary presentation thus present a combination of difficulties which can be ignored only in an approximate analysis, for it is almost impossible to devise a meter which, for a large discharge range, will be both short enough to make the energy loss inappreciable and long enough to insure nearly parallel flow at some definite section of the contraction. Not only will the true relationship between the actual depth of approach and the theoretical critical depth have to be determined by laboratory or field calibration, but the critical section for parallel flow—if it exists at all—will shift in position along the crest or throat with change in discharge.

The actual surface profile in a meter similar to that shown in Fig. 15 may be approximated by sketching in by eye a smooth reverse curve connecting the two computed portions, as shown. Obviously, no one profile can be established which will hold for every discharge, for the section at which this surface line intersects the line of critical depth for parallel flow will move in one direction or the other as the discharge varies. Its form will depend upon the proportions of the meter and upon the slope of the energy line, each of which geometrical relationships must be considered to change with the depth of the approaching flow.

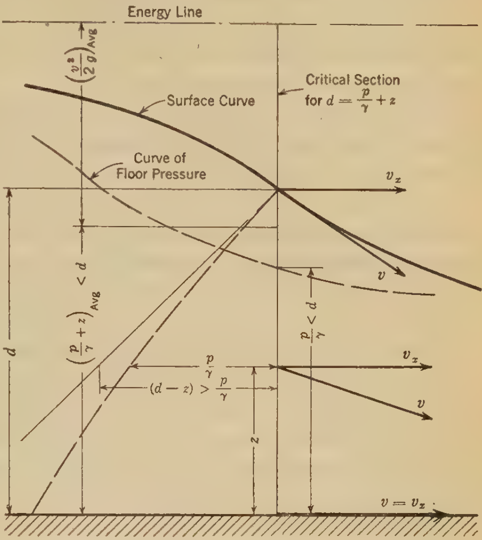

Energy Line
$\left(\frac{v^2}{2g}\right)_{Avg}$
Surface Curve
Critical Section for $d = \frac{p}{\gamma} + z$
Curve of Floor Pressure
v_x
v
d
$\left(\frac{p}{\gamma} + z\right)_{Avg} < d$
$\frac{p}{\gamma} < d$
$\frac{p}{\gamma}$
$(d - z) > \frac{p}{\gamma}$
z
v_x
v
$v = v_x$

FIG. 17.

As an illustration of the actual energy distribution in the throat of the meter, reference is made to Fig. 17, in which it is assumed, for the time being, that the total head is the same at all depths of flow. Although the depth at this section happens to be exactly equal to the computed critical depth, this really has no significance whatsoever, since the pressure distribution is not static; on the contrary, the pressure head, $\left(\frac{p}{\gamma}\right)$, at any elevation, z, above the floor, is considerably less than the distance, $(d - z)$, below the free surface, as a result of the vertical acceleration of the fluid particles at this section. Although the average velocity of flow as found from the equation of continuity, $V = \frac{q}{d} = \frac{1}{d}\int^{d} v_x \, \delta d$, is commonly used in the Bernoulli theorem, the latter may be written properly only in terms of the velocity vector, v, at any point and not its component, v_x, in the direction of flow; thus the true velocity head at any elevation, for the conditions shown, is everywhere greater than the assumed mean value, with the sole exception of the point at the water surface itself. In general, it may be said that in curvilinear flow both $\frac{v^2}{2g}$ and $\frac{p}{\gamma}$ will vary with z, so that the correct form of the Bernoulli theorem must be the following:

$$\epsilon = \frac{v^2}{2g} + \frac{p}{\gamma} + z \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

To the foregoing discrepancies in the assumptions traditionally made in dealing with such basic types of flow, there must be added further errors arising from variation in total head from floor to surface; from additional curvature due to vertical as well as horizontal contraction; from separation (change of actual flow section) and eddy formation due to abrupt inlet and excessive angle of divergence; and, finally, from partial drowning of the throat because of back-water. Even if many of these difficulties could be avoided through the use of a carefully constructed flume, attempts to produce a "critical-depth" meter of this kind with a constant theoretical coefficient under all passable discharges are quite futile; no meter of this type will yield a flow that is dynamically similar under different heads, for the geometrical relationship of the meter profile, water surface, and energy line cannot remain constant with changing depth.

When an attempt is made to analyze the physical state of flow, as undertaken by the authors, it is impossible to remain content with approximate methods; furthermore, it is not at all justifiable to claim theoretical completeness when it has not been attained. If the essential accuracy of the authors' experimental work is indicated by their practice of measuring static pressure closely enough simply by turning a Pitot tube sidewise, it is obvious that further refinement is quite needless; but if hydraulicians intend to use the Venturi flume in the future for more fundamental research, it is to be hoped that they will not follow blindly such approximate and empirical paths.

One must not conclude from these pages that the simple methods of attack are without value. On the contrary, if the full significance of the elementary facts contained in this discussion were more fully realized, the Boussinesq number would not have appeared in hydraulic literature on Venturi flumes, nor would much of the search for a true critical depth meter have been undertaken.

Dr. Ing. F. V. A. E. Engel[17] (by letter).—In comparison with other measuring devices, such as weirs, for example, the investigation of the Venturi flume meter extends over only a short period of about twenty years. This paper, therefore, is welcome as a further contribution on the subject. It certainly merits the careful study of hydraulic engineers who are confronted by problems relating to the measurement of large quantities of fluid flow.

The writer has been engaged for several years in investigating and designing Venturi flume meters and would like to offer the following comments. In connection with Equation (14), for a free discharging Venturi flume with a rectangular throat, reference should have been made to the investigations by Mr. A. H. Jameson. In 1925, Jameson[18] published equations relating to throat sections of different shapes; that is, a rectangle, triangle, parabola, and a trapezium. Definite progress was made by Jameson[19] in a paper published in 1930, but this work does not seem to have become generally known. The equations developed make it possible to determine the dimensions of a Venturi flume of rectangular section by a direct method, which seems to be much simpler than to solve Equation (14) graphically. Particularly in the case of a rectangular section of both the throat of the flume and the channel, the Jameson method for obtaining the various dimensions is so straightforward that it is the only one to be recommended.

Possibly the accuracy obtained by the authors with their method of calibration would have been greater had they been able to conduct the tests on the Venturi flume in a hydraulic laboratory. The results they obtained by comparing their flume with the Parshall meter are not very convincing, as an accuracy of the order of 5% would appear rather unsatisfactory.

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[17] Wembley, Middlesex, England.

[18] "The Venturi Flume and the Effect of Contractions in Open Channels", by A. H. Jameson, *Transactions*, Inst. of Water Engrs., Vol. 30 (1925).

[19] "The Development of the Venturi Flume", by A. H. Jameson, *Water and Water Engineering*, Vol. 32 (1930), pp. 105 to 107.

To the writer's knowledge, the Parshall meter has not been tested in a hydraulic laboratory and, therefore, its discharge coefficient is not known with sufficient accuracy. The discharge coefficients of the Parshall meter and the design by Messrs. Palmer and Bowlus may have quite different characteristics and it seems a rather doubtful method, therefore, to calibrate one Venturi flume meter against another. The only reliable method of calibration would appear to be either by weighing or by the determination of water-volume passing through the flume in a measured time.

A further point which does not seem very satisfactory is that the calibration only covered a range of Q-values equal to about 1 to 5, whereas a range of about 1 to 8 or 1 to 10 is usually demanded.

TABLE 6.—DISCHARGE COEFFICIENTS, VENTURI FLUME; THROAT LENGTH, 20 INCHES

Test No.	Rate of flow, Q, in liters per sec* (1)	DEPTH OF FLOW, IN CENTIMETERS †		WIDTH, IN CENTIMETERS †		Discharge coefficient C_f (6)	Deviation from the mean value of C_f (percentages) (7)	Discharge coefficient, C'_f (8)	CONSTANTS, RELATED TO CHANNEL SECTION UP STREAM FROM FLUME	
		Up stream, d_2 (2)	Throat, d_3 (3)	Up stream, b_2 (4)	Throat, b_3 (5)				Boussinesq number (9)	Froude number (10)
(a) FLUME NO. 4, 1932; WIDTH RATIO, $\frac{b_3}{b_2} = 0.294$										
1	37.171	39.44	31.27	31.30	9.17	0.9433	−0.52	0.9676	0.203	0.153
2	30.764	34.65	27.19	13.30	9.18	0.9468	−0.15	0.9638	0.194	0.154
3	24.205	29.49	22.91	31.30	9.19	0.9482	0	0.9648	0.185	0.154
9	24.208	29.49	22.97	31.30	9.19	0.9483	+0.01	0.9648	0.185	0.154
10	24.211	29.50	22.92	31.30	9.19	0.9481	−0.01	0.9645	0.185	0.154
33	24.204	29.62	23.07	31.30	9.14	0.9465	−0.18	0.9634	0.184	0.153
11	18.430	24.60	18.83	31.30	9.19	0.9472	−0.11	0.9640	0.175	0.154
12	13.365	19.76	14.78	31.30	9.21	0.9516	+0.36	0.9690	0.165	0.155
19	13.344	19.77	14.81	31.30	9.21	0.9495	+0.14	0.9668	0.164	0.155
20	13.344	19.77	14.79	31.30	9.21	0.9495	+0.14	0.9668	0.164	0.155
41	13.232	19.69	14.77	31.25	9.17	0.9512	+0.32	0.9687	0.164	0.155
40	13.111	19.57	14.69	31.25	9.17	0.9517	+0.37	0.9686	0.164	0.155
21	8.446	14.58	10.58	31.30	9.26	0.9445	−0.39	0.9611	0.152	0.155
22	4.471	9.49	6.67	31.05	9.28	0.9486	+0.04	0.9665	0.141	0.157
(b) FLUME NO. 3, 1932; WIDTH RATIO, $\frac{b_3}{b_2} = 0.533$										
22	51.897	31.96	25.18	31.20	16.6	0.9526	−0.50	1.0149	0.363	0.294
23	49.538	31.12	24.45	31.20	16.6	0.9469	−1.10	1.0081	0.358	0.293
1	46.88	29.82	23.32	31.20	16.6	0.9547	−0.28	1.0171	0.356	0.294
3	46.88	29.82	23.30	31.20	16.6	0.9547	−0.28	1.0171	0.356	0.294
7	46.88	29.80	23.30	31.20	16.6	0.9554	−0.21	1.0182	0.356	0.294
40	46.294	29.66	23.10	31.20	16.6	0.9598	+0.25	1.0128	0.354	0.294
24	39.885	26.78	20.61	31.20	16.6	0.9542	−0.33	1.0169	0.344	0.294
59	35.369	24.64	18.91	31.20	16.6	0.9553	−0.22	1.0188	0.336	0.295
25	30.892	22.53	17.05	31.20	16.6	0.9575	+0.01	1.0208	0.327	0.296
8	25.253	19.61	14.59	31.10	16.6	0.9619	+0.47	1.0276	0.318	0.298
11	25.181	19.63	14.63	31.10	16.6	0.9585	+0.12	1.0231	0.317	0.297
12	25.241	19.64	14.63	31.10	16.6	0.9601	+0.28	1.0248	0.318	0.298
13	25.301	19.64	14.64	31.10	16.6	0.9616	+0.44	1.0273	0.318	0.298
48	25.225	19.67	14.70	31.10	16.6	0.9574	0	1.0219	0.317	0.297
26	19.887	16.70	12.22	31.10	16.6	0.9640	+0.69	1.0295	0.305	0.300
63	16.407	14.67	10.65	31.10	16.6	0.9657	+0.87	1.0318	0.296	0.300
27	13.436	12.86	9.25	31.10	16.6	0.9633	+0.62	1.0288	0.286	0.299
53	8.824	9.74	7.28	31.10	16.6	0.9613	+0.41	1.0256	0.269	0.298
54	8.824	9.74	7.24	31.10	16.6	0.9613	+0.41	1.0256	0.269	0.298
16	8.732	9.69	7.15	31.00	16.6	0.9584	+0.10	1.0228	0.270	0.298
17	8.732	9.69	7.15	31.00	16.6	0.9584	+0.10	1.0228	0.270	0.298
21	8.725	9.71	7.17	31.00	16.6	0.9549	−0.26	1.0189	0.266	0.297
28	7.365	8.68	6.43	31.00	16.6	0.9542	−0.33	1.0176	0.262	0.296
29	6.140	7.74	5.72	31.00	16.6	0.9462	−1.17	1.0075	0.255	0.294

* 100 liters per sec = 3.54 cu ft per sec. † 1 cm = 0.3937 in.

It is the opinion of the writer that the Venturi flume meter, if correctly designed, is one of the most accurate of measuring devices, and that it has great advantage in comparison with the weir method.

Table 6 contains some test results on Venturi flume meters obtained during previous investigations[20] in the hydraulic laboratory of the City and Guilds (Engineering) College, in London, England, in 1932. Both the Venturi flumes were installed in a rectangular channel and had a rectangular throat section. The discharge equations used were:

$$Q = \left(\frac{2}{3}\right)^{\frac{3}{2}} C_f \sqrt{g} \; b_3 \left(d_2 + \frac{V_2^2}{2g}\right)^{\frac{3}{2}} \qquad \text{(46)}$$

and,

$$Q = \left(\frac{2}{3}\right)^{\frac{3}{2}} C'_f \sqrt{g} \; b_3 d_2^{\frac{3}{2}} \qquad \text{(47)}$$

in which C'_f is the product of C_f and the velocity-of-approach factor. Further data may be obtained from the previous publications by the writer[21] on the subject. For the flume with a width ratio of 0.294 (Table 6(*a*)), the average coefficient of discharge is 0.9482 (see Column (6)). The rate of flow was varied over a measuring range of about 1 to 8. It is of interest to note that there is only one test for which the discharge coefficient is more than 0.5% from the average. All the other test results are within the limits of ± 0.5 per cent. For a width ratio of 0.533, the test results of which are given in Table 6(*b*), there are only two test points outside the limit of ± 1%, whereas 19 of the total of 24 were within the limits of ± 0.5%, the average coefficient being 0.9574. Seldom under actual working conditions could an accuracy like this be obtained, but one can assume that the accuracy of a Venturi flume meter in a rectangular channel and using dimensions similar to those used in the writer's investigations, would be in the order of ± 1.5 per cent.

The section entitled "Energy Losses", seems rather misleading. The problem treated only refers to the dissipation of energy in the convergent entrance section and the authors' conclusion is that there is practically no dissipation of energy—a point which is evident from previous research. The more important point regarding the head losses due to the Venturi flume was totally neglected by the authors, whereas this is of the greatest practical value.

In designing Venturi flume meters, it is absolutely necessary to ascertain whether free discharge conditions prevail for certain down-stream conditions. There is a minimum head loss for which the flume will discharge without banking up the up-stream level for a given rate of flow. For a Venturi flume with a width ratio of 0.533, the writer found that in a channel with a horizontal bottom, it was possible to increase the down-stream depth without changing the state of free discharge, to a level corresponding to 92% of the up-stream depth. In a flume of a width ratio of 0.294, however, the down-stream depth was between 74% and 90% of the up-stream depth for the limit of free discharge conditions.

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[20] "Non-Uniform Flow of Water: Problems and Phenomena in Open Channels with Side Contractions", by F. V. A. Engel, *The Engineer*, Vol. 155 (1933), pp. 392 to 394, 429–430, and 456–457.

[21] "The Venturi Flume", by F. V. A. E. Engel, *The Engineer*, Vol. 158 (1934), pp. 104 to 107 and 131 to 133.

In order to determine head losses, the writer has developed a certain criterion based on the Boussinesq number[21], [22]. The critical Boussinesq number determines the head loss, which may be about 8%, or a value between 10 and 30 per cent. The critical Boussinesq number for the second flume is 0.358, and from this it will be seen that only for Test No. 22 (Table 6(*b*)) will the head loss be greater than 8 per cent. In the case of the flume with the narrow throat, the critical Boussinesq number is 0.147 and from Column (9), Table 6(*a*), it is evident that in all cases, except the last line, the head losses will be greater than 8 per cent.

The critical Froude number determines whether the flume is discharging freely, and the test results are quite in accordance with the critical numbers obtained by calculation. The critical Froude number for the flume with the width ratio of 0.294 is 0.154, referred to the up-stream section, whereas for the second flume, the critical Froude number is 0.296. The last columns of Table 6 show also a characterizing feature of both the Froude and Boussinesq numbers, namely a constant value of the Froude number and a comparatively large variation in the Boussinesq number.

J. C. Stevens,[23] M. Am. Soc. C. E. (by letter).—A very practical application of the use of critical flow for measuring the discharge in conduits is demonstrated in this paper. If critical flow can be produced in a conduit by any kind of constriction whatever, the depth above that constriction immediately becomes a stable index of discharge.

Caution must be observed, however, in designing the control so that it is not submerged for the range of discharges that may obtain. Whenever the tail-water level is more than about 65% of that of the head-water over the crest of the control, the head-water rises with the tail-water and loses its stability as an index of flow. Two depths must then be observed and discharge determination from them becomes more complex and uncertain. Recorders with totalizing equipment also lose their efficacy at this point of submergence.

In circular conduits the writer has used a "hump" control on the bed of the conduit to produce critical flow over it. The flow is readily computed from the criterion that for critical flow the velocity head equals one-half the mean depth. This applies to any shape of conduit whatever as long as width is a continuous function of the depth.

Fig. 18 shows the type of control used in a circular conduit, which is believed to have some advantages over the trapezoidal control used by the authors and certainly involves less energy losses. It is readily constructed of concrete without forms; it may even be pre-cast and placed while the conduit is in service. Fig. 18 also shows the rating curves for five heights of the control section. Discharges are given for depths in the conduit just above the control and both depths and discharges are in terms of the inside diameter of the pipe.

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[22] "Venturi-Kanalmesser: Die messtechnischen Eigenschaften in Abhaengigkeit von den Stroemungsarten", von F. V. A. E. Engel, *Archiv für Technisches Messen*, V. 1253–2, March, 1935, No. 45.

[23] Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.

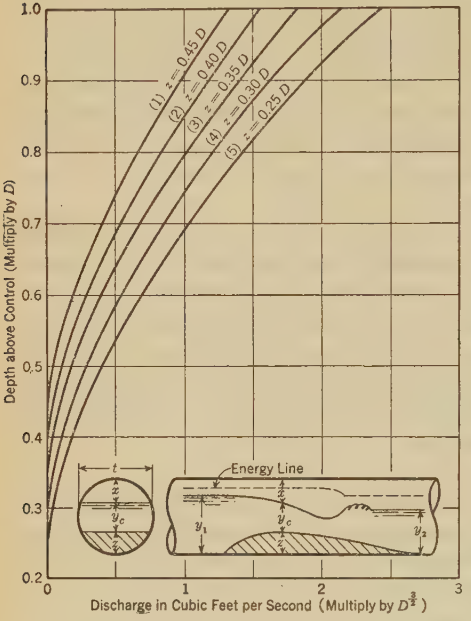

FIG. 18.

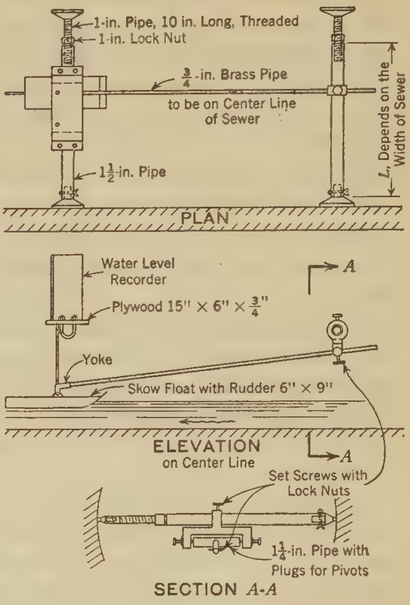
1-in. Pipe, 10 in. Long, Threaded
1-in. Lock Nut
$\frac{3}{4}$-in. Brass Pipe
to be on Center Line of Sewer
$1\frac{1}{2}$-in. Pipe
L, Depends on the Width of Sewer
PLAN
Water Level Recorder
Plywood 15" × 6" × $\frac{3}{4}$"
Yoke
Skow Float with Rudder 6" × 9"
A
ELEVATION on Center Line
Set Screws with Lock Nuts
$1\frac{1}{4}$-in. Pipe with Plugs for Pivots
SECTION A-A

FIG. 19.

Table 7 is a sample of the computations for one of the curves, assuming D equal to unity. To obtain corresponding values for any other diameter, the tabular values are multiplied by the function of the diameter indicated at

TABLE 7.—SAMPLE COMPUTATIONS FOR CURVE 4, FIG. 18

$$\left(\frac{z}{D} = 0.30;\ \text{AND}\ \frac{Z}{D^2} = 0.198\right)$$

$\frac{y_c}{D}$ (1)	$\frac{z + y_c}{D}$ (2)	$\frac{X}{D}$ (3)	$\frac{X}{D^2}$ (4)	$\frac{0.785 - (Z + X)}{D^2} = \frac{Y_c}{D^2}$ (5)	$\frac{t}{D}$ (6)	$\frac{Y_c}{t} \div D = \frac{y_m}{D}$ (7)	$\frac{1}{2} y_m \div D = \frac{h_c}{D}$ (8)	$\frac{\sqrt{2 g h_c}}{\sqrt{D}} = \frac{V_c}{\sqrt{D}}$ (9)	$\frac{Y_c V_c}{D^{\frac{3}{2}}} = \frac{Q}{D^{\frac{3}{2}}}$ (10)	$\frac{y_c + h_c + Z}{D} = \frac{\epsilon}{D}$ (11)	$\frac{A^*}{D^2}$ (12)	$\frac{Q}{A} \div \sqrt{D} = \frac{V_1}{\sqrt{D}}$ (13)	$\frac{V^2_1}{2 g} \div D = \frac{h_1}{D}$ (14)	$\frac{\epsilon_h - h_1}{D} = \frac{y_1}{D}$ (15)
0	0.3	0.7	0.587	0	0.916	0	0	0	0	0.300	0.198	0	0	0.30
0.1	0.4	0.6	0.492	0.095	0.980	0.097	0.098	1.77	0.168	0.448	0.341	0.49	0.004	0.44
0.2	0.5	0.5	0.393	0.194	1.000	0.194	0.097	2.50	0.485	0.597	0.489	1.01	0.016	0.58
0.3	0.6	0.4	0.293	0.294	0.980	0.300	0.150	3.11	0.914	0.750	0.632	1.45	0.032	0.72
0.4	0.7	0.3	0.198	0.389	0.916	0.425	0.212	3.70	1.439	0.912	0.752	1.91	0.057	0.86
0.5	0.8	0.2	0.112	0.475	0.800	0.594	0.297	4.37	2.075	1.097	0.785	2.64	0.109	0.99

*A_e = area corresponding to ϵ_e.

the head of each column. A table of areas and chord lengths of circular segments is all that is required. Supplementing the notation used by the authors, let z = maximum height of the control "hump"; Z = area of the control hump at maximum height; $x = 1 - (y_c + z)$; X = area of a segment with a rise of x; y = depths (y_c, critical depth; y_m, mean depth, etc.); Y = area corresponding to depths, y; t = top width of water surface = chord of a segment with a rise of $z + y_c$; h = velocity head (h_c for critical flow, etc.); ε = energy head above the pipe invert; and A = area above the control corresponding to energy head, ϵ. All symbols are expressed in terms of the pipe diameter, D.

The depth, y_1 (see Fig. 18), is obtained by deducting the velocity head at this point from the energy head. The trial method used by the authors can be eliminated by dividing the discharge by the area corresponding to the energy head, to obtain the pipe velocity, V_1. This is shown in the last four columns of Table 7. The error involved from assuming no energy loss from y_1 to y_c and that from using the area corresponding to the energy head, ϵ, tend to compensate each other.

The method of computing the rating curve directly illustrated in Table 7 is equally effective for the trapezoidal control described by the authors.

In large sewers in which only the sanitary flow is to be measured, or at manholes over smaller ones, the equipment illustrated in Fig. 19 has been used very effectively. Two screw struts are held in the sewer by expanding them against the sides of the pipe or the manhole. One strut carries a small platform on which the recorder is mounted. The other supports an adjustable pivot from which trails a float that can be adjusted both up stream and down stream and laterally until it is properly adjusted to connect with the style of the recorder.

A scow-shaped float with a rudder was found to have decided advantages over other types. One advantage of this equipment is the ease with which it can be moved from place to place.

PROF. ING. FILIPPO ARREDI[24] (by letter).—The sole purpose of this discussion is to suggest a variation in the authors' method of calculating discharge for the given case, which is a little different from that described in the paper and, possibly, slightly more practical.

Equations (8) and (10) of the paper apply to the throat section, if the water is assumed to flow at critical velocity. For the section of the channel above the throat Equation (2) applies, in which ϵ and Q have the same values as in the throat, and d and A are the depth of water and the area of the wetted section of the channel, respectively.

The method of calculating the Venturi flume which the authors propose is as follows: Taking a value for d_c, the values of A and B are calculated for the throat, and the values of Q and ϵ are then obtained from Equations (8) and (10). Then, from Equation (2), substituting these values of Q and ϵ, d can be obtained by trial, taking account of the relations between A and d.

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As it will usually be necessary to study throats of various types in order to make a choice among them, it will also be necessary to make calculations for many values of Q, and considerable work, requiring a large amount of time, may be needed. For this reason, it may perhaps be useful to supplement the authors' method by a graphical device which may prove more rapid in that it does not require "trial-and-error." The relation,

$$\frac{V^2}{2\,g} = \frac{Q^2}{2\,g}\,\frac{1}{A^2} \quad \text{.........................(48)}$$

applies for both the channel and the throat. If a value is assumed for $\frac{Q^2}{2\,g}$ and arbitrary values are taken for A, the corresponding values of $\frac{V^2}{2\,g}$ may be obtained easily from Equation (48). The computation is repeated for various values of $\frac{Q^2}{2\,g}$. Curves corresponding to each value assumed for $\frac{Q^2}{2\,g}$ are then drawn in a diagram (see Fig. 20) in which the values of $\frac{V^2}{2\,g}$ are ordinates and those of A are abscissas.

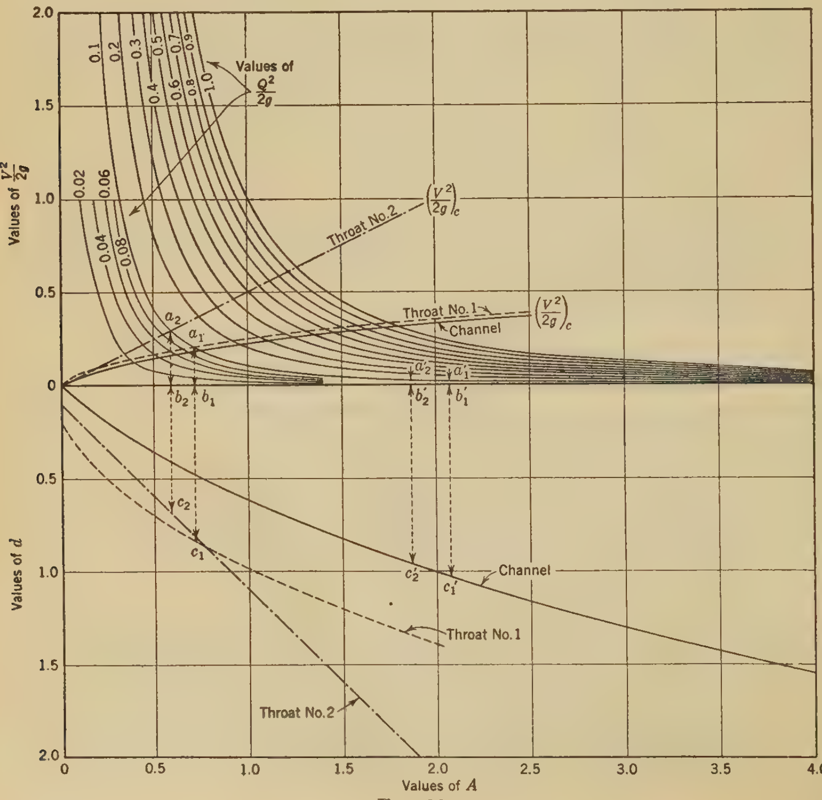


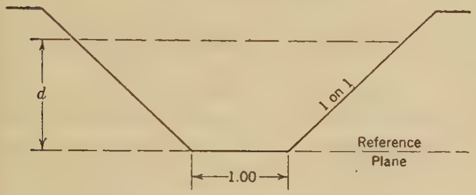
FIG. 20.

Knowing the section of the canal, and assuming the sections of one or more throats, the relation between the values of A and the depth, d, becomes known for each case, d being taken with respect to a common datum. In Fig. 20 the curve that expresses A in terms of d, for either channel or throat, is drawn downward from the zero line.

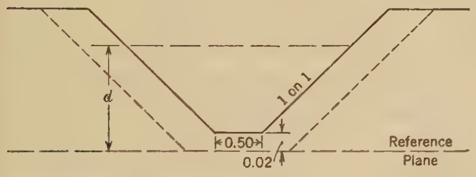
For critical conditions, the relation,

$$\left(\frac{V^2}{2\,g}\right)_c = \frac{A}{2\,B} \quad \text{.........................(49)}$$

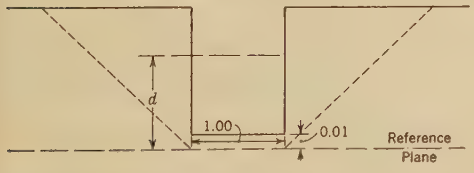
applies. Assuming values of d_c in the throat, corresponding values of A and B are calculated, and $\left(\frac{V^2}{2g}\right)_c$ is computed by means of Equation (49), and the curve showing $\left(\frac{V^2}{2\,g}\right)_c$ as a function of A can be drawn intersecting the curves already drawn. It may be useful, although not necessary, to draw a similar curve for the channel also (see Fig. 20).



(a) SECTION OF CHANNEL



(b) SECTION AT THROAT NO. 1



(c) SECTION AT THROAT NO. 2

FIG. 21.

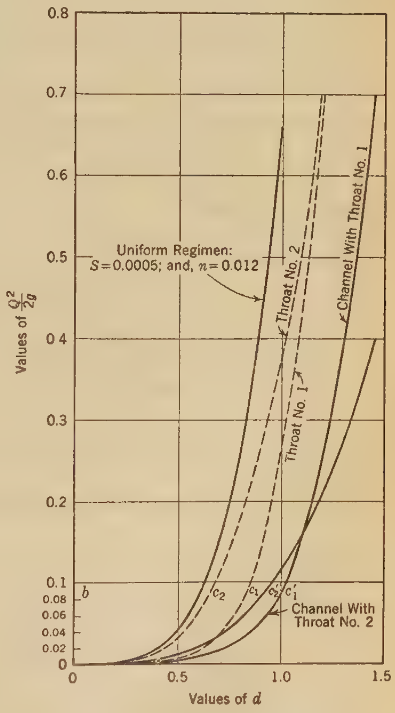


FIG. 22.

Assuming, for example, that a discharge, Q, such that $\frac{Q^2}{2\,g} = 0.1$, flows through Throat No. 1 in the critical condition, it is then easy to see that the ordinate, $a_1\,b_1$, represents the velocity head. In Fig. 20, a_1 is the point of intersection of the $\frac{V^2}{2\,g}$- curve, plotted as a function of A (for $\frac{Q^2}{2\,g} = 0.1$), with the curve of $\left(\frac{V^2}{2\,g}\right)_c$ for Throat No. 1. The dimension, $b_1\,c_1$, represents the depth of water, c_1 being the intersection of the vertical through a_1, with the A-curve plotted in terms of d for Throat No. 1; and the length, $a_1\,c_1$, being the total energy head in the throat.

The next step is then to find a vertical segment, $a'_1\,c'_1$, which will be equal in length to $a_1\,c_1$ and which will have its upper end on the $\frac{V^2}{2\,g}$- curve in terms of A for $\frac{Q^2}{2\,g} = 0.1$, and its lower end on the d-curve in terms of A for the channel. Then, $a'_1\,b'_1$ will be the velocity head, $b'_1\,c'_1$, the depth of water, and $a'_1\,c'_1$, the total energy head above the throat. In a similar manner, it is seen that the segment, $b_2\,c_2$, represents the depth of water in the throat and $b'_2\,c'_2$, the depth of water in the channel, if the same discharge flows through Throat No. 2 in the critical condition, $a'_2\,c'_2$ being equal to $a_2\,c_2$.

It is evident that the curve of $\frac{V^2}{2\,g} = f(A)$ for any constant, $\frac{Q^2}{2\,g}$, will be the same, whatever the channels and throats may be. The other curves in Fig. 20 were drawn by assuming channels and throats such as those in Fig. 21. The depth, d, is always referred to the plane through the channel bottom, at a given section. The detailed formulas by which these curves were computed are not reported.

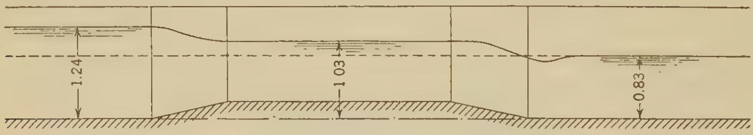


FIG. 23.—PROFILE, THROAT NO. 1; $\frac{Q_2}{2\,g} = 0.3$.

In order, rapidly, to sketch the profiles of the water surface in the flume, it will be useful to make diagrams such as those in Fig. 22, in which the values of $\frac{Q^2}{2\,g}$ are entered as ordinates and the values of d (computed as described previously), as abscissas. The curves of $\frac{Q^2}{2\,g}$, as a function of d, are drawn for the throats and the channel above the throat. Then, in Fig. 22, at $\frac{Q^2}{2\,g} = 0.1$, are recorded the ordinates, $b c_1$, $b c_2$, $b c'_1$, $b c'_2$, obtained from Fig. 20, which represent the depth, d, for Throats Nos. 1 and 2 and for the water in the channel above either Throat No. 1 or Throat No. 2. By varying $\frac{Q^2}{2\,g}$, it becomes possible to obtain a sufficient number of points to trace such curves.

It will also be useful to draw the curve of $\frac{Q^2}{2\,g}$ as a function of d for the uniform regimen that is present below the throat. In Fig. 22, this curve has been drawn on the basis of Manning's formula, assuming a slope, $S = 0.0005$ and $n = 0.012$. Fig. 23 represents a profile with Throat No. 1, relative to a discharge, Q, such that $\frac{Q^2}{2\,g} = 0.3$, the values for d of 1.24 above the throat, 1.03 at the throat, and 0.83 below the throat having been obtained from Fig. 22.

By the same method by which the curves of d were drawn in Fig. 22, it is easy to draw those for ϵ as a function of the $\frac{Q^2}{2\,g}$- curves which may be considered equal to those drawn by the authors in Figs. 8 and 10. It would then be easy to select from the throats studied the one best adapted to a particular case, more satisfactorily than under the conditions recorded by the authors in their paper.

HAROLD K. PALMER,[25] M. AM. SOC. C. E. AND FRED D. BOWLUS,[26] ASSOC. M. AM. SOC. C. E. (by letter).—From the many constructive discussions received, the writers are encouraged to hope that, more and more, water will be measured by Venturi flumes developing critical depths. Especially will this be true when more accurate verification can be obtained, in hydraulic laboratories, of some of the principles involved, and more simplified calculations of flow can be made, as suggested by some of the discussers.

Mr. Hopkins has developed excellent formulas for calculating co-ordinates on the quantity-energy head curves. Since these curves form the basis of all Venturi flume rating curves, the writers suggest that all notations referring to the frictional loss, varying with the shape of the throat, and to velocity head, varying with the velocity, be omitted from the terms of the formulas offered by Mr. Hopkins. This would mean substituting ε for Z in his formulas. In order that these may be readily available, the formulas for the more common throat sections are summarized as follows: For a flume of rectangular cross-section, Equation (26) becomes:

$$Q = 3.09\ b\ \varepsilon^{1.5} \qquad (50)$$

for a flume of V-shaped cross-section, Equation (28) becomes:

$$Q = 2.297\ p\ \varepsilon^{2.5} \qquad (51)$$

and, for a flume of trapezoidal cross-section, Equation (31) becomes:

$$Q = 3.06\ (b + 0.72\ p\ \varepsilon)\ \varepsilon^{1.5} \qquad (52)$$

Professor Arredi has suggested a most welcome graphical method of determining the proper size for a Venturi flume throat in any given installation and for drawing the rating curve, thus obviating the more laborious method advanced by the writers.

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Mr. Stevens describes a slab, curved in longitudinal section (which he has already used in circular conduits), based on somewhat the same principle as that of the writers. This would probably add some refinement by reducing frictional losses which, however, might be smaller than the recording device could register. The writers believe that the insertion of a level section in the bottom slab will be positive assurance that critical depth is obtained with parallel flow, as proved by their early experiments. They do not agree with Mr. Stevens as to the preference of the simple flat slab over the trapezoidal form of throat. The Los Angeles County Sanitation Districts had a 6-in. slab with a 3-ft level section (6 ft including transitions), installed in a 54-in. sewer, but found it necessary to change to a trapezoidal flume for greater accuracy. One difficulty with a slab is that a thickness required to give good results during peak flows may cause deposits of silt or sludge at low flows. For very small flows in a sewer, a flume with side contractions and no bottom slab would be better.

The writers also disagree with Mr. Stevens' statement that whenever the depth of tail-water below the throat of a Venturi flume is more than 65% of that above the throat, correct calculations of the flow can be obtained only by recording depths at the two points. The writers have recently been able to investigate this point by installing a flume in an irrigation ditch of the Arroyo Ditch and Water Company (see Fig. 24), and believe that within the range of accuracy of a recorder the flow can be calculated from a single head-water depth above the flume when the tail-water depth is approximately 90% of the head-water depth. Dr. Engel has found this percentage to be as high as 92. A simple analysis of losses and depths of flow through a rectangular Venturi flume installed in a conduit with a grade so flat as to cause a low velocity, tends to substantiate the correctness of the latter figure (92%). In this case the depth of water in the throat at the point of critical velocity is practically two-thirds the head-water depth, or the velocity head equal to one-third the head-water depth (neglecting the small velocity head above the throat). As the water passes from the restricted throat section into the normal cross-section of the conduit below the flume, a condition obtains similar to that in a pipe enlargement, in which case it is the usual practice to consider that the loss is 25% of the difference in velocity heads. Neglecting the slight velocity head of the tail-water, only 25% of the velocity head in the throat would be lost, or one-twelfth the total energy head. This amounts to 8% and thus would leave the tail-water depth at about 92% of the head-water depth.

The writers have observed also that if the jump, or undulations, occur in the transition section below the throat, there is no effect on the water depth above the meter, but when the undulations occur in the throat and are backed up to a point near the section of critical depth, the head-water depth may be affected. It is best, therefore, to design the throat so that the jump will occur in the lower transition, or with the tail-water depth less than 90% of the head-water depth.

The writers are indebted to Mr. Stevens for the details of his float mechanism which should give accurate readings without being affected by floating débris.

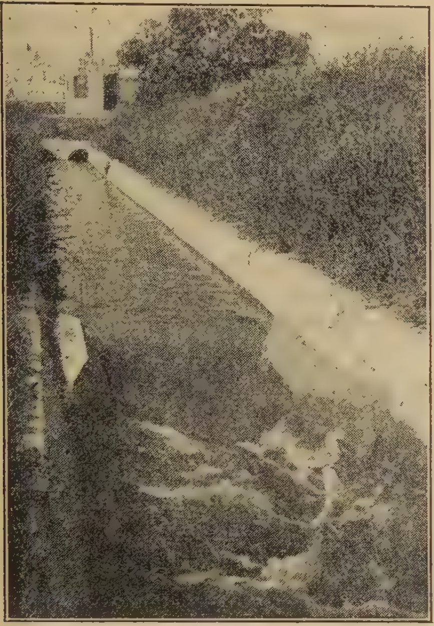

FIG. 24.—VENTURI FLUME IN WHICH JUMP OCCURS BELOW THROAT WITH NO APPARENT BACK-WATER EFFECT. VELOCITY ABOVE THROAT IS ABOUT 2.5 FEET PER SECOND, RESULTING IN SURFACE WAVES.

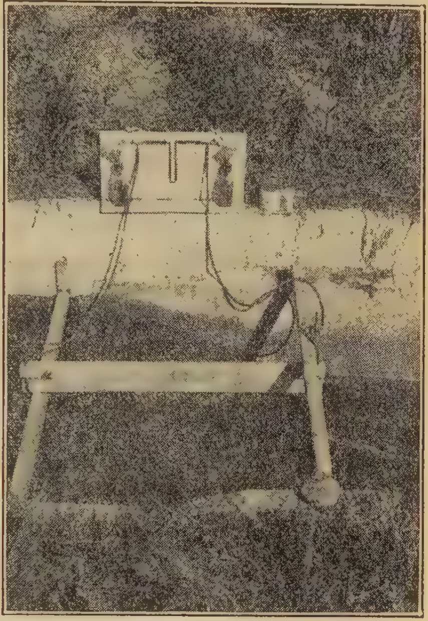

FIG. 25.—DIFFERENTIAL HEAD METER SET TO REGISTER LOSS OF HEAD AT ENTRANCE TO FLUME. IN THIS CASE LOSS WAS 0.01 FOOT FOR 10 CUBIC FEET PER SECOND.

Dr. Engel claims, in his English experiments, an accuracy for Venturi flumes of 99.5%, but unfortunately in the two cases for which he published the data, the water did not flow at critical depths in the throat and, therefore, the results are not applicable to the present case. The accuracy attained seems to vary inversely as some power of the velocity above the throat. Dr. Engel questions the accuracy of the Parshall meter, but in the United States it is generally accepted as a standard method of measuring water, and the device has been legalized by the Division of Water Rights of the California Department of Public Works. The writers have apparently been misunderstood in the test comparison with the Parshall meter. As the wooden Venturi flume was installed in an irregular channel the comparative results with the Parshall meter were only submitted to show the general adaptability of the flume throughout a considerable range in flow, in a sewer where it would have been impossible to have installed a weir. Dr. Engel states that the important point regarding losses due to the installation of a Venturi flume in a conduit has been entirely neglected by the writers. In the writers' method of computing a rating curve, the loss of energy up stream from the throat affects the rating curve, whereas the loss down stream has no effect as long as it does not cause a backing up of the water. In the formula for the Venturi flume as given by Dr. Engel, the loss of head up stream is taken care of in the constant, C_f, but it is probably a function of the length of the throat and flow, being negligible for smaller throats and also for small flows.

Mr. Rouse has ably discussed several of the theoretical limitations to the proposed method of using these flumes, which apply in cases where circumstances permit more than the ordinary accuracy of measuring the depth of water, or in very large installations. The writers had some of these limitations in mind when they suggested measuring the loss of energy at the entrance to the throat. He rightly questions its suitability as a control meter in hydraulic laboratory experiments as it would be necessary to calibrate each installation of that kind. Under such cases the weir would serve just as well and would be much easier to install, but in the field where allowable grade loss and funds for an elaborate metering station are limited, and especially where the need for measuring the flow arises after the completion of a conduit, the Venturi flume appears superior.

His most serious objection is that the exact location of the critical depth section is unknown, and varies with the flow; this is true, but it is the minimum energy rather than the critical depth that the writers use; and small variation from critical depth, or some curvature of surface, may occur without an appreciable effect on the energy-head depth. There is a difference in the height of the energy head at the control section in the throat and the point of measurement, depending on the hydraulic conditions in this section of the conduit; but it is small and can be measured easily and allowed for within the degree of accuracy of the recording device (see Fig. 25). Although, theoretically, it is a drawback, practically the error is less than the sensitivity of the recording devices now available.

The following case illustrates some of the difficulties met in field installations. In a 24-in. trapezoidal flume set in a 54-in. sewer having a grade of 0.15%, a flow averaging 21.5 cu ft per sec produced a record with a blurred line nearly 0.25 in. in width, due to small surface waves superimposed on longer waves with a period of a minute or more. The flow varied from 20.6 to 22.4 cu ft per sec and the mean could be obtained only by estimating the middle of the trace.

Mr. Rouse takes exception to the writers' attempt at measuring flow over a great range by the method proposed. This difficulty is one which must be met in sewage-flow measurements where the daily maximum may be three times the minimum, and during a storm a much greater range is to be expected. If the sewerage system is growing, the daily maximum ten years hence may easily be ten times the minimum of to-day. In storm drains the flow is even more erratic. This precludes the use of the hook-gauge for measuring depths and substitutes the stage recorder; therefore, in considering the degree of accuracy to be attained, it must be borne in mind that refinements too small to be shown on the record need not be considered.

Mr. Rouse feels that this investigation should have been performed in a laboratory, with which the writers fully agree, but no laboratory was available, and there was great need of finding some method of installing meters in constructed sewers where weirs had been found inadequate. After the investigation had developed a practical meter, it was discussed with other engineers in the Southwest who showed great interest in it and urged that discussions and results be published at once. The writers are confident that flumes carefully constructed in accordance with their recommendations will give accurate results.

However, it is hoped that laboratory work will be undertaken to investigate some of the following uncertain points: (*a*) Proper length of throat; (*b*) formula for energy loss in head-water; (*c*) maximum ratio of tail-water depth to head-water depth; (*d*) proper angle of transition to axis of conduit; and, (*e*) accuracy of measurement as a function of velocity of approach.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 1949

THE STRESS FUNCTION AND PHOTO-ELASTICITY APPLIED TO DAMS

By John H. A. Brahtz,[1] Esq.

With Discussion by Messrs. I. K. Silverman, Fred L. Plummer, Arshag G. Solakian, Lars R. Jorgensen, Elmer O. Bergman, D. P. Krynine, and John H. A. Brahtz.

Synopsis

The object of Part I of this paper is to familiarize engineers with the use of the Airy stress function for the solution of problems in plane stress and plane strain when ordinary engineering methods fail to give even approximate results.

The object of Part II is to familiarize the engineer with the photo-elastic phenomenon and its application to civil engineering structures and to compare results obtained in this way with the theoretical results obtained in Part I.

A brief popular outline of the theory, accompanied by a description of the apparatus developed at the California Institute of Technology, at Pasadena, Calif., is given. Methods of evaluating stresses are explained and applied to experiments on Morris Dam.

Introduction

Ordinarily, the theory of the Airy function found in textbooks[2] is based on the assumption that body forces (weight and inertia) can be neglected. This is entirely inadequate in civil engineering structures in which the stresses due to weight often are greater than those caused by boundary forces. Furthermore, in texts in which the body forces are included the definitions of stresses are generally such that the form of the function is not invariant to a change in co-ordinates. The stress definitions given herein, including both

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Note.—Published in September, 1935, *Proceedings*.

[1] With the U. S. Bureau of Reclamation, Denver, Colo.

[2] See, for example, "Applied Elasticity", by J. Prescott.

boundary and body forces, are such that the stress function will be of the same form in rectangular and polar co-ordinates.[8]

The stress function is restricted to isotropic materials which follow Hooke's law in both compression and tension. It may also be applied, however, to concrete masonry structures if the resulting stresses are those of compression, or very slight tension, even if the elastic constants are not strictly constant for all stresses. It is generally conceded that a slight variation in the elastic modulus has little effect on the final stress distribution, except, of course, at singular points or at points of high stress concentration. With these assumptions the stress function may then be applied to a slice of a gravity dam. The question of uplift is not considered in this treatment, but it is assumed: (1) That there is sufficient resultant average compression at all points of the dam to overcome any internal pore pressures that may exist; and (2) that the pores are so small that the average stress distribution at a point may still be found as in isotropic material.

This paper, together with theoretical and experimental work done by others, shows that for purposes of analysis the triangular gravity dam on an elastic foundation may be divided conveniently into three regions: (1) The upper two-thirds of the dam proper; (2) the lower one-third, including the base region of the foundation; and (3) the foundation proper. Finally, special investigation must be made in the regions close to the heel and toe whether they are sharp corners or fillets, and near the crown. In the present application, stress functions have been derived for four cases and a twofold purpose is served: (*a*) To show the methods of deriving stress functions; and (*b*) to obtain specific results applicable to the gravity dam on an elastic foundation. In most cases the derivations have been omitted, due to the limitation of space. The original manuscript is on file in Engineering Societies' Library, in New York, N. Y., and at the California Institute of Technology, at Pasadena, Calif.

Application I.—The stress functions, stresses, and deflections valid in the upper part of triangular dams are derived for hydrostatic, and are given for body, forces with computed results plotted in the case of Morris Dam. In addition, the stress functions are given for a number of special loadings.

Application II.—The stress functions applicable in the foundation are given with stresses and deflections for concentrated and distributed loads, the computed results being plotted for a study of Grand Coulee Dam.

Application III.—The "corner-function" applicable at sharp re-entrant corners is derived. Two methods of procedure are given for the determination of stresses in the lower part of gravity dams. The results are plotted for a study of Grand Coulee Dam.

Application IV.—An approximate method is derived for the determination of stresses at re-entrant sharp and rounded corners. Examples are computed for Morris Dam.

It should be emphasized that even if the gravity dam is used for illustrative examples of application, the aim is not to advance new design criteria. The writer hopes, however, that the methods described will help to obtain a closer estimate of the stresses that actually occur. It is worthy of note that a state of stress computed by the Airy stress function is in equilibrium and compatible with Hooke's generalized law.

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[8] "Notes on the Airy Stress Function", by John H. A. Brahtz, *Bulletin*, Am. Math. Soc., June, 1934.

ACKNOWLEDGMENT

Acknowledgment is freely given to Theodor von Kármán, Elwood Mead, S. B. Morris, J. L. Savage, and R. F. Walter, Members, Am. Soc. C. E., and to Professor H. Bateman, of the California Institute of Technology, Pasadena, Calif., for their valuable suggestions and co-operation in the preparation of this paper. The photo-elastic experiments on Morris Dam were made possible through the financial support of the Pasadena Water Department.

PART I.—THE THEORY OF THE AIRY STRESS FUNCTION

In this section the stress function is defined in rectangular and polar co-ordinates. Convenient forms of boundary conditions are treated.

A two-dimensional elastic system under plane stress or plane strain is in equilibrium if the stress components are defined as follows:

By rectangular co-ordinates:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} - g_x\, x - g_y\, y \qquad \text{(1a)}$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} - g_x\, x - g_y\, y \qquad \text{(1b)}$$

and,

$$\tau_{x,y} = -\frac{\partial^2 F}{\partial x\, \partial y} \qquad \text{(1c)}$$

in which σ_x and σ_y = components of normal stress parallel to the X-axis and the Y-axis, respectively; $\tau_{x,y}$ = shear stress in the direction of the X-axis or the Y-axis; F = a stress function; and g = the total body force per unit volume, with components, g_x and g_y along the X and Y-axes.

In the polar co-ordinates:

$$\sigma_r = \frac{\partial^2 F}{r^2\, \partial \theta^2} + \frac{\partial F}{r\, \partial r} - g r \cos(\theta - \beta) \qquad \text{(2a)}$$

$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2} - g r \cos(\theta - \beta) \qquad \text{(2b)}$$

and,

$$\tau_{r,\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial F}{r\, \partial \theta}\right) \qquad \text{(2c)}$$

If, now, F is so restricted that it is a solution to the differential equation:

$$\nabla^4 F \equiv \frac{\partial^4 F}{\partial x^4} + 2\,\frac{\partial^4 F}{\partial x^2\, \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \qquad \text{(3a)}$$

or,

$$\nabla^4 F \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \cdot \partial r} + \frac{\partial^2}{r^2\, \partial \theta^2}\right)^2 F = 0 \qquad \text{(3b)}$$

the stress defined in Equations (1) and (2) will also be compatible with the generalized Hooke's law. Problems in plane stress or plane strain have thus been reduced to the determination of the functions, F, that satisfy Equations (3) and such that the stresses determined by Equations (1) or Equations (2) agree with the given force distributions over the boundaries of the structure. The first part is very simple because it can be verified that both the real and imaginary parts of the expression:

$$F = A_1 f_1(z) + A_2 x f_2(z) + A_3 y f_3(z) + A_4 r^2 f_4(z) \ldots\ldots\ldots (4)$$

satisfy Equations (3). A_1, A_2, A_3, and A_4 are arbitrary constants; f_1, f_2, f_3, and f_4 are arbitrary analytic functions of the complex variable, $z = x + iy$; $r^2 = x^2 + y^2$; and $i = \sqrt{-1}$.

The second part of the problem (to satisfy the boundary conditions) is usually difficult, and must be solved for each individual case. It is important to realize that as soon as the stress function is known the stresses are also known at all points by Equations (1) or Equations (2).

BOUNDARY CONDITIONS

Referring to Fig. 1 the boundary conditions to be satisfied by $F(x, y)$ may be formulated in terms of well-known engineering concepts, as follows: Let the area enclosed by the hatched boundary curve represent a plane elastic body of unit thickness, held in equilibrium by a system of forces, P_n (concentrated or continuously distributed), acting on the boundary, ABC, and body forces, g, per unit volume; g_x and g_y = components of g along the co-ordinate axes; and H and V = the forces along the X-axis and Y-axis per unit length of the boundary.

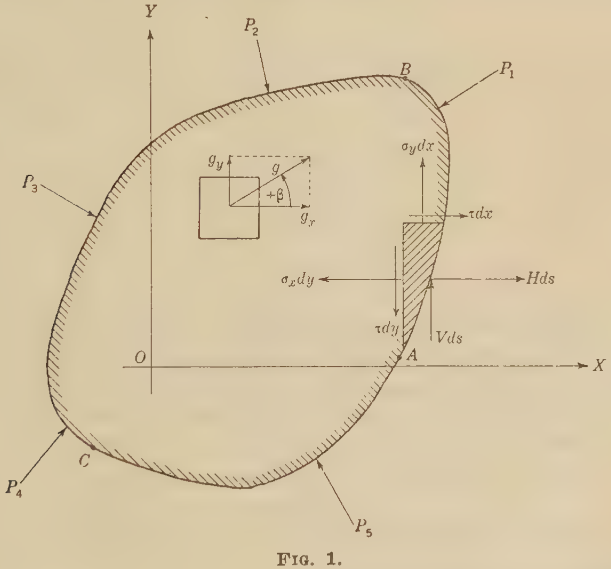

FIG. 1.

Consider an element, ds, at the boundary with co-ordinates (x, y). It is in equilibrium under the stresses and the boundary forces, $H\,ds$ and $V\,ds$, (Fig. 1). By projecting all forces on the co-ordinate axes and taking moments about a point, B, of the boundary with co-ordinates, $x_B\,y_B$, the conditions of equilibrium are:

$$\sigma_x\,dy - \tau\,dx = H\,ds \dots\dots\dots\dots (5a)$$

$$\sigma_y\,dx - \tau\,dy = -V\,ds \dots\dots\dots\dots (5b)$$

and,

$$(\sigma_x\,dy - \tau\,dx)\,(y_B - y) + (\sigma_y\,dx - \tau\,dy)\,(x_B - x) = dM \dots (5c)$$

in which dM indicates the moment of $H\,ds$ and $V\,ds$ about Point B, positive in direction (counter-clockwise about Point O, Fig. 1). The body forces of the element are of higher order of magnitude and need not be considered.

By substituting the stresses as defined by Equations (1) and integrating the equations from Point A, Fig. 1, along the boundary in the positive direction to Point B, the following equations are obtained:

$$\left(\frac{\partial F}{\partial y}\right)_B - \left(\frac{\partial F}{\partial y}\right)_A = X + g_x \int_A^B x\,dy + g_y \int_A^B y\,dy \dots\dots (6a)$$

$$\left(\frac{\partial F}{\partial x}\right)_B - \left(\frac{\partial F}{\partial x}\right)_A = -Y + g_x \int_A^B x\,dx + g_y \int_A^B y\,dx \dots (6b)$$

and,

$$F_B - F_A - (x_B - x_A)\left(\frac{\partial F}{\partial x}\right)_A - (y_B - y_A)\left(\frac{\partial F}{\partial y}\right)_A$$

$$= M + \int_A^B (x g_x + y g_y)\Big[(x_B - x)\,dx + (y_B - y)\,dy\Big] \dots\dots\dots\dots (6c)$$

in which X and Y are the projections along the X-axis and Y-axis; and $M =$ the moment about B of all boundary forces between A and B. By a suitable choice of co-ordinate axes Equations (6) can be simplified somewhat. From the definition of the stresses in Equations (1), etc., it follows that a constant and terms that are linear in x and y, added to the stress function, contribute no stress. Therefore, by choosing the origin at Point A, Fig. 1, it is always possible to arrange matters so that $F_A = \left(\frac{\partial F}{\partial x}\right)_A = \left(\frac{\partial F}{\partial y}\right)_A = 0$. The boundary equations then become:

$$\left(\frac{\partial F}{\partial y}\right)_B = X_B + g_x \int_0^B x\,dy + \frac{1}{2}\,g_y\,y^2_B \dots\dots\dots\dots (7a)$$

$$\left(\frac{\partial F}{\partial x}\right)_B = -Y_B + \frac{1}{2}\,g_x\,x^2_B + g_y \int_0^B y\,dx \dots\dots\dots (7b)$$

and,

$$F_B = M_B + g_x \int_0^B (y_B - y)\, x\, dy + g_y \int_0^B (x_B - x)\, y\, dx$$
$$+ \frac{1}{6} x^3_B\, g_x + \frac{1}{6} y^3_B\, g_y \text{ .... }(7c)$$

In the definitions of X_B, Y_B, and M_B, it must be remembered that the integrations are to be carried out from the origin (Point A, Fig. 1) in the positive (X-Y) direction. Hence, if Point B is located in the negative direction from the origin, the signs of X_B, Y_B, and M_B, as defined in connection with Equations (6), must be reversed, exactly as in taking moments, thrust, and shear at a section of a beam in bending. This formulation of the boundary conditions will be found especially useful when approximate solutions are desired; that is, in case the boundary conditions are to be satisfied only at a few points.

In exact solutions (that is, if the boundary conditions can be satisfied at all points of the boundary), only two of the three sets of formulas Equations (5), (6), and (7) need to be considered, the third will be satisfied automatically. If no body forces exist (that is, if $g_x = g_y = 0$) Equations (5), (6), (7) become especially simple and instructive. In this case the following simple interpretations of the stress function, F_P, and its partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, are evident by Equations (7).

The value of F_P (the stress function for the given boundary forces) at any point, B, of the boundary is equal to the moment of all forces acting on the boundary between the origin and Point B. The value of $\frac{\partial F_P}{\partial y}$ at Point B is equal to the projection on the X-axis of the same forces and $\frac{\partial F_P}{\partial x}$ equals the negative projection on the Y-axis.

It is often convenient to determine two functions, F_P and F_g, the first representing the boundary forces and the second the body forces; then, $F = F_P + F_g$. If no mass forces exist the boundary conditions can be given a simple form on the portions of the boundary where no forces are acting, namely:

$$F = 0 \text{ .... }(8a)$$

and, the derivative normal to the boundary,

$$\frac{\partial F}{\partial n} = 0 \text{ .... }(8b)$$

When the stress function, F, is known, the displacements, u, parallel to the X-axis, and v, parallel to the Y-axis, can be determined in the case of plane stress by the expressions:

$$E\,u = -(1 + \mu)\frac{\partial F}{\partial x} - \frac{1 - \mu}{2}(g_x x^2 - g_x y^2 + 2\, g_y\, x\, y) + f_1 + A\, y + B \text{ .. }(9a)$$

and,

$$E\,v = -(1+\mu)\,\frac{\partial F}{\partial y} - \frac{1-\mu}{2}\,(g_y\,y^2 - g_y\,x^2 + 2\,g_x\,x\,y) + f_2 - A\,x + C \;.\;(9b)$$

in which, E is Young's modulus; μ is Poisson's ratio; f_1 = real part of $\int f(z)\;dz$; f_2 = imaginary part of $\int f(z)\;dz$; and, $f(z)$ is the analytic function of the complex variable:

$$z = x + i\,y = r\,e^{i\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,(10)$$

such that $\nabla^2 F$ = the real part of $f(z)$. The constants A, B, and C in Equations (9) determine the reference axis for u and v and only effect a solid-body rotation and translation.

In polar co-ordinates the radial and tangential displacements, u_r and v_θ, are given by:

$$E\,u_r = -(1+\mu)\,\frac{\partial F}{\partial r} - \frac{1-\mu}{2}\,g\,r^2\cos(\theta-\beta) + f'_1 + B\cos\theta + C\sin\theta \;.\;(11)$$

and,

$$E\,v_\theta = -(1+\mu)\,\frac{\partial F}{r\,\partial\theta} - \frac{1-\mu}{2}\,g\,r^2\sin(\theta-\beta) + f'_2 + A\,r$$
$$+ C\cos\theta - B\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,(12)$$

in which f'_1 = real part of $e^{-i\theta}\int f(z)\,dz$; f'_2 = imaginary part of $e^{-i\theta}\int f(z)\;dz$; and, $f(z)$ is defined as in connection with Equations (9). In Equations (11) and (12), the constants, A, B, and C, have the same meaning as in Equations (9). If no mass forces occur, $g = 0$. Displacements in the case of plane strain are found by Equations (9), (11), and (12), by replacing E with $\frac{E}{1-\mu^2}$ and μ with $\frac{\mu}{1-\mu}$.

Application I.—Stresses in the Upper Part of Triangular Gravity Dams

In order to indicate the general procedure for determining the stress function in the case of two-dimensional structures for a given loading the case of the infinite wedge with hydrostatic pressure on one side, and no body forces, will be developed in some detail. The solution was first offered by M. Levy.[4]

Assume the hydrostatic pressure on the up-stream face, $y = 0$, to be p at a unit distance from the top measured along the face; then, by Equations (1) with $y = 0$, the boundary conditions are (remembering that no mass forces occur so that $g_x = g_y = 0$):

$$\sigma_y \equiv \frac{\partial^2 F}{\partial x^2} = -\,p x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,(13)$$

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[4] *Comptes Rendus*, Vol. 127, 1898, pp. 10–15.

and,

$$\tau \equiv -\frac{\partial^2 F}{\partial x\, \partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

When $y = x \tan \gamma = x K$; and, therefore, $\frac{dy}{dx} = K$: By Equations (5a) and (5b) (remembering that no boundary forces exist on the rear face):

$$\frac{\partial^2 F}{\partial y^2} K + \frac{\partial^2 F}{\partial x\, \partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

and,

$$\frac{\partial^2 F}{\partial x^2} \frac{1}{K} + \frac{\partial^2 F}{\partial x\, \partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (15b)$$

It is evident that the polynomial,

$$F_p = A x^3 + B y^3 + C x^2 y + D x y^2 \ldots\ldots\ldots\ldots\ldots (16)$$

satisfies Equations (3), and, therefore, F_p is an Airy stress function. If the arbitrary constants A, B, C, and D, can be determined such that the boundary conditions are satisfied, Equation (16) is the solution.

By substitution of F_p into Equations (13) and (14) and with $y = 0$: $6\, A x = -p\, x$. Consequently, $A = \frac{-p}{6}$; $-2\, C x = 0$; and, $C = 0$. By substitution of F_p into Equations (15) and with $y = Kx$; $6\, B\, x\, K^2 + 4\, D\, x\, K = 0$; $-\frac{px}{K} + 2\, D\, x\, K = 0$; $B = -\frac{p}{3 K^3}$; and $D = +\frac{p}{2 K^2}$.

With these values of A, B, C, and D, introduced into Equation (16) the stress function for the hydrostatic forces becomes:

$$F_p = -\frac{p}{6 K^3} \left(x^3 K^3 + 2\, y^3 - 3\, x y^2 K \right) \ldots\ldots\ldots\ldots (17)$$

By the definitions expressed as Equations (1), with $g_x = g_y = 0$, the stresses are:

$$\sigma_x = \frac{p}{K^3} \left(x K - 2 y \right) \ldots\ldots\ldots\ldots\ldots\ldots (18a)$$

$$\sigma_y = -\, p x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18b)$$

and,

$$\tau = -\frac{p}{K^2}\, y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18c)$$

The displacements corresponding to Equation (17) are derived in some detail in order to illustrate the use of the general equations (9); thus: $\frac{\partial F_p}{\partial x} = -\frac{p}{2 K^3} \left(K^3 x^2 - K y^2 \right)$; $\frac{\partial F_p}{\partial y} = -\frac{p}{K^3} \left(y^2 - K x y \right)$; and,

$$\nabla^2 F \equiv \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{p}{K^3} \left(x K - x K^3 - 2 y \right) \equiv a x + b y$$

in which $a = \frac{p}{K^3}(K - K^3)$; and, $b = -\frac{2p}{K^3}$. Referring to Equation (10), $\nabla^2 F$ = real part of $f(z) = f(x + i\,y)$. Hence, $f(z) = az - ibz = (a - ib)z$; and,

$$\int f(z)\,dz = \frac{1}{2}(a - ib)\,z^2 = \left[\frac{a}{2}(x^2 - y^2) + bxy\right] + i\left[axy - \frac{b}{2}(x^2 - y^2)\right] \quad \text{(19)}$$

Then, referring to Equations (9), $f_1 = \frac{a}{2}(x^2 - y^2 + bxy$ = real part of Equation (19); and, $f_2 = axy - \frac{b}{2}(x^2 - y^2)$ = imaginary part of Equation (19). Substituting these quantities into Equations (9):

$$u = \frac{p}{2\,E_1\,K^3}\left[(1 + \mu_1)(K^3 x^2 - Ky^2) + (K - K^3)(x^2 - y^2) - 4xy + 2\,Ay + B\right] \quad \text{(20a)}$$

and,

$$v = \frac{p}{E_1 K^3}\left[(1 + \mu_1)(y^2 - Kxy) + x^2 - y^2 + (K - K^3)\,xy - Ax + C\right] \quad \text{(20b)}$$

In Equations (20) E_1 and μ_1 are Young's modulus and Poisson's ratio, respectively, for the dam in case of plane stress ($\sigma_z = 0$). In case of plane strain, E_1 and μ_1 are found, as explained in connection with Equations (11) and (12).

Mass Forces.—Let the components of the mass forces (weight and earthquake forces) be g_x and g_y along the X-axis and Y-axis, and assume that no forces are acting on the faces of the dam. It may be verified that the function:

$$F_g = \frac{1}{6K^2}\left[g_x K^2 x^3 + (g_x K + g_y K^2 - 2\,g_y)y^3 + 3\,g_y K\,xy^2\right] \quad \text{(21)}$$

will satisfy the boundary conditions expressed by Equations (5) or Equations (7). By Equations (1) the stresses are:

$$\sigma_x = \frac{1}{K}\left[(g_y - g_x K)\,x + \left(g_x - \frac{2g_y}{K}\right)y\right] \quad \text{(22a)}$$

$$\sigma_y = -g_y\,y \quad \text{(22b)}$$

and,

$$\tau = -\frac{y g_y}{K} \quad \text{(22c)}$$

The displacements corresponding to Equation (21) are found by substituting,

$$\frac{\partial F_g}{\partial x} = \frac{1}{2K} \left(K g_x\, x^2 + g_y\, y^2\right) \dots\dots\dots\dots\dots\dots (23a)$$

$$\frac{\partial F_g}{\partial y} = \frac{1}{2K^2} \left[\left(g_x K + g_y K^2 - 2 g_y\right) y^2 + 2K\, g_y\, x\, y\right] \dots\dots (23b)$$

$$f_1 = \frac{1}{2\,K} \left(K g_x + g_y\right) \left(x^2 - y^2\right) + \frac{1}{K^2} \left(K g_x + K^2 g_y - 2\, g_y\right) xy \,.. (23c)$$

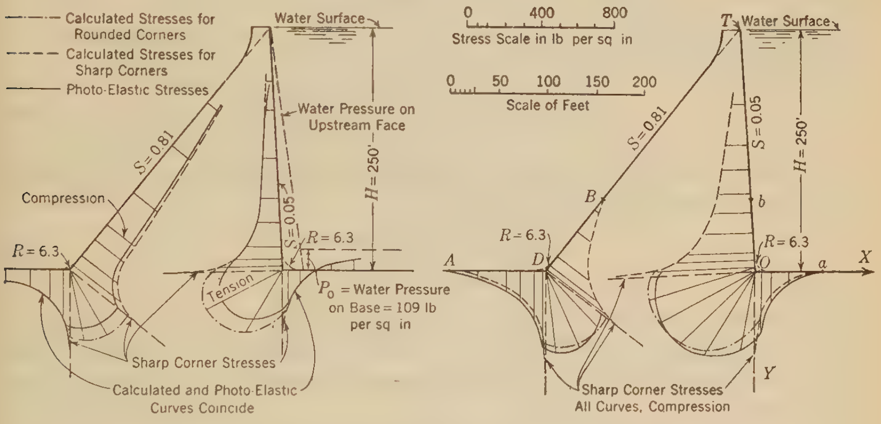
and,

$$f_2 = \frac{1}{K} \left(K g_x + g_y\right) xy - \frac{1}{2K^2} \left(K g_x + K^2 g_y - 2 g_y\right) \left(x^2 - y^2\right) .. (23d)$$

into Equations (9).

It will be seen by Equations (18) and (22) that all stresses are linear along any straight line in the triangular dam of infinite height. This agrees with the assumptions of engineering practice. It will be shown subsequently that this is not true near the base in a dam of finite height.

The boundary stresses computed by Equations (17) and (21) for Morris Dam, in California, are plotted in Fig. 2.



(a) HYDROSTATIC LOAD ONLY (b) WEIGHT OF DAM ONLY

FIG. 2.—PRINCIPAL STRESSES ALONG BOUNDARY.

Additional Solutions in Polar Co-Ordinates[5] *for the Triangular Dam of Infinite Height.*—Simple expressions are obtained when the bisector of the top angle, $\gamma = 2\alpha$, is chosen as the X-axis, and the Y-axis is positive down stream; as, for example:

(1) When the full hydrostatic pressure, pr, is exerted on the up-stream face, and with no forces on the down-stream face:

$$F_p = \frac{pr^3}{24} \left[\left(3 \sin^2 \alpha - \sin^2 \theta\right) \frac{\sin \theta}{\sin^3 \alpha} - \left(3 \cos^2 \alpha - \cos^2 \theta\right) \frac{\cos \theta}{\cos^3 \alpha} \right]$$

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[5] For greater detail refer to Carothers' "Plain Strains in a Wedge, with Applications for Masonry Dams", *Proceedings*, Royal Soc., Edinburgh, Vol. 53 (1913), pp. 292–303.

(2) When a uniform pressure, p, is exerted on both faces:

$$F = -\frac{1}{2} pr^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

(3) When a uniform pressure, p, is exerted on the up-stream face and uniform tension, p, is exerted on the down-stream face:

$$F = \frac{pr^2 (\sin 2\theta - 2\theta \cos 2\alpha)}{2 (\sin 2\alpha - 2\alpha \cos 2\alpha)} \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

(4) When a uniform pressure is exerted on the up-stream face, with no force on the down-stream face:

$$F = \frac{1}{4} pr^2 \frac{\sin 2\theta - \sin 2\alpha - (2\theta - 2\alpha) \cos 2\alpha}{\sin 2\alpha - 2\alpha \cos 2\alpha} \ldots\ldots\ldots (26)$$

(5) With a pressure, pr, on the up-stream face and tension, pr, on the down-stream face:

$$F = \frac{pr^3}{6} \frac{\cos\alpha \sin 3\theta - 3 \cos 3\alpha \sin\theta}{\cos\alpha \sin 3\alpha - 3 \cos 3\alpha \sin\alpha} \ldots\ldots\ldots\ldots (27)$$

(6) With a pressure, pr, on both faces use Equation (27), substituting "sin" for "cos" and "cos" for "sin" throughout.

(7) With a moment, M, per unit of the Z-axis (positive in direction $+\theta$) acting at the vertex:

$$F = -\frac{M}{2} \frac{\sin 2\theta - 2\theta \cos 2\alpha}{\sin 2\alpha - 2\alpha \cos 2\alpha} \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

(8) With a force, P, per unit length of the Z-axis acting at the vertex and directed along the positive Y-axis:

$$F = P \frac{r\,\theta \cos\theta}{2\alpha - \sin 2\alpha} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

(9) With a pressure, p, directed along the positive X-axis, use Equation (29), substituting "$-\sin\theta$" for "$\cos\theta$", and "$+\cos 2\alpha$" for "$-\sin 2\alpha$"; and

(10) With body forces, g, per unit volume, directed at an angle, β, with the X-axis:

$$F_g = \frac{gr^3}{12} \left[\frac{\cos\beta}{\cos^2\alpha} (3 \cos^2\alpha \cos\theta - \cos^3\theta) + \frac{\sin\beta}{\sin^2\alpha} (3 \sin^2\alpha \sin\theta - \sin^3\theta) \right] \ldots\ldots\ldots\ldots\ldots (30)$$

The stresses are obtained by Equations (2) with $g = 0$ if no mass forces exist.

APPLICATION II.—STRESSES IN FOUNDATIONS.[6]

Let a tangential line load, Q, act per unit thickness of the infinite half plane (foundation). With the co-ordinate system in Fig. 3 it is easily verified that the function, $F = \frac{-Q}{\pi} r\,\theta \sin\theta = -\frac{Q}{\pi} y \tan^{-1}\frac{y}{x}$, is an Airy function because Equations (3) are satisfied. It satisfies the boundary conditions, $F = 0$, when $\theta = 0$, and $\theta = \pi$. Therefore, by Equations (8) no normal forces act on the X-axis. Furthermore, $\frac{\partial F}{\partial y} = 0$ when $\theta = 0$, and $-Q$ when $\theta = \pi$. Therefore, by Equations (7) no tangential force distribution occurs when $r > 0$, or when $r < 0$, but a force, $+Q$, occurs at $r = 0$ because here $\frac{\partial F}{\partial y}$ makes a jump equal to $+Q$.

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[6] For a more detailed solution, see "Applications Potentiels à l'Etude de Equilibre et du Mouvements des Solides Elastiques", by J. Bouissinesq, Paris, 1885.

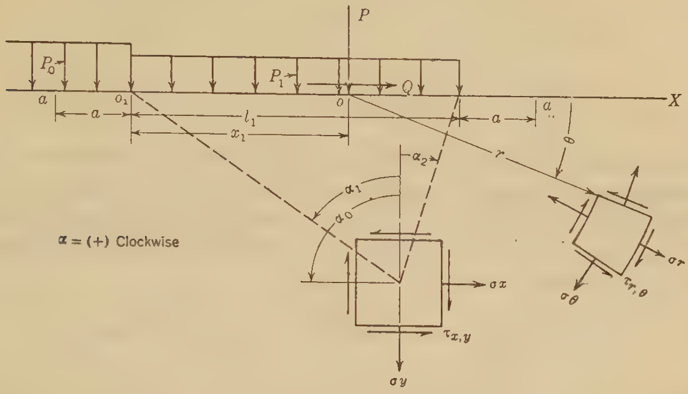

FIG. 3.

The stresses are found by Equations (2) with $g = 0$; thus:

$$\sigma_r = -\frac{2\,Q}{\pi}\,\frac{\cos\theta}{r} \quad\text{.....................}(31a)$$

and,

$$\sigma_\theta = \tau_{r\theta} = 0 \quad\text{.......................}(31b)$$

The rectangular stress components in the foundation may be found by the formulas:

$$\sigma_x = \sigma_r \cos^2\theta = -\frac{2\,Q}{\pi}\,\frac{\cos^3\theta}{r} \quad\text{................}(32a)$$

$$\sigma_y = \sigma_r \sin^2\theta = -\frac{2\,Q}{\pi}\,\frac{\cos\theta\sin^2\theta}{r} \quad\text{..............}(32b)$$

and,

$$\tau_{x,y} = \frac{\sigma_r}{2}\sin 2\,\theta = -\frac{2\,Q}{\pi}\,\frac{\cos^2\theta\,\sin\theta}{r} \quad\text{...........}(32c)$$

The influence function,

$$F = -\frac{1}{\pi}\,y\tan^{-1}\frac{y}{x - x_1} \quad\text{....................}(33)$$

gives the stresses in the foundation due to a unit tangential force acting at a point, $(x_1, 0)$, and directed along the positive X-axis in a positive direction. For a uniform distribution, q_1, over a distance, l_1, it will be found by integration of Equation (33) that the stresses are:

$$\sigma_x = \frac{q_1}{\pi} \left[+ (\log \cos^2 \alpha_1 - \cos^2 \alpha_1) - (\log \cos^2 \alpha_2 - \cos^2 \alpha_2) \right] \ldots (34a)$$

$$\sigma_y = \frac{q_1}{2\pi} \left[+ \cos 2\alpha_1 - \cos 2\alpha_2 \right] \ldots\ldots\ldots\ldots\ldots (34b)$$

and,

$$\tau_{x,y} = \frac{q_1}{2\pi} \left[+ (2\alpha_1 - \sin 2\alpha_1) - (2\alpha_2 - \sin 2\alpha_2) \right] \ldots\ldots (34c)$$

in which the angles, α_1 and α_2, are as shown in Fig. 3. Equations (34) are easily extended to a number of uniform distributions, q_1, q_2, q_3, etc.

At Point a of the surface, $\sigma_y = \tau = 0$ (see Fig. 2(b)) and σ_x by Equations (34) becomes invalid; $\sigma_x = \sigma_r$ is then obtained by direct integration of Equations (31); thus:

$$\sigma_x = \mp \int_a^{a+l_1} \sigma_r \, dr = \mp \frac{2q_1}{\pi} \int_0^{a+l_1} \frac{dr}{r} = \mp \frac{2\,q_1}{\pi} \log \frac{a + l_1}{a} \ldots (35)$$

in which minus is to be used if a is on the positive side of the distribution, q_1.

By Equations (9) (remembering that $g = 0$) the displacements due to Equation (33) are:

$$u = -\frac{1}{\pi\, E_2} \left\{ (1 + \mu_2) \frac{(x - x_1)^2}{(x - x_1)^2 + y^2} + \log\left[(x - x_1)^2 + y^2\right] - Ay + C \right\} \ldots (36a)$$

and,

$$v = \frac{1}{\pi\, E_2} \left\{ (1 + \mu_2) \left[y \frac{x - x_1}{(x - x_1)^2 + y^2} + \tan^{-1} \frac{y}{x - x_1} \right] \right.$$

$$\left. - 2 \tan^{-1} \frac{y}{x - x_1} + Ax + B \right\} \ldots\ldots\ldots\ldots\ldots\ldots (36b)$$

Next, let a normal compression, P, act per unit thickness. The stress function for this load, $F = + \dfrac{P}{\pi} r\, \theta \cos \theta = + \dfrac{P}{\pi} x \tan^{-1} \left(\dfrac{y}{x}\right)$, will satisfy the boundary conditions. The stresses by Equations (2), with $g = 0$, are: $\sigma_r = -\dfrac{2\,P}{\pi} \dfrac{\sin \theta}{r}$; $\sigma_\theta = 0$; and $\tau = 0$.

The rectangular components of stress due to P are:

$$\sigma_x = -\frac{2P}{\pi} \frac{\sin \theta \cos^2 \theta}{r}; \quad \sigma_y = -\frac{2P}{\pi} \frac{\sin^3 \theta}{r}; \text{ and, } \tau_{x,y} = -\frac{2P}{\pi} \frac{\sin^2 \theta \cos \theta}{r}$$

The influence function,

$$F = \frac{1}{\pi} (x - x_1) \tan^{-1} \left(\frac{y}{x - x_1} \right) \ldots\ldots\ldots\ldots\ldots (37)$$

will give the stresses in the foundation due to a unit normal pressure at the point, $(x_1, 0)$.

For a uniform distribution, p_1, over the distance, l_1 (Fig. 3), the stresses are:

$$\sigma_x = \frac{p_1}{2\pi}\,[+\,(2\alpha_1 - \sin 2\alpha_1) - (2\alpha_2 - \sin 2\alpha_2)] \ldots\ldots (38a)$$

$$\sigma_y = \frac{p_1}{2\pi}\,[+\,(2\alpha_1 + \sin 2\alpha_1) - (2\alpha_2 + \sin 2\alpha_2)] \ldots\ldots (38b)$$

and,

$$\tau_{x,\,y} = \frac{p_1}{2\pi}\,[+\,\cos 2\alpha_1 - \cos 2\alpha_2] \ldots\ldots\ldots\ldots (38c)$$

Equations (38) can be extended to several uniform distributions, p_1, p_2, p_3, etc. The displacements due to Equation (37) are:

$$u = \frac{1}{\pi\,E_2}\left\{(1 + \mu_2)\left[\,y\,\frac{x - x_1}{(x - x_1)^2 + y^2} - \tan^{-1}\frac{y}{x - x_1}\right] + 2\tan^{-1}\frac{y}{x - x_1} + Ay + B\right\}$$

and,

$$v = -\frac{1}{\pi\,E_2}\left\{(1 + \mu_2)\,\frac{(x - x_1)^2}{(x - x_1)^2 + y^2} + \log\,[(x - x_1)^2 + y^2] + Ax + C\right\}$$

An interesting case occurs when p_0 is the uniform reservoir pressure on the foundation above a dam. If the X-axis is chosen positive down stream, the stresses are:

$$\sigma_x = \frac{-\,p_0}{2\,\pi}\,[\pi + 2\,\alpha_1 - \sin 2\,\alpha_1] \ldots\ldots\ldots\ldots (39a)$$

$$\sigma_y = \frac{-\,p_0}{2\,\pi}\,[\pi + 2\,\alpha_1 + \sin 2\,\alpha_1] \ldots\ldots\ldots\ldots (39b)$$

and,

$$\tau_{x,y} = \frac{-\,p_0}{2\,\pi}\,[1 + \cos 2\,\alpha_1] \ldots\ldots\ldots\ldots\ldots (39c)$$

The corresponding stress function is $F = +\,\dfrac{p_0\,r^2}{4}\,(\sin 2\,\theta - 2\,\theta)$, with the origin at the heel. The corresponding displacements are:

$$u = \frac{p_0}{\pi\,E_2}\left\{(1 + \mu_2)\,x\tan^{-1}\frac{y}{x} + 2\,x\tan^{-1}\frac{y}{x} - y\log\,(x^2 + y^2) + Ay + B\right\} \ldots (40a)$$

and,

$$v = \frac{p_0}{\pi\,E_2}\left\{(1 + \mu_2)\,y\tan^{-1}\frac{y}{x} - 2\,y\tan^{-1}\frac{y}{x} + x\log\,(x^2 + y^2) - Ax + C\right\} \ldots (40b)$$

Finally, the stresses in the foundation due to the weight, g, per unit volume are: $\sigma_x = \sigma_y = -\,gy$; and $\tau_{x,\,y} = 0$.

Application III.—Stresses in the Region of the Base of the Gravity Dam

In the third application of the theory, polar co-ordinates will be used to determine "corner-functions" for a wedge or a corner with an angle, γ. The "corner-function" is defined such as to give no forces on the straight boundaries of the wedge or the corner. For the present purpose it is convenient to choose the co-ordinate system shown in Fig. 4. These functions appear in connection with the exact solutions by the writer[7] for stresses in two-dimensional corners. H. M. Westergaard, M. Am. Soc. C. E., in a communication to Professor von Kármán, has called attention to the existence of such functions, which he used for the investigation of stresses in the region of the toe and heel of dams, cracks in concrete, etc.

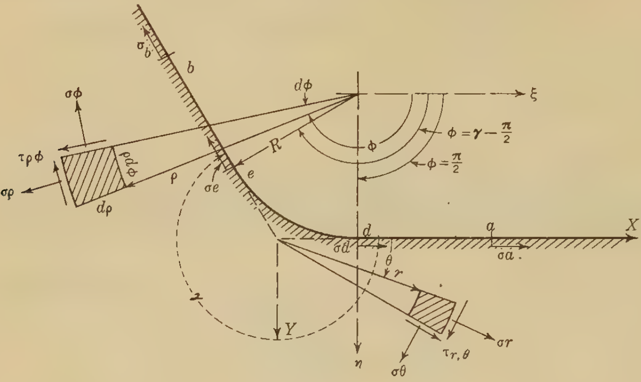

Fig. 4.

The definition of this function is expressed by Equations (8); hence, when $\theta = 0$ and $\theta = \gamma$:

$$F = 0 \qquad \text{(41a)}$$

and,

$$\frac{\partial F}{\partial n} = \frac{\partial F}{r\,\partial \theta} = 0 \qquad \text{(41b)}$$

Consider the real part of the following function which, by Equation (4), is known to be a solution of Equations (3):

$$F = [(B - i\,A)\,x + (C - i\,D)\,y]\,z^n \qquad (42)$$

in which $x = r\cos\theta$; $y = r\sin\theta$; $z = r\,e^{i\theta} = r\,(\cos\theta + i\sin\theta)$; and, $z^n = r\,e^{in\theta} = r^n\,(\cos n\,\theta + i\sin n\,\theta)$. By substitution of these expressions into Equation (42) and taking the real part only:

$$F = r^{n+1}\,[B\cos n\,\theta\cos\theta + A\sin n\,\theta\cos\theta + C\cos n\,\theta\sin\theta + D\sin n\,\theta\sin\theta] \qquad (43)$$

---

[7] Presented to the California Institute of Technology in 1932, in partial fulfillment of the requirement for the degree of Doctor of Philosophy; also contained in *Technical Memorandum No. 420*, U. S. Bureau of Reclamation, Denver, Colo.; excerpt published in *Applied Mechanics*, April–June, 1933, Vol. 1, No. 2; and in *Physics*, Vol. 4, No. 2, February, 1933.

It is to be noted that, by starting with a complex function of the type of Equation (4), Airy functions are easily produced and the task of verifying Equations (3) is avoided.

The constants, A, B, C, D, and n, must be determined by the boundary conditions expressed in Equations (41) and it will be found that the "corner-function" (Equation (43)) becomes:

$$F = A' r^{n+1} \left[n \cos n\,\theta \sin\theta + m \sin n\,\theta \sin\theta - \sin n\,\theta \cos\theta \right] \ldots. (44)$$

in which,

$$n \sin\gamma = \pm \sin n\,\gamma \ldots\ldots\ldots\ldots\ldots (45)$$

and,

$$m = \cot\gamma - n \cot n\,\gamma \ldots\ldots\ldots\ldots\ldots (46)$$

The stresses produced by Equation (44) are found by Equations (3) with $g = 0$. The displacements corresponding to Equation (44) are:

$$\begin{aligned} u = A' \frac{r^n}{E_1} \Big\{ & (1 + \mu_1) \left[n \cos\theta \sin (n-1)\,\theta + \sin n\,\theta - n^2 \sin\theta \cos (n-1)\,\theta \right. \\ & \left. - m\, n \sin\theta \sin (n-1)\,\theta \right] - 2 \sin n\,\theta - 2\, n \sin n\,\theta \\ & + 2\, m \cos n\,\theta \Big\} + C\, r \sin\theta + D \end{aligned}$$

and,

$$\begin{aligned} v = A' \frac{r^n}{E_1} \Big\{ & (1 + \mu_1) \left[n \cos\theta \cos (n-1)\,\theta - n \cos n\,\theta - m \sin n\,\theta \right. \\ & \left. + n^2 \sin\theta \sin (n-1)\,\theta - m\, n \sin\theta \cos (n-1)\,\theta \right] \\ & + 2 \cos n\,\theta + 2\, n \cos n\,\theta + 2\, m \sin n\,\theta \Big\} - C\, r \cos\theta + E \end{aligned}$$

in which C, D, and E are arbitrary constants used to fix the reference system for u and v. The values of n that satisfy Equation (45) will be referred to as "corner-values." It will be found that when $\pi < \gamma < 2\,\pi$, there are two significant real roots, n_0 and n_1, the others are complex; when $\gamma = \pi$, all roots are real: 1, 2, 3, etc.; when $\gamma = 2\,\pi$, all roots are real: $\frac{1}{2}, 1, \frac{3}{2}, 2$, etc.; and, when $0 < \gamma < \pi$, all roots are complex. The real roots, n_0 and n_1, for various values of γ are given in Fig. 5, based on the formulas:

$$\sin n_0\,\gamma = -\, n_0 \sin\gamma \ldots\ldots\ldots\ldots\ldots (47a)$$

and,

$$\sin n_1\,\gamma = +\, n_1 \sin\gamma \ldots\ldots\ldots\ldots\ldots (47b)$$

The complex roots may be found as follows: Let $n = a + i\,b$, which, substituted in Equation (45) gives:

$$\pm (a + i\,b) \sin\gamma = \sin a\,\gamma \cosh b\,\gamma + i \cos a\,\gamma \sinh b\,\gamma$$

Equating real and imaginary parts, the following simultaneous equations are obtained:

$$\sin a\,\gamma \cosh \gamma\, b = \pm\, a \sin \gamma \qquad (48a)$$

and,

$$\cos a\,\gamma \sinh b\,\gamma = \pm\, b \sin \gamma \qquad (48b)$$

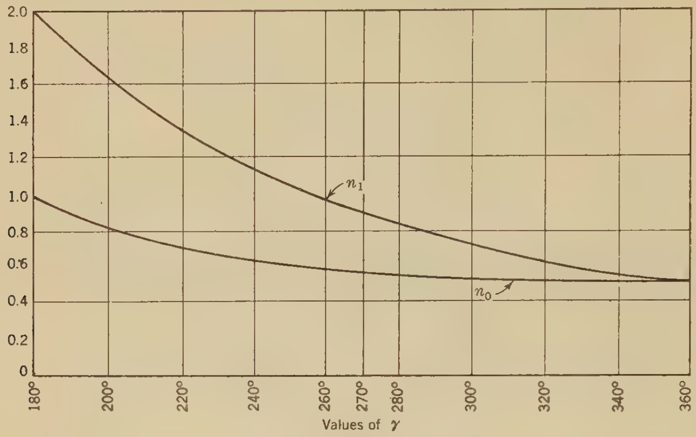

FIG. 5.

When γ is known, approximate values for a and b are found by:

$$a_k = \pm \frac{\pi\,(2\,k - 1) - \epsilon_k}{2\,\gamma} \qquad (49a)$$

and,

$$b_k \doteq \pm \frac{1}{\gamma} \log \left\{ \frac{\pi\,(2\,k - 1) - \epsilon_k}{\gamma} \sin \gamma \right\} \qquad (49b)$$

in which,

$$\epsilon_k = \frac{4 \log \left(\dfrac{\pi\,(2\,k - 1) \sin \gamma}{\gamma} \right)}{\pi\,(2\,k - 1)} \qquad (49c)$$

and, $k = 2, 3, 4, 5$, etc.

The values, a_k and b_k, obtained by Equations (49a) and (49b) will not satisfy Equations (48) exactly; therefore, they must be further adjusted.[8]

When n_k is known the corresponding value of m_k is found by Equation (46). Let $m = d + i\,e$ which, substituted into Equation (46), separating real and imaginary parts and making use of Equations (48), gives $e = b \tan a\,\gamma$; and $d = \cot \gamma - a \cot a\,\gamma$.

The various quantities in Equations (44) have now been expressed in terms of γ so that, when the angle, γ, is known, the "corner-function" is fully determined except for the arbitrary constant, A'. Each complex value of n will furnish a real and imaginary part of Equation (44). Both parts will satisfy the required boundary conditions; therefore, two functions are obtained for each value of n, each with an arbitrary constant, A and B, respectively. Furthermore, the sum of the functions obtained for all values of n will satisfy the boundary conditions. This function may be written:

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[8] Adjustment demonstrated in record manuscript, filed for reference in Engineering Societies Library, New York, N. Y.

$$F_\gamma = \text{real part of} \sum_{0,\,1,\,2,\,3,\,\text{etc.}}^{k=} (A_k - i\,B_k)\,F_k \ldots\ldots\ldots\ldots (50)$$

in which F_k is given by Equation (44).

The function, F_γ, contains a double infinity of arbitrary constants, A_k and B_k, and is constituted so that it corresponds to the case in which there are no forces acting on the straight boundaries of the corner, γ, irrespective of the values of A_k and B_k. At points inside the corner the stresses will not be zero, but the stress distribution will be such that the resultant of the stresses acting on any line terminating at points of the boundaries will be zero. In other words the function, F_γ, will only affect the distribution of the stresses. It is important that the full significance of F_γ is realized. It is extremely useful for the investigation of stresses at sharp corners and for the correction of stresses in structures if the boundary lines differ from those assumed in the original computations.

Application of F_γ for the Determination of Stresses Near the Base of a Straight Triangular Gravity Dam on an Infinite Elastic Foundation

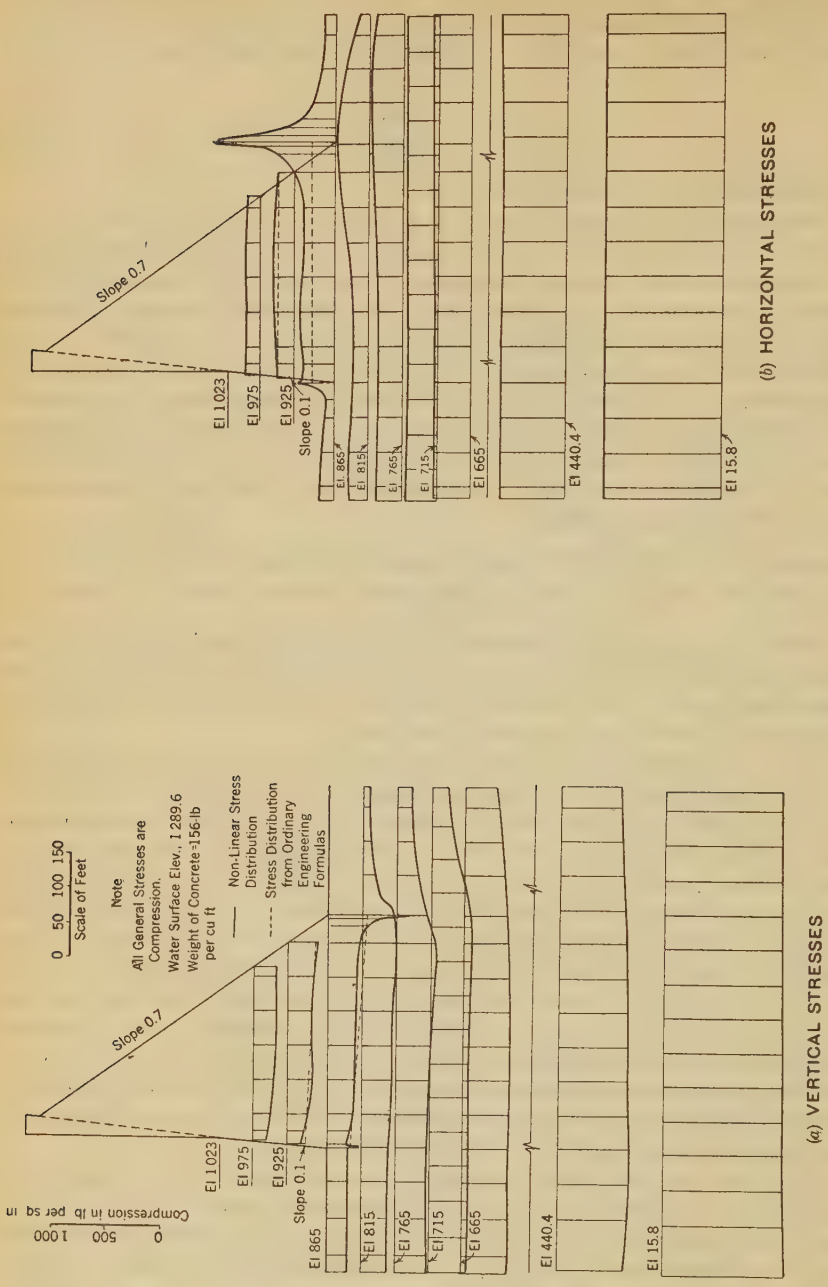
Two cases will serve to demonstrate the foregoing theory. In Case 1, the elastic properties of the dam and foundation are assumed to be the same throughout; and, in Case 2, they are different in the dam and foundation, but constant throughout in each of the two.

Case 1.—Consider an infinite elastic homogeneous plate of unit thickness. On this plate imagine the boundary lines of the cross-section of the dam and its foundation. These lines will divide the plane into two separate regions one of which is a slice of the dam and its foundation. The problem is to determine a function, F, that will deliver stresses between the two regions equal to the loads on the dam and its foundation. This function then will deliver the correct stresses throughout. Incidentally, it is interesting to note that F will give the stresses in both regions of the infinite plate.

Select the height of the dam (see Fig. 2), measured along the up-stream face, as the unit of length. Assume a full hydrostatic load on the up-stream face and constant pressure, p, on the up-stream foundation. Let the uniform mass forces (weight and inertia) be g per unit volume acting at an angle, β, with the X-axis. It is assumed that the forces, if any, acting on the rear face and the down-stream foundation are known.

Then the boundary conditions which must be satisfied by the stress function, F, are: (1) When $\theta = 0$; $y = 0$; (up-stream face): $\sigma_\theta = -p\,(1 - r)$; and, $\tau = 0$; (2) when $\theta = \gamma$; $y = x \tan\gamma = x K$; (up-stream foundation): $\sigma_\theta = -p$; and $\tau = 0$; and, (3) the rear face and down-stream foundation will be considered later. Let,

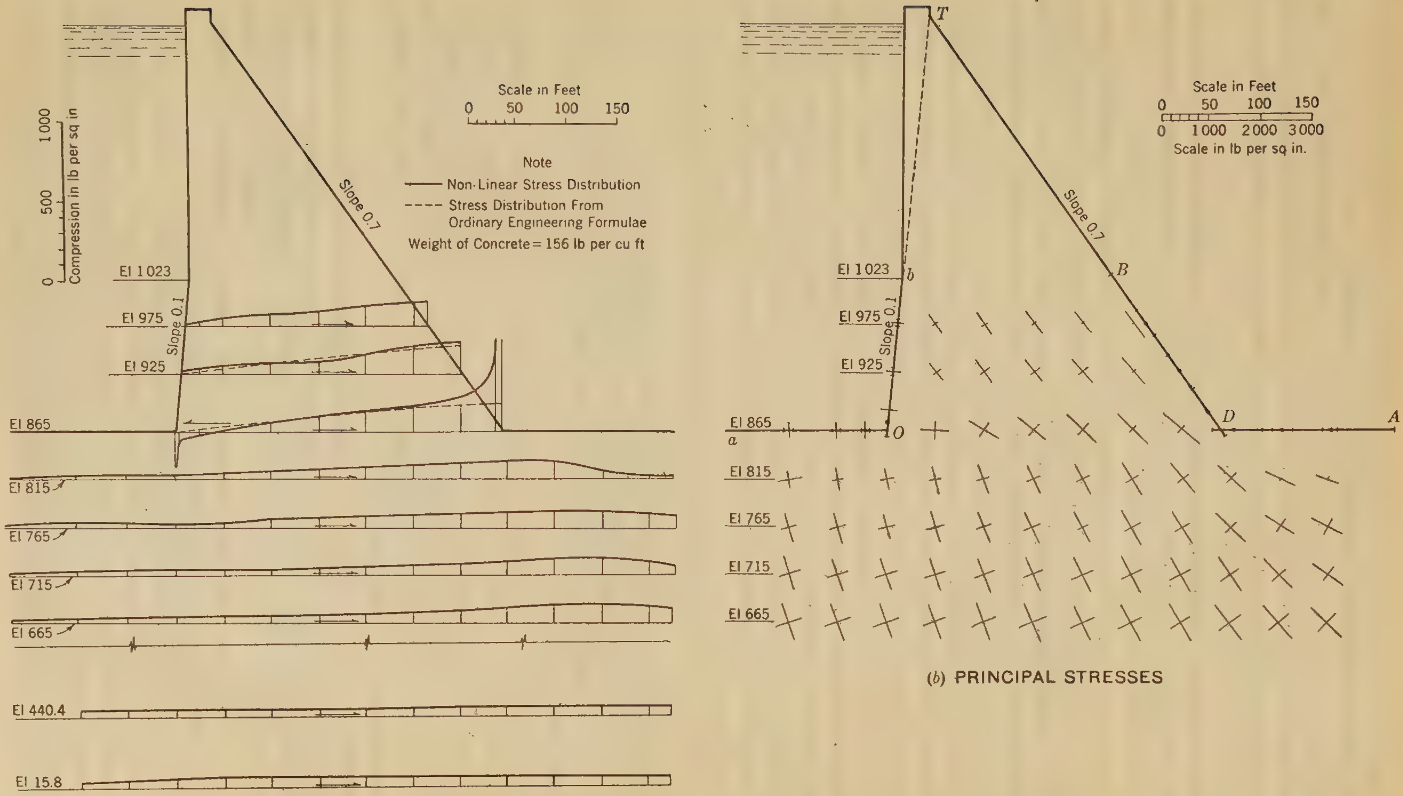
$$F = -F_p + F_0 + F_g + F_\gamma \ldots\ldots\ldots\ldots\ldots\ldots (51)$$



(a) VERTICAL STRESSES

(b) HORIZONTAL STRESSES

FIG. 6.


Compression in lb per sq in
0
500
1 000
Scale in Feet
0 50 100 150
Note
Non-Linear Stress Distribution
Stress Distribution From Ordinary Engineering Formulae
Weight of Concrete = 156 lb per cu ft
Slope 0.7
Slope 0.1
El 1 023
El 975
El 925
El 865
El 815
El 765
El 715
El 665
El 440.4
El 15.8
T
B
D
A
O
a
b
Scale in Feet
0 50 100 150
0 1000 2000 3000
Scale in lb per sq in.

(a) SHEARING STRESSES

(b) PRINCIPAL STRESSES

FIG. 7.

in which F_p, F_g, F_0, and F_γ are given in Equations (17), (21), (24), and (50); γ is the re-entrant angle at the heel ($\gamma > \pi$); and, $K = \tan \gamma$.

By adding the stresses when $y = 0$ and $y = K x$, respectively, due to the various functions in Equation (51), it will be found that Conditions (1) and (2) are satisfied exactly. (It is to be noted that F_γ was determined so as to give no contribution on these lines.)

There remains the matter of satisfying the conditions along the line, *T B D A*, in Fig. 2(*b*). For this purpose the form of the boundary conditions derived in Equations (7) is utilized.

In Equation (51), F is determinate with the exception of the arbitrary constants, A_k and B_k, in the function, F_γ. This double infinity of constants is used to satisfy Equations (7*a*) and (7*b*), at all points of the line, *T B D A*. The right-hand sides of Equations (7*a*) and (7*b*) are completely known and can be computed at all points when the forces on the dam are known. A double infinity of simultaneous equations is obtained with a double infinity of unknowns, A_k and B_k. This problem is solved, mathematically, by expanding the right-hand sides of Equations (7*a*) and (7*b*) in terms of the "corner-functions," F_γ. In the practical solution of the problem it is only necessary to satisfy Equations (7) for a finite number of points. In this case it is better to include all three of Equations (7). It will be found that if they are satisfied at three points, *B*, *D*, and *A* (Fig. 2(*b*)), Equations (7) will be closely satisfied at intermediate points. This can be checked by plotting F, $\frac{\partial F}{\partial x}$, and $\frac{\partial F}{\partial y}$, along the line, *T B D A*, and comparing the results with the values computed by the right-hand sides of Equations (7).

Three equations of conditions are thus obtained at each point, *B*, *D*, and *A*, a total of nine equations. This means that nine constants, A_k and B_k, must be included in F_γ.

When only a finite number of points, *B*, *D*, *A*, etc., are used, it is necessary to compute the stress functions for both the heel and the toe; that is, for the two different angles, γ. As soon as the constants, A_k and B_k are determined, F_γ is known by Equation (50). Thus, the total stress function valid in the region of the corner is known by Equation (51) and the stresses can be determined by Equations (2).

The stresses computed in one of the preliminary studies of Grand Coulee Dam[9] are given in Figs. 6 and 7. The stresses in the foundation were found by Equations (34), (35), (38), and (39), after the normal stress and shear distributions along the base had been computed. The stresses in the upper two-thirds of the dam are not shown. They may be computed by Equations (18) and (22), or by the usual engineering methods.

It will be seen that all stresses are compressive and tend to become infinite at the heel and toe. Of course, this is not possible because the material will become plastic before such magnitudes are reached and redistributions will occur near the corners, which must be considered as singular points where the elastic assumptions do not hold. The redistribution cannot be computed by the elastic theory, but it is evident that, after the plastic flow takes place, the stresses will be decreased near the corner and slightly increased a short distance from the corner. It will be noticed that the theoretical stresses are extremely high over only a minute distance so that the actual forces involved are very small. The effect of a redistribution, therefore, can only be slight.

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[9] The details are contained in *Technical Memorandum No. 403* by Chief Designing Engineer J. L. Savage, M. Am. Soc. C. E., U. S. Bureau of Reclamation, Denver, Colo.

In order to be able to predict the elastic stresses at the heel and toe it would be necessary to fillet the corner with a large enough curvature to keep the theoretical stresses within the elastic limit at all times. This was done in other studies of this dam.

Case 2.—Referring to Fig. 2, consider the dam as a triangle with elastic modulus, F_1, and Poisson's ratio, μ_1, resting on an elastic foundation which is considered as an infinite half plane with modulus, E_2, and Poisson's ratio, μ_2. A slice, one unit wide, across the dam and into the foundation is considered. It will be assumed that no separation or slippage will occur between the base and the foundation. This implies that, when all effects are considered, the resulting normal stresses are compression, and that the shear stresses are within the permissible relation to the normal stresses.

All forces acting on the dam and on the foundations are known, except the reactions between the two. The problem is to determine these reactions such that the elastically deformed base and foundation make a perfect fit when equilibrium exists. This is expressed by demanding that the relative displacements of the dam and the foundation must be zero along the base.

With the co-ordinate system as in Fig. 4, let the stress function for the dam be $F = F_p + F_g + F_\gamma$, corresponding to full hydrostatic pressure on the up-stream face, and the mass forces, g, per unit volume. Functions F_p and F_g are given by Equations (17) and (21); and F_γ is the "corner-function" derived in Equation (50) with γ equal to the top angle of the dam ($\gamma < \pi$). In this case F_γ may be considered as a correction to the basic function, $F_p + F_g$, for the infinite wedge. It has already been indicated that F_γ does not alter the force system which acts on the faces of the dam. Its value is known except for the arbitrary constants, A_k and B_k, which must be determined by making the deformed base of the dam congruent with the deformed foundation line. By this method it is possible to determine: (1) The effect of a variation of the ratio, $\frac{E_1}{E_2}$, on the stresses, in which E_1 and E_2 are Young's modulus for the dam and foundation, respectively; (2) the effect of uniform shrinkage of the dam relative to the foundation; and (3) it includes Case 1 when $E_1 = E_2$ and $\mu_1 = \mu_2$.

APPLICATION IV.—APPROXIMATE BOUNDARY STRESSES AT SHARP AND ROUNDED CORNERS OR FILLETS

Formulas are developed herein for the boundary stresses at sharp and rounded corners of elastic structures, using only the real "corner-values," n_0 and n_1, given in Fig. 5. The method is applicable to any structure with any loading. Numerical examples are computed in the case of a triangular dam.

Theory Pertaining to Sharp Corners.—Assume that the body forces and the forces acting on the boundaries, $\theta = 0$ and $\theta = \gamma$, of the corner in Fig. 4 is represented by a function, F_2, and that the stresses, σ_a and σ_b, at the points, $r = a$ and $r = b$, of the boundaries are known, or have been computed by ordinary engineering methods. It is required to determine a function, F, to represent the state of stress at the corner such that, when $\theta = 0$:

$$F = F_2 \qquad \text{(52a)}$$

and,

$$\frac{\partial F}{\partial \theta} = \frac{\partial F_2}{\partial \theta} \qquad \text{(52b)}$$

Then by Equations (2),

$$\left[\frac{\partial^2 F}{r^2 \partial \theta^2} + \frac{1}{r}\,\frac{\partial F}{\partial r} - r\,g \cos(\theta - \beta)\right]_{\binom{\theta = 0}{r = a}} = \sigma_a \qquad \text{(53)}$$

On the boundary, $\theta = \gamma$, Equations (52) and (53) apply likewise except, of course, that here $\theta = \gamma$, $r = b$, and σ_a is replaced by σ_b. Let,

$$F = F_2 + F_\gamma \qquad \text{(54)}$$

Function F_γ is the "corner-function" for γ given by Equation (50), with $k = 0$ and 1; that is, only the real values, n_0 and n_1, found by Fig. 5 are involved. Hence, $F_\gamma = F_0 + F_1$, in which F_0 and F_1 are given by Equation (44) for $n = n_0$ and $n = n_1$.

It will be seen that F defined by Equation (54) satisfies Equation (52). It only remains to determine the constants, A_0 and A_1, such that Equation (53) will be satisfied.

By substituting F into this formula and re-arranging:

$$\left[\frac{\partial^2 F_\gamma}{r^2 \partial \theta^2} + \frac{1}{r}\,\frac{\partial F_\gamma}{\partial r}\right]_{\binom{\theta = 0}{r = a}} = \sigma_a - \left[\frac{\partial^2 F_2}{r^2 \partial \theta^2} + \frac{1}{r}\,\frac{\partial F_2}{\partial r} - r\,g \cos(\theta - \beta)\right]_{\binom{\theta = 0}{r = a}} \qquad \text{(55)}$$

and, the analogous equation for $\theta = \gamma$, $r = b$, in which every term is known except A_0 and A_1. The brackets on the right side of Equation (55) are the stresses σ_r, computed from the function F_2 by Equations (2). The derivation of F_2 will be given subsequently herein. Denote these terms by $Q_{0,\,a}$ and $Q_{\gamma,\,b}$. In general, let

$$Q_{r,\,\theta} = \frac{\partial^2 F_2}{r^2 \partial \theta^2} + \frac{1}{r}\,\frac{\partial F_2}{\partial r} - r\,g \cos(\theta - \beta) \qquad \text{(56)}$$

and let subscripts denote the co-ordinates of the point at which the quantities are to be computed.

By partial differentiations of F_γ given by Equation (44):

$$\left[\frac{\partial^2 F_\gamma}{r^2 \partial\,\theta^2} + \frac{1}{r}\,\frac{\partial F_\gamma}{\partial r}\right]_{\theta=0} = 2\,A_0\,n_0\,m_0\,r^{n_0-1} + 2\,A_1\,n_1\,m_1\,r^{n_1-1} \ldots\ldots (57a)$$

and,

$$\left[\frac{\partial^2 F_\gamma}{r^2\,\partial\,\theta^2} + \frac{1}{r}\,\frac{\partial F_\gamma}{\partial\,r}\right]_{\theta=\gamma} = 2A_0\,n_0\,m_0\,r^{n_0-1} - 2\,A_1\,n_1\,m_1\,r^{n_1-1} \ldots\ldots (57b)$$

Substituting back into Equation (55), the following equations are obtained:

$$2\,A_0\,n_0\,m_0\,a^{n_0-1} + 2\,A_1\,n_1\,m_1\,a^{n_1-1} = \sigma_a - Q_{0,a} \ldots\ldots\ldots (58a)$$

and,

$$2\,A_0\,n_0\,m_0\,b^{n_0-1} - 2\,A_1\,n_1\,m_1\,b^{n_1-1} = \sigma_b - Q_{0,b} \ldots\ldots\ldots (58b)$$

In most cases it is possible to choose $a = b$, and, if so, further simplification is possible: Placing $a = b$ in Equations (58) and solving:

$$A_0 = \frac{\sigma_a - Q_{0,a} + \sigma_b - Q_{\gamma,b}}{4\,n_0\,m_0\,a^{n_0-1}} \ldots\ldots\ldots\ldots\ldots\ldots (59a)$$

and,

$$A_1 = \frac{\sigma_a - Q_{0,a} - \sigma_b + Q_{\gamma,b}}{4\,n_1\,m_1\,a^{n_1-1}} \ldots\ldots\ldots\ldots\ldots\ldots (59b)$$

The stress function, $F = F_2 + F_\gamma = F_0 + F_1 + F_2$, is now completely known and the stresses can be found by the definitions in Equations (2); thus:

$$(\sigma_r)_{\theta=0} = \frac{\sigma_a - Q_{0,a} + \sigma_b - Q_{\gamma,b}}{2}\left(\frac{r}{a}\right)^{n_0-1}$$

$$+ \frac{\sigma_a - Q_{0,a} - \sigma_b + Q_{\gamma,b}}{2}\left(\frac{r}{a}\right)^{n_1-1} + Q_{0,r} \ldots\ldots\ldots\ldots (60a)$$

and,

$$(\sigma_r)_{\theta=\gamma} = \frac{\sigma_a - Q_{0,a} + \sigma_b - Q_{\gamma,b}}{2}\left(\frac{r}{a}\right)^{n_0-1}$$

$$- \frac{\sigma_a - Q_{0,a} - \sigma_b + Q_{\gamma,b}}{2}\left(\frac{r}{a}\right)^{n_1-1} + Q_{\gamma,r} \ldots\ldots\ldots\ldots (60b)$$

The function, F_2, will now be given for a dam. Let $F_2 = F_p + F_g$, in which F_p is the part of F_2 due to the external forces acting on the boundaries, $\theta = 0$ and $\theta = \gamma$ (Fig. 4); and, F_g is the part of F_2 entirely due to the body forces, g, per unit volume.

Case 1.—In this case there are no external forces acting on the boundaries, $\theta = 0$ and $\theta = \gamma$, on the stretch, a to b. Hence, $F_p = 0$, and by Equation (56), $Q_{r,\theta} = 0$.

Case 2.—There is a constant pressure, p, on the boundary $\theta = 0$, and a pressure $(1 - r)\ p$ on $\theta = \gamma$. By a co-ordinate transformation in Equation (17) and superposing $-\frac{1}{2}\,p\,r^2$, it will be found that,

$$F_p = \frac{pr^3 \sin^2\theta}{12 \sin^3\gamma}\ [3 \cos\theta \sin 2\gamma + 6 \sin\theta \sin^2\gamma - 4 \sin\theta] - \tfrac{1}{2}\,pr^2 \ldots (61)$$

will satisfy the conditions of this case. Then by Equation (56):

$$Q_{0,\,r} = \frac{pr \cos\gamma}{\sin^2\gamma} - p \ldots\ldots\ldots\ldots\ldots\ldots (62a)$$

and,

$$Q_{\gamma,r} = -\frac{pr \cos^2\gamma}{\sin^2\gamma} - p \ldots\ldots\ldots\ldots\ldots\ldots (62b)$$

Case 3.—Only mass forces are considered, and the function, F_g, must be such that σ_θ and $\tau_{r,\theta}$ are zero along the straight boundaries, $\theta = 0$, and $\theta = \gamma$. Substituting polar co-ordinates into Equation (21) and placing $g_x = g \cos\beta$ and $g_y = g \sin\beta$:

$$F_g = \frac{g\,r^3}{6}\left[\frac{\cos\beta}{\tan\gamma}\sin^3\theta + \frac{3\sin\beta}{\tan\gamma}\cos\theta\sin^2\theta - \frac{2\sin\beta}{\tan^2\gamma}\sin^3\theta + \sin\beta\sin^3\theta + \cos\beta\cos^3\theta\right] \ldots\ldots\ldots\ldots\ldots\ldots (63)$$

By Equation (56):

$$Q_{0,r} = -\frac{rg \sin(\gamma - \beta)}{\sin\gamma} \ldots\ldots\ldots\ldots\ldots\ldots (64a)$$

and,

$$Q_{\gamma,\,r} = -\frac{rg \sin\beta}{\sin\gamma} \ldots\ldots\ldots\ldots\ldots\ldots (64b)$$

It will be seen in Fig. 5 that n_0 is always less than unity for angles between 180 and 360 degrees. This means that the stresses found by Equations (60) will approach infinity as the radius vector, r, approaches zero. In other words, a mathematically sharp re-entrant corner cannot occur in a structure without having the stresses exceed the elastic limit, contrary to the assumptions of the mathematical theory of elasticity. The material becomes plastic and the stresses in the immediate neighborhood must be redistributed.

Theory Pertaining to Fillets.—In order to predict the elastic stresses it is necessary to abandon the sharp corner and assume a finite curvature. In this case the preceding formulas no longer are valid in the immediate neighborhood of the corner. Photo-elastic experimentation has shown, as might be expected, that a slight curvature only will effect the stresses in the immediate neighborhood (that is, at distances of the order of magnitude of the radius of curvature). The stresses, σ_r, at the tangent points, d and e, Fig. 4, may still be computed (on the side of safety) as if the corner was sharp.

With the foregoing assumption an approximate method is developed herein for the determination of the stresses on the boundary of the fillet. The origin is chosen in the center of curvature and the axes, ξ and η, are parallel to Axes X and Y. The notations, ρ, ϕ, are used for the polar co-ordinates in the new system.

Assume that the stresses, σ_d and σ_e, have been computed by the methods shown (such as, by Equations (60)), and assume that a constant pressure is exerted on the curved boundary, $\rho = R$. If there is no pressure, $p = 0$. Now, consider the stress function:

$$F = (B_1 \cos\phi + B_2 \sin\phi)\left(\frac{1}{\rho} + \frac{2\,\rho\log\rho}{R^2}\right) - \frac{1}{2}\,\rho^2\,p$$

$$+ \frac{1}{2}\,gR^2\,\rho\,\phi\sin(\phi - \beta) \ldots\ldots\ldots\ldots\ldots\ldots(65)$$

With Equations (2) applied to Equation (65) it is easily verified that when $\rho = R$, $\sigma_\rho = -p$; and $\tau_{\rho,\phi} = 0$ for any value of ϕ.

The arbitrary constants, B_1 and B_2, are now to be determined such that when $\rho = R$, and $\phi = \frac{\pi}{2}$:

$$\sigma_\phi = \sigma_d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(66a)$$

and when $\rho = R$, and $\phi = \gamma - \frac{\pi}{2}$:

$$\sigma_\phi = \sigma_e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(66b)$$

By applying Equations (2) to Equation (65):

$$\sigma_\phi = (B_1\cos\phi + B_2\sin\phi)\left(\frac{2}{\rho^3} + \frac{2}{\rho R^2}\right) - p - \rho g\cos(\phi - \beta)\ldots(67)$$

Hence, by the conditions expressed in Equations (66):

$$B_1 = [\sigma_e + p + (\sigma_d + p)\cos\gamma + gR\cos\beta\sin\gamma]\,\frac{R^3}{4\sin\gamma}\ldots(68a)$$

and,

$$B_2 = [\sigma_d + p + gR\sin\beta]\,\frac{R^3}{4}\ldots\ldots\ldots\ldots\ldots(68b)$$

Substituting back into Equation (67) with $\rho = R$:

$$(\sigma_\phi)_{\rho=R} = [\sigma_e + p + (\sigma_d + p)\cos\gamma]\,\frac{\cos\phi}{\sin\gamma} + (\sigma_d + p)\sin\phi - p\ \ldots(69)$$

The method of procedure is best shown by a few typical examples.

EXAMPLES COMPUTED BY APPROXIMATE THEORY

The underlying assumptions are based, to a large extent, on results of photo-elastic experiments conducted in connection with the Morris Dam (see Part II). It is natural, therefore, to base the example on computations for this dam, because verification is possible. A cross-section of the Morris Dam is shown in Fig. 2. Theory and experiments have shown that the stresses in the upper two-thirds of the dam can be computed accurately by Equations (17) and (21). The height of the dam is taken as the unit of length. All stresses must be multiplied later by the actual height, H.

Up-Stream Corner, Hydrostatic Loading.—Consider first the up-stream corner with full hydrostatic pressure, $p\,(1 - r)$, on the up-stream face and constant pressure, p, on the foundation above the dam. The body forces will be considered subsequently; that is, $g = 0$.

For this loading the value of $F_2 = F_p$ is given in Equation (61). Points a and b (Figs. 2 and 4) are chosen as $r = a = b = 0.3$; the stress, σ_a, can be computed with reasonable accuracy by superposing Equations (35) and (39a); q_1 in Equation (35) is the total horizontal water pressure on the dam uniformly distributed over the base l_1; and p_0 in Equation (39a) is the pressure, p, on the up-stream foundation.

In the present application, a is an up-stream point; $q_1 = \dfrac{p}{2\,l_1}$; and, $l_1 = 0.86$. The total stress, then, is $\sigma_a = + \dfrac{p}{0.86\,\pi} \log \dfrac{1.16}{0.3} - p = - 0.500\,p$. Point b is sufficiently far from the base so that σ_b may be computed as σ_x by Equation (18a). In the present application, $K = \tan 41°\,52' = 0.8963$; $x = 0.7$; $y = 0$; and $\sigma_b = + 0.871\,p$.

The up-stream corner angle, γ, $= 267°\,08'$. By Fig. 5, $n_0 = 0.55$; and $n_1 = 0.93$; and, by Equation (46), $m_0 = 0.895$; and $m_1 = - 0.318$. The quantities, $Q_{0,\,a}$ and $Q_{\gamma,\,b}$, are now found by placing $a = b = 0.3$ in Equations (62); thus: $Q_{0,\,r} = - (1 + 0.05\,r)\;p$; $Q_{\gamma,\,r} = - (1 + 0.0025\,r)\;p$; $Q_{0,\,a} = - 1.015\,p$; and $Q_{\gamma,\,b} = - 1.001\,p$. By substitution in Equations (60);

$$(\sigma_r)_{\theta=0} = \left[0.6945\left(\frac{1}{r}\right)^{0.45} - 0.6785\left(\frac{1}{r}\right)^{0.07} - 0.05\,r - 1\right]p \;\ldots(70a)$$

and,

$$(\sigma_r)_{\theta=\gamma} = \left[0.6945\left(\frac{1}{r}\right)^{0.45} + 0.6785\left(\frac{1}{r}\right)^{0.07} - 0.0025\,r - 1\right]p \;..(70b)$$

In Equations (70), it is to be noted that: r is measured in units of height of dam and p is the actual hydrostatic pressure at the base. By Equations (70), σ_r becomes infinite as $r \rightarrow 0$. To eliminate this condition assume a finite radius of curvature, $R = 0.025$, at the corner with tangent points at $d = e = R \cot 0.5\,\gamma = 0.0238$ (see Figs. 2 and 4).

Now, compute σ_d by Equation (70a) and σ_e by Equation (70b) for $r = e = d = 0.0238$; thus, $\sigma_d = + 1.943\,p$; and $\sigma_e = + 3.523\,p$. Substitute in Equation (69) with $g = 0$:

$$(\sigma_\phi)_{\rho=R} = [- 4.3813 \cos\phi + 2.943 \sin\phi - 1]\;p \ldots\ldots\ldots(71)$$

The maximum occurs when $\phi = 146°\,07'$; and σ_ϕ (max.) $= + 4.278\,p$. The stresses computed by Equations (70) and (71) are plotted in Fig. 2(a).

Up-Stream Corner, Mass Forces.—Now, consider mass forces only; that is, let $p = 0$. Only vertical forces are assumed to exist; hence, $\beta = 90°$; $g_x = 0$; and $g = g_y =$ weight per unit volume. The stress, σ_a, computed by

Equation (39*a*) is zero; σ_b is computed as σ_x by Equation (22*a*) (for $x = 0.7$; $y = 0$, and $K = 0.8963$); and, therefore, $\sigma_a = 0$, and $\sigma_b = -0.660\,g$.

Now, compute the quantities, Q, etc., by Equations (64), with $\beta = 90°$; $\gamma = 267° 08'$; and $a = b = 0.3$; thus: $Q_{o,r} = +0.0501\,r\,g$; $Q_{\gamma,r} = +1.001\,r\,g$; $Q_{o,a} = +0.015\,g$; and $Q_{\gamma,b} = +0.3003\,g$.

By substitution in Equations (60) with $p = 0$, and n_0 and n_1 as before:

$$(\sigma_r)_{\theta=0} = \left[-0.2837\left(\frac{1}{r}\right)^{0.45} + 0.4345\left(\frac{1}{r}\right)^{0.07} + 0.0501\,r\right]g \quad \ldots(72a)$$

and,

$$(\sigma_r)_{\theta=\gamma} = \left[-0.2837\left(\frac{1}{r}\right)^{0.45} - 0.4345\left(\frac{1}{r}\right)^{0.07} + 1.001\,r\right]g \quad \ldots(72b)$$

in which r is measured in units of H. Again, the stresses become infinite for the sharp corner when $r \rightarrow 0$. The radius of curvature, $R = 0.025$, is assumed as before, and $d = e = 0.0238$. By Equations (72) with $r = d = e = 0.0238$; $\sigma_d = -0.9596\,g$; and $\sigma_e = -2.0657\,g$. Substitute in Equation (69) with $p = 0$, thus:

$$(\sigma_r)_{\rho=R} = [+2.0202\cos\phi - 0.9695\sin\phi]\,g \ldots\ldots\ldots\ldots(73)$$

The maximum occurs when $\phi = 154° 36'$; and maximum $\sigma_\phi = -2.237\,g$. The stresses computed by Equations (72) and (73) are plotted in Fig. 2(*b*). For the section under consideration let $p = 108.5$ lb per sq in., and $g = 2.5\,p = 271$ lb per sq in., corresponding to 156-lb concrete.

In computing the stresses for the down-stream corner the procedure is exactly the same as for the up-stream corner. The origin is chosen at the toe with the X-axis along the down-stream foundation and the Y-axis positive into the foundation. If no pressures act on the down-stream face and foundation, $F_p = 0$; and F_g is again given by Equation (63), with γ equal to the re-entrant corner angle of the toe, and $\beta = 90°$, if the foundation is horizontal and if inertia forces do not exist. Point b (Fig. 2 (*b*)) is chosen on the down-stream face and Point a on the down-stream foundation, such that $r_a = r_b = -\dfrac{0.3}{\sin\gamma}$. The radial stress, $\sigma_b = \sigma_x + \sigma_y$, is computed by Equations (18) in the case of hydrostatic loading and by Equations (22) in the case of mass forces. The stress, σ_a, is computed by Equation (85) and becomes negative. The results are plotted in Fig. 2 for the Morris Dam, assuming a fillet radius of $0.025\,H$ as at the up-stream corner.

The Morris Dam is not designed with fillets, but it was necessary to manufacture the models used for the photo-elastic experiments (scale, 1:600) with a $\frac{1}{8}$-in. radius (0.025 × height) at the corners to prevent failure when hydrostatic load was applied in the absence of body forces. For this reason the same radius was used in the computations for stresses in the fillets.

The approximate method of analysis is recommended only for preliminary work, in order to fix suitable dimensions in the base region of the section. In the final analysis one of the methods explained in Application III should be used.

Based on the present computations and those for other radii of fillets it would seem that, for stable gravity dams, radii of 0.04 H to 0.08 H would be suitable at the heel and 0.10 H to 0.20 H at the toe, depending on the height of the dam and the quality of the masonry. The up-stream fillet is determined for the case of an empty reservoir, the down-stream one for the case of a full reservoir.

PART II.—PHOTO-ELASTIC EXPERIMENTS IN CONNECTION WITH THE MORRIS DAM, IN CALIFORNIA

Brief Outline of the Photo-Elastic Phenomenon

Although the principles involved in photo-elastic stress analysis may be found in treatises on photo-elasticity[10] a short description is given herewith, for the sake of completeness.

About 125 years ago the English physicist, Sir David Brewster, discovered that isotropic media become doubly refracting when placed under stress. They then behave optically as crystalline media. The planes of principal stress correspond to the crystal planes. If a plane-polarized monochromatic ray (see Fig. 8), is sent normally through a plate stressed in its own plane, the ray will be resolved into two component rays each vibrating in a plane of principal stress. The two ray components will usually have different velocities in the two planes; therefore, while passing through the plate the wave front of one ray will get ahead of the wave front of the other ray a small distance, δ, which is called the relative optical displacement of the two ray components. If, after leaving the plate, the two components are combined so as to vibrate again in the same plane, interference will take place due to the relative displacement, δ. Complete interference will occur when $\delta = 0$, 1, ... n, wave lengths. This furnishes a means of measuring the displacements. It has been shown theoretically by Clerk Maxwell[11], and verified experimentally that the difference in magnitudes of the two principal stresses, σ_P and σ_Q, at the line of passage is in direct proportion to the relative displacement. The latter can be measured in terms of wave lengths of the light by counting the order, n, of interference, hence:

$$\sigma_P - \sigma_Q = n\ C \dots\dots\dots\dots\dots\dots\dots\dots (74)$$

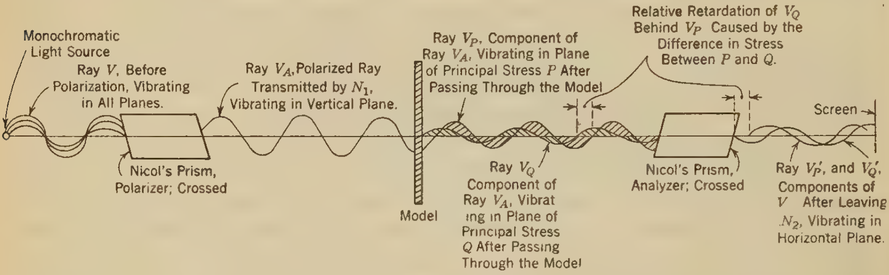
Monochromatic Light Source
Ray V, Before Polarization, Vibrating in All Planes.
Ray V_A, Polarized Ray Transmitted by N_1, Vibrating in Vertical Plane.
Ray V_P, Component of Ray V_A, Vibrating in Plane of Principal Stress P After Passing Through the Model
Relative Retardation of V_Q Behind V_P Caused by the Difference in Stress Between P and Q.
Screen
Nicol's Prism, Polarizer; Crossed
Model
Ray V_Q Component of Ray V_A, Vibrating in Plane of Principal Stress Q After Passing Through the Model
Nicol's Prism, Analyzer; Crossed
Ray V_P', and V_Q', Components of V After Leaving N_2, Vibrating in Horizontal Plane.

Fig. 8.

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[10] "Photo-Elasticity", by Coker and Filon.

[11] "On the Equilibrium of Elastic Solids", by Clerk Maxwell, *Transactions*, Royal Soc. Edinburgh, Vol. XX, Pt. I; also, "Photo-Elasticity", by Coker and Filon, p. 198.

n which C is a constant which depends on the type of material, the thickness of the plate, and the wave length of the light. It will be shown later ow C is determined. (It is to be noted in passing that the number of fringes s directly proportional to the thickness of the plate.)

The principal stress difference, $\sigma_P - \sigma_Q$, is equal to twice the maximum shear at the line of passage; hence, this quantity is known when the fringe order, n, is observed. Furthermore, $\sigma_P - \sigma_Q$ will change continuously from point to point in the plate; therefore, the optical displacements, δ, will do likewise, and equal integral values of n will occur along continuous interference bands called "isochromatics" which then may be defined as lines of equal maximum shear.

The plane polariscope is shown schematically in Fig. 8, with explanatory notes. The two Nicol's prisms, the polarizer and the analyzer, are "crossed"; that is, their polarizing planes are set at right angles so that no light will reach the screen, when the plate is not in the field or when it is unstressed.

As already explained alternate dark and light bands, "isochromatics", will appear on the screen when the stressed plate is placed in the path of the light. Along the edges of the model one of the stresses, σ_P or σ_Q, is known, and the fringe order, n, can be observed so that the other stress (σ_Q or σ_P) may be computed by Equation (74) when C is known.

CALIBRATION

The simplest way to determine the value of C is to subject a beam of rectangular cross-section to uniform bending. The beam must be cut from the model material. The fringe pattern obtained in a bakelite test beam is shown in Fig. 9. The dark band at mid-height is the isochromatic of zero order ($n = 0$), or the neutral axis. The constant bending moment and the dimensions of the beam section can be measured so that the horizontal stress at the extreme fibers can be computed. In the present experiment, $\sigma_P = 1\,055$ lb per sq in. No load occurs at mid-span; that is, $\sigma_Q = 0$. The fringe order, n, is counted at the top or bottom (in this case, 6), the neutral axis being of zero order. Hence, by Equation (74),

FIG. 9.—ISOCHROMATICS IN CALIBRATION BEAM.

$$C = \frac{\sigma_P - \sigma_Q}{n} \quad \text{.........................(75)}$$

or, $\frac{1\,055}{6} = 176$ lb per sq in. The even spacing of the parallel fringes shows that the engineering assumption of linear stress distribution holds exactly for a beam of uniform thickness stressed with constant bending moment. The thickness of the beam was 0.346 in. and white light with a blue Wratten filter (about 4100 Angstrom) was used. This filter and models, 0.346 in. thick, were used in both the experiments to be described. If the thickness, t, of the model were different from that of the test beam, T, merely multiply C in Equation (75) by $\frac{T}{t}$.

It may happen that Ray V_A (Fig. 8) is vibrating in the plane of the principal stress, σ_P, at the line of passage. In this case Ray V_A (= Ray V_P) will pass unmolested through the plate and the orthogonal component, V_Q, will be zero. Therefore, nothing will happen on the screen and darkness will prevail in the image of the point of passage, the Nicol's prisms being "crossed". When the polarizing plane of the polarizer is set at a known angle the rays automatically will seek the points in the model at which one of the principal stresses is parallel to the polarizing plane. The stress directions in a model will generally vary continuously (except at singular points) so that the dark points just described will also form a continuous band in the image known as an "isoclinic" or locus of points having their principal stresses parallel to the known plane of polarization. If the polarizer and analyzer are now rotated together through a certain angle, another isoclinic will appear, etc. In this manner the direction of the principal stresses can be found throughout the model.

Although the location of the isoclinic changes with each prism setting, the system of isochromatics will remain stationary and, therefore, are easily distinguished.

By using circularly polarized light, instead of plane polarized light the isoclinics can be eliminated. In this experiment the light vector, V_A, is made to rotate with a high, uniform, angular velocity while passing through the model and thus it can have no directional preferences; therefore, isoclinics cannot occur. Circularly polarized light is produced by inserting what is known as a "quarter wave plate" between each Nicol prism and the model. These crystalline plates are usually obtained by splitting mica to a thickness corresponding to a relative retardation of the two component rays, equal to one-fourth the wave length used in the experiment. The theory of this phenomenon may be found in treatises on passage of light through crystalline media.[12]

By using materials of low optical sensitivity (that is, a very large C in Equation (74)) such as plate glass, the isoclinic lines which are only a function of the stress direction can be made to appear sharply long before the model is stressed sufficiently for the isochromatics to show except as a slight illumination of the glass. The latter effect makes the isoclinic stand out clearly as a black line on a light background (see Fig. 13). Glass without initial stresses must be used, and is distinguished by holding the unstressed glass plate in the field of the polariscope before cutting it into the desired model.

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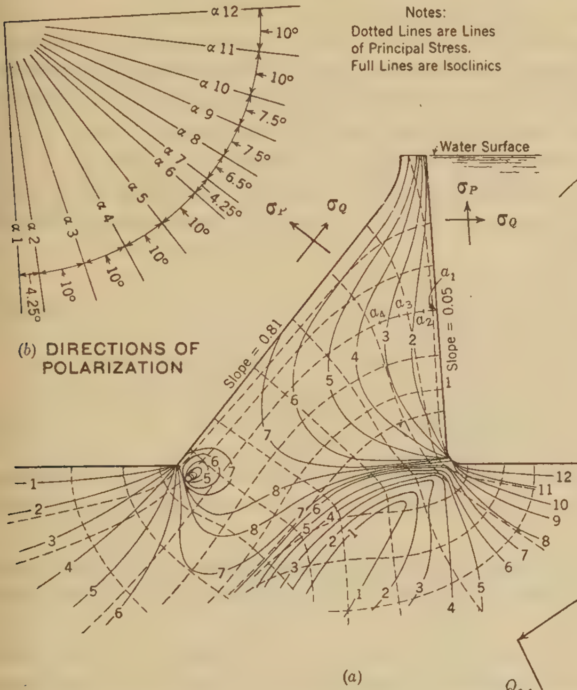
[12] "Photo-Elasticity", by Coker and Filon, p. 73.

The bakelite plates used for the isochromatics generally had to be annealed before planing, polishing, and modeling, by heating them in a well controlled oven to about 80° C for a couple of hours and then cooling them gradually to room temperature over a period of about 10 hr. A newer type of bakelite usually does not need to be annealed.

EVALUATION OF STRESSES

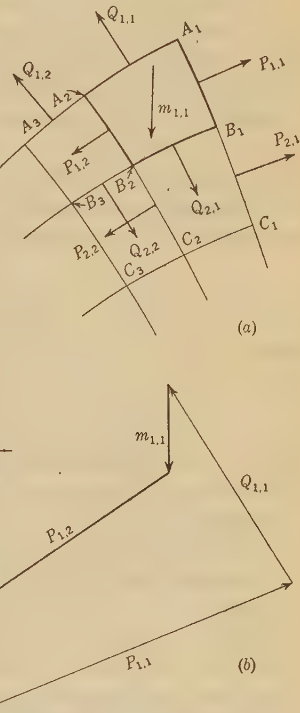
The lines of principal stress can be drawn when the system of isoclinic lines is known as explained subsequently under the heading, "Hydrostatic Loading; Isoclinics" (see, also, Fig. 10). The magnitude of stress along the lines of principal stress can then be integrated by use of the isochromatic lines, the known boundary forces, and Equation (74). Methods for this work are given in treatises on photo-elasticity.[13]


Notes:
Dotted Lines are Lines of Principal Stress.
Full Lines are Isoclinics

(*b*) DIRECTIONS OF POLARIZATION

(*a*)
α 12
α 11
α 10
α 9
α 8
α 7
α 6
α 5
α 4
α 3
α 2
α 1
10°
10°
7.5°
7.5°
6.5°
4.25°
10°
10°
10°
10°
4.25°
Water Surface
σP
σQ
σP
σQ
Slope = 0.81
Slope = 0.05
a1
a2
a3
a4
1
2
3
4
5
6
7
8
9
10
11
12

FIG. 10.—ISOCLINICS AND LINES OF PRINCIPAL STRESS; HYDROSTATIC LOADING.


(*a*)

(*b*)
Q1.1
Q1.2
A1
A2
A3
P1.1
P2.1
m1.1
B1
B2
B3
P1.2
P2.2
Q2.1
Q2.2
C1
C2
C3
m1.1
Q1.1
P1.2
Q2.1
P1.1

FIG. 11.—GRAPHICAL INTEGRATION OF PRINCIPAL STRESSES.

In many types of problems, even when body forces are to be dealt with, the following method is convenient for determining the magnitudes, σ_P and σ_Q, along given lines of principal stress. The method is essentially structural and was developed by considering that the structure is composed of two systems of orthogonal arches bounded by the lines of principal stress. These arch strips are interlaced in such a way that only normal forces (no shear) exist on any element or block bounded by the four principal stress lines; that is, two from each system. The integration problem is then reduced to the determination of the force polygon, being given the funicular curve and certain reactions.

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[13] "Photo-Elasticity," by Coker and Filon, pp. 143–146.

Consider the element, A_1, B_1, A_2, B_2, bounded by the principal lines of stress shown in Fig. 11(*a*). The only forces acting on the element are $P_{1,1}$, $P_{1,2}$, $Q_{1,1}$, and $Q_{2,1}$, normal to Surfaces A_1, B_1, etc., and perhaps the mass force, M_1. If $P_{1,1}$, $Q_{1,1}$, and M_1 are assumed to be known in magnitude and direction, then $P_{1,2}$ and $Q_{2,1}$ can be found by constructing the force polygon.

Proceed to the element, $A_2\ B_2\ A_3\ B_3$, etc., until all the forces, $P_{1,n}$ and $Q_{2,n}$, are found. It is necessary to start the construction at the boundary, A_1, A_2, A_3, etc., where the forces, $Q_{1,n}$ and P_1, are known or can be found by the isochromatics and boundary forces. When all the forces, $Q_{2,n}$, are found, consider B_1, B_2, B_3, etc., as a new boundary, apply the now known forces, $Q_{2,n}$, reversed, and the known forces, $P_{2,1}$, and find the forces, $Q_{3,n}$ and $P_{2,n}$, by construction, etc. The average stresses over the elementary sides are found by dividing the forces by the corresponding sides. Finally, the principal stresses at B_2, say, will be:

$$\sigma_Q = \frac{1}{2}\left[\frac{Q_{2,1}}{\overline{B_1\ B_2}} + \frac{Q_{2,2}}{\overline{B_2\ B_3}}\right] \dots\dots\dots\dots\dots (76a)$$

and,

$$\sigma_P = \frac{1}{2}\left[\frac{P_{1,2}}{\overline{A_2\ B_2}} + \frac{P_{2,2}}{\overline{B_2\ C_2}}\right] \dots\dots\dots\dots\dots (76b)$$

It will be seen that Equations (74) furnish a check on $\sigma_P - \sigma_Q$ at all points, which should be made after the completion of the stresses along each line of principal stress.

The construction is easily arranged in a force diagram so that duplications of lines are avoided. Note that if the lines curve rapidly small elements must be used.

The integration of stresses along principal lines of stress is tedious and often not very accurate in complicated stress patterns. Consequently, efforts have been made to obtain other methods.

A two-dimensional state of stress is determined completely when the principal stresses, σ_P and σ_Q, and their directions are known at all points; $\sigma_P - \sigma_Q$ is known throughout by the isochromatics and by Equation (74). The directions are known by the isoclinics; hence, if $\sigma_P + \sigma_Q$ can be found, the state of stress is determinate; $\sigma_Q + \sigma_P$ can be obtained by measuring the lateral dilations of the model with an extensometer,[13] or by the membrane analogy.[14]

BODY FORCES

Body forces — weight and inertia — are difficult to apply correctly in photo-elastic experimentation. It is possible to examine isolated regions in the model by substituting the surrounding mass forces by equivalent external loads as long as these forces are placed far enough away from the region in question to permit proper distribution. This method was used in the experiment shown in Fig. 2(*b*) and Fig. 12(*a*).

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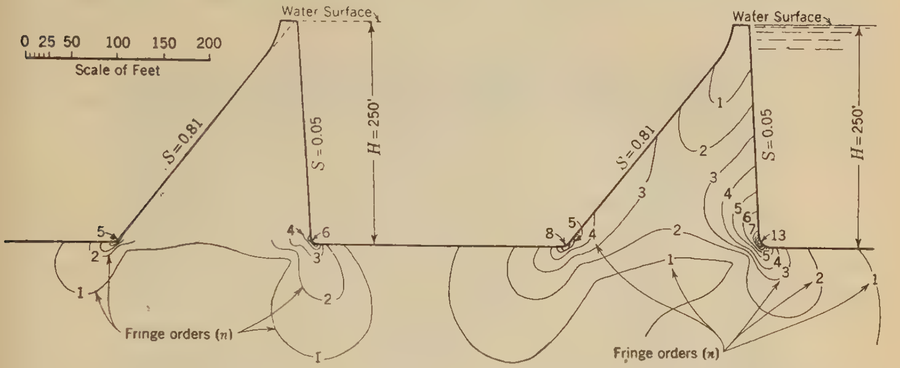
[14] See p. 927.



(*a*) DUE TO WEIGHT OF DAM CORRECT IN FOUNDATION ONLY

(*b*) FOR HYDROSTATIC PRESSURE ON UPSTREAM FACE AND FOUNDATION

FIG. 12.—ISOCHROMATICS.

The weight of the model itself is too slight to have any effect unless it is magnified many times. The easiest method would be to rotate the model and observe it whenever it passes through the optical field. This method has not yet been attempted, and it will require a steady rotor mechanism, free of vibration, to obtain a good photographic exposure of the photo-elastic lines. The polariscope used by the writer would be admirably suited for such an experiment. The model could be attached to one end of a rigid rotor and the photographic plate to the other end, each passing through the optical field simultaneously with the same relative angular velocity. A new method has been developed by the writer in connection with experimentation on the Grand Coulee Dam, by which only the corrections to the straight-line stress distribution are involved. An interesting equivalent boundary loading has recently been pointed out by M. A. Biot.[15]

Stresses have been determined experimentally for the section shown in Fig. 2, under all critical loading conditions, including earthquake accelerations and uniform shrinkage relative to the foundation.

Hydrostatic Load on the Up-Stream Face and Foundation: Isochromatics.—A bakelite model, with blue filter and quarter-wave plates, was used to produce isochromatics. The model scale was 1:600; the photographic scale, 1:900; and the loading factor, 4.6, which means that the hydrostatic pressure at any point in the model was 4.6 times as great as at the corresponding point of the actual dam. The up-stream face was loaded linearly by three levers, each applying the correct force through knife-edges to a steel shoe. The uniform distribution on the up-stream foundation was applied by a single lever acting on a heavy steel shoe. Soaped rubber paddings were used between the shoes and the edge of the model, in order to avoid shear forces and local concentrations.

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[15] *Transactions*, A. S. M. E., Vol. 2, No. 2, June, 1935.

The isochromatics are plotted in Fig. 12(*b*) and, from these, the maximum shear in the model, in pounds per square inch, is obtained directly by Equation (74); thus: $\tau_{\text{max.}} = \dfrac{\sigma_P - \sigma_Q}{2} = \dfrac{n\,C}{2} = 88\,n$, C being equal to 176 by Equation (75). The shear stresses at corresponding points of the actual dam will then be, in pounds per square inch,

$$\tau_{\text{max.}} = \frac{88\,n}{\text{Load factor}} = \frac{88\,n}{4.6} = 19\,n \ldots\ldots\ldots\ldots\ldots (77)$$

in which n is the fringe order given in Fig. 12(*b*). The actual boundary stresses are also determined by Equation (74):

$$\sigma_P - \sigma_Q = \frac{176\,n}{\text{Load factor}} = 38\,n \ldots\ldots\ldots\ldots\ldots (78)$$

On the rear face and foundation, $\sigma_P = 0$; hence:

$$\sigma_Q = -38\,n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

On the up-stream face and foundation of the dam, the hydrostatic pressure is, in pounds per square inch, $\sigma_Q = \dfrac{-62.5}{144}\,h' = -0.434\,h'$, in which h' is the depth, in feet, to the point in the prototype, $= \dfrac{h \times 900}{12} = 75\,h$, and h is the depth, in inches, to the corresponding point in the photograph. Hence, by Equation (78),

$$\sigma_P = 38\,n - 32.55\,h \ldots\ldots\ldots\ldots\ldots\ldots\ldots (80)$$

in which n is the fringe order at the point in question.

The experimental results obtained by substituting the fringe order, n, along the boundaries of Fig. 12(*b*) into Equations (79) and (80) are plotted in Fig. 2(*a*) with the theoretical values obtained in Part I of this paper.

It will be noticed that in the upper two-thirds of the dam the stresses obtained by Equations (18) agree with experimentation. In the base region the boundary stresses are in close agreement with the approximate methods discussed under the heading, "Application IV", in Part I. It can be shown that the isochromatics for the infinite wedge are concentric ellipses with their centers at the vertex, agreeing with the upper isochromatics in Fig. 12(*b*). A slight disagreement between the computed and experimental values may be expected due to the finite crown width, which is neglected in Equation (17).

Hydrostatic Loading: Isoclinics.—A plate-glass model and white light without the quarter-wave plates were used to produce isoclinics. The photographic scale again was 1:900. It is to be noted that the isoclinics are independent of the loading factor as defined previously, except that these lines become sharper with increased factor. In order to cover the entire region it is only necessary to obtain the isoclinics for various prism settings, α, through a range of 90°, as shown in Fig. 10(*b*). A separate photograph is necessary of course for each prism setting, a sample of which is shown in Fig. 13 corresponding to the prism setting, α_3, in Fig. 10(*b*). The isoclinics are plotted in Fig. 10(*a*), and the lines of principal stress are obtained in the following manner: Through a_1 of Fig. 10(*a*) draw $a_1 a_2$ perpendicular to α_1 of Fig. 10(*b*); then draw $a_2 a_3$ perpendicular to α_2, etc. The points, $a_2 a_3$, etc., are taken approximately midway between the corresponding isoclinics, 1, 2, etc.

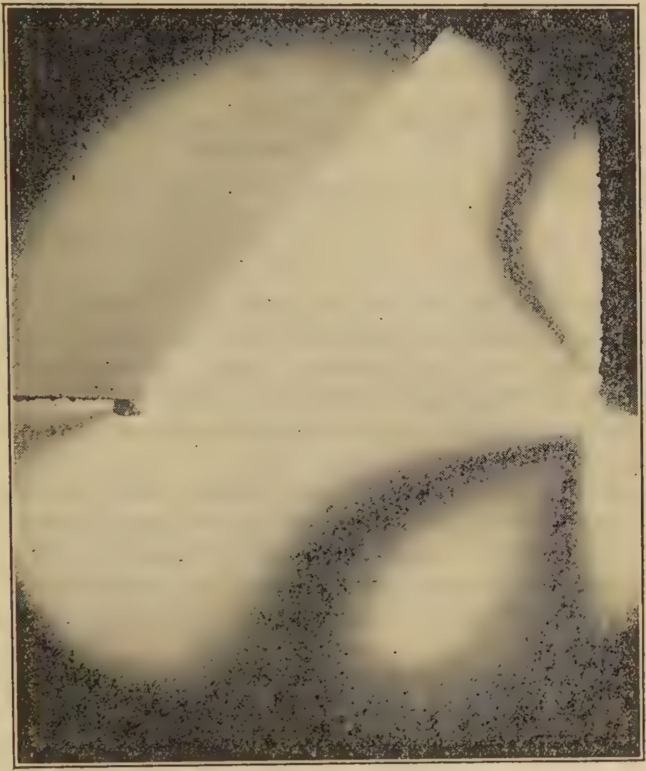

FIG. 13.—ISOCLINIC LINES FOR PRISM SETTING, α_3, IN FIG. 10(*b*).

It can be shown that the isoclinics for the wedge are straight lines radiating from the vertex, agreeing approximately with the upper two-thirds of Fig. 10(*a*). The isoclinics perhaps show more clearly than any other argument the effect of the foundation on the stresses in the base region. The corners in the models were rounded; with sharp corners, all isoclinics would be represented at each corner.

Weight Forces (Correct in Base Region Only).—In this experiment the stresses were obtained near the base due to the weight. Owing to the loading difficulties only the stresses near the base and in the foundation were determined. The forces representing the weight of the dam above the base were applied through a single lever distributing the weight to four points in such a manner that the resultant was correct at the base. The isochromatics are plotted in Fig. 12(a) near the base and in the foundation. The photographic scale was 1:900, and the loading factor was 1.75, based on 156-lb concrete.

The stresses in the actual dam would be (see Fig. 12(a)):

$$\tau_{\max.} = \frac{\sigma_P - \sigma_Q}{2} = \frac{176\,n}{2 \times \text{load factor}} = 50\,n \ldots\ldots\ldots\ldots (81)$$

On the boundaries, $\sigma_P = 0$; and hence,

$$\sigma_Q = -\,100\,n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (82)$$

The experimental results near the base obtained by substituting the fringe order, n, at the boundaries of Fig. 12(a) into Equation (82) are plotted in Fig. 2(b) together with the theoretical values. It will be seen that the stresses obtained in Part I agree fairly well with the experimental results. It is of interest to note that the compression at the down-stream corner is of considerable magnitude.

The results in Fig. 2 can be applied to a dam of similar proportions, but of height, H', merely by changing the stress scale in the ratio, 250 : H'. If the concrete in the new dam weighs w lb per cu ft, the stress scale in Fig. 2(b) is changed further in the ratio, 156 : w. The stress concentration in the base region at the down-stream face, due to weight, extends only a short distance into the dam.

The foundation was clamped between rubber gaskets to the steel frame along three edges. This arrangement allowed elastic deformation to take place so as to produce, as closely as possible, the effect of an infinite half-plane. The photo-elastic results in the foundation, of course, are only valid in the region near the base.

In the interpretation of experimental results from model to prototype it is important to note that in two-dimensional problems: (1) The stress distribution is independent of the elastic properties; and (2) the stresses in the prototype are directly proportional to the stresses at corresponding points in the model.

The discrepancy between the experimental and theoretical curves of Fig. 2(a) may be due partly to incorrectness in the pressure, Q, at the fillet of the model and partly to a slight shop error in the radius of the fillet. Other experiments with larger fillet radii showed much better agreement between experimental and theoretical values of fillet stresses. Very small radii were used in these experiments in order to produce as closely as possible the effect of sharp corners.

Based on the comparison between theory and photo-elastic experimentation the writer concludes that results obtained by the foregoing application of the mathematical theory for isotropic media which behave in accordance with Hooke's law are reasonably correct. It remains only to justify the application to materials, such as concrete masonry, which do not strictly follow Hooke's law. Of course, this can only be done by observations in the field and in testing laboratories. A great many such data are already available, but it is beyond the scope of this paper to treat on this subject. A well placed system of electrical strain-gauges was installed in a section of Morris Dam which, in time, will throw further light on this point, on the question of shrinkage effects, and on lateral stresses in long gravity dams.

DISCUSSION

I. K. SILVERMAN,[16] JUN. AM. SOC. C. E. (by letter).—The first solution for the stresses in a gravity dam based on the equations of the theory of elasticity was given by Lévy.[17] However, this solution holds only for an infinite wedge and does not take into account the elastic action of the base. Lévy's solution coincides with the ordinary engineering methods for the determination of the stresses in triangular dams and, at the same time, gives the law for the distribution of the shearing stresses. Until recent times very little experimenal work has appeared in English or American engineering literature on this important question. The experiments[18] of Ottley, Brightmore, Wilson, and Gore made in 1907, on this type of structure, were outstanding. On the other hand, there have been many theoretical investigations and among the outstanding ones are those of Wolf,[19] Vogt,[20] and Kalman.[21] The latter investigators have shown definitely that the influence of the foundation cannot be neglected and that the stresses in the lower third of the dam are greatly affected by its elastic behavior. The author has pointed to this fact and has furnished a valuable tool in the form of the function given by Equation (50), for the determination of the qualitative effect of the restraining action of the base. This function is important since it furnishes an infinite number of free constants which may be used to satisfy any number of elastic or stress conditions. Thus, it is to be seen that the stresses may be determined to any degree of mathematical accuracy. It is to be noted that Equation (43) is a solution of Equation (3*b*) and, therefore, describes a state of stress which is in equilibrium and compatible with Hooke's law.

This very important function may be determined by the usual methods for the solution of partial differential equations when the form of the solution is indicated by the nature of the problem. Consider the case of a rectangular plate[22] with its sides parallel to the OX and OY-axes in Fig. 1. The stresses on the sides, $y = \pm c$, are described as follows:

For $y = + c$:

$$\sigma_y = - B \sin \frac{m \pi x}{l} \ldots\ldots\ldots\ldots\ldots\ldots (83a)$$

and for $y = - c$,

$$\sigma_y = - A \sin \frac{m \pi x}{l} \ldots\ldots\ldots\ldots\ldots\ldots (83b)$$

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[16] With U. S. Bureau of Reclamation, Denver, Colo.

[17] *Comptes Rendus*, Vol. 127, 1898, pp. 10–15. For a complete discussion concerning the stresses in gravity dams, see "Stresses in Gravity Dams by the Principle of Least Work", by B. F. Jakobsen, and the discussion thereon, *Transactions*, Am. Soc. C. E., Vol. 96 (1932), pp. 489–591.

[18] *Minutes of Proceedings*, Inst. C. E., Vol. CLXXII, 1907–08 Session, Pt. II, 1908.

[19] "Zur Integration der Gleichumg $\nabla^4 F = 0$ durch polynome im Falle des Staumauer Problemes" *Sitzungberichte* der Kaiserlichen Akademie der Wissenschaften in Wien, 1914. Vol. 123, No. 2.

[20] "Ueber die Berechnung der Fundamentdeformation", Det Norske Videnkaps Akademi, Oslo, 1925.

[21] "Sulla validità del regime Levy nelle dighe del tipo di gravità", *L'Energia Elettica*, March and April, 1927.

[22] "Theory of Elasticity", by S. Timoshenko, Eng. Societies Monographs, McGraw-Hill. 1934, p. 44.

In both cases, $\tau_{xy} = 0$. From the nature of these given stresses and of the shape of the body, it is logical to assume that the Airy function is of the form:

$$F = \sin \frac{m \pi x}{l} Y \dots\dots\dots\dots\dots\dots (84)$$

in which Y is a function of y only. Inserting Equation (84) into Equation ($3a$) an ordinary differential equation for Y is obtained, as follows:

$$\alpha^4 Y - 2 \alpha^2 \frac{d^2 Y}{dx^2} + \frac{d^4 Y}{dx^4} = 0 \dots\dots\dots\dots\dots\dots (85)$$

in which $\alpha = \frac{m \pi}{l}$. The solution of Equation (85) contains four constants of integration which may be determined from the boundary conditions expressed by Equations (83) and (84). The solution of Equation (85) is:

$$Y = C_1 \sinh \alpha y + C_2 \cosh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y \,.. (86)$$

The Airy function is:

$$F = \sin \alpha \times [C_1 \sinh \alpha y + C_2 \cosh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y] . (87)$$

Similarly, when the structure dealt with is bounded by two concentric circles or parts of concentric circles and the loads on the boundaries are expressed as functions of the angle, θ, the form of the Airy function is $F = R \sin n \theta$, or,

$$R \cos n \theta \dots\dots\dots\dots\dots\dots\dots\dots (88)$$

in which R is a function of the variable, r, only. Inserting Equation (88) into Equation ($3b$) an ordinary differential equation is obtained for R. The solution of this equation yields the following Airy function.[23]

$$\begin{aligned} F = {} & a_0 \log r + b_0 r^2 + c_0 r^2 \log r + d_0 r^2 \theta + \frac{a_1}{2} r \theta \sin \theta + a'_0 \theta \\ & + (b_1 r^3 + a'_1 r^{-1} + b'_1 r \log r) \cos \theta - \frac{c_1}{2} r \theta \cos \theta \\ & + (d_1 r^3 + c'_1 r^{-1} + d'_1 r \log r) \sin \theta \\ & + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a'_n r^{-n} + b'_n r^{-n+2}) \cos n \theta \\ & + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c'_n r^{-n} + d'_n r^{-n+2}) \sin n \theta \dots\dots (89) \end{aligned}$$

in which the constants of integration are to be determined from the boundary conditions.

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[23] "Theory of Elasticity", by S. Timoshenko, Eng. Societies Monographs, p. 114.

The most suitable function for the stresses at a sharp corner, of the type shown in Fig. 4, is of the form:

$$F = r^{n+1} f \qquad (90)$$

in which f is a function of the variable, θ, only. Insertion of Equation (90) into Equation (3*b*) gives the following ordinary differential equation for f:

$$\frac{d^4 f}{d\,\theta^4} + 2\,(n^2 + 1)\,\frac{d^2 f}{d\,\theta^2} + (n^2 - 1)^2 f = 0 \qquad (91)$$

The solution of Equation (91) is:

$$f = c_1 \cos\,(n-1)\theta + c_2 \sin\,(n-1)\theta + c_3 \sin\,(n+1)\theta + c_4 \cos\,(n+1)\theta . \quad (92)$$

The constants of integration and the values of n that remain to be determined are obtained from the boundary conditions on the faces, $\theta = 0$ and $\theta = \gamma$. Expanding the terms of Equation (92), the resulting formula is identical with Equation (43) in which A, B, C, and D are constants of integration replacing c_1, c_2, c_3, and c_4 of Equation (92).

For the boundary conditions — $\theta = 0$; $\sigma_\theta = 0$; $\tau_{r\theta} = 0$; and $\theta = \gamma$; $\sigma_\theta = 0$; $\tau_{r\theta} = 0$ — a set of homogeneous linear equations is obtained, namely;

$$B = 0 \qquad (93)$$

$$A\,n + C = 0 \qquad (94)$$

$$D \sin n\,\gamma + A \sin n\,\gamma \cos \gamma + C \cos n\,\gamma \sin \gamma = 0 \qquad (95)$$

and,

$$D\,(n \cos n\,\gamma \sin \gamma + \sin n\,\gamma \cos \gamma) + A\,(n \cos n\,\gamma \cos \gamma - \sin n\,\gamma - \sin \gamma)$$
$$+\ C\,(\cos n\,\gamma \cos \gamma - n \sin n\,\gamma \sin \gamma) = 0 \qquad (96)$$

The simultaneous solution of Equations (93) to (96) gives only relations between the constants, namely, $B = 0$; $\frac{C}{A} = -\,n$; and, $\frac{D}{A} = -\,(\cot \gamma - n \cot n\,\gamma)$. Assuming that $A = -\,1$, then $C = n$; and $D = (\cot \gamma - n \cot n\,\gamma) = m$, which values are identical with those of the author. Furthermore, in order that the constants may differ from zero, the following relation must hold true:

$$n^2 \sin^2 \gamma = \sin^2 n\,\gamma \qquad (97)$$

By means of Equation (97) any number of functions of Equation (43), each with a free constant, may be obtained to satisfy conditions on boundaries other than $\theta = 0$ and $\theta = \gamma$, as shown by Equation (50).

The function expressed by Equation (43) has appeared from time to time as special cases in engineering literature, but no general form has been given. H. M. Westergaard, M. Am. Soc. C. E.,[24] has applied this type of function when $\gamma = 360$ degrees. The function took the following form,

$$F = \sum_{n=1,2,3\ldots}^{n} K_n \frac{r^{n+1.5}}{2\,(n+0.5)\,(n+1.5)} \Big[(n-0.5) \sin (n+1.5)\,\theta - (n+1.5) \sin (n-0.5)\theta \Big] \ldots\ldots\ldots\ldots\ldots\ldots (98)$$

in which the coefficients, K_n, are free constants.

FRED L. PLUMMER,[25] ASSOC. M. AM. SOC. C. E. (by letter).—It is unfortunate that so few practicing engineers are able to make use of many of the more fundamental and more generalized stress functions. In illustrating so effectively one application of the Airy stress function the author has performed a difficult but exceptionally valuable service to the profession. The writer wishes to discuss Part II of the paper which describes the use of photo-elastic studies in determining the distribution of principal and shearing stresses in dams. This method of analysis has many proper applications in the field of structural engineering and should be more generally understood and used by designing engineers. In many cases, quite crude apparatus will give qualitative results of great value.

The author calls attention to the difficulties encountered in attempting to simulate body forces (due to weight and inertia) in the model. The writer has met this problem in two ways. In one study a model was rotated together with a slotted wheel giving a stroboscopic effect. This method is similar to that suggested in the paper and gave satisfactory results. The apparatus is somewhat elaborate, however, and requires careful adjustment.

A much more direct solution may be effected by the use of models made of gelatin. This material may be secured in either "ground" or "sheet" form, at little cost. It should be soaked in cold water for at least 1 hr and then heated to a temperature of not more than 150° F. The concentration of the solution can be varied from 2% or 3% to about 50% (by weight), giving a wide range to the strength and stiffness of the mass which solidifies upon cooling. The stress-strain-optical properties are such as to make the material quite suitable for photo-elastic studies when body forces are of importance. These properties vary, however, as the material ages, and it is necessary, therefore, to make calibration tests at the time of the model tests. The writer has found concentrations of from 10 to 20% most suitable for models of gravity dams. Such models may have thicknesses as great as 6 in., as compared with a fraction of an inch which is common for models made of bakelite. This fact, together with the greater stress-optical sensitivity of the material, makes easy the determination of the stresses caused by the weight of the material itself. The use of much larger models is also made practical.

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[24] "Stresses at a Crack, Size of the Crack, and the Bending of Reinforced Concrete" by H. M. Westergaard, *Journal*, Concrete Inst. November–December, 1933, p. 93.

[25] Assoc. Prof., Structural Eng., Case School of Applied Science, Cleveland, Ohio.

The writer has made several studies using this type of model. A brief description of one such study conducted in co-operation with the Zanesville, Ohio, Office of the U. S. Corps of Engineers may be of general interest. Major J. D. Arthur, Corps of Engineers, U. S. Army, District Engineer, T. T. Kappen, Assoc. M. Am. Soc. C. E., Chief of the Engineering Division, A. L. Alin, M. Am. Soc. C. E., Chief of the Dams and Reservoirs Division, and R. R. Philippe, Director of the Soils Laboratory, were responsible for the planning and conduct of the tests.

The structure under consideration was a gravity type of concrete dam. A typical cross-section of the spillway section is shown in Fig. 14. Two series of photo-elastic model studies were made. Bakelite models were used in the first series, the procedure being very similar to that outlined by the author, except that vertical loads were applied so as to simulate, successively, the proper intensity of weight forces at three different elevations (Elevation 850, Elevation 870, and Elevation 890). Gelatin models constructed to a scale of 1:60 were used for the remaining studies. The weight of the prototype was represented accurately by the weight of the model while the forces due to water pressure were reproduced by the use of a liquid having a weight, in pounds per cubic foot, bearing the same relation to 62.4 as that of the unit weight of the gelatin as tested had to the assumed unit weight of the concrete. Liquid pressures were applied directly to the model through a thin rubber diaphragm. It is believed that the external forces as well as the body forces were thus represented accurately. It was possible to study the changes in the combined stresses as the water level might rise in the prototype from empty reservoir to maximum flood level estimated in this case as Elevation 931, which corresponds to 15 ft of water over the top of the spillway. By varying the relative strength and stiffness of the mass of gelatin representing the foundation material, it is possible to study the effect of foundation materials of varying strength characteristics upon the stresses near the base of the dam.

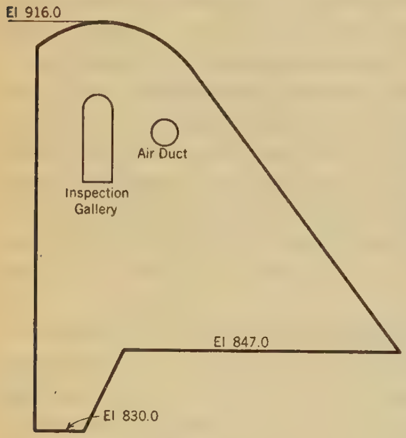

Fig. 14.

In this case it was especially desired to determine the stresses in the material around the openings for the air duct and inspection gallery, and in the cut-off anchor wall which was provided because of horizontal layers of soapstone that occurred in the foundation material. The results of the tests were highly satisfactory, the final design of steel reinforcement at these points being based on the test results.

Arshag G. Solakian,[26] Esq. (by letter).—A description of the theoretical and experimental analysis of stresses in dams due to body forces and hydrostatic pressure is contained in this paper. Undoubtedly, it serves "to familiarize engineers with the use of the Airy stress function for the solution of problems in plane stress and plane strain * * *".

Much has been published in the United States concerning the optical principles involved in the art of stress analysis by means of polarized light and transparent materials, especially as applied to the determination of stresses in structural and machine elements. In this respect, the paper does not add considerably to the technique of photo-elastic measurements. Furthermore, the only experimental results offered are the boundary stress distribution curves and a photograph of an isoclinic band (which only serves to give the direction of the principal stresses along its path (Fig. 13)). The most useful patterns of stress fringes, which give a concrete idea as to the distribution of stresses in the model under specific types of loading, are omitted from the paper.

The method of conducting the experimental investigation (that is, of substituting body forces in the model by equivalent external loads) is not sufficiently accurate to justify its general adoption. It is for this reason that Mr. Brahtz finds that in certain cases (see following Equation (8)), in the upper two-thirds of the dam the stresses obtained agree with experimentation, and, in other cases, in the base region the boundary stresses are in close agreement with the approximate methods used in Part I. This indicates that for no special case of loading do the stresses in the entire field agree with those obtained by theory. An important objection to the superposition method of finding the resultant stresses (due to (1) the weight of the dam; (2) the vertical pressure of water over the top of the foundation; and (3) the lateral pressure of water against the face of the dam in contact), is that the principal stress orientations for the three different types of loading are different. Consequently, the investigator must go through a tedious analytical and graphical solution, which leads to results of questionable precision.

As Mr. Brahtz suggests, "the easiest method would be to rotate the model and observe it whenever it passes through the optical field". This is an exact method and has already been attempted with success in a similar problem, by making use of the centrifuge-polariscope apparatus developed at Columbia University[27], in New York City. A box containing the model of a retaining wall[28], with sand acting as back-fill, is allowed to rotate at constant speed to produce increased gravitational effects. A steady image of the stress fringes in the model is projected on the screen by means of a stroboscopic polarized light arrangement. The screen is stationary as in an ordinary polariscope of transmission type. The model and the sand are oriented in the box (see Fig. 15) in such a manner as to produce in the structure the resultant effect

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[26] Lecturer in Civ. Eng., Columbia Univ., New York, N. Y.

[27] "Centrifugal Method of Testing Models", by P. B. Bucky, A. G. Solakian, and L. S. Baldin, *Civil Engineering*, May, 1935, p. 287.

[28] *Loc. cit.*, Fig. 3, p. 288.

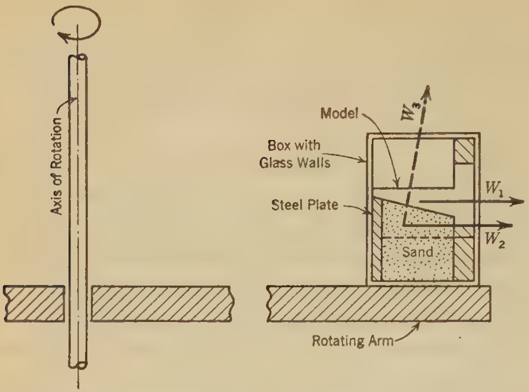

Fig. 15.—Arrangement of Model and Sand in the Centrifugal Machine.

(as to magnitude and direction of principal stresses) of the weight of the dam, W_1, the vertical pressure of the sand, W^2, and the lateral thrust of sand, W_3. A fringe pattern[27] of the stresses obtained by this method, with a bakelite model, and sand back-fill rotating at a speed of 1 500 rpm in a centrifugal field having a gravitational strength of 485, has been published elsewhere[29]. Because of the limited speed and length of the rotating arm of the centrifugal machine and the size of the model made of bakelite having a moderate optical sensitivity to applied stress, the number of fringes produced in the pattern are not sufficient for an exact determination of the stresses. A new photo-elastic material[30], having an optical sensitivity about three times greater than that of bakelite, would have permitted sufficient data to be secured from a pattern obtained under similar conditions, for an exact solution of the problem.

Lars R. Jorgensen,[31] M. Am. Soc. C. E. (by letter).—In adding to the knowledge of gravity dam design the author has done an immense amount of work. Except, perhaps, for the multiple-arch, and the Ambursen types, the design of dams is not yet an exact science, and there is actual need for more work in this field.

The actual safety of a gravity dam is made up principally of the following items: The weight acting downward on the foundation opposing any tendency to slide; the capacity of the material to withstand vertical tension at and near the up-stream face; and the shearing strength of the material, generally concrete. Since no gravity dam is known to have failed by crushing of the material, dangerous compressive stresses apparently do not develop in that type; this also checks with theory.

Since the vertical tensile strength of the bond between horizontal joints is a very uncertain quantity, just as is the shearing strength of such joints, although perhaps to a lesser extent, these items are ordinarily neglected, as they should be, when computing the safety factor of a gravity dam. For such a structure, the weight times the coefficient of friction must be greater than the water load, in order to give a factor of safety greater than one, when the structure is acting strictly as a gravity dam.

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[29] *Civil Engineering*, May, 1935, p. 290.

[30] "A New Photoelastic Material", by A. G. Solakian, *Mechanical Engineering*, December, 1935.

[31] Cons. Hydro-Elec. Engr., Berkeley, Calif.

The writer has recently had an opportunity to make studies on a proposed gravity dam, 173 ft high, of triangular cross-section (top width, 20 ft) having a suggested base width of 75% of the height. In this case, the water load divided by the weight of the section was 0.55. Since the coefficient of friction may be between 0.6 and 0.75, this section acting as a gravity dam may have a factor of safety between 1.09 and 1.36, provided there is no uplift and no back-water. Such low factors of safety would not be considered in any other kind of a structure. On most well-built dams the factor of safety may be higher than the foregoing values indicate, but if it is, that fact will be due to the tensile and shearing strength of the concrete and not to gravity action.

Tests on models[32] have shown that stresses in dams are not linearly distributed as is generally assumed, but that tension does exist in the up-stream face of a loaded dam at and toward its junction with the foundation. These particular models had a base width of 77% of the height. The existence of tensile stresses in the up-stream face of a loaded gravity dam was also shown theoretically by Mr. B. F. Jakobsen[33], at least approximately, and by Ivan E. Houk, M. Am. Soc. C. E.[34]

A practical demonstration of the more than probable existence of tensile stresses in the up-stream face of a loaded so-called gravity dam is contained in the record of the failures of 68 gravity dams, with or without curvature, during the last 136 yr, or at the average rate of 1 every 2 yr. The best method for preventing such failures in the future is to accept the existence of these tensile stresses and provide gravity dams with a wider section than is customary at present.

The author shows a section of the Morris Dam on the San Gabriel River, in California. This dam would scarcely have any tension in the up-stream face at and near the foundation, since its base is about 86% of its height. With its Levy cross-section, this dam is perhaps the most conservative gravity dam in California, and, undoubtedly, is safe under any condition, since the factor of safety of such a section always will remain greater than one, without having to depend upon any help from the tensile strength or shearing strength of the concrete.

How far one can go below a Levy section is a matter of judgment, but any reduction below such a section will be at the expense of the safety factor. Many engineers seem to have an unlimited faith in any dam called a gravity dam; undoubtedly, that confidence is well placed when the base width is 86% of the height, but for base widths of 66.6% of the height—which a considerable number of dams have—that confidence is badly placed. The actual factor of safety of such a section will depend upon the capacity of the concrete to take tension in the up-stream face. Gravity action alone will not produce a factor of safety worth mentioning.

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[32] "Experimental Investigation of the Stresses in Masonry Dams", by J. S. Wilson and William Gore, *Minutes of Proceedings*, Inst. C. E., Vol. CLXXII, 1907–08.

[33] "Stresses in Gravity Dams by Principle of Least Work", *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 489.

[34] *Western Construction News*, April 10, 1932, p. 190.

Whether arch action increases the stability of a gravity dam arched in plan, and to what extent, at least in winter, is a debatable question and one not easy to answer. The question is even disputed internationally. The French regulations for building dams discourage the practice of curving gravity dams in plan, whereas the Italian regulations encourage it. The Italian regulations even go so far as not to require contraction joints, whenever a gravity dam is curved in plan.

The latest dam failure was on the Orba River, in Italy, where a gravity dam about 40 ft high went out on August 13, 1935, causing a great loss of life and property below it. A cloudburst brought down more water than the spillways could take care of and forced the surplus to flow over the crest. The section of the dam was quite ample for the normal water load, but not for the increased water load, which was estimated at 50% above the normal. This shows that the normal factor of safety could not have been more than 1.50 from all sources, since this factor of safety was completely cancelled by a 50% increase in the water load.

When a dam fails, there is generally much more involved than just the cost of the structure. It is especially important to satisfy the worst condition of stress distribution that can exist in the structure, rather than the best. Several of the recent gravity dam failures point to the fact that tension does exist in the up-stream face at, and near, the bottom of a loaded dam of less than the Levy section. The assumption of linear stress distribution, or of the author's stress distribution, is not on the safe side, although no fault can be found with the mathematics in the paper. If it was safe, several of the recent gravity dam failures would not have happened.

The earlier French regulations for building gravity dams contained the provision that, at all points of the up-stream face, such dam sections must have a pressure at least equal to the water pressure. This requirement produces the so-called Levy section having a base width of about 85% of the height. The new rules allow smaller sections, but for full reservoir there must still be compression in the concrete at the up-stream face.

Especially when it is subject to high-water pressure, concrete deteriorates with age, this deterioration being most marked in the up-stream section, where it is water-soaked. In many dams, deposits of lime have been shoveled out of the inspection galleries. This material can have come from no other place than the section of dam lying up stream of the gallery. The deterioration is most pernicious in its effect on the tensile strength, which is most needed relatively in this portion. If the section is so slim that the factor of safety depends in any great degree on the tensile strength of the concrete, then the dam is likely to be in distress at a fairly early age. The best means to avoid tensile stresses in the up-stream face is to provide an ample section. If such a section is much less than a Levy section, especially on the side-hill part of the dam, the factor of safety will only be an uncertain amount greater than one, acting as a gravity dam. Sixty-eight failures of gravity dams bear ample witness to this fact.

Elmer O. Bergman,[35] M. Am. Soc. C. E. (by letter).—The treatment of the stresses due to the weight of the structure, as proposed in this paper, is a particularly valuable contribution to the subject. The author is to be commended, furthermore, for his able presentation of the Airy stress function and its application to the analysis of civil engineering structures.

In the past the Airy stress function has been applied mainly to the solution of problems of interest to the mechanical engineer in which the stresses due to weight can be neglected in comparison with those set up by the applied loads. Another important point of difference between the problem treated by the author and those dealt with in the literature of elasticity lies in the disparity of cost between a dam and the ordinary structural and machine elements. Because the solution of all but the simplest problems of elasticity is involved and laborious, an effort is made to obtain a design formula that is relatively easy to apply and is on the side of safety. This necessarily results in the use of more material which, in a large dam, might result in a considerable increase in cost. The author has developed an approximate solution in Application IV, but he recommends that its use be limited to preliminary work.

Any variable which is a function of two independent space variables can be represented by a surface. The surface represented by the Airy stress function is called the Airy surface. The curvatures and twists of such a surface using a small scale for F are given by Equations (1) and hence afford a geometrical representation of the stresses. It should be noted that, in this interpretation, the curvature of the surface along any section is related to the normal stress acting across that section. Thus, the curvature at a point in the direction parallel to x represents the stress, σ_y, at that point.

The condition that Equations (6) reduce to Equations (7) is that the x, y plane be tangent to the Airy surface at the origin. When no body forces are acting, the value of the Airy stress function, F, at any point (by Equation 7(c)), represents the moment of the forces between the origin and the point about the point. Thus, if the origin is taken as in Fig. 1, the moment at B is the moment about B of the internal and external forces acting along any line connecting the origin with B.

The Airy surface, therefore, is a two-dimensional analogue of the familiar bending moment diagram. The slope of the tangent line to the surface at any point represents the sum from the origin to the point of the forces perpendicular to the tangent line. Thus, the Airy surface furnishes a helpful means of visualizing the moments, shears, and stresses in the slab.

The writer has been connected with the testing and analysis of several models of dams constructed of a compound of commercial building plaster and diatomaceous earth, a material very similar in action to concrete, but having a much lower modulus of elasticity. The models consisted of thin slices from the central part of the dam where they can be considered as being in a state of plane strain. As tested, the models were in a state of plane stress. Under these conditions there is direct correspondence between the stresses in the model and prototype, but not between the displacements and the strains. (For the relations between displacements, see the author's discussion relating to Equations (9) to (12).) The strains due to dead load and live load were measured on four intersecting gauge lines at points in the dam and its foundation with optical strain-gauges.

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Vertical, horizontal, shearing, and principal stresses were computed from these strains and the elastic constants of the material. The results of these tests can not be compared directly with those of the author because they were not on the same section, but they show the same trend of action as indicated in Figs. 6 and 7, except that, due to plastic flow in the material of the model under high stress, the high concentration of stress at the heel and toe of the dam did not appear.

D. P. KRYNINE,[36] M. AM. SOC. C. E. (by letter).—The use of the Airy stress function in dam engineering is discussed in this highly interesting and important paper. The writer's attention has been especially attracted by the determination of stresses in foundations (Application II).

Limits of Application of the Stress Function to Dam Foundations.—As stated in the "Introduction" the use of the stress function is restricted to elastic isotropic materials. Granite or basalt rock may be considered as elastic and, to a certain extent, isotropic; and, consequently, elastic theories may be applied in the case of such foundations. The situation is more complicated if the foundation material is sandstone, for example, since it is questionable whether in all cases such a material satisfies the conditions of both elasticity and isotropy. On the other hand, a clay foundation would justify the application of the stress function, whereas, in the case of glacial drift, such an application would be misleading since the actual stress distribution is far different from the patterns given by the theory of elasticity.

The use of the stress function is always limited to the two-dimensional problem. In other words, satisfactory results may be expected from the application of the stress function to sections at some distance from the ends of the dam, but not otherwise. Near the ends, the problem is three-dimensional, and the stress function is not applicable.

Within these limitations Mr. Brahtz has demonstrated, in a comprehensive and intelligent manner, that stresses in a dam foundation can be determined by means of the stress function. It still remains to be proved that they should be thus determined and the writer is rather pessimistic in this respect.

Stress Distribution Under a Foundation.—It can be demonstrated that the value of the stress, σ, caused by a concentrated force, *P*, acting at the boundary of a semi-infinite earth mass (Fig. 16), is:

$$\sigma = \frac{\nu\, P}{2\,\pi\,\rho^2}\sin^{\nu-2}\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (99)$$

Equation (99) has been proposed by John H. Griffith[37], M. Am. Soc. C. E., and Dr. O. K. Fröhlich.[37] It can be developed without making use of elastic theories. Hence, it is valid for any isotropic or statistically isotropic body. The coefficient, ν, is the so-called "concentration factor", and is equal to 3,

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[37] "Pressures Under Substructures", by John H. Griffith, M. Am. Soc. C. E., *Engineering and Contracting*, March, 1929, pp. 113–119; also, "Drukverdieling in Bouwgrond", by Dr. O. K. Fröhlich, *De Ingenieur*, April 15, 1932, p. B–52.

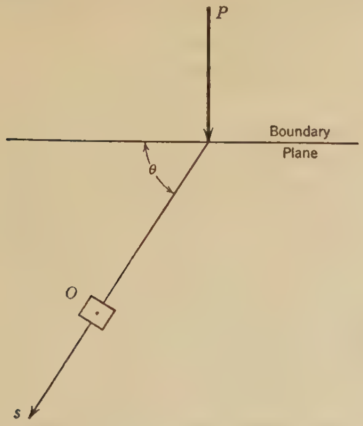

FIG. 16.

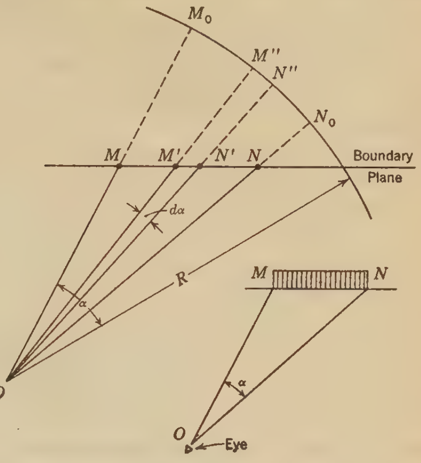

FIG. 17.

for the case of an isotropic elastic body in which the value of the reciprocal of Poisson's ratio, m, is equal to 2. For this case Equation (99) becomes:

$$\sigma = \frac{3\,P}{2\,\pi\,\rho^2} \sin\theta \quad \text{......................} (100)$$

Equation (100) may be obtained from those developed by Boussinesq[6], assuming that $m = 2$. A two-dimensional analogue to Equation (100) is Michell's expression for radial distribution[38]:

$$\sigma = \frac{2\,P}{\pi\,\rho} \sin\theta \quad \text{......................} (101)$$

The plane stress distribution does not depend on the value of Poisson's ratio, μ, and Equation (101) holds in the case of any value of that ratio.

Concentrated Forces at the Boundary of a Mass.—The problem in connection with Fig. 3 of the paper, in the case of a two-dimensional elastic body, may be treated in the form of a general case (that is, for any three-dimensional body, isotropic or statistically isotropic), by applying Equation (99). In the case of a horizontal force, Q, the solution may be written directly[39]:

$$\sigma_x = -\frac{\nu\,Q}{2\,\pi\,\rho^2} \cos^{\nu}\theta \quad \text{..................} (102a)$$

$$\sigma_y = -\frac{\nu\,Q}{2\,\pi\,\rho^2} \cos^{\nu-2}\theta \sin^2\theta \quad \text{............} (102b)$$

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[38] "Theory of Elasticity", by S. Timoshenko, p. 82, 1934.

[39] "Druckverteilung im Baugrunde", by O. K. Fröhlich, p. 27, 1934.

and,

$$\tau_{x,y} = -\frac{\nu Q}{2 \pi \rho^2} \cos^{\nu-1} \theta \sin \theta \ldots\ldots\ldots\ldots (102c)$$

A two-dimensional analogue to Equations (102) for the particular case in which $\nu = 3$ leads directly to Equations (32) of the paper.

In the same manner the stresses caused by a vertical force, P, acting at the boundary are:

$$\sigma_x = -\frac{\nu P}{2 \pi \rho^2} \sin^{\nu-2} \theta \cos^2 \theta \ldots\ldots\ldots\ldots (103a)$$

$$\sigma_y = -\frac{\nu P}{2 \pi \rho^2} \sin^{\nu} \theta \ldots\ldots\ldots\ldots (103b)$$

and,

$$\tau_{x,y} = -\frac{\nu P}{2 \pi \rho^2} \sin^{\nu-1} \theta \cos \theta \ldots\ldots\ldots\ldots (103c)$$

For the case of a two-dimensional elastic mass the three-dimensional formulas (Equations (103)) may be easily reduced to those given by Mr. Brahtz immediately preceding Equation (37).

Uniform Load Distribution at the Boundary of a Two-Dimensional Mass; "Angle of Visibility".—The problem of a uniformly distributed load acting at the horizontal boundary of a mass may be also solved by applying the general formula, Equation (99). For the purposes of this discussion, a two-dimensional elastic mass only will be dealt with and Equation (101) will be applied. In Equations (38) and (39) Mr. Brahtz gives the values of the stresses, σ_x, σ_y, and $\tau_{x,y}$, as functions of two angles, α_1 and α_2. It is more convenient to use a single angle, $\alpha = \alpha_1 + \alpha_2$ (see Fig. 17). The angle, α, may be termed the "angle of visibility" since this is the angle at which a hypothetical observer placed at the given point, O, of the mass, would see the foundation if the mass were transparent. The writer understands that the conception of the "angle of visibility" was first introduced by Professor N. M. Gersevanov.[40] Tracing a circle of an arbitrary radius, R, with Point O as a center, and considering an elementary angle of visibility, $d\alpha$, corresponding to a loaded element, $M' N'$, at the boundary, the elementary stress, $d\sigma$, at Point O, would be, using Equation (101):

$$d\sigma = -2p \left[\frac{\rho \, d\alpha}{\cos \theta}\right] \frac{1}{\pi \rho} \cos \theta = -\frac{2p}{\pi} d\alpha \ldots\ldots\ldots (104)$$

Were the mass cylindrical (that is, circular in cross-section) with a radius, R, and were the arc, $M_0 N_0$, loaded normally with the unit load, p, the corresponding elementary stress, $d\sigma$, at Point O, would be:

$$d\sigma = -2p \, [R \, d\alpha] \frac{1}{\pi R} = -\frac{2p}{\pi} d\alpha \ldots\ldots\ldots (105)$$

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[40] "Principles of Dynamics of an Earth Mass", by N. M. Gersevanov, p. 153 (printed in Russian, in 1933).

since in this case the force, $p\ [R\ d\alpha]$, acts normally to the circular boundary of the mass, and the angle, θ, equals zero. Integrating Equations (104) and (105) through the entire angle, α, it may be stated that the sum of the principal stress, σ, at Point O, is proportional to the "angle of visibility", α:

$$\sigma = -\frac{2\,p}{\pi}\,\alpha \dots\dots\dots\dots\dots\dots\dots\dots (106)$$

It is also obvious that the sum of principal stresses at a point does not depend on the shape of the loaded surface provided the "angle of visibility", α, of the loaded surface remains the same, and the surface in question is loaded uniformly and normally.

It may be seen from Fig. 17 that using the "angle of visibility" an asymmetrical loading problem (loaded surface, $M\,N$) is reduced to a symmetrical problem (loaded surface, $M_0\,N_0$). This fact permits one to state directly that the direction of the major principal stress coincides with the bisector of the "angle of visibility." It is well known that the proof of this simple statement, as given in textbooks on elasticity, is rather complicated.

Projecting all the elementary stresses, $d\sigma$, on the normal to a plane, and integrating, the pressure normal to that plane is obtained. Thus:

$$d\sigma_x = -\frac{2\,p}{\pi}\,d\alpha\,\cos^2\theta = -\frac{p}{\frac{1}{2}\,\pi\,R^2}\,[R\,d\alpha\,\cos\theta]\,[R\,\cos\theta] \dots (107a)$$

and,

$$d\sigma_y = -\frac{2\,p}{\pi}\,d\alpha\,\sin^2\theta = -\frac{p}{\frac{1}{2}\,\pi\,R^2}\,[R\,d\alpha\,\sin\theta]\,[R\,\sin\theta] \dots (107b)$$

It may be seen from Fig. 18(a) that $[R\ d\alpha\ \sin\theta]$ and $[R\ d\alpha\ \cos\theta]$ are projections of the arc element, $R\ d\alpha$, on horizontal and vertical planes, respectively. In the same manner $[R\ \cos\theta]$ and $[R\ \sin\theta]$ are projections of the

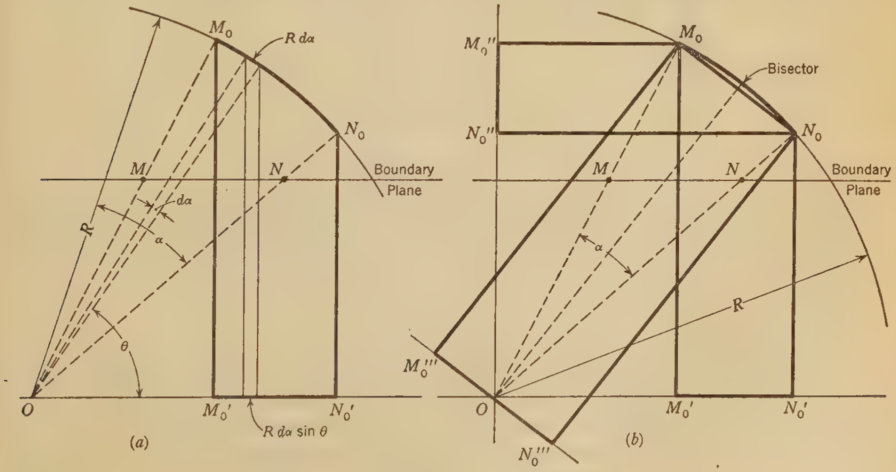

FIG. 18.

radius, R (also on horizontal and vertical planes, respectively). Designating the half area of a circle of a radius, R, by A, and integrating (Fig. 18(b)):

$$\sigma_x = -p \frac{\text{Area } M_0 M''_0 N''_0 N_0}{A} \quad \text{..............(108}a\text{)}$$

and,

$$\sigma_y = -p \frac{\text{Area } M_0 M'_0 N'_0 N_0}{A} \quad \text{..............(108}b\text{)}$$

Projecting now the arc, $M_0 N_0$, on the normal to the bisector of the "angle of visibility", the values of the principal stresses, σ_1 and σ_2, may be obtained. The minor principal stress, σ_2, is proportional to the area between the chord, $M_0 N_0$, and the arc, $M_0 N_0$. The major principal stress is proportional to the sum of that area and the rectangle, $M_0 M'''_0 N'''_0 N_0$. Shearing stresses may be easily obtained in the same way. Thus, the determination of stresses is reduced to a simple measurement of areas by planimeter. This method may be extended to the case of non-uniformly distributed vertical loads, which represents a problem that is almost impossible to attack by the use of the stress function method. Fig. 19(a) illustrates the determination of the stress, σ_y, in such a case. Each ordinate of the area, $M_0 M'_0 N'_0 N_0$, is to be multiplied by the ratio, $\frac{p_1}{p}$, of the actual unit load, p_1, at a given point of the foundation, to a certain standard unit load, p.

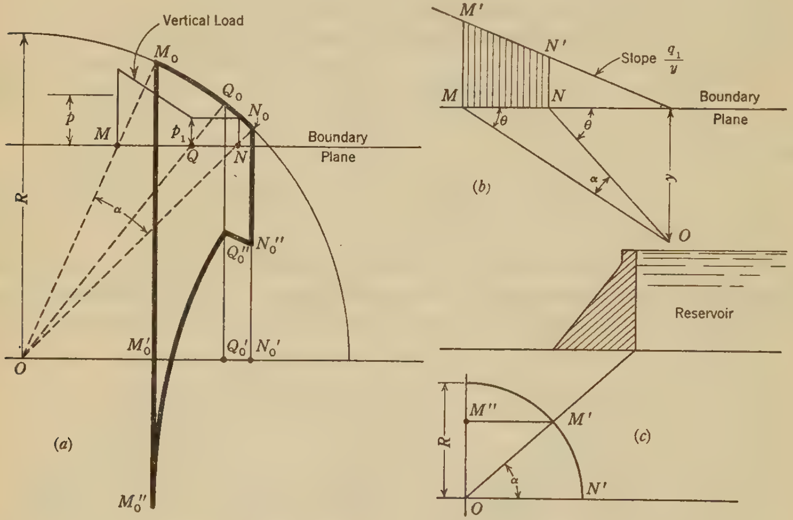

FIG. 19.

A horizontal force, Q, and a vertical, $P = Q \cot \theta$, produce equal stresses at a given point which follows from a simple inspection of the formulas expressing the stresses, σ_x, σ_y, and $\tau_{x,y}$. The problem of uniform distribution of a horizontal force, q_1, per unit of length of the distance, l_1, may be thus reduced to that of a non-uniform distribution of the vertical unit force,

$q_1 \cot \theta$, over the same distance, l_1. A straight line is drawn with its origin, O', at the boundary of the mass (Fig. 19(b)). The slope of this line is $\frac{q_1}{y}$, in which y is the depth of the given point, O. The "angle of visibility", α, would determine an area, $M\,M'\,N'\,N$, showing the law of distribution of the equivalent vertical load over the distance, $M\,N = l_1$. The problem is afterward solved according to Fig. 19(a) thus avoiding Equations (34). The problem of a non-uniform distribution of a horizontal force, q_1, may be also solved in a similar manner.

Reservoir Pressure.—This pressure determined by Mr. Brahtz analytically, may be readily found by graphics. Fig. 19(c) shows the graphic determination of the stress, σ_x, in the case of a long reservoir when one of the sides of the "angle of visibility" is practically horizontal. The stress in question is proportional to the area, $O\,N'\,M'\,M''$. The value of the radius, R, is arbitrary.

Value of the Poisson Ratio of a Dam Foundation and Displacement Formulas.—The author introduces the symbol, μ_2, as the Poisson ratio of the material comprising the dam foundation. One of the basic principles of soil mechanics is the incompressibility of soil particles. In other words, a soil particle is assumed to be capable of a change in its shape, but not its volume, in which case the value of Poisson's ratio would be 0.5. On the other hand, earth masses under action of their own weight produce lateral pressure, and the ratio of the lateral pressure to the vertical, or the "coefficient of pressure at rest", K, may equal 0.25 to 0.20, for example. If the elastic formula, $\kappa = \frac{\mu_2}{1 - \mu_2}$, is used for determining this coefficient of pressure at rest, the value of μ_2 would be one-fifth or one-sixth. Thus, a substance would have two different values of Poisson's ratio, which is obviously impossible. In the writer's opinion the conception of a "Poisson ratio" as applied to foundations, conveys a correct idea only in the case of elastic, dense rocks, such as some granites or basalts, and should not be used in other cases except, possibly, in the case of stiff clay foundations.

As to the displacement formulas, such as Equation (36) or Equation (40), in addition to containing an indefinite value of the Poisson ratio, μ_2, they are objectionable from the following points of view:

First.—They have been developed under the assumption that the earth mass is weightless and obeys Hooke's law unconditionally. An earth mass, however, has been under the action of its own weight perhaps for millions of years; elastic deformations have long since occurred, and the construction of a dam involves not merely the application of a relatively light load for which Hooke's law holds, but a further considerable increase in stress which, at some depth, probably exceeds the elastic limit. At such high ranges of loading, deformations increase more rapidly than the corresponding loads, even in perfectly elastic bodies, and in the given case there is a danger of under-estimating the value of the displacement if Equation (36) or Equation (40) is used. Such an under-estimation may be termed a "positive error."

Second.—A displacement connotes a movement, or translation, of matter. In order to move even a molecule a certain minimum of energy is required. Hence, it is logical to assume that, at a certain distance from the load, no displacement occurs which is contrary to the conditions of Equations (36) and (40). This fact may cause a "negative error."

Third.—It is known from every-day practice that elastic displacements in earth masses are accompanied by irreversible movements to such an extent that the elastic rebound after unloading a mass, sometimes reaches only 10% or 15% of the original displacement. No means has yet been devised for separating elastic, from inelastic, deformations; and only laboratory experiments furnish some bases for judgment as to the extent of the possible displacement. Positive and negative errors may compensate, and, in some cases, the observed settlement is found to agree closely with the value computed by elastic formulas; but by no means should this phenomenon be accepted as proof that such formulas would furnish satisfactory results in so far as displacements under any conceivable circumstances are concerned.

Shearing Stresses; Isochromatics.—Danger from shear is mostly a surface problem, and shear stresses are to be determined in the upper layers of the mass, close to its boundary surface. There is no danger of shear in deeper strata, where either compression or consolidation occurs. What matters in this connection is the maximum shearing stress, τ_{max}, as determined from the computed values of principal stresses or from isochromatics (see Fig. 12). The values of the shearing stress, $\tau_{x,y}$, acting in the horizontal plane (see heading "Application II.—Stresses in Foundations") are but of little interest for the designer of a foundation.

Attention should be called to Fig. 12(*a*) in which the shape of isochromatics reveals the overloading of the foundation of a dam near its edges. This diagram checks with photo-elastic experiments by other investigators when a punch is pressed against a two-dimensional elastic mass. It should be borne in mind, however, that loading experiments with rigid earth masses (such as sand) reveal overloading at the center of the base of the structure. This induces one to believe that isochromatics, as shown in Fig. 12(*a*), have a restricted meaning only and should be used when there is absolutely no doubt as to both elasticity and isotropy of the foundations, which evidently is not the general case. The same is true of Fig. 12(*b*).

Conclusion.—Applied mechanics, including the theory of elasticity, has been developed for the most part to find the answer to practical questions. It has happened that the attention of the Engineering Profession is principally directed toward machines and framed structures; hence the unusual development of mechanics in the study of such subjects as beams, columns, shafts, plates, disks, etc. The study of "masses" or bodies possessing three great dimensions has been relatively neglected. The masses, in turn, may be subdivided into "unlimited" or semi-infinite and "limited" masses. In the study of stress distribution in semi-infinite masses there have been the works of such investigators as Boussinesq, Michell, Carothers, and, in the past few years, of a number of research workers in soil mechanics. In the study of "limited" masses, such as concrete or masonry, notable progress has occurred lately, mostly due to research incidental to the design of dams, including photo-elasticity. In the latter category, the paper by Mr. Brahtz is a valuable contribution to the study of "limited" masses, and there is every reason to believe that his work along this line and his methods will be used to advantage not only by the builders of dams but also by workers in the allied fields of engineering science, such as soil mechanics.

JOHN H. A. BRAHTZ,[41] ESQ. (by letter).—In closing this paper, the writer wishes to express his appreciation for the many interesting discussions that have appeared. All have been of a high caliber, and illustrate the fact that engineers are getting away from antiquated rules of thumb. With a fuller realization of the importance of two-dimensional stress problems, they are attacking such problems with open minds, more and more leaning toward the theory of elasticity and its co-partner, photo-elasticity. This inquisitive and open-minded attitude is decidedly encouraging. Only thus can the profession advance.

In reviewing the discussions, those pertaining to theory will be considered first.

Mr. Silverman refers to, and very clearly demonstrates, the usual method for obtaining solutions to the differential equation (Equations (3)) as a product of two functions, each of which is a function only of one of the independent variables (x, y) or (r, θ), respectively. As stated in discussing Equation (44) of the paper, this method was purposely not employed, in order to demonstrate the use of the general expression, Equation (4). In separating Equation (4) into its real and imaginary parts, both of which will be solutions to Equations (3), either Cartesian or polar co-ordinates may be used simply by placing the complex variable, z, equal to $x + i\,y$, or $r\,(\cos\theta + i\sin\theta)$, respectively. In fact, the stress components and displacements are often obtained most easily by carrying out the differentiations of the stress function in complex form before taking real or imaginary parts. In this connection, the following expressions are useful. Let $f = U + i\,V$ be a holomorphic complex function and consider the real part only; then:

$$\frac{\partial U}{\partial x} = \text{real part of}\left(\frac{df}{dz}\right) = \mathsf{R}\left(\frac{df}{dz}\right) \;\ldots\ldots\ldots\ldots(109a)$$

$$\frac{\partial U}{\partial y} = \text{imaginary part of} -\left(\frac{df}{dz}\right) = -\,\mathsf{I}\left(\frac{df}{dz}\right) \ldots\ldots(109b)$$

$$\frac{\partial U}{\partial r} = +\,\mathsf{R}\left(e^{i\theta}\,\frac{df}{dz}\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(109c)$$

and,

$$\frac{\partial U}{r\,\partial\theta} = -\,\mathsf{I}\left(e^{i\theta}\,\frac{df}{dz}\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(109d)$$

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Altogether, the theory of functions of a complex variable has proved extremely useful and expedient in two-dimensional stress analysis.

It is to be noted that the first function of Equation (4) is holomorphic, the last three are not, and must be treated as products of holomorphic functions and x, y, or r^2, respectively, in carrying out the differentiations.

It is again emphasized that many methods are available for finding solutions to Equations (3). The trick is to find the proper combination of solutions which satisfies the given boundary conditions. Unfortunately, no general method is available for this purpose. On the other hand, if such a solution has been found in some manner, it is fortunate to know that it is unique.

In other words, to find the solution for a given two-dimensional elastic problem it is necessary to try this or that combination of known biharmonics (those which satisfy Equations (3)). If the solution is finally obtained—often after days or weeks of hard labor—the writer generally must resort to a phrase, such as "consider the following function", and then merely prove that it does satisfy all the conditions of the problem without being able to explain just how he obtained the function. This often leaves the reader with the impression that the writer possesses some mysterious technique, which is not at all the case.

The value of the Airy stress function to the practical engineer begins with an intimate knowledge of the few known simple exact mathematical solutions and the ability to combine them in such a manner as to give a good approximate solution of the problem at hand. As a rule, an exact mathematical solution to a practical problem is so unwieldy as to be of little or no practical use. Indeed, the assumptions that must be made as to the physical properties and behavior of the materials involved in practical constructions are generally such as to preclude an exact solution of any practical problem. It should be understood that at best only a close estimate of the stresses in the actual structure can be expected even from the most careful mathematical analysis or photo-elastic experimentation.

The solutions and formulas presented by the writer hold strictly only under the assumptions stated in the "Introduction." In practice these assumptions are never completely fulfilled. In fact, in some instances, it is impossible ever to determine the real properties. This is especially true in the case of foundations.

Professor Krynine treats this subject in his excellent discussion and points out the various materials for which elastic statistical isotropy may be assumed and others in which it may not. This should serve as a valuable guide for the designer of foundations.

The writer has dealt only with elastic media, in plain stress or plain strain. Professor Krynine has made a valuable contribution in extending the theory of foundations to three-dimensional plastic media. The apparent ambiguity in the value of Poisson's ratio, μ_2, for the foundation treated by Professor Krynine ceases to exist when it is again remembered that the formulas only hold for elastic media in plain stress or strain. In that case, no assumption is made as to the incompressibility of soil particles.

In computing stresses in a semi-infinite (three-dimensional) solid due to its own weight, to a uniformly distributed surface loading, or to a combination of both loadings, it would be proper to specify the horizontal strains equal to zero. This gives rise to the formulas[42]:

$$\sigma_x = \sigma_z = K\,\sigma_y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (110)$$

in which σ_y is the vertical stress equal to the total load above the point, and $K = \dfrac{\mu_2}{1 - \mu_2}$.

The displacement formulas presented by the writer are useful for the analysis of a long straight gravity dam on an elastic or quasi-elastic foundation, even if the elastic modulus of the dam itself differs from that of the foundation. The method of procedure of this problem was outlined under Case 2 in the paper. Numerical results have been developed as shown graphically in Fig. 20. The curves were obtained in connection with a study of Grand Coulee Dam (440 ft high), with an up-stream slope of 0.15, a down-stream slope of 0.8, and with concrete weighing 156 lb per cu ft. The curves indicate qualitatively the tendencies of distribution at the base as the foundation stiffens relative to the dam. The analysis did not take into consideration the stress concentrations near the heel and the toe. It is of interest to notice the general agreement between the curves for rigid foundation ($n = \infty$), and those obtained by Mr. B. F. Jakobsen[43].

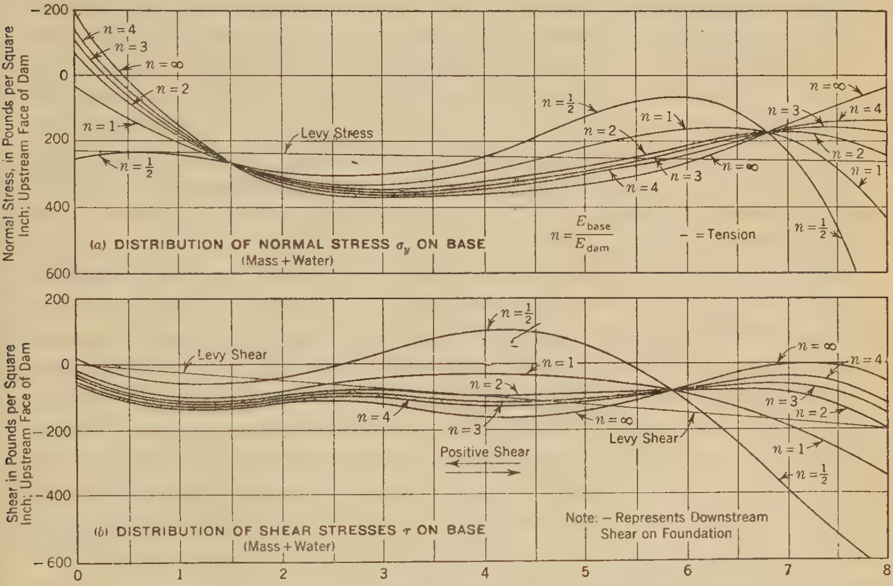

FIG. 20.—STRESS DISTRIBUTION, GRAND COULEE DAM.

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[42] *Bulletin 117*, Iowa Eng. Experiment Station, pp. 39–43.

[43] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 489.

Professor Bergman points out in a lucid manner the physical interpretation of the curvatures in the Airy surface. These properties are used in the "slab analogy", an experimental method by which stresses in a two-dimensional problem can be obtained by measuring curvatures and twists in a warped elastic slab. This is possible since the differential equation (Equations (3)), of the Airy elastic stress function, $\nabla^4 F = 0$, has the same form as the differential equation for the normal displacements, z. of a warped elastic slab, $\nabla^4 z = 0$, when the curvatures are held to a low magnitude. This means that the curvatures must be measured by means of microscopes or correspondingly sensitive measuring devices. In the analogy, the curvature of the slab at any point is proportional to the normal stress in a perpendicular direction in the prototype, and the twist at any point is proportional to the corresponding shear. Naturally, the slab must be warped by twists and curvatures at the boundaries that are themselves proportional to the boundary forces acting as the prototype.

Professor Bergman has been closely connected with model experiments conducted on dams by the U. S. Bureau of Reclamation, at the University of Colorado, under the direction of Ivan E. Houk, M. Am. Soc. C. E. It is indeed gratifying to contemplate the testimony of Professor Bergman that good general agreement exists between the theoretical results presented by the writer and this type of experimentation. That the stress concentrations at sharp corners were, to a large extent, relieved by plastic flow was to be expected. In deriving the approximate formulas for fillets the writer did so in order to be able to design a fillet with a safe elastic stress distribution without recourse to plastic time-flow of which very little is known under alternating load conditions.

In the "Introduction" to the paper the writer stated specifically that the intention was not to set up design criteria, and that the question of uplift was disregarded, except to assume that sufficient compression must exist to counteract the effect of uplift or liquid pore pressure. It is correct, therefore, to apply the theoretical results only to dam sections in which this is true, or at least where the resulting tension is not sufficient to cause rupture. In connection with this question, Mr. Jorgensen cites several dam failures and shows that, in all cases, the total effective slope (up-stream plus down-stream slope) was far less than the slopes called for by the Lévy criterion. No doubt, the question of internal liquid pressures in concrete is of great importance. Theoretical studies are now being made by the writer for the purpose of determining the effect of pore pressure on the stresses in hydraulic structures. No real headway can be made in application to practical structures until the question of "contact area" between particles has been settled experimentally for various concrete mixtures. Experiments to that end have been inaugurated at the laboratories of the Bureau of Reclamation. Some work has already been done in this field by various experimenters[44], but there seems to be a wide difference in opinion; in fact, the value given for effective contact area varies from 2 to 40 per cent. Until the question is settled beyond a doubt it would seem necessary, then, to assume the contact area to be practically zero, which means that full uplift must be assumed to exist. Assume, for example, that triangular uplift pressure exists at the base, equal to the full reservoir head at the heel and zero at the toe (which is reasonable for a nearly impervious dam on a pervious foundation), and apply this to the study of Grand Coulee Dam as shown in Fig. 6. It will be found that the vertical normal stresses (Fig. 6(*a*)) near the up-stream face and the horizontal normal stresses in the same region of the foundation, Fig. 6(*b*)) both become tension. The final section of Grand Coulee Dam has been designed with an up-stream slope of 0.15 and a down-stream slope of 0.8.

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[44] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1052.

It will be noticed in Fig. 6(*a*) that the vertical normal compression stress near the heel is considerably less than that computed by the straight-line formula (Lévy). This would more than bear out Mr. Jorgensen's contention, and would indicate that, due to the restraint of the foundation, a section even heavier than the Lévy cross-section is needed in cases of rather rigid foundations if the experiments confirm the evidence that full uplift must be applied to the dam. This contention is still more true if earthquake accelerations are to be considered and the situation is further aggravated by shrinkage stresses.

Fig. 20(*a*) would indicate that the ratio, n, between the elastic moduli in the foundation and the dam has an important influence on the stress situation at the heel. Further studies along this line should be made so that this seemingly important effect can be taken into consideration in the design of the minimum section on a given foundation. The approximations made in the analysis underlying Fig. 20 were such as to indicate that the curves for $n = 0.5$ deviate slightly too much from the Lévy line.

A theoretical study of the situation around the heel, in which an open crack was assumed to exist under full uplift, demanded an effective total slope of 0.87 (with 150-lb concrete) for stability.

Mr. Jorgensen also touches upon the subject of curved gravity dams. The problem is three-dimensional, so that the individual case must be dealt with separately. The writer has no opinion to offer on the subject. Studies made by Mr. Jakobsen seem to justify the impression that the only point in favor of a curvature without arch action is public psychology.

Parts I and II of the paper were originally presented in two separate manuscripts. It was finally deemed advisable to combine them, and with the limitation set on a single paper, it became necessary to reduce both. Consequently, only sufficient description was retained of the photo-elastic material to show general methods and give the necessary checks on the theoretical results obtained in Part I. Furthermore, the photo-elastic work was conducted as early as 1931–1932 and much improved technique has been introduced since then. Reference to Fig. 21 will show that large spherical reflectors were used instead of the much more expensive lenses, and that the polarizing unit is offset from the analyzing unit, thus providing a convenient large space for the model and operator[45].

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[45] *Review of Scientific Instruments*, February, 1934, pp. 80–83.

The important features of the two discussions dealing with Part II are the methods for the determination of stresses due to body forces. It is of interest to note that Professor Plummer and Mr. Solakian have developed the centrifugal method independently. Professor Plummer finds that the method gives good results, but that it requires elaborate apparatus and very careful adjustments, and he prefers the direct-loading method described in his very excellent discussion and developed in co-operation with the U. S. Corps of Engineers, at Zanesville, Ohio. Professor Plummer gives new and valuable information in the preparation of gelatin models. If the optical creep in this very optically sensitive substance will not prevent a reliable calibration, the writer believes it will be of inestimable value in photo-elastic experimentation, especially on foundations and related problems.

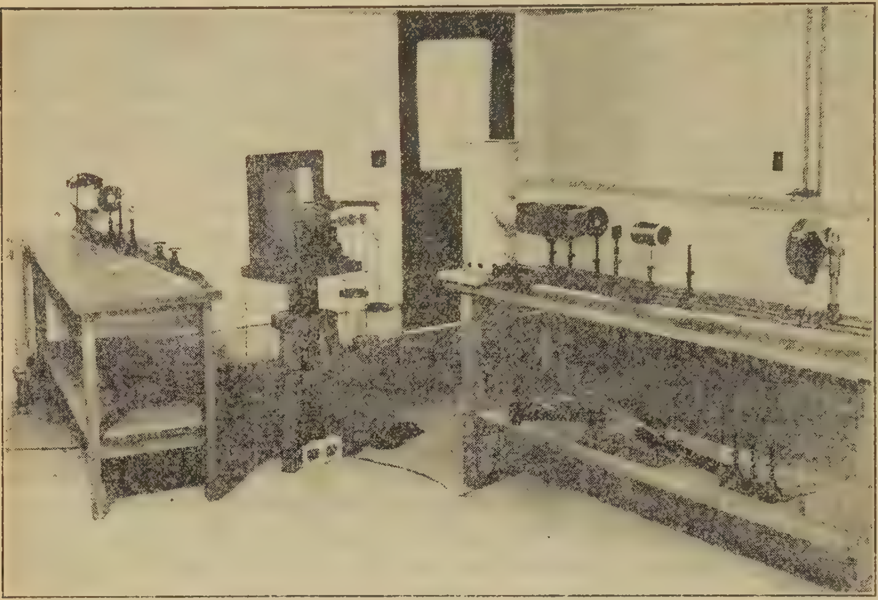

FIG. 21.—VIEW OF PHOTO-ELASTIC LABORATORY, CALIFORNIA INSTITUTE OF TECHNOLOGY.

The writer has adopted the ingenious method, suggested by Dr. Biot, of loading the boundaries with normal force distributions equal to $K y'$, in which K is equal to the weight of the material in the prototype, and y' is the distance from the point of the boundary measured in the direction of the body force to a convenient line perpendicular to that force. This loading will deliver the correct isochromatics and isoclinics due to body forces.

Let the principal stresses obtained from the foregoing loading be σ'_1 and σ'_2; then, the actual stresses at a point due to body force will be.

$$\sigma_1 = \sigma'_1 - K y \dots\dots\dots\dots (111a)$$

and,

$$\sigma_2 = \sigma'_2 - K y \dots\dots\dots\dots (111b)$$

in which y is the ordinate to the point measured in the same system as y'. The correctness of this method can be proved directly by use of the Airy stress function. It has been used extensively and very successfully in the photo-elastic laboratory of the U. S. Bureau of Reclamation.

Mr. Solakian objects to the method of superposition of water loads and body forces. As a matter of fact, in the case of dams and many other civil engineering structures, both sets of stresses are usually required separately, either for design purposes or if strains are to be measured in the prototype due to water load (live load) only.

In conclusion, the writer wishes to announce the construction of a new photo-elastic interferometer for the Bureau of Reclamation by which the individual principal stresses at a point can be found directly. It will mark a new advance in technique which promises considerable time-saving. The apparatus was designed by John Soehrens, Jun. Am. Soc. C. E., in the photo-elastic laboratory, and was built under his supervision.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 1950

FLOOD AND EROSION CONTROL PROBLEMS AND THEIR SOLUTION

By E. Courtlandt Eaton,[1] M. Am. Soc. C. E.

With Discussion by Messrs. Arthur G. Pickett, R. W. Davenport, C. S. Jarvis, Harry F. Blaney, W. P. Rowe, J. B. Lippincott, E. I. Kotok and C. J. Kraebel, Donald M. Baker, and E. Courtlandt Eaton.

Synopsis

The flood and erosion control problems treated in this paper occur in highly developed areas where there is a relatively low seasonal run-off; these areas, however, are subject to brief, although violent, torrential storms, that result in floods of exceptionally high intensity. When fires denude the sparse vegetation from steep mountain slopes the flood intensities increase and added débris hazards occur, due to erosion.

Basic precipitation and run-off records are given in this paper, as are engineering methods of constructing hydrographs of expected floods, and the regulation needed for control. Similarly, measurements of erosion quantities are presented with suggested solutions of control and a method of avoiding unnecessary capital expenditures in advance of requirements. The incidental conservation of flood waters for domestic use and for irrigation is discussed.

Introduction

Los Angeles County in California, with a population of about 2 250 000 (of whom more than 85% are residents of cities), covers an area of 4 115 sq miles. Its population increase has been nearly 180% since the last major flood (1914), and assessed property values rose, in the same period (1914 to 1931), from $850 000 000 to more than $4 000 000 000. The County's petroleum, agricultural, motion picture, and manufacturing industries have an

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Note.—Published in September, 1935, *Proceedings*.

[1] Cons. Engr., Los Angeles, Calif.

aggregate annual value of more than $1 000 000 000. It is estimated that only 8% of the present population have experienced, or have a realization of, the 1914 flood.

Property valued at more than $300 000 000 and containing 380 000 persons, is subject to possible inundation due to floods.[2] Hazards may be classified under two main heads: The main valleys and coastal plains are subject to inundation by either the overflow of present channels or by the departure of the rivers from existing to new locations; and, in the foot-hill areas, the perils are from sudden flash floods, and débris flows, particularly following water-shed denudation by fires. Valley inundation may be anticipated, by at least hours, from distant mountain rainfall records. Foot-hill débris flows may result if burned catchment areas, after becoming saturated with water, are subjected to high-intensity storms. Such flow occurs without warning and may follow storms of high intensity or cloudbursts, almost immediately.

The factors that combine to menace life and property in this region are: (*a*) Exceptionally precipitous gradients; (*b*) short mountain streams; (*c*) the unstable character of deep weathered mountain cover held from erosion by sparse vegetation; and (*d*) the characteristic violent torrential storms. Protection from floods may operate to conserve the average annual 120 000 acre-ft of flood water that is now wasted.

The paper describes physical conditions and methods of solving problems, especially those dealing with the relatively new engineering field of erosion control. Necessarily, full details cannot be included, but the foot-note references serve as a bibliography of public reports.

PHYSICAL CONDITIONS

Of the 4 115 sq miles that constitute Los Angeles County, 70%, or 2 758 sq miles, is included in the south slopes which drain to the Pacific Ocean (see Fig. 1). The area comprises 1 589 sq miles of mountain water-shed (809 sq miles of which is in Federal forests); 285 sq miles of small watersheds; and 884 sq miles of valley plains sloping toward the ocean. About 210 000 acres of valley land is under irrigation, and 235 000 acres is devoted to industrial and domestic purposes.

The high mountain areas form a northern boundary of approximately 45 miles, crest length, from which the two major collecting rivers—the San Gabriel and Los Angeles—originate and, after traversing the valleys, they converge and discharge into the ocean at points only six miles apart. Mountain peaks rise to Elevation 10 080 (United States Geological Survey datum). The main rivers are forced to converge by the foot-hills that project on to the plains—the Santa Monica hills from the west and the Puente and San José hills from the east.

The San Gabriel River has a tributary drainage of about 800 sq miles and the Los Angeles River one of about 900 sq miles. The largest catchment of the San Gabriel River is 213 sq miles of the San Gabriel water-shed; that of the Los Angeles River is the Big and Little Tujunga watersheds with 137 sq miles. The remaining drainages of both rivers are made

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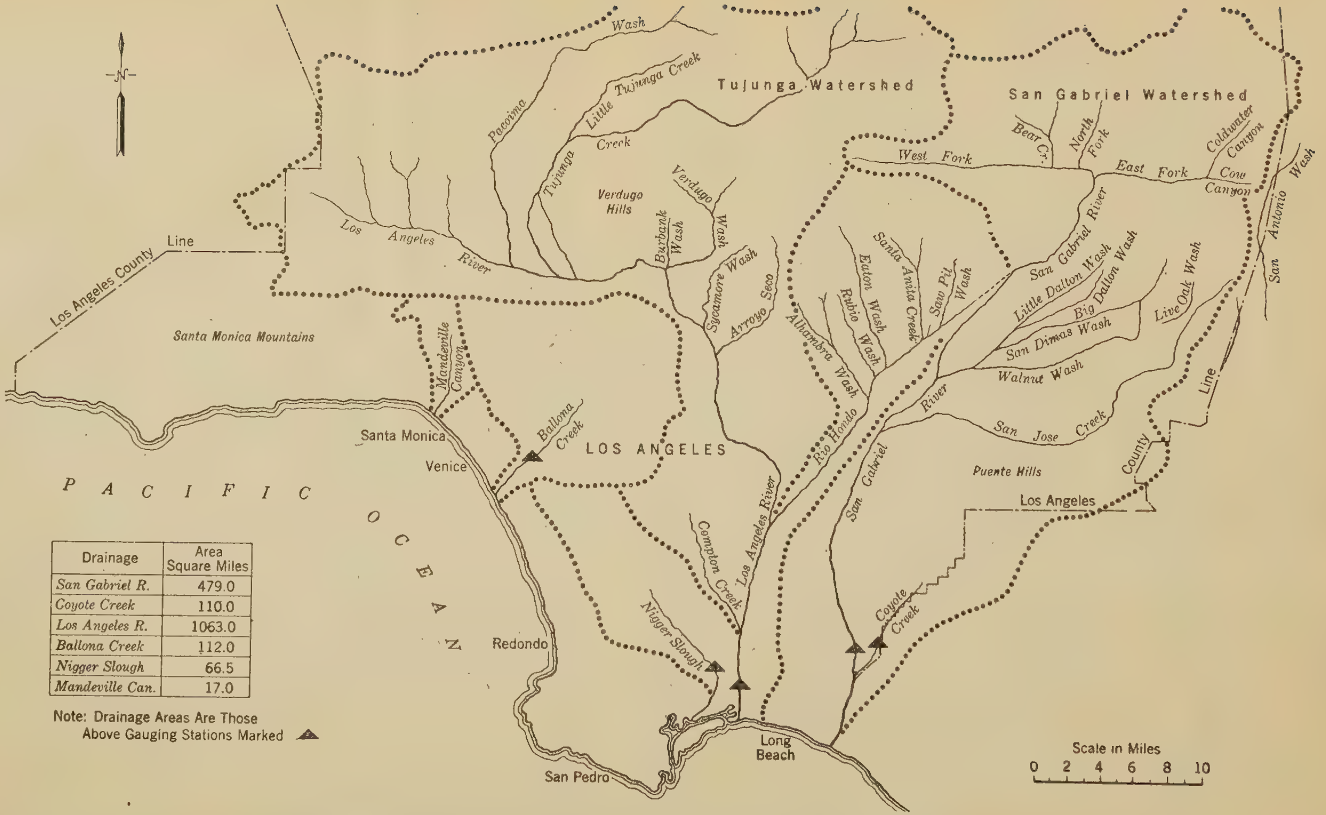
[2] "Comprehensive Plan for Flood Control and Conservation," by E. C. Eaton, 1931.


N
Tujunga Watershed
San Gabriel Watershed
Wash
Pacoima
Little Tujunga Creek
Tujunga Creek
Verdugo Hills
Verdugo Wash
Burbank Wash
Los Angeles River
Sycamore Wash
Arroyo Seco
Alhambra Wash
Rio Hondo
Eaton Wash
Rubio Wash
Santa Anita Creek
Saw Pit Wash
West Fork
Bear Cr.
North Fork
East Fork
Coldwater Canyon
Cow Canyon
San Antonio Wash
San Gabriel River
Little Dalton Wash
Big Dalton Wash
San Dimas Wash
Live Oak Wash
Walnut Wash
San Jose Creek
Puente Hills
County Line
Los Angeles
Coyote Creek
Los Angeles County Line
Santa Monica Mountains
Mandeville Canyon
Ballona Creek
LOS ANGELES
Compton Creek
Nigger Slough
Santa Monica
Venice
Redondo
San Pedro
Long Beach
PACIFIC OCEAN
Scale in Miles
0 2 4 6 8 10

Drainage	Area Square Miles
San Gabriel R.	479.0
Coyote Creek	110.0
Los Angeles R.	1063.0
Ballona Creek	112.0
Nigger Slough	66.5
Mandeville Can.	17.0

Note: Drainage Areas Are Those Above Gauging Stations Marked ▲

FIG. 1.—MAP OF LOS ANGELES COUNTY: DRAINAGES TO PACIFIC OCEAN.

up of numerous small streams originating in more than 100 mountain and foot-hill water-sheds ranging in size from 0.5 sq mile to 30 sq miles.

A third drainage system is Ballona Creek fed from 134 sq miles, its principal supply being the run-off from the streets of Los Angeles, Hollywood, and Beverly Hills, transported to Ballona Creek through storm drains. Its ocean outlet is near Venice, Calif. A fourth drainage is Nigger Slough, similarly deriving its main supply from impervious areas. Its discharge point is near San Pedro, Calif.

Along the coast from Venice westward to the Los Angeles County line about twenty individual streams flow directly into the Pacific Ocean. The largest are Malibu Creek, with a drainage area of 67 sq miles; Topanga Creek (20 sq miles), and Mandeville, Rustic, and Sullivan Canyons (12 sq miles), the latter group passing through Santa Monica, Calif.

Mountain and foot-hill drainages have well-defined canyon channels on precipitous grades of 20 to 30%, sharply flattening from 8 to 10% where they debouch on to unstable delta areas. In many instances cones show little evidence of active channels, and, during the dry cycle (1914 to 1931), home sites were sold readily in foot-hill territories by real estate agents who were themselves possibly ignorant of hazards. Normally, mountain and foot-hill watersheds are covered with a chaparral type of vegetation, efficiently functioning as retarders of flood flows and controlling erosion.

HISTORIC FLOODS

The most recent major floods occurred in 1914 and 1916. In 1914 there were two storm periods: One on January 25, with a maximum intensity of 2.6 in. in 24 hr; and the other on February 18, with an intensity of 4.26 in. Exclusive of harbor damages the estimated loss from these two floods was $10 000 000. Conservatively estimated representative flood peaks were[3] as shown in Table 1. Many lives were lost, thousands of people were made homeless, thirty-five bridges were destroyed, and, for six days, there was little communication with the outside world.

TABLE 1.—REPRESENTATIVE PEAK FLOWS

Stream	Drainage area, in square miles	Peak run-off, in cubic feet per second per square mile
San Gabriel River	229	117
Big Tujunga	118	115
Arroyo Seco	39	366
Sawpit	7	550

Engineers concede that this was by no means a record flood. During previous floods within a 70-yr period, main rivers have materially changed their courses, the Los Angeles River changing from westerly to southwesterly, and the San Gabriel River cutting its new channel from 3 to 6 miles south-easterly. A new river, the Rio Hondo, cut its way and connected the San Gabriel and the Los Angeles River.

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[3] Rept. of Board of Engrs. on Flood Control to the Los Angeles County Board of Supervisors, January 25, 1917, by H. Hawgood, Charles T. Leeds, J. B. Lippincott, and F. H. Olmsted, Members, Am. Soc. C. E.

Early settlers testify to the occurrence of severe floods in 1815, and again in 1825, and, successively, five major flood periods are recorded in 1861–62, 1884, 1886, 1889, and 1914.[2]

During the unparalleled dry period, 1916 to 1931, rapid development altered conditions radically, crowding former channel and ponding areas. The increased development of paving and impervious areas alone, totaling more than 55 sq miles, has increased the rates and quantities of run-off, and decreased the natural absorption and consequent replenishment of subterranean storages.

A curve prepared in 1931 (see Fig. 2) shows past trends of seasonal rainfall. After this curve was plotted a prediction was made of an uptrend into a wet cycle with increasing wet periods, comparable to the period, 1881–90, in which interval one or more major floods would occur.[2]

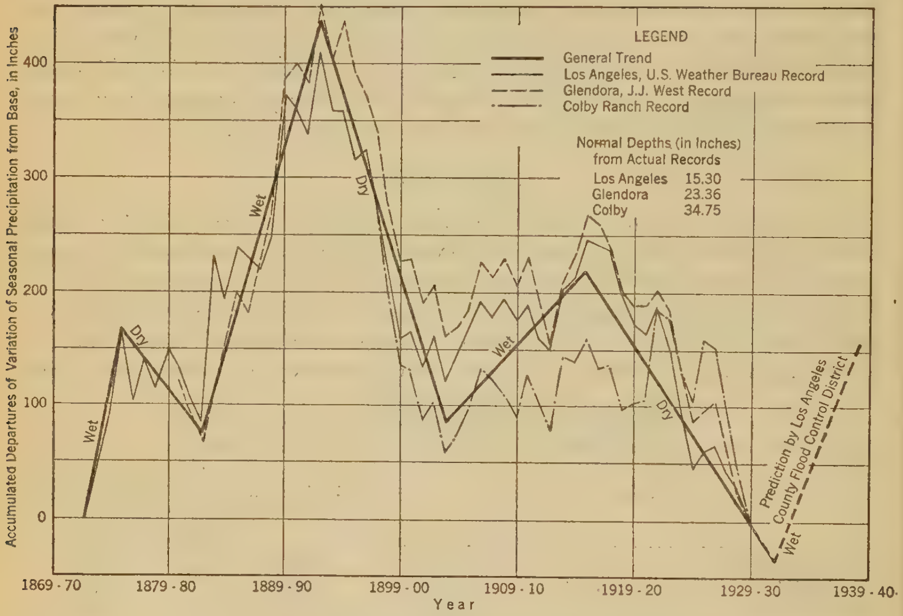

Fig. 2.—General Trend, Seasonal Rainfall.

Rainfall—Run-off

A co-ordinated system of 237 rainfall and 197 run-off stations was installed by the Los Angeles County Flood Control District commencing in 1927. The more important stations were equipped with recorders. The year, 1931–32, not a year of general flood, had valley rainfall totaling 114% of the normal, that in the mountains totaling 113% of its normal. The seasonal run-off from the San Gabriel River exceeded the combined run-off of the three previous years. Summarized records from three drainages follow: (*a*) San

Gabriel-West Fork—area, 49 sq miles; (*b*) Little Tujunga Creek—area, 21 sq miles; and, (*c*) Ballona Creek—area, 112 sq miles.

The San Gabriel and Tujunga water-sheds are mountain catchments with cover in good condition. The San Gabriel water-shed was burned in 1924, but its brush growth is now (1935) restored. The Tujunga water-shed has not been burned since 1919. Ballona Creek is of a different character, draining flat valley territory, including street drainage from Los Angeles, Hollywood, and Beverly Hills.

The season (1931–32) contained two storm periods, one in December, 1931, and the other in February, 1932. Table 2 gives the distribution of precipitation, maximum 24-hr intensities, and the resulting run-off. The distribution of the San Gabriel run-off was affected by snow. There was 6 to 8 in. of snow above Elevation 4 000 during December, 1931; and 4 to 8 in., during January, 1932; the snow disappeared by February 26, 1932.

TABLE 2.—DISTRIBUTION OF PRECIPITATION, 1931–32

Period		Days of precipitation	Total precipitation, in inches	Maximum precipitation in 24 hours, in inches	Total run-off, in inches	Ratio: Run-off to rainfall, percentage
From:	To:					
(*a*) SAN GABRIEL WATER-SHED						
October 1	December 24	18	14.10	3.50	0.35	2.5
December 25	January 2	7	8.20	5.04	1.12	13.7
Total for first storm period....		25	22.30	....	1.47	6.6
January 3	February 7	7	10.36	5.30	1.13	....
February 8	February 28	8	12.45	4.54	5.06	....
March 1	May 1	5	1.23	0.67	1.89	....
Total for second storm period.		20	24.04	....	8.08	33.7
(*b*) LITTLE TUJUNGA WATER-SHED						
October 1	December 24	11	5.15	1.55	Practically none	
December 25	January 2	4	7.00	2.75	0.12	1.7
Total for first storm period..		15	12.15	....	0.12	1.0
January 3	February 7	8	5.64	2.10	0.12	2.1
February 8	February 28	6	4.41	1.15	1.38	31.3
Total for second storm period.		14	10.05	....	1.50	15.0
(*c*) BALLONA CREEK WATER-SHED						
October 1	December 24	11	4.85	1.95	0.87	18.0
December 25	January 2	4	3.40	2.09	0.88	26.0
Total for first storm period...		15	8.25	....	1.75	21.0
January 3	February 7	10	3.43	1.94	0.54	15.7
February 8	February 28	8	4.55	1.95	1.23	27.0
Total for second storm period.		18	7.98	....	1.77	22.0

Both mountain catchments (particularly the Little Tujunga, the flow of which was not affected by snowfall), show the relatively large proportion of annual precipitation required to produce sufficient saturation to permit run-off. Mountain and foot-hill catchments are deeply covered with weathered material, ranging from the porous disintegrated granites to the less porous soils with higher clay content; consequently, separate values for any watershed under consideration must be determined. The flood flows will depend upon: (1) Slopes, character of soils and cover (organic growth, litter, etc); (2) the quantity and rate of rainfall required to produce saturation; (3) the 24-hr and 1-hr intensities; and (4) how closely the high-intensity storms follow upon the saturation period.

On the Little Tujunga water-shed from February 6 to 10, 1932, the rains totaled 5.13 in. with 24-hr intensities, on three consecutive days, of 2.10, 1.10, and 1.15 in. The run-off in the succeeding five days totaled 1.05 in., or a ratio of run-off to rainfall, of −20 per cent.

On both mountain catchments a total of 10 in. of rainfall was required before appreciable run-off occurred. Contrasted with this is the relatively quick response of run-off to rainfall on Ballona Creek. A total rain of 0.17 in. was followed (November 15, 1931) by a 24-hr rain of 0.96 in., of which 17% appeared in a few hours as run-off. Table 3 shows the rainfall-run-off relation for maximum month, day, and hour, for the three catchments.

TABLE 3.—COMPARISON BETWEEN RAINFALL AND RUN-OFF, 1931–32

Water-shed	MAXIMUM MONTH			MAXIMUM DAY			MAXIMUM HOUR		
	Rainfall, in inches	Run-off, in inches	Percentage of run-off	Rainfall, in inches	Run-off, in inches	Percentage of run-off	Rainfall, in inches	Run-off, in inches	Percentage of run-off
San Gabriel...	19.74	5.8	29	3.80	1.58	41	0.69	0.08	16
Little Tujunga	7.71	1.47	19	2.10	0.48	23	....	....	....
Ballona Creek	5.95	1.36	23	2.09	0.60	28	0.27	0.07	26

A representative number of scattered records of 24-hr and corresponding 1-hr intensities is given in Table 4.

TABLE 4.—REPRESENTATIVE TWENTY FOUR-HOUR AND CORRESPONDING ONE-HOUR RAINFALL INTENSITIES

Station	Years of record	Elevation, in feet (U. S. Geological Survey)	TWENTY-FOUR HOUR RAINFALL		One-hour rainfall, in inches
			Date	In inches	
Opids Camp...............	17	4 480	4– 5–26	12.30	2.00
Mt. Wilson................	30	5 850	12–19–21	11.26	1.10
Alder Creek...............	..	.....	12–19–21	7.40	0.80
San Gabriel Intake.........	..	1 250	4– 5–26	6.76	1.16
Haines....................	17	2 500	4– 5–26	5.95	1.10
Valley Forge...............	..	3 400	2–15–27	5.58	0.87
Sister Elsie...............	..	.....	4– 5–26	4.77	0.62
Coldbrook Camp...........	..	3 300	4– 5–26	6.44	0.76

FLOOD HYDROGRAPH DETERMINATION

Studies of the Big Tujunga water-shed are given herein, showing a method of computing possible flood flows, the results, and the regulation required. This catchment, as yet only partly controlled, is the largest individual tributary of the Los Angeles River drainage. Its control is a necessity because of the flood hazard to the thousands living adjacent to the Tujunga Wash and to those living along the Lower Los Angeles River. That most of these people have not, personally, observed conditions causing or accompanying a major flood makes the hazard none the less real.

The Big Tujunga drainage area above the U. S. Geological Survey gauging station ($3\frac{1}{2}$ miles up stream from the mouth of the canyon) is 106 sq miles; to the mouth of the canyon it is 113.5 sq miles; and to a dam site farthest down stream (6 miles below the canyon mouth), 124.5 sq miles. Its average width is 7 miles. The range of elevation is from 1 250 ft at the canyon mouth to the highest peak, Pacifico Mountain, 7 078 ft (U. S. Geological Survey).

Rainfall records are available at thirteen stations in, or applicable to, the drainage. The longest dates back to 1872; the shortest to 1902. The U. S. Geological Survey has gauged the stream only since October 28, 1916, but, fortunately, long-time records are available on the contiguous San Gabriel water-shed. In computing the probable flood peak the procedure was as described in the following paragraphs.[4]

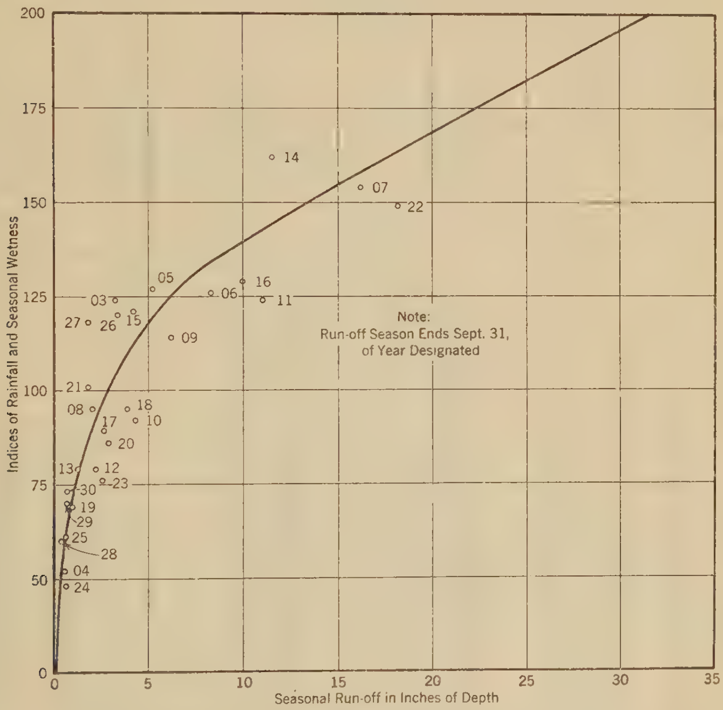

Fig. 3.—Big Tujunga Water-Shed—Indicated Seasonal Run-Off, in Inches of Depth.

A correlation was prepared from the longer rainfall-run-off records on the San Gabriel water-shed. The results, together with actual Tujunga records, were plotted against the master indices of mean seasonal rainfall and are shown in Fig. 3 in which the seasonal run-off is expressed as depth, in inches, on the water-shed.

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[4] Rept. on Big Tujunga Creek Flood Control and Conservation Possibilities, by Franklin Thomas, M. Am. Soc. C. E., September 22, 1931.

Table 5 is a summarization of the results. In its preparation, the curve, Fig. 3, was used to provide the values in Column (5) only when actual, or the correlated, records were not available. As summarized, Table 5 gave the following:

Seasonal Run-Off, in Inches of Depth:	
Fifty-eight year average	5.8
Maximum (1883–84)	47.7
Minimum (1876–77)	0.2
Seasonal Rainfall, in Inches of Depth (Composite for Tujunga Water-Shed):	
Fifty-eight year average	27.3
Maximum	65.8
Minimum	6.8
Ratio of Run-Off to Rainfall, in Percentage:	
Fifty-eight year average	21
Maximum	72
Minimum	2

Rainfall indices for years of largest flows are shown on Fig. 4. In arriving at the expectancy of a major flood it was decided to use the flood of 1883–84 as a criterion for protection, because all indications pointed to that year as the one of major flood in the 58-yr period.

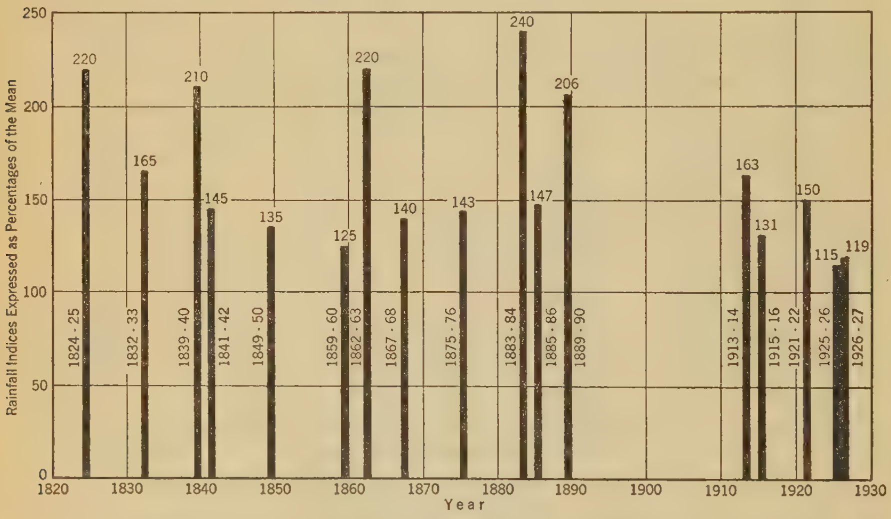


FIG. 4.—RAINFALL INDICES FOR YEARS OF LARGE FLOODS.

The mountain soil cover is disintegrated to great depth; practically no rain falls between May and October, both inclusive; and hydrographic experience has shown that on this area a total of about 10 in. of rainfall causes saturation. If this is closely followed by high-intensity storms, floods are inevitable, the magnitude rising rapidly with increasing intensities.

TABLE 5.—INDICATED PRECIPITATION AND RUN-OFF, BIG TUJUNGA WATER-SHED; PERIOD OF 1872–73 TO 1929–30, INCLUSIVE

(Flow at Dam Site No. 5; Drainage Area, 124.5 Square Miles)

Season (1)	Index of seasonal wetness (2)	Precipitation depth, in inches, October 1 to September 30 (3)	Ratio of run-off to rainfall, percentage (4)	Run-off depth, in inches. (5)	Index of seasonable run-off (6)	Estimated seasonal run-off in acre-feet (7)	Seasonal Discharge, United States Geological Survey Gauging Station: Correlated from the San Gabriel, United States Geological Survey (8)	Seasonal Discharge, United States Geological Survey Gauging Station: Measured (9)
1872–1873	97	26.5	10	2.8	48	18 600	....	.....
1874	155	42.3	36	15.2	263	100 930	....	.....
1875	142	38.8	28	10.8	187	71 700	....	.....
1876	175	47.8	47	22.3	386	148 000	....	.....
1877	25	6.8	2	0.2	3	1 330	....	.....
1878	111	30.3	14	4.2	73	27 890	....	.....
1879	56	15.3	3	0.5	9	3 320	....	.....
1880	117	32.0	15	4.9	85	32 540	....	.....
1881	69	18.8	5	1.0	17	6 640	....	.....
1882	69	18.8	5	1.0	17	6 640	....	.....
1883	70	19.1	5	1.0	18	6 640	....	.....
1884	241	65.8	72	47.7	827	316 700	....	.....
1885	51	13.9	3	0.4	7	2 660	....	.....
1886	143	39.1	3	11.2	194	74 370	....	.....
1887	86	23.5	8	1.9	33	12 620	....	.....
1888	109	29.8	13	3.9	67	25 900	....	.....
1889	131	35.8	21	7.5	130	49 800	....	.....
1890	216	59.0	64	37.9	657	251 660	....	.....
1891	91	24.9	9	2.2	38	14 600	....	.....
1892	74	20.2	6	1.2	21	7 970	....	.....
1893	152	41.5	34	14.1	244	93 620	....	.....
1894	47	12.9	2	0.3	5	2 000	....	.....
1895	117	32.0	15	4.9	85	32 530	....	.....
1896	54	14.8	3	0.4	7	2 660	....	.....
1897	109	29.8	13	3.9	68	25 900	....	.....
1898	43	11.8	3	0.3	5	2 000	....	.....
1890	33	9.0	2	0.2	3	1 330	....	.....
1900	54	14.8	3	0.4	7	2 660	....	.....
1901	106	29.0	12	3.6	62	23 900	....	.....
1902	63	17.2	4	0.7	12	4 650	....	.....
1903	124	33.9	10	3.32	58	22 040	18 790	.....
1904	52	14.2	4	0.61	11	4 050	3 428	.....
1905	127	34.7	15	5.25	91	34 860	29 700	.....
1906	126	34.4	24	8.31	144	55 180	47 000	.....
1907	154	42.1	38	16.21	281	107 600	91 650	.....
1908	95	26.0	8	2.13	37	14 000	12 034	.....
1909	115	31.4	20	6.20	107	41 000	35 077	.....
1910	92	25.1	17	4.31	75	28 600	24 350	.....
1911	124	33.9	33	11.07	192	73 500	62 565	.....
1912	79	21.6	10	2.25	39	14 900	12 750	.....
1913	79	21.6	6	1.29	22	8 500	7 266	.....
1914	162	44.3	25	11.58	200	76 500	65 490	.....
1915	121	33.1	13	4.22	73	28 000	23 905	.....
1916	129	35.2	28	9.99	172	66 000	56 500	.....
1917	89	24.3	11	2.65	46	17 500	....	15 000
1918	95	26.0	15	3.93	68	26 000	....	22 200
1919	69	18.9	5	1.01	17	6 700	....	5 730
1920	86	23.5	12	2.94	51	19 500	....	16 600
1921	101	27.6	7	1.86	32	12 200	....	10 500
1922	149	40.7	45	18.22	313	120 000	....	103 000
1923	76	20.8	12	2.60	45	17 200	....	14 700
1924	48	13.1	5	0.65	11	4 300	....	3 670
1925	61	16.7	4	0.65	11	4 300	....	3 700
1926	120	32.8	11	3.45	60	22 900	....	19 500
1927	118	32.2	6	1.91	33	12 600	....	10 794
1928	60	16.4	3	0.43	7	2 800	....	2 441
1929	70	19.1	4	0.73	13	4 800	....	4 130
1929–1930	73	20.0	4	0.77	13	5 000	....	4 350
Summary:								
Total	...	....	..	.....	.....	2 222 290	490 505	236 315
Mean	100	27.3	21	5.8	100	38 315	35 036	16 880
Maximum 1883–84	241	65.8	72	47.7	827	316 700	91 650	103 000
Minimum 1876–77	25	6.8	2	0.2	3	1 330	3 428	2 441

Although it would be possible, under conditions of concentrated rainfall of high intensity, for a flood to occur in a season of index of less than 100, ordinarily the distribution is such that years having indices less than 120 would seldom produce large floods; those with indices as high as 130 to 170 would almost invariably produce large floods; and indices above 200 represent so much rainfall that major floods would be inevitable.

In computing probable flood flow quantities it is necessary, therefore, to examine not only daily precipitation records but hourly intensities. Results of studies of probable maximum 24-hr precipitation of a frequency of once in 50 yr are shown as shaded zones in Fig. 5. Like studies of maximum sustained intensities, in inches per hour, are designated by lines and quantities also in Fig. 5.

A value for the possible 24-hr rainfall averaging 10.4 in. over the catchment, was adopted as well as hourly intensities ranging from 1.8 to 2.7 in. in various parts of the same water-shed. Isolated instances exist in small zones of 24-hr precipitation in excess of 12 in., but it was not expected that this quantity would be general over the entire water-shed.

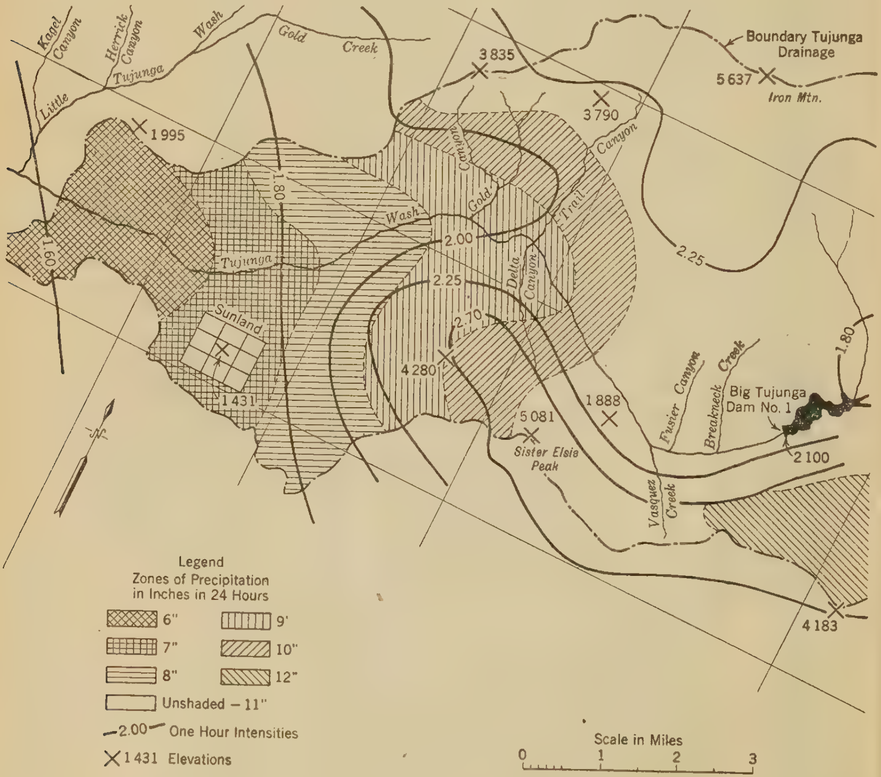

Kagel Canyon
Herrick Canyon
Little Tujunga
Wash
Gold Creek
3835
Boundary Tujunga Drainage
5637
Iron Mtn.
3790
Canyon
1995
1.80
Gold Canyon
Trail
Wash
2.00
2.25
1.60
Tujunga
Sunland
Delta Canyon
2.70
4280
1431
5081
Sister Elsie Peak
1888
Fusier Canyon
Breakneck Creek
Big Tujunga Dam No. 1
2100
1.80
Vasquez Creek
4183
Legend
Zones of Precipitation in Inches in 24 Hours
6"
7"
8"
9"
10"
12"
Unshaded – 11"
2.00 One Hour Intensities
1431 Elevations
Scale in Miles
0
1
2
3

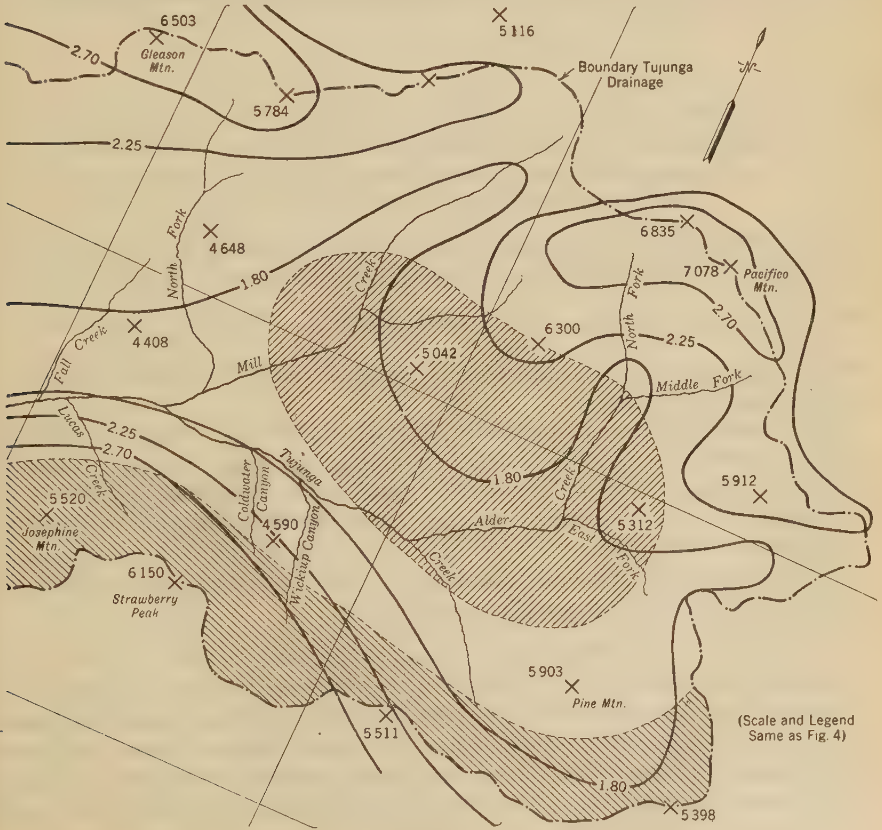
FIG. 5.—BIG TUJUNGA WATER-SHED: ZONES

Further experience showed that a critical 4-day storm may be expected, and that the run-off during that storm will not exceed 50% of the total seasonal run-off. Percentage ratios of maximum 4-day flows to seasonal run-off for the same years, are:

Year	Percentage ratio	Year	Percentage ratio
1906........	23	1921........	5
1914........	12	1926........	35
1916........	25	1927........	51

The flows from 1906 to 1916, inclusive, are derived by correlation with San Gabriel records, and the remainder are from actual records.

In a determination of desirable flood-regulating capacity it is necessary to know the effect of the lesser peak flows that will immediately precede the major flood peak. Daily discharges were plotted for critical periods during seven successive days on a number of water-sheds where rainfall-run-off records were available, which indicated that a 4-day hydrograph would involve as extensive a period as would materially affect flood storage; from such records the relation of successive peak flows was derived. The estimated distribution of run-off during a 4-day period on the Big Tujunga water-shed is given in


6 503
Gleason Mtn.
2.70
5 784
5 116
Boundary Tujunga Drainage
2.25
North Fork
4 648
1.80
Creek
4 408
Fall Creek
Mill
5 042
6 835
7 078
Pacifico Mtn.
2.70
6 300
North Fork
2.25
Middle Fork
Lucas
2.25
2.70
Creek
5 520
Josephine Mtn.
Coldwater Canyon
Tujunga
4 590
Wickiup Canyon
1.80
Creek
Alder
East Fork
5 312
5 912
Creek
6 150
Strawberry Peak
5 903
Pine Mtn.
5 511
1.80
5 398
(Scale and Legend Same as Fig. 4)

OF PRECIPITATION AND LINES OF INTENSITIES.

TABLE 6.—Estimated Distribution of Total Run-Off During Four-Day Period

Flood day	Run-off, in acre-feet	Rainfall, in inches	Run-off, in inches	Ratio: Run-off to rainfall
Maximum day	30 026	10.4	4.5	0.43
One day preceding	23 236	8.7	3.5	0.40
Two days preceding	13 316	6.7	2.0	0.30
Three days preceding	5 884	4.5	0.9	0.20
Total	72 462	30.3	10.9	0.36

Table 6. The successive peaks are shown on Fig. 6. The quantities refer to estimated flows at a point six miles down stream from the mouth of the canyon at Dam Site No. 5.

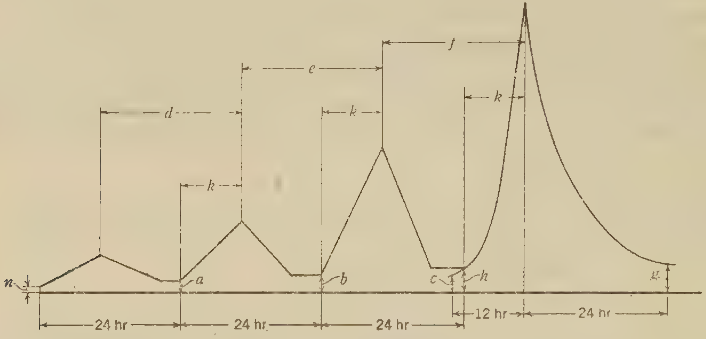


Fig. 6.—Construction of Four-Day Hydrograph.

The 30 000 acre-ft in one day is equivalent to 241 acre-ft per sq mile of water-shed; the corresponding peak is 48 300 cu ft per sec, or 388 cu ft per sec per sq mile; and the ratio of 4-day to seasonal run-off is $\frac{72\,462}{316\,700}$, or 23 per cent. All the values appear reasonable and quite readily obtainable during a year of excess precipitation.

Five reservoir sites are available in the canyon and below its mouth. Operation studies were made of all combinations resulting in a decision that regulating storage totaling 27 360 acre-ft would reduce the flood peak of 48 300 cu ft per sec to 14 500 cu ft per sec, at a capital cost of about $152 per acre-ft, equivalent to $125 per cu ft per sec, of regulation affected. This regulating storage can be created by dams at three locations, developing capacities of 6 240, 7 170, and 13 950 acre-ft, respectively.

Erosion and Erosion Control

Erosion, in this instance, is defined as the washing of loose material from weathered mountain and foot-hill areas, and its deposition upon the lower valley floors. It results from the same, still active, natural forces that have been building up, gradually, the valleys to many thousands of feet in depth. The removed material when being transported from its original source and after deposition is commonly called "débris."

Rates of débris movements (where mountain slopes are steep) depend primarily on the condition of natural mountain vegetative cover. Destruction of this protective cover by fires, or otherwise, will permit enormous débris flows to result from rainfall intensities which would create, normally, but small débris movement. This abnormal condition diminishes with re-growth of Nature's protective covering, which is often a slow process, since erosion has partly removed the top-soil capable of supporting vegetation.

Hazards from Débris Movements

Excessive débris waves first affect the lives and property of foot-hill residents. Run-off from smaller tributaries responds more quickly to rainfall than run-off in main rivers and, consequently, a débris-laden stream, upon entering a larger main channel, deposits temporary débris cones, forming barriers of such proportions that the delayed peak of the main river cannot immediately remove them, and river overflow may result. To design an efficient channel under conditions of annual stream flow varying from 3 to 850% of means, is most difficult; when the problem is complicated by heavy débris flows construction costs increase enormously and erosion control at or near the origin of the débris becomes essential.

Records of Water-Shed Fires

Financial considerations govern in the matter of fire prevention and although Federal, State, and County agencies are entirely efficient, at present (1935), with the funds allocated, far from 100% effectiveness in fire prevention has been attained, as shown by forest fire records during the fourteen years since 1919. Major fire records are summarized[5] in Table 7.

TABLE 7.—Burned Water-Shed Areas—1 000 Acres or More—1919 to 1933

No.	Year	Area burned, in acres	No.	Year	Area burned, in acres
1	1919	60 000	10	1925	5 650
2	1919	60 000	11	1925	4 000
3	1921	4 000	12	1927	4 630
4	1921	3 500	13	1927	9 000
5	1921	3 300	14	1927	2 960
6	1921	1 500	15	1928	40 000
7	1922	20 000	16	1930	23 000
8	1924	52 000	17	1933	4 860
9	1924	22 000	....	....	....
Total	....	......	....	....	320 400
Average per fire	....	......	....	....	18 900

What expenditures are warranted for fire prevention varies widely in different areas, depending upon population concentrations and values menaced. The foot-hill water-sheds of Los Angeles County are made up of many hundred individually small catchments, aggregating 285 sq miles. Of this, areas totaling nearly 60 sq miles, as yet not denuded, are located above incorporated territory having a combined population of more than 100 000 persons and assessed values of more than $200 000 000.

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[5] Rept. on New Year's Foothill Débris Flood, by E. C. Eaton, March 19, 1934.

Erosion with Normal Water-shed Cover

Where water-sheds have not been disturbed by fires, roads, earth-slides, etc., erosion is relatively slight. Quantitative records of erosion from Los Angeles County water-sheds that have a normal cover are not as plentiful nor as accurate as those from water-sheds from which the débris flows are more readily measured; therefore, only estimates are available.

One estimate (cited as ample) of the extent to which the storage on the San Gabriel water-shed would be depleted, was 3 200 cu yd per sq mile annually.[6] This value probably includes assumptions of occasional burnings since the estimate was made in 1927, within the 3-yr critical period after more than 30% of the area was burned in 1924.

The measured débris deposited in Sweetwater Reservoir, San Diego County, is reported as averaging 1 400 cu yd annually per sq mile in 39 yr[7], and Gibraltar Reservoir, in Santa Barbara County, averaged 1 380 cu yd per sq mile in its 11-yr period.[7] A major fire in 1923, burning over most of the water-shed, makes this value high. Twelve scattered records collected by the writer average 190 yd per storm per sq mile, ranging from 30 to 500 yd.

No single rate is applicable to all the catchments of Los Angeles County and, although admittedly based upon insufficient data, it is the writer's opinion that, with a water-shed undisturbed by fires for 10 yr or more, 1 500 yd per sq mile is a maximum quantity.

Débris Quantities from Burned Areas

One quantitative example of débris movement is that given by the case of Sunset and Brand Canyons[8] which debouch directly on the streets of Burbank, a city of 16 600 population and more than $28 000 000 assessed valuation in 1931. Brand Canyon discharges on a part of Glendale which has a total population of 60 000 persons and a total assessed property valuation of more than $80 000 000. The catchment area of Sunset Canyon is 1.10 sq miles; its length is about 1.5 miles, and its width $\frac{2}{3}$ mile; and in elevation, it ranges from 886 to 3 126 (U. S. Geological Survey). Brand Canyon has a similar catchment of 1.03 sq miles. A fire that burned a total of 4 600 acres occurred from December 3 to 5, 1927, and included all of Sunset and Brand Canyons as well as adjacent canyons. One year later (October, 1928), the first rains occurred, recording a total depth of 0.40 in. During November, 2.10 in. of rain fell, of which more than 65% occurred in one day; a 0.59-in. rain of November 13 was followed on November 14 by a downpour with a sustained 1-hr intensity of 0.43 in., between 8:30 and 9:30 A. M., making a total for that day of 1.41 in. Intensities of 10-min duration were recorded as high as 0.35 in.

Canyon observers reported the resulting débris flows to have followed closely after the high-intensity period, with two distinct flash flows. The first was composed mostly of water, which rose 3.5 ft in a channel 55 ft wide; and this was succeeded by a second flow estimated, by observers, to flow at a rate of 20 to 30 ft per sec. The débris flow lasted 30 min.

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[6] Rept. on Flood Control and Conservation, San Gabriel River, by F. H. Fowler, M. Am. Soc. C. E., C. D. Marx, Past-President and Hon. M. Am. Soc. C. E., and C. H. Paul, M. Am. Soc. C. E., March, 1927.

[7] "The Silt Problem", by J. C. Stevens, see p. 207.

[8] Rept. on Sunset and Brand Park Canyons Flood of November 14, 1928, by E. C. Eaton.

From all available information, including cross-sections, it was estimated that 29 acre-ft of combined débris and water flowed from Brand Canyon in a 30-min period, of which 17 acre-ft (or nearly 60%) was débris, and 12 acre-ft was water, an equivalent run-off depth of 0.22 in. The average velocity was computed at about 6 ft per sec. The débris flow was equivalent to 26 000 cu yd per sq mile of water-shed, a quantity that checked closely with records of truck removals from streets and lawns. Débris removal costs by public agencies averaged $1 per cu yd, exclusive of the cost to residents for removals from cellars and lawns.

Another example was afforded by the Arroyo Sequis, on the western County boundary, which has a water-shed of 11.4 sq miles, discharging directly into the ocean. It is roughly triangular in shape and ranges in elevation from sea level to 3 060 (U. S. Geological Survey). An intense fire from October 29 to November 6, 1930, burned 90% of this area, leaving its decomposed granite slopes, of 300 to 500 ft per thousand, exposed to rain.

The nearest recording rain-gauge was 12 miles inland, at Elevation 600. Since storms approach from the west they strike the coast water-shed an hour earlier than the gauge and it is probable that the actual rainfall on the burned area was greater in quantity and intensity than that recorded. Rainfall records are listed in Table 8. Beginning at 1:00 P. M., on January 7, 1931, and continuing to 4:00 P. M., the intensities for 10-min periods, expressed in equivalent inches per hour, are listed in Table 9. The total rain falling in the 170 min. was 1.24 in., or an average rate for the period of 0.44 in. per hr.

TABLE 8.—RAINFALL RECORDS, ARROYO SEQUIS WATER-SHED

Day	Month and year	Rainfall, in inches	Cumulative rainfall, in inches
....	October, 1930	0.16	0.16
....	November, 1930	2.09	2.25
....	December, 1930	0.00	2.25
1	January, 1931	1.11	3.36
5	January, 1931	0.47	3.83
7	January, 1931	1.30	4.13

TABLE 9.—ARROYO SEQUIS WATER-SHED; TEN-MINUTE RAINFALL INTENSITIES

Time, January 7, 1931	Rate of rainfall for 10-min periods, in inches per hour	Time, January 7, 1931	Rate of rainfall for 10-min periods, in inches per hour	Time, January 7, 1931	Rate of rainfall for 10-min periods, in inches per hour
1:00 P. M.	0.00	2:00 P. M.	0.30	3:00 P. M.	0.54
1:10 P. M.	0.12	2:10 P. M.	0.33	3:10 P. M.	0.60
1:20 P. M.	0.06	2:20 P. M.	0.24	3:20 P. M.	0.24
1:30 P. M.	0.09	2:30 P. M.	0.36	3:30 P. M.	0.42
1:40 P. M.	0.15	2:40 P. M.	0.30	3:40 P. M.	0.63
1:50 P. M.	0.18	2:50 P. M.	0.45	3:50 P. M.	1.38
				4:00 P. M.	1.15

Quantitative measurements could not be made since much débris passed to the Pacific Ocean, but estimates based on mud marks on highway bridge piers (which were as much as 15 in. higher than adjacent bank marks), gave velocities of between 6 and 9 ft per sec. At least 50% by volume was solid matter and the combined débris and water flow was estimated as 700 cu ft per sec per sq mile of water-shed.

A third example—Delta Canyon, a 0.63-sq mile tributary, of Big Tujunga Wash—has a particularly large cone of débris at its mouth (see Fig. 7). In elevation it ranges from 1 650 to 4 500 ft (U. S. Geological Survey); its length is 1.9 miles from crest to mouth; its average width, 0.3 mile; and the grade increases from the minimum slope of 625 ft per mile at the lower end. The canyon forks ¾ mile from the mouth, and at the end of the small fork, running west, is a steep mountain peak which is considered as having contributed the major portion of the débris. A large slip along the entire side of this mountain was first noted in 1913, but some sliding probably occurred previous to that date. The area is badly faulted and the slopes are in an unstable condition.

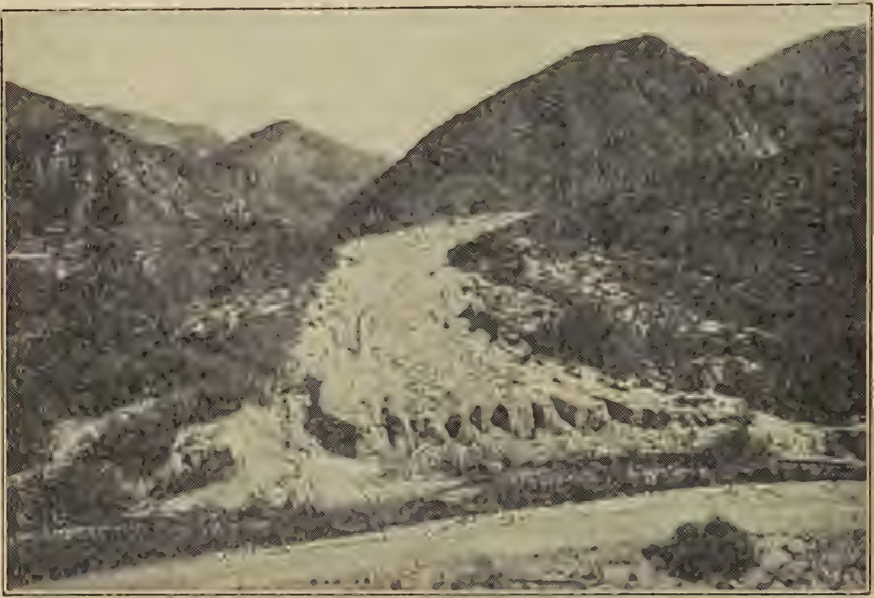

FIG. 7.—DÉBRIS CONE AT MOUTH OF DELTA CANYON.

Fire destroyed 80% of the canyon cover on September 13, 1913, and, during the 1914 flood, débris was deposited at its mouth, shifting the former stream bed to the west. Although slips had been noted prior to 1914 there is no record of débris at its mouth prior to the rains. The heavy rains of 1926 brought down additional deposits. No differentiation can be made between the relative quantities transported by the two débris flows, and a part of the delta has been removed by channel cutting in the main arm of the Tujunga River. However, the volume transported in the two storms is estimated, conservatively, at 123 000 cu yd per sq mile, or more than 60 000 cu yd per sq mile per storm.

The La Crescenta-Montrose débris flow, on January 1, 1934, furnishes the best and most accurate, as well as the largest unit débris flow of authentic record.[5] In this case seventeen contiguous water-sheds have a combined crest length of 6 miles and a total area of 7.5 sq miles. The catchment areas in this region are steep and rugged, rising 2 500 ft in 1.75 miles. The catchments range, in size, from Pickens, the largest, with an area of 1.6 sq miles, to several as small as 0.2 sq mile. They debouch, through sixteen separate channels, on to developed foot-hill, city, and urban areas below the canyon mouths, aggregating about 7 sq miles. The area, which is mainly residential, extends from Tujunga City, the largest and at the western boundary, easterly through the unincorporated cities and towns of Highway Highland, La Crescenta, and Verdugo City to La Canada, the easterly boundary.

The aforementioned water-sheds were completely denuded by fire from November 21 to 24, 1933. They had not been burned previously since 1878. The early storms that began December 14, at 5:00 P. M., and ended on December 15, yielded 4 in. of rain which packed the ash residue into the surface pores of the heavy soil cover, built up during decades of undisturbed brush. Recording rainfall records are given in Fig. 8(*a*) and Fig. 8(*b*) from Flintridge Fire Station at Elevation 1 325 (U. S. Geological Survey). Following this preliminary saturation, came succeeding storms culminating with December 31, on which date successive 1-hr intensities increased from 0.5 in. per hr at noon to 0.78 in. at 1:00 P. M.; 1.8 in. at 2:00 P. M.; and 1.14 in. at 3:00 P. M. After 3:00 P. M. the rainfall subsided to 0.3 in. per hr until midnight when it increased suddenly to 1.28 in. per hr. A flash, 5-min intensity at a rate of 2.16 in. per hr began at 11:47 P. M. Three separate recording gauges, distributed over the area, registered sustained 1-hr intensities at midnight of 1.28, 0.85, and 0.87 in. per hr, respectively; and it is probable that a 1-in. intensity was representative of the entire area. No particular 1-hr intensity was a maximum, all-time record, but the successive cycles of sustained high intensity were without known parallel.

Beginning almost exactly at midnight on December 31, 1933, and lasting for an hour, the resulting succession of débris flows caused 30 deaths; 483 homes were either completely swept away or rendered uninhabitable; and the total property damage aggregated $5 000 000. All the loss of life and 80% of the damage occurred in the eastern part below Pickens and Hall-Beckley Canyons. The largest city, Tujunga, was completely protected by a débris basin at the mouth of Haines Canyon (see Figs. 9 and 10). Another basin was under construction at the head of Verdugo Wash, the main collector at which the smaller tributaries eventually terminate.

Falling on a completely saturated soil, the rain had caused sheet erosion over practically the entire mountain area concurrently with a slippage of numerous masses from steep canyon sides of weathered rock, into the narrow canyon channels. Mass movements originated from canyon sides as high as 200 ft above stream bed, some of the masses being as much as 50 to 150 ft wide. Trees 2 ft, or more, in diameter slipped with the masses and were swept into the streams. When sufficient run-off was impounded behind the earth barriers thus created, to saturate them, they moved down stream until they either drained sufficiently to retard progress, or were stopped by other slides farther down stream. As these movements were repeated, débris was collected in increasing volumes and the masses moved with increasing momentum. Simultaneously, the side tributaries attained peak flows and the combined effect was to release débris and water from canyons in waves 15 ft or more in height. Reaching open territory these waves flattened to

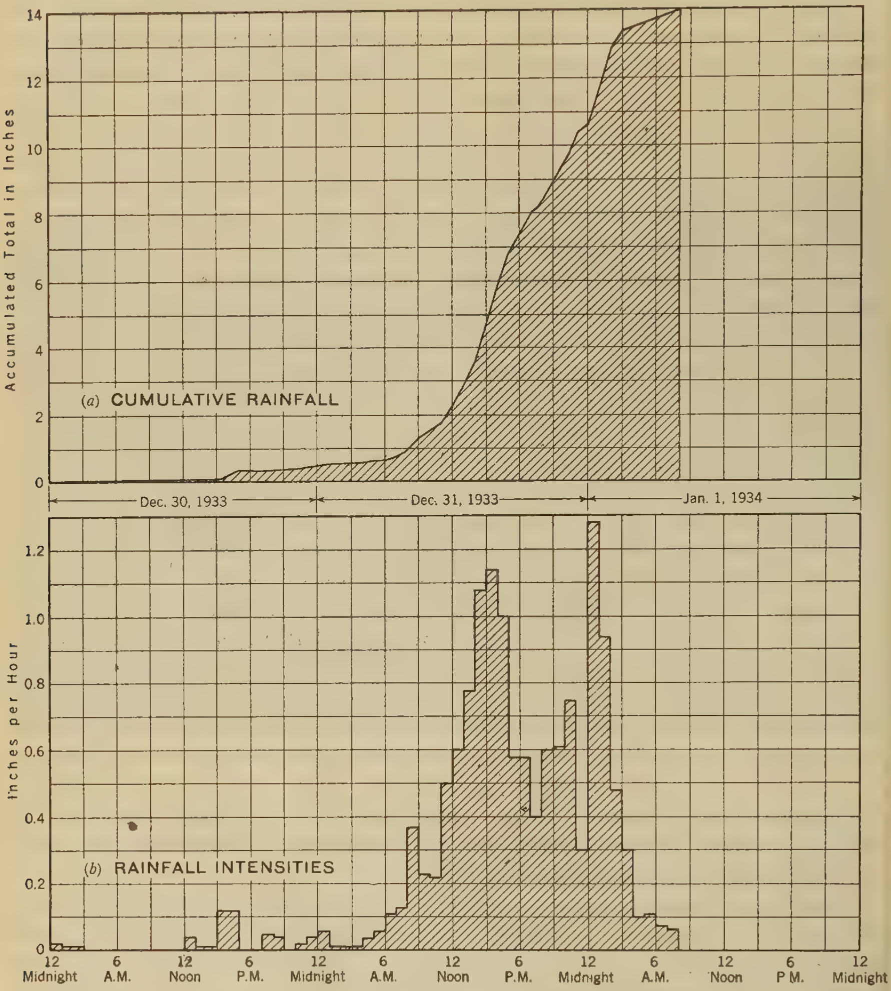

FIG. 8.—RAINFALL RECORDS, STORM OF DECEMBER 30, 1933, TO JANUARY 1, 1934.

5 or 6 ft in height and spread to widths of 100 to 200 ft scouring new channels and thus picking up new débris loads. In canyons velocities were sufficient to carry along many 10-ton boulders. One $59\frac{1}{2}$-ton boulder was rolled on to a paved street at the mouth of Dunsmuir Canyon.

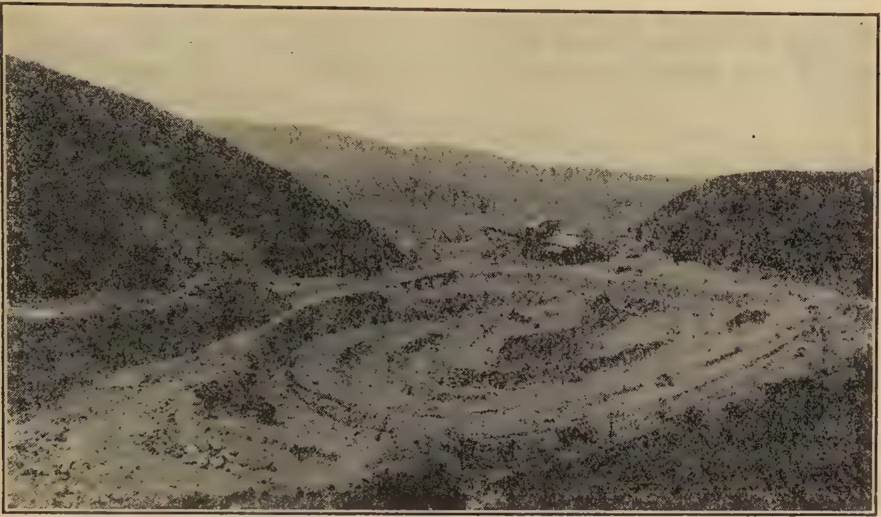

FIG. 9.—VIEW OF HAINES CANYON DÉBRIS BASIN, BEFORE STORM OF JANUARY 1, 1934.

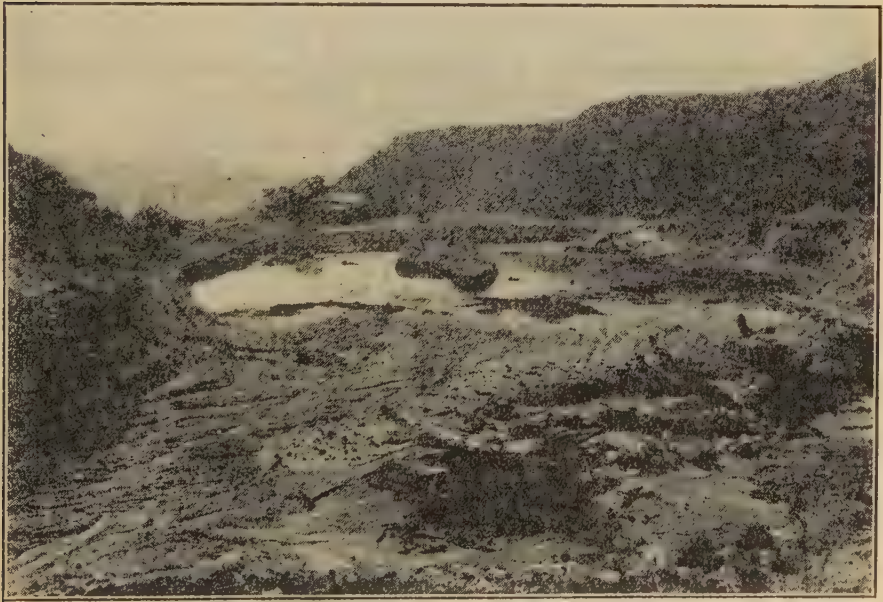

FIG. 10.—VIEW OF HAINES CANYON DÉBRIS CONE, AFTER STORM OF JANUARY 1, 1934.

Based on carefully computed quantities of deposits in the two débris basins, and on field and airplane surveys, débris deposits on property (in cubic yards) were:

In Haines Débris Basin	32 000
In Verdugo Débris Basin	92 000
On property and streets	535 000
Total	659 000

The foot-hill and urban area below the canyon mouths covers about 7 sq miles; of this area 1 040 acres, or 23%, were affected by débris (see Fig.11).

Practically all the débris was deposited within the 1-hr period after midnight, occurring in a succession of about fifteen sharp débris flows, moving at velocities estimated at from 5 to 10 ft per sec. Canyon velocities were considerably greater. Each sharp peak of major débris flow would be succeeded by a rapidly moving stream which contained less débris, cutting a channel to one side or through the deposited débris masses. In some instances these masses would drain out and, successively, become saturated by stream flow to a point at which they would again begin to move; and, encountering a rapidly moving stream, they would be carried along in a succession of surges. It is estimated that the flows containing maximum percentages of débris occupied about 30 min.

The east half of Haines Canyon (the westerly limit of the burned area) had been denuded. Practically the entire débris movement from this source originated from 0.47 sq miles, indicating a débris rate of 67 000 cu yd per sq mile scoured out by a single storm. Débris on property and streets and in the Verdugo Débris Basin came from 7.08 sq miles, equivalent to nearly 90 000 cu yd per sq mile.

An analysis of smaller sizes of débris collected from streets, is presented in Table 10.

TABLE 10.—ANALYSES OF DÉBRIS IN MONTROSE SECTION
(Average weight per cubic foot dry = 103 lb.)

Screen size	SAMPLE NO.								
	1	2	3	4	5	6	7	8	9
1½-in	.....	4.9	15.0	4.2	6.8	......	3.2	13.5	....
¾-in	5.5	7.7	5.7	10.9	10.3	3.8	4.8	11.9	2.1
⅜-in	4.1	9.9	5.7	6.6	8.9	7.6	4.0	6.4	2.9
No 4	5.8	6.8	7.1	5.4	4.1	6.8	5.6	4.8	5.7
No. 8	7.7	10.2	10.0	7.9	6.1	9.9	10.5	7.1	8.6
No 14	15.2	16.7	15.0	14.5	11.0	14.4	17.8	11.9	16.4
No. 28	19.4	19.1	16.5	19.4	15.1	16.7	19.4	13.5	21.5
No. 48	17.3	14.7	12.2	18.8	16.4	15.1	14.5	11.9	18.6
No. 100	12.1	6.6	7.1	9.1	11.0	12.1	9.7	7.9	12.1
Pan	12.9	3.4	5.7	4.2	10.3	13.6	10.5	11.1	12.1
Total (percentages)	100	100	100	100	100	100	100	100	100
Weight, in Pounds per Cubic Foot:									
Dry	109.6	110.7	98.0	96.3	101.3	107.0	105.0	95.3	104.0
Wet	127.5	127.5	118.1	116.5	126.6	123.7	128.7	126.0	126.0
Percentage of voids	33	32.5	39.1	39.2	35.0	32.0	34.3	36.9	35.1

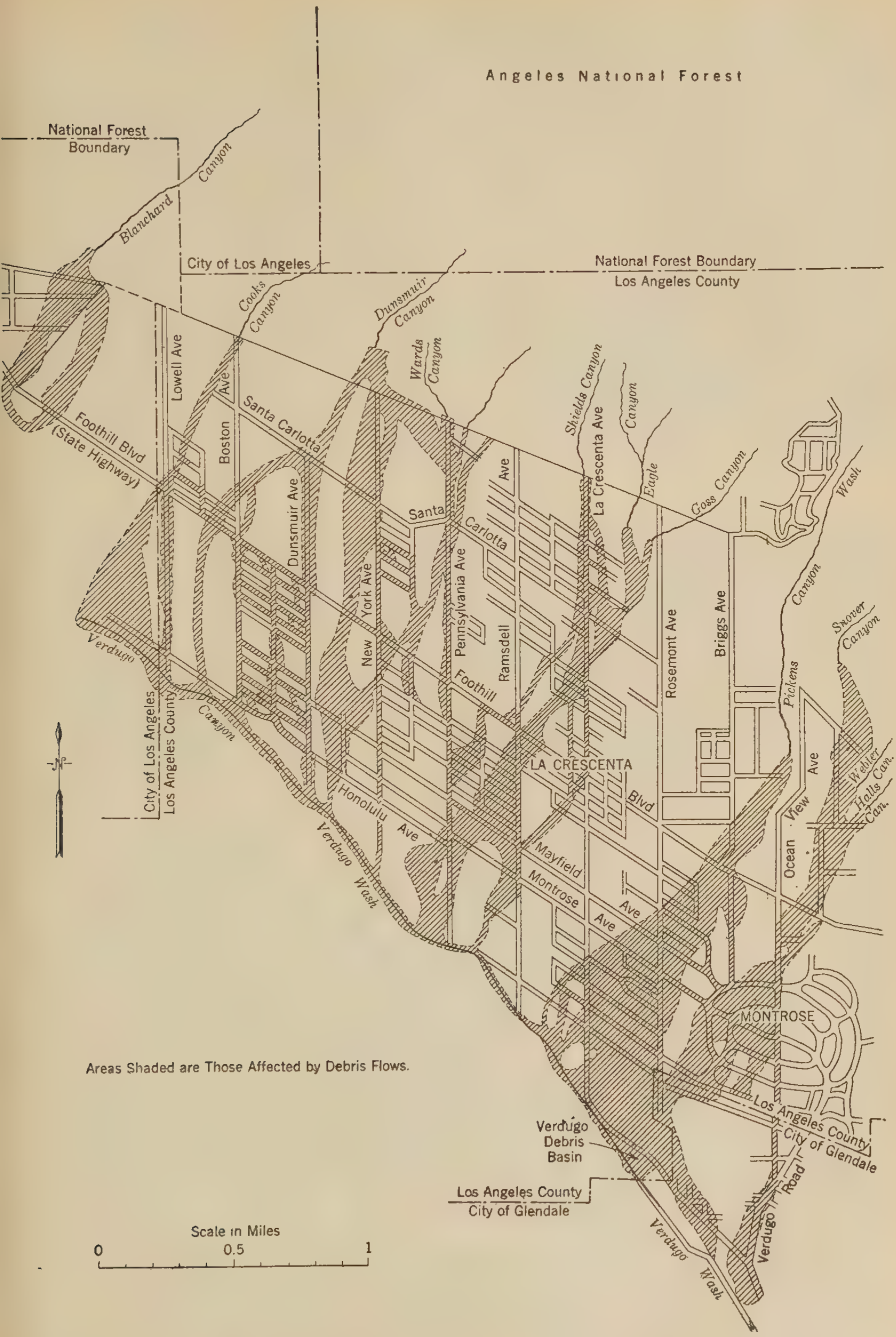
Angeles National Forest
National Forest Boundary
Blanchard Canyon
City of Los Angeles
National Forest Boundary
Los Angeles County
Cooks Canyon
Dunsmuir Canyon
Wards Canyon
Shields Canyon
Canyon
Eagle
Goss Canyon
Wash
Lowell Ave
Ave
Boston
Santa Carlotta
Foothill Blvd (State Highway)
Dunsmuir Ave
Santa
Carlotta
Ave
La Crescenta Ave
New York Ave
Pennsylvania Ave
Ramsdell
Foothill
Rosemont Ave
Briggs Ave
Canyon
Snover Canyon
Pickens
Verdugo
Canyon
City of Los Angeles
Los Angeles County
LA CRESCENTA
Blvd
Honolulu Ave
Verdugo Wash
Mayfield Ave
Montrose Ave
Ocean View Ave
Webber
Halls Can.
Can.
MONTROSE
Areas Shaded are Those Affected by Debris Flows.
Verdugo Debris Basin
Los Angeles County
City of Glendale
Los Angeles County
City of Glendale
Road
Verdugo
Verdugo Wash
Scale in Miles
0
0.5
1
N

Fig. 11.—La Crescenta-Montrose District; Areas Affected by Débris Flood of January 1, 1934.

Cross-sections were taken, following the flood, of the principal canyon channels. A typical section showing Pickens Canyon is given in Fig. 12. The highest water or mud marks were left at moments when the débris flows were either practically stationary or were moving only slowly; but the final lowest channel sections as left resulted from cutting by the later and receding stream flows of higher velocities, which ground the stream bed far below its original grade and below that at the time of the making of the highest mud marks. Computations of peak flows based upon these highest marks and the final channel cross-sections must be modified by judgment, of course, as gained from those witnessing the flows at the time of their activity.

The writer's conclusions were that a sustained 1-hr run-off occurred, which was equivalent to an average depth of 0.9 in., or 580 cu ft per sec per sq mile, and that the maximum débris flow contained 70% of saturated débris to 30% of water by volume, flowing at an average velocity of 5 ft per sec. The relative quantities of débris and water discharged in the 1-hr period were estimated as 48 acre-ft of water and 55 acre-ft of débris.

Débris Control

For major débris movements, positive protection requires direct storage. The best protection thus far devised is the débris basin, which is admittedly expensive and not sightly. It is positive, however, and its first cost is often less than that of one débris removal from streets.

A case in point is the Haines Canyon Débris Basin (see Figs. 9 and 10). Its low capital cost of about $0.25 per cu yd was due mainly: (1) To its location in a Federal Forest area, consequently involving no cost for right of way; and (2) to the basin excavation created by an operating rock plant. Immediately after it was filled, the débris was removed with an average 500-ft haul, for a cost of $0.30 per cu yd. Débris basins at more than one hundred locations will be needed ultimately. Their capital costs per cubic yard of débris capacity will range from $0.30 to $1.00 per yd, depending mainly on excavation costs. Their physical locations are closely fixed, between the canyon mouths and the deltas, at points sufficiently toward the canyon mouths to prevent channel-cutting around them and as far down on the deltas as feasible toward the flatter slopes where excavation costs will be least.

Their storage capacity is obtained, in part, by cut-and-fill excavation in which considerable waste is necessary due to the safety requirement that the deepest part toward the outlet or spillway should be in solid cut. The sides of the basin may be leveed from part of the excavated material. Where feasible, side-channel spillways are advisable, thus forcing the débris flows to enter from the upper end and follow circular paths, facilitating maximum deposition of débris. Sufficient adjacent areas should be provided for future deposition of material.

Where water-sheds are unburned, the operating costs of débris basins are negligible, because little or no débris removal is necessary. After a fire and succeeding high-intensity storms, the basins must be emptied promptly.

The necessity of locating the basin largely in cut, results in excavation costs ranging from 50 to 75% of the total, the other costs being for spillways

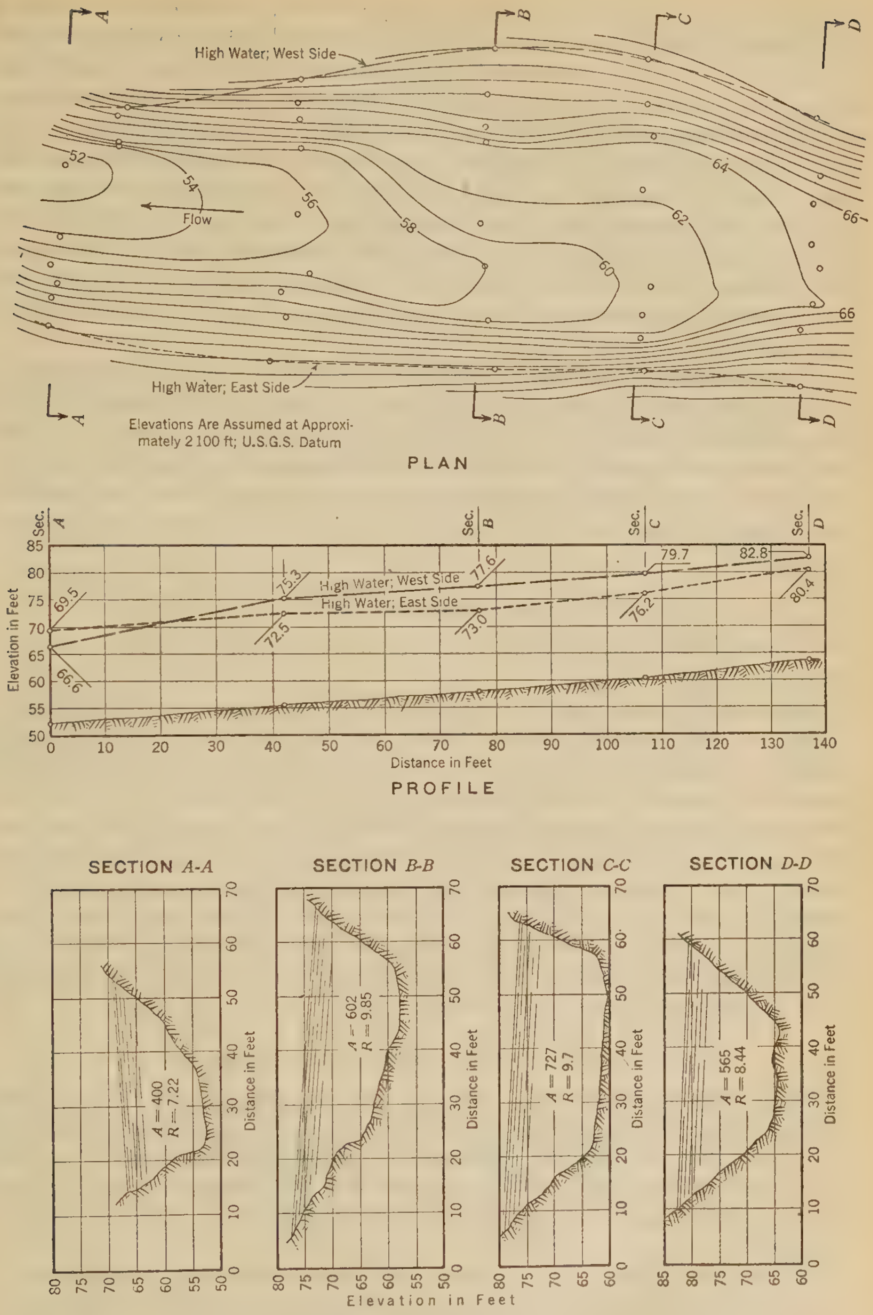
A
B
C
D
High Water; West Side
52
54
56
58
60
62
64
66
66
Flow
High Water; East Side
Elevations Are Assumed at Approximately 2 100 ft; U.S.G.S. Datum
PLAN
Sec. A
Sec. B
Sec. C
Sec. D
Elevation in Feet
85
80
75
70
65
60
55
50
69.5
66.6
75.3
72.5
77.6
73.0
79.7
76.2
82.8
80.4
High Water; West Side
High Water; East Side
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
Distance in Feet
PROFILE
SECTION A-A
A = 400
R = 7.22
SECTION B-B
A = 602
R = 9.85
SECTION C-C
A = 727
R = 9.7
SECTION D-D
A = 565
R = 8.44
Distance in Feet
0
10
20
30
40
50
60
70
80
75
70
65
60
55
50
85
Elevation in Feet

FIG. 12.—PICKENS CANYON: PLAN AND PROFILE OF CANYON, JANUARY 1, 1934.

and auxiliary structures connecting the basin to the head-works of channels. The spillway generally occupies limited space and requires considerable time to construct. On the other hand, the basin excavation covers a large area (1 acre to 2 acres, or more), and thus permits the use of large equipment and rapid construction. Two large power shovels, supplemented by large trucks, can excavate, on a conservative estimate, 20 000 cu yd per three-shift day. Under emergency conditions, a basin for protection from an area of 2 sq miles may be excavated in from 15 to 30 days. Although it is urgently needed during the 3 to 5-yr period following a fire, a débris basin will be inactive over the period of years when no fires have occurred. At best, it cannot be made attractive in appearance and must be located not far from residence property.

These factors suggest the possibility of saving in capital investment by setting aside definite areas, preparing detailed plans, and constructing spillways, channels, and auxiliary structures, but deferring the excavation of pits until fires have occurred. The large excavation equipment can be used later for prompt removal of débris deposits. After mountain cover has regrown, basins may be permitted to fill until fires again indicate that they should be re-excavated.

WATER CONSERVATION INCIDENTAL TO FLOOD REGULATION

Conservation of flood wastes will result from flood regulation for protective purposes.

Increased drafts on local water supplies have been accentuated by decreases in original replenishment due to paving and building activities which have changed former pervious surfaces to impervious surfaces. River channels have been improved reducing thus the original wetted areas. The over-draft from wells tapping the basins between 1914 (the last major flood year) and 1931, has created a present underground storage space of more than 2 250 000 acre-ft. The effects of pumping in excess of replenishment is especially marked in the basins of the coastal plain where serious intrusions of salt water have resulted. Here, water from wells on an area of more than 57 sq miles shows a salt content of more than 15 grains per gal, which is considered to be the limit for human consumption.

SURFACE HOLD-OVER CONSERVATION EXPENSIVE

To provide conservation by surface hold-over storage under Los Angeles County water-shed conditions would run into enormous costs, particularly since hold-over of floods for five years or more would have to be provided. High surface storage costs are here due to the steep slopes and large number of individually small drainages that are characteristic of the mountain territory.

As indicating the costs involved, the twelve reservoirs constructed or under construction by the Los Angeles County Flood Control District will regulate the floods from 400 sq miles, having a combined capacity of 109 000 acre-ft, at an average capital cost of $200 per acre-ft. Under such conditions capacities must be limited to those needed for regulation alone, and, following storm periods, the reservoirs must be emptied progressively at rates permitting percolation into underground basins.

Fortunately, there are available unparalleled natural reservoirs of high capacity. Created by the deposition of sand, gravel, and boulders through the ages, these basins are twenty-six in number. Dikes and geologic faults have created underground barriers, thus forming a series of reservoirs which retard underflow, the water passing over and through these dikes into succeeding basins down stream. Void spaces in the gravels range from 20 to 10%, the larger values applying to the main valley basins, the smaller to the coastal areas.

The logical solution is to conserve the present wastes by regulation and passage into the underground basins. Measured 1931-32 flood-water waste to the Pacific Ocean from 1 841 sq miles of major drainages, was 101 578 acre-ft, indicating conservation possibilities when flood regulation has been provided sufficient for percolation into existing underground basins. The distribution of this flood-water waste is shown in Table 11.

TABLE 11.—MEASURED FLOOD-WATER WASTES TO THE OCEAN, 1931–32

Stream	Drainage area, in square miles	Waste, in acre-feet	Waste, in acre-feet per square mile	Waste, depth in inches of run-off
Los Angeles River	1 063	50 958*	47.9	0.94
San Gabriel River	479	6 555	13.7	0.26
Ballona Creek	112	21 785	194.5	3.80
Malibu Creek	103	14 665	142.2	2.78
Nigger Slough	66	4 029	60.7	1.18
Topanga Creek	18	3 586	199.0	3.88
Total	1 841	101 578	54.9	1.07

* Includes part of San Gabriel run-off transported to Los Angeles River *via* Rio Hondo.

SPREADING TO INCREASE PERCOLATION

Regulated flood flows may be encouraged to percolate in increasing quantities by extending the wetted areas on overlying gravel cones by "spreading." A representative set of percolation measurements made in 1930–31 and 1932 at fifty-three different locations is given in Table 12.

Rates of percolation through surface gravels are mainly proportional to areas wetted and only to a minor degree upon depth of water. Flood flows, even when the main débris has been removed, contain sufficient fine suspended matter, with collodial content, so that a depth of deposit of from $\frac{1}{16}$ to $\frac{1}{8}$ in. is sufficient to reduce the percolation rates 50%, or more, or practically to seal the surface. In stream-bed areas the "scarification" of a few inches is sufficient to increase percolation rates markedly.

Scarification by plowing and harrowing of test sections on the San Gabriel River and Rio Hondo gave results indicated in Table 13.

Increased percolation may be effected by the construction of "spreading works." In their design, ponding alone, or the creation of large shallow basins, does not furnish the complete solution for the percolation of waters containing sediment. It has been proved, however, that by keeping the streams of water moving and by flushing certain areas, the top gravel layers receive a scrubbing action, and high percolation rates can be maintained. The velocities necessary range between 2 and 4 ft per sec and must be supplemented by flushing immediately before, and immediately after, a run. The quantity necessary to waste by flushing should not exceed 5% of the total quantity spread. Spreading works include head-gates from which main and lateral ditches take off through gates in effect similar to an irrigation system. The object of spreading is to provide a wetted area of water in motion covering as large a total area as possible.

TABLE 12.—REPRESENTATIVE PERCOLATING RATES

Stream	Date made	Flow, in cubic feet per second per wetted acre	Stream	Date made	Flow, in cubic feet per second per wetted acre
Rio Hondo	March 25, 1930	2.71	Big Tujunga Creek	February 18, 1932	3.17
	May 6, 1930	1.73	San Dimas Creek	March 31, 1930	1.29
	May 7, 1930	1.81		May 12, 1930	5.05
	February 5, 1931	2.25		February 26, 1931	1.72
	December 31, 1931	4.24		January 6, 1932	1.93
	March 16, 1932	1.89		January 7, 1932	1.25
	March 17, 1932	2.64		March 9, 1932	1.45
	April 6, 1932	1.72	San Antonio Wash	March 14, 1931	1.38
	April 15, 1932	0.91	Big Santa Anita Wash	February 18, 1931	1.42
	August 4, 1931	2.41	Arroyo Seco Wash	January 12, 1932	1.80
	August 4, 1931	1.70		January 14, 1932	1.39
San Gabriel River	March 25, 1930	3.18	Walnut Creek	May, 1930	0.50
	April 2, 1930	2.45	San Fernando Creek	September 29, 1930	2.02
	May 8, 1930	1.85	Pacoima Creek	September 30, 1930	1.86
	February 6, 1931	2.28	Sawpit Creek	March 7, 1932	4.00
	April 30, 1931	1.66			
	December 30, 1931	2.66	Average		2 02
	February 3, 1932	1.75			
	March 16, 1932	1.69	Spreading Grounds:		
	March 17, 1932	1.88	San Gabriel River	January 22, 1931	6.96
	April 6, 1932	1.40	San Gabriel River	May 5, 1932	3.38
Little Dalton Wash	March 22, 1930	1.36	Big Dalton Wash	February 26, 1931	7.00
	May 14, 1930	0.95		March 11, 1931	8.15
Big Dalton Wash	March 22, 1930	1.44	San Antonio Wash	April 27, 1931	2.43
	May 15, 1930	1.57		April 27, 1931	7.45
	February 26, 1931	1.65		March 22, 1932	3.07
Eaton Wash	May 5, 1929	1.19		April 18, 1932	3.27
	December 29, 1931	0.81			
Little Tujunga Creek	February 2, 1932	1.89	Average		5.23
	March 25, 1932	5.00			

TABLE 13.—PERCOLATION RATES IN THE SAN GABRIEL RIVER AND RIO HONDO

Length of section, in feet	Percolation Rates, in Cubic Feet per Second per Wetted Acre		
	Before	After	Percentage increase
2 700	1.75	1.90	10
2 610	0 52	0.88	69
4 270	4.21	4.77	13
2 020	2.15	1.99	28
2 400	1.91		
6 030	2.72	5.50	102

Under a carefully planned spreading system it is possible to provide a net wetted area of from 25 to 35% of the gross area at construction costs ranging from $200 to $300 per gross acre of spreading grounds. Operating costs will range from $0.15 to $0.50 per acre-ft of water percolated, the quantity varying according to the area, its shape, and the layout of the ditch systems.

As a practical average value, from 0.5 to 1 cu ft per sec per gross acre spreading area may be spread, depending upon the location of the area, and the character and layout of the spreading works.

Conclusions

Under physical and weather conditions as described the conclusions are, as follows:

Flood Flows.—On the larger mountains with vegetation undisturbed, a total of 10 in. of rainfall will produce sufficient saturation to permit marked run-off. Storms of intensities of 1 in. per hr and more, if closely following the saturation period, will produce floods, their magnitude rising rapidly with increasing intensities. Twenty-four hour intensities may be expected, varying from 3 in. on the coastal plains to 13 in. on the mountains, with sustained 1-hr intensities of 1 in. to 3 in. Under the higher intensities, following saturation, flood flows of 400 acre-ft per day per sq mile are possible. With the characteristic sharp peaks of these areas this quantity will represent a 1-hr peak of 400 cu ft per sec per sq mile. The regulatory storage needed will depend upon the economic balance between channel costs and regulation costs. In Los Angeles County its range is from 150 to 300 acre-ft per sq mile.

On the lower foot-hills and valley areas with high percentages of impervious surfaces, little or no saturation is needed to produce an immediate response of run-off to rainfall. It is necessary to anticipate future development both domestic and industrial in designing drainage channels in such areas and to set aside adequate areas for the ultimately required drainages.

Débris Flows.—After a normal water-shed cover has been destroyed, erosion rates will increase from 50 to 100 times that with undisturbed vegetation. The best protection from débris is the normal vegetative cover. After denudation, any protective program is a secondary, far less efficient, and much more costly, defense. Erosion of a water-shed removes portions of the soil capable of supporting vegetation. Sufficient regrowth to afford protection will take from 5 to 10 yr, depending upon weather conditions and the extent to which the soil has been removed. With vegetation destroyed protection requires direct débris storage; this can be afforded by débris basins at or near the mouths of the canyons. Following denudation, erosion may produce from 50 000 to 100 000 cu yd of débris per sq mile, the quantity depending upon slopes and intensities. The larger quantity may result from intensities as low as 1 in per hr. Little preliminary saturation is necessary and may be as low as 5 in. if occurring in a short period.

In designing débris basins, flows of 75 000 to 150 000 cu yd per sq mile of drainage should be provided for, arrangement should be made for prompt removal, and adjoining areas should be provided for deposition.

Burning of water-sheds is not always of human origin. Lightning is one cause. Even with the best possible fire-prevention measures occasional fires will occur. Even a foot-hill water-shed 1 sq mile in extent may produce sufficient débris to cause heavy loss of life and immense property damage, when it is located above developed territory. In such areas, basins should be at least set aside; complete detailed plans prepared for their construction; and the auxiliaries of spillways, etc., should be built, so that in emergencies they may be constructed promptly.

Conservation.—Provision for flood regulation for protective purposes will give capacities in excess of that required for conservation of the flood waters that will result from increased percolation, which is principally in proportion to the stream-bed areas wetted. Where increased percolation above that of the natural channel is needed, spreading systems may be built at costs of $200 to $300 per gross acre, with percolation capacities of from 1 to 2 acre-ft per day per acre.

Acknowledgments

The writer desires to express his appreciation to Franklin Thomas, M. Am. Soc. C. E., who made, for the Los Angeles County Flood Control District, the report on the Big Tujunga water-shed, quoted in this paper. Appreciation is expressed to the many members of the District staff who collected the basic data from which this paper has been prepared; and, in particular, to F. H. Hay, Assoc. M. Am. Soc. C. E., Chief Hydrographer, and to R. S. Goodridge and F. J. Cornick, Assoc. Members Am. Soc. C. E., of the Hydrographic Department; also, to Mr. E. C. Kenyon, Jr., in charge of Erosion Control.

DISCUSSION

ARTHUR G. PICKETT,[9] ASSOC. M. AM. SOC. C. E. (by letter).—In the latter half of his paper, Mr. Eaton has demonstrated conclusively that the primary cause of débris flows hazardous to communities of the Los Angeles coastal plain, is the denudation by forest fires of the brush cover of adjacent foot-hill and mountain water-sheds. These fires, if followed by the type of heavy semi-tropical rains that frequently occur in Southern California, result in sheet erosion and débris flows from the mountain sides on to the unprotected communities below. Such flows will not cease until the native brush cover has been fully restored.

No engineer familiar with the topography and geology of these water-sheds, who observed the results of the La Crescenta-Montrose disaster, will question the possibility of a débris flow during such a storm, of from 50 000 to 100 000 cu yd per sq mile of water-shed burned.

Mr. Eaton presents factual data to the effect that the La Crescenta-Montrose disaster resulted in the loss of thirty lives, the destruction of 483 homes, and a total property damage of $5 000 000, all of which was the result of one storm upon a burned area of 7.5 sq miles. To this loss must be added the cost of débris removal and the necessary construction of débris basins and flood-control works adequate in size to prevent similar damage which may occur at any time in the next 5 or 10 yr, or until the native brush cover is sufficiently restored to retard excessive erosion.

Based upon data presented in the paper, the cost of débris removal following the disaster, probably exceeded $425 000, or more than $57 000 per sq mile of water-shed burned. Los Angeles County is admittedly entering a wet cycle, or period of more than average rainfall, during which several storms of greater intensity than the one which caused the disaster may be expected. Consequently, the cost of construction and maintenance of débris basins in the area during the next five years may, according to Mr. Eaton's data, exceed $1 000 000.

The writer is thoroughly in accord with most engineers, who agree that if the mountain water-shed had not been denuded by forest fire, little or no damage would have occurred in the foot-hill communities of this flood area. It would seem, therefore, that the possibility of successful suppression of brush and forest fires adjacent to such communities should receive the primary consideration of engineers interested in flood and débris control.

Similar fires in the Hollywood Hills have been fought successfully by the Fire Department of the City of Los Angeles. R. J. Scott, Chief of the Fire Department, has found that, if trained fire fighters, properly equipped, can reach a fire within 15 min of its inception, and if 10 000 gal of water per min can be placed at their disposal for a short time, they can almost invariably suppress the fire without serious loss of water-shed cover; whereas without this water available, a loss of hundreds or even thousands of acres may be expected. Table 7 of the paper, shows that seventeen fires fought, without the advantage of an adequate water supply, resulted in a loss of 320 400 acres of water-shed in Los Angeles County, an average of 18 900 acres per fire.

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[9] Civ. Engr., Los Angeles County Surv., Los Angeles, Calif.

Although the personnel of the Federal Forest Service and County Forestry Departments is thoroughly organized and trained, limited appropriations have prevented the installation of water systems, as well as the acquisition of other much needed facilities for forest fire protection. Until recently, such facilities have been regarded by many as an unwarranted expense, even in the relatively small areas, adjacent to heavily populated communities, where more than 90% of such fires originate.

Following the La Crescenta-Montrose disaster, plans and specifications were prepared for the construction of a project, under the work relief program, to provide a water system, motorways, firebreaks, trails, and similar facilities necessary for the fire protection of the extremely dangerous water-shed above the communities of Altadena and Pasadena.

This project was the first unit of a comprehensive plan to provide such facilities for approximately 16 000 acres of valuable but dangerous mountain water-shed, hazardous to the communities of Altadena, Pasadena, South Pasadena, Sierra Madre, Arcadia, San Marino, San Gabriel, and Rosemead, with a group population of more than 300 000.

The only feature of the plan, not included in similar projects now under construction (1935), is the water system. The quick, economical delivery of the large quantities of water required, presented many new and unusual problems which were complicated by the steep grades and topography encountered. These problems were solved, however, and an adequate water system was designed which would enable fire-fighters to reach any point in the area with heavy streams of water under gravity pressure, by the extension of not more than 500 ft of hose.

As designed, the main storage reservoir of the system is located above the highest point in the area to be protected, and will have a capacity of 1 000 000 gal. Water will be delivered to this reservoir, either from a rainfall catchment area, or by gravity flow through 4 000 ft of 6-in. pipe line constructed into Millard Canyon, where an adequate, reliable stream of water is always available.

From this reservoir, 6-in. steel pipe lines will carry the water down the main ridges to eight secondary reservoirs of the system, which are necessary for pressure regulation and additional storage. From these reservoirs, 6, 4, and 3-in. pipe lines will extend to roads, trails, motorways, and other strategic points throughout the area. One hundred and fifty fire hydrants, fitted with adapters to permit instant connection of either the 1½-in. hose used by the Forest Service, or the 2½-in. hose used by County Fire Departments, will be located along the various water lines.

The design will permit full delivery of water from each hydrant, even if all the other hydrants of the system are in operation at the same time. Sufficient water storage is provided to assure an adequate supply for the fighting of fire throughout the area. At points of unusual hazard, where even a city fire stream might not prove effective, small monitors or hydraulic giants have been provided to operate under higher pressures and with larger quantities of water than would otherwise be possible.

Specifications provide that all reservoirs shall be constructed in excavation, lined with reinforced concrete to prevent leakage, and covered to reduce evaporation; and that all pipe shall be Grade A steel, mill-tested to 1 100-lb pressure.

An area of 1 600 acres would be protected in this manner by the project, all of which is located in the "high risk zone," where Forest Service officials expect 90% of all fires to start. The total cost of materials and supplies required for the construction of this unit of the water system is $69 600. Commenting on the project publicly William V. Mendenhall, Supervisor of the Angeles National Forest, stated that with such facilities available, his men would almost guarantee to suppress any fire in the area with a loss of not to exceed 1 acre of water-shed.

In the higher mountains adjacent to this project, the danger of fire being started by human carelessness is reduced materially. If fires do occur, conditions are such that Forest Service officials believe smaller quantities of water will be required for their suppression. Consequently, fewer water lines are required in these areas of secondary risk, and the cost of the water system per square mile is reduced materially.

Before the disaster, the La Crescenta-Montrose water-shed included both "high" and "secondary" risk zones. A similar water system, designed to protect the entire burned area, could have been constructed at any time previous to the fire, at a cost for materials and supplies of $150 000. The cost of relief labor, or that of labor by the Civilian Conservation Corps, could not be considered as a true charge against the project. However, if constructed by contract, the installation of all facilities necessary for fire protection would not have cost as much as the removal of débris immediately following the disaster.

Unquestionably, sites for débris basins should be purchased and other protective measures, as outlined by Mr. Eaton, constructed for the protection of communities from débris flow in case the water-shed is denuded by fire. It would also seem logical to spend an equivalent sum of money to insure against such a catastrophe, by the installation of water systems and other facilities necessary to combat, successfully, the many forest fires that do occur. This is especially true when the chances are 100 to 1 that, sooner or later, the water-shed will be destroyed unless such facilities are provided.

R. W. Davenport,[10] M. Am. Soc. C. E. (by letter).—In studying this paper, the writer has been interested in examining, rather critically, the section entitled "Flood Hydrograph Determination", and some of the incidental observations are submitted herewith.

Fig. 3 shows the run-off for the basin for the years ending September 30, 1903 to 1930, plotted against the respective indices of seasonal wetness, or the estimated mean annual precipitation over the basin expressed in percentage of the mean as determined for the period, 1872–1930. The run-off from 1916 to 1930 was estimated from the record of the Sunland gauging station of the U. S. Geological Survey evidently by application of the drainage area ratio (124.5 sq miles ÷ 106 sq miles), and, from 1903 to 1915, the run-off was deduced from the record of the neighboring San Gabriel Basin. These data give the twenty-eight points plotted in the diagram.

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[10] Hydr. Engr., U. S. Geological Survey, Washington, D. C.

In undertaking to express a general relationship between precipitation and run-off in a problem of this kind, it is a commonly accepted procedure to consider run-off as a residual after the precipitation has been subjected to losses by evaporation and transpiration, and possibly other losses, usually of relatively minor importance. These losses, like the precipitation, are thus to a large extent the resultant of the operation of meteorological agencies. It is not practicable, of course, to determine the factor representing the losses by direct observation, but it may be determined indirectly by subtracting the run-off of a basin from the precipitation for periods selected so as to eliminate or to minimize differences due to changes in surface or ground-water storage. It is the writer's observation that for humid regions the factor varies in a rather definite and systematic manner, dependent upon latitude, altitude, temperature, and other regional characteristics and that from year to year it has an amplitude of variation corresponding roughly to that for precipitation. In arid regions, where the rainfall is insufficient to meet the potential demands of evaporation, transpiration, etc., the losses, of course, will tend to be smaller and more variable than in regions where the rainfall is ample.

Table 5 has been examined with regard to the indicated losses from 1903 to 1930, and the results, for groups determined by the amount of the annual precipitation, are listed in Table 14. Over the period of years considered, these data show a definite tendency for the losses in the Big Tujunga Basin to increase as the precipitation increases. For rainfalls of 30.1 in., and more, the losses varied from 22.5 to 32.7 in. and averaged 27.4 in.—apparently not much different from what would be expected in any similarly located humid region of similar temperature and similar quantities of rainfall.

The curve drawn through the plotted points of Fig. 3 takes an upward direction, so that for an annual precipitation of 54.6 in. (index of seasonal wetness = 200), the losses would approximate 23.4 in., whereas for the maximum year (1883–84), with a precipitation of 65.8 in., the application of the derived relationship by extrapolation, gives losses amounting to only 18.1 in. However, the facts indicated by Table 14 and the interpretation of general experience elsewhere, as noted, suggest that the losses with 65.8 in. of precipitation would be materially more than 18.1 in. and might well be

TABLE 14.—PRECIPITATION, RUN-OFF AND LOSSES, BIG TUJUNGA DRAINAGE BASIN

Range of precipitation, in inches	Number of years in groups	Mean precipitation, in inches	Mean losses, in inches
20 and less	7	16.9	16.2
20.1–30.0	9	24.1	21.4
30.1 and more	12	35.7	27.4

assumed to be 27 in., or more. This assumption would lead to a reduction of the estimated seasonal run-off for the maximum year from 316 700 acre-ft to probably not more than 250 000 acre-ft. In view of the rather large difference thus obtained, the writer would be pleased to have the observations of the author regarding it.

The writer is especially interested in the forms of hydrographs shown in Fig. 6, because they seem to depart from the forms that he would expect from his experience with comparable streams. The hydrographs seem to show that for four successive days, substantially the entire run-off from rainfall on each day cleared the basin within a 24-hr period. General experience and a superficial examination of the rainfall and run-off conditions of the Big Tujunga Basin suggest that the run-off from the rainfall on a given day will continue in substantial, but gradually decreasing, amounts for two or three days after the time of maximum flow from that rainfall. Studies of characteristics of streams in this respect have been presented by LeRoy K. Sherman, M. Am. Soc. C. E.,[11] and Merrill M. Bernard, M. Am. Soc. C. E.[12] Therefore, the writer is considerably interested to know whether there are conditions in the Big Tujunga Basin that warrant the assumption of hydrographs as shown, which are seemingly somewhat at variance with the indications of experience elsewhere.

The writer has not examined what, if any, difference these considerations might make in the conclusions with regard to flood flows. In general, a momentary peak of 388 cu ft per sec per sq mile for the Big Tujunga drainage basin as adopted seems a reasonable and appropriately conservative estimate for the peak flow of a rare flood.

The compilation of data and experience on flood and erosion control problems in this paper present a most interesting and valuable contribution.

C. S. Jarvis,[13] M. Am. Soc. C. E. (by letter).—Although the problem outlined in the paper may be more nearly representative of extreme conditions than of average or usual flood and erosion control problems to be encountered in the United States, it follows that an adequate solution for the more difficult situation may be applicable, at least in part, for the simpler ones.

The observed and reasonably expected rates of precipitation for the foothill region around Los Angeles, Calif., are far from the maxima observed elsewhere for both long and short periods, particularly along or near the Gulf Coast. Likewise, the rates of run-off per square mile of drainage area thus far observed in that foot-hill region are considerably less than have been recorded for numerous small streams. The outstanding or unique features of the situation described by the author comprise the following: (1) Steep slopes on the windward side of relatively high mountain ranges, and near the coast; (2) warm or moderate all-year temperatures except on the higher elevations; (3) seasonal precipitation occurring mainly during the winter;

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[11] "Streamflow from Rainfall by Unit-Graph Method", *Engineering News-Record*, April 7, 1932.

[12] "An Approach to Determinate Stream Flow", *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 347.

[13] Hydr. Engr., Soil Conservation Service, Washington, D. C.

(4) droughts occurring every summer, thus limiting the types of surviving vegetation and resulting in recurrent fire hazards; (5) unconsolidated débris and soil materials depending for their stability, on the steeper slopes, largely on the roots of vegetation; (6) soil surface partly protected by plant foliage, and incapable of maintaining itself without such protection; (7) metropolitan and urban developments on the detrital cones and flood-plains; (8) local water resources deficient, especially during the seasons of maximum demand; (9) justification for storage developments to utilize all the normal run-off, thence to facilitate recovery and repeated use; (10) ground-water supplies heavily overdrawn, with consequent intrusion of salt water to menace and further deplete the remaining ground-water reserves; (11) unusual opportunities for flood regulation by spreading, and by storage in basins partly bounded by loose, permeable strata, for the double purpose of local protection of lives and property and prompt replenishment of underground storage; and, (12) property values capable of supporting a comprehensive program for flood regulation and protection, water conservation, and its maximum utilization.

It is readily granted that other localities may be characterized by one or more of the aforementioned features which occur in combination in the Los Angeles District; but nowhere else in this country are all the factors represented to such a degree. Therefore, the measures economically justified for the Los Angeles situation may be applicable only in part elsewhere.

It has been widely observed among the head-waters of streams in both foot-hill and mountainous regions that minor landslides are going on progressively during any normally wet season, and occasionally during periods of deficient rainfall. The planes of contact between ledge rock and products of weathering are usually planes of weakness and instability; this is also true of rock débris on inclined beds of clay or other relatively impervious material. Water trickling along such planes naturally acts as a lubricant, inducing motion, cleavage, subsidence, and occasionally landslides.

Such surface disturbances among head-waters are so common and so unimportant as to escape prominence. The remoteness from the main channel of the stream, the relatively small volumes of water yielded from the disturbed areas, and the correspondingly small amount of erosion resulting therefrom, all combine to keep such phenomena in the background. However, when they occur along the lower courses of streams, where lives are endangered and property threatened, they assume considerable prominence. It appears that some of the unstable masses on such slopes are susceptible of treatment to insure their permanency in present positions; or, occasionally, to accelerate their stabilization by inducing movement to flatten the slope to a natural grade of repose, corresponding to most severe conditions expected to be encountered.

A parallel is afforded in the treatment of sites for flumes, pipe lines, or other structures near the foot of a slope. It is advisable and customary to clear the upper slopes of boulders or other rock masses that might be loosened or detached by weathering processes; or, occasionally, it is deemed expedient to repair or connect protruding ledges to serve as barriers for the protection of the lower slope.

Recent practice in the construction of trails and access roads among scenic mountain areas deserves more than passing notice. To prevent accumulations of water on the trails or roadways, diverting channels have been constructed to discharge into basin-like depressions, such as the borrow-pits from which surfacing material has been obtained. In such cases, where several feet of depth and potential hydrostatic pressure are provided, and where coarse materials, such as gravel or rock waste and other loose materials are exposed, the rapid percolation and the storage combine to dispose of surface runoff. The ease with which a large number of such percolation and storage facilities may be provided seems to make up for their restricted capacity, taken separately.

Several municipal water supply or irrigation projects, as well as hydroelectric power projects, have proved failures due largely to leakage from the reservoirs. Where comprehensive developments are undertaken for the benefit of all legitimate interests on a river system, it may well prove to be a profitable venture to seek reservoir sites that will feed underground storage by leakage or percolation. Where test borings disclose coarse, pervious materials at reasonable depths below the valley floor, it may be advisable to trench or cut through the more nearly impervious overlying material to induce underground flow and storage.

Modifications or adaptations of the foregoing methods may well be adopted in connection with flood and erosion control problems. Particularly, the provision for many avenues of access to strata of coarse, open texture, and multiple small storage basins for both water and débris may measurably relieve the burden to be sustained by the larger reservoirs down stream, helping to maintain their original capacity and thus adding to their effectiveness.

Doubtless, general recognition of existing facts would prove that nearly every locality is confronted with flood and erosion control problems, each meriting individual treatment and capable of a best solution; or, where misguided by inexperienced or over-confident advisers, a community may appropriate and spend generously without measurable progress toward a satisfactory solution. The author has done much to clarify the view as to potential dangers from those natural phenomena, and as to remedial steps which have proved effective in the Los Angeles District. Mr. Eaton has rendered a distinct service by his able presentation of the subject-matter.

Harry F. Blaney,[14] M. Am. Soc. C. E. (by letter).—Some excellent data on débris flows in Southern California are presented by Mr. Eaton. The writer and Colin A. Taylor, Assoc. M. Am. Soc. C. E., made a brief survey[15] of damage resulting from movement of débris out of canyons during the storm of December 30, 1933 to January 1, 1934, described by the author. This incident offers the opportunity to draw several object lessons and to take steps to guard against a recurrence of such losses.

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[14] Irrig. Engr., Bureau of Agricultural Eng., U. S. Dept. of Agriculture, Los Angeles, Calif.

[15] Unpublished report, "Damage Resulting from Movements of Débris Out of Canyons During the Storm of December 30, 1933, to January 1, 1934, in Southern California", by Colin A. Taylor.

The movement of large masses of detritus out of mountain canyons as mud flows is not an unusual geologic occurrence. Large débris fans have been built up and are steepest and most rugged at the mouths of small canyons in which the stream flow is intermittent. The increasing development of the country has brought more and more of these areas under cultivation, and many of the most desirable residential sections are located high on the fans. However, in many cases, unregulated subdivision has caused the development of entire fans so that the natural watercourses are encroached on before adequate protection is provided.

In mud flows saturated masses of soil furnish the transporting medium that carries huge rocks out of the canyons. The rocks may originate in the same slide with the soil, or they may be picked up in the canyon bottom when temporary dams formed by slides break loose.

A viscous mass of mud slipping from a canyon wall may contain just enough water to make it flow, and progress may be relatively slow in the flatter slopes of the canyon bottom. In this condition, huge rocks are carried along without rolling and the movement has the appearance of a lava flow, and, in this condition, may move slowly out of the canyon on to the fan.

When a large quantity of surface run-off occurs at the time the saturated soil slips from the canyon walls, the water may first pass over the top of the slower moving mud flow. Water may also pile up behind the temporary dams and then, as they give way, mix with the soil and make the mass less viscous. As the proportion of water increases the mass becomes more fluid and the rocks tend to settle and roll or slide along the bottom, while the finer material mixed with the water moves over the top of the mass with increasing velocity. The puddled mass that moves the larger boulders drains very slowly.

Eight samples were collected at 4:00 P. M., on January 2, 1934, of the fine material in the wash under some large boulders at Foothill Boulevard, north of Montrose, Calif. (see Fig. 11). The average moisture content was 18.4% by weight, and after screening out the material larger than 2 mm in diameter, the percentage of moisture was 27.8, based on the oven-dry weight of the material smaller than 2 mm. The percentage of water in a saturated soil as it lies in place before it first starts to slip would be approximately 25% by weight, or 33% by volume. This saturated fine material acts as a lubricating medium and assists in the transportation of the larger boulders.

In the canyon bottom mechanical mixing with more water takes place, and the percentage of water must be higher except in the slower moving mud flows. The water content of the material must be continually varying through wide ranges so that the evaluation of flows from gauge heights is most uncertain. There is also the further complication caused by the underlying rocks and viscous layers of material moving much more slowly than the more fluid material on top. Furthermore, the churning bores of flood crests also contain air which tends to increase the maximum gauge height or high-water marks. From this, it follows that the estimation of flows of this character by gauge height record at the ordinary gauging station is most impractical as no record is obtained of the water content of the flow, nor can the velocity be considered as a water equivalent.

It would appear that the most practical method of measuring discharge rates under conditions such as prevailed in these canyons is to gauge the rate of rise in the débris basin at the canyon mouth and also to establish a gauging station in the channel below the débris basin.

One of the most important functions of brush cover in controlling run-off is the aeration of the soil. Decayed roots and worm holes furnish channels through which air is readily moved from the lower soil depths. Rain does not ordinarily run down these channels as is popularly supposed, but moves into the soil as capillary water. The function performed by the roots and worm holes is to release air back to the surface as it is displaced by the downward moving moisture. This release of air prevents the development of back pressure, which would materially reduce the rate of infiltration of rain water, particularly in long-continued steady rains.

The burning of the water-shed increases the potential danger many-fold, but mud flows will occur on certain soil types with the vegetative cover still intact as witnessed by the flows from Neusbickel Canyon, near San Dimas, Calif., during the same storm. In this case, the soil is a clay loam and the moisture could not be transmitted into the more compact underlying subsoil fast enough to prevent saturation of the top soil. Slips of the top soil then occurred on the steeper slopes. The soil in Neusbickel Canyon supports a dense growth of mustard that starts each winter and dies the following summer. There are also occasional patches of brush. The canyon sides are very steep—60° and more in many places. A 5-in. rain on December 12 to 14, 1933, filled the top soil to field capacity, and the fresh growth of mustard was less than ½ in. high at the start of the storm on December 30, when 15 in. of rain fell within 30 hr. saturating the top soil so that it slipped down the slopes, at many points forming great arrow-shaped scars on the hillside. In some cases, the movement was slow so that the material was carried away as fast as it dropped into the stream bed. In others, the evidence shows that great masses avalanched down and temporarily blocked the canyon. At the end of the storm, however, the stream bed was swept clean to bed-rock and the débris was deposited in a fan-shaped mass at the mouth of the canyon.

The control of moving masses of detritus has been a serious problem in the Western States for many years, and the protection of the irrigated agricultural areas from floods has been the concern of governmental agencies. The problem has been studied in Utah, and control methods have been devised by engineers of the U. S. Bureau of Agricultural Engineering. The results of the Utah investigations were published in 1933.[16] Similar methods of control might be effective in some areas of Southern California for the protection of life and property in the foothill areas.

Mr. Eaton discusses "Spreading to Increase Percolation" and, in Table 13, contributes some valuable information on the effect of scarification by plowing and harrowing on the rates of percolation in the San Gabriel River and Rio Hondo. In this connection, the U. S. Bureau of Agricultural Engineering has been conducting experiments on water-spreading at the mouth of

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[16] "The Barrier System for Control of Floods in Mountain Streams", by L. M. Winsor, Assoc. M. Am. Soc. C. E., *Miscellaneous Publication No. 165*, U. S. Dept. of Agriculture.

San Gabriel Canyon for several years. This investigation, which is in charge of Mr. A. T. Mitchelson, has produced some interesting data on the stimulating effect of undisturbed native vegetation by root activity on the rate of percolation. In 1930, an experimental plot of 0.38 acre was diked and water was applied uniformly in such a way as not to interfere with the healthy growth of the vegetation. Continuous measurement was made of the water flowing on to the plot and of the run-off. In 1931, a second plot of like size, shape, and soil type was established adjacent to the first, but from it the vegetation was removed, the roots grubbed out, and the surface plowed, harrowed, and furrowed. It was operated simultaneously with the other plot. In 1932, a third plot was established, which was also adjacent to and of the same size as the first, and was operated simultaneously with the others. The third plot, however, was operated on the basin method, a head of about 6 in. of water being held over the wetted area.

In the season, February to June, 1930, during a continuous run of 87½ days, the plot in native vegetation absorbed water at rates averaging 3.98 acre-ft per acre per day and ranging from 3.03 to 5.33 acre-ft per acre per day. In January to March, 1931, during a run of 82 days, this plot absorbed at rates averaging 6.50 acre-ft per acre per day and ranging from 2.13 to 7.92 acre-ft per acre per day, while on the furrowed plot the daily absorption rates averaged only 2.53 acre-ft per acre and ranged from 1.15 to 4.16 acre-ft per acre. In the season, January to June, 1932, the daily percolation rates for the first plot again were highest, averaging 5.03 acre-ft and ranging from 2.12 to 8.04 acre-ft per acre. On the furrowed plot, the rates ranged from 1.66 to 4.22 acre-ft per acre and averaged 2.93 acre-ft per acre per day, while on the basined plot the minimum, maximum, and average rates were, respectively, 2.22, 5.08, and 3.76 acre-ft per acre per day.

W. P. Rowe,[17] Assoc. M. Am. Soc. C. E. (by letter).—A very complete outline of the flood-control problems of Los Angeles County is presented in this paper. The information contained in Tables 12 and 13, dealing with percolation rates, is of particular interest as millions of dollars are now (1936) being spent in Southern California on projects designed to increase the supply of the underground water basins. In view of these enormous expenditures the writer believes that the economic side of this problem should receive more particular attention.

The paper contains certain statements relative to quantities of material eroded from mountain sides which the writer can not pass unchallenged. This is especially true in view of the widespread interest being displayed in erosion control and the propensities of certain erosion-control enthusiasts in quoting published data to support their claims without analyzing their accuracy.

In discussing this feature of the paper, the writer will use the author's definition, namely: "Erosion, in this instance, is defined as the washing of loose material from weathered mountains and foot-hill areas, and its deposition upon the lower valley floors." Four quantitative estimates of erosion from burned-over areas are cited by Mr. Eaton and he states: "The La Crescenta-Montrose débris flow, on January 1, 1934, furnishes the best and most accurate, as well as the largest unit débris flow of authentic record." Haines Canyon is included in his treatise on this area. Under "Erosion with Normal Water-Shed Cover", he states: "Where water-sheds have not been disturbed by fires, roads, earth-slides, etc., erosion is relatively slight", and immediately preceding Fig. 8: "Mass movements originated from canyon sides as high as 200 ft above stream bed, some of the masses being as much as 50 to 150 ft wide. Trees 2 ft, or more, in diameter slipped with the masses and were swept into the streams." It is true that the brush cover on the mountain sides was burned, but most of the trees and other canyon bottom growth escaped unharmed by the fire. This is characteristic of most mountain fires in Southern California. If one were to believe evidences of previous earth slippages in this area and the statement in the paper that this area had not been burned over since 1878, one must conclude that similar slippages would have occurred during the storm of January 1, 1934, regardless of the fire.

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[17] Cons. Engr., San Bernardino, Calif.

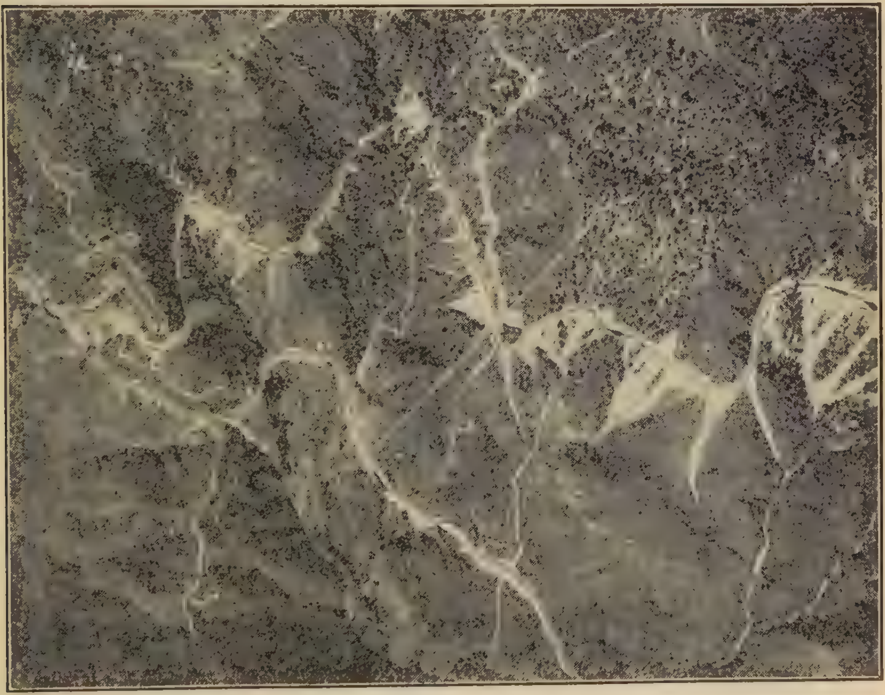

Fig. 13.—Erosion in the San Bernardino Mountains

It is almost impossible to find a water-shed in Southern California that conforms to the author's description of one on which erosion is relatively light because of the absence of previous "fires, roads, earth-slides, etc." Fig. 13 shows a portion of the San Bernardino Mountains north of San Bernardino. It depicts all the major man-made aids to erosion, namely, earth-slides created by cutting faulted ground, and earth-slides created by the overcast of materials excavated from side-hill cuts, fire-breaks, old roads, and new roads. Most of the area was burned over by fire in 1911 and parts of the same area have been visited by two other fires since that time. The débris removal from the mountain sides by storm waters after the fires was very slight in comparison with that due to the various construction activities. No evil effects from the fires are visible either on the ground, or from the air, and the escape of canyon bottom vegetation from fires is characteristic of Southern California brush fires. The apparent lines of erosion down the canyons is due to the movement of overcast material.

Under "Erosion and Erosion Control", the author states that the forces of erosion are still active and have built deposits of many thousands of feet of débris in the valleys and he apparently ascribes this deposition to the results of burning of water-shed cover. In his analysis of the quantities of erosion from the San Gabriel Canyon area (see "Erosion with Normal Water-Shed Cover"), he shows that the flood of 1927, one of the highest peak flows of record, caused the erosion of only 3 200 cu yd per sq mile, 3 yr after a major fire. With this as a basis one should assume that it is only a combination of heavy rainfall on a burned-over area that causes débris flow from an undisturbed water-shed. The coincidence of a season of heavy rain following a fire which might occur every 56 yr, as in the case of the La Crescenta-Montrose flood, is too remote to account for the deposition of the enormous valley fill in the La Crescenta-Montrose area, unless one were to reverse the old stand-by theory of the Forest Service propagandists and assume that forest cover no longer makes the rainfall, but that lack of forest cover is responsible for heavy rains. The logical explanation lies in a study of the geology of the water-shed by which it will be found that erosion has been active regardless of fires.

Following Fig. 8 the author, describing the movement of peak flows of débris and water below the canyon mouths, states: "Reaching open territory these waves flattened to 5 or 6 ft in height and spread to widths of 100 to 200 ft, scouring new channels and thus picking up new débris loads"; then (preceding Table 10): "Each sharp peak of major débris flow would be succeeded by a rapidly moving stream which contained less débris, cutting a channel to one side or through the deposited débris masses"; and, again (following Fig. 11): "* * * but the final lowest channel sections as left resulted from cutting by the later and receding stream flows of higher velocities, which ground the stream bed far below its original grade and below that at the time of the making of the highest mud marks."

The débris cones on which this channel cutting and shifting occurred are the result of previous erosions and the re-working of this material can not be classed as erosion under the author's definition. Nevertheless, he includes the quantity of this re-eroded material in his estimates of erosion per square mile of water-shed.

The author has failed to mention one very important factor in describing the shifting channels and later cutting on the La Crescenta-Montrose débris cones. Prior to the flood, a series of wire and rock check dams had been constructed across the active stream channels. The crests of these dams in many instances were less than a foot below the top of the banks on either side. When the flood flows from the mountain occurred, they overtopped these dams and scoured out the loose alluvial material below them and deposited it behind the dam next lower down stream. When the small basins behind these dams were filled to the crests of the dams, the water and débris were diverted laterally around the ends of the dams and on to the adjacent territory. The 59½-ton boulder mentioned by the author (see following Fig. 9) was deposited in this manner. It was not carried from the mountain side by this one flood, but rolled down the débris-filled channel until it was diverted by one of these check dams. The remnants of the check dams in this vicinity gave ample proof of this theory. The fact that the active channels on these débris cones usually ride on the highest part of the cones aided in this channel shifting.

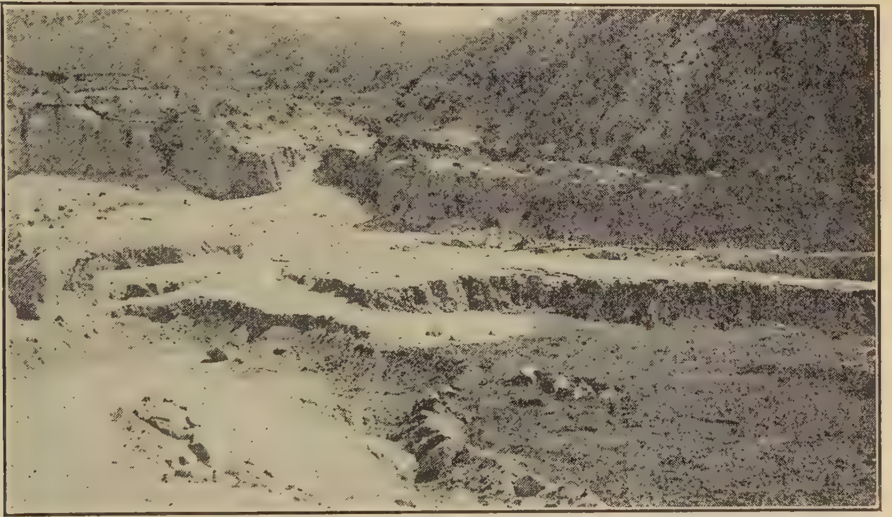

FIG. 14.—DÉBRIS BASIN IN HAINES CANYON ON MARCH 19, 1934.

The writer will agree with the author in the statement as to the effectiveness of the Haines Canyon débris basin in controlling débris flows, but disagrees when he attributes the 32 000 cu yd of débris caught in the basin to erosion from the mountain side under his definition. This basin was the result of the operation of a gravel company excavating at the head of the débris cone of Haines Canyon at its apex within the canyon walls. The original active channel had a slope of 10 per cent. The operations of the gravel company in excavating below the original grade of the stream resulted in a face at least 40 ft high at the upper end. This face was practically vertical and consisted of sand, gravel, and boulders. When the flood occurred, the water poured over this face and started to re-adjust the grade of its bed to fit this new condition. The result was a cutting of a deep channel up stream from the gravel pit, all the eroded material being deposited in the basin.

The classification of all the material deposited in the débris basin as erosion from 0.47 sq mile of burned-over water-shed and the assumption that, therefore, the rate of erosion from 1 sq mile of burned-over water-shed was 62 000 cu yd, would not conform to the author's own definition.

Fig. 14 is a view looking up stream across the Haines Canyon débris basin on March 19, 1934. It shows (1) the re-adjusted stream bed with banks from 15 to 30 ft high; (2) the débris cone deposited within the basin, but above the spillway level; (3) the original high-water line at the top edge of the recent excavation made after the storm of January 1; (4) a re-adjustment of grade on the old cone by stream flow by a storm after January 1; and (5) the extent of the most recent excavation. The spillway is to the right of the photograph, and its crest is about 10 ft below the level of the remnant of narrow-gauge railroad showing at the foot of the bank.

The importance of having the spillway of a débris basin in solid cut has been shown by the author; but of equal importance is the protection of the inlet channel so that the basin will not be filled from stream-bed deposits washing in as the stream re-adjusts its grade to fit the new conditions.

The evils of erosion are bad enough without undue magnification. The writer is probably old-fashioned in his beliefs, but he still maintains that the Grand Canyon was not the result of forest fires, draining of swamps, or any of the other multitudes of natural evils that have been blamed on the activities of mankind.

J. B. Lippincott,[18] M. Am. Soc. C. E. (by letter).—Many interesting data have been collected in this paper with reference to floods and their resulting erosion. Of these two problems the greater is erosion. This information is of value because, since the formation of the Los Angeles County Flood Control District in 1915, there have been few if any engineering reports published containing the physical data collected, or of construction progress, other than through the daily press, despite the request of the local engineers for such information. By some peculiar twist of the legal mind the District Attorneys ruled that bond funds could not be so expended. This paper, therefore, is welcome to engineers. A total of $46 308 044, from various sources, had been spent on this work prior to the fall of 1935. Since then $10 000 000 more have been allotted to the District from Federal sources.

Since the appointment of C. H. Howell, M. Am. Soc. C. E., as Chief Engineer on January 30, 1935, a comprehensive document entitled "Rainfall and Runoff Report, Seasons 1932–33 and 1933–34" was issued under date of June 1, 1935, which contains hydrographic data for that period. Especially in view of the difficulties of the problems involved it is hoped that, in the future, reports will be currently presented relative not only to hydrography, but also to the engineering features, including the reports of boards of engineers and geologists. This is in accordance with ordinary practice for large public works.

Bonds were voted by the Los Angeles County Flood Control District for two purposes: (1) The prevention of flood damage; and (2) the conservation of water by spreading floods over absorbent valley fills for underground storage. The general plan contemplates regulating reservoirs in the mountains, and flood-control channels across the valleys and plains, supplemented by spreading basins by means of which the water could be saved. The building of the dams has been difficult because of the geological character of the rocks of the Sierra Madre Range. These rocks have been shattered by earth movements and largely decomposed. The foundations and abutments for these dams have been poor. The local engineers are not alone in having difficulties with these adverse conditions. Engineers and geologists from numerous other parts of the United States have been consulted, with much the same experience. Twelve dams have been built successfully but there have been several disappointments. Important features of the general problem are yet unsolved. The dense population of Los Angeles County and the high property values (both of which are threatened) create an insistent demand for protection. Very little has been done in building spreading basins although the Supreme Court of California has pronounced this as one of the main features of the project.

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[18] Cons. Hydr. Engr., Los Angeles, Calif.

Because of the steepness of the mountain drainage basins, violent, flashy floods are discharged on to the valleys at intervals averaging about once in five years. These floods are followed by long periods of low flow. As described by the author, they carry surprisingly large quantities of débris. During an occasion of this nature the rolling and grinding of the boulders can sometimes be heard for several miles.

The gradient of the San Gabriel River, which is the principal stream of Los Angeles County, indicates the difficulty of the problem of disposing of this débris-laden water. Its smaller tributary basins have stream-bed slopes of from 125 to 400 ft per mile, and the main channel of the river in its mountain section, has a slope of 60 ft per mile. Below the mouths of the canyon the gradient across the San Gabriel Valley for a distance of 12 miles is 25 ft per mile. The river then flows through a pass on to the Coastal Plain where for a distance to the sea of 20 miles the gradient is 6.5 ft per mile. The hydraulic properties of these channels, even when leveed, become more unfavorable at points nearer the ocean. In the case of the Verdugo Wash (north of the City of Los Angeles), the gradients across its débris cones at La Crescenta (Montrose) are 500 ft per mile; thence, through the City of Glendale, the channel has a grade of 80 ft per mile, discharging into the Los Angeles River in the City of Los Angeles, where the channel has a slope of 25 ft per mile. The discharge of mountain floods ranging from 500 to 1 000 sec-ft per sq mile, eroding the shattered and decomposed rocks, has resulted in the building up of débris cones of great volume at the mouths of the canyons, and a general delta plain toward the sea. These débris and delta formations are of universal occurrence. Topographically, many of the cones are surprisingly symmetrical. All the channels are unstable and shifting. The steep slope and the great volumes of the débris cones indicate that the process of their formation has extended through long periods of time and that these deposits will continue.

Because of the commanding elevations of these cones and their attractive climate they have been extensively occupied and improved and, in many instances, towns are located in these exposed locations. The author shows that where the small mountain basins have been recently denuded by fire (which cannot be entirely avoided during the long dry summers) the volumes eroded from the smaller and steeper basins amount to extremes of from 40 000 to 50 000 cu yd per sq mile during single floods. If the floods are diverted at the mouths of the canyons, into lined channels, as is being done, with the foregoing gradients, the place where the débris will be dropped depends on the flattening of the gradient. The deposit must occur somewhere before the flood reaches the sea. As an indication of this last-named process apparently the mouth of the Los Angeles River has changed its position 18 miles, the San Gabriel River, 9 miles, and the Santa Ana River, in Orange County, 10 miles.

Efforts to Solve the Problem.—As stated previously, the population and property value of Los Angeles County is such that efforts must be continued to control both the flood and the débris. The first general plan of attack was made in 1917 when $4 450 000 in bonds was voted for the building of dams on the Arroyo Seco and San Dimas River, together with the diversion of the flood waters of the Los Angeles River from Los Angeles Harbor.

This was followed in 1924 by a plan that was hurriedly presented, without mature engineering study, and against the protest of the Los Angeles Section of the Society. It involved an expenditure of $35 300 000 for building dams, channel improvements, and spreading basins, the announced program being to regulate the floods of the major streams and to put the regulated flows into underground storage.

Despite the recommendation of two boards appointed subsequent to the bond election (relative to the original San Gabriel Dam), the Courts held that the type, size, and location of the structure defined in the bond election proceedings had to be followed rigidly. After foundation difficulties had developed in connection with the construction of this major structure, the California Supreme Court subsequently held that, because of "changed conditions", modification in the plans could be adopted, an essential feature in the decision being that the resulting decreased surface storage should be replaced by an increase in the quantity of water to be stored underground.

Although to date (1936) twelve dams have been built and are performing service of value, other efforts to build large dams have been unsatisfactory in some instances.

As the smaller reservoirs are practically débris basins in which to impound eroded material, the cost of some of them as such is of interest. The costs in Table 15 are high but they are less than the expense of $1.00

TABLE 15.—COSTS OF SMALL RESERVOIRS

Name of dam	Reservoir capacity, in acre-feet	STORAGE COST	
		Per cubic yard	Per acre-foot
Live Oak	250	$0.47	$760
Sawpit	470	0.83	1 340
Little Santa Anita	110	0.79	1 280
Big Dalton	1 290	0.49	790

per cu yd for cleaning up débris that was deposited at the rate of 26 000 cu yd per sq mile of drainage basin in October, 1928, on to streets and yards below the mouth of Brand Canyon, in Glendale, Calif., as stated by the author. His description of these débris flows is especially interesting and presents data that are new. It should be remembered that the great débris flows occur only when the burning of the cover of the drainage basin is followed by torrential rains. Although this combination may occur only at long intervals (say, 10 to 15 yr), the menace is present constantly.

Check Dams.—During the early years of the District there was a popular demand for the building of "check dams." This idea extended throughout Southern California. These check dams at first consisted of dry rock-fills, about 5 to 10 ft in height, built in the mountain channels without being founded on bed-rock. Frequently, they were only a few hundred feet apart, the idea being that their small basins would fill with débris which would absorb the flood water and permit of slow percolation later in the season, and also that they would step down the steep gradients of the channel. These structures promptly failed with the first floods. They were followed by a modified construction, wrapping loose rock in wire mesh; 4 060 of these check dams were built by the Flood Control District. They were also placed a few hundred feet apart in the canyons and on débris cones. The rolling of large rocks by floods cut the wires, and the floods also washed around their ends. To date, approximately 2 000 of these dams have failed under flood action. Accumulated material behind them contributed to débris that was projected on to underlying areas. It was stated that "check dams" had been used successfully in the Alps. Observations in Switzerland and Japan where flood and débris problems prevail, are that such small dams are substantial structures built of cut stone or cement masonry, and are usually founded upon bed-rock. The Los Angeles County check dams were built largely against the protests of the engineers in charge of flood-control work.[19]

Although there are 1 589 sq miles of mountain drainage area in the Flood Control District, the writer knows of no comprehensive estimate of the cost of controlling floods in this area by check dams. Some cost figures, therefore, may be interesting. The flood-control engineers have reported[20] that "the cost per cubic yard of debris stored by check dams varies widely, ranging from as low as 45 cents per cubic yard of debris up to as high as $6.00." If a cost of $3.00 per cu yd for such storage is assumed, this is equivalent to $4 839 per acre-ft of capacity. The following is also given from a report[20] written in 1931: "The actual measurements from a single moderate storm where fires have denuded the watershed have shown over 25 000 cubic yards of débris from one square mile of foothill area with a possible recurrence equal in amount for each storm during the period required for regrowth of vegetation." This re-growth requires fifteen years. Horse Canyon, a tributary of the San Gabriel River having 2.15 sq miles of drainage was completely "check dammed" in the summer of 1932. Twenty-three rock and wire check dams were built, and 137 loose rock dams, which is a total of 160 check dams, or 1 check dam for each 6.25 acres. The cost per square mile of this work was $10 445, and the cost per cubic yard of débris storage, 51 cents. An estimate of the cost of controlling 13.76 sq miles in this basin was $256 082, or an average of $18 610 per sq mile.[21] The conclusion is reached that the concrete dams and reservoirs, such as those referred to previously, none of which has failed, are more substantial and no more expensive than the "check dams." Studies have been made in Santa Barbara County, California, by the U. S. Forest Service for controlling the débris on the Santa Ynez, which confirm these conclusions.

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[19] Rept. of E. Courtlandt Eaton, M. Am. Soc. C. E., August 17, 1927.

[20] Rept. by E. Courtlandt Eaton, M. Am. Soc. C. E., and Frank Gillelen, Assoc. M. Am. Soc. C. E., May 22, 1931.

Flood-control work in Switzerland by means of larger and more substantial check dams on Lambac Creek, near Brienze, having a drainage area of 1.66 sq miles, shows an expenditure between 1898 and 1913 of $216 000, of which 60% was for channel construction below the mouth of the canyon. This amounts to a cost of $130 000 per sq mile. At the time of this cost estimate the work had not been completed. Where suitable reservoir and dam sites exist on larger drainage areas, the construction of the larger and more substantial dams is justified.

Débris Basins.—The next effort by the Flood Control District in the development of a plan to control débris from small mountain drainage areas was by means of "débris basins." These basins are excavated in the débris cone, with overflow weirs discharging into lined channels. The basins are built at the mouths of canyons at the apex of the débris cone. The general theory is to obtain a storage capacity of 100 000 cu yd of débris for each square mile of tributary drainage basin. The excavated material consists of sand, gravel, and large boulders. The plan is to discharge the floods through these basins in which the larger débris will be deposited and build concrete-lined channels of large capacity from the outlet of the basin down the slopes to the larger drainage lines and thence through other larger lined channels to the rivers. To January 20, 1936, twenty-three of these basins have been built.

The following is a description by the Flood Control engineers of one of these basins in the Montrose District, called Haines Canyon: The drainage area, in square miles, is 1.50. Of the basin, 60% has been burned. The present water capacity of the basin as built is 24 700 cu yd. Assuming that the débris in the basin itself takes a slope of approximately one-half the natural slope of the channel, its débris capacity is estimated at 87 250 cu yd. It is proposed to enlarge this débris capacity ultimately to 150 000 cu yd. What is termed the "bulked *Q*", of water and débris, in cubic feet per second, is 200% of the maximum peak water flow calculated for the drainage area above the basin. This bulked *Q* is given as 10 000 cu ft per sec. This will allow bulking by débris of 100%, whereas the estimates of débris flow for the January 1, 1934, storm in the La Crescenta area, indicated a bulking of 50% by débris. What is called the "clear-water *Q*" of 5 000 sec-ft is the design *Q*, for the first section of the channel below the basin. In designing the inlet and outlet structure of the basin, the bulked *Q* is used.

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[21] Rept. by E. Courtlandt Eaton, M. Am. Soc. C. E., October 31, 1932.

As the author well shows the débris brought down by floods is enormously increased when such floods follow soon after the burning of the brush cover of the basins. He states that following a fire, the floods of January 1, 1934, discharged from 7 sq miles of drainage north of La Crescenta (Montrose) 659 000 cu yd on to property and streets, amounting to more than 90 000 cu yd per sq mile. Other instances are given by the author of greater débris flow from other burnt areas. Referring to the cost of these débris basins he states: "Their capital costs per cubic yard of débris capacity will range from $0.30 to $1.00 per cu yd, depending mainly upon excavation costs." It will be noted from these figures that the cost of débris-basin storage does not differ greatly from the costs of similar storage behind the four concrete dams previously cited.

The probable life of the débris basin as well as of the small reservoir will be short. In the case of the débris basins the plan is to re-excavate them when they are partly or entirely filled. When the smaller reservoir is filled the probable plan may be to provide another on the same stream. These figures are presented not in criticism of the plans of the Flood Control District, but rather as indicative of the expensive nature of the problem involved and which should be solved. These great débris flows occur only when floods follow fires, a combination of events that does not occur frequently. Nevertheless, they are a menace that should be provided against.

Assuming maximum flood discharges of 500 to 1 000 cu ft per sq mile delivered into concrete-lined channels having gradients of more than 10%, the further question arises as to whether the exceedingly high-water velocities of fully 40 ft per sec which may occur, will flow around curves in the channels irrespective of their débris. The opportunity to observe this phenomenon has not yet occurred.

Brush and Forest Cover.—The beneficial effect of forest or brush cover in reducing floods and erosion has been amply demonstrated in California by observation on adjacent experimental plots both burnt and unburnt in two mountainous regions, and also by careful observations on neighboring drainage basins of similar exposure some of which have been denuded by fire. These measurements (which are in accord) have been conducted by the U. S. Department of Agriculture. To give the details would be to repeat the views of the writer which have been presented[22] previously. They show in one series of test plots that the run-off from burnt areas, with rain at the rate of 2.4 in. per hr, was 200 cu ft per sec per sq mile and that there was "no surficial run-off measured" from the covered plots. Eroded material from the burnt plots was 3 cu yd per acre and that from the unburnt area was zero. With a 12-in. rainfall during the storm of December 30, 1933, to January 1, 1934, the peak run-off from the Verdugo Wash of 19.3 sq miles (which included the Montrose area, 6 sq miles of which was steep, absorbent, canyon floor and largely burnt over), was 350 cu ft per sec per sq mile and the erosion, 50 000 cu yd per sq mile, whereas, from the unburnt mountainous adjacent Arroyo Seco Basin of 16.24 sq miles with brush cover, the run-off was 58 cu ft per sec per sq mile and the erosion was relatively insignificant. Other basins show greater contrasts.

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[22] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 330.

These examples show the great value of the dense brush on the Southern California mountains. If this cover can be maintained, it relieves the flood and débris problem. If fires are prevented it affords a much simpler and less expensive process than the building of costly engineering works. This solution resolves itself into the effort to prevent fires. The Federal, State, and County organizations have combined to prevent fires and have succeeded substantially in reducing their hazard. Lookouts with telephone connections have been established throughout the mountains. Fire-breaks have been built along ridges. The system of roads being built, will permit of the rapid transportation of men and equipment to fires. If a fire can be reached in its incipiency, it can be extinguished. If, however, it becomes general, the task is well-nigh hopeless; the effort then is to control it within drainage basins. In the writer's opinion relatively greater benefits can be obtained from efforts to prevent fires than by the construction of engineering works, although both are necessary. A most rigid patrol and control of the use of the Forest Reserve is justified.

The occupation of the menaced débris cones should be discouraged and should have been prohibited. However, they are already (1936) largely occupied. As stated by the author this occupation invites disasters such as that which occurred on January 1, 1934, when 30 people lost their lives and $5 000 000 property damage occurred at Montrose. It is unfair to require the public to pay the costs for their protection which may be at a money cost in excess of assessed values. The location of these improvements in many instances prevents the spreading of flood water for conservation purposes on the cones as originally planned.

Conclusions.—In conclusion, the writer expresses his appreciation of the difficulties involved in the handling of the flood problems of the Los Angeles County Flood Control District. The task appears well-nigh impossible of complete solution on the one hand and a necessity on the other. The presentation of basic data in this paper is valuable. Further reports on the engineering problems of the District should be made generally available in order that the minds of many engineers can give it thought. The engineers in charge of the work are entitled to helpful co-operation. Opportunities for conserving the valuable flood water by its underground storage are yet available in the larger valleys and on the Coastal Plain and should not be neglected as this is one of the main features of the project. It is a cheaper conservation method in Los Angeles County than the storing of water in surface reservoirs. However, each drainage basin requires a separate study and plan.

E. I. KOTOK,[23] ESQ., AND C. J. KRAEBEL,[24] ESQ. (by letter).—A great volume of data on rainfall and run-off relations in Los Angeles County, California, has been accumulated by the County Flood Control District and various Federal agencies. A smaller amount of information on erosion has been similarly gathered. Mr. Eaton has done a signal service in selecting some of the most significant facts and estimates and presenting them with his excellent analyses. The engineering methods described for flood and débris control have not only proved successful where they have been used, but are indicated as indispensable for the protection of communities contiguous to mountain water-sheds which have been denuded of vegetation.

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[23] Director, California Forest and Range Experiment Station, U. S. Forest Service, Berkeley, Calif.

[24] Senior Silviculturist, California Forest and Range Experiment Station, in Charge of Studies in Water-Shed Management and Erosion Control, U. S. Forest Service, Berkeley, Calif.

In stating that the natural cover of vegetation affords the "best protection from débris", Mr. Eaton expresses a fact which has been so frequently and spectacularly demonstrated in California by the sequence of fire and flood that it is now quite generally accepted. The formulation of adequate plans for water-shed management, however, still requires the obtaining of quantitative data on the degree of influence exerted by different types and conditions of forest cover, not only upon soil movement, but also upon water movement. Such planning requires, furthermore, a study of the effects of manipulating or "improving" the cover by cutting, planting, or otherwise, and the development of methods of restoring vegetative control quickly after accidental denudation.

In these problems the forester's interest is vital, since in California nearly all the major water-sheds lie within National forests. In Southern California where water is the limiting factor of human development, the primary objective of forest management is to produce from mountain catchment areas the maximum annual yield of useful water—with emphasis on the word "useful" and with due regard for the increasingly important secondary use of the forest in this region, namely, recreation.

The California Forest and Range Experiment Station, representing the research branch of the U. S. Forest Service, is engaged in a comprehensive program of studies in water-shed management[25], including both field and laboratory attacks and giving first priority to the aforementioned problems. A large part of this work is centered in the San Dimas Experimental Forest, an area of 17 000 acres within the Angeles National Forest, set aside for research purposes by special order of the Chief Forester, U. S. Forest Service. The area embraces the entire water-sheds of Big Dalton Creek (7.5 sq miles) and San Dimas Creek (18.3 sq miles), situated in the San Gabriel River drainage and shown in Fig. 1, both commanded near their canyon mouths by large dams built in 1928 and 1922, respectively, by the Los Angeles County Flood Control District for flood control and water conservation purposes. During the three years, 1933, 1934, and 1935, with the help of unemployment relief labor, the Experimental Forest has been equipped with approximately 350 rain-gauges, 17 stream-gauging stations, 9 débris measuring reservoirs, numerous climatic stations, sample plots, and the necessary roads, trails, telephone lines, laboratory, and other service buildings. All engineering design and construction has been done under the direction of the Regional Engineer of the Forest Service, and at all stages, the staff has invited criticism and suggestions from local engineers, water officials, and other interested persons, in the effort to make of the Experimental Forest an adequate laboratory of water-shed management and erosion control problems for Southern California.

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[25] The program is described briefly by Mr. E. I. Kotok, and some representative data are presented by Mr. C. J. Kraebel, in the Report on Progress Conference on Water Conservation, Los Angeles, Calif., March 13–14, 1935, by the Committee on the Conservation of Water, Irrigation Division, Am. Soc. C. E. (mimeographed December, 1935).

In discussing the 1934 flood flow from Pickens Canyon, the auther touches on the difficulty of calculating such a flow from high-water marks and channel sections after the flood. A characteristic of such flows is their scouring action by which the stream beds may be lowered several feet during a single storm. Both the exposed bed-rock of the scoured channels and the roughly stratified structure of the great outwash fans at their mouths constitute evidence that such scouring has occurred periodically through past centuries. A survey of Pickens Canyon made by members of the Experiment Station after the flood revealed soil loss from ½ mile of the canyon bottom at the rate of 20 000 cu yd per lin mile of channel. Fig. 15 depicts a typical section of the channel, photographed after the Montrose flood of January 1, 1934, which illustrates the difficulty encountered in estimating run-off rates of débris-laden flows. The upper tape, *A*, indicates the highest water stage, the middle tape, *B*, the pre-storm channel bed, and the lowest tape, *C*, the present scoured channel as it was left by the flood. Erosion of the stream bottom was progressive. The question, for which there is no definite answer, is: "What was the channel profile when the water surface was at the highest level?". It is quite possible for the maximum run-off to have occurred when the channel was only partly scoured and the water surface had dropped below its highest mark.

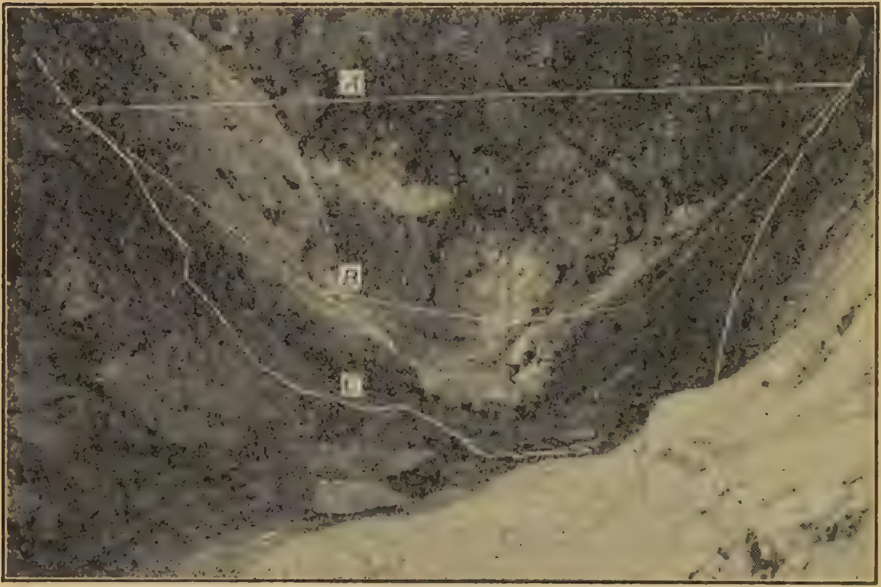

FIG. 15.—CHANNEL OF PICKENS CANYON, AFTER MONTROSE FLOOD OF JANUARY 1, 1934.

In the effort to find a positive means of measuring débris-laden stream flows, the staff of the Experimental Forest Station has found the Parshall flume unsatisfactory and is developing a critical-depth flume with parallel sides and sloping floor. Tests of this new meter have thus far been very encouraging.[26]

Regarding estimates of erosion with normal water-shed cover the author rightly states that information is not plentiful. A probable reason for this lack of data is that débris production from well-covered chaparral water-sheds is so slight that it ordinarily escapes notice. The erosion rates given for the San Gabriel, Gibraltar, and Sweetwater water-sheds (see heading "Erosion with Normal Water-Shed Cover") do not represent erosion from well-covered areas because all these water-sheds had been severely burned over, and the Sweetwater water-shed had also been over-grazed for many years. There are so many variables involved which influence erosion of burned areas and also the recovery of the burned vegetation that it is unwise to generalize about either. The author's estimate of 1 500 cu yd per sq mile annually from areas unburned for 10 yr, or more, seems a generous maximum.

From experience in the Experimental Forest the writers submit some rates of erosion, in Table 16(*a*), based upon actual catchments in tight reservoirs for the entire rain season of 1934–1935. From the great New Year's storm ending January 1, 1934, the rates measured in the same experimental water-sheds were as shown in Table 16(*b*). These extremely low run-off and erosion rates, when compared with the rates cited by Mr. Eaton for the same storm on the Mt. Lukens burned area, indicate that the quantitative influence of good vegetative cover is considerable. The burned water-sheds received an average of 12.5 in. of rain, yielded a maximum run-off of 580 cu ft per sec per sq mile (Pickens Canyon), and an erosion rate of approximately 67 000 cu yd per sq mile (Haines Canyon). Thus, the run-off ratio, as of covered to burned area, is 1:23, and the erosion ratio, 1:1 600.

TABLE 16.—EROSION RATES AND RUN-OFF IN CHAPARRAL WATER-SHEDS, 1934–1935

Water-shed	Area, in square miles	Elevation, in feet	(*a*) THE ENTIRE RAIN SEASON, 1934–35			(*b*) THE GREAT STORM, JANUARY 1, 1934		
			Precipitation, in inches of depth	Erosion rate, in cubic yards per square mile	Condition of water-shed	Precipitation, in inches of depth	Maximum run-off, in cubic feet per second per square mile	Erosion, in cubic yards per square mile
Bell No. 3......	0.097	3 500	33	500*	Burned in 1919	11.8	34	62
Fern No. 1.....	0.055	5 000	41	30	Not burned †.	12.6	26	65
Fern No. 2.....	0.063	5 000	41	10	Not burned †.	12.7	22	19
Fern No. 3.....	0.083	5 000	41	12	Not burned †.	12.6	19	24
Average.....	.....	.....	......	....	....	12.4	25	42

* This high rate resulted from a small local landslip probably started by construction disturbances.

† Not burned for fifty years or more.

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[26] "Experiments with Critical Depth Flumes for Measurement of Flow in Débris-Laden Streams", by H. G. Wilm, J. S. Cotton, and H. C. Storey, California Forest and Range Experiment Station, presented at the Pacific Coast Meeting of the Am. Geophysical Union (Section of Hydrology), at the California Inst. of Technology, Pasadena, Calif., February 1, 1936.

Loss of soil by erosion in the burned area shown in Fig. 16 approached 100 000 cu yd per sq mile. The inter-shrub spaces proceeded to form gullies almost unhindered, because of the wide spacing between burned shrubs, their slow sprouting, and the sparseness of ground-cover vegetation between the shrub clumps. Ground-cover plants are lacking because the soil-stored seeds which might have escaped burning, were subsequently washed down the slope by sheet erosion.

The unburned water-sheds cited in Table 16(*a*) may be considered too small to be significant. Erosion measurements for large unburned water-sheds are not available, but run-off records from such water-sheds for the great storm clearly reflect a regulatory influence of vegetative cover similar to that on the small water-sheds (see Table 17). The run-off rates from these water-sheds, lying in the same mountain range as the burned area, receiving the same or heavier rainfall, and having similar geologic structure and topography, nevertheless yielded peak run-off only one-tenth (or less) as high as that from the burned area. The logical deduction that the controlled run-off from the forest-covered areas could have produced proportionately little erosion is substantiated by the fact that the water, observed during the storm, was almost clear of suspended matter.

TABLE 17.—RUN-OFF FROM LARGE UNBURNED WATER-SHEDS, STORM OF JANUARY 1, 1934

Water-shed	Area, in square miles	Rainfall, total, in inches	Run-off, in cubic feet per second per square mile
Arroyo Seco	16.4	11 to 17*	58
Eaton	6.5	13.8	51
Big Santa Anita	10.5	13 to 19*	54
San Dimas	18.3	10.8†	53

* Range of catchment in various rain-gauges.

† Catchment determined by statistical analysis of the catch in 150 rain-gauges distributed over the area.

As a means of reducing sheet erosion from burned water-sheds, the Forest Service has adopted from agriculture the idea of sowing a quick cover-crop and has found the lowly mustard to be the plant best suited to the purpose. The cost per acre, when sown by hand at the rate of 5 lb of seed per acre, averages $1.25. Inaccessible country is sown by airplane at $0.90 per acre, but the work is dangerous and the seed distribution is likely to be irregular. Quantitative data on the soil-holding effects are not yet available, but Fig. 17 shows a typical mustard stand, six months after sowing, in which the mustard with a density of 6 plants per sq ft, has filled the inter-shrub spaces with soil-binding roots, while the tops protect the soil from driving rain and wind. The oven-dry weight of mustard plants, returned as protective litter to the burned soils, ranges from 2 000 to 6 000 lb per acre the first year, and the seed production by this first crop is from 200 to 1 500 lb per acre. It is expected that the mustard will ultimately be crowded out by the recovery of the chaparral. At that stage there is a remote possibility that sufficient seed may remain stored in the soil for many years to produce a protective cover-crop after some future fire.

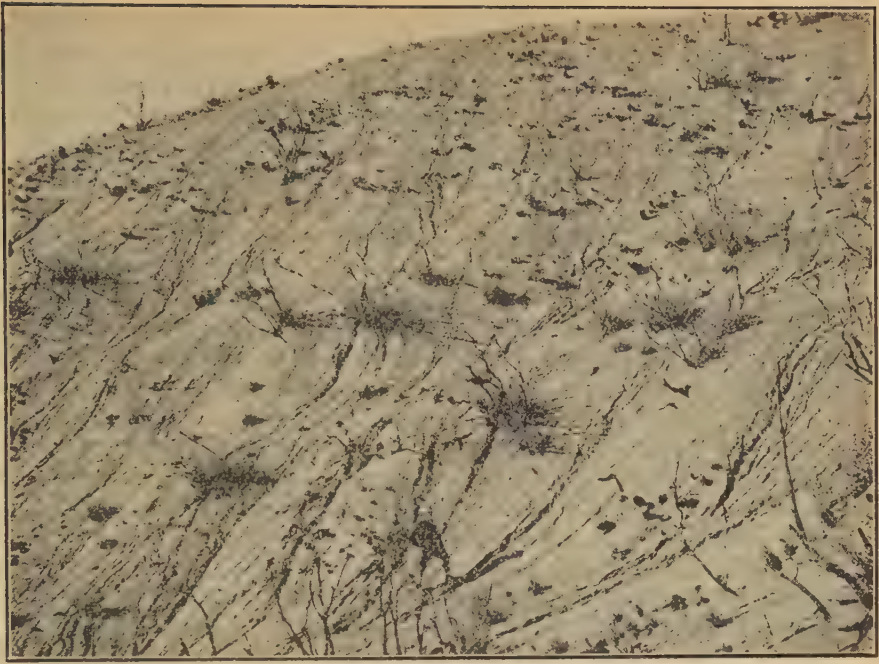

Fig. 16.—Loss of Soil by Erosion in Burned Area (Photographed Second Winter After Fire) Approached 100 000 Cubic Yards per Square Mile.

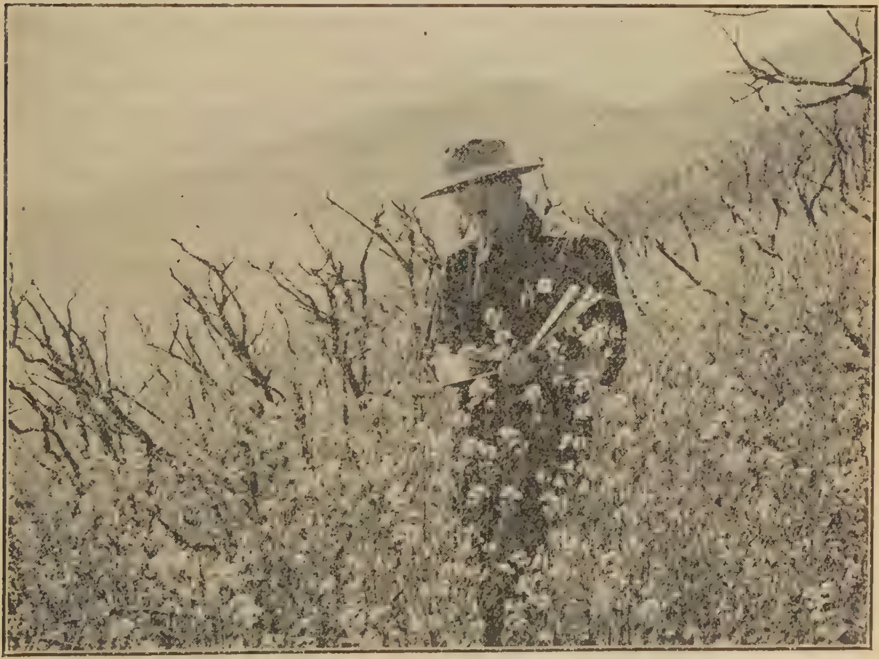

Fig. 17.—Denuded Areas Sown to Mustard at Rate of 5 Pounds of Seed per Acre.

DONALD M. BAKER,[27] M. AM. SOC. C. E. (by letter).—The flood and erosion control problem in Los Angeles County is complex in the extreme. It may be considered from the following aspects: Physical, meteorological, hydrological, recreational, cultural, historical, financial, and political.

Physical.—Settlement occurs principally upon the coastal plain, 1 000 sq miles in extent, lying between the Pacific Ocean and the base of the San Gabriel Range, where the elevation reaches about 1 000 ft. The mountains then rise abruptly within a short distance to elevations of from 6 000 to 10 000 ft, their slopes covered with a deep soil, easily eroded, which supports a cover of heavy brush and timber. This cover constitutes a serious fire hazard, and when destroyed, heavy flood run-off and débris flows result. Slopes of streams draining the south side of the range are steep, and available reservoir sites are of small capacity, while the geologic structure of the range affords very unsatisfactory foundation conditions. About half way between the San Gabriel Range and the ocean, and extending parallel to it, a series of low hills traverse the coastal plain, with two openings, through which pass the Los Angeles and San Gabriel Rivers. Three large ground-water basins—the San Gabriel Valley, San Fernando Valley, and the coastal plain—supplied by percolation of streams draining the mountains, form valuable sources of water supply for overlying lands.

Meteorological.—Mean precipitation, concentrated principally in the period from December to March, inclusive, increases from 10 to 12 in. near the ocean to 20 to 25 in. at the base of the mountains, and to 35 in, and more at their crest, with annual variations of from 40% to 250% of the mean, and cyclic fluctuations extending over long periods.

The long dry season extending from the spring until the beginning of winter, with low humidity which continues for weeks at a time, creates a condition in the brush and tree cover which makes it very susceptible to fire.

Hydrological.—Damaging floods of county-wide extent have occurred singly, or in groups, at intervals of about twenty-five years. Unit rates of run-off from such floods are high.

Recreational.—Proximity of the mountainous areas to the large population on the coastal plain invites use of the mountains for recreation, which, in turn, requires that they be made accessible. This has been done through the construction of many trails and, recently, of automobile roads. This enhanced accessibility has greatly increased the fire hazard with its threat of removing protective brush cover, and many serious fires have occurred, which have further added to the flood run-off and erosion.

Cultural.—The population of the coastal plain has increased from 500 000 in 1916, when the last major flood occurred, to about 2 250 000 in 1936, or 350 per cent. Mr. Eaton's estimate that only 8% of the present population has experienced or has a realization of a major flood is evidence of the difficulty met in obtaining public interest in the problem. Much settlement and development have taken place on areas subject to flood menace without an appreciation of the menace either on the part of the developers, settlers, or of public officials. The average population density of the area is in excess of 2 000 persons per sq mile, and high values of land and improvements exist, many of such values occurring in areas subject to flood menace.

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[27] Cons. Engr., Los Angeles, Calif.

Historical.—Local attention was focused upon flood-control problems following the 1914 flood, the first to cause serious damage. Major floods occurring previous to that date, in the 1880–90 decade, although greater in magnitude than those of 1914 and 1916, caused far less damage because of the relatively small population and the lack of intensive development. Following 1914, a flood-control district was formed, embracing practically the entire County south of the mountains. The plan of flood control first adopted consisted primarily of channel rectifications and improvement with little attention given to water conservation. A bond issue of about $4 500 000 was voted in 1917 to carry out this plan.

The great increase in population occurring in the 1920–30 decade focused attention upon the need for conservation of local water supplies, and, in 1924, a plan, hastily conceived and prepared without adequate hydrologic and other data, was devised for which a bond issue of $35 000 000 was voted. In this plan conservation of water was emphasized, the bulk of expenditure being for storage dams in the mountains, foremost of which was a gigantic concrete dam in the San Gabriel Canyon estimated to cost $25 000 000. Some of the smaller dams proposed in this plan were built. The large dam in the San Gabriel Canyon was abandoned, due to poor foundation conditions, and plans for two smaller dams with much less storage capacity were substituted. With the advent of a bountiful supply of water from the Colorado River, interest in conservation of flood water has waned, and the present construction program, amounting to nearly $20 000 000 and carried on by the Corps of Engineers, U. S. Army, through the assistance of a large Federal grant, is again directed entirely toward channel rectification and improvement.

Financial.—The cost of the work done by the Flood Control District is met by a district tax upon land and improvements only. Since the assessed valuation of the City of Los Angeles, which is within the District, has ranged from 60% to 70% of that of the entire District, the City has been in the past (and will be in the future) called upon to meet this percentage of the total cost of the program, except that done under Federal grant. Except for some channel improvement in the Los Angeles River and the Pacoima and Tujunga Dams, the bulk of the money spent for construction work has been on structures and projects outside the city limits, although under the present program involving Federal aid, a considerable portion of the funds are allocated to projects within the City of Los Angeles. The latter has a very serious stormwater problem within its boundaries, and until a few years ago it had made large annual expenditures for the construction of storm drains, the costs being raised by local assessment.

Political.—The Board of Supervisors of Los Angeles County acts as the legislative body of the Los Angeles County Flood Control District, and various other County officials, such as the Assessor, Tax Collector, Auditor, County Counsel, also act for the District. The latter has its own engineering personnel which, however, since the District is a separate political entity, is not subject to County Civil Service regulations. This form of organization, originally proposed in the interest of economy, has not been satisfactory, as flood-control administration and policy have been submerged in general County issues, and popular opinion has not found an opportunity to express itself on flood control alone.

Proper control measures group themselves into the following:

(1) Water-shed management, including the preservation of existing vegetal cover, renewal of destroyed cover, and prevention of erosion;

(2) Disposition or conservation of flood waters (or both disposition and conservation), including construction of regulation and storage reservoirs, spreading works, and flood channels; and,

(3) Control of land utilization in flood-menaced areas, including the reservation of land for reservoir and channel easements and the control of land utilization and improvement in menaced areas.

Recently, much excellent work has been accomplished by Federal and County forestry agencies in water-shed management. Motor roads to afford accessibility for fire-fighting have been constructed, and equipment and organization for such fighting have been provided. Work is progressing in restoration of burnt-over areas and in prevention of erosion from bare slopes. An extensive program for collecting basic hydrologic data, lacking at the time previous plans were made and bond issues voted, has been under way for some years past.

An attempt was made in the 1935 Session of the State Legislature to amend the Flood Control Act, creating for the Flood Control District a legislative body separate from the County Board of Supervisors, providing for the preparation and adoption of a comprehensive plan of flood control before any further construction work was done, and placing the personnel of the District's staff under civil service. Although such revision had the hearty support of all civic bodies interested in the subject, political opposition in certain quarters prevented its accomplishment.

Widespread differences of opinion exist among those hoping to benefit by the conservation features of the District's activities as to the title to the water which will be developed by such features, and until this subject is settled and a policy adopted governing the utilization of such water, a threat of extensive litigation will continue.

Nothing so far has been done toward the control of land utilization in flood-menaced areas, although it would appear that, once such areas were designated after proper study and investigation, legal authority probably exists to prevent such development and utilization through regulation of the sub-division of land by the planning commissions of the County and of the various cities within the District.

Neither has any accomplishment been had in the development of a comprehensive plan of flood and erosion control for the District in which all the various phases and aspects of the problem—regulation, conservation, land use, and financing—are considered and co-ordinated. Until such a comprehensive plan has been developed and adopted and the opportunistic and drifting policy which has characterized past efforts to solve the problem has been abandoned, future occurrence of events which are related by the author of this paper may be expected.

E. Courtlandt Eaton,[28] M. Am. Soc. C. E. (by letter).—Mr. Pickett states correctly that fire-prevention methods on water-sheds are the cheapest and best insurance against the débris menace. Next in order of economy are provisions for quick access to incipient fires and provisions for adequate supplies of water to assist the fire fighters. The quantity of water required is not large since modern "fog nozzles", or high-pressure nozzles creating a fine spray as a protection to men and to increase the humidity locally, have been found to be very effective. In line with the plan advanced by Mr. Pickett, it has been proposed to build submerged and covered concrete tanks in the mountains at accessible locations which may be filled from rainfall, by collector channels, or by adjacent paved areas, in readiness for the fighting of possible fires during a succeeding dry season. As stated by Mr. Pickett, limited appropriations have prevented fulfillment of these plans.

If measures for prevention and suppression fail, débris basins must be resorted to as the last line of defense at costs which may reach capital expenditures as great as $150 000 per sq. mile of water-shed.

Mr. Davenport mentions the accepted procedure, in determining annual run-off quantities, of considering run-off as a residual after subtracting evaporation, transpiration, and other losses. This method is entirely proper, of course, where the primary purpose is to establish a relation between rainfall and run-off, and are particularly applicable to localities where rainfall is less flashy in character and water-sheds are not so subject to physical changes as is the case in Los Angeles County.

Under Los Angeles County weather and physical conditions a given quantity of rainfall in a season may produce widely varying amounts of run-off, depending upon the intensities and the previous condition of saturation of the water-shed, and detail study of daily and hourly precipitation must be made. In the studies of Big Tujunga water-shed the problem was mainly that of arriving at the regulating capacity required and the expected peak flows during the critical 4-day period rather than to determine the seasonal run-off. Table 5, therefore, is merely a general and preliminary study based upon the indices of seasonal wetness method and was used as a guide in the selection of a critical season for detailed analysis. It is possible that the total run-off for that season, under normal water-shed conditions, would be close to 250 000 acre-ft. However, there are historical records indicating that the water-shed had not fully recovered from a previous fire so that the actual run-off may have been even greater than that shown in Table 5. Since all historical and rainfall records pointed to 1884 as the year of major flood in a 50-yr period, that year was selected for detailed analysis.

The details of the method used in computing the 4-day peak flows are too long for reproduction herein, but in brief the procedure included: (1) The measurement by planimeter, of sub-areas from a map similar to Fig. 5; (2) the distance from the point of delivery to the next succeeding point down stream was measured and the channel slope computed; (3) times of concentration were determined for the sub-areas; and (4) a coefficient of run-off was selected for the average typical soil and cover characteristics as determined by field investigations.

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[28] Cons. Engr., Los Angeles, Calif.

The flow was then computed from the rational expression, $Q = C\ I\ A$, in which Q = run-off, in cubic feet per second; C = coefficient of run-off; I = average intensity of rainfall during the time of concentration of the sub-areas, in inches per hour; and, A = area of sub-area, in acres.

The relation between the daily peaks in the maximum 4-day flood period was derived from co-related studies of the San Gabriel River water-shed on which there are considerably longer rainfall and run-off records than on the Big Tujunga water-shed.

Mr. Jarvis states that the conditions in Los Angeles County are extreme and that the County has a combination of physical features and high property values not commonly met elsewhere. The writer agrees. Nevertheless, similar erosion problems to a lesser degree exist elsewhere in the West although, fortunately, high concentration of population and property values do not as yet lie within the paths of débris flows.

Mr. Blaney mentions the excellent experimental work done by Mr. Mitchelson, of the U. S. Bureau of Agricultural Engineering, from which much valuable information has been obtained. The relatively high rates of percolation obtained by Mr. Mitchelson as compared with those of the writer in Table 12 are attributable to the relatively clear water available for Mr. Mitchelson's experiments. The percolation rates given by the writer are in the main those during receding flood-flow periods when the water was cloudy. Were it possible, in spreading operations, to count upon handling a reasonably clear stream flow the "basin" method of spreading could be used at a considerable saving in land costs and spreading works. Under conditions where the water carries considerable suspended matter, the ditch system, in the writer's judgment, is preferable.

Mr. Rowe infers that the coincidence of a heavy rain following a fire is remote. The writer cannot agree. A storm of high intensity, coming any time within a 3-yr, or more, regrowth period after a fire will cause a débris flow; in fact, a second high-intensity storm within that period may produce even a greater débris flow than occurred with the first storm due to the previously loosened water-shed condition. Mr. Rowe mentions check dams as a possible indirect source of some of the débris. A thorough study made of the canyons immediately following the débris flows does not support his theory. Mr. Rowe also attributes a portion of the débris flow in Haines Canyon débris basin to the operation of a gravel pit which formed the nucleous of the aforementioned basin. This theory is borne out neither by the examinations made of the water-shed to trace the sources of débris, nor by the analyses of débris in the basin, much of which consisted of earth and top-soil originating from water-shed slopes.

Mr. Lippincott mentions one of the difficulties under which the Los Angeles County Flood Control District has labored. Although there are in the District's files and on file as public records, many complete reports containing valuable engineering data, legal restrictions have prevented the financing of their reproduction for general distribution and a lack of general knowledge other than that contained in the daily press has resulted.

Mr. Lippincott mentions the publication of a "Rainfall and Runoff Report, Seasons 1932–33 and 1933–34." In fairness to previous administrations many rainfall data were obtained extending back almost as far as the formation of the District and complete rainfall-run-off reports in the aforementioned form, have been issued each year since 1927.

Mr. Lippincott draws a comparison in Table 15 between the four dams of highest unit cost of the twelve constructed by the District. The average cost of all dams built by the District has been about $200 per acre-ft, ranging from $35 to $1 300 per acre-ft. The average is equivalent to about 13 cents per cu yd of débris capacity.

Undoubtedly, dams, even at the highest cost mentioned, are more economical than débris basins. However, dam sites are limited and the débris basin takes its place where no adequate dam sites are available.

Incidentally, considering flood control and conservation features, a statement of cost per acre-foot of storage capacity alone is not indicative of flood-regulating values, since the reservoirs may regulate in a single season amounts three to four times their nominal storage capacity and the conservation or hold-over storage features are present in the underground basins fed by the regulated flows.

Messrs. Kotok and Kraebel have mentioned the work being done under their direction by the California Forest and Range Experiment Station of the U. S. Forest Service. From their results will be obtained invaluable quantitative data that will furnish a basis for future planning of remedial and preventive measures. The writer agrees that his value of 1 500 cu yd of débris per sq mile annually from unburned water-sheds may be high.

Fig. 15 shows graphically the difficulty in estimating rates of débris from the high-water mark, *A*, and taking the cross-section, *C*, after the recession of the flood. The section at the time of maximum flow might be at any place between *B* and *A*.

Mr. Baker mentions the difficulty in obtaining public interest in flood-protection problems. Fundamentally, the Flood-Control Act itself is responsible. Under the Act, the members of the County Board of Supervisors, only a fraction of whose time can be given to flood-control matters, act as directors without additional salary for that service. Furthermore, the Act provides that the County Counsel, County Auditor, and County Purchasing Agent act in their several capacities for the District without additional compensation for this service. Each of these reports directly to the Board of Supervisors. Thus, there is no semblance of centralized control or responsibility. This condition, in conjunction with the failure to finance, legally, the general dissemination of reports and information, or the lack of legal authority to finance a public relations bureau, has placed the District in a defensive position.

Although only partly completed as a completely connected unit, the works built thus far have safeguarded thousands of lives and have prevented, and are preventing, destruction to property that might otherwise mount to sums several times the expenditures made for protection.

Incidentally, a Comprehensive Plan Report was issued by the District in 1931, drawing attention to the menaced areas and outlining needed protective works. As stated by Mr. Baker attempts to amend the Flood Control Act have so far been unsuccessful.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1951

THE RELATION OF ANALYSIS TO STRUCTURAL DESIGN

By Hardy Cross[1], M. Am. Soc. C. E.

With Discussion by Messrs. L. J. Mensch, Russell C. Brinker, Marshall G. Findley, I. K. Silverman, Bruce G. Johnston, Harold E. Wessman, N. M. Newmark, L. E. Grinter, F. P. Shearwood, and Hardy Cross.

Synopsis

Confusion sometimes exists in structural design as to the use to be made of analyses. The designer soon realizes that precision is futile in some cases and important in others, and the experienced designer realizes fully that analysis of the conventional type is frequently a poor guide to proper proportions. That analysis shows a certain member to be overstressed commonly indicates that the member should be made larger; but the over-stress sometimes has little importance and may be disregarded. In some cases where the over-stress is serious, the best solution is not obvious; sometimes the structural layout should be changed entirely.

Critical study soon leads to recognition of important differences between load-carrying stresses and stresses which produce no appreciable resistance to the applied loads. The latter may be due either to external movement of abutments or to internal distortions, or they may be due to deformation induced in one part of a structure as a result of that in another part. The load-carrying stresses may also be divided into two groups. The distinction in this case, however, is based upon response to changes in design; these sub-groups are not always clearly distinguishable, but they have characteristics so widely different in certain cases as to force their differentiation.

A classification is presented herein, with the idea of suggesting a convenient arrangement of certain familiar characteristics rather than with any wish to define the groups formally. The designations suggested, therefore,

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Note.—Presented at the meeting of the Structural Division, New York, N. Y., on January 17, 1935, and published in October, 1935, *Preceedings*

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are for convenience of reference only. The non-load-carrying stresses will be distinguished as Deformation Stresses and Participation Stresses; and the load-carrying stresses as those normal in their characteristics and those which are hybrid.

Deformation stresses are a consequence of strain and strain is a consequence of internal or external movements not due to stress in the structure, such as abutment movements, shrinkage, or the effects of temperature change.

Participation stresses are similar to deformation stresses, but they are due to a quite different cause. They include what are known as "secondary stresses" in bridge trusses and "participation stresses" in bracing systems as special cases. The designation is used herein for want of a better one, but the term is used in a wider sense than usual.

The primary action of most structures is such that the stress in any one part is independent, or nearly independent, of that of the other parts. This is termed normal, structural action. The group indicated includes all structures statically determined and, for good reasons, it includes also most of the forms of indeterminate structure that experience has shown to be useful.

There is a type of structure in which one member cannot be designed separately but must be designed with due consideration for its effect on other members. Such action is referred to as "hybrid," because it has some of the characteristics both of normal structural action and of participation action. The group of structures seems to be quite large and to have characteristics of great importance to designers.

Although the designations assigned herein may be new, the concepts involved are not new. What the writer wishes to do is to classify and arrange certain views of structural design which have an honored tradition in American practice. The paper is not intended to be quantitative, except in so far as quantitative statements may help in defining qualitative action. The classification proposed has some value in reconciling discordant views held by practical designers and those held by theoretical analysts, and seems, further, to have value in reconciling conflicting views held by theoretical students in the field. It is also of value in anticipating the characteristics of proposed structural types.

I.—Types of Structural Action

Deformation Stresses.—The outstanding characteristic of deformation stresses is that the strains in the structure are fixed and that the stresses are deduced from the strains. The stresses themselves are not fixed at all, but depend entirely upon the stress-strain relation of the material.

Another important characteristic is that strength has nothing, or practically nothing, to do with the problem. The strains are fixed by the over-all dimensions of the structure and by the amount of deformation to be accounted for. The designer finds it convenient to predetermine such strains, reduce them to equivalent stresses, and deduct these stresses from the working stresses available for load-carrying capacity.

Participation Stresses.—This term is used to include what are commonly known as secondary stresses in bridge trusses, participation stresses in bracing systems, participation stresses in cross-frames due to unequal deflection of trusses or girders, cross-flexure in the vertical members of trusses, secondary flexure in slender columns of bents, secondary flexure in slender spandrel columns of open spandrel arches, and secondary flexure in slender columns of buildings.

Participation stresses have many of the characteristics of deformation stresses: The strain is fixed; the stress is a consequence of the strain; and the strain, in general, can be affected in an important way only by changing over-all dimensions and not by changing the strength of the member. Thus, it is generally known that increasing the moment of inertia without changing the depth of truss members will affect the secondary stresses only as the primary stresses are affected.

Another important characteristic of these stresses is that they do not increase in proportion to the load up to rupture, but increase less rapidly after the yield point of the material has been passed. They are then clearly somewhat less dangerous than primary load-carrying stresses.

The interest of the designer in these secondary stresses is not to determine their value exactly but rather to be sure that this value is not too high. In order to find what value is too high, it is not sufficient to approach the problem from the analytical viewpoint. The values permissible in design vary with the material used, with the importance of the member involved, and with the type of failure that would result. At present, secondary stresses are accepted as a necessary evil. Try to keep them within a reasonable figure, and otherwise forget about them. Any effort to change this viewpoint represents a radical departure in thinking in the field of structural design. (In 1934, a valuable paper on the "Effect of Secondary Stresses Upon Ultimate Strength," was presented[2] by John I. Parcel, M. Am. Soc. C. E., and Eldred B. Murer, Jun. Am. Soc. C. E. Surely, the idea of discounting secondary stresses is not new in America; but accurate data as to the amount that may be discounted are much needed.)

An important difference between participation stress and deformation stress lies in their relation to the properties of the material. The two are affected in the same way by departures from Hooke's law for, in both cases, it is the strain that is fixed, the stress being a consequence of the strain. In so far as the ratio of stress to strain, however, changes with time (time flow of concrete) the effect is pronounced and direct in the case of deformation stresses, whereas time flow as distinguished from plastic flow has no effect at all in the case of participation stresses because, although the stress for a given participation strain is less as time goes on, the strain itself is in the same ratio greater because of flow under primary stress.

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[2] See p. 289.

If the participation strain is linear, as in the case of cross-bracing, the designer can control the stress only by changing the length of the member, which usually means that he cannot control it at all. If the strain is angular, the designer can control the maximum participation stress by changing the length of the member (which is usually impracticable), by changing the depth in the plane of flexure, and, to some extent, by changing the form or variation of section along the member.

Participation stresses are dangerous to the extent that they impair the primary load-carrying capacity of the member. Any generalization about them that does not consider the nature of failure from primary load is misleading.

Normal Structural Action.—The term is used herein to describe the action of those structures or structural parts in which it is possible to determine at the beginning the approximate magnitude of the forces in action, and in which the magnitude of these forces is affected comparatively little by the relative stress intensity in the parts of the structure.

The most obvious example is the ordinary statically determinate structure. In this type primary stress in one member is not affected at all by the stresses in any other member. All stresses are determined directly by statics, and the members are then proportioned for the forces which act upon them.

Most of the classical forms of indeterminate structures act normally, although their action is less definite than for structures statically determinate. A good example is the ordinary continuous truss, which, in practice, is often designed at once without any exact analysis being made after the design is complete. The experienced designer knows that for these structures such analyses will not indicate any important changes in his design.

Other examples are ribbed arches and spandrel-braced arches of steel, the so-called "rigid frame" bridges hinged at footings so far as the forces and moments at the knee are concerned. Most cases of continuous girders are included in this type.

In these structures it is possible to follow the procedure recommended in the textbooks. The engineer is told to guess at the sections, analyze, revise, and re-analyze. A bad first guess does not make much difference; the series of designs converges rapidly. In these cases, however, the designer can do much better than a bad first guess. Preliminary studies of pressure lines and of the properties of influence lines will enable him to make a good first guess—so good that the revision is trivial.

Hybrid Structural Action.—The term is used herein to mean structural action in which two or more parts participate in carrying loads to such an extent that if the strength of one part is changed the forces acting on other parts are largely affected. Clearly, there is some interaction in all indeterminate structures; the difference here indicated between normal and hybrid action is one of degree, and the two classes merge into each other. Participation stresses also are directly affected by changes in the primary stresses, but the relation is not reciprocal.

Hybrid structural action may be divided into two classes from the viewpoint of the designer's knowledge: Those in which the nature of the structural action can be foreseen; and those in which it cannot be foreseen. This says nothing except that the designer either does or does not know what he is doing; but the distinction needs to be made.

This type of action may be further divided into two classes, depending on whether the structure can or cannot be designed efficiently as laid out. It is very important to know this at once, but the knowledge depends on the designer's understanding of the problem.

A very simple example of what herein is called hybrid structural action occurs where two parallel beams are connected so that their center deflections must be the same under a single load at the center. If they are of the same span, depth, and material, they share the load in proportion to their strengths, and load can be assigned to one or the other at will. Even if the spans are different, this is still true if the depth-length ratio is the same for the two beams.

In this case the designer can foresee the action. He then designs as he chooses and knows that the stresses will be as assumed without further analysis; the analysis precedes any designing. Moreover, the structure can be designed efficiently since the stresses in the two beams will be the same. If the depth-length ratio is different for the two beams, however, they cannot be designed efficiently. No matter what the designer does, the relative stresses will be proportional to this ratio; but he can still foresee this fact. This case is simple; frequently the action of the structure is not easily predetermined and, in many cases, the efficiency of the design must be considered for several conditions of loading.

Near the center of long trusses having two diagonals in each panel, the designer may predetermine the stress intensities in these diagonals pretty accurately. In any panel the stress intensities in the diagonals are in inverse ratio to the squares of the lengths of these diagonals if there is no stress in the posts and if the stress intensities in the chords are equal and opposite. For dead load, these conditions in chords and posts are nearly fulfilled; for isolated loads, they are only approximately correct.

Knowing this and considering for the present only dead load, the designer can assign areas to the diagonals at will. If the chords are parallel, the diagonals can be designed efficiently; if the chords are not parallel, one diagonal must be inefficient. Influence lines may be helpful, but they fail to furnish at once a very illuminating picture of the essential structural action. For single concentrated loads the effect of the stresses in the posts and of unequal stresses in the chords may be pronounced. If single moving concentrated loads dominate the design, exact proportioning seems hopeless.

In these cases—parallel beams and double diagonals—the textbook recommendation to guess at a section, analyze, and re-design, to convergence will not work very well. If the layout is efficient, analysis will show the first guess to be right—and will show any guess to be right—but if it is inefficient, repetition of analysis and design will eliminate the inefficient member—after many repetitions—and result in a simpler structure.

Similarly, the king post truss cannot be designed for given working stresses in beam and sag rods except for a small range of the ratio of beam depth to sag depth, which ratio can be predetermined. The chief interest in the king post truss is that, goemetrically, it is so obviously the familiar problem of a truss subjected to secondary stresses. Looking at the truss from this viewpoint, the engineer would first design it; then he would reduce the depth of the beam until the secondary stress in it was within safe limits, or he could, by adjusting the sag rods, produce initial stresses to offset the secondaries. If, however, he attempts to utilize these secondaries, to put them to work in carrying the load, an entirely different structural problem is presented, and new methods of study are required.

Hybrid structural action also occurs in the queen post truss, in systems of intersecting beams, in Vierendeel girders, probably in most slabs (at least where variation of depth is involved), in some problems of continuous beams, and in many problems of continuous frames. It is discussed subsequently, in the case of bents, of rectangular wind frames, and of arches integral with their spandrel structures.

Study of a Two-Legged Bent Illustrating the Types of Structural Action.—The distinctions herein presented are well illustrated by studies of a two-legged bent carrying vertical loads. It is assumed that the members are rectangular and homogeneous. It is proposed to study the flexural stresses in the columns. The girder section is assumed to be the same throughout the discussion but first the depth, and then the width, of the column is varied.

If the column is very narrow the girder acts practically as a beam simply supported. The rotation of the top of the column must be the same as that of the end of the girder. The angular strain in the column, therefore, is fixed, and the flexural stress varies almost directly with the depth. If the column is extremely rigid it takes nearly the full fixed-end moment in the girder and the flexural stress varies nearly inversely as the square of the column depth. Between these conditions is one in which the flexural stress in the column is nearly independent of the depth; the column is picking up moment as fast as it can take it.

If the width of the column is increased, it will be found that in the first stage (column slender) the flexural stress in the column is scarcely affected at all; in the last stage (column stiff) this stress varies inversely as the width; and in the transition stage, the increase in width reduces the flexural stress somewhat but not at all in proportion to the increase in strength.

The first stage represents a structure essentially statically determined (post and lintel or column and beam), but with participation stresses in the columns. When the column becomes very stiff the structure—so far as the columns are concerned—is normal in its action, the stresses in the columns being determined for a fairly definite moment. In the transition stage the structural action is hybrid and does not respond readily to ordinary design procedure; increase in depth may either increase or decrease the flexural stress or leave it unchanged; and increase in width (which amounts to increase in moment of inertia) produces comparatively little effect.

Deformation stresses would be produced in this structure by change of temperature. As the depth of column is increased, the deformation stresses at the top of the column would vary at first almost directly as this depth, but later would be relieved by flexure of the girder.

Perhaps the most important fact revealed in this case is that if the flexural stresses indicated in the hybrid stage are dangerously high, there does not seem to be very much that can be done effectively, as long as the girder section is constant. This column may be thrown from the hybrid stage into the normal stage of action, however, by reducing the stiffness of the girder. In the rigid-frame bridge this is done by reducing the center depth of the girder.

II.—General Remarks on Hybrid Structural Action

It is difficult to identify hybrid action in an unfamiliar structural type, but after one has come to recognize the type he begins early to suspect its existence in certain cases. Probably the chief identifying characteristic of the type is that it responds sluggishly or erratically to traditional methods of structural design. Successive cycles of design and analysis may indicate a trend, but produce only slowly a definite and satisfactory conclusion. If there are discontinuities in this design procedure the traditional process may be quite misleading.

Traditional processes are not very helpful in this field, although they still have their place. In these cases there are usually many variables and the curves of variation present maxima and minima. It should not be necessary to point out to scientific men the extreme difficulty—the grave danger—of applying purely empirical methods to such problems. It is impossible, in such a case, to generalize or extrapolate beyond the range of data presented and it is almost impossible to classify the data for study no matter how numerous these data are, unless such arrangement is based on an adequate theory.

It makes a good deal of difference what the designer wants to do. Where the action is normal for given proportions, there is usually only one answer and that is easily approximated at once and easily determined accurately by cut and try. Where the action is hybrid there are many possible structures; the designer must make his choice. In a sense he tells the structure what he wants it to do, and the structure will try to do it. If, however, it is something that the structure cannot do at all, the designer has erred; if it is something that the structure cannot do efficiently, the design is penalized. To apply the more erudite terminology of mechanics to this conception, the fiber stresses desired may involve incompatible strains.

This type of structural action is often best approached from a direct study of the fiber stresses. H. V. Spurr, M. Am. Soc. C. E., has done this in his wind-frame studies,[3] and, without discussing herein whether his method of design is necessary, sufficient, or invariably satisfactory, the writer feels that his contribution to the direct method of attack is of great value.

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[3] "Wind Bracing", by H. V. Spurr, McGraw-Hill Book Co., 1930.

Rigid-Frame Bent Subject to Vertical Loads.—Consider a bent of the so-called rigid-frame type sometimes used for building frames consisting of a roof girder, curved or polygonal, carried by two columns. The pressure line for dead load in this structure may lie anywhere between that for a simple curved beam supported on columns and that for a three-hinged arch (hinges at the center of the girder and at the bases of the columns).

The action of the structure may be studied by either of two methods. Choose some pressure line which is expected to give a good distribution of material, determine the moments for this pressure line, and then vary the moment of inertia along the section so as to satisfy the conditions of continuity. The depths of the sections may then be varied by choice. By this method the designer tells the moments where he wants them and then decides whether he likes the result.

As an alternative procedure which has some advantage, he chooses the desired pressure line, selects working stresses along this line, and then varies the depths to provide the requisite conditions of continuity. This requires judgment, but may be done quite well by eye. The moments of inertia may then be determined for these moments, stresses, and depths. The important fact is that the design may be predetermined—and predetermined over a wide range.

If at three points of the structure the moments of inertia are intentionally reduced compared with the sections elsewhere, the structure now becomes a normal structure (a three-hinged arch) with participation stresses at the weakened sections, which now act as hinges.

Rectangular Wind Frames.—The problem of analyzing rectangular frames for horizontal forces due to wind or to earthquake accelerations continues to occupy an important place in structural literature. It seems particularly illuminating to discuss the problem from the viewpoint proposed herein. It is assumed that the columns have been designed for vertical loads and that their sections will not be changed; discussion, then, is directed entirely to the design of the girders. The effect of offsets is not considered.

It is not intended to discuss the participation stresses induced in the girders of upper floors by departures from planarity at floor levels. The writer's studies indicate that the problem is neither so important nor so difficult as some have indicated.

Assume that the girders have been designed by some of the conventional methods. An analysis is now made and it is found that some girders are overstressed and some under-stressed. Tradition indicates re-design by increasing the size of the overstressed girder and decreasing that of the girder understressed. However, analysis will show in many cases practically the same stresses as before; in other cases, it may show a slight improvement in the stress distribution which will be further improved by repeated re-design to a certain point; but numerous cycles of re-design may be necessary to produce much improvement.

The girders of a symmetrical rectangular frame, carrying horizontal loads, may be designed on the following assumptions: (*a*) Points of inflection are at the mid-points of all members; (*b*) column shears are proportional to moments of inertia of columns; and (*c*) design stresses in girders are proportional to depth-length ratios of girders. If so designed, it will act as designed except near the fixed base. The analysis precedes and dictates the design; after design, no analysis is needed.

If points of inflection in the columns and distributions of column shears are assumed throughout the building, the designer can, by working from irrotational footings, determine relative stiffness, or K-values for the girders all the way up the building. Negative values for K would indicate impossible assumptions and positive values of K might be unsatisfactory because, for a given working stress, too deep a girder would be indicated or, for a given depth, too great a fiber stress would result.

The object is not to recommend this method of design, although it has value, but rather to indicate that as an alternative to the procedure, now popular, of assuming a structure and, by analysis, determining the stresses in it, presumably with a view to re-design, the designer may predetermine the function that the bracing is to perform, design directly, and then see whether the design is satisfactory as regards girder depths or working stresses. Of the two procedures, the second is often the more satisfactory. Each method is useful for certain purposes.

Note that the sensitiveness of the procedure depends upon the relative stiffness of columns and girders. If the columns are very stiff relative to the girders the moments can be applied almost anywhere without disturbing the desirable condition of inflection points very near the mid-points of the girders; the moments in the girders will be proportional to their K-values.

It seems futile to ask which of several conventional methods of analysis most nearly conforms to the results of so-called exact analysis; the answer depends on the proportions of the structure. Special attention is directed, therefore, to the dangers of induction from specific data whether obtained by computation, from models, or from tests in a laboratory, unless the range of the data covers all variables.

Arches Integral with Their Spandrel Structures.—An important example of hybrid structural action occurs in arches that are integral with their spandrel construction. It has long been recognized that there is interaction of the rib and the spandrels, but such interaction is commonly neglected in design.

In special cases applying to such structures, the engineer can see certain controlling relations. If the columns of an arch having open spandrels are quite flexible and very closely spaced, angular changes along the arch rib must be the same as those along the deck girder, since the vertical deflections of the two are every where the same. In this case, then, the entire structure could be designed at once as an arch made up of two members placed side by side as in a flitch beam. The relative flexural stresses in the rib and deck can be predicted and may be controlled by varying their relative depths.

Clearly, in this case, the designer may put all flexural resistance in the rib, or in the deck girder; or he may divide this flexural resistance between the two members at will; arches have been built of all three types—a stiff arch without stiffening girder; a flexible arch with stiffening girder; and a stiff arch reinforced with a stiffening girder. Of course, in any case, there must be some flexural resistance in the flexible member and some flexural resistance in the columns. These, however, will have the well-defined characteristics of participation stresses.

If the columns are not closely spaced the simplicity of the picture is marred both because the flitch-beam picture is less simple and also because the deck girder now has primary stresses as a continuous beam.

As the columns become quite stiff, however, the picture becomes very complicated. Such a structure is clearly hybrid in its action and the nature of the inter-relations is not at once apparent. Methods of studying such cases have been developed by Nathan M. Newmark, Jun. Am. Soc. C. E.[4]

It is important to question whether it is good practice to change from normal structural action with participation stresses to hybrid action with obvious reduction in the factor of safety of the rib, with little promise of economy, and with much complication and uncertainty in design. There is little, if any, evidence that the participation stresses in these structures are dangerous or even objectionable. The idea of putting secondary stresses to work is not usually very promising.

III.—General Remarks on Indeterminacy

Thirty-four years ago Mr. Frank H. Cilley presented a paper under the title "The Exact Design of Statically Indeterminate Frameworks. An Exposition of Its Possibility but Futility."[5] The main thesis was that "statical indetermination in a structure is always to be regarded as self-interference with efficiency." The paper followed a previous paper by the same author,[6] and revived a discussion of long standing as to the relative advantages of determinate and indeterminate systems. Two discussions of the paper were presented by distinguished American engineers and several by foreign engineers.

There is little question that Mr. Cilley's views represented those of many, and probably of a large majority, of the leading American structural engineers of his day. To-day, on the other hand, literature contains numerous articles extolling the virtues of indeterminacy and some writers even go so far as to attribute to indeterminate structures virtues which appear to be contradictory. They are referred to as "reservoirs of resilience," their rigidity is praised, as are their economy and their strength.

Since 1900, many indeterminate steel trusses have been built in America with claims, apparently well supported, of considerable economy. More important still, a new material—concrete—which is more conveniently made continuous, has come into general use.

The rather awkward methods of analysis current at the beginning of this century undoubtedly delayed the development of continuous structural types.

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[4] Some phases of Mr. Newmark's extensive studies are reported in "Interaction Between Rib and Superstructure in Concrete Arch Bridges", by Nathan M. Newmark, Jun. Am. Soc. C. E. Thesis presented to the University of Illinois in 1934, in partial fulfillment of the requirement for the degree of the Doctor of Philosophy.

[5] *Transactions*, Am. Soc. C. E., Vol. XLIII (1900), p. 353.

[6] "Some Fundamental Propositions Relating to the Design of Frameworks", by Frank H. Cilley, *Technology Quarterly*, June, 1897.

At that time the analysis of some of the more complicated types now proposed was impracticable. The profession has made progress in this field. It may be well now to divert the attention of structural designers from the endless elaboration of analytical technique to the more important matter of interpretation of analyses.

It appears that Mr. Cilley's paper was directed too much to a consideration of what is herein termed hybrid structural action. The degree of self-interference in normal structures is quite negligible. Each normal indeterminate structure usually has a determinate analogue in comparison with which it has certain virtues and which has certain virtues in comparison with it. No generalization is possible, but the indeterminate structure has won consideration and is often indicated.

Structures in which hybrid action predominates are also sometimes indicated. If their action can be clearly foreseen and if they are designed for one controlling load condition, they may be designed economically. Often they are indicated for reasons entirely apart from structural efficiency, as in the case of rectangular wind-bracing; but where their action cannot be clearly visualized, conventional procedures of analysis by computation or by model may furnish little help.

IV.—Concluding Remarks

The paper has indicated four types of structural action the characteristics of which make their separate discussion worth while. These are (*a*) the action that produces deformation stresses; (*b*) that which produces participation stresses; (*c*) normal structural action; and (*d*) hybrid structural action. Deformation stresses and participation stresses have many characteristics in common, but are essentially different in cause and sometimes in action. Hybrid structural action represents a transition stage between participation stress and normal structural action.

Deformation stresses cannot be avoided except by avoiding the deformation that produces them. They may be slightly modified so that a little more strain takes place at one point and a little less at another, but about the same total strain—angular or linear— will inevitably occur. Linear strain due to angular strain may be reduced by decreasing the depth.

The designer's interest in participation stresses is that they shall not be too high; he does not want their exact values. If they are too high, he changes either the over-all dimensions or the details of construction. What is "too high" will always remain a matter of judgment. In normal structural types only is the traditional procedure, of first computing the forces and then designing for them, applicable.

Hybrid structures may be designed in many ways. In order that analysis may guide to design, it should precede design so that the designer may see in what ways the structure can act. Then, in a quite literal sense, he tells it how to act and makes it act in that way. The difficulty is that he may blunder in trying to have it act in a way in which it cannot possibly act or that, of the many ways in which it can act, he chooses an inefficient one.

Structures characterized by hybrid action are difficult to design and are often inefficient in any case. Study of them is made difficult by the inadequacy of traditional methods; it is almost hopeless, and may be dangerous, to study them by emprical methods.

Generalizations as to relative advantages of determinate and indeterminate systems are difficult in any case. Some conflicting opinions in the literature may be reconciled by recognizing the distinction indicated.

DISCUSSION

L. J. Mensch,[7] M. Am. Soc. C. E. (by letter).—The gradual increase in scientific observations of engineering structures has led to an increasing attention to details of theory. The normal action, or what may be more correctly termed the idealized action, of structures as described and taught by engineering textbooks occurs rarely, and very often other influences cause deformation and participation stresses that becloud the issues in actual practice.

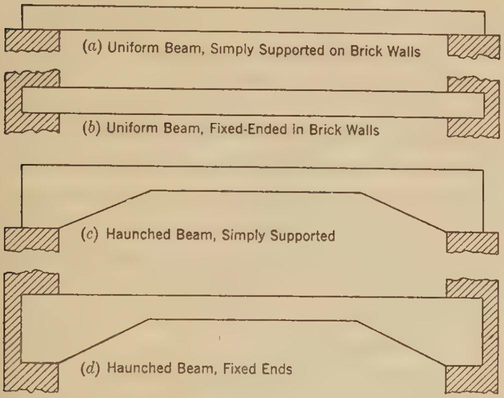
(a) Uniform Beam, Simply Supported on Brick Walls
(b) Uniform Beam, Fixed-Ended in Brick Walls
(c) Haunched Beam, Simply Supported
(d) Haunched Beam, Fixed Ends

Fig. 1.

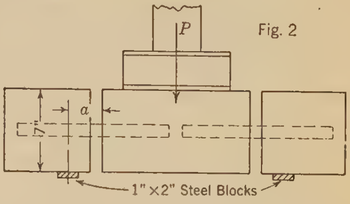
P
a
1" × 2" Steel Blocks

Fig. 2

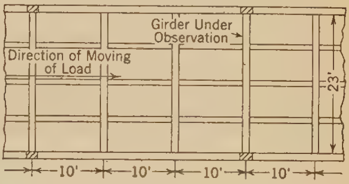
Girder Under Observation
Direction of Moving of Load
23'
10'
10'
10'
10'

Fig. 3.

A good illustration of this point is afforded by tests made by Dr. F. von Emperger on a large number of girders for the Austrian Concrete Committee.[8] In Fig. 1, (*a*) represents girders loosely supported on brick walls; (*b*) represents girders well embedded in brick walls; (*c*) represents girders with brackets loosely supported on brick walls; and (*d*) represents girders with brackets well embedded in brick walls. All girders had the same span of about 13 ft, all had the same cross-section of $6\frac{1}{2}$ by $8\frac{1}{2}$ in., and the same reinforcement, except that some beams had brackets at the ends, as shown. Girders (*c*) failed at 75% more load than Girders (*a*), although according to the normal action as defined by textbooks the ultimate strength should have been the same. Girders (*b*) failed at 250% more load than Girders (*a*); and Girders (*d*) failed at 575% more load than Girders (*a*). The participation stresses as classified by Professor Cross were of a favorable nature in this case.

A similar effect was observed[9] by the writer in 1934 when having tests made on dowels connecting concrete blocks, as shown in Fig. 2. The stresses in the dowels at ultimate load, when computed by the formula, $s = \frac{P\,a}{2}$, were from 150 000 to more than 200 000 lb per sq in. As the re-rolled steel was probably of less strength than 100 000 lb per sq in., the effect of the friction at the supports was to increase the strength of the dowels 100 per cent. Both favorable and unfavorable participation and deformation stresses are found in concrete pavements, due to friction on the subgrade. Friction causes the concrete to crack when the pavement shortens, due to temperature and moisture variation, but when a concentrated load deforms the under side of the slab, the friction causes a moment of opposite direction to the moment of the concentrated load.

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[7] Civ. Engr. and Constructor, Chicago, Ill.

[8] "Mitteilungen über Versuche", Heft IV, 1913.

[9] "Joints for Concrete Pavements", by L. J. Mensch, 1935, p. 33.

A very thorough investigation of a similar nature may be found in the report of the French Committee on Reinforced Concrete[10] The Committee tested to destruction, a reinforced concrete floor of the Science and Art Building of the Paris Exhibition of 1900. As shown in Fig. 3, the girders were of 23-ft span, placed 10 ft on centers; they were connected by small beams of 6 by 8-in. stems, and covered by a 4-in. slab. Concentrated loads of from 10 000 to 42 000 lb were pulled on a small truck in the center line of the building, and the deformation and deflection of one girder were observed when the concentrated load was at various distances from the longitudinal axis of the girder under observation. At a distance of the concentrated load from the girder of 30 ft, 20 ft, and 10 ft, the deflection of the girder was 1%, 10%, and 47% of the deflection observed when the concentrated load was exactly over the girder, and the Committee concluded from its many observations that the stresses produced in the last case were 50% of those of a single girder, disconnected from the adjoining girders.

One can readily infer from these examples that secondary stresses often help the structures to carry higher loads, and so-called inventors have often misled the public by taking advantage of this fact.

The writer cannot agree with Professor Cross that the post and lintel structure is a statically determinate system. The lintel will bend under load and throw the upper inside edges of the posts into heavy compression, thus producing an eccentricity of the load on the posts for nearly one-half their depth. As a consequence, the columns may and may not bend sufficiently to produce at the top an inclination equal to the rotation of the lintel at the ends and thereby offer an even bearing to the lintel. The analysis of such a system has been given by the writer elsewhere[11].

All properly designed bents with either slender or stiff columns may be classified as good engineering if they can be defended on architectural or economical grounds. One may consider all deformation and participation stresses, the deflections of the girders and columns, the flow of concrete due to dead weight, the flow of concrete due to the interaction of concrete and steel during shrinkage, temperature effects, active and passive earth pressure, and possible movements of the abutments. Nevertheless, to make a structure lasting and free from serious blemish, there is still lacking a very important point to the beginner, and this is practical experience and critical observation of existing, similar structures.

In ordinary reinforced building construction engineers are accustomed to design the highly indeterminate girders and slabs by the bending moment formulas: $M = \frac{wL^2}{8}$; $M = \frac{wL^2}{10}$; $M = \frac{wL^2}{12}$; and $M = \frac{wL^2}{16}$, as given by the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete[12]. This Joint Committee did not give any rules for determining the bending moment on inside columns, and, therefore, many engineers believe that it is good practice to omit them from consideration. Exponents of scientific analysis of indeterminate structures may object to such formulas, violating as they do every rule of continuity and compatibility. They may object, furthermore, on the ground that such formulas do not seem to be based on logical reasoning; if they were logical, the coefficients would be different for dead load and for live load and would depend on the relative stiffness of the adjoining members. It is a fact, however, that only in rare cases (especially when there are very large differences of spans in the adjoining members) will a so-called exact analysis make a great change in the moments at the supports and the moments in the center of the girders will be found considerably smaller than those given by the Joint Committee. No great harm is done when the girders are made stronger than necessary in the central section and the adoption of these formulas has saved hundreds of millions of dollars to building owners. The introduction of this type of formula is to the credit of F. Hennebique, and dates from 1894. Structural steel engineers did not have such a leader and they are not in the habit of reducing the bending moment in girders of skeleton buildings from the moment of a freely supported girder.

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[10] "Experiences, Rapports", etc., 1907.

[11] *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–20), p. 1682.

Unless similar simple formulas are offered to engineers for the design of statically indeterminate structures such as those with two or more legged bents, rigid frame, and continuous bridges, old and new types of arched bridges, and single and multiple-arched bents for roof constructions, the writer is convinced that no truly great progress will be made in this field in the United States. The writer has seen designs of that nature which cost considerably more than statically determinate structures. Collections of such formulas have appeared in handbooks in Europe and represent the labors of many conscientious workers, but they need much elaboration before American engineers will have sufficient confidence to use them often. The fact that only uniform sections are used for the various members works against their use for long spans; deflections are not given; proper details and examples are presented only sparingly and theoretical spans instead of clear spans are used in order to facilitate the computations. The work of A. Strassner, published more than twenty years ago, marks a great progress as it shows how to deal with members of varying sections. It marked the first attempt at improvement; it is still very theoretical, and the assumptions and the tables still contain errors. These tables have been used by American engineers for the construction of tables of other coefficients; the latter may contain additional errors and caution is necessary in using them.

Such a handbook as the writer has in mind cannot be written by a single expert or a college professor and his most capable assistants, but must be the work of a great many experts and their associates, and each problem must be investigated by at least four men, working, independently, in groups of two. There is no question but that the solution of statically indeterminate structures requires much tedious work. The most skillful mathematician is likely to make many errors, but two men working together can save time and avoid most errors. At present, only two methods are known for the solution of statically indeterminate structures: The mathematical theory of elasticity and the mechanical solution by apparatus such as those of Professor Beggs, Gottschalk, Nubupest, and others. The mathematical theory of elasticity states neither more nor less than that: Every particle of the interior of a body must be in equilibrium; every particle of the boundary surface must be in equilibrium; stress must be compatible with strain in every point; and, of course, the entire body must be in equilibrium under the action of the loads and other forces such as reactions and surface tensions (frictions and shear from adjoining bodies). It is practically impossible to solve the simplest case of a structural member according to the strict theory, and simplifications are necessary. A particle is in equilibrium when the sum of the components of all forces (stresses) acting on this particle parallel to the X, Y, and Z axes are zero. One simplification most often used is to disregard the stresses parallel to the Y and Z-axes and to confine the investigation to the stresses parallel to the X-axis, which may be assumed to be in the longitudinal direction of the member. Another simplification which is often necessary is to investigate the stresses or conditions of equilibrium at only one or two sections of the member, and to hope and believe that nothing of a dangerous nature will happen in other sections or directions which are not investigated according to the rules of elasticity. A good example is Clapeyron's theory of continuous girders. Clapeyron established the conditions of equilibrium at sections over the supports by making the stresses parallel to the direction of the girder on both sides of the section equal and of opposite direction. In Professor Cross' language he established the principle of continuity, but committed the inexcusable error of not considering the stresses due to the width of the support, which error changes the result by 20% in most cases.

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[12] *Proceedings*, Am. Soc. C. E., October, 1924, Papers and Discussions, p. 1153.

In the traditional treatment of a fixed arch, the conditions of statical and elastic equilibrium are generally applied to only one section, and most easily to one abutment. The arch is considered as a cantilever fixed at the other abutment, and, under the assumed loading, the other abutment which is being investigated is assumed to deflect horizontally a distance, h, and vertically a distance, v; and it is assumed to rotate through an angle, b. The problem is find the horizontal force, H, the vertical force, V, and the moment, M_a that would cause the deflections, h and v, and the rotation, b, to vanish. In order to find H, V, and M_a, the cantilever is treated as a common girder and the horizontal and vertical deflections and the rotation of the free end abutment are computed for the loads and the unknown forces, H, V, and M_a. Then, the so-called elastic equations are formed by: Equating the sum of all horizontal deflections to zero; equating the sum of all vertical deflections to zero; and equating the sum of all rotations of the end to zero.

Not many careful tests have been made to establish how correct the designer is when he uses the formulas for the deflection and rotation of a straight girder of uniform section for the computation of the deflections and rotations of girders of curvilinear form with markedly varying section, knees-braces, or other irregularities. In extreme cases the error is probably of the order of 25 per cent. The identical equations can be obtained by the theory of least work, by the principle of virtual velocities, or by Maxwell's equations. The latter are simply rules for obtaining deflections and rotations; intrinsically they are not more scientific than the foregoing procedure and do not imply an "exact" theory of elasticity.

There are many methods of obtaining the deflections: By arithmetical summation, by integration, by the moment-area method, the slope-deflection method, and the shear-area method, to name only a few. These methods are not new theories of elasticity as some writers would like to have them called, but are more or less cumbersome methods of finding deflections. New methods are re-invented by every new generation of engineers in the desperate search for finding the nearly impossible—a simple method of computing deflections. Statically indeterminate structures would be used more often and more scientifically and would be much better understood if only one method had been taught in the last seventy years, and if that method had been thoroughly elaborated. The many methods described recently in literature are only a confusion to engineers.

In order to shorten the analysis of typical structures, engineers have computed the deflection and rotation of certain points on these structures for unit loads placed at various positions and have presented the results of their labors in the form of diagrams and tables, which are not yet simple enough to be of great use.

In the case of highly indeterminate systems with hybrid action, as termed by Professor Cross, such as trusses with double diagonals, skeleton girder and column constructions, and series of curved roof bents, the designer must deviate still further from the teachings of the theory of elasticity before he can find a way to analyze the structure. He must make new guesses or assumptions, and one of the oldest and best is that of Saint-Venant, who assumed that the local disturbance caused by a load at a particular point (or member) dies out at a short distance from the point of application. This guess, rule, or principle (some even call it a theory), when applied to a girder of a skeleton building, can be expressed as follows: Consider the girder to be analyzed as connected only to the adjoining spans and columns, the next adjoining spans and stories being considered as having no influence on the girder. The far ends of the adjoining girders and columns may be considered fixed or rotated at an angle suitable to the case. The surprisingly simple results were given by the writer many years ago.[11]

The mathematical difficulties imposed by the exact theory of elasticity (and very often by its simplified form) being insuperable, the analysis of most statically indeterminate structures is beyond the realm of science and becomes an art, in which experience, skill, imagination, guesses, and hypotheses, often of obscure origin, play a greater rôle than calculus, and where charlatans may run riot and false prophets may shine to the utter confusion of the student. A properly conceived handbook would elevate the art, which is still at a very low ebb.

Russell C. Brinker,[13] Jun. Am. Soc. C. E. (by letter).—A logical classification of the types of structural action and the stresses producing the different types is presented in this paper. Further emphasis might easily be placed upon the question of when secondary stresses may or may not be lightly neglected. The "dishing out" of floor-beam webs and the cracking of stringer-connection angles on some old railway bridges give evidence of secondary stress effects neglected in their designs.

The factor of safety, or more often the "factor of ignorance", allowed in the spread between the specification stresses and the elastic limit in steel bridge design to cover secondary stress and other effects could be put on a more rational basis if the secondary stresses were given greater consideration. Members having low percentages of secondary stresses compared to their primaries, are heavily penalized by giving them the same basic allowable unit stress as those members having high percentages. The fact that the secondary stresses have little effect on the ultimate strength of members does not mean that they should be allowed to produce greatly unbalanced total stresses below the elastic limit because this is the range of usefulness of most structural members. Truly more data on the question of when—or how—to discount these secondaries are needed.

Much refinement in analysis has been made in the decade, 1925–1935, but needed improvements in specifications covering unit stresses appear to have received little attention. It is obvious, that analysis is directly tied up with design stresses, as is the question of details (which is also somewhat behind in progress). In 1923 there was considerable discussion as to the validity of the unit stresses being proposed in specifications for railway bridges[14]. Many prominent engineers looked with disfavor upon the policy of using the same low unit stress for dead load as is used for live load to cover future increases, uncertain impact, etc., but little was accomplished. It is to be hoped that, delving into the types of stresses involved in structural action, Professor Cross will also revive interest in how to evaluate them in terms of more applicable design stresses.

Marshall G. Findley,[15] Assoc. M. Am. Soc. C. E. (by letter).—In this paper Professor Cross has made a notable contribution to the literature of structural design. The problem discussed is not one of detail method, but of general balance, of plan of action, and of criteria for determining the choice of method. It might almost be said that the author presents a pragmatic conception of structural design. The problem involved needs discussion. So much literature of structural analysis has arisen that any one supervising design work is continuously faced with the need for decisions as to how far to go in the use of analytical methods. There is no question but that, on the one hand, certain practical problems are analyzed by mathematics too completely and too well. On the other hand, other problems are too little analyzed—solved too much by rule-of-thumb methods. Only rarely is analysis checked by common sense so as to keep it within bounds while making full use of its possibilities. The personal element complicates the practical problem, one designer enjoying the mathematics of theory too much, while another relies too much on experience, which may be imperfect and misleading.

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[13] Easton, Pa.

[14] *Transactions*, Am. Soc. C. E., Vol. LXXXVI (1923), pp. 532 *et seq.*

[15] Structural Engr., Water Purification Plant, City Engr.'s Office, Milwaukee, Wis.

The term, "hybrid", as used by Professor Cross, however, is very unfortunate. The writer refuses to use this term, and will use instead "monolithic" as being more appropriate. The criticism is that monolithic action is natural and fundamental in some structures, whereas the term, "hybrid", suggests that it is a combination of two types of action. Professor Cross has recognized that the basic nature of monolithic action is his method of moment distribution, wherein every computation starts by assuming perfect fixation at each end of each member. At the same time, the use of this inappropriate term, "hybrid", seems to be bound up with a fundamental misconception. In general, scarcely a structure has been built in which true and natural action is not in some sense "hybrid". It would be impossible to avoid the design of structures that do not somewhere, or in some manner, deserve this term of opprobrium. The real point at issue is not whether "hybrid" structures can or cannot be avoided; it is rather how best to take account of actual monolithic action and its effects.

The criticism of this term is only incidental, of course; but in order to clarify the misunderstanding that it suggests, and also to lead the way to an examination of certain detail problems, the writer presents first some general considerations. These thoughts are partly a development of ideas expressed in Professor Cross' paper—one concerning the need of reconciling discordant views held by practical designers and by theoretical analysts; the other that the designer often tells a structure what to do, and then must see to it that it can do it efficiently. Professor Cross makes a division and then pays very little attention to deformation stresses, to participation stresses, or to so-called normal structural action. He stresses the increasing importance of monolithic action, and in this point all will agree. It is because of this increasing importance that structural analysis is becoming more widely used and, at the same time, more complex.

Two Conceptions of Structural Analysis.—Structural analysis is often considered an exact science. The aim under this conception is to secure an exact picture, by means of computation or of graphic analysis, of action of the beam or structure under stress; of its deflection at each point; of its tension, compression, or shear at this point; or of its dangers of cracking, twisting, or crushing, etc. This conception is very commendable within limits, but its limits are narrow. Only beams simply supported, simple continuous beams, deep trusses, simple frames, and a few other structures can be analyzed in this complete and thorough manner; and, usually, these structures will not be so analyzed for a practical reason—that is, the lack of time. Where such complete analysis is possible, imagination and experience usually make most of it unnecessary.

A larger conception of analysis must be sought, partly on account of current lack of knowledge, and partly on account of one's inordinate ambition to build complex things and to meet complex situations. Columns—simple and compound, flat-slab systems, complex frames, and various other structures that cannot be completely analyzed at present—must be designed. Design cannot wait for perfect analysis; as a matter of fact, it never has done so. Design has always led the way for analysis. Properly understood, design requires the larger conception of analysis. Analysis is no longer a complete and exhaustive prophecy of stresses and deformations. Instead, it must be relied upon as a basis of rational comparison whereby complex and unusual conditions may be assimilated to simple and ordinary conditions.

As a matter of fact, the actual progress of structural analysis is such that even approximate prophecy of action in many structures is impossible. Especially is this the case where pressures against, or produced by, earth are involved. No one knows exactly what happens in a complex footing or in a wharf bulkhead. Only on the larger conception of design is it reasonable to use mathematical analysis in connection with such structures. Analysis is then a guide to rational comparison between a proposed design and an existing structure somewhat similar. Such structures include columns, footings, flat-slab floor systems, steel and timber sheet-piling, and monolithic frames of concrete. The computation is a guide to design, but cannot be considered exact prophecy.

This larger conception of analysis, as the basis of comparative design, as the rational method of transfer of experience from shorter spans to longer spans, from narrow panels to wider panels, from light loads to heavy loads, from static forces to moving forces, etc., needs discussion and elaboration. The outmoded prejudice against indeterminate frameworks, still often expressed verbally, is clearly a consequence of the conception of analysis as exact prophecy. The current interest in frame analysis, in shrinkage and secondary stresses, and in soil mechanics, is a sign that the conception of analysis as a basis of rational comparison is in the ascendency. This interest sometimes leads to the assumption of impossible tasks, with resultant discouragement. Perhaps a discussion of certain definite problems, from the broader point of view, criticizing methods in the light of purposes, may be of some benefit.

The Simple Beam.—The outstanding case of "normal structural action" is perhaps the beam of one span simply supported at each end. In this case. when loads are given, shear and moment and deflection diagrams can be drawn quickly, and, superficially considered, prophecy of action might be considered quite definite. Theoretically, however, prophecy is not at all exact, and even practically there may be some question.

The fundamental assumption of beam theory, that a plane section before bending remains plane after bending, is only approximately true. In a rectangular metal section, it might be quite accurate, but in steel **I**-beams or channels, in timber with its grain and flaws, and in reinforced concrete with its two different materials, this assumption is only justified by the practical need for it. Shear computations are very rough. Even in a homogeneous rectangular section, it is necessary to use an approximation to estimate the relation of maximum unit shear to average unit shear. Furthermore, deflection computations usually neglect shear deformation entirely, whereas in some types of beams it is of greater magnitude than moment deformation; that is, even in a simple beam all the computations for shear, moment, and deflection are based on assumptions which are usually approximately true, but which, in exceptional cases, may be very far from the truth.

This fundamental assumption of beam theory is only supposed to hold true when no stresses exceed the elastic limit of the materials. This elastic limit is itself a more or less uncertain value, more variable in some materials than in others. Hooke's law of elasticity, that stress varies as strain, may become invalid in particular cases for various reasons, such as: (1) Overloading a member; (2) heterogeneity of a material (as in timber); or (3) a gradual departure from the straight-line ratio (as in concrete).

Certain practical conditions may also arise where the difficulty is more serious. Consider particularly the matter of lateral stiffness. Girders carrying rails across track hoppers, for instance, require more attention to lateral bracing than to routine moment and shear computations. From the designer's point of view the lateral forces acting are due to chance and cannot be estimated except empirically. Cross-bracing or diaphragms introduced to stiffen such girders make them "hybrid" structures, which must act monolithically. In an ordinary floor the beams are braced at their top flanges by the slab. That this occurs is recognized in practical design by certain rule-of-thumb devices (cross-bracing, tie-rods, top cross-bars, etc.). The effect of these devices is to make a "hybrid" structure of even ordinary floors.

Thus, in general, even the simplest types of structure may exist under conditions that make them complex in action, and unsafe to design as by "normal structural action". Eccentric loads, the mechanics of buckling, and the relation between shear and tension, are not very well understood in fundamental theory, and must be dealt with by constant watchfulness on the part of the practical designer. "Hybrid" action which must be taken into account in some manner exists in many unsuspected details. With this warning in mind, moment, shear, and deflection diagrams serve as very satisfactory guides for the design of simple beams. However, they are only approximations. Materials do not conform thoroughly to the assumptions of design, even in the most favorable cases. Lack of homogeneity in material may result in eccentricity and, therefore, in so-called "hybrid" action, where the assumptions are practically perfect in every other element, and the designer must be on the lookout for such possibilities, even in the simplest cases.

Even a strut or tie designed for axial stress only presents many uncertainties when examined inch by inch (who knows the exact details of internal stresses in a tension member around the rivet holes, for instance?); so that, in general, in plain tension or compression members, as well as in simple beams, there are uncertainties of fundamental assumption, uncertainties of homogeneity, uncertainties of rigidity, uncertainties of local stress, and uncertainties of perfect elasticity which make the theoretical part of the design very imperfect, considered as exact prophecy. In this case a generalization may be made. Even in the simplest cases, theoretical structural analysis may be considered successful only when, and because, it is a guide to safe, economical, and efficient design.

The Long Column.—The foregoing uncertainties apply to complex structures just as surely as to simple structures; but there are also other uncertainties of many kinds. One of these uncertainties appears in the case of the long column. Briefly, the best method of analysis seems to be an indirect one; namely, the selection of a section such that under chance small eccentricity, the tendency to bend does not increase with time. Other methods of analyzing long columns are not truly methods of analysis, but rather methods of empirical design.

In this case the rational purpose of design is that chance deformation stresses shall be controlled so that they do not essentially tend to increase with time. Here, clearly, if one small part of a column is weakened unduly, the result will be to throw enormous stresses into other parts of the column. Returning to the Cross definition of hybrid action, then, one may conclude that any long column is a "hybrid" structure, which, of course, is simply another way of stating that prophecy is difficult, as regards the exact action of a long column.

The Monolithic Frame.—The uncertainty of prophecy is complicated by an entirely different type of difficulty in the case of the monolithic frame. Assume for the present that all external forces are definite, known, and static (which is rarely the case, as, for instance, on a track culvert). The argument as to whether clear opening or center distance shall be used for each span will perhaps never be decided finally. Theoretically—that is from the point of view of prophecy—this is a very vital question. If one decision is made, end moments and shears will be computed, which are very large and which cannot possibly develop; and, consequently, the ends of the moment and shear diagrams must be adjusted or neglected. If the other decision is made, the question must be more or less arbitrarily settled as to the treatment of certain axial and eccentric forces acting on frame walls.

Practically, this question is not such a serious one, but only because the point of view of prophecy must necessarily be abandoned. Whichever decision as to method he makes, the designer follows that method through various structures, and learns what adjustments to make in order to obtain a satisfactory design. His analysis is of value as a basis of comparison, not as a prophecy. Enough structures of the kind have been designed, built, and tested to serve as a starting point for the comparison.

Deformation Stresses in Concrete.—Setting, shrinkage, and temperature changes normally affect all concrete structures; ordinarily, of course, the object is to distribute deformations, assigning them by definite joints to definite locations.

In certain cases, however, such changes cause participation stresses, thereby approaching "hybrid" action. Such a case arose in the design of certain structures for the Water Purification Plant, at Milwaukee, Wis. In this case, water-tightness requires continuity of certain walls and slabs, whereas the fact that certain slabs frame into a wall at different elevations means that temperature changes in the slabs will throw bending stresses of considerable magnitude into this wall. Only by a systematic comparison of some kind between temperature deformations in these slabs, and unit-moment deformations in the wall, can the resulting moment in the wall be determined. The stiffer the wall, the greater will be the resistance to temperature deformations, and the greater the moment in the wall and the tensile stress in the slab.

The principle involved, that of comparison of deformations, of course, is fundamentally the principle of all monolithic action. All methods of indeterminate structural analysis, whether graphic or algebraic, whether based on conjugate joints or the slope-deflection principle or the theory of least work, are methods of discovering distributions of stresses for which certain deformations are equal and consistent. If the writer understands correctly, Professor Cross looks upon simple and obvious uses of the principle of deformations as cases of "normal structural action", whereas more complex uses of this principle are covered by the term, "hybrid action".

To this point, elements of uncertainty that must be taken into account by structural analysis, even including the so-called "hybrid action", have been discussed; but all the cases examined thus far have assumed known and definite loads acting on the structures in question. Other elements of uncertainty exist, which the writer wishes to include in this discussion, in which external loads are increased or decreased by the action of the structure analyzed. Every designer has analyzed some of these cases, sometimes by guess, sometimes by rough assumption, sometimes by more careful analysis which, however, is always open to question as to its meaning. The writer believes that analysis of these cases is usually practically possible, and, in general, should be undertaken; but such analysis is clearly wasted time and effort if considered as exact prophecy.

Simple Footings.—The assumption commonly made is that earth pressure is uniform over the entire area of the footing (for centered loads) or that it varies uniformly from one corner or side to the opposite one. In the design of footings this assumption is nearly always on the safe side, so it is used very often although known to be erroneous. The more accurate assumption, that intensity of earth pressure varies as the deformation of its surface plane, is difficult to use in many cases, starting as it does with a differential equation of the fourth order. Practically, extreme theoretical accuracy is not required. Consider, for instance, a simple wall footing, with a centered load. The theoretical curve of earth pressures is a sine curve, but a parabola or an ellipse may be used without serious error; that is, the resulting shear and moment curves will be the same for practical purposes. In fact, a straight-line diagram may be drawn, which is also sufficiently close as an approximation.

In more complex cases, with two or more columns on a footing, or the introduction of wind or impact forces, an exact earth pressure diagram becomes practically, although not theoretically, impossible to draw. In some cases, practical approximations for design purposes can be made quite satisfactorily; in other cases apparently no amount of study and discussion will obtain even approximate earth-pressure diagrams, and the designer must analyze the structure by the method of limits, assuming several possible worst cases.

The situation is further complicated, of course, by the difference in action between sand, clay, hardpan, and rock, and by the usual lack of homogeneity in the foundation material. In this case, the essential point is that stress in the footing and stress in the foundation material are related to each other by means of deformation. The stress in each is affected by the capacity of the other to take stress, and not in small degree, but vitally. Thus, the load distribution cannot be known until the design is settled. This is really another type of structural action not included in the paper, and should be given a separate name in generalized discussion. Perhaps the term "reactive" action might be coined for this type of structural action.

The footing is by no means the most complex form of "reactive" action. There are at least two definite forms in which design analyses are being carried on where such "reactive" action exists in more complex form.

Sheet-Piling.—Especially as used in anchored cantilever bulkheads, for quays, and slips, the sheet-pile is loaded by active horizontal forces due to water and earth pressures, and is held in place by the passive resistance of the earth near the bottom, and by an anchorage at the top. The anchorage at the top, in turn, often is held in equilibrium by developing passive earth pressures. Separating the items of loading and of structural action, then, the following items remain to be considered: (1) Water pressure, which is perfectly definite, and can be computed without appreciable error; (2) active earth pressure, which is definite but more difficult to determine exactly, and nevertheless can usually be approximated sufficiently well for practical purposes (like water pressure, this pressure exists independently of the deformation of the movement of piles); (3) passive earth pressure, usually greatest near the bottom of the sheet-pile (in some cases passive earth pressure can be estimated very closely by the conditions of static equilibrium, and, in other cases, it depends on Item (4)); (4) bending action in the sheet-pile, which must be considered because only by deformation or movement can passive pressure be developed; (5) deformation at the top of the pile, due to give of anchorage, axial stress in anchor rods, and bending of wales; and (6) passive pressure developed by the anchorage, which mutually depends upon and affects all these other items.

Theoretical methods have been proposed, which take certain of these items into account. Empirical methods have been used, which take all of them into account, although often by the cut-and-try method. It is clearly a difficult type of analysis, because pressures that can themselves be estimated primarily only by methods of indeterminate structural analysis determine, and in turn are modified by, pressures in which a secondary analysis by means of deformations must be made.

The term, "complex reactive" action, may be applied to this type of action. Design analysis of this type might be considered an attempt at prophecy of action, but as such is certainly not exact.

Roadway Slabs.—An element of contingency enters into the design of tall buildings to allow for lateral resistance in regions subject to hurricanes and earthquakes, or in the selection of sizes for culvert drains in regions subject to heavy rainstorms. The heaviest conceivable winds or earthquakes cannot be resisted economically; and the heaviest rains must be expected to cause floods. First, the designer must decide where to draw the line between high cost and awkward structure on one hand, and greater convenience and economy with less safety on the other.

A similar element of contingency occurs in the design of roadway slabs, in addition to the consideration of moving loads, impact, and "reactive" action. If all sub-grades were compacted perfectly and remained so, there would be little bending stress in any roadway slab. Modern construction methods attempt to produce a perfect sub-grade which will remain perfect. Consequently, although the designer may feel very sure that there will be failures of sub-grade, as a matter of experience, he cannot prophesy where a failure will occur, or how far it will extend. The assumption usually made is that a circular failure of a certain diameter will occur in the sub-grade, and that the slab should be reinforced to carry expected concentrations across such holes, whether central or at the edge of a slab.

In the selection of the diameter of assumed sub-grade failure, the designer is again drawing the line of contingency. In this case design analysis must cope with a third type of "reactive" action which might be termed, "contingent reactive" action. Such design analysis is not prophecy even approximately, but only a basis of comparative design which may be standardized after testing by use.

Criticisms.—After consideration of these rather ordinary cases, it is obvious that Professor Cross' very excellent introduction to the subject needs some modification. He emphasizes two points about which there can be no question, and which are very enlightening and suggestive. The first is the generalization as to participation stresses, which, if not new, is certainly worded effectively, so that deductive reasoning can draw out again from it the fruits of experience which have gone into its formation. The second is the pragmatic conception of design where so-called "hybrid" action exists. Of course, the personal element enters in; nevertheless, it often occurs that the designer "tells a structure what he wants it to do, and the structure will try to do it."

It is because of the characteristic of monolithic and complex structures herein indicated, that safety in structural design can be obtained so often without lengthy theoretical analysis, and with only slight loss in economy. By the method of limiting cases, which is an attempt to prophesy several of the worst conditions that might occur at opposite extremes, it is often possible to design, safely and quickly, a very complex monolithic structure.

The writer has previously criticized the author's use of the term, "hybrid". Now, it seems in place to comment upon another use of words, namely, "normal structural action". This phrase is not so much of a unity as it seems because the author includes in it not only statically determinate structures, but also monolithic structures of varying degrees of complexity, similar only in that the experienced designer can approximate their true action without much study. In other words, design analysis as a method of prophesying actual action is applicable.

Conclusion.—In summary the writer wishes to emphasize several points which have been raised. The first is that "normal structural action" is not a convenient term; even the simplest static action includes many elements of uncertainty. Furthermore, it should be subdivided into static action, and simple monolithic action, which are quite different in their essential nature. "Hybrid" action is perhaps an excellent term for complex indeterminate action when a definite load is divided according to relative stiffnesses of two or more independent or partly independent structures or systems of members.

Certain new classes have been proposed which should be included, resulting in a revision of the classification by Professor Cross, as follows:

I.—Stresses Determined by Strains:
    (*A*) Deformation stresses; and,
    (*B*) Participation stresses.

II.—Types of Action for Supporting Known Superimposed Loads:
    (*C*) Static action;
    (*D*) Simple monolithic action;
    (*E*) Complex frame action;
    (*F*) Hybrid or interactive action; and,
    (*G*) Empirically analyzed action (columns, flat slabs).

III.—Type of Action in Which Loading Is Affected by Action ("Reactive"):
    (*H*) Simple reactive action;
    (*J*) Complex reactive action; and,
    (*K*) Contingent reactive action.

The writer agrees with the author very heartily in his suggestion that further distinctions are needed in the general discussion as to the relative advantages of determinate and indeterminate systems; and hopes that Professor Cross will concur, at least in a general way, with the further distinctions suggested herein.

I. K. SILVERMAN,[16] JUN. AM. SOC. C. E. (by letter).—Character of stress rather than its magnitude is the subject of this paper. Since most of the papers concerning structural analysis appearing in the technical literature of the last ten years have dealt with the technique of stress determination, this paper deserves special attention.

From the author's point of view structural analysis is suggested as a measuring device for efficiency. Thus, if loads were fixed, a statically determinate structure which was built along the pressure line corresponding to the given loads would be as efficient as possible as far as structural action was concerned.

The fundamental assumption made in structural analysis is to the effect that under the action of loads the geometry of the structure remains unchanged as regards span, depth, angles, etc. Under this assumption both statically determinate and indeterminate structures are analyzed for their primary stresses. In the analysis of a statically determinate system it is unnecessary to know the make-up of the members or their physical properties, except to be sure that they are such that deformations caused by the loads do not have an appreciable effect on the dimensions of the structure. It would be difficult, for instance, to determine by ordinary methods the stresses in a structure made of rubber if the loads were such as to cause large deformations. For indeterminate structures with constant moduli of elasticity the make-up of the members must be known since "hybrid" action is present in practically all cases. The writer is acquainted with a truss, continuous over three spans, in which the make-up of the lower chord members adjoining the intermediate piers had a marked effect on the stresses in other members.

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The fixed arch, which is a very common structure, exhibits hybrid action (as defined by the author) to a high degree. Under the action of temperature the stresses at the springing are greatly effected by the thickness at the crown. Fig. 4 illustrates the case in which an increase in the depth of the crown of an arch did not necessarily lead to a reduction in unit stress.[17] This structural action is one which ordinarily cannot be predicted. In a circular arch it is found that temperature stresses can be reduced more effectively by increasing the central angle than by changing the strength of the member.

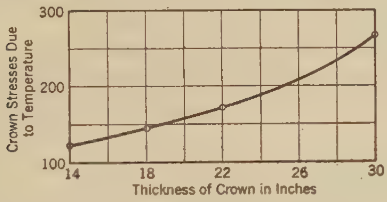

FIG. 4.

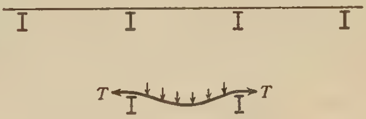

FIG. 5.

A type of action that may be classified under participation stresses, although the stresses produced are load-carrying stresses, is that which is illustrated by the modern suspension bridge in which advantage is taken of the increase in the sag of the cable in computing the maximum cable pull. Another case of the same type of action is shown in Fig. 5 for the continuous plate subjected to a normal load and supported as shown.

A study of the development of structural forms furnishes a fertile field of illustrations of the types of action classified by the author, and the history of the development of these forms illustrates how structural analysis has helped in the production of frames that have primarily normal structural action. A study of this development would be as valuable a part of the curriculum of engineering schools as are those parts which deal with the technique of stress determination.

BRUCE G. JOHNSTON,[18] JUN. AM. SOC. C. E. (by letter).—A logical and concise classification of the factors underlying the relation between structural analysis and design has been presented by the author. The paper deals principally with determinate and indeterminate frames, but the same logical thought applies as well to the study of any structural part, or detail, however small. The "participation stresses", as defined by the author, become somewhat analogous to localized or concentrated stresses in a structural detail, to which increasing attention has been drawn in recent years.

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[17] "Design of Symmetrical Concrete Arches", by Charles S. Whitney, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 88 (1925), p. 998.

[18] Instr., Dept. of Civil Eng., Columbia Univ., New York, N. Y.

Every structural element is designed to perform a definite service without failing or deflecting too far under the working load that it carries. Obviously, the safe working load on the structure should be a specified fraction of the load at which the "limit of structural usefulness" is reached. The "limit of structural usefulness" is usually not a function of the maximum stresses alone, although it may be so in special cases. Many studies have been made with regard to theories of failure under combined stress at a point[19], with none as yet generally satisfactory; but the results of such a theory, even if available, would usually have only an indirect relation, if any, to the general failure of any particular type of structure as a whole.

Studies of the stress distribution in a mathematically idealized elastic body present varied and interesting mathematical problems. Although solutions of many of these problems form the basis and historical background of structural analysis, the continued search for new solutions leads often into byways of small practical significance, and the subject of "stresses" may receive disproportionate attention at the expense of a more complete consideration of general structural behavior up to the point of failure.

The "limit of structural usefulness" of any structure may be determined experimentally by plotting the load-deflection diagram and determining, by one of the various arbitrary means available, the yield point of the structure as a unit. If every element of material in the structure is still in the elastic range as the "limit of structural usefulness" is approached—then an exact and complete mathematical analysis gives a close approach to the solution of the particular problem. However, in many types of structures, made wholly or in part of a ductile material such as structural steel, a slight "giving" of the material in local parts often effects a favorable redistribution of stresses. In such cases the local deformations may cause a relatively negligible permanent deflection in the structure as a whole. Although this fact is well known among structural engineers, its importance is sometimes lost sight of by those who may be interested only in elasticity or photo-elasticity. In some cases, however, as exemplified by both the first and second failures in the design and erection of the Quebec Bridge, the neglect of secondary factors caused failure. The conclusion must be that mathematical analysis and engineering judgment are inseparable factors, one without the other being of no great use.

The following outline gives examples of some of the principal zones of elastic, or plastic action within which the limit of structural usefulness may fall:

A.—Limit of structural usefulness reached within, or at the limit of, the elastic range of the material:

(1)—Limited by elastic stresses;
(2)—Limited by elastic deflections; and
(3)—Limited by elastic buckling.

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[19] "Failure Theories of Materials Subjected to Combined Stresses", by Joseph Marin. Jun. Am. Soc. C. E., see p. 1162.

B.—Limit of structural usefulness not reached until after parts of the structure have passed the elastic limit:

(1)—Plastic yielding localized and not significant;
(2)—Plastic yielding localized at a critical point, leading to failure; and,
(3)—Limited by plastic buckling.

Numerous examples of different structural types may be found to fall under each of the preceding headings.

Both the allowable working load and the limit of structural usefulness may be further conditioned by any of the following:

(1)—The hazard and type of use to which the structure is to be put;
(2)—The type of loading—static, dynamic, repeated, or alternating;
(3)—Physical properties and uniformity of quality of material; and
(4)—Precision of fit and type of fabrication.

Continued research is needed to further the correlation of these factors by studying the entire "load history" of all types and variations of structural elements which are used in current engineering practice. Further development of the theory of elasticity in combination with a consideration of plasticity will greatly increase the effective usefulness of many precise theoretical analyses.

HAROLD E. WESSMAN,[20] ASSOC. M. AM. SOC. C. E. (by letter).—Many readers accustomed to looking for strings of equations and tables of numerical data as earmarks of distinction will fail to realize the value of the structural classification proposed in this extremely important paper. It is well to keep in mind that the paper although qualitative and philosophical in nature, betrays extensive quantitative research of a high order. Such research, involving both analysis and design, must necessarily support the significant conclusions reached by the author.

The inclusion of definite numerical data might confuse or minimize the importance of the basic philosophy. Nevertheless, in view of the conclusions about analysis made by the Committee of the Structural Division, on Wind-Bracing in Steel Buildings[21], the writer hopes that the author will reinforce his remarks on rectangular wind frames by specific examples. In certain types of statically indeterminate structures, analysis and design cannot be considered apart from one another with any assurance of intelligent conclusions. Structures are generally built in accordance with design specifications that impose certain controlling conditions. Moreover, grades, clearances, construction methods, connecting details, and variations in the physical constants of the material, also establish limitations that must be considered in analysis if one wants a structure to act the way it is figured to act.

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[21] *Proceedings,* Am. Soc. C. E., March, 1936, p. 397.

As the author states the concepts involved in his classification of types of structural action are not new. Nevertheless, the profession is greatly indebted to Professor Cross for proposing a logical classification that tends to clarify the relation of analysis to structural design. Regardless of a difference of opinion as to terminology or designation, one cannot help but feel that the author has given structural engineers a "shot in the arm" that will lift many of them out of the rut of conventional thought.

Undoubtedly, experienced designers have sensed the existence of "hybrid" structural action, but for the most part they have had neither the time nor the inclination to go into that phase of analytical research which brings out the full significance of interaction between various units. More often than not the practicing engineer is hurried in the preparation of plans for a definite structure. If computations show that certain members are overstressed, the tendency is to increase areas or moments of inertia in proportion to the over-stress, without taking time for a final check analysis. This is quite satisfactory if normal structural action emphatically predominates. Unfortunately, many indeterminate structures are of such a nature that a change in area or in moment of inertia of a member also means a change in the load or in the moment resisted by that member.

The writer recalls difficulties encountered in the design of some of the columns supporting the continuous plate-girder viaducts built by the South Park Commissioners over the Illinois Central Railroad tracks along the Lake Front, in Chicago, Ill. The maximum width of the columns in a direction parallel to the longitudinal axis of the bridge was limited by horizontal track clearances established by the Illinois Central Railroad Company. The minimum width was limited by the slenderness ratio for primary compression members in the governing specifications. The columns were rigidly connected to the continuous girders and were subject to dead and live load reaction, dead and live load moment, temperature moment, traction moment, and a transverse wind moment.

A preliminary design section based upon a preliminary analysis was found to be overstressed when all factors were combined. Conventional procedure indicated that more section was needed. The area was increased in proportion to the over-stress, but when a check analysis was made, the column was still overstressed; therefore, the effect of the various factors was then studied separately.

When area was added to the cross-section the unit stress from normal dead load plus live load was decreased; but the unit stress from the bending moment caused by dead load and live load remained practically the same, because increasing the area increased the stiffness, thereby drawing more moment to the column. Moreover, as long as the width of the column was kept constant, the unit stress from temperature moment remained constant regardless of any change in the area of the column. The effect of traction was very small. Decreasing the width of the column was considered. This decrease would have lowered the unit stress, but on the other hand it would also lower the allowable limiting stress because of an increase in slenderness ratio.

Evidently, curves showing the relation between unit stress and area, moment and moment of inertia, or unit stress and width of column, are important aids in selecting the final design. Other relationships may be dictated by the needs of the particular problem.

The author points out the existence of hybrid structural action, particularly in two-legged bents, rectangular wind frames, and reinforced concrete arches integral with their spandrel structures. The writer wishes to add the two-hinged suspension bridge of the usual type (not self-anchored) to the list. It may seem a far cry from a suspension bridge to a flitched beam; nevertheless, the two have much in common.

The cable and the stiffening truss participate in resisting total external bending moment at any section. If an unstiffened cable is used, the external bending moment at any section due to dead load and live load is equilibrated by $(H_w + H)\ (y + \eta')$, in which $H_w + H$ is the total horizontal component of cable stress and $y + \eta'$ is the total cable ordinate to the deflected cable at the section considered; η' represents the deflection of the unstiffened cable away from normal position. Since the cable itself is flexible, the cable shape must conform to the equilibrium polygon for the load acting on the span.

When a stiffening truss is added to the cable, the deflection, η', at any section is reduced to a value, η, depending upon the stiffness of the truss. If the truss has infinite stiffness, the cable will be forced back through the entire distance, η', to the normal position under dead load and mean temperature. The change in moment at the section is then $(H_w + H)\ \eta'$. To preserve equilibrium at the section, this expression will then be the moment in the stiffening truss—if the truss is infinitely stiff.

For a flexible truss, the cable will be forced back only a part of the distance, η', and the moment in the truss will be some proportion of $(H_w + H)\ \eta'$. If cable stretch and side-span interaction are ignored, the proportion is given by Equation (1):

$$M = \frac{M_t}{M_t + (H_w + H)\ \eta'}\ (H_w + H)\ \eta' \qquad (1)$$

in which M_t is the moment that would be induced in the stiffening truss if it were bent to the deflection curve of the unstiffened cable. In general terms, M_t is given by Equation (2):

$$M_t = C_1\ E\ I\ \eta' \qquad (2)$$

in which C_1 is a constant depending upon loading conditions.

The moment given by Equation (1) does not include the moment in the truss caused by cable stretch from stress and temperature and change in center sag due to side-span interaction. This additional moment is expressed by Equation (3), also in general terms:

$$M = C_2\ \frac{E\ I}{l^2}\ df \qquad (3)$$

in which C_2 is a constant depending upon the variation in suspender loading, and df is the change in center sag due to cable stretch from stress and temperature, and to side-span interaction.

From Equations (1) and (3), quite accurate values of the maximum moment in the stiffening truss at the quarter-point and at the center may be found for preliminary design purposes. The derivation and application of these equations are not given in this discussion.

Equation (1), particularly, emphasizes the existence of hybrid action. As the stiffness of the truss is increased, the moment taken by the truss increases. If greater stiffness is obtained by keeping the chord area constant and increasing the depth of the truss, the unit stress may actually increase. This seems paradoxical but computations bear it out. Note what happens to the maximum positive moment in the main span truss of the Manhattan Suspension Bridge when the moment of inertia of the truss is varied.

Fig. 6 shows this graphically. The abscissa labeled, I, corresponds to 43 900 $\text{in}^2\text{-ft}^2$, the equivalent moment of inertia of the main span truss as designed. The corresponding moment is 119 000 000 ft-lb. If the value of I is doubled, the maximum moment increases. On the other hand, if I is halved or quartered, the moment decreases rapidly. The relation is not linear, however.

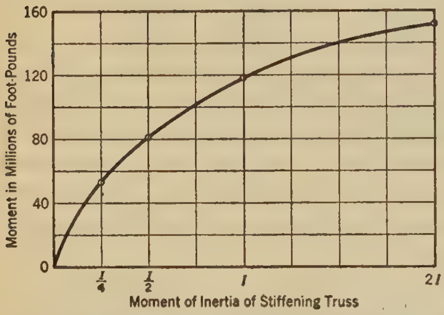

FIG. 6.—RELATION BETWEEN MAXIMUM POSITIVE MOMENT AND MOMENT OF INERTIA OF MAIN SPAN STIFFENING TRUSS OF MANHATTAN BRIDGE.

If the chord areas are kept constant and a variation in I is obtained by changing the depth of the truss, the curve in Fig. 6 may supply the basis for a new curve in which unit stress is plotted as an ordinate and depth of truss as an abscissa; or, depth may be kept constant and a variation in I obtained by varying the area of the cross-section. A curve of unit stress against chord area may then be plotted.

Fig. 6 emphasizes the fact that the most economical stiffening truss is no stiffening truss whatever. What shall be done about local grade changes and curvatures in the roadway, however? The question "stares one in the face"; so a stiffening truss is put on the cable. How stiff shall it be made? What is the limiting grade change or curvature? Who knows? Here is a field for extensive research.

Fig. 6 reveals that there is no one design for the stiffening truss of a suspension bridge. There are any number of possible designs. Conventional analysis fails almost completely as a guide to proper proportions. Studies such as these and the author's "Conclusions" reinforce a growing conviction on the part of the writer that one of the most fertile fields of research in structural engineering is the development of accurate and rapid preliminary design methods. It is most certainly a field that appeals to the practicing engineer.

N. M. NEWMARK,[22] JUN. AM. SOC. C. E. (by letter).—Perhaps this paper by Professor Cross will serve to divert attention from the elaboration of analytical technique to the development of a technique of design. Certainly the author is to be commended for raising the question of the proper interpretation of analyses and for pointing out a practical and useful philosophy of design. The classification of structural action presented in the paper is the first attempt of which the writer knows to establish a basis for a rational design procedure.

Whether or not the traditional process of structural design is practicable depends on how the forces and moments in a member are affected by revisions of the dimensions of that member and of other members in the structure. For example, a typical relation between strain (angular or linear) and size of member, for a particular member of a given structure under a constant condition of loading, is shown in Fig. 7. Curve (*a*) shows the relation for one set of given constant dimensions of all other members, and Curve (*b*) that for another set.

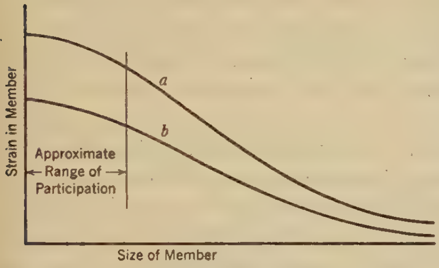

FIG. 7.—TYPICAL RELATION BETWEEN SIZE OF MEMBER AND STRAIN IN MEMBER FOR GIVEN CONDITION OF LOADING AND FOR TWO DIFFERENT SETS OF DIMENSIONS OF OTHER MEMBERS IN THE STRUCTURE.

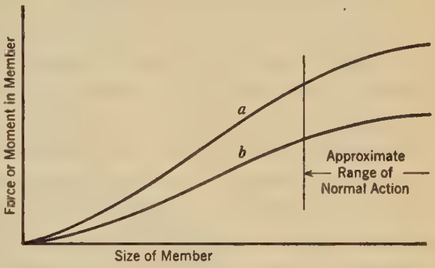

FIG. 8.—TYPICAL RELATION BETWEEN SIZE OF MEMBER AND FORCES ACTING ON MEMBER FOR GIVEN CONDITION OF LOADING AND FOR TWO DIFFERENT SETS OF DIMENSIONS OF OTHER MEMBERS IN THE STRUCTURE.

In Fig. 8 are shown curves for the relation between total force or moment in the same member under the same conditions as those for which the curves of Fig. 7 were drawn. The approximate ranges of "participation" and of "normal" action are indicated in the diagrams. Action in the intermediate range is characterized as "hybrid" action.

If, for practical variations in other members of the structure, Curves (*a*) and (*b*) are close to each other, the structure as a whole may be considered a "normal" structure; otherwise, it is a "hybrid" structure. The matter of efficiency or inefficiency depends upon whether or not the members in the structure can act together at the desired working stresses. This generally depends on the over-all proportions of the structure as well as on the sizes of the individual members.

For a condition in which the structure is constrained to undergo a fixed deformation, a set of curves somewhat similar to those in Figs. 7 and 8 are obtained, but in general the "deformation" stresses are of a type corresponding to "participation" stresses. The fundamental difference between load stresses and deformation stresses may be stated as follows: For given applied deformations, the strains throughout the structure are independent of the absolute values of the modulus of elasticity; and for given applied loads, the forces in the individual members are independent of the absolute values of the modulus of elasticity.

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Importance of Sources of Stress.—The relative importance of the various sources of stress must be considered in any attempt to revise the proportions of a structure or of its members. As an interesting example consider the case of a filled-spandrel arch bridge. Any effect of the spandrel walls may be ignored since this is purely an illustrative example. In general, dead load is by far the most important source of stress in such a structure. If the arch axis fits the pressure line for dead load, the dead load forces at any section of the barrel are practically independent of the variations in depth of the barrel. The structure is then a "normal" structure for dead loads; it remains "normal" even when the live load effect is considered, since the variation in depth to account for the live load stresses will be fairly small, and the effect of such variation on the maximum total stress will be practically negligible.

However, if such a structure were to be designed for live load only, it would very likely be "hybrid" since a variation in depth in one part of the barrel would effect the stresses in other sections to a fairly considerable extent.

Another question in this connection is worth raising. Consider the design of a continuous beam for relatively heavy concentrated live loads. If the designer limits himself to the choice of prismatic sections in each span, he may possibly arrive at a design in which the maximum fiber stresses in each span are equal. If, however, he attempts to choose proportions such that the maximum stresses at all points in all spans are equal to the design stress, the problem of design, by any existing procedure, becomes almost hopelessly complicated.

Significance of Over-Stress.—The significance of over-stress in a structural member is a matter which has not received as much attention, perhaps, as it deserves. Certainly the literature contains few references to this topic. The author's classification seems to offer a basis for the study of the question. Professor Cross suggests that over-stress due to participation or deformation is not always dangerous. In some hybrid structures, also, over-stress loses its significance when the ultimate strength and nature of failure of the structure are considered, as in the case of the trussed beam discussed subsequently. Apparently, a great opportunity is available in this field for experimental research.

The classification offers a convenient method of estimating the possibility of the significance of chance factors in the properties of the materials, and in the manner of construction. In a "normal" structural type such factors have comparatively little influence. In a "hybrid" structure, they may be of considerable importance. Studies of the effect in an arch rib of chance variations of modulus of elasticity have been reported by Professor Cross.[23] The results seem to show that there is only a relatively small effect of chance factors on the stresses in an arch rib for the usual case in which the dead load is comparatively large. When the dead load is very small compared with the live load, chance variations might cause some concern. However, the effect of chance variations cannot be discussed independently of the significance of over-stress.

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[23] "Dependability of the Theory of Concrete Arches", by Hardy Cross, *Bulletin No. 203*, Univ. of Illinois Eng. Experiment Station, 1930.

The Trussed Beam.—The problem of the trussed beam is of interest in throwing light upon some of the aspects of the use of the classification in design. Certainly, if the sag rods are quite small, the structure is "normal", with the beam "carrying the load", and with "participation" stress in the rods. Again, if the beam is very shallow and has an area of the same order as the sag rods, the structure is "normal", and acts as a truss with secondary or "participation" stresses in the beam. For intermediate cases, the structure is "hybrid."

For the case of a concentrated load at the center of the beam, the hybrid structure is "efficient" when the sag depth is approximately 1.5 to 2 times the beam depth (for the beam and sag rods both of the same material). For lesser sag depths, the sag rods will be understressed compared with the beam for all proportions of the structure. For greater sag depths the sag rods will be overstressed compared with the beam unless the beam is so small that the structure approaches a truss. When the beam is fairly small, it will be overstressed compared with the sag rods unless its depth is adjusted so as to reduce the flexural stresses. If the beam must carry loads at other points than at the center, the foregoing results are changed quantitatively but not qualitatively.

Of course, actually, the stresses in beam and truss may be adjusted either by putting initial tension in the sag rods or by slacking off on them, when the sag depth is less than, or greater than, the efficient sag depth, respectively.

Merely as an illustration, consider the case of a 30-ft beam carrying a concentrated load of 44 000 lb at its mid-point. It is desired to truss the beam to help support the load. Adjustments of the sag rods or stresses due to dead load are not considered. It is assumed in the following calculations that the vertical post is fairly large so that it does not deform appreciably under the load.

Problem 1.—The beam is an 18-in., 90-lb I-beam; and the sag depth is chosen arbitrarily as 36 in. from the center of the beam. What area of sag rod is required for working stresses of 18 000 lb per sq in.?

Discount about 2 000 lb per sq in. in the beam for direct stress. At a flexural stress of 16 000 lb per sq in., the beam will carry about 56% of the load. Designing the sag rods for the remaining stress, one finds that the area required is about 2.75 sq in. A review with this design indicates that the total stress in the beam is 17 600 lb per sq in., and in the rods, 18 200 lb per sq in. Evidently, an efficient sag depth was chosen.

Problem 2.—Under the same conditions as in Problem 1, the sag depth is taken as 54 in. A first estimate of the area required in the sag rod is 1.67 sq in.; but a review indicates a stress of 23 900 lb per sq in. in the rods, and a total stress of 14 800 lb per sq in. in the beam. Apparently, the sag depth chosen is inefficient.

By use of the traditional design procedure, the next trial design is an 18-in., 70-lb **I**-beam, and an area of 2.22 sq in. in the rods. This combination, if moments and forces are not changed, should give a stress of 18 000 lb per sq in. in the rods, and 20 100 lb per sq in. in the beam. However, a new analysis gives stresses of 21 900 and 15 900 lb per sq in. in rods and beam, respectively.

Repetitions of the traditional procedure lead very slowly to a balanced design. The preceding results suggest for a next trial a 54.7-lb **I**-beam and an area of 2.70 sq in. in the rods. Limiting the depth to 18 in. the designer finally ends with a 48.2-lb beam, and rods of an area of 3.0 sq in., for which the stresses are calculated to be 18 600 and 16 200 lb per sq in. in rods and beam, respectively.

If the problem had been to reinforce the original beam, the designer would have found eventually that approximately the same rod area would have been required as in Problem 1, namely 2.7 sq in. The beam, however, would have been working at a stress of 12 300 lb per sq in., compared with 17 300 lb per sq in. in the rods.

In this problem the question may well be raised as to the significance of over-stress in the sag rods when the beam is understressed. If the stress in the rods reaches the yield point the beam will begin to take a larger share of the load. The structure might adjust itself in a manner analogous to slacking off the tension in the sag rods by increasing their length slightly. The writer does not believe that it is well to depend on such stress relief in general, but the question does deserve some consideration.

Problem 3.—With other conditions the same as in Problem 1, consider a 24-in., 79.9-lb **I**-beam, and a sag depth of 36 in.

In the same manner as in Problems 1 and 2 a first estimate of the area required in the sag rods is 1.57 sq in. The calculated stresses, however, are 18 800 lb per sq in. for the beam and 16 000 lb per sq in. for the rods. If the rods are reduced in area and the beam is increased in size, as is indicated by the traditional procedure, the ultimate effect will be to eliminate the rods entirely, and the entire load will be carried by a larger beam.

If it is a matter of reinforcing the given beam, the designer will have to resign himself to the use of a lower working stress in the rods than in the beam. With an area of 1.90 sq in. in the rods, the stresses will be 15 000 and 18 200 lb per sq in. in rods and beam, respectively.

One could have anticipated the final designs obtained in these three problems, however. It is not difficult to estimate what simultaneous stresses are required for continuity with given dimensions of sag rod and beam.

Arches Continuous with Their Superstructure.—Certain limiting cases of the arch with integral spandrel structure are "normal", but practically all such structures of the proportions usually built are hybrid.

An arch with a comparatively heavy rib and flexible columns and a deck supporting relatively light live loads is a normal structure with participation stress in the deck and columns added to the normal stress in these members due to their action in the continuous deck frame.

Another normal type is the structure consisting of a flexible rib with a stiffening girder. There will be flexural participation stress in the rib, in addition to the normal stress due to the direct thrust in the rib.

A third type of structure having "normal" characteristics in certain cases is what might be termed an arched Vierendeel truss. This structure must have quite rigid columns. The corresponding simply supported Vierendeel truss would be normal, but fixing the supports introduces some complications. In this connection, it is important to note that the fixed arch rib has hybrid characteristics for live loads, as was pointed out previously.

Other normal arch structures may be secured in various ways by the manner of construction or design involving the use of articulations and flexible sections. Such structures have well marked participation stresses at such articulations and pseudo-hinges. Of course, not all structures with hinges and points are "normal."

In special cases applying to certain other arch structures the controlling relations are comparatively clear, as Professor Cross has indicated. The flitch-beam concept applying to the structure with flexible columns has considerable value. Use was made of this concept by Rankine[24] and Considére.[25] An analysis preliminary to design may be developed along the lines suggested by Professor Cross' treatment as follows: The deck girder will have primary stresses due to its action as a continuous beam. These stresses are due to the dead load of the deck only, and the entire live load. The arch rib will take all the thrust, most of which is due to dead load. Then the difference between the working stress and the primary girder stress is available to resist interaction flexure in the girder. The difference between the working stress and the stress due to direct thrust in the arch rib is available to resist flexure in the rib. The stress due to interaction flexure in the girder is to the flexural stress in the rib approximately as the depth of the girder is to the depth of the rib. The rib and deck resist the moment, except for primary moments, approximately in proportion to their moment of inertia.

A few rough figures will enable an estimate to be made as to whether a tentative combination of depths of girder and rib is efficient or inefficient, and will give some idea as to what can be done about it. For example, if the girder is found to be overstressed, it must be remembered that increasing the depth of the girder may increase the interaction stress although it decreases the primary stress.

For the general case in which the columns must be of intermediate stiffness there seems to be no convenient method of estimating in advance whether the structure can act as it is assumed to act in a preliminary analysis. Use of the traditional procedure of successive analysis (usually by means of a model) and design does not appear to be very promising. Such a procedure might easily be misleading, particularly where a re-design is based on an analysis which is not repeated for the new design. It is open to question whether a better structure is obtained by such a procedure than by neglecting interaction entirely.

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[24] See "Civil Engineering", Seventh Edition, 1871, pp. 313–314; and other editions at approximately the same page numbers.

[25] See, for example, "Cours de Beton Armé", by A. Mesnager, Paris, 1921, p. 198.

Other Problems.—The fundamental philosophy presented by the author has wide applicability. Some of its possible uses have been mentioned, but there are many other cases in which it is useful. As an illustration consider the problem of the stresses in a bearing-plate supporting a concentrated load on an elastic solid.

If the plate is very flexible compared with the solid, the latter deforms in the same manner as under a load directly applied. The stresses in the solid are of the nature of "normal" stresses. The plate, in conforming to the configuration of the solid, has stresses of the nature of participation stresses. If the plate is very stiff, the pressure between it and the solid approaches a certain limiting distribution, and the stresses are "normal" in both plate and solid. If the bearing-plate is neither very flexible nor very stiff, the pressure distribution under it varies considerably with any change in thickness, and the stresses are certainly "hybrid."

The classification has applicability to many other problems concerned with what is sometimes called "internal stress", as well as to problems concerning so-called "frame" structures.

Concluding Remarks.—The profession is indebted to the author for his presentation of a practical philosophy of design. It is to be hoped that his paper will awaken interest in methods of design. Perhaps in the future more energy will be devoted to securing quantitative data for the design of structures rather than to the development of variations in existing procedures for their analysis.

L. E. Grinter,[26] Assoc. M. Am. Soc. C. E. (by letter).—To his important contributions on the theoretical analysis of continuous frames, Professor Cross has added an enlightening paper concerning the relation of analysis to structural design. His subdivision of all structures into normal and hybrid types cuts ruthlessly through the field with little regard for the classical arrangement into statically determinate and statically indeterminate groups. This point of view is at least refreshing, and it seems to be considerably more than that.

Statically Determinate Structures.—The writer has often wondered whether he has ever actually seen a practical structure that could be properly classified as statically determinate. Simple beams are decidedly unusual. What is commonly termed a simple beam (that is, a steel floor joist or roof joist) is frequently encased in concrete poured continuously with other beams through the connection of a reinforced concrete floor-slab or roof-slab. Thus, this structure, which is analyzed as statically determinate, may be one of the most complex of all statically indeterminate constructions—a continuous flat slab with partly discontinuous stiffening ribs supported in part by the columns and in part by restrained edge beams. A search for a statically determinate cantilever structure will not lead to much more satisfactory results since in common with three-hinged arches they become not only statically indeterminate, but mathematically indeterminate when corrosion produces an unknown pin friction. They, too, are usually constructed so that there is partial continuity through a floor system.

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[26] Prof. of Structural Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

This comparison of structural types could be carried on indefinitely. Surely the lower chord of a pin-connected through truss ought to be a statically determinate member; but, instead, the writer recalls making a railway bridge test for live loading for which numerous strain-gauge measurements on the eye-bars of a lower chord member showed an average stress of little more than one-half the theoretical value. One could explain this phenomenon in part by the self-evident fact that the floor system was functioning as a part of the lower chord, but this general observation would be of little value in an attempt to analyze or to design a similar structure. The fact is that this simple span truss was acting in a most complex manner, that it was internally statically indeterminate, and that the manner of fabrication and erection had a most important effect upon its stresses.

From these and other observations, one might conclude that the classical distinction between statically determinate and statically indeterminate structures has its main justification in the classroom where its value is largely a matter of convenience to the student and the teacher. For the practical designer this distinction is dangerously convenient and, therefore, of questionable importance. Hence, the subdivision suggested by the author is at least worthy of consideration.

Impressed Distortions and Induced Distortions.—Professor Cross has made a division between what he terms deformation stresses and participation stresses. A comparable physical distinction that the writer has been in the habit of making depends upon whether the corresponding distortion is impressed upon the structure or induced in the structure by its own action. Evidently, temperature effects are impressed upon the structure from an outside source. Similarly, the tower of a suspension bridge may have a bending deflection impressed upon it because of a reduced sag of the cable in the side spans when the center span is loaded. At least, this action is external to the tower itself and may thus be classified as an impressed distortion. On the other hand, ordinary secondary stresses in truss members are caused by joint rotation in the plane of the truss which is an induced distortion. Apparently, however, the division line is less clear when one considers such effects as rib-shortening in arches. This action is quite clearly the same as temperature effect or spread, both of which are produced by impressed distortions. Nevertheless, rib-shortening stresses are secondary flexural stresses produced by the shortening of the rib from primary compression, and thus should be designated as "induced stresses." Hence, one observes that impressed and induced distortions merge one into the other.

Professor Cross has attempted to clarify the aforementioned difficulty by observing that time flow will eliminate or reduce a deformation stress (the consequence of an impressed distortion), but that a participation stress (the consequence of an induced distortion) will be unaffected by such plastic flow. In general, this distinction holds true although it may be desirable to add that the distribution of all stresses can be affected by such plastic flow.

These observations explain the writer's preference for the terms, "induced stress" and "induced distortion", as more descriptive of the physical picture than the terms, "participation stress" and "participation strain." Certainly these secondary stresses participate very slightly in the job of supporting the loads; instead, they are induced by primary strains much to the designer's disquietude.

Two statements of the author will be repeated for study: (1) "Thus, it is generally known that increasing the moment of inertia without changing the depth of truss members will affect the secondary stresses only as the primary stresses are affected"; and (2), "the designer finds it convenient to predetermine such strains [impressed distortion], reduce them to equivalent stresses, and deduct these stresses from the working stresses available for load-carrying capacity." Although the writer thinks that Professor Cross has been too generous in concluding that the information referred to in Statement (1) is generally known, he certainly agrees to its importance. Statement (2), although simple in itself, must be complicated by the fact that only the total distortion is evident and that the unit strain is dependent upon the shape of the axis of the member and upon the variation of its cross-section. Hence, for the general case one must admit that even final deformation stresses (impressed distortions) cannot be found until the design is completed. This is just another of those cases in which the designer cannot "foresee the action" and falls back upon a cut-and-try procedure.

Visual Structural Action.—The writer would mention another classification of structural action which in many points agrees with the subdivision suggested by Professor Cross. However, a few structures will be found to have been shifted from the complex to the simpler classification, or *vice versa*. A comparison of the two points of view seems particularly useful since each presents factors for consideration that the other misses.

The subdivision to be suggested is made dependent upon a criterion which can be stated forcefully in the words of Professor Cross, "that the designer either does or does not know what he is doing." In the writer's words this becomes a subdivision into "visual structural action' and "non-visual or concealed structural action." As another criterion that will be of some use, it may be stated that this classification of visual structural action will include all academic statically determinate types and, in general, those indeterminate types that involve, at least theoretically, no more than three redundants. In some few cases a structure having more than three redundants may possibly be arranged to fit into the simple classification of visual structural action. Examples will be given.

The limitation of three redundants is suggested because it is quite evident to the instructor that the limit of visual study commonly is reached when three interacting influences exist. The interaction of two redundants is relatively simple but the number of possible interwoven patterns is about tripled with three redundants. A visual study of the interplay of four influences is not ordinarily within the grasp of other than a highly trained mind. As an example of the usefulness of this classification, one notes that the problem of two crossing beams (deflected equally) which were placed in the complex subdivision by Professor Cross fall into the simple subdivision in this discussion. The analysis and design of such beams, although not feasible by the "guess and revise procedure", is possible in a direct manner, as suggested by the author.

Non-Visual or Concealed Structural Action.—In this classification is found those structures and parts of structures for which an adequate design procedure does not now exist. It is no great credit to the profession, however, that it includes so many ordinary structural types. Hence, it is worth calling attention to the fact that in this subdivision there is included many concrete arches, most tall building frames, flat slabs, continuous footings, bridge abutments built continuously with their wing-walls, arched dams, and even many of the so-called statically determinate structures because of the partial continuity introduced by common methods of construction. In other words, with few exceptions, an adequate design procedure to-day is available only for those structures in which the interaction of the structural parts can actually be visualized. The designer will do well to select and construct structures within this classification until direct procedures for the design of complex structures are made available. When he cannot do this, he should "go slow" in making his design.

Parallel Beams.—Two cases of parallel beams seem worthy of investigation. In wind-stress studies, parallel beams of different depths and perhaps even of different lengths may be forced to rotate through equal end angles. Then, since θ varies as $\frac{M\,L}{E\,I}$ and $M = \frac{S\,I}{c}$, it is evident that for equal θ-values the relative stresses in the two beams are defined as follows:

$$\frac{S_1}{S_2} = \frac{\frac{L_2}{c_2}}{\frac{L_1}{c_1}} = \frac{\frac{d_1}{L_1}}{\frac{d_2}{L_2}} \quad \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

in which $\frac{d}{L}$ is the depth-length ratio of the beam. Evidently, the only geometrical shape that will permit such beams to be stressed equally is that of equal $\frac{d}{L}$ - values.

When the deflections rather than the end slopes are identical for the two beams, the deflection varies as $\frac{M\,L^2}{E\,I}$ and the relative stresses in the two beams are defined as follows:

$$\frac{S_1}{S_2} = \frac{\frac{L^2_2}{c_2}}{\frac{L^2_1}{c_1}} = \frac{\frac{d_1}{L^2_1}}{\frac{d_2}{L^2_2}} \quad \dots\dots\dots\dots\dots\dots\dots\dots (5)$$

Hence, these beams must have equal $\frac{d}{L^2}$ - values in order to be stressed equally. This would be the writer's definition of efficient design. In any case such beams furnish an illustration of a statically indeterminate structure properly classified as hybrid under the author's subdivision, but as visual structural action by the writer.

The Vierendeel Truss with Parallel Chords.—This is another structure properly classified as hybrid by Professor Cross, but still easily analyzed and readily designed because of the visual characteristics of its structural action despite the fact that it is highly indeterminate. A few common sense basic assumptions can be made for the purpose of a preliminary analysis and design by statics. Then, in the usual case, an analysis will show that the first design was satisfactory. The few necessary revisions are easily determined because of the visual characteristics of its structural action.

The distinction between the Vierendeel truss and a building frame is that the truss is essentially a regular structure, whereas the building frame is in most cases irregular. Commonly, there are no reasonable assumptions that can be made to lead to a preliminary design of a highly irregular wind-frame that will not necessitate serious revision of sections subject to the difficulties inherent in the revision of any structure the action of which cannot be visualized. Of course, the Vierendeel truss can be hopelessly complicated by the use of a floor system such that the resultant structural action will be beyond visualization. A curved top chord will introduce additional complications.

Rectangular Wind Frames.—The author proposes an interesting design procedure for a regular wind frame. His discussion apparently is intended to apply to a building of medium height since it is assumed that the column cross-sections will be unchanged by the effect of wind. If such a building frame has been designed by a conventional method, it is likely that overstresses will be found when an analysis is performed by a more exact method. However, one need not run headlong into the difficulty pointed out by Professor Cross. Naturally, an increase in size (depth and moment of inertia) increases the stiffness of the overstressed girder, causes it to resist more wind moment, and increases its fiber stress still more. One frequently finds it necessary to reduce its depth, thus reducing the fiber stress for a given moment. At the same time, it may be desirable to stiffen adjacent girders to relieve the overstressed girder of its excess moment.

Of course such a procedure is difficult to apply to a highly irregular structure, but there is no particular reason why the design of a structure as complex as an irregular tall building frame should not consume considerable time. The author's proposal that the design should be based upon assumed points of contraflexure, assumed shear distribution, and design stresses proportional to the depth-length ratios of the girders (equal end slopes) would be practicable for a very regular frame. However, the important need is for a design method that will make the design of an irregular building frame less tedious.

Conclusion.—The writer has found in this paper much material that will require slow digestion. The inference is clear that direct design is the ideal toward which one should strive. The cut-and-try or guess-and-check procedure fails in many cases unless applied with more than ordinary intelligence. Finally, hybrid structures, in which the type of structural action is concealed, in which "self interference" exists, or in which one attempts to "put secondary stresses to work", offers little promise of economy, but far more promise of confusion and poor design.

F. P. SHEARWOOD,[27] M. AM. SOC. C. E. (by letter).—Practical designers, who are called upon to plan for structures on a commercial basis will find this paper of great assistance, since it helps them to visualize the function of the strains and stresses that are caused by the multitudes of indeterminate and unnoticed strains that exist in almost every structure.

When an exact stress analysis of almost any riveted frame is contemplated, the designer is faced with the problems of its irregular secondary lacing, its eccentric connections on all planes, its varying areas of cross-section at the splices and other points, the stiffening effect of the gusset-plates, the participation of the bracing, floor system, and many other resistances which custom allows to be neglected or overlooked. With all these problems to solve (or to overlook), he is likely to be thankful for the fact that he works to a liberal factor of safety. If he strolls through the fabricating shop and watches the straightening rolls, the "bull dozers", the punch shears, and other tools straining the material, he is again grateful that steel is ductile. In his final survey he watches the latest type of fabrication—fusion welding—and reflects on the strains that must be set up by the cooling of that thin molten streak; he then asks himself if there is any margin left between the temperature strains that are set up and the elastic limit. These facts appear to conflict with the rigid limitation placed on the unit stresses and strains allowed in the calculations for proportioning the material, and they suggest that designers may have allowed the fascination of mathematical exactitude, based on assumptions and on Hooke's law, to outweigh and over-awe actual engineering experiences.

It is likely to be forgotten that, although the ultimate strength of steel is 60 000 lb per sq in, its ultimate strain, as measured in calculating stresses from distortion, is about 6 000 000 lb per sq in. (that is, 20% elongation with a modulus of 30 000 000). Since the ductility of steel is utilized during fabrication, why has it been ignored by designers when proportioning steelwork which has more than one path of resistance, and in which the loading is more or less constant? There are many continuous spans, trusses with several systems of web-bracing, rigid frames, as well as many other forms, which, in the event of some part of one path of resistance becoming strained above its elastic limit, will bring the other paths into proportionately greater resistance, and thus insure a redistribution of the combined resistance before the weak member is strained to a dangerous degree.

The ductility of structural steel can be relied upon to provide safely for far more strain from secondary distortions than is usually allowed by standard specifications; moreover, the relative participation of the various paths of resistance to these deformations, is not always wholly dependent on the modulus of steel, but is frequently influenced and relieved by the cross-bending of connection angles, the slip of the rivets, and other relieving deflections. These deflections, although very minute, are relatively great when compared with the direct elastic deformation of the steel. They will alter the distribution materially, therefore, and relieve the extreme fiber stresses.

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[27] Chf. Engr., Dominion Bridge Co., Ltd., Montreal, Que., Canada.

Limiting the strain in structural steel (after fabrication) to a value below the elastic limit is in sharp contract to the common practice in reinforced concrete, in which the concrete is actually allowed to fracture, and the steel reinforcement is still counted on to function by adhesion to the fractured concrete.

With mature judgement, appropriate calculations for guidance and the knowledge that, in structural steel, the strain increases much faster than the stress when overloading occurs, designers should count, to a greater extent, on the sum total strength of the several resisting paths in indeterminate structures, rather than limit their use to the calculated stress of the weakest member.

Hardy Cross,[28] M. Am. Soc. C. E. (by letter).—Discussion of the paper may be divided into that dealing with the adequacy or correctness of analysis and that dealing with the use to be made in design of an analysis assumed to be correct. It is generally recognized that flaws may be found in many conventional analyses and most designers understand that assumed conditions of loading or of restraint may not exist. The writer, however, did not attempt to discuss these matters but restricted himself to the question of the interpretation of analytical procedures.

The discussion by Mr. Mensch points out the uncertainties inherent in structural analyses. It also suggests the confusion existing as to the procedures used in analysis. The writer may remark that he does not understand in what sense "the analysis of most statically indeterminate structures is beyond the realm of science and becomes an art"; it seems necessary to accept analysis as a science and design as an art. The procedures of analysis and design are different; that the science is defective and the art often clumsy is another matter. Probably neither analysis nor design will be much improved by more handbooks.

Mr. Mensch discusses the statical determinacy of post and lintel construction. The writer stated that a girder resting on narrow columns "represents a structure essentially statically determined (post and lintel or column and beam), but with participation stresses in the columns." Surely most engineers will agree that a girder resting on two relatively narrow studs should be designed as simply supported. A better example of the difference between the point of view of the analyst and that of the designer would be difficult to find.

In connection with the remarks of Mr. Brinker as to the relative importance of secondary stresses, attention may again be called to the paper by

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[28] Prof., Structural Eng., Univ. of Illinois, Urbana, Ill.

Messrs. Parcel and Murer.[2] Mr. Brinker clearly points out that a uniform stress cannot be chosen and applied indiscriminately to the sum of all stresses from whatever source, and the writer heartily concurs; in fact, to bring this out was one of the main purposes of the paper.

The very interesting remarks of Mr. Findley deal chiefly with the uncertainties rather than with the interpretation of analyses. He puts his finger on the main purpose of the paper, however, when he refers to what he calls the writer's pragmatic concept of structural action. To a considerable extent the attitude of the writer toward design is pragmatic. His point of view may be briefly re-stated as follows: Omit from the picture certain stresses of the type described as participation and deformation stresses—accept for the present the fact that they do not and are not intended to carry any appreciable portion of the load. Assume, now, the pragmatic attitude that when one tells the structure what to do it will try to do it. Questions now arise: Can it do it? Can it do it efficiently? How can the designer determine how to make it do it? Sometimes repeated analysis gives a clue to the action of the structure; often it does not. It is not a question of monolithic construction or of statical determinacy, but of normal response to successive analysis and revision. The term, hybrid, was chosen for those types which do not normally respond because these types partake of the action in part of those which respond at once and in part of those exposed to participation. The writer would like better designations than "normal" and "hybrid", but "statically determinate" and "monolithic" will not serve the purpose.

Mr. Newmark clearly states the main thesis. If a member is overstressed, what can be done about it (the superstructures of arches)? How is the condition best remedied (the trussed beam)? Can anything at all be done about it except to change the type of structure?

Mr. Grinter also wishes to modify the terminology. The writer holds no brief for the terminology chosen; but the discussion by Mr. Grinter brings out clearly the importance of an understanding on the part of the designer of the responsiveness of the structural type to revision—of its response to persuasion.

Mr. Silverman calls attention to temperature stresses in an arch as an example of deformation stresses. Although convenient because it fits into routine conceptions, the treatment of temperature stresses usually given is rather misleading. It is the temperature strains that are fixed, and in arches (and suspension bridges) the most important factor in fixing them is the ratio of depth of rib to rise of arch.

Mr. Wessman discusses the hybrid action of rectangular bracing frames in buildings, a matter discussed also by Mr. Grinter. The writer finds himself puzzled by the attitude of some writers as to the so-called "exact" analyses of wind stresses. The problem of how best to go through the analysis which purports to give the exact moments (and which would give them for given wind loads if it were not for the interaction of partitions, of girder encasements and floors, and for the uncertainty of bracket action and rivet slip), seems a study appropriate for the graduate seminar rather than for the design office. In cases of regular framing the designer knows pretty well how his moments go. In cases of irregular framing he does the best he can, telling the moments in any story where he wants them to go. Analysis will now give some indication as to whether they are likely to go where directed; but analysis—conventional analysis—will not tell at once what allocation of moments is impractical, or within what limits other allocations are satisfactory. In the writer's opinion, this is the important field of research in wind-bracing.

It is scarcely possible to choose a better illustration of the utility of the point of view presented in the paper than the field of suspension bridges to which Mr. Wessman calls attention. In large bridges where the cable has great resistance to distortion the stresses in the truss due to live load on that truss are a function chiefly of its depth and not very largely of its strength; in some trusses the action is "hybrid", both depth and strength are factors; relatively light bridges are "normal" in their action, and the "elastic theory" gives at once a pretty satisfactory design. However, note that, in any case, temperature stresses are of the nature of deformation stresses, the strains being a function of the ratio of truss depth to cable sag, as in arches.

A comparison of the curve (Fig. 6) given by Mr. Wessman, showing variation of moment in stiffening truss with moment of inertia, with a corresponding curve showing variation in moment in one partner of a two-part flitch beam, is very illuminating.

Mr. Shearwood clearly points out the uncertainties produced in steel by fabrication methods and the basis for valuing their importance. His statement of the strength of mild steel in terms of ultimate strain expressed as equivalent strength seems especially happy, emphasizing the importance of ductility. Ductility plays a very different part in participation stresses—where the strain is fixed—and in normal structural action—where the desired strength is approximately fixed; but in many of the complicated structural types it plays a rather complicated part. The total strength is fixed but, at the same time, there is an inter-relation of strains. Perhaps this may serve to justify to Mr. Findley the term "hybrid."

Mr. Johnston and Mr. Newmark have discussed the significance to be attached to over-stress. The question has special interest in the case of hybrid structures—structures in which the load is "carried" in several ways. In this connection Mr. Johnston uses the term, "limit of structural usefulness." The subject needs extensive discussion. To quote from the paper, "The values of stress permissible in design vary with the material used, with the importance of the member involved, and with the type of failure that would result." Thus it has often been argued that it is not very important how moment is distributed between the positive and the negative in concrete beams if provision is made for the total moment. Some engineers seem to overlook the danger of secondary failure in bond or in diagonal tension if the negative steel is seriously overstressed.

The writer appreciates the courtesy of those who have discussed the paper. Their comments open up many fields of research in structural design which are being neglected in the current deluge of algebraic investigations.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 1952

TUNNEL AND PENSTOCK TESTS AT CHELAN STATION, WASHINGTON

By Ellery R. Fosdick, Esq.[1]

With Discussion by Messrs. W. A. Hill, S. P. Wing, J. N. Bradley, and H. P. Evans, Jr., W. S. Merrill, and Ellery R. Fosdick.

Synopsis

A thorough and detailed analysis of flow-line losses was computed from data obtained by tests on the unusually long flow line leading to the hydro-electric generating plant at Chelan, Wash. This paper contains the results of that analysis, and includes:

(1) The losses that occur in each part of the flow line;

(2) The coefficients of roughness that are applicable to the large concrete and steel lined pipes of this installation;

(3) An unusual and complete analysis of the losses that occurred in the wye-branch; and

(4) A segregation of the diversion losses (see heading, "Definitions of Symbols") in the wye-branch and lower penstock bends from the losses due to frictional resistance, as a means of checking laboratory tests and research work that was being conducted along this line.

The tests were made and a report was prepared as a means of facilitating the most economical operation of the generating station and to supply additional data in the furtherance of engineering knowledge and its application to future designs involving these features.

Introduction

In 1927, after the hydro-electric generating plant at Chelan, Wash., had been completed and put into service, it was found desirable (because of the unusually long flow line) to conduct a series of tests to determine the flow-

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Note.—Published in October, 1935, *Proceeding.*

[1] Asst. Engr., State of Washington, Dept. of Public Service, Spokane, Wash.

line losses. By means of these tests it was expected to obtain more or less detailed information concerning: (1) The total flow-line loss from which it would be possible to compute, easily and accurately, the water consumption of the turbines; (2) coefficients of roughness that apply to the flow of water in large concrete and steel pipes; (3) an analysis of the losses that occur in the wye-branch; and (4) the laws governing diversion and eddy losses in the pipe bends.

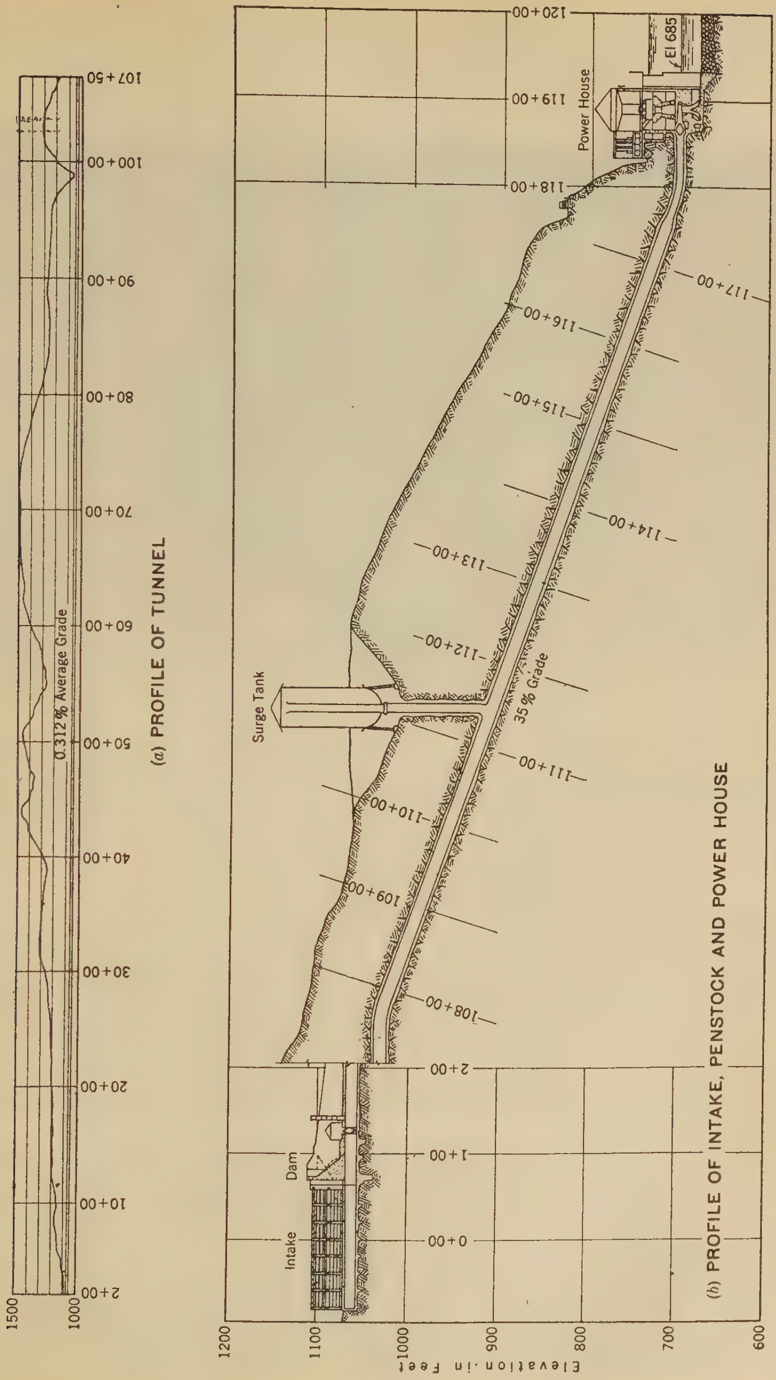
Item (1) was desired for use in operating the plant and Items (2), (3), and (4) were for the purpose of adding to the available knowledge of hydraulic losses that occur in large conduits, thereby facilitating their future design. Practically nothing of a definite nature had been done at the time these tests were made toward analyzing the losses that occur in a wye-branch and little more had been done toward an understanding of the losses in pipe bends.

Description of the Chelan Development

The Chelan hydro-electric development utilizes Chelan Lake as a storage reservoir, the potential head between this lake and the Columbia River being about 400 ft. The lake has been formed by a glacial fill across the lower end of the former Chelan Gorge. Its outlet, the Chelan River, flows through what now remains of the Chelan Gorge for a distance of about 2.5 miles to the Columbia River.

The dam that regulates the level of Chelan Lake, is built across the river about 0.5 mile from the lake. The intake of the flow line leading to the power house, situated near the junction of the Chelan and Columbia Rivers, is incorporated in the dam.

Fig. 1 is a profile drawing of the flow line and penstock from the intake to the tail-race. The intake to the flow line is of a commercial, draft distributor type from which the water enters a concrete pressure tunnel, 14.00 ft in diameter, constructed to a 0.312% grade for a distance of 10 578 ft. (The concrete tunnel was cast around steel forms which produced a smooth regular finish on the interior surface.) At the end of this distance the tunnel lining is changed from concrete to a double-butt, strap-riveted, steel shell and the grade is changed to 35% for a distance of 397.79 ft, at which point a 14-ft steel-lined pipe rises vertically to the surge tank. Square-edged butt-straps and button-head rivets were used to fasten adjoining sections of steel together, as may be seen in Fig. 2. The interior surface was protected only with the customary factory coat of red lead paint. Consequently, the surface was somewhat rough at the time these tests were run because the steel began rusting soon after the flow line was filled with water. Since the tests were run the interior has been cleaned and repainted with a bituminous enamel, but no observations have been made to ascertain what effect this has upon the losses. The penstock grade continues the same as before, for 604 ft more, to the wye-branch, the diameter converging in this distance from 14.00 ft to 12.50 ft, where it divides through the wye-branch into two 8.83-ft, double-butt strap-riveted, steel-lined, penstock pipes, as shown in Fig. 2. At a point 40 ft below the wye-branch the two diverging penstocks bend in opposite directions through an angle of 29° 13′ 26″, so that they are parallel to each other in a


1500
1000
2+00
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100+00
107+50
0.312 % Average Grade
1200
1100
1000
900
800
700
600
Elevation in Feet
Intake
Dam
0+00
1+00
2+00
108+00
109+00
110+00
111+00
112+00
113+00
114+00
115+00
116+00
117+00
118+00
119+00
120+00
Surge Tank
35 % Grade
Power House
El 685

(a) PROFILE OF TUNNEL

(b) PROFILE OF INTAKE, PENSTOCK AND POWER HOUSE

FIG. 1.—SECTIONAL PROFILE, CHELAN DEVELOPMENT

horizontal plane. The total length of each penstock branch is 138 ft, the lower end connecting to a hydraulically operated needle-valve from which the water passes through a reducing section which converges to a diameter of 6.50 ft at the point where it connects to the turbine scroll-case.

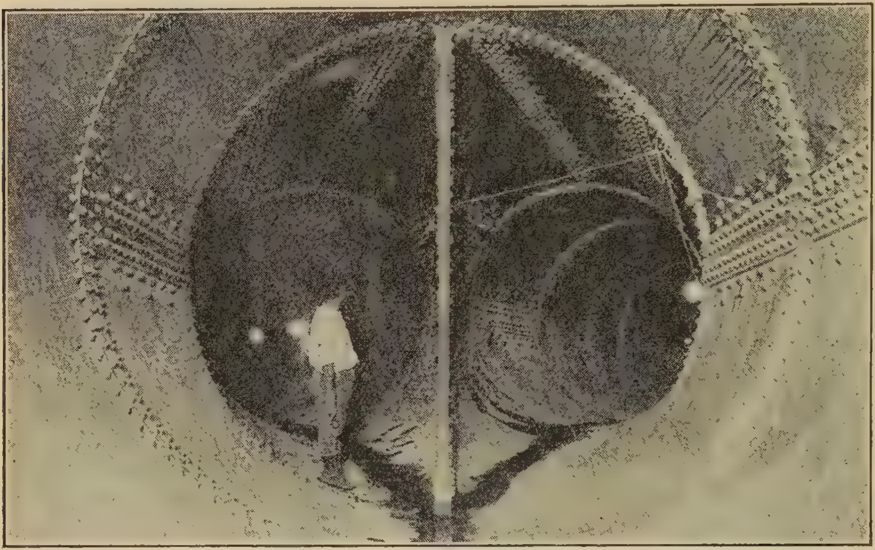

FIG. 2.—INTERIOR VIEW OF PENSTOCK AT STATION 117 + 20, SHOWING THE WYE.

The water from each turbine flows through a spreading draft-tube and out into the tail-race in the Chelan River. The two turbines are identical, each being of the vertical reaction type rated at 34 600 hp, and each using approximately 1 100 cu ft per sec of water when operating with the turbine gates wide open.

EQUIPMENT USED FOR TESTS

Data for the tests were obtained from pressure elevations recorded at eight different observation stations situated at critical points along the tunnel and penstocks from the forebay to the tail-race. The elevation of the water in the forebay, and also in the vent in the tunnel a short distance below the intake, was recorded with specially-built, high-speed, water-level recorders. These instruments were commercial, curve-drawing recorders with the driving mechanism rebuilt to give a chart speed of 0.1 in. per min, which is sixty times faster than normal.

Mercury manometers to be used for recording pressure heads were fastened to piezometer connections in the tunnel at two places between the vent and the surge tank in adits that were excavated for access to the tunnel during construction, at the surge tank, and on each of the converging sections just ahead of the turbine scroll-cases, and a staff gauge was used for reading the elevation of the tail-race. Each of the manometers, and the staff gauge in the tail-race were read by observers during the tests at predetermined intervals corresponding to different stable conditions of flow in the conduit, although it was not necessary to use the data obtained from the readings in the adits as the tunnel lining is concrete throughout the distance from the vent to the surge tank and the total losses for this section were the only data required.

Instead of using the conventional U-tube manometer at the observation points on the turbine scroll-case in the power house, it was necessary (because of the high head which, in some instances, supported a mercury column approximately 28 ft high), to use a single riser for each manometer, which was connected to a specially designed mercury pot in which the water pressure was transmitted to the mercury. The mercury pots were constructed so that a relatively large surface of the mercury was exposed to the water, resulting in a relatively small change of the surface elevation of the mercury in the pot for relatively large changes in water pressure.

Test Runs

The test runs were varied so as to cover the three following conditions of load distribution between the two units: (1) One unit operating with the turbine gates wide open, the other increasing its load by steps until it was wide open; (2) two units with balanced loads decreasing the loads by steps to zero; and (3) one unit only, increasing its load by steps until wide open. To prevent governor hunting and the attendant variation of flow, the governors were cut out, and the turbines were controlled manually throughout these tests.

Definition of Symbols

The symbols introduced in this paper are defined as follows:

e = a subscript denoting "due to eddies";

h = head loss; h_e = head loss due to eddies above the wye-branch tie-plate; h_a = loss of head that occurs in flowing water when the stream is diverted laterally from a straight path, as in the case of water flowing through a pipe bend or a wye-branch (referred to as "diversion" head loss); h_v = velocity head in Q when measured in the main penstock above the wye; h_{v1} and h_{v2} corresponding to Q_1 and Q_2; Δh_v = velocity-head difference between the flow in the main penstock and in Branch No. 2;

v = a subscript denoting "velocity";

C = Chezy's coefficient; C_w = the Williams-Hazen coefficient of roughness;

H = total head loss;

K = a constant = $1.668\ C_w^{1.852}\ R^{1.167}$;

Q = rate of discharge = total combined flow to Units Nos. 1 and 2; Q_1 and Q_2 = rate of flow to Units Nos. 1 and 2, respectively;

R = hydraulic radius;

V = average velocity at a cross-section;

a = a subscript denoting "due to diversion of flow";

Δ = "difference between".

Computation of Results

The pressure heads observed at the various points were increased to total energy heads by adding the velocity heads existing in the conduit, after which the total energy heads were converted into equivalent energy-head elevations for comparison with the forebay, vent, and tail-race elevations.

The elevation of Chelan Lake was disturbed by a seiche while these tests were being run. It was necessary, therefore, to correct slightly, all the observed elevations for this disturbance, except those in the tail-race. In addition, the manometer readings at the surge tank were corrected for an error introduced by the angularity of the riser to the penstock; and the manometer readings at the turbine scroll-cases were corrected for an error in the value of the density of the mercury. The latter errors were determined from an evident displacement of the origin on the loss curve of the wye-branch, the cause for which would not otherwise be explained.

The curves in Figs. 3, 4, 5, and 6 are plotted, therefore, from the observed losses after they have been adjusted slightly to compensate for certain inherent errors, making it possible to draw smoother and more accurate curves through the test points.

The water consumption of the turbines for different loads when operating under various heads had previously been determined by means of tests using the Gibson method.[3] These data were used in determining the flows corresponding to the loads which were carried by the units during the tunnel and penstock tests.

ENTRANCE AND TUNNEL LOSSES

The observed losses that occur through the gathering tube and trash racks have been plotted in Fig. 3 and as may be seen are very small, reaching a maximum of only 1.75 ft for the maximum flow of 2 200 cu ft per sec. The total losses in the flow line between the vent and the surge tank, a distance of 10 578 ft, are shown in Fig. 4.

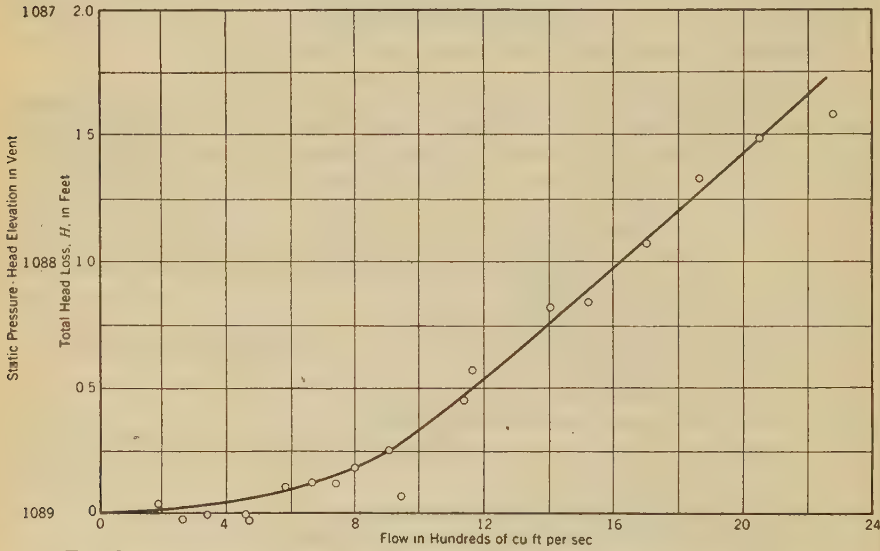

FIG. 3.—ENTRANCE LOSSES THROUGH THE GATHERING TUBE AND TRASH RACKS.

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[3] "Pressures in Penstocks Caused by the Gradual Closing of Turbine Gates", by Norman R. Gibson, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–20), p. 707.

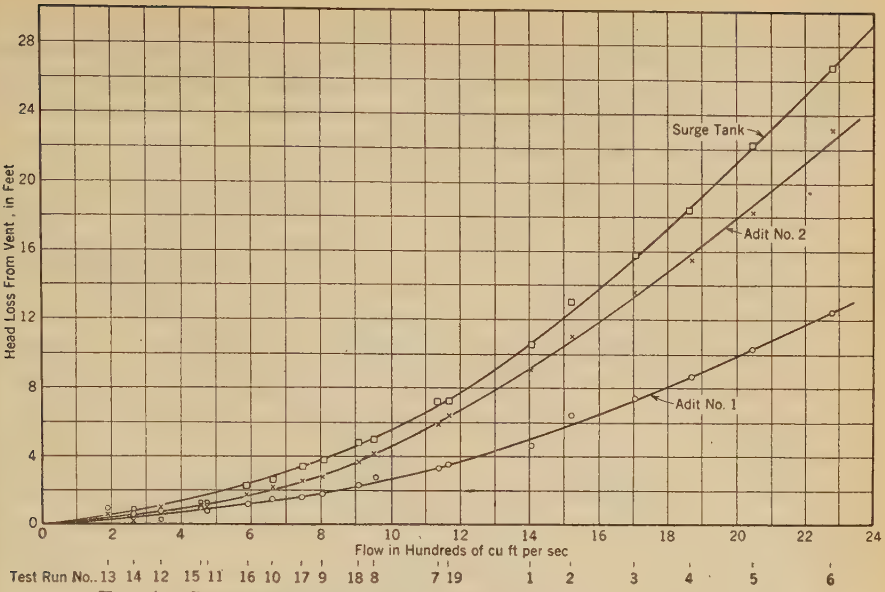


FIG. 4.—CORRECTED FLOW LINE LOSSES FROM VENT TO SURGE TANK.

BRANCH PENSTOCK LOSSES

From turbine tests which were previously conducted using the Gibson method,[2] the losses through the lower part of each penstock branch had been determined by differential pressures. In order to determine the total head loss through each penstock branch the frictional resistance component of this loss was increased proportionately to obtain the head loss due to friction for the entire length of each penstock branch. To this was added the diversion loss component which was present as a result of the bend in the lower part of each penstock branch and which obviously remained unchanged since the additional section of pipe was straight. Thus, the total loss in each penstock branch was the sum of the proportionally increased frictional resistance and the unchanged diversion losses.

Fig. 5(*a*) shows graphically the variation of the diversion losses in each of the lower penstock bends. The fact that the diversion losses through Branch No. 1 are less than in Branch No. 2 is probably due to the unsymmetrical arrangement of the spirals in the turbine scroll-case with reference to the angles through which the penstocks turn. The branches separate as they leave the wye; that is, they bend in opposite directions from the axis of the penstock above the wye whereas both the scroll-cases are spiraled in the same direction.

The equations for the curves of diversion losses in the two penstock branches are similar in character, which demonstrates that in each case the same basic laws govern these losses, although they are different in each penstock for identical flows. The equation for diversion losses in Branch No. 1 is $Q^{2.20} = 330\ H$, whereas that for diversion losses in Branch No. 2 is $Q^{2.20} = 210\ H$ (see Fig. 5(*a*)).

The total head losses in each of the penstock branches, due to frictional resistance and diversion losses combined, are shown in Fig. 5(*b*). The variation of the total head loss with the flow, in Penstock No. 1, may be expressed as $Q^2_1 = 45.5\ H$, and the variation of the total losses in Penstock No. 2 is expressed by $Q^2_2 = 40.8\ H$ (see Fig. 5(*b*)).

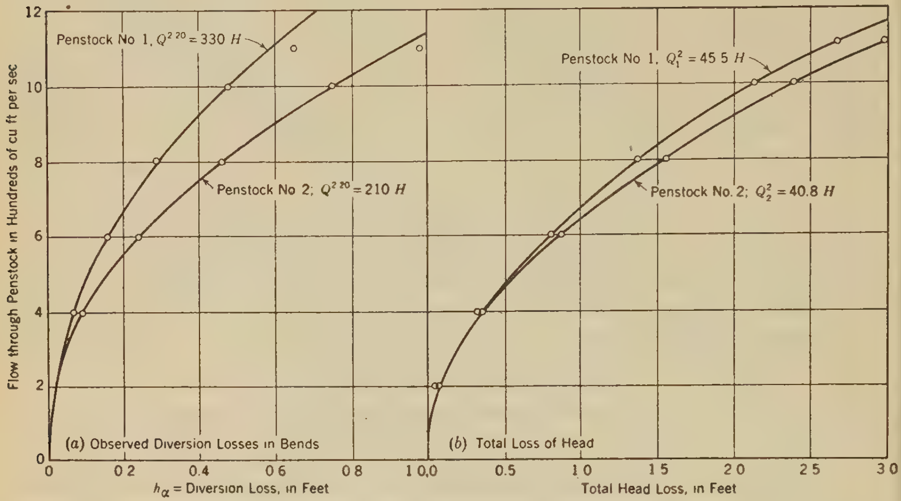


FIG. 5.—BRANCH PENSTOCK LOSSES.

The loss in head that occurred between the surge tank and points in either penstock just below the wye-branch was obtained by subtracting the head loss between the wye and either scroll-case from the corresponding total loss between the surge tank and either scroll-case.

LOSSES FROM THE SURGE TANK THROUGH THE WYE

For the purpose of studying the characteristics of the losses in the wye and penstock branches, the test loads were varied as follows:

Group (1): Unit No. 1 wide open; and Unit No. 2 increasing its load by steps until it was wide open, as shown by Test Runs 1 to 6, inclusive (see Fig. 6);

Group (2): Unit No. 1 and Unit No. 2 carrying balanced loads and decreasing loads by steps to zero as shown by Test Runs 7 to 12, inclusive; and,

Group (3): Unit No. 2 shut down and Unit No. 1 increasing its load by steps until wide open, as shown by Test Runs 13 to 19, inclusive.

To simplify the discussion that follows, these three groups of test runs will be considered in the order of Items (3), (1), and (2), instead of the order in which they actually were run.

Observed losses from the surge tank through Penstock Branch No. 2 to a point just below the wye during Group (3) of the tests, were composed of penstock friction losses from the surge tank to the wye and eddy losses that occurred above the tie-plate in the wye. It is evident that there could be no diversion losses in this branch since no water except that which leaked through the turbine gates was flowing through it. These losses are shown by the curve in Fig. 6 passing through Test Points 13 to 19, inclusive, for Wye Branch No. 2 the ordinates of which are h_{e1} distant from the curve of friction losses that were present between the surge tank and the wye. The manner of determining the position of this latter curve and the resulting values of eddy losses, h_{e1}, will be discussed subsequently herein.

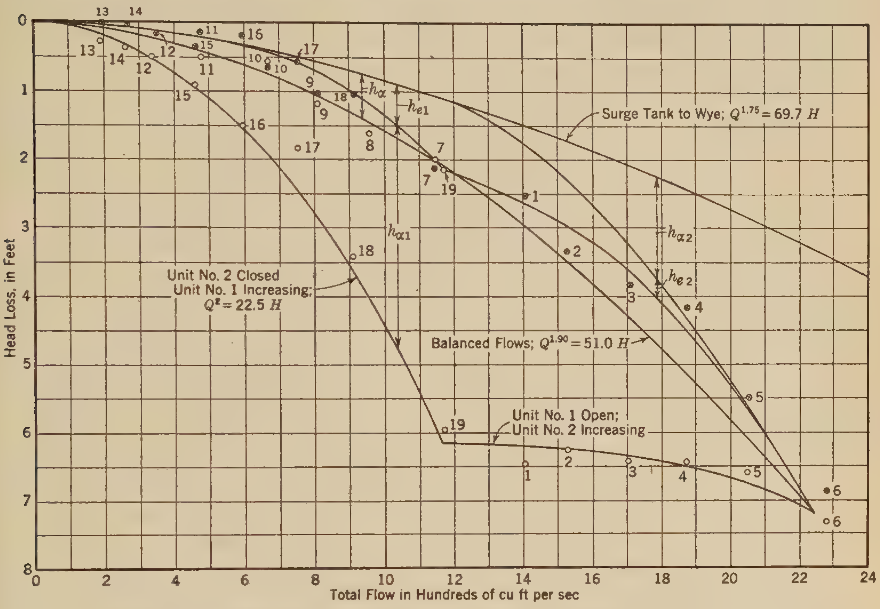


FIG. 6.—LOSSES FROM SURGE TANK THROUGH BOTH WYE BRANCHES.

During this same group of tests the observed losses through the penstock and Branch No. 1 to a point just below the wye, as in the case of losses through Branch No. 2, include (1) penstock friction losses from the surge tank to the wye; (2) eddy losses in the wye-branch above the tie-plate; and (3) the losses incurred when the water from the main penstock was diverted through an angle of 22° 30′ into Branch No. 1. The curve in Fig. 6 that passes through Test Points 13 to 19, inclusive, for Branch No. 1, with the ordinates, $h_{e1} + h_{\alpha 1}$, distant from the curve of friction losses from the surge tank to the wye, shows these losses.

It becomes evident, then, that the difference between the observed losses in Branch No. 2 and Branch No. 1 is the result of diversion losses that occur in Branch No. 1. These losses have been replotted in Fig. 7 and their relationship to the flow in Wye Branch No. 1 has been found to be $Q^{1.75} = 18.5\, h_\alpha$.

Observed losses from the surge tank through Branch No. 2 during Group (1) of the tests, with Unit No. 1 open and Unit No. 2 increasing its load, as in Group (3), consist of (1) penstock friction losses between the surge tank and the wye; (2) eddy losses above the wye; and (3) diversion losses that occur in Branch No. 2. Diversion losses occurred during this group of tests because Unit No. 2 was running.

In Test Run Group (3) the eddy losses increased as the unbalance of flow increased between the two penstock branches, reaching a maximum at Test Run 19 when Unit No. 1 was wide open and Unit No. 2 was shut down. As the eddy losses increased when the unbalance of flows increased, during Test Run Group (3), so likewise did they decrease when the unbalance of flows decreased during Test Run Group (1). They became zero when the flows became balanced, which occurred with a flow slightly less than that at Test Run 6 when both units were wide open.

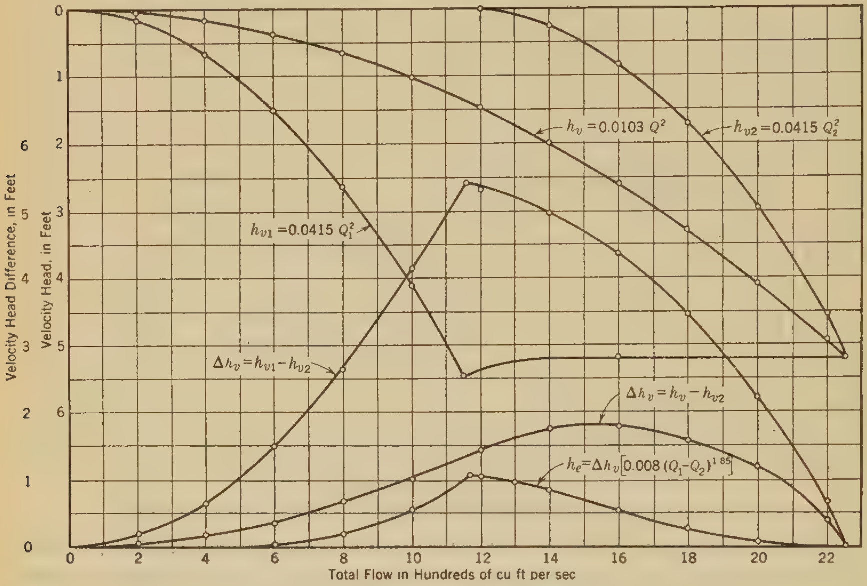


FIG. 7.—EDDY AND DIVERSION LOSSES IN WYE BRANCHES.

Diversion losses in Penstock Branch No. 2, during Group (1) of the test runs, were considered to be the same as those observed for corresponding flows in Branch No. 1 during the test runs in Group (3). The curve of observed losses from the surge tank through the wye in Branch No. 2 for Group (1) of the tests is shown on Fig. 6 as having ordinates of $h_{a2} + h_{e2}$, below the curve of friction losses from the surge tank to the wye. This curve passes through Test Points 1 to 6, inclusive, for Branch No. 2.

Observed losses from the surge tank through Wye Branch No. 1 during Group (1) of the tests are shown in Fig. 6 by the curve passing through Test Points 1 to 6, inclusive, for Branch No. 1. This curve is a continuation of the one that is distant, $h_{e1} + h_{a1}$, below the curve of friction losses from the surge tank to the wye. It slopes downward from Point 19 by an amount equal to the difference between the rate at which the friction losses between the surge tank and the wye increase with the flow of water through the penstock and the rate at which eddy losses (h_{e1}), above the wye decrease as the flows in the two penstock branches approach a balanced condition. It is evident from the downward slope of the curve that, with the unbalance of flows that occurred during Group (1) of these tests, the friction losses from the surge tank to the wye increased more rapidly than the eddy losses above the wye decreased.

Observed losses from the surge tank through either wye-branch during the tests of Group (2) when the units were carrying equal loads, were found in each case to be approximately equal. These losses are plotted as the balanced flow curve in Fig. 6 which passes through Test Points 7 to 12, inclusive, for both penstock branches; it also passes near Point 6. With balanced flows there were no eddy losses above the wye and, consequently, the curve represents the sum of penstock friction losses from the surge tank to the wye and the diversion loss in either wye-branch. It can be reproduced by the expression, $Q^{1.80} = 51.0\,H$.

When comparing this curve with that for observed losses in Branch No. 1 during Group (3) of the test runs, with Unit No. 2 shut down and Unit No. 1 carrying increasing loads, it should be remembered that for equal flows in the penstock above the wye-branch the flow through each branch of the wye with balanced loads would be only one-half as much as that through Branch No. 1 when it was carrying all the water with Unit No. 2 shut down. Thus, the diversion losses in each branch would be 28.6% of the total, since they vary as $Q^{1.75}$. The ordinates between the curve of friction and diversion losses from the surge tank through the wye for balanced loads, and the desired curve of friction losses from the surge tank to above the wye, are equal, therefore, to 28.6% of $h_{\alpha 1}$, which is the diversion loss for corresponding flows in Wye Branch No. 1 when no water was flowing in Wye Branch No. 2.

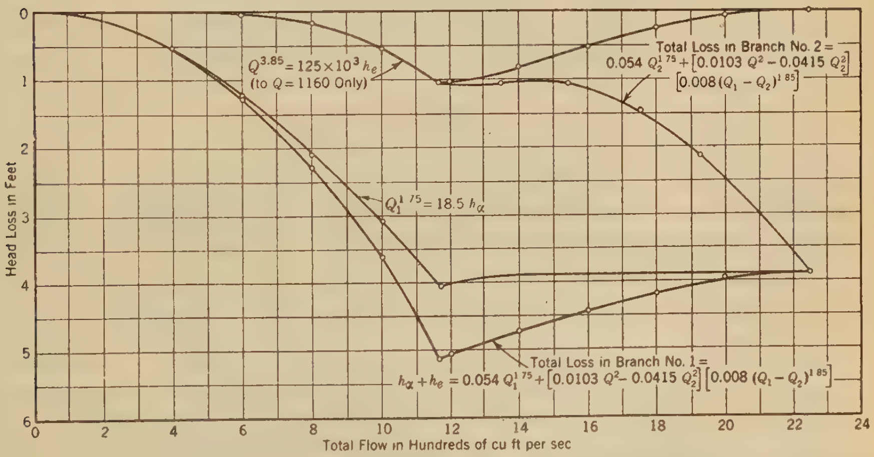


FIG. 8.—RELATION OF EDDY LOSSES TO THE DIFFERENCE IN THE VELOCITY HEAD IN THE WYE

It is possible, therefore, to locate a series of points for the curve of friction losses that occur between the surge tank and the wye for flows up to and including a total of 1 167 cu ft per sec, by subtracting the diversion losses from the observed losses for balanced flows. However, from this value to (but not including) a flow of 2 250 cu ft per sec, no points can be located in this manner. At 2 250 cu ft per sec the ordinate between these two curves is equal to the diversion losses that occurred when 1 125 cu ft per sec was flowing through Penstock Branch No. 1 with Unit No. 2 shut down. The curve of friction losses between the surge tank and the wye has been drawn through the points located in this manner. The equation is $Q^{1.75} = 69.7\,H$.

Eddy losses above the wye-branch tie-plate varied with, but not in proportion to, the unbalance of flows in the two branches of the wye. To study this relationship further, curves of velocity heads in each wye-branch and in the penstock above the wye and the differences between each under the test conditions have been plotted in Fig. 8. A curve of eddy losses was likewise plotted on this sheet which shows at once, that the eddy losses are closely related to the difference between the velocity head in the penstock above the wye and in the wye-branch carrying the smaller quantity of water, which, in the case of this group of tests, was Branch No. 2. As shown in Fig. 7 the eddy losses with one unit running were found to vary as $Q^{3.85} = 12.5 \times 10^3\,h_e$.

The difference between the velocity head in the main penstock above the wye and the velocity head in Wye Branch No. 2 is:

$$\Delta\,h_v = h_v - h_{v2} = 0.0103\,Q^2 - 0.0415\,Q_2^2 \ldots\ldots\ldots\ldots (1)$$

Dividing h_e by $\Delta\,h_v$, a factor is obtained by which $\Delta\,h_v$ may be multiplied to obtain h_e; thus:

$$\frac{\Delta\;h_v \left(\dfrac{Q^{3.85}}{12.5 \times 10^3}\right)}{0.0103\,Q^2 - 0.0415\,Q_2^2} = \Delta\,h_v \times 0.008\,Q^{1.85} \ldots\ldots\ldots\ldots (2)$$

Equation (2) equals h_e when $Q_2 = 0$. Since the eddy losses were found to vary with the unbalance of flow rather than with the total flow, Equation (2) becomes,

$$h_e = \Delta\,h_v\;[0.008\;(Q_1 - Q_2)^{1.85}] \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

when Q_2 is less than Q_1. Expanding Equation (3) to include the expression for the difference in velocity head, the equation for eddy losses above the tie-plate in the wye-branch for any diversion of flow between the two branches in the wye, when Q_2 is less that Q_1 is:

$$h_e = [0.0103\,Q^2 - 0.0415\,Q_2^2]\;[0.008\;(Q_1 - Q_2)^{1.85}] \ldots\ldots\ldots (4)$$

Eddy losses computed by Equation (4) agree closely with the test results plotted in Fig. 7. Care must be taken when using this equation to interchange the positions of Q_1 and Q_2 when Q_2 is greater than Q_1. The total loss, H, in the wye-branch is the sum of eddy losses and diversion losses and may thus be expressed for Branch No. 1 when Q_2 is less than Q_1:

$$H = h_e + h_a = [0.0103\,Q^2 - 0.0415\,Q_2^2]\;[0.008\;(Q_1 - Q_2)^{1.85}] + 0.0540\,Q_1^{1.75} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The expression for Branch No. 2, when Q_2 is less than Q_1, is:

$$H = [0.0103\,Q^2 - 0.0415\,Q_2^2]\,[0.008\,(Q_1 - Q_2)^{1.85}] + 0.0540\,Q_2^{1.75} \ldots (6)$$

When Q_1 is less than Q_2 the expression for Branch No. 2 is:

$$H = [0.0103\,Q^2 - 0.0415\,Q_1^2]\,[0.008\,(Q_2 - Q_1)^{1.85}] + 0.054\,Q_2^{1.75} \ldots (7)$$

Equations (5), (6), and (7), may be used to compute the total losses in Branch No. 1 and Branch No. 2 of the wye with any division of flow between the two units. The total losses that were observed in the two branches of the wye during Groups (1) and (3) of the test runs, are plotted in Fig. 7. Equations (5) and (6) apply to these curves.

TOTAL FLOW-LINE LOSSES

With the losses in each part of the flow line and the laws governing them known, it was next desired to combine their values to determine the total flow-line losses for different divisions of flow through the two penstocks. This has been done in Table 1 for the two conditions of loading: (1) Equal flows through both turbines; and (2) unequal flows through the turbines (maximum condition of unbalance).

TABLE 1.—TOTAL FLOW-LINE LOSSES FROM FOREBAY TO SCROLL-CASE

FLOW, IN CUBIC FEET PER SECOND			LOSSES OF HEAD, IN FEET							
Turbine No. 1 (1)	Turbine No. 2 (2)	Total (3)	Entrance (4)	Tunnel to surge tank (5)	Surge Tank Through Wye: No. 1 (6)	Surge Tank Through Wye: No. 2 (7)	Penstock: No. 1 (8)	Penstock: No. 2 (9)	Total: No. 1 (10)	Total: No. 2 (11)
(a) UNEQUAL FLOWS THROUGH TURBINES										
200	17	217	0.02	0.50	0.21	0.05	0.08	0	0.81	0.57
400	17	417	0.05	1.40	0.75	0.17	0.38	0	2.58	1.62
600	17	617	0.10	2.40	1.66	0.37	0.86	0	5.02	2.87
800	17	817	0.18	3.90	2.90	0.76	1.48	0	8.46	4.84
1 000	17	1 017	0.32	5.50	4.13	1.44	2.30	0	12.68	7.26
1 125	17	1 142	0.46	7.00	5.88	2.02	2.95	0	16.29	9.48
1 125	200	1 325	0.66	9.40	6.23	2.45	2.95	0.08	19.24	12.59
1 125	400	1 525	0.90	12.50	6.26	2.95	2.95	0.38	22.61	16.73
1 125	600	1 725	1.21	15.90	6.38	3.73	2.95	0.86	26.44	21.70
1 125	800	1 925	1.34	19.60	6.57	4.84	2.95	1.48	30.46	27.26
1 125	1 000	2 125	1.56	23.60	6.93	6.22	2.95	2.30	35.04	33.68
1 125	1 125	2 250	1.76	26.20	7.22	7.22	2.95	2.95	38.13	38.13
(b) EQUAL FLOWS THROUGH TURBINES										
100		200	0.02	0.50	0.08		0.04		0.64	
200		400	0.05	1.40	0.30		0.08		1.83	
300		600	0.10	2.40	0.53		0.21		3.24	
400		800	0.18	3.90	1.06		0.38		5.52	
500		1 000	0.32	5.50	1.58		0.60		8.00	
600		1 200	0.53	7.70	2.20		0.86		11.29	
700		1 400	0.75	10.55	2.97		1.15		15.42	
800		1 600	0.98	13.75	3.83		1.48		20.04	
900		1 800	1.20	17.25	4.78		1.85		25.08	
1 000		2 000	1.42	21.10	5.83		2.30		30.65	
1 100		2 200	1.65	25.05	6.96		2.83		36.49	

From the data in Table 1, Fig. 9 is plotted as a working curve to be used for determining flow-line losses and net heads on the turbines. Knowing the net head it is easy to compute the turbine-water consumption. This information is much used for storage and general hydrological computations.

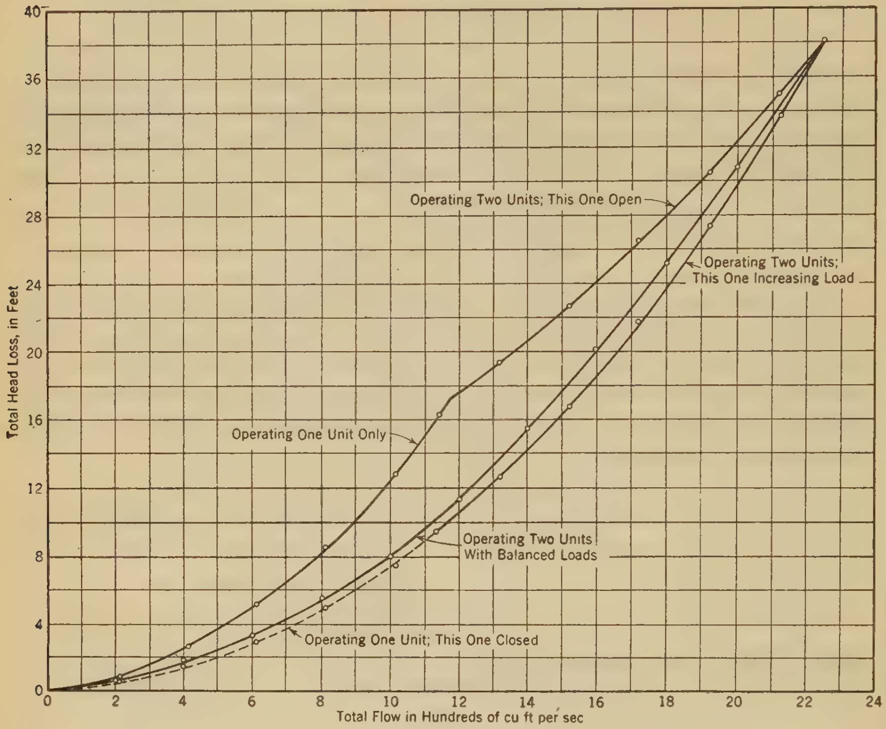


FIG. 9.—TOTAL FLOW LINE LOSS FROM FOREBAY TO SCROLL-CASE.

WILLIAMS-HAZEN AND CHEZY COEFFICIENTS OF ROUGHNESS

The final step in this part of the research was first to reconcile Kutter's formula and the Williams-Hazen formula for friction losses in concrete and steel-lined tunnels to the losses actually observed in these tests, and, then, to ascertain how well the coefficients commonly recommended for use in the formula fitted this particular flow-line installation.

To do this it was first necessary to compute the coefficients of roughness which were applicable to the converging steel-lined penstock between the surge tank and the wye in which section the losses had been determined. The coefficients of roughness that were obtained in this manner were used in computing the losses which must have occurred in the steel-lined penstock from the end of the concrete-lined tunnel to the surge tank. They were also used in computing the losses in the steel-lined section of the tunnel just below the vent. The sum of these two losses was removed from the total observed losses between the vent and the surge tank, leaving those losses which occurred in that part of the tunnel with a concrete lining. The coefficient of roughness applicable to the concrete-lined tunnel was computed from these losses by using the Williams-Hazen formula:

$$H = \frac{V^{1.852}}{1.668\, C_w^{1.852}\, R^{1.167}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

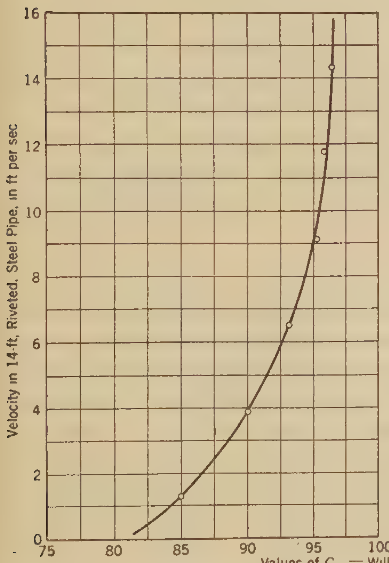
For the purpose of a comparison between the observed and the computed losses in a pipe, Equation (8) may be written:

$$H = \frac{V^{1.852}}{K} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

The coefficients of roughness which were obtained for the steel-lined penstock were found to vary from about 82 for very low velocities of flow to 96.5 for high velocities, as shown in Fig. 10, instead of remaining practically constant for all velocities. This variation is the result of the difference between the Williams-Hazen exponent of 1.852 which is applied to the mean velocity of flow and the exponent of 1.75 which was computed from the observed friction losses between the surge tank and the wye, the equation for these observed losses being:

$$H = \frac{V^{1.75}}{K} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

The coefficients of roughness which were computed for the concrete-lined tunnel were found to remain nearly constant for all velocities, averaging



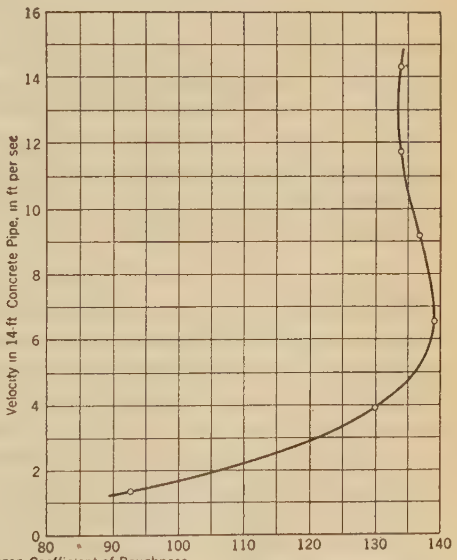


FIG. 10.—VARIATION IN WILLIAMS-HAZEN COEFFICIENT OF ROUGHNESS, C_w, WITH VELOCITY FOR PENSTOCK BETWEEN SURGE TANK AND WYE.

FIG. 11.—VARIATION IN THE WILLIAMS-HAZEN COEFFICIENT OF ROUGHNESS, C_w, WITH VELOCITY IN CONCRETE PRESSURE TUNNEL.

about 134, which is the value most commonly used. The computed values of these coefficients are shown in Fig. 11 where it is seen that, except for the lower velocities of flow, they varied within comparatively narrow limits.

In a similar manner the value of the Chezy coefficient, C, that was applicable to the concrete tunnel and steel penstock for different velocities of flow, was computed from the observed losses. The coefficient of roughness, n, that would be used to compute these values of C was determined from Kutter's formula. These data are plotted in Fig. 12.

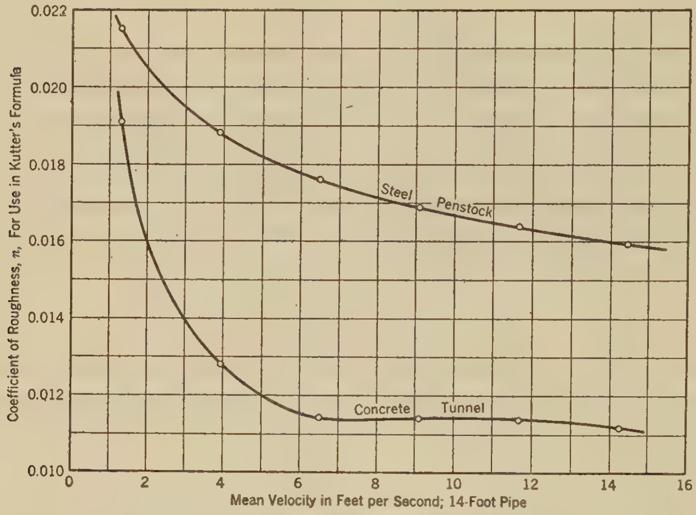


FIG. 12.—VARIATION OF KUTTER'S COEFFICIENT OF ROUGHNESS WITH VELOCITY, FOR CONCRETE TUNNEL AND STEEL PENSTOCK.

The values of n for the steel penstock were found to vary from about 0.022 for very low velocities of flow to 0.016 for high velocities. Likewise, the values of n for the concrete tunnel varied from about 0.020 for very low velocities of flow to 0.011 for high velocities.

CONCLUSIONS

The entrance losses were found to be small and varied almost exactly in proportion to Q^2. The losses that occurred in the concrete-lined tunnel were found to be the same as those which would be computed by the Williams-Hazen formula when using a coefficient of roughness equal to 134, which is the value commonly recommended for concrete-lined tunnels, thus proving the correctness of the formula for use on this particular part of the installation. When computing the losses in the concrete tunnel by means of the Chezy formula, it was necessary to vary Kutter's coefficient of roughness from 0.016 to 0.011 for different velocities, as compared to a value of 0.011 that is commonly used for designing.

The observed losses in the steel-lined penstock were found to vary with $Q^{1.75}$ instead of with $Q^{1.852}$ as would be the case when using the Williams-Hazen formula. As a consequence, in order to compute the true losses for this particular part of the installation when using the formula, it was necessary to vary the coefficient of roughness from a value of about 82 for the lower velocities to a value of approximately 96 for the higher ones. Thus, the formula was not applicable to the steel-lined tunnel of this installation when using a constant value for the coefficient of roughness. When using the Chezy formula for computing the losses in the steel penstock, it was necessary to vary Kutter's coefficient of roughness, n, between 0.020 and 0.016 for different velocities, as compared to a value of 0.016 that is commonly used for designing.

The losses in the wye-branch, which were found to be relatively high, were composed of eddy losses occurring above the tie-plate, and diversion losses which were created as a result of the curvature in the flow line at this point. These losses varied according to definite laws which were determined as a result of these tests.

Losses in the penstock branches were found to be composed of friction losses and diversion losses in the lower penstock bends. They were of a greater magnitude than would normally be the case, due to the presence of the diversion losses.

The diversion losses in the lower penstock bends were found to vary with the 2.20 power of the flow, while those in the wye-branch were found to vary with the 1.75 power. A comparison between the diversion losses at these points is of little significance, however, because the shape of the wye-branch was determined mainly by the requirements for mechanical strength rather than by the hydraulic efficiency that could be attained. Consequently, the cross-sectional shape and curvature of each branch in the wye are not comparable to the bend in each penstock branch.

The total flow-line losses from the forebay to either turbine scroll-case were found to increase when the loads on the two units were unbalanced. This was due chiefly to the eddy losses that occurred in the wye-branch. The losses in the flow line were found to be less when operating the two units with balanced, rather than with unbalanced, flows. However, this should not be done to the exclusion of allowances for turbine and generator efficiency under conditions of loading where their effect dominated the over-all efficiency of the plant.

The unsymmetrical arrangement of the spirals in the turbine scroll-case was found to produce unequal diversion losses in the two lower penstock bends. This inequality was relatively small and unimportant with regard to flow-line efficiency, but served to demonstrate the far-reaching effect of the eddies that resulted from the diversion losses.

ACKNOWLEDGMENT

The field work for this investigation was conducted under the supervision of Mr. W. A. Hill, of the Washington Water Power Company, in Spokane, Wash.

DISCUSSION

W. A. Hill,[3] Esq. (by letter).—An important addition to the data available concerning the hydraulic behavior of large pressure conduits in actual operation, is contained in this paper. Although it is to be regretted that no piezometer connection existed immediately above the wye branch, for the purpose of checking the wye-branch losses and, at the same time, affording a direct means of determining the friction losses in the steel-lined section of the tunnel, the conformity of the test points to the theory of the nature and location of the wye-branch losses is such that the validity of the analysis can scarcely be questioned.

The values of the coefficient, C_w, as derived, furnish an important confirmation of those already in use for the concrete section of the pressure tunnel; but the apparent exponential discrepancy, with respect to the steel-lined section, is unexplained. The additional point of observation, previously mentioned, would have aided materially in substantiating these results.

The most illuminating part of the analysis is that which deals with the losses occurring in the vicinity of the wye branch. Little work has been done with this special type of structure in the past, with a view to determining its behavior under varying conditions of flow and the magnitude of the losses involved. Mr. Fosdick segregates these losses into two classes, "eddy" losses and "diversion" losses; the former take place above the point of bifurcation and affect both penstock branches alike, whereas the latter occur beyond this point and reflect the energy losses in the individual branches only. Due to the destruction of directional velocity head, both are essentially impact losses, but their respective values are widely disproportional to the velocity heads involved. A comparison of velocity-head changes produced (see Table 2)—with corresponding losses observed, throughout the range of one unit shut down and the other increasing load—will illustrate this point clearly. For the range selected, $\Delta h_v = h_v$ (since $h_{v2} = 0$), and $\Delta h_{v1} = h_{v1} \sin^2 \alpha$ ($\alpha = 22° 30'$).

TABLE 2.—Comparison of Velocity-Head Changes and Corresponding Losses.

Q	Δh_v	h_e	Factor	Δh_{v1}	h_{α}	Factor
200	0.04	0.01	0.03	0.02	0.10	5 0
400	0.16	0.02	0.10	0.10	0.50	5.0
600	0.37	0.08	0.22	0.21	1.20	5.7
800	0.66	0.26	0.39	0.38	2.10	5.5
1 000	1.03	0.59	0.57	0.60	3.10	5.2
1 160	1.38	1.03	0.74	0.80	3.90	4.9

A glance at the respective factors for eddy and diversion losses in Table 2 reveals that the former never equal unity, indicating a partial recovery of velocity head; in the case of the diversion losses, however, the approximate factor of five indicates a loss far in excess of the total theoretical impact values.

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[3] Hydr. Engr., The Washington Water Power Co., Spokane, Wash.

The obvious disparity in the relationship of these losses may lie, conceivably, wholly in the fact that all values of velocity heads used in the computations are based on average conduit velocities, whereas the observed effects result from the actual velocities at the points of varying influence. The velocity changes which produce the "eddy" losses may occur largely near the wall of the conduit, the higher interior velocities being gradually diverted toward the branch carrying the larger flow. That an opposite condition exists with respect to the "diversion" losses is a certainty, for inspection of the design shows that the greatest and most abrupt diversion of flow occurs on the produced axis of the pressure tunnel, where the velocity is at its maximum value. The conclusion to be drawn from these facts is that attempted computation of losses in structures of this character should be made only after giving careful consideration to the matter of velocity distribution.

The magnitude of the losses evident in this particular installation arrests the attention indeed. When a total head loss of 5 ft for one unit wide open, or of almost 4 ft for full plant load, occurs in such a short length of flow line, it is certainly time to give some attention to the matter of design. A quick computation of the horse-power equivalent of this loss will give some idea of its economic importance. However, as long as the problem is left solely in the hands of the structural designer, little, if any, improvement may be expected.

The "eddy" losses, characteristic of unbalanced flow, can never be eliminated entirely although their maximum effect can probably be considerably reduced; the "diversion" losses, which are by far the more serious, can be minimized by adequate care in design. The structural problem involved can be solved in several ways, the method to be used depending on such physical characteristics as internal pressure, size of conduit, and location of structure.

One method of combining hydraulic efficiency with structural adequacy, which is seldom, if ever, used in the United States, is that developed by the French engineer, M. Ferrand; for simplicity and effectiveness it is unexcelled. The device consists, essentially, of a manifold, of proper hydraulic design, which is entirely inclosed in a steel jacket of appropriate structural capacity. This jacket may be of spherical form, of combined cylindrical and hemi-spherical form, or of combined conical and hemi-spherical form, depending upon the dimensions of the inner structure to be housed. By means of open ports in the main penstock, the conduit pressure at any instant is immediately communicated to the otherwise closed chamber lying between the outer shell and the manifold proper. In this manner a uniform, and always balanced, support is furnished for the interior or water-conducting shape. It will be evident that the duplication of metal involved is of little moment, for the interior structure need be only of sufficient weight to withstand unbalanced kinetic pressures.

In view of the fact that the science of hydraulics is, and ever must be, in a large measure dependent upon empiricism, it is to be hoped that a keener appreciation of the certain benefit to be derived from the equipping and testing of future installations, will lead those responsible for the design and construction of hydraulic projects to incorporate in their plans provision for further work of this character. As a general rule, the expense involved in adapting a hydraulic structure to the rôle of laboratory model is so trifling, as compared with the value of the data that may be secured in this manner, that failure to do so can be construed only as a mark of professional indifference.

S. P. Wing,[4] M. Am. Soc. C. E., and J. N. Bradley,[4] and H. P. Evans,[4] Jr., Juniors, Am. Soc. C. E. (by letter).—Most hydraulic formulas are essentially empirical and can only be used with confidence within the range for which their coefficients have been determined experimentally. The author's tests of a 14-ft penstock, with velocities as high as 14 ft per sec, give very welcome information at the extreme limit of existing experimental data.

It is to be regretted that the author failed to give his field observations. This has tended to confuse observational with derived results which, combined with a somewhat unusual method of analysis, has made it difficult to compare these tests with the work of others. Since the results of this test, together with those from a very few others, form the basic data from which coefficients for larger installations must be extrapolated, it is thought that a few notes concerning the accuracy of the tests and a brief analysis of the results in a form more readily comparable with the work of others will be of value.

The experiments and analyses consist essentially in the determination of the hydraulic losses in a long pressure conduit with various fittings. Piezometric measurements were made on five sections as follows, lengths being given in pipe diameters: (1) Intake to vent, 18 diameters; (2) vent to Adit 1, 385 diameters; (3) Adit 1 to Adit 2, 274 diameters; (4) Adit 2 to surge tank, 98 diameters; (5) and surge tank to scroll-case.

Section (1) consists of the trash rack, 10.8 diameters of concrete-gathering tube, 4.1 diameters of 14-ft steel pipe to the center line of a horizontal butterfly valve, and 1.1 diameters of steel pipe from the valve to the vent. High-speed water-level recorders measured the hydraulic grade at the lake and the vent.

Section (2) consists of 1.7 diameters of steel pipe, and 383.3 diameters of 14-ft straight concrete-lined tunnel.

Section (3), from Adit 1 to Adit 2, consists of 274 diameters of 14-ft straight concrete-lined tunnel. The hydraulic grades at both adits were measured by mercury manometers connected to single taps in the conduit.

Section (4), from Adit 2 to the surge tank, consists of 98 diameters of 14-ft concrete-lined tunnel, followed by 28.5 diameters of 14-ft single-butt, strap-riveted, steel pipe which contains a 30° bend with a radius of 4.20 diameters. The surge-tank riser, 25 diameters down stream from the bend, is 14 ft in diameter and makes a vertical angle of approximately 19° with the penstock. Water elevations were read by a single piezometer connecting into the riser.

Section (5), from the surge tank to the scroll-case, consists, first, of 712 ft of steel pipe, varying in diameter from 14 ft at the surge tank to 12 ft 6 in. at the wye connection (average, 54 diameters of 13.25-ft pipe). Part of the pipe is single-butt and the remainder is double-butt, strap-riveted. At the symmetrical wye, which has a central angle of 45°, the pipe is reduced from 12 ft 6-in. to two 8 ft 10-in. branches. From the center line of the wye to the center line of a 29° bend with a radius of 7.5 diameters, is a distance of about 6.2 diameters, and from the bend to the center line of the balanced valve is 7.6 diameters, the total length from the wye to the center line of the valve being 13.8 diameters of 8 ft 10-in. pipe. A ring piezometer is connected to a dampened mercury manometer, 15 ft (1.7 diameters) from the center line of the valve. The piezometer connections are 18 in. up stream from the 6 ft 6-in. opening to the turbine scroll-case in the wall of the reducer which follows the valve.

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[4] Engrs., U. S. Bureau of Reclamation, Denver, Colo.

Prior to the hydraulic tests reported in the paper, efficiency tests were made on the turbines using the Gibson method to determine the discharge. For this purpose a differential manometer was connected to a single tap at a point about 2 diameters down stream from the end of the wye and 1.0 diameter up stream from the bend, and to a single tap at a point about 4.5 diameters down stream from the end of the bend and 0.5 diameter ahead of the valve, the length of included pipe being 8.8 diameters. Simultaneously with the differential pressure readings, turbine head readings were made at the ring piezometer as well as such electrical measurements as were necessary to compute the generator efficiency. Each unit was tested with the other unit closed down, variations in load being obtained by controlling the wicket-gate openings of the unit being tested. From data thus obtained discharge curves were plotted for varying generator outputs, heads, and gate-openings. These curves were then used in the later tests to determine the mean velocities.

Since the author does not give the details of the efficiency tests, it is difficult to estimate their accuracy. As previously described the locations of the pressure taps were near fittings which caused abnormal pressure and velocity distributions in the flowing water, thus making high precision difficult.

The total energy passing a given cross-section of a conduit per second is expressed by: $w \int \left(z + \frac{p}{w} + \frac{v^2}{2\,g} \right) v \, da$. Where unusual conditions exist, such as near fittings, the law expressing the variation of each of these quantities throughout the cross-section is unknown; for example, the ring piezometer, from which the head on the turbine was obtained, is only about 7 diameters below a combined vertical and horizontal bend, 1 diameter below the balanced valve, and only 0.22 diameter up stream from the scroll-case. At this section. the stream lines would not be parallel to the pipe walls.[5] If actual velocity and pressure traverses were to be made across the section the following might represent the magnitude of the corrections required in the nominal values to obtain the true energy grade: $\pm$ 1.0 ft + 0.1 h_v + 0.2 h_v = 0.3 h_v $\pm$ 1.0 ft.

For the maximum h_v measured (5.3 ft), this correction equals 0.6 ft to 2.6 ft, and is from 0.16% to 0.70% of the head on the turbine. Although this correction is nothing but an estimate, certain of the tests discussed later give confirmatory evidence. It should be noted that such a percentage correction if necessary also applies directly to the efficiency determination.

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[5] "Strömung in Spiral Gehausen", by Kranz, V. D. I., *Forschungs*, Heft 370.

Considering the difficulties under which the discharge measurements were obtained originally and the errors involved later in transferring electrical measurements at various gate-openings to velocities, it is believed that for the hydraulic tests an error of at least 3% in the velocities can be expected.

The difficulties which have just been discussed in determining the true energy grade at a point where the stream lines are not parallel and where the velocity and pressure distribution throughout the flowing stream are unknown, apply to three other of the piezometric locations. The effect of this condition will be discussed in connection with the tests.

Section (1).—Intake to Vent.—This section contains trash rack, gathering tube, and friction losses plus an apparent drop in pressure across a butterfly valve. The vent at which the hydraulic grade was measured is only 0.5 diameter down stream from the leaf of the open gate. From Italian tests[6], in which the differential drops in hydraulic grade across butterfly valves were measured, the drop across the valve may be estimated as 0.45 h_v for the author's test, the velocity head being that for the 14-ft pipe. The friction losses in the intake section may also be estimated as 0.1 h_v. Subtraction of the sum of the friction loss and valve drop from the nominal energy loss indicates that the net trash rack and gathering tube entrance losses certainly do not exceed 0.05 h_v and are probably less. The losses given in Fig. 3, which approximate 0.60 h_v, are, therefore, not net intake losses, but consist largely in the drop across the butterfly valve.

Sections (2) and (3).—Vent to Adit 2.—The writers consider that the tests made on Sections (2) and (3), which include 659 diameters of a straight 14-ft concrete conduit, are by far the most accurate and valuable of the author's data and are deserving of more discussion than he has given them. There are few tests of large pipe at high velocities, as free from errors due to the effects of bends and fittings as these tests. Two methods of treating the data are possible: Either Section (3) alone can be analyzed, the difference between the readings of Adit 1 and Adit 2 giving the energy loss directly, as at these sections the velocity distributions are the same; or Section (2) can be combined with Section (3), and the loss over a much greater length of pipe can be used. In this latter case a correction must be applied to the drop in the hydraulic grade as measured to allow for the influence of the butterfly valve. Either of the foregoing two methods yields substantially the same results. However, since errors in piezometer readings affect the accuracy inversely proportional to the length of section in which the friction loss is determined, Sections (2) and (3) combined appear to give the more reliable results, even when the additional error involved in estimating the valve correction is considered. The method of making this estimate is as follows:

If to the reading at the vent (1.0 h_v + 0.45 h_v) is added, the energy grade up stream from the valve is obtained. The energy loss due to the valve, which, of course, is less than the piezometer drop across it, occurs in Section (2) and may be estimated as 0.25 h_v. At Adit 2 the velocity profile is fully developed, so that the true energy head may be approximated by adding 1.05 h_v to the piezometer reading. The net energy head loss in Sections (2) and (3) thus equals the difference in readings between the vent and Adit 2, plus 0.15 h_v. This valve correction may only be within ± 0.15 h_v of the correct value, but since the measured loss approximates 7.0 h_v, the percentage error would only be of the order of 2 per cent. The probable error of the mean of sixteen field determinations of the friction drop is computed to be about 1 per cent. Combining all errors it is believed that the determinations of the friction factor have a probable error of not more than 4 per cent.

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[6] *Energia Electtrica*, November, 1933, p. 922.

The best method for estimating and comparing hydraulic losses in pipes, which the writers have found, consists in plotting experimental results on logarithmic paper, using friction factors, f, equal to $\frac{2\,g\,D\,H}{L\,V^2}$, as ordinates and Reynolds' numbers, $\frac{V\,D}{\nu}$, as abscissas. Good discussions of the theoretical basis for using these non-dimensional parameters have appeared elsewhere.[7] A very practical reason for using such plots, however, is that they provide an excellent means of assembling and comparing visually the entire range of experimental data. From them the errors probable in extrapolation, or in using formulas, may be easily estimated.

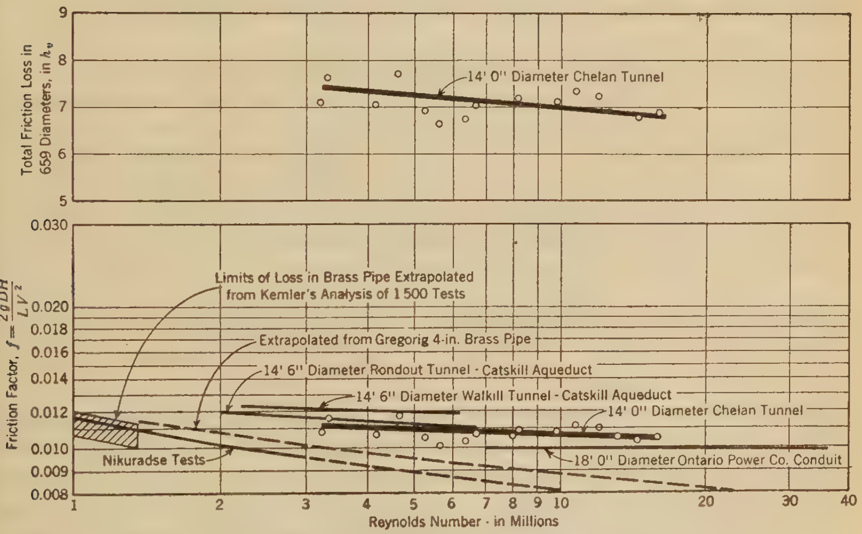

FIG. 13.—FRICTION LOSS IN CHELAN TUNNEL.

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[7] "Modern Conceptions of the Mechanics of Fluid Turbulence", by Hunter Rouse, Assoc. M. Am. Soc. C. E., *Proceedings*, Am. Soc. C. E., January, 1936, p. 21; also, "A Study of the Data on the Flow of Fluids in Pipes", by E. Kemler, *Transactions* A. S. M. E., Hyd. 55–2.

Fig. 13 is the plot of the experimentally measured losses between the vent and Adit 2 of the Chelan Tunnel, both the total measured loss and the friction factor, *f*, being plotted against the Reynolds' number. For comparison with this test the results of equally reliable experiments made on two of the tunnels of the Catskill Aqueduct and on the 18-ft tunnel of the Niagara Power Company, as given by Kemler, have been added. These data, obtained from straight concrete conduits, for which the concrete was poured against steel forms, approximating a mortar finish the surface of which was free from form marks, show a consistent trend of losses over a considerable range of Reynolds' numbers. They become of great value, therefore, for the purpose of extrapolating to Reynolds' numbers of from 30 000 000 to 80 000 000, a range which covers flow in pipe from 20 to 50 ft in diameter.

To aid in extrapolation, tests on smooth brass pipe at high Reynolds' numbers have also been plotted. As concrete pipe becomes larger and larger, its surface relatively tends to approach in smoothness that of the smaller brass pipe. The lower limit for losses in large pipe, therefore, may be expected to approach the losses found for brass pipe at similar Reynolds' numbers.

In his analysis of the losses in the concrete pipe, the author apparently used the measured loss between the vent and the surge tank and, from this, subtracted the estimated loss in the steel pipe, making no other correction. Table 3 indicates the differences in the two methods of treatment.

TABLE 3.—COMPARISON OF LOSSES IN PIPES

Velocity, in feet per second	Reynolds' number	Kutter's *n*		Hazen-Williams' *C*	
		Author	Present analysis	Author	Present analysis
14	15 250 000	0.0112	0.0116	134	124
4	4 360 000	0.0128	0.0120	131	133

Although the author's values are within 2 to 9% of those given by the writers, the differences are large enough to be of significance in connection with the design of large pipe lines.

Section (4).—Adit 2 to Surge Tank.—This section contains losses due to friction in both concrete and steel pipe, a 30° bend, and an apparent pressure change due to expansion into the surge-tank riser. At Adit 2 the energy grade may be obtained by adding 1.05 h_v to the piezometer reading. At the surge-tank riser neither the average pressure nor the velocity head of the flowing water is known, therefore, the energy grade at this point cannot be computed directly. As an estimate the author assumed that, due to the acute angle between the riser and the penstock, the water level in the riser would be lowered about 0.1 h_v. Therefore, he obtained the nominal energy grade by adding 1.1 h_v to the surge-tank reading. The correctness of this assumption may be checked roughly by estimating the friction and bend losses in the steel pipe, adding the friction losses, as obtained from the experiment of the concrete pipe, and subtracting the sum from the energy grade at Adit 2. The following is the computation using the data from the author's Test No. 4:

At Adit 2:

Elevation of hydraulic grade, in feet		= 1 069.73
Mean velocity, 12.16 ft per sec; 1.05 h_v, in feet		= 2.42
Energy grade, in feet		= 1 072.15

Loss Between Adit 2 and Surge-Tank Riser:

Experimentally determined friction loss in concrete pipe, 98D × 0.0106 h_v	= 1.04 h_v	
Estimated friction loss in steel pipe, 28.5D × 0.0147 h_v	= 0.42 h_v	
Estimated friction loss in bend	= 0.06 h_v	
Total energy loss, in feet	= 1.52 h_v	= 3.50

At Surge Tank:

Energy grade at Adit 2 less loss, in feet		= 1 068.65
Velocity head, in feet	1.05 h_v	= 2.42
Computed hydraulic grade, in feet		= 1 066.23
Measured hydraulic grade, in feet		= 1 067.04
Re-gain of head, in feet	0.34 h_v	= 0.81

The average re-gain of velocity head at the surge tank, as determined by similar computations for all tests having a velocity of 1 ft, or more, was 0.15 h_v. Instead of a re-gain, the author assumed there would be a drop due to suction. The result is that in all his computations of the losses in Section (5) he used a surge-tank energy grade 0.2 h_v above that which the writers consider applicable.

Section (5).—Surge Tank to Scroll-Case Piezometer Ring.—This section contains the following losses: (1) Sudden contraction loss from surge-tank riser; (2) friction loss in 54 diameters of 13.25-ft. steel pipe; (3) wye loss; (4) friction loss in 13.8 diameters of 8.83-ft. pipe; (5) 30° bend loss; and (6) balanced valve drop. Since the velocity and pressure distributions at the surge tank and the piezometer ring are both different and unknown, correction factors must be applied to the hydraulic grade drop in order to obtain the total true energy loss in this section. According to previous estimates the true energy grade at the surge tank is about 0.2 h_v less than, and that at the scroll-case, 0.3 h_v higher than, the grades assumed by the author, the combined corrections reducing the total loss as given by the author by about 25 per cent. These figures are applicable to the condition of equal discharge in each of the penstock branches.

Turning next to the author's methods of distributing the total energy loss from the surge tank to the scroll-case into its six component parts, and from these data deriving laws for the losses in a wye, in a bend, and in a steel pipe, the writers consider that in spite of the ingenious methods which were used to make the most of the meager information available, the conclusions must be considered to apply only to the particular installation tested with, moreover, the reservation that piezometric drops are involved and not true energy losses. Even to obtain experimentally the true losses for this single installation, six additional measurements would have been required. For example, to evaluate the losses due to the bends in the wye branch the author made use of the differential pressures measured in the turbine efficiency tests. As already indicated, in this case the absolute magnitude of the pressure drop obtained depended both on the particular location of the piezometers on the pipe sections and on the conditions of flow both up stream and down stream from the piezometer locations. The pressure drop actually measured approximated 0.15 h_v for the higher range of velocities tested. This is three times the energy loss estimated from various bend experiments which have been made on small pipes[8]. Although the results from the latter cannot be extrapolated with any confidence to a bend of the size tested by the author, on the other hand, results obtained under such unfavorable field conditions as existed in these tests can only be regarded with extreme caution.

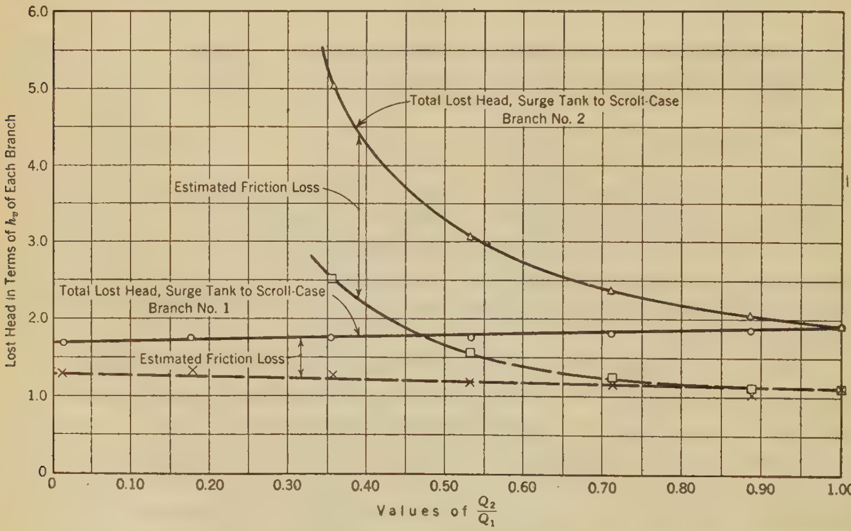

FIG. 14.—LOST HEAD FROM SURGE TANK TO SCROLL-CASE FOR FLOW OF 17 TO 1 125 CUBIC FEET PER SECOND, IN BRANCH NO. 2, AND FLOW IN BRANCH NO. 1 UNIFORM AT 1 125 CUBIC FEET PER SECOND.

It is believed the author's treatment of "eddy losses in the wye branch above the tie-plate" and of the losses through the balanced valve is inadequate. Perhaps a piezometric drop up stream from the tie-plate may be expected, but the major part of any energy loss must occur down stream from any obstruction. The author estimated the energy loss through the valve on the basis of the computed friction loss. An additional loss would be expected due to the curvature of the stream lines. On the whole, the writers consider that the best use which can be made of the author's field measurements in Section (5) is for the purpose of comparing the measured nominal total loss in energy due to a wye, 30° bend, and valve with that of a usual estimate.

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[8] *Transactions*, Munich Hydr. Inst., *Bulletin 3.*

On Fig. 14 the nominal total lost energy head from the surge tank to the piezometer ring, as obtained from Table 3, in terms of the velocity head existing in each of the 8.83-ft branches, has been plotted for various values of the ratio of the discharges in the two wye branches. Subtracting from these total losses, the friction loss in the steel penstocks, as estimated by the writers, two curves are obtained which give the sum of the contraction loss at the surge-tank riser, the wye loss, the 30° bend loss, and the balanced valve loss.

It will be noted, from these curves, that for increasing values of $\frac{Q_2}{Q_1}$, the losses in Branch 2 decrease on account of the great reduction in the deceleration losses through the wye. When $\frac{Q_2}{Q_1} = 1$, the nominal loss is $1.1 \frac{V^2}{2g}$. The following is the writers' distribution of the energy losses for this case, expressed in terms of the velocity head of the branch:

Entrance loss at surge-tank riser................	$0.05\ h_v$
Wye loss	$0.20\ h_v$
Bend loss	$0.05\ h_v$
Apparent loss in valve...........................	$0.25\ h_v$
Total	$0.55\ h_v$
Errors in estimates of losses and differences between true and nominal energy gradients at ends of section................................	$0.55\ h_v$
Total	$1.10\ h_v$

In conclusion, the writers feel that although the author's experiments did not provide adequate data from which true energy losses in bends, wyes, or steel pipe could be determined, they did give required operating data, and information on the friction losses in large concrete pipe which long will be of great value to other engineers.

The courtesy of the author in supplying the writers with information additional to that in the original paper has been much appreciated. To him and to the Washington Water Power Company are due the thanks of all hydraulic engineers for the expenditure of the time and money required in making and analyzing field tests on a large conduit and presenting the results to the profession.

W. S. MERRILL,[9] M. AM. SOC. C. E. (by letter).—Hydraulic engineers are indebted to Mr. Fosdick for the data on friction losses given in his paper describing the Chelan Tunnel and Penstock tests. Data on friction losses in steel penstocks of various sizes and with different types of joints are quite abundant, but similar data for large, concrete-lined tunnels are comparatively rare. The quite complete data given in this paper are, therefore, all the more valuable.

The value of $n = 0.0114$ in Kutter's formula is relatively low and indicates an excellent lining job, probably considerably better than the average to be found in concrete-lined tunnels. Engineers using this or a slightly higher, value in the design of concrete tunnels, must be sure that the figure they adopt is consistent with the kind of concrete surface that may reasonably be expected.

Figs. 10, 11, and 12, showing how the roughness coefficient varies with the velocity of the water in the conduit, are of interest. The relatively constant coefficient for velocities greater than 6 ft per sec in the tunnel should be noted for both the Kutter and the Hazen and Williams formulas. Hydraulic engineers, including Kutter himself, have realized that n is not an exact, unvarying constant. In spite of this, most textbooks, diagrams, and tables make no mention of this fact, and it probably is unknown to many engineers. Corresponding friction loss data for another long, concrete-lined tunnel probably will be of interest. The Waterville Tunnel[10], of Carolina Power & Light Company, is approximately 32 000 ft long, with three long-radius bends in plan. The upper 21 000 ft of the tunnel is on an unbroken grade of 0.285 per cent. At the lower end of this section, the tunnel drops vertically about 500 ft in a shaft 14 ft in diameter, and then extends the remainder of the distance to the wye on grades of 0.20 and 1.5 per cent. In section, the tunnel is horseshoe-shaped with an area of approximately 158 sq ft. The lining, with the exception of the invert, was placed monolithically around movable steel forms and is considered to be an example of excellent work, both as regards alignment and smoothness of the surface. The invert was screeded to shape and is not quite as smooth as the remainder of the lining.

Measurements of head loss in the tunnel were made in 1930, in connection with the efficiency tests of the units, which were conducted by the Gibson method[11]. To determine the exact loss of head in the tunnel for the various flows, it was necessary to make allowance for entrance head losses and for the losses of head at the two 90° bends. Based on the measured water flow and the head lost, the value of n in Kutter's formula was found to be approximately 0.014 for the velocities measured. This was the value used in the design.

ELLERY R. FOSDICK,[12] ESQ. (by letter).—The loss tests that were made in the long flow line at Chelan Station seem to have supplied a welcome addition to this kind of engineering information, judging by the excellent discussions presented. The scarcity of information on this subject is not altogether due to a lack of existing installations upon which suitable tests might be run, but from a lack of research activity. Adequate facilities enabling such tests to be made are not usually found on the older installations and are frequently not included in the newer ones in spite of the negligible cost involved. It would seem expedient and advisable, therefore, that, in the future, engineers recommend strongly that hydraulic installations be equipped with properly designed connections for use in conducting hydraulic loss tests, even if they are not contemplated at the time of construction.

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[9] Hydr. Engr., Ebasco Services Incorporated, New York, N. Y.

[10] *Engineering News-Record*, June 6, 1929.

[11] *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–20), p. 707.

[12] Asst. Engr., State of Washington, Dept. of Public Service, Spokane, Wash.

The method of combining improved hydraulic efficiency with structural adequacy by means of the manifold and shell type of construction that is mentioned by Mr. Hill is one that has long been known to hydraulic engineers but has been seldom used. It is particularly suitable for use in reducing the fluid losses that result from conduit bends and branches, and with the present knowledge that is available regarding these losses it is likely that in some future installations designers will incorporate this type of construction in a manner that will materially improve the operating characteristics of flow lines and penstocks.

Great progress has been made in other fields of mechanics but, until recently, the science of hydraulics has been largely based upon empiricism. The use of a dimensionless parameter as a means for comparing the hydraulic performance of various conduits was first made by Osborne Reynolds in the last part of the Nineteenth Century. For many years little or no attention was given to this discovery but, more recently, considerable progress has been made in the use of this basis, which is commonly referred to in terms of the Reynolds number.

The value of the Reynolds number as a basis for comparing the losses in conduits having a smooth concrete lining is well demonstrated in Fig. 13 by Messrs. Wing, Bradley, and Evans. As a result of being able to correlate these data it is possible to extrapolate the results with a considerable degree of accuracy for a range of Reynolds' numbers covering the flow in pipes much larger than any yet tested. The losses occurring in conduit bends and branches may likewise be compared by this method.

The hydraulic properties of the various types of conduits and equipment have long been, and still are, the subject of much investigation and study. As a result of this work the efficiency of flow lines and hydraulic equipment has been improved. However, no material changes have been made in the design of conduit bends and branches so as to decrease the losses that result from them, and these appear to be among the few remaining types of hydraulic structures that have sufficient losses in some instances to justify material changes in design.

Several investigators, both in the United States and in Europe, have made extensive studies of the losses in conduit bends. Sufficient information is now available to permit the construction of bends that will materially decrease the losses resulting from this type of structure.

The laws governing the losses in conduit branches are closely related to those governing losses in bends, but studies of these losses have received comparatively little attention by investigators. Some additional work along this line could very well be done to advantage.

A comparison of the loss factors as determined for the 22° 30′ wye in the Chelan penstock, with the loss factors that were found by Franz Peterman[8] in laboratory tests of a 45° wye when extrapolated for the same ratio of branch area to main-line area as the Chelan wye, are shown in Fig. 15. It will be observed that the shape of the curves for the relation of the loss factors to the ratio of the flow in the branch as compared to the flow in the main line is similar and that the loss factor for the 22° 30′ wye when $\frac{Q_1}{Q} = 1$, or with all the flow passing through one branch, is approximately 23% of the loss factor for the 45° wye. This is about the same relationship that is normally found between the loss factors in single-branch miter bends of 22° 30′ and 45°, respectively.

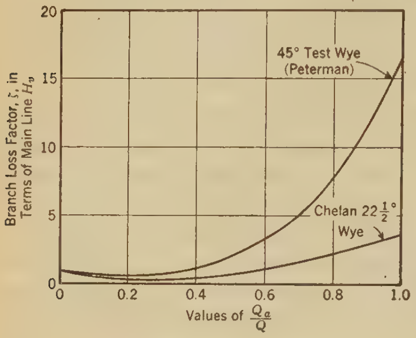

Fig. 15.—Relation Between Loss Factors and Division of Flow for 45-Degree and 22.5-Degree Sharp-Edged Wye Branches.

In Fig. 16, has been plotted the loss factors for a 30° miter bend for Reynolds numbers from 70 000 to 250 000 as determined by Werner Schubart[8].

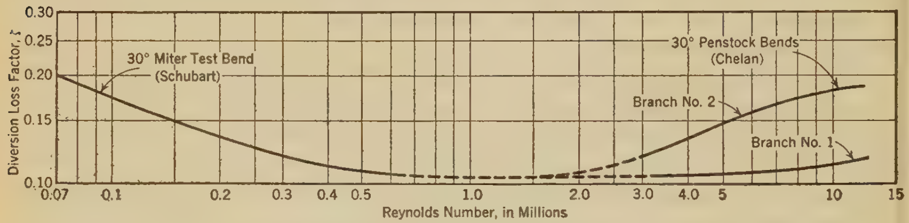

Fig. 16.—Comparison of Loss Factors for Chelan Penstock Bends with Results of Laboratory Tests.

The loss factors for the lower 30° bend in the Chelan penstocks for Reynolds' numbers from about 3 000 000 to 13 000 000 have also been plotted on this diagram.

The loss factors for miter bends have been found by investigators to remain practically constant for values of Reynolds' numbers greater than 250 000, and it will be seen from Fig. 16 that the curve of loss factors for Chelan Penstock Branch No. 1 falls approximately on the extension of the curve of loss factors for the 30° miter bends that were tested in the laboratory. The curve of loss factors for Penstock Branch No. 2 has an upward slope which appears to be in error and for which no satisfactory explanation can be given. The loss factors in each of the penstock bends are undoubtedly influenced to some extent by the disturbance to linear flow originating in the wye-branch which is located about six diameters above, and these factors, therefore, should be used only as approximate values for the purpose of extrapolating the more precise laboratory factors for use with the larger Reynolds' numbers.

The fairly close agreement that exists between the results of laboratory tests and the losses that were observed in the Chelan wye-branch and the penstock bends, that have been previously cited, indicates that these data are fairly reliable and have an accuracy that falls within the limits normally obtainable by field tests of this kind when subject to the various disturbances encountered. The losses that were estimated for certain parts of the Chelan flow line by Messrs. Wing, Bradley, and Evans may be subject to considerable error when applied to this particular installation and should be so regarded when noting differences between their estimates and the observed data.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 1953

PROPOSED IMPROVEMENT OF THE CAPE COD CANAL

By E. C. Harwood,[1] Esq.

With Discussion by Messrs. C. S. Jarvis, George R. Rich, and E. C. Harwood.

Synopsis

The Cape Cod Canal has been open to navigation for twenty-one years, but could accommodate only 20% of the North Atlantic coastwise shipping in 1934 because of limited depth and width. Improvement is economically justified. For several years following the purchase of the canal by the Federal Government, plans for improvement have been under consideration. The most important question involved has been whether there should be locks, or a wider, open canal. The engineering and other difficulties involved in each case are discussed, together with the reasons for selecting the open canal. The new bridges which have been constructed are of unusual interest, one having the longest vertical lift span in the world. Construction methods include excavation by land methods as well as by dipper and by hydraulic dredging. The tide and current conditions are important because of their effect on navigation. The writer has attempted to predict the maximum current to be expected.

Introduction

The Cape Cod Canal is situated in Southeastern New England at the narrow neck where Cape Cod joins the mainland of Massachusetts. It is approximately 50 miles south of Boston and connects Cape Cod Bay on the east with Buzzards Bay on the west. The general location of the canal and the ocean routes from which the shipping through the canal is drawn, are shown in Fig. 1.

Note.—Presented at the meeting of the Waterways Division, New York, N. Y., on January 16, 1935, and published in October, 1935, *Proceedings*.

[1] Capt., Corps of Engrs., U. S. Army, Boston, Mass.

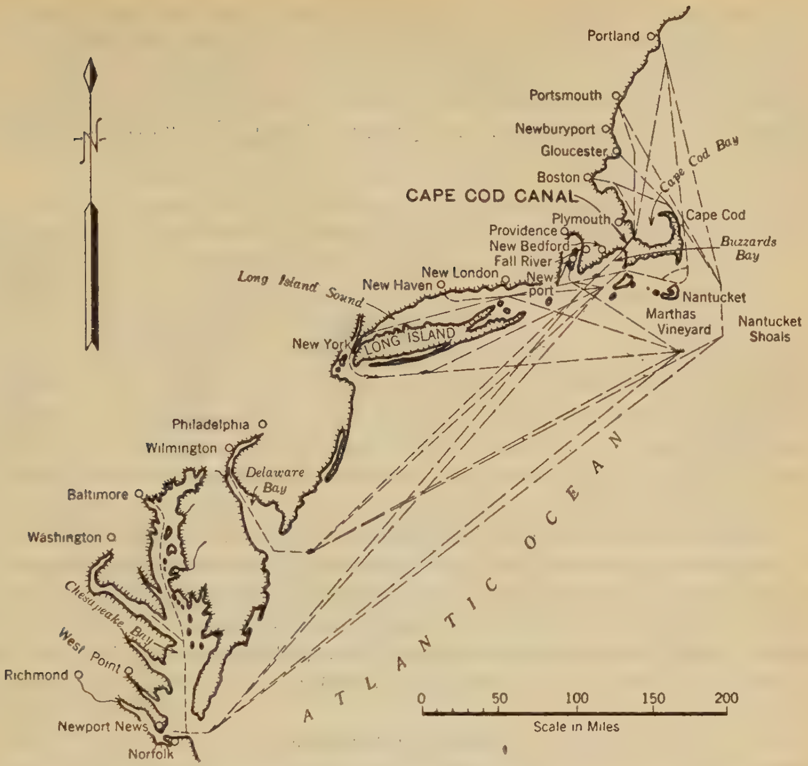
N
Portland
Portsmouth
Newburyport
Gloucester
Boston
Cape Cod Bay
CAPE COD CANAL
Plymouth
Cape Cod
Providence
New Bedford
Fall River
Buzzards Bay
New London
New-port
Nantucket
Long Island Sound
New Haven
Marthas Vineyard
Nantucket Shoals
New York
LONG ISLAND
Philadelphia
Wilmington
Delaware Bay
Baltimore
Washington
Chesapeake Bay
West Point
Richmond
Newport News
Norfolk
ATLANTIC OCEAN
0 50 100 150 200
Scale in Miles

Fig. 1.—Cape Cod Canal and Ocean Routes from Which Traffic Is Drawn.

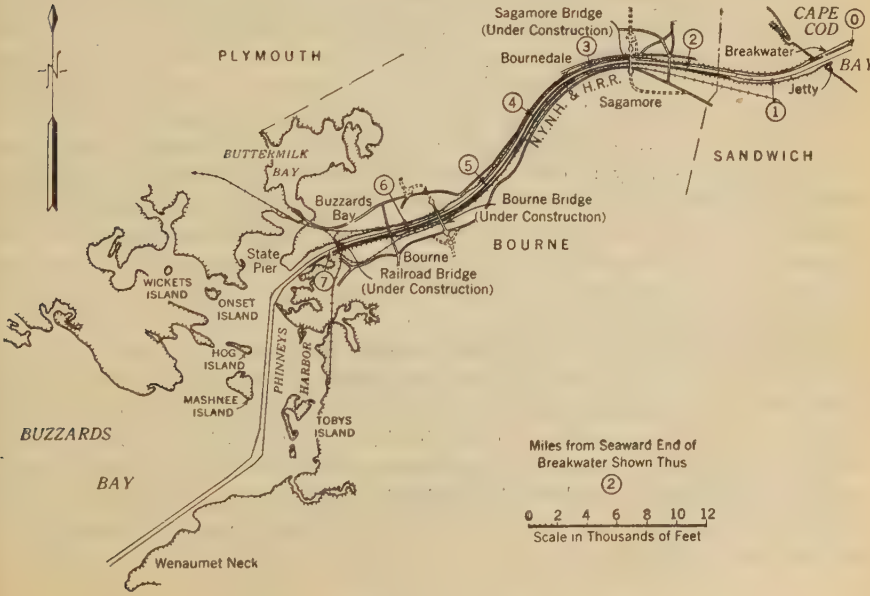
N
Sagamore Bridge (Under Construction)
CAPE COD BAY
0
PLYMOUTH
3
2
Bournedale
Breakwater
Jetty
N.Y.N.H. & H.R.R.
Sagamore
1
4
SANDWICH
BUTTERMILK BAY
5
6
Buzzards Bay
Bourne Bridge (Under Construction)
BOURNE
State Pier
Bourne Railroad Bridge (Under Construction)
7
WICKETS ISLAND
ONSET ISLAND
HOG ISLAND
PHINNEYS HARBOR
MASHNEE ISLAND
TOBYS ISLAND
BUZZARDS BAY
Miles from Seaward End of Breakwater Shown Thus
2
0 2 4 6 8 10 12
Scale in Thousands of Feet
Wenaumet Neck

Fig. 2.—General Map of Cape Cod Canal.

Including its approach channels in Buzzards Bay, the canal has a total length of 13 miles. Beginning at the Cape Cod end, the approximate widths at various sections of the canal (see Fig. 2) are stated in Table 1. The project depth of the canal is 25 ft but this depth has not actually been maintained for several years. At present (1935), the controlling depth is approximately 22 ft.

TABLE 1.—APPROXIMATE CHANNEL WIDTHS

Location	BOTTOM WIDTH, IN FEET		Length of section, in feet
	East end of section	West end of section	
Cape Cod approach	300	300	2 500
Land cut	300	300	2 000
Land cut	300	205	1 000
Land cut to a point west of railway bridge	205	205	35 800
To Buzzards Bay	205	250	200
Through Buzzards Bay	250*	250*	29 000

* At turns, the width is 450 ft.

Until June, 1935, the canal was crossed by three drawbridges, each having a horizontal clearance of 140 ft. Necessarily, these structures limited the effective width of the canal, and it was generally regarded as safe for one-way traffic only—at least as far as the larger vessels were concerned. The new bridges (which are discussed in greater detail elsewhere in this paper) have a horizontal clearance of approximately 500 ft and a vertical clearance of 135 ft.

HISTORICAL SUMMARY

The first agitation for a canal across Cape Cod followed the establishment of the old Dutch trading post, shortly after the Pilgrims settled at Plymouth, Mass. It is reported that Miles Standish ascended the Scusset River and then crossed the intervening land to the head-waters of the Monument River, there meeting the Dutch traders from New Amsterdam. The commercial use of this narrow neck of the peninsula dates from that historic time.

As early as 1697, the General Court of Massachusetts adopted a resolution providing for a survey to determine whether or not a canal would be practicable. There is no record of the report, which undoubtedly followed, and it was nearly eighty years later that the Colony of Massachusetts again authorized a study of the problem. The Revolutionary War interfered with the completion of the survey, and, in 1791, another committee was appointed to examine and report on the project.

In 1824, Congress passed an Act directing that surveys be made and the report of Maj. P. H. Perault, U. S. Topographical Engineer, was printed in 1830, after being approved by the State of Massachusetts. The project considered at that time was for a canal having a bottom width of 36 ft and a depth of 8 ft, with locks. It is interesting to note that the general public expected the canal to be built immediately thereafter, and that it was even described abroad as one of the great public works projects about to be undertaken.

In 1860 the canal project was reviewed once more, this time by the Governor of Massachusetts. The Legislature ordered a committee of engineers to report on it, and the resulting document is the first comprehensive report of the entire project.

This committee suggested a canal with a bottom width of 120 ft and a depth of 18 ft. Although locks were contemplated, it is interesting to note that apparently the members of the committee were among the first to realize that it might be possible to construct a canal without locks. This suggestion was stated more definitely in 1870 by Brevet Maj. Gen. J. G. Foster, of the Corps of Engineers, U. S. Army, who pointed out at that time, that in spite of the difference in head between the water levels at the two ends of the canal, the resulting currents would not be sufficiently great to require locks.

Subsequently, various charters were granted to individuals or companies that hoped to construct the long discussed canal. It was not until 1883, however, that any earth was actually excavated at the site. Mr. Frederic A. Lockwood, who was something of a mechanical genius, had invented a crude hydraulic dredge. Through his efforts, a company was formed and a channel nearly 1 mile long, 15 ft deep, and about 100 ft wide, was actually dug.

Fortunately, the land for the right of way was obtained at that time, and through the foresight of Col. Thomas L. Livermore it was held until the work was begun in earnest. Had it been permitted to revert to the original owners, there is no question but that the subsequent efforts to obtain a Cape Cod Canal would have found progress hampered by excessive costs for the land involved.

At successive intervals, the Massachusetts Legislature granted other charters to companies which proposed to build the Cape Cod Canal, but it was not until 1904 that financial interests of sufficient strength to carry the project through took an active interest in the work. At that time, Mr. DeWitt C. Flanagan presented the project to the late August Belmont, of New York, N. Y., and obtained a promise of action when the general financial outlook improved.

In 1909, Mr. Belmont decided to begin construction and a construction company was organized for that purpose. The details of the organization, the charter, and the procedure followed, have been well described by the late William Barclay Parsons, Hon. M. Am. Soc. C. E.[2]

OPERATION EXPERIENCE

The canal was opened to traffic in 1914, at which time the depth was approximately 15 ft and the bottom width about 100 ft. Traffic never reached the anticipated tonnage, due in part to the fact that the depth was insufficient and in part to the currents existing in the canal, which made navigation hazardous in its narrow width.

During the World War, the operation of the canal was taken over by the Federal Government. Of course, traffic increased greatly during that period because of the fear of German submarines on the outer route. After the World War, the private interests that had built, and formerly operated, the canal refused to accept its return by the United States Government, and efforts were made to persuade Congress that it should be purchased with Federal funds. After protracted negotiations and condemnation proceedings, Congress finally authorized its purchase by the United States for the sum of $11 500 000 in January, 1927.

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[2] "The Cape Cod Canal", by William Barclay Parsons, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXII (1918), p. 1.

Prior to acceptance by the United States, dredging to the full project dimensions was required so that a bottom width of 100 ft and a depth of nearly 25 ft were available throughout.

From 1928, when the Government assumed ownership, to the present time (1935) more than $1 000 000 has been spent in maintenance dredging, but the material removed is believed to have come primarily from the banks of the canal. Apparently, there has never been any tendency for sand to enter the canal at either end.

Modern Widening Operations

Since 1933, funds have been available for further widening at the bends and a 70-ft cut was completed early in December, 1934. Subsequently, two large dredges made an additional 35-ft cut into the south bank. The canal now (1935) has a bottom width of 205 ft throughout and a controlling depth of approximately 22 ft. (At the old bridge locations it has been impossible to obtain the full 205-ft bottom width for the time being. That work will be done when the old bridges have been removed.)

The canal is used by a large volume of shipping, both passenger and freight, as well as the slow-moving tows of barges. Some conception of the volume of shipping can be obtained from the record for 1934, which totaled 2 792 000 cargo tons and 11 500 vessels.

Why Improve the Canal?

There are two principal reasons for considering further improvement of the Cape Cod Canal. In the first place, the hazards of the outside route around Cape Cod are well known and by many are considered more serious than those of Cape Hatteras. These dangers may be classified under a number of different headings, as follows: Exposure, lack of refuge, shoals, zigzag courses, currents, storms, ice, and fogs.

In attempting to round Cape Cod a vessel is necessarily exposed to the elements from the time it leaves the vicinity of Martha's Vineyard to Provincetown, a distance of nearly 100 miles, and there is little or no shelter provided by the low-lying land of the Cape itself. Practically all the seriously dangerous storms in the vicinity come from either the northeast or the southwest, and the course followed is especially exposed to northeast storms. In contrast with the route through the Cape Cod Canal, which is sheltered for approximately 60% of the distance between New York and Boston, the outside route involves an equal percentage of the journey without any shelter.

There are absolutely no harbors of refuge along the outside route. Provincetown might be so regarded, but the harbor is open to the southwest storms and, therefore, is not an adequate harbor of refuge. Other minor harbors offer only 15-ft depths so that they cannot be used by the modern larger sea-going barges, motor ships, and steamers.

The shoals on the outer route have long been known to be particularly dangerous because of their shifting character and the impossibility of maintaining a straight channel through them. Although there is some shoaling in the canal itself, maintenance dredging has been sufficient to preserve a reasonable project depth, and there is every reason to believe that, when the banks have been properly riprapped, the constant shoaling that now occurs will greatly decrease.

Much has been heard of the currents in the Cape Cod Canal itself, and it is sometimes forgotten that the tidal currents over the shoals east of Cape Cod although comparatively weak, often change in direction and make the following of a given course, under some conditions, difficult indeed. At certain places on the outside route a rotary current, with a velocity of approximately 2 knots, has been observed. In the vicinity of Chatham, where it is necessary for the ships rounding the Cape to follow a zigzag course, the tidal currents are especially strong and hazardous to navigation.

In view of the foregoing difficulties, it should be obvious that fog offers a far greater hazard to vessels on the route around Cape Cod than to ships well away from land on the open sea. Furthermore, careful study extending over a long period of years has shown that the fogs on the outside route in the vicinity of the Atlantic Ocean are more frequent and of longer duration than those in the vicinity of Buzzards Bay and the Cape Cod Canal.

In navigating the shoals in Nantucket Sound it is necessary to follow a zigzag course, the angles of which at times exceed 90 degrees. During recent years a straighter channel has been dredged by the Corps of Engineers from Stonehorse Lightship northeast to the slue between Pollock Rip Shoal and Bears Shoal. Although this route is somewhat shorter and less tortuous than the outer one, it still involves several changes in direction which, in time of thick fog, means hazardous navigating.

Ice conditions are not usually such as to offer an obstacle to easy navigation, but there are occasionally very severe winters when the ice floes reach large proportions. At such times the shoal areas in the narrow channels are likely to be covered with large ice floes which introduce added hazards to the outside route. It is true that the very shallow, northern part of Buzzards Bay is even more likely to freeze, with resulting dangers to navigation. However, the proposed new straight channel will obviate much of this difficulty, and it is certain that if ice must be encountered, it is far better to grapple with it in the protected waters of Buzzards Bay than in the broader stretches of the outside route.

The outside route, furthermore, is exposed to the full fury of the northeast storms that sweep across the Northern Atlantic. These storms are widely known as among the most severe to occur on any of the oceans of the world, and the loss of property and lives that has resulted during many years past testifies to their destructive power. For example, the records of wrecks for many years past is mute evidence of the dangers of the outside route. Although the data available do not include full information as to cargo actually lost, or even as to the number of lives, the evidence is sufficiently complete to indicate that the annual cost has been approximately $500 000, and that the number of lives lost has averaged at least 15 annually.

COMMERCIAL REASONS

In addition to the desirability of saving lives and avoiding the losses due to wrecks, there are other more definitely commercial aspects of the problem. At present (1935), the value of the shipping that passes through the canal is only a little more than one-fifth of that which might do so if ships were provided with a reasonable depth and freedom from the present hazards of canal navigation.

The traffic available, which might use the canal, can best be shown by reference to the record of coastwise cargoes entering and leaving the Port of Boston. In 1933, this amounted to 12 300 000 tons, whereas the cargo tonnage through the canal was only 2 804 000 tons. Fig. 3, with Table 2, presents the record[8] of passages through the canal from the date of its opening to, and including 1934.

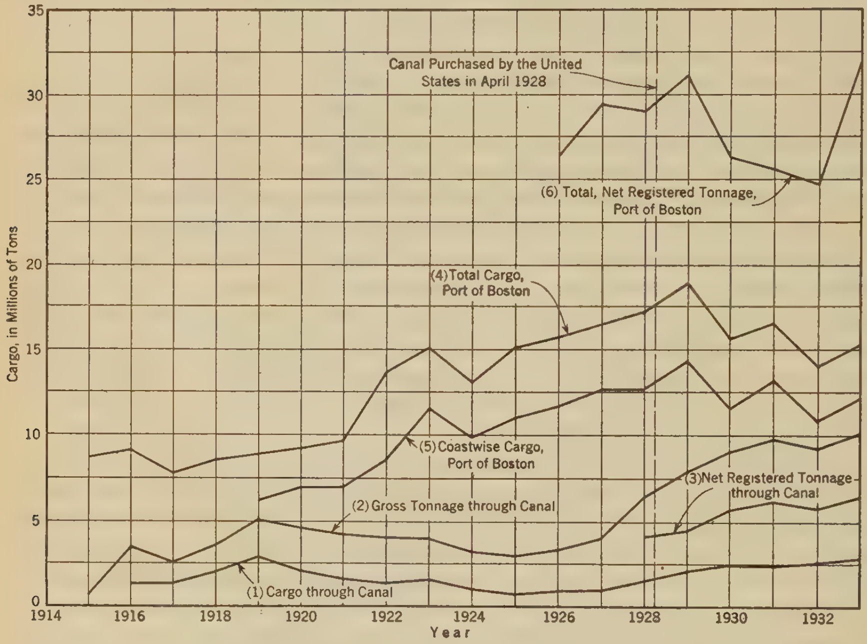

FIG. 3.—COMMERCE THROUGH AND AROUND THE CAPE COD CANAL.

The canal was in operation only two years before the World War. When it first opened, late in 1914, it was not completed to full project dimensions and attracted only a small volume of traffic (see Curve 1, Fig. 3). During the war, many ships used the canal to avoid the submarine dangers of the outside route. The increase continued until 1920, after which traffic dropped until

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[8] H. R. Doc. No. 795, 71st Congress, 3d Session.

1928. In April, 1928, the canal was taken over by the Federal Government. The volume of traffic then rose rapidly, reaching a peak in 1930, so far as the number of ships are concerned (see Table 2). However, both gross and net registered tonnage have increased steadily in recent years, except for the one year, 1932 (see Fig. 3, Curves 2 and 3). There appear to be several reasons for this increase: First, the elimination of tolls undoubtedly attracted many ships; second, the prejudice of mariners against the canal has been overcome to a large extent; third, the canal is being maintained to a depth of more than 21 ft at mean low water; and, fourth, the widening to 205 ft facilitates passage through the canal. That ships of deeper draft have been using the canal in recent years is shown in Table 3.

TABLE 2.—Traffic Through the Cape Cod Canal

Year (1)	Number of:		Year (1)	Number of:		Year (1)	Number of:		Year (1)	Number of:	
	Vessels (2)	Passengers (3)		Vessels (2)	Passengers (3)		Vessels (2)	Passengers (3)		Vessels (2)	Passengers (3)
1914	582	*	1919	7 452	112 128	1925	5 010	160 184	1930	11 464	253 727
1915	2 685	*	1920	8 140	119 088	1926	5 259	166 787	1931	11 291	230 707
1916	4 635	139 917	1921	7 013	112 731	1927	5 745	202 849	1932	10 417	189 275
1917	3 330	117 109	1922	7 180	113 318	1928†	9 312	231 426	1933	11 022	178 642
1918	4 738	81 998	1923	6 771	116 309	1929	10 317	232 824	1934	11 516	201 520
			1924	5 489	133 117						

*No statistics available. †First three months under Canal Company.

TABLE 3.—Analysis of Traffic

Year	Vessels Drawing 18 and 20 Feet			Average Net Registered Tonnage		
	Steamships	All others	Total	Steamships	Barges	Total
1929........	0	1	1	580	543	1 123
1930........	143	47	190	1 039	552	1 591
1931........	193	55	248	1 080	561	1 641
1932........	260	26	286	1 025	558	1 583
1933........	492	21	513	1 083	550	1 633

It is evident that, to a large extent, mariners have ceased to fear the canal. Furthermore, the widening to 205 ft has attracted shipping, and it is logical to assume that any further improvement would attract still more traffic.

More than 80% of the total coastwise trade in and out of the Port of Boston passes around or through the Cape Cod Canal. Furthermore, most of the coastwise traffic passing Cape Cod is with the Port of Boston. Curves 4, 5, and 6, Fig. 3, show commerce in the Port of Boston for the years indicated.[4] The general trend during the years 1929 to 1933 is similar to that found in the case of the Cape Cod Canal.

A large volume of traffic which might use the canal still passes around Cape Cod. Some of it could use the canal at present. Other traffic is unable to use it because of its limiting depths. Table 4 shows the draft and number of trips for vessels entering and leaving Boston Harbor during the calendar years, 1927 to 1933, inclusive. Fig. 4 illustrates the relation between drafts and the capacities of vessels entering and leaving Boston Harbor in the years 1929 and 1933.

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[4] H. R. Doc. No. 795, 71st Congress, 3d Session, p. 19.

TABLE 4.—Summary of the Draft and Number of Trips of Vessels, Domestic and Foreign, Entering and Leaving Boston Harbor During the Calendar Years, 1927 to 1933. (Arrival and Departure Count as Separate Trips.)

Draft	1927	1928	1929	1930	1931	1932	1933
(*a*) Vessels of a Draft Suited to the Project Depth (25 Feet) of the Present Canal							
Less than 18 ft.	13 338	10 655	10 528	10 090	9 732	9 690	11 499
18 to 20 ft.	2 115	2 548	2 648	2 664	2 320	2 305	2 283
20 to 22 ft.	1 296	1 419	1 475	1 356	1 290	1 268	1 186
Total	16 749	14 622	14 651	14 110	13 342	13 263	16 154
Percentage of all shipping	(89.6)	(87.7)	(87.0)	(87.6)	(87.0)	(87.9)	(88.9)
(*b*) Vessels of a Draft too Great for the Project Depth (25 Feet) but Suited to a 30-Foot Canal Depth							
22 to 24 ft.	512	546	677	679	522	480	610
24 to 26 ft.	500	485	513	462	438	369	450
Total	1 012	1 031	1 190	1 141	960	849	1 060
Percentage of all shipping	(5.4)	(6.2)	(7.1)	(7.1)	(6.3)	(5.6)	(5.8)
(*c*) Vessels of a Draft Suited to a 35-Foot Canal Depth							
26 to 28 ft.	685	743	727	580	671	705	863
28 to 30 ft.	206	193	233	218	275	240	267
Total	891	936	960	798	946	945	1 130
Percentage of all shipping	(4.8)	(5.6)	(5.7)	(5.0)	(6.2)	(6.3)	(6.2)
(*d*) Vessels of a Draft Suited to a 40-Foot Canal Depth							
30 to 32 ft.	15	48	28	39	74	30	8
32 to 35 ft.	18	28	8	12	2	1	0
Total	33	76	36	51	76	31	8
Percentage of all shipping	(0.2)	(0.5)	(0.2)	(0.3)	(0.2)	....	....
(*e*) Vessels of a Draft too Great for Safety with a 40-Foot Depth, Except with Favorable Tides							
More than 35 ft.	6	2	7	4	2	1	0

The increase in canal passages for 1933 was 605 vessels (see Table 2), whereas the increase in shipping around Cape Cod during the same period was 350 vessels. Of the total increase, 955 vessels (63%) went through the canal and 37% went around Cape Cod. Reference to Table 5 will show that commerce is using larger vessels. A table for Boston Harbor similar to Table 5 gives approximately equivalent results. With the increase in size and draft of ships, a demand for an increase in the depth of the canal may be expected.

It is apparent from the foregoing that the popularity of the Cape Cod Canal is increasing rapidly and that the commerce which now uses the canal, as well as that which may use it in the future, is growing in volume. It is unquestionably significant that in 1933 supposedly one of the worst years of the depression, net registered tonnage passing through the Cape Cod Canal was the greatest in its entire history, except for a short period during the World War when the fear of submarines forced all ships that could do so to use the canal. When it is remembered that the net registered tonnage entering and leaving the Port of Boston was likewise at an all-time high in 1933, and that much of this commerce will use the canal as soon as it is improved, the future usefulness of that waterway is seen to be unquestionable.

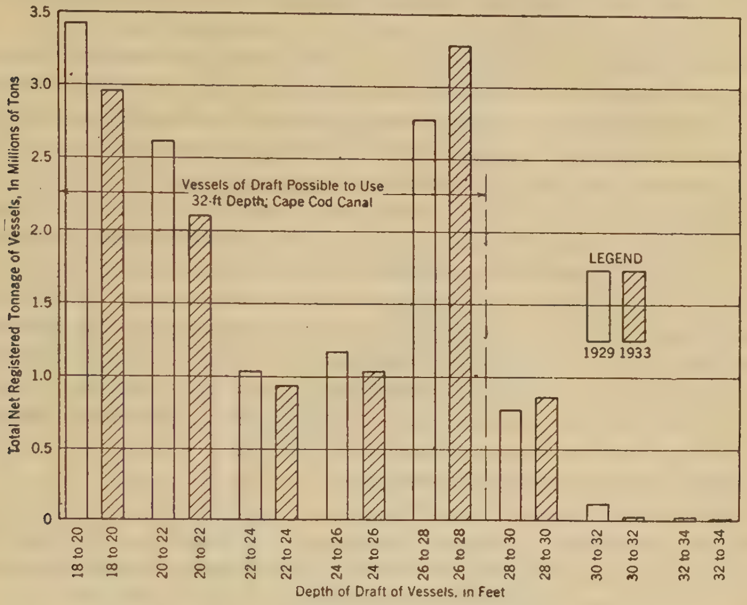

FIG. 4.—DRAFTS AND CAPACITIES OF VESSELS ENTERING AND LEAVING BOSTON HARBOR.

TABLE 5.—AVERAGE NET REGISTERED TONNAGE OF VESSELS PASSING THROUGH CAPE COD CANAL

Type of vessel	1929	1930	1931	1932	1933
Steamships . . .	900	1 190	1 350	1 138	1 365
Motorships . . .	90	100	121	137	148
Barges	720	770	810	790	752

On account of the great diversity of freight and of kinds of ships involved, it would be difficult to make an accurate estimate of the savings to shippers using the canal. However, it is known that the total coastwise cargo in and out of Boston during 1933 was 12 250 000 tons. Approximately 80% of this total will probably use the canal when adequate improvements have been made.

Only 2 805 000 cargo tons were passed through the canal in 1933, leaving 6 995 000 tons which might have gone through. The total costs of all improvements contemplated, including the new bridges, is estimated at about \$31 075 000. At 3%, the annual interest charges would be \$932 000, and this is only 13 cents per cargo ton, or 0.20 cent per ton-mile, even on the basis of commerce during a depression year. (The saving in distance by using the canal is taken as 65 miles. It is true that a small portion of the coastwise tonnage in and out of Boston may never use the canal. However, any decrease in the possible economic benefits derived on that account would probably be more than counterbalanced by coastwise tonnage, particularly coal and petroleum products, to ports north of Boston, and by the fact that a large proportion of the ships using the canal will save more than 65 miles in distance traveled). In recent years, the average vessel has been about 1 200 net registered tons, of which the cargo is approximately 45 per cent. This would mean an average cargo of 540 tons, and at 13 cents per ton the cost chargeable to each vessel would be about \$70 per passage. Fuel costs alone for steamers would probably be between \$50 and \$200, depending on the ship, for the additional distance involved if the canal were not used. For large tows the saving in fuel by use of the canal route would be less, but for them the safety of that route is of even greater importance.

In considering the project as a whole and not merely the justification for the proposed improvement, it would be necessary to include in the total economic benefits the annual losses by wrecks on the passage around Cape Cod. This has averaged more than \$500 000 over a period of several years.

The Problem of Improving the Canal

Hearings held in 1934 and earlier, as well as repeated contacts with masters of ships and pilots by officers and civilian employees on duty in Boston and at the canal, make it possible to state the nature of the improvement desired by interests using the canal. Briefly, it is that the channel should be deepened and widened to permit two-way traffic and that the width should be sufficient to remove the risk of being caught during a falling tide with the bow of the ship on one bank and the stern on the other; or, that the canal should be widened somewhat so as to permit two-way traffic and that locks should be installed in order to still the currents, thereby removing the risk of stranding a vessel cross-wise in the channel.

In considering this aspect of the problem, it is necessary to keep in mind the fact that there are two quite different classes of shipping which use the canal. They are the motor and steam ships on the one hand and the tows of barges on the other. In recent years, the importance of the latter has considerably diminished due to the utilization of powerful motor barges for a large part of the traffic. The shipping that operates under its own power may be further subdivided into the fast passenger-carrying vessels and the slower freight carriers, including motor barges.

The passenger carriers are naturally interested in making the passage as rapidly as possible since they operate on a schedule which gives them little or no allowance for delays. The interests involved desire the improvement to be along the lines first mentioned (that is, without locks) in order that these ships may lose a minimum of time. The slower freight ships, including the motor barges, are not so concerned about maintaining time schedules and, therefore, would not be so strongly opposed to a lock canal, provided there was sufficient width throughout for two-way traffic. The barge proportion of the traffic, which is steadily declining in importance and now constitutes less than one-third the registered tonnage, travels at an even slower rate. It is possible that, on the average, a lock canal might somewhat reduce their time of passage. However, time is of less importance to them than to the other classes mentioned and, although the somewhat faster current in a 500-ft canal might make it necessary for tows to wait for a favoring tide, or else be taken through one barge at a time, it is certain that those tows which had the power could move on through regardless of currents. (At present (1935), even these tows must break up and go through one barge at a time because of the narrow width and the difficulty of keeping tows in line.) Furthermore, favorable currents would be encountered at least 50% of the time, and tows could proceed through without breaking up if the canal were wide enough to permit maneuvering when necessary. It has been reported that no new barges have been constructed for several years and that the trend is emphatically toward the use of motor ships. Obviously, too much consideration must not be given to a class of shipping that is becoming obsolete, which is a decreasing fraction of the whole, and which should benefit substantially under either type of improvement in any case.

The project dimensions on which the estimates are based are a depth of 32 ft and widths of 250 ft and 540 ft for the lock and open canal, respectively. This particular depth was selected because it will accommodate practically all the vessels now entering and leaving the Port of Boston which may be expected to use an improved canal. Taking into consideration the number of ships of the different drafts and their cargo capacity, it is found that this is the economic depth. That such is the case is readily seen by reference to Fig. 4, where the cargo-carrying capacity of vessels entering and leaving Boston Harbor is shown for each of several drafts. (There should be at least 4 ft of over-depth to allow for minus tides, for "squat" in a restricted channel, and for the fact that the channel bottom will not be soft material, but will be thickly strewn with boulders.) It is apparent that a depth of 30 ft would not be sufficient for the relatively large volume of cargo carried in vessels having a draft of from 26 to 28 ft, whereas 32 ft would be deep enough for these ships. The additional 2 ft of depth gives a great increase in cargo-passage capacity. On the other hand, it is equally apparent that a further increase in depth—say, to 35 ft—would result in only a small increase in cargo-passage capacity and that it would not be economically justified at this time. A width of 540 ft is considered desirable for the open canal for reasons already mentioned, and in order that the further deepening to 40 ft at some future time may be made, giving a bottom width of 500 ft, without disturbing the side slopes and bank protection.

THE LOCK *Versus* AN OPEN CANAL

Since the publication of H. R. Document No. 795, important new information has been obtained and experience has been gained which necessitates a revision of earlier ideas. New factors to be considered are: (1) Ice; (2) pile-driving; (3) excavation costs; (4) storms; (5) currents; (6) shoaling; (7) delays in passing through locks; and (8) dredging difficulties.

(1).—Ice.—The most important new development was the experience with ice conditions during the winter of 1933–34. The conditions of that period proved definitely that great difficulty would be experienced from time to time in a still-water canal. There appears to be no practicable means of avoiding ice under those conditions, and it would be most undesirable to have the canal closed for an extended period during the winter months when presumably it would be most needed because of storm hazards on the outside route.

(2).—Pile-Driving.—Another development of primary importance is the experience with the foundations for the new bridges. Great difficulty was encountered in driving the steel sheet-piling to grade because of boulders. Presumably, cut-off walls of sheet-piling would be required under the lock walls and miter-sills in order that the locks might be unwatered when necessary to make repairs. It is now believed probable that the difficulties encountered in driving such walls of sheet-piling would be extremely expensive to overcome and that the cost of locks would be increased materially for this reason.

(3).—Excavation Costs.—Apparently, the possibility of making a 500-ft canal was not previously given much consideration because of the high costs for earth removal which were formerly used as a basis for estimates. In view of the contract dredging completed in 1934, and because of the acquisition of new equipment, including very large dipper dredges (one with a capacity of 26 cu yd began working at the canal in December, 1934), it appears that costs will be lower than was considered possible a few years ago. Furthermore, there is the possibility of still further lowering excavation costs by doing a substantial portion of the work in the dry.

(4).—Storms.—Recent experience has shown conclusively that the difficulties of navigation and the hazards peculiar to the canal have been primarily due to the narrow width. Considered solely from the viewpoint of navigability, it is believed that the 500-ft open canal will be definitely superior to a 250-ft or a 300-ft lock canal. One of the hazards which will exist in either case is the fog that is frequently found in that vicinity (although not so frequently as on the route around Cape Cod). The 500-ft open canal will allow ample room for passing ships even under the worst conditions, permitting the movement of vessels. Likewise, the strong northeast gales that accompany the severe storms in this region, should be considered. In a lock canal, some difficulty would probably be experienced, especially in the case of the larger ships when loaded lightly. Because it is necessary to approach a lock very slowly, a ship would be less responsive to its rudder, and the force of the wind would be more effective. On the other hand, the most severe gales would introduce no hazard to navigation in a 500-ft open canal.

(*5*).—*Currents.*—Although it has been assumed, to be on the safe side, that currents in the 500-ft canal will be increased in accordance with accepted hydraulic formulas, the actual currents found when the improvement is made will probably be somewhat less. It appears that the volume of water passing through the 100-ft canal has somewhat affected the level of the water in Buzzards Bay, with consequent reduction of the difference between the water levels at the ends of the canal. It is only reasonable to suppose that the volume of water passing through a 500-ft canal, about five times as much, will affect the water level in Buzzards Bay to an even greater extent. The result will be to reduce the head, or the difference in level at the two ends of the canal, thereby diminishing the currents.

(*6*).—*Shoaling.*—Further study of the erosion that has occurred in the 100-ft channel, and the consequent shoaling which requires maintenance dredging, leads to the belief that most of the shoaling has been due to wave wash on the banks and the resulting caving in of large quantitites of sand. It follows that adequate bank protection in the form of rip-rap can be expected to remedy this situation. In any event, the wave wash in a 500-ft canal will be less than in the old 100-ft width. Any hazards to navigation due to shoaling and the maintenance dredging made necessary thereby will probably be of no importance in a 500-ft canal.

(*7*).—*Delays in Passing Through the Locks.*—In addition to the foregoing, there are several factors which make the lock canal less desirable. The delays incident to passage of the locks would be most objectionable to the fast passenger-carrying ships. The only shipping that would be delayed at all in the case of an open canal would be that portion of the tug-and-barges combinations, or tows, which lacked power to move against the current. However, this traffic moves slowly at best, and saves twelve or thirteen hours by using the canal. Inasmuch as the average delay involved would be less than two hours, and since the currents would expedite passages when favorable, it is clear that what the tows gained by a lock canal would be more than offset by delay to more important shipping which constitutes the largest and a growing portion of the whole. Still another consideration is that locks are vulnerable to attack in the event of war, whereas an open 500-ft canal could scarcely be blocked for any long period.

(*8*).—*Dredging Difficulties.*—A few years ago it was feared that it would be impossible to dredge in the 100-ft open canal, but this can now be forgotten, inasmuch as the canal has been widened to 170 ft throughout its length, and dredging has started on a further widening to 205 ft minimum width. Apparently, the prevention of shoaling is largely a question of proper bank protection against wave wash and that would be necessary in either type of canal.

The alternate plans of improvement which it was finally decided to consider in detail are the project recommended in H. R. Document No. 795, but with twin locks, *versus* an open canal of adequate width to eliminate the hazard of grounding across the channel during a falling tide. Estimates of cost are given in Table 6. As a result of the studies made, it is believed that

TABLE 6.—COST ESTIMATES, CAPE COD CANAL, 1934

Location (1)	Excavation, in cubic yards (2)	Flow measure (3)	Unit costs, in cents per cubic yard (4)	Total cost (5)
(*a*) LOCK CANAL, 32 BY 250 FEET, TWIN LOCKS AT STATION 72 (SEE FIG. 2)				
Prism Excavation, Including 1-Ft Over-Depth at Cape Cod Bay to Locks and 3-Ft Over-Depth from Locks to 32-Ft Contour in Buzzard's Bay:				
Cape Cod Bay to locks	3 300 000	In place	25	$825 000
Locks to Station 400 (see Fig. 2)	7 240 000	In scow	45	3 258 000
Station 400 to Wings Neck	12 438 000	In scow	45	5 597 000
Wings Neck to 32-ft contour	1 146 000	In scow	45	516 000
Sub-total	(24 124 000)			$10 196 000
Locks complete				11 500 000
Other Items:				
Buildings, real estate, dolphins, water supply, highway, wind break, closing dam, etc.				600 000
Bank protection				1 516 000
Lighting				100 000
Sub-total				$23 912 000
Engineering and contingencies, 15%				3 587 000
Total				$27 499 000
(*b*) OPEN CANAL, 32 BY 540 FEET				
Dredging, Including 1-Ft Over-Depth from Cape Cod Bay to Station 60 (see Fig. 2) and 3-Ft Over-Depth from Station 60 to 32-Ft Contour in Buzzard's Bay:				
Cape Cod Bay to Station 60 (see Fig. 2)	3 786 000	In place	25	$946 000
Station 60 to Station 400 (see Fig. 2)	17 446 000	In scow	45	7 851 000
Station 400 to Wings Neck	12 438 000	In scow	45	5 597 000
Wings Neck to 32-ft contour	1 146 000	In scow	45	516 000
Sub-total	(34 816 000)			$14 910 000
Excavation in the dry, from Station 60 to Station 400	(8 785 000)	In place	25	2 196 000
Sub-total				$17 106 000
Other Items:				
Mooring basins and harbor of refuge				320 000
Bank protection				1 516 000
Lighting				100 000
Land takings, administration buildings, etc.				900 000
Railway and highway relocations				208 000
Sub-total				$20 150 000
Engineering and contingencies, 15%				3 022 000
Total				$23 172 000
(*c*) HARBOR OF REFUGE AT BUZZARD'S BAY				
Dredging East Mooring Basin to 32-ft depth, with a 1-ft over-depth allowance	567 000		25	$142 500
Dredging West Mooring Basin to 32-ft depth, with a 3-ft over-depth allowance	266 000		45	120 000
Dredging channel into Onset Bay to 15-ft depth, 100 ft wide, with a 1-ft over-depth allowance	78 000		45	35 000
Mooring dolphins, East Basin	(30 each)		(500)	15 000
Mooring dolphins, West Basin	(15 each)		$(500)	7 500
Total				$320 000
(*d*) BANK PROTECTION				
One-man stone	175 150		($7)	$1 226 000
Crushed rock, or quarry spalls	58 000		($5)	290 000
Total				$1 516 000

a large part of this excavation, approximately 8 785 000 cu yd, can be done in the dry at a great saving in cost. If funds were made available only in small sums over a long period, it would probably be best to widen by dredging alone, but if the project is adopted and funds are made available so that the work can be done in the most efficient manner, a saving of approximately $2 703 000 may be expected. On this basis, the open canal would be $4 327 000 less expensive than the lock canal.

Other Features of the Improvement

In addition to the deepening and widening, other minor betterments should be provided. The additional volume of shipping anticipated will require enlargement of existing facilities and other work which is discussed briefly herein.

At times, during northerly or northeasterly gales, Cape Cod Bay becomes so rough that tows and smaller craft are unable to continue the voyage out of the canal. Therefore, it will be necessary to provide a mooring basin where vessels thus delayed may tie up until such time as they can proceed with safety. Such a mooring basin should be about 2 000 ft long and extend inshore about 300 ft from the edge of the channel (see Fig. 5). Each end of the mooring basin should be tapered so that the width required (200 ft) is reached in a distance of from 300 to 500 ft. It is planned that additional model experiments will be undertaken, and that one object of the investigation will be to ascertain what size, shape, and location of mooring basin will result in the least interference with navigation through its effects on currents and shoaling.

There are no special difficulties due to weather at the Buzzards Bay end of the canal. The waters are protected and are safe for all but the smaller craft. However, at times, it is necessary to tie up tows at that end while the tug traverses the canal with part load. A mooring basin will be needed to accommodate this traffic, and would occasionally be useful to other vessels in foggy weather. Such a basin should preferably be located northeast of Hog Island and southeast of the new straight channel. It should be about 1 000 ft long and 300 ft wide, with ends tapered. Mooring facilities should be provided.

A harbor of refuge should be provided at the Buzzards Bay end of the canal for smaller craft. For this purpose, a 15-ft channel into Onset Bay would be suitable. The estimated cost of this improvement is listed in Table 6(*c*).

Lighting.—The present lighting system in the canal consists of a series of 60-cp electric lights on piles, spaced 500 ft apart. The alignment is very poor due to erosion of the banks. This system does not meet requirements during adverse weather, because of low visibility. A system of lights should be installed which may be increased in intensity during foggy weather, or while there is vapor in the canal. The lights should be about 25 ft above the water level on poles spaced 300 ft apart. Colored lights placed at critical places in the canal may be used to indicate curves and bridges. The estimated cost of such an installation is $100 000.

Bank Protection.—It will be necessary to protect the canal banks against erosion due to wave wash and currents. This protection should extend for 5 ft (measured vertically) below mean low water and to 5 ft above mean high water. Experience with various forms of rip-rap has shown that a blanket of one-man stone, 1.5 ft thick, on a covering of spalls about 6 in. thick, is the best form. The spalls preferably should be of two sizes, the smaller on the bottom. In this connection, mention should be made of the fact that at one or two points along the banks of the present canal, peculiar difficulties have been encountered in attempting to protect the banks. The sand at these points is extremely fine and either washes out because of eddy currents and boils, or is displaced by a ground-water flow.

The estimated cost of such protection—based on a slope of 1 on 2.5 below, and 1 on 2 above, mean low water, the protection to extend from 5 ft above mean high water to 5 ft below mean low water, and assuming a mean tide range of 6.5 ft—is listed in Table 6(*d*).

Land Takings.—Either the lock or open canal will require additional land, but the total area involved is only a small part of the whole. On the basis of appraisals made and negotiations in progress, it is estimated that the total cost of additional land will not exceed $592 000.

Railway and Highway Relocation.—The widening of the canal to 500 ft will necessitate relocating about 2 miles of highway on the north bank, about 3 miles of railway on the south bank, and short stretches of highway relocation which must be co-ordinated with the railway work. The estimated cost of the work to be done by the Federal Government is $208 000.

Regulating Works in Buzzards Bay.—Consideration has been given to the possibility of constructing regulating works in Buzzards Bay. Such works would be in the form of dikes that would lengthen the channel, thereby reducing the slopes and currents in the open canal. Certain theoretical studies have been made, but it is clear that insufficient data are available on which to base sound conclusions. Although the open 500-ft canal will be an excellent and satisfactorily navigable waterway, the project might be modified later to include regulating works, if experience with the enlarged waterway should indicate their desirability. The project as a whole should not be delayed pending a solution of this relatively minor technical question.

TECHNICAL ASPECTS OF THE PROBLEM

The technical aspects of the problem naturally divide into the two major items of excavation and hydraulics of the proposed canal. Three bridges span the present canal. There are two highway bridges of the rolling lift type which are practically identical. Their substructures consist of isolated piers under each load, stiffened laterally with reinforced concrete girders. These piers were sunk by pneumatic caissons and, in the case of the bridge at Sagamore, piling was used.

The superstructure of these bridges consists of two movable spans of the double-leaf rolling lift type. The elevation of the roadways was about 40 ft above mean sea level and the spans were about 160 ft. Originally, each of these bridges carried a single-track electric railway, but those railways were

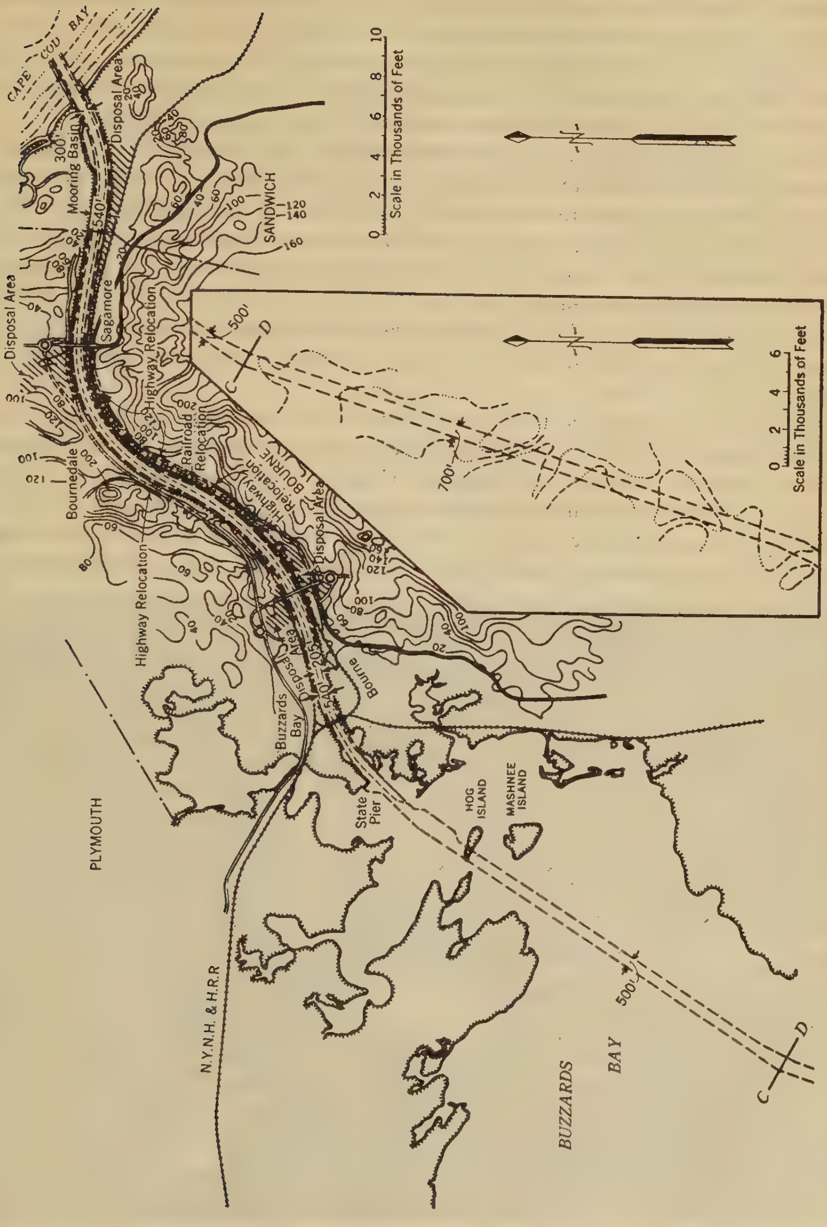
CAPE COD BAY
Disposal Area
300' Mooring Basin
540'
Disposal Area
Sagamore
Highway Relocation
Railroad Relocation
Bournedale
BOURNE
Highway Relocation
Disposal Area
Highway Relocation
Disposal Area
205'
540'
Buzzards Bay
Bourne
State Pier
HOG ISLAND
MASHNEE ISLAND
SANDWICH
PLYMOUTH
N.Y.N.H. & H.R.R
BUZZARDS BAY
500'
700'
C
D
Scale in Thousands of Feet
0 2 4 6 8 10
Scale in Thousands of Feet
0 2 4 6

Fig. 5.—Proposed Improvement, Cape Cod Canal.

abandoned long ago and the tracks have been removed. The clearance between the fenders protecting the main piers is about 140 ft and this is the present limiting width of the channel. The existing railway bridge consists of a double-track, trunnion, bascule superstructure. The piers are of conventional solid concrete on piles with granite facing. The length of this bridge is 160 ft and the clearance between fenders is approximately 140 ft.

When the major improvements contemplated for the Cape Cod Canal were first under consideration, plans were prepared involving a single combination of highway and railway bridge. This was to have a 550-ft span and a vertical clearance above high water of 150 ft; a 6-lane highway for traffic was provided. Since those plans were prepared, however, the substantial growth in summer automobile traffic and the obligation of the United States to protect the interests of the towns near-by, have made it desirable to construct two bridges instead of one for the highways, and a separate railroad bridge.

The highway bridges are similar although the one near Bourne has four approach spans, in addition to the continuous truss extending over three spans which is found in each of the highway bridges. The three-span trusses used are of the Warren continuous type with an arched center span. The floor is supported on the upper chord of the outside spans and is suspended below the arched center spans. Each of the highway bridges has a 40-ft roadway and a sidewalk in addition. Fig. 6 is an excellent view of the Bourne Bridge. The Sagamore Bridge is of similar design, except for the absence of approach spans. The substructures for these bridges consist of piers having massive

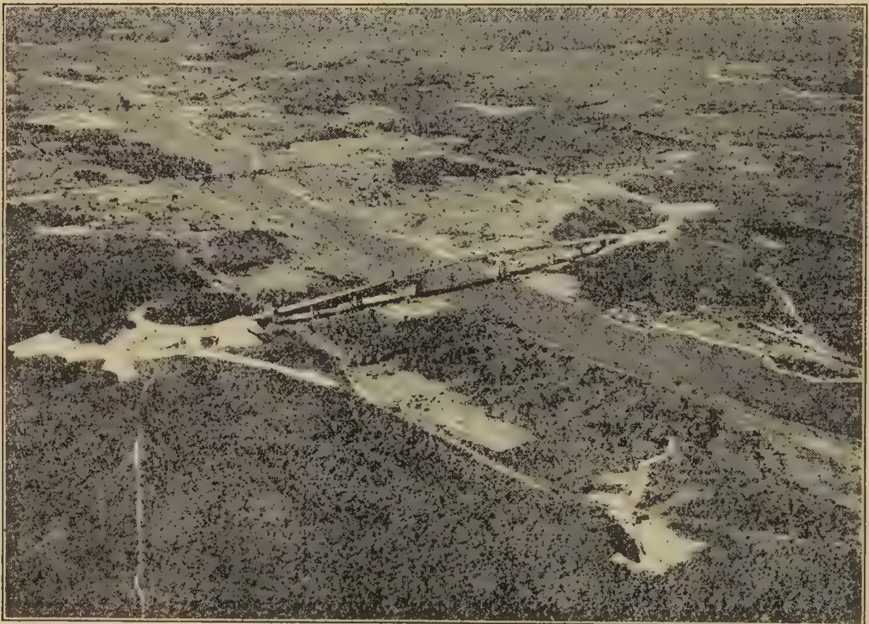

Fig. 6.—New Bourne Highway Bridge and Approaches.

concrete bases with two concrete shafts connected at the top by a concrete diaphragm. The concrete shafts are heavily reinforced and the end abutments are hollow reinforced concrete, with massive pylons.

In the case of the channel piers, coffer-dams were constructed, and, at first, it was expected that the piers would rest on the natural soil. However, due to the fact that there was difficulty with boulders and that it was almost impossible, therefore, to drive the steel sheet-piling as deep as originally intended, and because of the action of the saturated sand under the unequal pressures existing during construction, piles were used in the case of three of the four highway channel piers. They were driven in the wet, but even under those circumstances considerable difficulty was encountered with the blowing in of sand, because the foundation material lacks any substantial quantity of binder, such as clay.

The approaches to these highway bridges are by way of traffic circles at each end, except at the south end of the Sagamore Bridge. A special underpass has been constructed in the approach to the north end of the Bourne Bridge. The railway bridge (see Fig. 2) is the longest vertical lift bridge in the world, with a 544-ft span between end bearings. The towers are 265 ft high, and the vertical clearance of the bridge in the raised position, of course, is the same as in the case of the highway bridges, namely, 135 ft. The truss itself is of the Warren type with vertical members, the trusses being 27 ft, center to center.

This bridge is raised and lowered by four 150-hp induction motors, two of which are used to drive, and two to synchronize, the movement of the two ends of the truss. In addition, emergency motors are provided and emergency gasoline generator sets furnish energy if the commercial power facilities fail.

This structure is the first of this type in the United States to be equipped with roller bearings. Under ordinary operating conditions, it is raised or lowered through 130 ft in approximately 2 min. The bridge is normally in the raised position in contrast with the older bridge, which is normally down and has to be raised as ships approach. Parenthetically, it may be mentioned that this feature will greatly lessen the possibility of a ship striking the bridge. Ship captains will expect to find it up and when they to not find it in the raised position it is to be expected that they will take the necessary precautions to insure that the current will not sweep them into the bridge before it can be raised. Under present conditions, it is normally expected by the ship's captain that the bridge will be raised as he approaches. Consequently, when there have been difficulties with the operating mechanism, this fact has not been realized by the master of an approaching ship until too late to prevent the bridge being struck if the current made it impossible to stop within a short distance.

A few of the difficulties encountered in constructing the new bridges are of special interest. In the first place, the number of boulders which are found in the Cape Cod soil near the canal made it extremely difficult to sink the steel sheet-piling. The foundation material in some cases proved to be very loose and, although it was not compressible if held in position, it flowed easily due to the fineness of the sand and the extremely high water content. Although piles were not contemplated originally, it was believed desirable to use them, and the coffer-dam sheeting was left in place in order to confine the sand. The wooden piles were driven under water in the case of the highway bridges, whereas the coffer-dams were unwatered before driving in the case of the railroad bridge piers. Divers cut the wooden piles used for the highway bridges, and the concrete was placed under water by bottom dump buckets. Another interesting feature of the coffer-dam work at the railroad bridge was the unwatering by the well-point method. Well-points were driven around the coffer-dam and the ground-water flow was discharged into the canal.

The north coffer-dam for the railroad bridge offered many difficulties. The material on that side of the canal seemed to be especially fluid with the result that "blow-ins" of minor proportions were frequent occurrences. Furthermore, the shifting of the earth in the immediate vicinity tended to distort the coffer-dam sheeting and place severe strains on some of the wooden brace members. At one time, it was thought that it might be necessary to place additional bracing to such an extent that driving piles in the quantity desired would have been impossible (see Fig. 7). Finally, a sufficient number of piles were driven so that the first few feet of concrete could be placed, and it was with a feeling of relief that the contractors, as well as the Government engineers, saw the first 3 ft of concrete sluiced into place. Of course, as soon as it had set, the danger of the complete destruction of the coffer-dam was reduced to minor proportions.

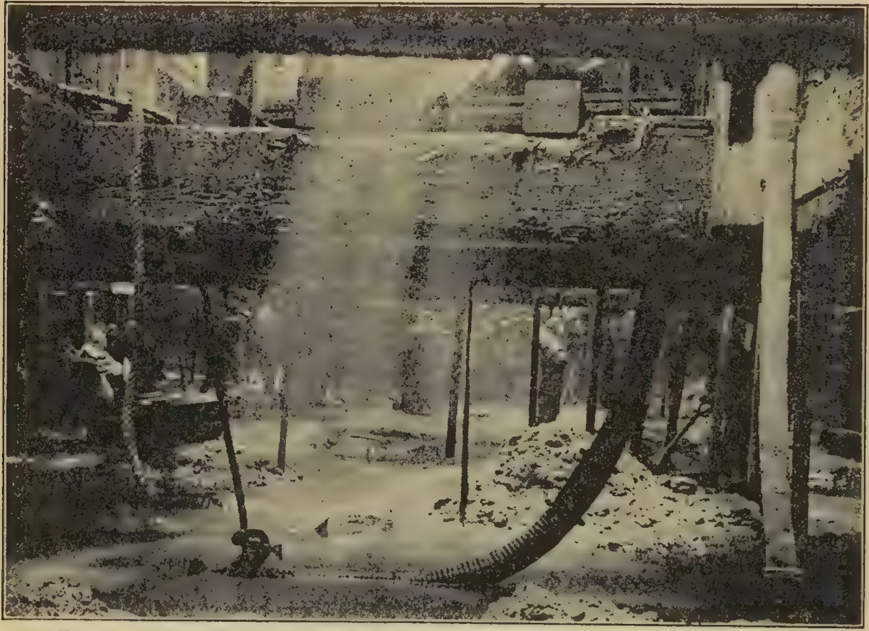

FIG. 7.—COFFER-DAM UNWATERED, NORTH MAIN PIER OF NEW RAILROAD BRIDGE.

EXCAVATION

When the canal was first constructed, it was expected that practically all the excavation could be done by hydraulic methods. Actually, however, it was found that although somewhat more than a mile of the channel from the Cape Cod end could be dredged hydraulically, the larger part of the work could not be handled by that type of equipment.

General Parsons has described[2] at great length, the attempts to operate hydraulic dredges in the land cut. When the original contractors found that they could not make further progress by hydraulic methods, they purchased second-hand dipper machines which happened to be available. Unfortunately, these machines lacked sufficient power to handle the large boulders encountered in dredging the canal channel, and, finally, it became necessary to have two 10-yd dipper dredges. The dredges were assembled at the canal and proved to be entirely satisfactory.

General Parsons had previously urged the contractors to consider the use of steam shovels and industrial railway equipment. This was done with very satisfactory results. It was found possible to excavate in the dry to a depth of 20 ft below the ground-water level. Handling the boulders with such equipment where they could easily be broken up by blasting proved to be far more practicable than the use of even the better type of dipper dredge finally obtained. Under modern conditions, with superior shovel equipment and the mobile and highly adaptable truck for hauling away material, it seems certain that excavation in the dry where practicable will be even more advantageous than it was at the time of the original construction of the canal.

Fortunately, recent contracts let in connection with the highway relocation for the new bridges make it possible to estimate closely the cost of dry excavation. Prices of 24 cents per cu yd have been obtained, even where the haul was in excess of 1 000 ft. It seems certain that on such a large volume as will be involved in the proposed widening it will pay to make use of the modern drag-line equipment, or the large steam shovels and 8 to 10-yd trucks. The material is such that no difficulty will be encountered in making roads for the truck transportation, and there are some places where heavy drag-line equipment with a sufficiently long boom can remove the dirt to be excavated in the dry without any need for haulage. With this in mind, further studies have been made in order to ascertain what depth might be reached by excavation in the dry. Tests were made of the permeability of the soil based on samples taken at various points along the length of the canal. It is believed that the seepage from the present canal can easily be cared for by pumping, even if the excavation in the dry is carried to about 18 ft below mean high water. It will be necessary to leave a substantial dam between the present canal and the new excavation, and one having a top width of 30 ft at mean high water will be used.

In addition to the advantage of handling the boulders in the dry and the fact that earth removal will be somewhat cheaper by this means, there is also a great advantage in being able to place the rip-rap in the dry. As every one who has had such experience knows, placing rip-rap under water is a somewhat uncertain process and is likely to be wasteful of material. On the other hand, by placing the rip-rap blanket in the dry, it will be possible to have it exactly as desired with the least cost.

With a large part of the excavation in the dry, the problem of providing disposal areas is introduced. After a careful study of the territory in the immediate vicinity of the canal certain sites have been selected as indicated in Fig. 5. These sites provide more than sufficient disposal areas for the nearly 9 000 000 cu yd which can be excavated in the dry.

The excavation at the east end of the canal, with the exception of sufficient excavation in the dry for the purpose of placing the rip-rap, can be done by hydraulic methods. West of Station 70 (see Fig. 5), however, it will be necessary to make use of the most modern high-powered dipper-dredge equipment. The analysis of borings that have been taken along the site of the proposed straight channel in Buzzards Bay indicate that it may be possible to excavate a major portion of the material there by hydraulic methods.

PROGRAM OF CONSTRUCTION

Dredging operations on the new straight channel through Hog Island can be started at the earliest practicable date. This can continue to completion without interruption if funds are provided. The approximate maximum rate at which funds can be used efficiently for this part of the work is $3 000 000 the first year, and $3 113 000 the second year. At this rate, the work would be completed in two years.

In the land cut, excavation in the dry will be the most economical. It is estimated that approximately 8 785 000 cu yd of material can be removed by means of drag-line, shovels, trucks, and other land equipment. Much of the necessary equipment would be available in this region and carrying on the work in this manner would provide jobs for the unemployed of near-by cities. The cost of such excavation will probably be little more than one-half the estimated cost of dredging through the major portion of the land cut. Furthermore, by excavating in the dry it will be possible to place the rip-rap for bank protection on dry land instead of under water. This will be more economical and in the long run will give better results. Several contracts for excavation in the dry can be projected simultaneously, and each should include placing the required rip-rap in the section excavated. Areas for the disposal of the material excavated have been selected tentatively. It is estimated that funds for this part of the work can be used efficiently at the maximum rate of $2 000 000 the first year and $1 712 000 the second. Two years will be required to complete this part of the work, including the placing of rip-rap. It is desirable that this part of the improvement be started at the earliest practicable date.

If the excavation in the dry is begun promptly, it will be possible to begin the dredging in the land cut at those sections where the excavation in the dry and the placing of rip-rap shall have been completed. Approximately 21 232 000 cu yd will have to be removed from the land cut by dredging. It is estimated that funds can be used efficiently at the rate indicated in Table 7.

By utilizing dry excavation as far as practicable, the foregoing program would probably effect a saving of about $2 703 000 as compared to the all-dredging method. However, if the enlargement is to be spread over a long period of years, the more costly dredging method will probably be advisable, because each expenditure would provide a more immediate benefit to navigation.

TABLE 7.—Maximum Rates at Which Funds Can Be Used

Description of work	First year	Second year	Third year	Fourth year	Fifth year
Hog Island channel	$3 000 000	$3 113 000	...	...	...
Land, highway, and railway, etc.	1 000 000	108 000	...	...	...
Dry excavation and rip-rap	2 000 000	1 712 000	...	...	...
Dredging land cut	...	1 000 000	$3 000 000	$3 000 000	$1 797 000
Moorings, lighting, etc.	...	...	100 000	100 000	220 000
Totals	$6 000 000	$5 933 000	$3 100 000	$3 100 000	$2 017 000
Engineering and contingencies, 15%	900 000	890 000	465 000	465 000	302 000
Grand total	$6 900 000	$6 823 000	$3 565 000	$3 565 000	$2 319 000

HYDRAULICS

As a very distinguished engineer has expressed it,[2] the problem presented by the flow of water in the Cape Cod Canal is "an extremely complex one in hydrodynamics, being the analysis of the motion of water in a canal of considerable magnitude connecting two seas, the tides in which differ to a great extent, both as to phase and amplitude." In comparative length and cross-section, as well as in the tidal conditions at the ends, the Cape Cod Canal differs to such an extent from the few other similar artificial waterways that methods of analysis which have been applied to some of them cannot be readily applied to it. For example, the "Reflected Wave Theory" developed by Earl I. Brown, M. Am. Soc. C. E.,[5] and applied by him to the Chesapeake and Delaware Canal is difficult to apply to the Cape Cod Canal.

Observations of tides and currents in the 100-ft canal yielded data by which the maximum velocity during a tide and the time of its occurrence could be predicted with considerable accuracy. The United States Coast and Geodetic Survey publishes tables of such values in the yearly "Current Tables." The prediction of the velocities to be anticipated in the proposed enlarged canal presents a difficult problem for which only an approximate analytical solution may be expected, since the flow in the canal is not only non-uniform (surface slope not parallel to bottom), but it is variable (continually changing with time).

The tide in Buzzards Bay (west end of the canal) precedes the tide in Cape Cod Bay (east end of the canal) by approximately 3 hr, one-half the tidal interval. The rise at the Buzzards Bay end is thus well under way while the water is still falling at the other end, and Buzzards Bay begins to fall some hours before Cape Cod Bay has reached its peak (see Fig. 8). The mean range of tide in Buzzards Bay is approximately 4 ft; in Cape Cod Bay, it is approximately 9 ft. The phase relationship is such that the "tide head" (defined as the maximum simultaneous difference in elevation of the water surfaces occurring during any one tide at the recording gauges near the ends of the canal) occurs on the average within 1 hr after high water and within 1 hr after low water in Cape Cod Bay and averages nearly 5 ft (over-all slope, 0.000146). The greatest tide head ever recorded was 9.5 ft (over-all slope, 0.000275), but such a head is most unusual, since less than 1% of the tide heads exceed 7.5 ft.

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[5] "Flow of Water in Tidal Canals", by Earl I. Brown, *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 749.

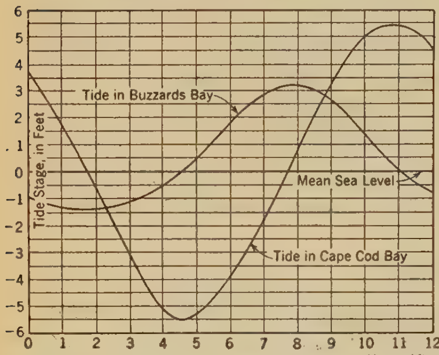

FIG. 8.—TYPICAL TIDE CURVES, FROM OBSERVATIONS, SEPTEMBER AND OCTOBER, 1932.

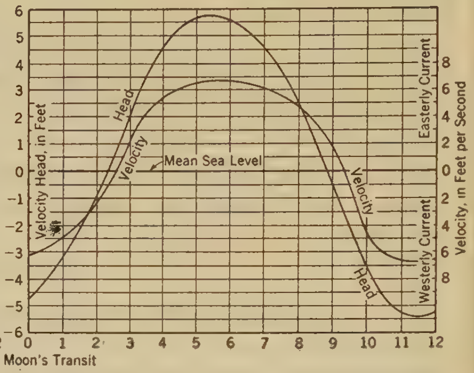

FIG. 9.—TYPICAL CURRENT CURVES, FROM OBSERVATIONS, SEPTEMBER AND OCTOBER, 1932.

Because of inertia effect, maximum velocity and maximum head during any tide do not occur simultaneously; nor do slack-water and zero head (see Fig. 9). The interval of time by which maximum velocity follows maximum head varies widely with individual tides (and is difficult to determine closely because the velocity changes are negligible for a considerable period near the time of the maximum), but it is generally less than 30 min. The interval of time by which slack-water follows zero head also varies with individual tides, and is also generally less than 30 min. For all practical purposes, slack-water may be considered to occur at virtually the same time throughout the length of the canal.

As a result of the tidal conditions at the ends, the water in the canal flows in one direction for approximately 6 hr, reverses, and flows in the opposite direction for approximately 6 hr. The current, therefore, changes direction four times a day. The maximum velocity in each direction is of approximately the same magnitude. The head between the ends of the canal and the velocity of the current in the canal are constantly changing, increasing from zero to a maximum and decreasing to zero in 6 hr. In the following comments the maximum head (tide head) and maximum velocity for any tide will be the quantities most frequently discussed. It should be understood, however, that this maximum velocity persists for a relatively short time. The general relationship between tides, head, and velocity are shown in Fig. 10.

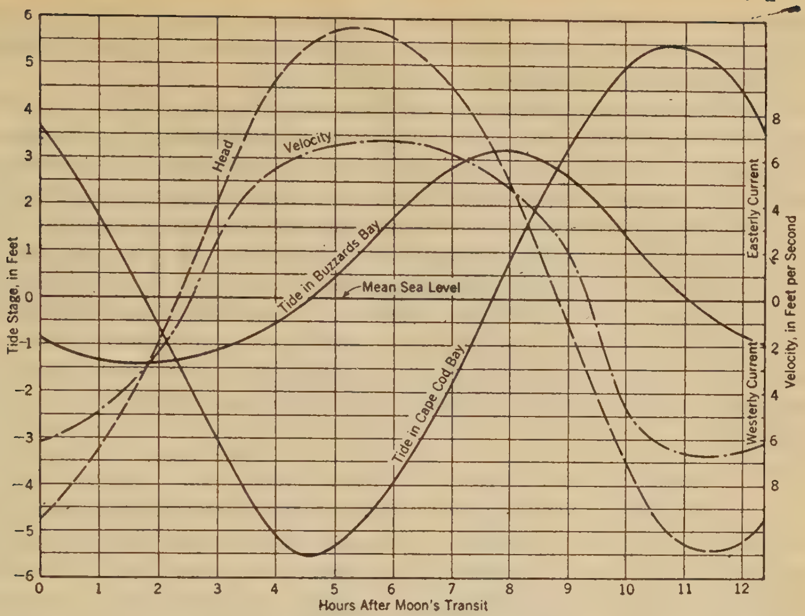

FIG. 10.—TYPICAL TIDE AND CURRENT CURVES, FROM OBSERVATIONS, SEPTEMBER AND OCTOBER, 1932.

EARLY STUDIES

Prior to the beginning of construction in 1909, a number of investigations, surveys, and reports were made. From the time Thomas Machin was commissioned, in 1776, to make surveys and plans, until 1862, all proposals for the canal involved locks. The report of the Massachusetts Joint Committee on the proposed Cape Cod Canal in 1862 contained the first recorded official discussion of the possibility of constructing a canal without locks, as has been mentioned. The Committee was advised concerning technical matters by an Advisory Council (the late Joseph G. Totten, Brigadier-General, U. S. Topographical Engineers, Hon. M. Am. Soc. C. E., the late A. D. Bache, Hon. M. Am. Soc. C. E., and Captain C. H. Davis, U. S. Navy) for which the late Henry Mitchell, M. Am. Soc. C. E., then an Assistant in the Coast and Geodetic Survey, investigated the tidal phenomena involved. The reports of the Advisory Council and of Mr. Mitchell were widely quoted in subsequent reports, and practically all investigators from that time until General Parsons began his tidal observations in 1907, based their computations and discussions of currents on Mr. Mitchell's observations of tides during one month in 1860. Unfortunately, Mr. Mitchell assumed mean tide level in Buzzards Bay to be the same as mean tide level in Cape Cod Bay, and the error was not discovered until nearly fifty years later, when it was pointed out by General Parsons. Mr. Mitchell's observations indicated that the tide head producing flow in one direction would be far from equal to that producing flow in the opposite direction, a result which would tend to make velocities computed from his data higher, in one direction, than those computed from the correct data (a fact which may partly explain the opposition of many prominent investigators to the construction of an open canal).

Nevertheless, between the years 1870 and 1880, two eminent engineers sponsored the proposition that a canal without locks would be feasible. In his "Report on the Cape Cod Canal",[6] Gen. J. G. Foster, U. S. Army, gave results of calculations of current velocity in a proposed open canal 198 ft wide by 23 ft deep, as indicating a maximum probable velocity therein of 4 miles per hr. The late Clemens Herschel, Past-President and Hon. M. Am. Soc. C. E., supported, with independent computations, General Foster's conclusion that an open canal was entirely feasible. From this time until General Parsons was appointed Chief Engineer of the Company which finally constructed the canal, predictions of current velocities to be anticipated in an open canal of various cross-sectional dimensions were made by many other engineers.

PARSONS' INVESTIGATIONS

Tidal observations were begun by General Parsons in October, 1907, with the setting up of two, automatic, recording tide-gauges, one in Buzzards Bay, the other in Cape Cod Bay. From these observations (which continued until 1915) he found that, although the Cape Cod Bay tide is symmetrical above and below mean sea level, the Buzzards Bay tide is unsymmetrical with reference to that plane, mean high water being 2.1 ft above mean sea level, and mean low water being only 1.4 ft below it. Mean tide level, therefore, was above mean sea level and, consequently, the average values of the heads producing velocities in opposite directions do not differ appreciably, as would be the case if the Buzzards Bay mean tide level coincided with mean sea level as assumed in the 1860 tidal observations.

After the canal was completed, observation posts were selected at ten interior points in the land cut, and a series of tide and current observations were made under the direction of General Parsons[2] by observers and assistants at each post. Current velocities were measured by means of surface floats. In addition, current meter observations were made at Station 225 (nearly half way between the ends of the canal, see Fig. 2) to determine the relation between the mean velocity of the cross-section and the observed center-surface velocity. This factor was desired for the purpose of checking the results obtained from the application of various hydraulic formulas with the observed results. It was determined from these observations that: (1) The mean velocity of the entire section = 0.78 times the maximum surface velocity; (2) the maximum velocity in the section = 1.066 times the maximum surface velocity; and (3) the mean velocity of the entire section = 0.73 times the maximum velocity in the section.

Velocities computed by various methods were compared with those actually observed, and a method of solution by harmonic analysis was developed by General Parsons.[2] This method, based on the theory of tides in canals derived by Sir George Biddell Airy, as interpreted by Professor Maurice Lévy, yielded results which checked those observed closely. The method is rather complicated, however, and this consideration, together with the fact that all analytical results must be based on important basic assumptions which may or may not be true, led General Parsons to investigate various approximate methods of solving the problem. The methods included in this study were the formulas of Bazin, Eytelwein, Ganguillet-Kutter, the French Academy of Sciences, and the method of solving the formula for permanent non-uniform flow developed by Professor Bubendey. From this investigation he concluded that the Ganguillet-Kutter formula could be used for the evaluation of velocities in the Cape Cod Canal with sufficient accuracy for all practical purposes.

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[6] Senate Misc. Doc. No. 145, 41st Cong., 2d Session (1870).

OBSERVATIONS OF TIDES AND CURRENTS BY THE UNITED STATES CORPS OF ENGINEERS

Shortly after the United States purchased the canal in 1928, two automatic recording tide gauges were established in the same locations as those used from 1907 to 1915. These gauges have been maintained in continuous operation since August, 1928, for the purpose of collecting basic tidal data.

TIDE AND CURRENT OBSERVATIONS, 1932

A comprehensive series of tide and current observations was made in 1932 when the project dimensions were 100 by 25 ft, shortly before the beginning of the dredging for widening to 170 ft. The Survey Section of the 1st New York District, Engineers, U. S. Army, supplied the equipment and technical assistance to the Boston District for this series of observations. The field work was conducted by personnel of the Boston District U. S. Army Engineers, and the records were compiled and tabulated by the 1st New York District. The purpose of the observations was to determine the laws governing the flow in the canal as well as to secure a basis for comparison with conditions after future widening.

Observation posts were selected at intervals of approximately 5 000 ft throughout the canal, the locations being made to coincide as nearly as possible with those used by General Parsons in 1915. Two parties were employed in making the velocity observations, one party remaining at the master current station, at Station 225, making continuous observations for the full period of the survey (September 28 to October 6, 1932), the other party observing at each of the other posts for a period of approximately one day. The continuous record at Station 225 served as a basis for comparisons between the shorter records at the other points. A similar arrangement for observing tides was used, three master recording gauges being maintained for the full period of the survey while the intermediate tide staffs—one at each observation post—were read by the current observer during the period his party was in position at that post.

Current observations were made from launches anchored in the center of the canal, the measurement at each observation post being confined to a single vertical located at the axis of the deep-water channel, which was also close to the point of maximum velocity. Observations were made with two current meters simultaneously, one at three-tenths the depth, the other at seven-tenths the depth. The meters used were of the standard, large, Price type, mounted in an open rectangular iron frame. A sealed chamber of special design on the meter frame was used to protect the electrical connections against the effects of long-continued immersion in salt water. An automatic graphic recorder with pen arm was actuated by a magnet controlled by a magnetic accumulator relay connected in series with the electrical circuit of the meter. Each meter had its individual circuit with separate relay and recorder.

In order to eliminate minor fluctuations in the current velocities measured, average values over 15-min periods were taken from the records. After all these observed velocities were tabulated and platted, the observations of tides and current velocities at the subsidiary stations were reduced to mean values for the full period of the survey by reference to the master tide and current stations. For convenience, time was reduced to equivalent hours after the time of the moon's transit. From these reduced values, water-surface elevations and velocity were platted against time, as shown in Figs. 8, 9, and 10.

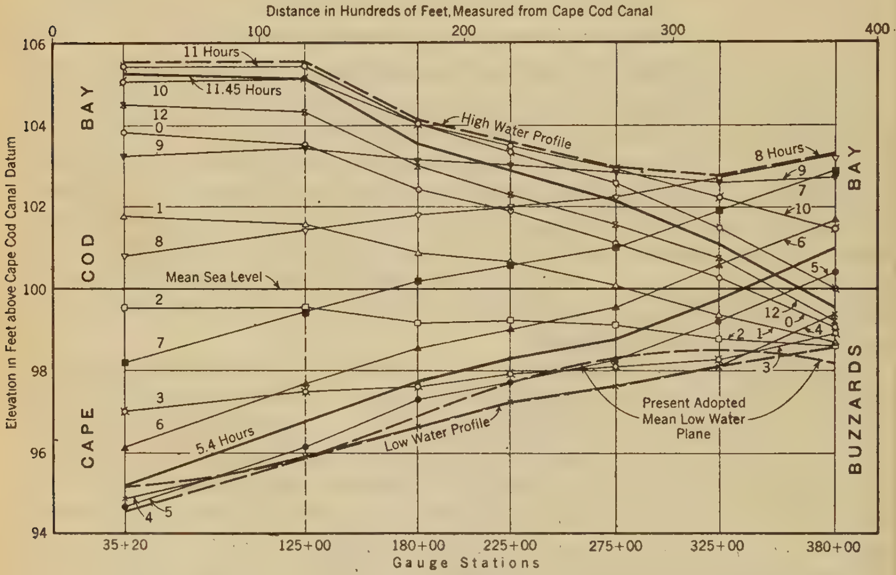

FIG. 11.—PROFILES OF WATER SURFACES: AVERAGE OF OBSERVATIONS, SEPTEMBER 28 TO OCTOBER 6, 1932.

The hourly profiles were also platted, as demonstrated by Fig. 11. The high-water and the low-water profiles were obtained by drawing lines osculatory to the hourly profiles.

As noted previously, the lag of the Cape Cod Bay tide behind the Buzzards Bay tide averages 3 hr. Therefore, high and low waters at one end of the canal occur nearly simultaneously with mean sea-level elevation at the other end. A noticeable feature, shown in Fig. 11, is the flattening of the water surface for a distance of more than two miles from the eastern end of the canal. The profiles near the time of high water especially reflect this effect which is probably due primarily to the propagation of the Cap Cod Bay tidal wave into the canal.

ANALYSIS OF RELATION BETWEEN HEAD AND VELOCITY

The data obtained from the 1932 observations were used as a basis for an investigation of the relation between head and velocity by Charles K. Panish, Assistant Engineer, 1st New York District, Engineers, U. S. Army.[7] In order to establish this relationship in simple form it was found desirable to distinguish between the observed head and the ideal, or effective, head producing velocity. The effective head is defined as that existing in a canal of equivalent uniform section under conditions of steady flow, and was determined by means of the following equations:

For decreasing heads:

$$H = H' + \Delta h + \Delta H \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and, for increasing heads:

$$H = H' + \Delta h - \Delta H \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which H = the effective head; H', the observed head; Δh is a correction due to the fact that the actual canal was not of uniform section; and, ΔH is a correction due to the acceleration in the current brought about by changes in the tide. The numerical values of these corrections were determined by Mr. Panish, thus: Values of ΔH vary from 0.0 to 0.5 ft, being 0 when the head is at a maximum during any tide; values of Δh vary from 0.0 to 1.1 ft, having, for the most common heads, the values listed in Table 8.

TABLE 8.—VALUES OF Δh

Head, in feet	VALUES OF Δh, IN FEET	
	Current from west to east	Current from east to west
4.0	0.50	0.68
5.0	0.63	0.82
6.0	0.78	1.00
7.0	0.92	1.05

The study made by Mr. Panish of the observed velocities and the corresponding effective heads indicated that for all values of the velocity important to this paper (that is, when turbulent flow occurs), the dominating forces are hydraulic, and that the velocities in the canal as it existed at the time of the observations, could be computed approximately from:

$$V = C\sqrt{H} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

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[7] "Hydraulics of the Cape Cod Canal", April 11, 1933. (Unpublished Departmental report.)

in which V is the velocity, in feet per second; H is the effective head corrected as described previously; and C has values as follows:

Definition of velocity, V	Value of C
Average over entire section	1.75
Average in the center vertical	2.24
Maximum in the section	2.40

These coefficients were determined for the canal as it was in 1932 (with project dimensions of 100 by 25 ft), and the proper values for a different cross-section can scarcely be determined analytically in advance of construction to that size of cross-section.

For predicting velocities in an enlarged canal, the method employed by Maj. Willis E. Teale, Corps of Engineers, U. S. Army, as developed from his analysis of both the Parsons' and the 1932 observations,[8] may be used. He concluded from this analysis that (with slope defined as the over-all slope determined from the head, corrected for acceleration only, between the recording gauges near the ends of the canal; and with R defined as the mean of the hydraulic radii of the interior sections of the canal) the Ganguillet-Kutter formula gives the then existing (1932) velocities in the canal with any required degree of accuracy when a value of $n = 0.0336$ is adopted. The correction for acceleration applied to the head between the recording gauges is $\frac{a\,L}{g}$, in which a is acceleration, in feet per second per second; L, the length of the section; and g, the acceleration due to gravity. For increasing velocities the acceleration head is subtracted from (for decreasing velocities added to) the over-all head. This correction for acceleration can be disregarded in computing the maximum velocities, inasmuch as the maximum head without acceleration correction is usually identical in value with the maximum corrected head.

Tide and Current Observations, 1934.—Dredging for widening the canal to a bottom width of 170 ft began October 31, 1932, the project depth remaining the same—25 ft. A much less extensive series of tide and current observations than the 1932 observations was made in September, 1934. At the time of the beginning of these observations the canal had a bottom width of at least 170 ft from the east end to Station 260, and a project bottom width of 100 ft from Station 260 to the west end. During the progress of the observations the west end of the dredged cut was advanced from Station 260 to Station 282 (see Fig. 2). With the canal 170 ft wide for about 70% and 100 ft wide for the remaining 30% of its length, it was naturally anticipated that the surface slopes would be steeper, and, in consequence, the current velocities higher, in the narrow part.

The observations were made in order to: (1) Determine the maximum velocity in the, as yet, unwidened part; (2) obtain data on the distribution of velocity within the cross-section with a bottom width of 170 ft; and

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[8] Cape Cod Canal—Analysis of September–October, 1932, "Velocity Observations", June 7, 1933. (Unpublished Departmental report.)

(3) determine the value of the roughness coefficient for the 170-ft section. Both the work in the field and the office computations were under the immediate direction of Donald F. Horton, Jun. Am. Soc. C. E.

The current observations were made from one launch from which were suspended two current meters of the identical type used in the 1932 observations. The Survey Section of the 1st New York District, U. S. Army Engineers, again supplied the equipment and technical assistance. Tide staffs were set up at the same locations as in 1932 and were observed as required to obtain data for the accomplishment of the specific objects outlined herein.

To determine the maximum velocity in the narrow section, observations were made at the center of the canal with one meter at three-tenths, and the other at seven-tenths, the depth. Two points had been selected as the probable locations at which the highest velocity would occur, and observations during two tides were made at each point. The highest velocities were observed at Station 346 and are as given in Table 9.

TABLE 9.—HIGHEST OBSERVED VELOCITIES AND TIDE HEADS (1934).

Direction of current	Tide head, between ends of canal, in feet	Velocity at 0.3 depth at center of canal, in feet per second
East to west	6.35	10.1
West to east	6.80	8.2

In order to eliminate minor fluctuations the velocities tabulated from the records were taken as averages over 15 min. The master current station of the 1932 observations (Station 225) was selected for the observations to obtain data on the distribution of velocity in the 170-ft cross-section. This location was chosen primarily because the canal alignment is a tangent for a considerable distance in each direction; it is also nearly mid-way between the ends of the canal. Nine measuring verticals were selected; one at the center, the others spaced 20 ft apart across the channel width. Velocity curves were observed at each of these verticals by means of the two current meters, one suspended constantly at three-tenths the depth, while, with the other meter, the velocity at each tenth the depth from surface to bottom was observed for not less than 7.5 min. The results obtained in each vertical, therefore, were a value of the velocity (average record of 7.5 min) at each tenth of the depth, and simultaneous values of the velocity at three-tenths the depth. Since the velocity is continually changing, the observed velocities at each tenth the depth were reduced to a common value of the velocity at three-tenths the depth by means of the simultaneously observed values. A vertical velocity curve was thus obtained for each of the nine verticals.

The determination of the variation in velocity across the channel was made by observations at three-tenths and seven-tenths the depth for not less than 7.5 min, successively, at each vertical. During the meter observations at each vertical the center surface velocity was determined by means of surface floats. The series of observations was made as expeditiously as possible during the period from 1 hr before, to 1 hr after, the time of maximum current, and was repeated the next day in the same manner when the water was flowing in the opposite direction. From these observations the average ratio of mean velocity of the entire cross-section to mean velocity in the center vertical was determined for four different elevations of the water surface and was found to be 0.87.

From these ratios and the relationship between the observed velocities and the mean velocity of the center vertical (which mean was determined from the vertical velocity curves observed at the center vertical), reduction factors for the convenient determination of mean velocity of the cross-section directly from the observed velocities were computed for use in:

$$V_m = F_1 V_{0.3} + F_2 V_{0.7} \qquad (4)$$

in which V_m = the mean velocity of the cross-section; $V_{0.3}$ = the observed velocity at 0.3 depth at center; $V_{0.7}$ = observed velocity at 0.7 depth at center; and, F_1 and F_2 are reduction factors, the average values of which, as determined for four different elevations of the water surface, were found to be $F_1 = 0.51$ and $F_2 = 0.31$. The average ratio, $\frac{V_m}{V_{0.3}}$, was determined to be 0.76.

For the determination of the roughness coefficient in the 170-ft canal, the reach from Station 180 to Station 275 (Fig. 2) was selected. Tide staffs were observed at Stations 180, 225, and 275, and current meter observations were made at the center of the canal at Station 225, with one meter at three-tenths, and the other at seven-tenths, the depth. These observations were made during the period from 1.5 hr before, to 1.5 hr after, the time of maximum velocity for both directions of the current. From these observed velocities (each an average of 15-min record), by application of the reduction factors already determined, the corresponding mean velocities of the section were computed. From the tide-staff observations, the average of the hydraulic radii within the reach under consideration, and the water surface slope corresponding to each of these mean velocities were determined. (The water surface slope was determined from the carefully observed difference in elevations between Station 180 and Station 275 corrected for acceleration.) By substitution of these known values in the Ganguillet-Kutter formula, a value of n for each of the twenty mean velocities determined from the observations was computed. The average of these values of n was 0.037. The methods used in analyzing the results of these observations followed, in general, very closely those used by Major Teale in his analysis of the 1932 observations.[8]

PREDICTION OF VELOCITIES IN AN ENLARGED CANAL

The flow of water in the Cape Cod Canal being variable (velocity and water surface continually changing), as well as varied (non-uniform), an exact analytical determination of the velocities to be anticipated in an enlarged cross-section cannot be made by any means known to the writer. Since the maximum velocity during a given tide is of paramount practical importance, the problem may be simplified in some respects by restricting it to the approximate determination of the maximum velocity resulting from a given tide head. Since the general equation[9] for variable-varied flow, namely,

$$S = \frac{V^2}{C^2 R} + \frac{d}{dx} \frac{(V^2)}{2g} + \frac{1}{g} \frac{\delta v}{\delta t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

differs from the general equation for varied flow only in the term, $\frac{1}{g} \frac{\delta v}{\delta t}$, which is zero at the instant of maximum velocity, the problem, as thus restricted, may be considered, for all practical purposes, one of steady varied flow in which the water surface elevations at the ends of the canal are given and the resultant discharge and velocity are desired. In Equation (5), S = slope; R = hydraulic radius; and t = time.

The ultimate size of cross-section which has been proposed is that with a bottom width of 500 ft, depth below mean low water of 40 ft (with 1 ft overdepth) and side slopes of 1 on 2.5. The maximum velocities to be anticipated at a point approximately midway between the ends of the canal (Station 225) with a cross-section of this size were computed by solution of the varied flow equation for a number of combinations of water surface elevations at the ends. The method solving the varied flow equation developed by Boris A. Bakhmeteff,[9] M. Am. Soc. C. E., was used for flow from west to east. In the other direction, flow takes place against an adverse or negative slope. In this case, the method developed by Arthur E. Matzke, Jun. Am. Soc. C. E.[10] (from equations derived by Professor Bakhmeteff), was used.

In the determination of the varied flow factors used in both these methods of solution, the Ganguillet-Kutter formula with coefficient of roughness of 0.035 was used.

The velocities first derived were obtained by dividing the computed discharge by the area of cross-section at Station 225, and, therefore, were mean velocities of the section. Mean velocity, however, is not of direct interest to navigators. The velocity at three-tenths the depth at the center—which roughly approximates in value the maximum velocity in the section—is believed (for most vessels which will use the canal) to be a better criterion of the extent to which a vessel may be affected by the current. The mean velocities first computed, therefore, were reduced to values corresponding to velocity at three-tenths the depth at the center by the application of the factor, 1.31, determined from the 1934 observations. The results are given in Table 10. These results agree closely with those obtained from the application of the method used by Major Teale previously described.[8]

Theoretically, since the surface slope is not parallel to the bottom slope, the highest velocity resulting from a given tide head would not occur at Station 225, but at one end of the canal, depending on the direction of the current. Actually, it is probable that this effect will be somewhat obscured, as it was observed to be in the existing canal, by unavoidable irregularities resulting from dredging operations.

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[9] "Hydraulics of Open Channels", by Boris A. Bakhmeteff, M. Am. Soc. C. E., Eng. Societies Monograph, 1932.

[10] "Varied Flow in Channels of Adverse Slope", by Arthur E. Matzke, *Proceedings*, Am. Soc. C. E., February, 1936, p. 193.

Assuming the same ratio between mean velocity of the section and velocity at three-tenths the depth at the center, as determined at Station 225, these highest velocities would be as given in Table 10(*b*).

TABLE 10.—Predicted Velocities in the Center of the 500 by 40-Foot Canal

Tide head, in feet	Maximum Velocity at 0.3 Depth				Maximum Velocity at 0.3 Depth			
	From West to East		From East to West		From West to East		From East to West	
	In feet per second	In knots	In feet per second	In knots	In feet per second	In knots	In feet per second	In knots
	(*a*) At Station 225				(*b*) Highest Velocities at Any Section			
6.0	8.8	5.2	9.2	5.4	9.2	5.4	10.2	6.0
6.5	9.2	5.4	9.4	5.6	9.6	5.7	10.7	6.3
7.0	9.4	5.6	9.9	5.8	10.0	5.9	11.1	6.5
7.5	9.8	5.8	10.2	6.0	10.5	6.2	11.5	6.8

Since a comparable velocity of 10.1 ft per sec occurred with a tide head of 6.4 ft in the narrow section of the existing canal during the widening to 170 ft, it appears that the maximum velocities to be anticipated in a 500 by 40-ft canal will not greatly exceed those which have been actually experienced with similar tidal conditions in the short restricted 100-ft section before the completion of the 170-ft widening program. Velocities in the proposed 540 by 32-ft canal should be approximately 7% less than in the 500 by 40-ft canal.

The frequency of tide heads of the magnitudes listed in Table 10 have been approximately as follows: Higher than 6.0 ft, 15% of tides; higher than 6.5 ft, 10% of tides; higher than 7.0 ft, 5% of tides; and higher than 7.5 ft, 1% of tides. The phase relationships of the tides at the ends of the canal are such that the effect of the increased discharge through an enlarged canal will be to decrease the frequency with which heads of a given magnitude occur. It is probable, for example, that with the 500 by 40-ft canal, tide heads higher than 6.5 ft will occur during less than 10% of the tides.

The computations described herein, of velocities to be anticipated in an enlarged canal, were based on the tidal data observed with the 100 by 25-ft canal, because the changes in tidal conditions resulting from the increased discharge through an enlarged canal cannot be determined analytically with certainty. Furthermore, the exact value of the coefficient of roughness (n in the Ganguillet-Kutter formula) is not known definitely. It is thus seen that, for these reasons, if for no others, the predicted velocities can be considered approximate only and are possibly in error by as much as 10 per cent.

MODEL STUDIES

Of practical importance in the control of the dredging operations for the enlargement of the canal is the locus of the plane of mean low water which will obtain after the completion of the enlargement, since the depth is to be determined with reference to that plane. It is anticipated that the increased discharge through an enlarged canal will change the elevation of mean low water in Buzzards Bay and in the canal proper, but no analytical method of determining this change quantitatively is known. Another problem of practical importance is the design of the proposed mooring basins, to eliminate as far as possible undesirable and dangerous eddies, which would not only menace shipping, but also would increase the difficulty and cost of maintenance. Both these problems are considered suitable for analysis by means of model studies. Therefore, it is believed desirable to undertake model studies of the enlarged canal for the purpose of studying these two problems. At the same time, determinations of velocity would be made as a check of the computed velocities, and with little additional cost, the effects of regulating works in Buzzards Bay may also be studied.

The proposed model of the enlarged canal would necessarily include enough of the upper end of Buzzards Bay to insure reaching a point at which the tide will not be affected by the increased discharge through the canal. No difficulty in this respect is anticipated for Cape Cod Bay since its characteristics are such that the increased discharge will not be large enough to affect the tide therein appreciably. Preliminary computations indicate that the area which should be reproduced will necessitate, with the scale tentatively selected, a model approximately 110 ft long.

During the investigation of a lock canal, two model studies were made. The first of these was undertaken by Thomas A. Lane, Jun. Am. Soc. C. E., and Lt. John L. Persons,[11] Corps of Engineers, U. S. Army. The purpose of the study was to determine the amount of depression below sea level of the water surface at the dock. This problem is of practical importance in the determination of the datum plane for dredging and of the design elevation of the lock sill. The study was made with a concrete model, approximately 40 ft long (scales: Horizontal, 1:1 000; vertical, 1:49). The results indicated that the depression of the water surface at the lock would be so small that for all practical purposes it could be neglected. This conclusion confirmed that reached by computations made in the office of the Boston District, U. S. Army Engineers, using the methods developed by Colonel Brown and General Parsons, to which reference has previously been made.

Model tests were also made to determine the most satisfactory design of the filling system for the lock. These tests were made by George R. Rich, M. Am. Soc. C. E.,[12] on a model to a scale of one-fortieth of a lock with a chamber 110 ft wide, 1 000 ft long, and 40 ft deep. Five different conduit designs were tested for filling time and effect on a vessel in lockage. The test ship used was a model of a standard cargo vessel, 600 ft long, with 75-ft beam, loaded to 35 000 tons displacement.

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[11] "An Experimental Study of Tidal Action in a Lock Canal". Thesis presented to Mass. Inst. Tech., in May, 1932, in partial fulfillment of the requirements for the degree of Master of Science.

[12] "Model Tests of the Navigation Lock for the Cape Cod Canal", July 9, 1932. (Unpublished Departmental report.)

DISCUSSION

C. S. JARVIS,[13] M. AM. SOC. C. E. (by letter).—On a project that required about 260 yr between its serious inception and the first construction work, and another 30 yr, or more, before it was opened to traffic at the outbreak of the World War, it is not surprising that many questions and problems require serious consideration before radical enlargements, revisions of alignment, and structural improvements of the Cape Cod Canal are authorized. The value of this project to navigation, as satisfactorily demonstrated during its first few years of operation, from the standpoint of time and distance saved as well as hazards avoided, led naturally to its being taken over by the Federal Government. To measure up to national standards for maintaining and promoting commerce by navigation, it was found necessary to enlarge the cross-section, revise the curvature and some other features of alignment, and to provide other measures of safety for vessels beset by fog and storm.

The restricted depth and width of the canal as originally constructed, together with the high velocities occasionally attained, due to tidal differences prevailing in Cape Cod and Buzzards Bays, were directly accountable for several mishaps to vessels undertaking to use the new waterway. Although the losses were relatively small as compared with the disasters avoided along the outside route, it became imperative to enlarge the cross-section and to improve the alignment of the canal. During the preparation of plans for this improvement, it developed that some vital differences of opinions existed among those responsible for the design, review, and supervision of the work; for example, it seemed to be regarded as axiomatic among some official circles, that the increase of cross-section would result in a considerable increase of velocity due to tidal action. To others, it was just as evident that there should be no pronounced change of velocity due to doubling or trebling the bottom width, with some increase in depth. In fact, the opinion was even ventured that if the widening were carried to great extremes, say 800 or 1 000 ft, the tides in Cape Cod Bay and Buzzards Bay would occur more nearly in synchronism, with decreased differences in water-surface elevation and velocities of flow. The apparent proof of this assumption is to be found in a consideration of what would be the condition if the two bays were connected by a very wide and unrestricted channel of great depth. Unquestionably, their tidal phases would thereafter be in fairly close agreement.

Investigations by Earl I. Brown, M. Am. Soc. C. E., and the late Harry Franklin Flynn, M. Am. Soc. C. E., in connection with the proposed New Jersey Section of the Intracoastal Waterway, to connect the Delaware and Raritan Rivers through a 31-mile land cut, demonstrated that enlarging the cross-section reduced the velocity of flow. The dimensions of the cross-sections compared were approximately those of the original and the recently authorized enlargement of the Cape Cod Canal, but somewhat greater in each case. If the aforementioned investigations and demonstrations are accepted as conclusive, it appears questionable whether the proposed enlargement to 540 by 32 ft will result in velocities already experienced in the original cross-section, or observed during the process of widening to the present 205-ft bottom width. From analogy, it seems to be fairly well assured that the wider section will accommodate a sufficiently greater volume of water interchanged from the two ends of the canal to reduce the tidal differences and the period of lag by measurable and considerable amounts, with a corresponding reduction in velocity of flow.

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[13] Hydr. Engr., Soil Conservation Service, Washington, D. C.

In view of the cost estimates published in Table 6, showing that the 540 by 32-ft canal section should cost approximately $4 300 000 less than a 250 by 32-ft canal with twin locks, it would be interesting to trace back among the earlier estimates in which the financial advantage was generally claimed for the lock canal. The project, the Engineering Profession, and the Federal Service are all greatly indebted to the Division Engineer of the North Atlantic Division, Col. G. M. Hoffman, Corps of Engineers, U. S. Army, who was largely responsible for the modification or reversal of prevailing opinion as to the proper procedure. His plea was for progress along lines that have been proved or that are readily susceptible of proof. As a result, the lock construction was delayed during the preliminary widening of the canal; and the intensive study of canal velocities as related to tidal differences in Cape Cod and Buzzards Bays resulted in significant data to prove that, with sufficient widening, the velocities would be found tolerable and the locks unnecessary. On the other hand, if the trend of evidence and opinion had continued to favor the construction of the locks after partial widening, then the structural work could have continued with more assurance, as the result of more mature consideration.

During the hydrologic observations of canal velocities simultaneously at designated stations, both before and after widening during recent years, notable progress was achieved in the art of gauging stream velocities with relation to water-surface elevations. In addition to the excellent controls introduced, as well described in the paper under appropriate headings, special notice should be accorded the practical use and value of the automatic graphic recorder specially devised in the Engineer Office for such work. It was the writer's privilege, after participating in the preliminary studies, to observe the apparatus, some of the records, and the methods of developing and correlating the data. Unusual skill and ingenuity were displayed by those responsible for the operation.

The orderly presentation of historical and technical data relating to the Cape Cod Canal, especially at this time, is of great value to the profession. The author is to be commended for his painstaking effort.

George R. Rich,[14] M. Am. Soc. C. E. (by letter).—The application of the harmonic theory advanced by the late General Parsons[2], or the reflected wave theory defined by Colonel Brown[5], to the determination of velocities and surface elevations at several selected cross-sections of an open waterway, such as the Cape Cod Canal, connecting two seas having substantial differences in tidal phase and amplitude, is both complicated and laborious. Moreover, as stated by Captain Harwood, such involved calculation is unwarranted in many cases because of the paucity of available data upon which to base the selection of the pertinent physical constants.

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[14] Hydr. Engr., U. S. Engr. Office, Boston, Mass.

However, in computing the action of a navigation lock, located some appreciable distance from the entrance to Cape Cod Bay, in raising the elevation of high tide and depressing the elevation of low tide at the lock site, relative to similar Cape Cod Bay levels, the writer has found that the basic equations of General Parsons may be adapted to give a rational yet comparatively simple means of estimating the hydraulic effect on the canal in advance of conducting confirmatory model tests. For example, the two general equations defining the oscillatory flow are:

$$\xi = e^{px}\left[C\cos\left(\frac{2\pi t}{T}+qx\right)+D\sin\left(\frac{2\pi t}{T}+qx\right)\right]$$

$$+\,e^{-px}\left[C'\cos\left(\frac{2\pi t}{T}-qx\right)-D'\sin\left(\frac{2\pi t}{T}-qx\right)\right]\ \ldots\ldots(6)$$

and,

$$h = -d\,e^{px}\left[(Cp+Dq)\cos\left(\frac{2\pi t}{T}+qx\right)+(Dp-Cq)\sin\left(\frac{2\pi t}{T}\right.\right.$$

$$\left.\left.+\,qx\right)\right]+d\,e^{-px}\left[(C'p-D'q)\cos\left(\frac{2\pi t}{T}-qx\right)\right.$$

$$\left.-\,(D'p+C'q)\sin\left(\frac{2\pi t}{T}-qx\right)\right]\ \ldots\ldots\ldots\ldots\ldots(7)^{15}$$

in which, in addition to the notation of the paper, ξ = the displacement, or distance traveled by a particle of water in time, t, in feet; h = the height of the water surface at any section, x, referred to the elevation of mean tide, in feet; t = the time, in seconds, referred to the time of mean tide preceding high water; and x = the distance of any cross-section from the entrance to Cape Cod Bay, in feet.

The following are physical constants, determined by the conditions of the given particular problem: T = the time, in seconds, between successive high tides; e = the base of the natural system of logarithms; A = the area of the average cross-section of canal, in square feet; w = the unit weight of the liquid, in this case 64 lb per cu ft for sea water; X = the wetted perimeter of the average cross-section of canal, in feet; d = the reduced depth of the canal, in feet; that is, the depth of a rectangular section having the same area and top width as the average actual section, measured from the elevation of mean tide; and f_0 = a coefficient such that the friction force, F, per unit of surface is proportional to the nth power of the velocity so that $F = f_0\,v^n$, or $F = f_0$ for unit velocity.

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[15] The second minus (—) sign from the end of this equation is erroneously given as plus (+) in *Transactions*, Am. Soc. C. E., Vol. LXXXII (1918), p. 113, Equation (12).

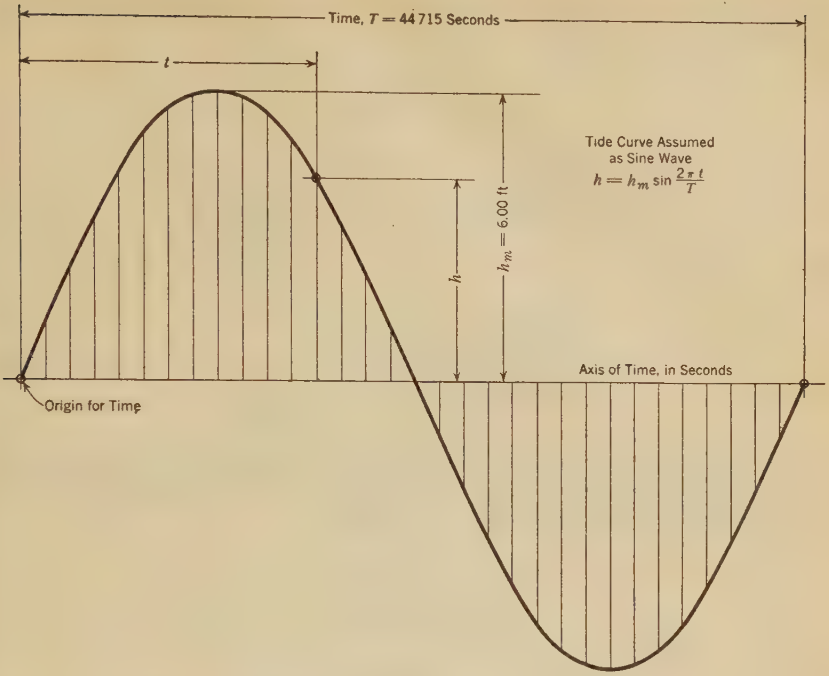

FIG. 12.—ASSUMED TIDE CURVE, CAPE COD BAY, AS A SINE WAVE, $h = h_m \sin \frac{2\pi t}{T}$.

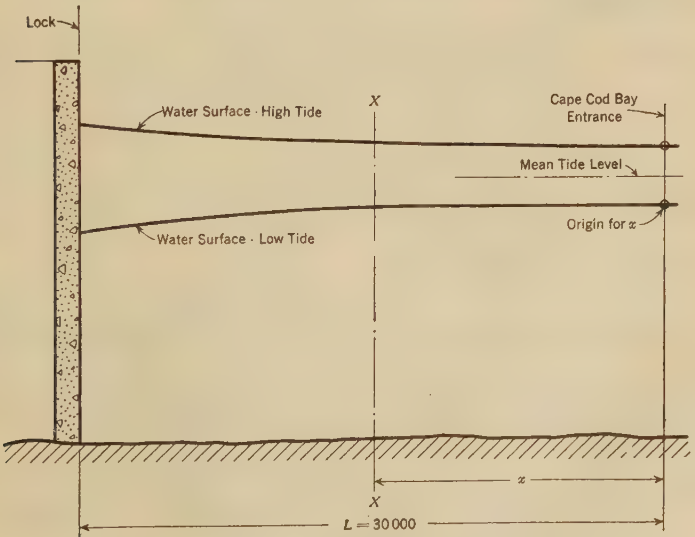
Lock
Water Surface · High Tide
X
Cape Cod Bay Entrance
Mean Tide Level
Origin for x
Water Surface · Low Tide
x
X
$L = 30\,000$

FIG. 13.—SCHEMATIC ELEVATION OF LOCKED CANAL.

For a locked canal with velocities of the order of 1 ft per sec, or less, $f_0 = 0.00756$ lb per sq ft; or,

$$f = \pm \frac{g\, f_0\, X}{w\, A} \quad \text{.........................(8}$$

$$p = \frac{\left(\frac{2\,\pi}{T}\right)}{\sqrt{2\,g\,d}} \sqrt{-1 + \sqrt{1 + \frac{f^2}{\left(\frac{2\,\pi}{T}\right)^2}}} \quad \text{..............(9}$$

and,

$$q = \frac{\left(\frac{2\,\pi}{T}\right)}{\sqrt{2\,g\,d}} \sqrt{1 + \sqrt{1 + \frac{f^2}{\left(\frac{2\,\pi}{T}\right)^2}}} \quad \text{..............(10)}$$

The symbols, C, C', D, and D', are constants of integration to be determined from the terminal conditions of each particular case. For a locked canal they are determined as follows (see Figs. 12 and 13):

When $x = 0$,

$$h = h_0 \sin \frac{2\,\pi\,t}{T} \quad \text{.........................(11)}$$

and, when $x = 30\,000$,

$$v = \frac{d\xi}{dt} = 0 \quad \text{..............................(12)}$$

Differentiating Equation (6) with respect to time, t, to obtain a general equation for velocity:

$$\frac{d\xi}{dt} = e^{p\,x} \left[-\frac{2\,\pi}{T}\, C \sin \left(\frac{2\,\pi\,t}{T} + q\,x\right) + \frac{2\,\pi}{T}\, D \cos \left(\frac{2\,\pi\,t}{T} + q\,x\right)\right]$$
$$+ e^{-p\,x} \left[-\frac{2\,\pi}{T}\, C' \sin \left(\frac{2\,\pi\,t}{T} - q\,x\right) - \frac{2\,\pi}{T}\, D' \cos \left(\frac{2\,\pi\,t}{T} - q\,x\right)\right] \text{...(13)}$$

The next step is to compute the physical constants (see Fig. 14). Let $X = 444$ ft; $A = 12\,250$ sq ft; $w = 64$ lb per cu ft; $g = 32.2$ ft per sec²; $d \doteq 28.5$ ft; $L = 30\,000$ ft; $\frac{2\,\pi}{T} = \frac{2\,\pi}{44\,715} = 0.00014$ radian per sec; $f_0 = 0.00756$ lb per sq ft; $f = 0.000138$; $p = 0.00000208$; and $q = 0.00000507$.

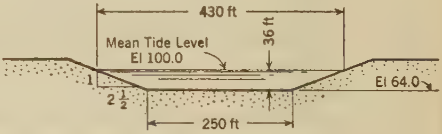

FIG. 14.—ASSUMED AVERAGE CROSS-SECTION LOCKED CANAL.

Substituting the values of the foregoing physical constants and making $x = 30\,000$ and $\frac{d\xi}{dt} = 0$ in Equation (13); then setting $\frac{2\,\pi\,t}{T}$ equal, successively, to 0 and $\frac{\pi}{2}$, two simultaneous equations are obtained involving C, C', D, and D'.

Next, substituting the values of the physical constants and making $x = 0$, $h = 6.00 \sin \frac{2\pi t}{T}$ in Equation (7); then setting $\frac{2\pi t}{T}$ equal, successively, to 0 and $\frac{\pi}{2}$, two additional equations are obtained in C, C', D, and D', giving $C = 1.77 \times 10^4$; $C' = -1.77 \times 10^4$; $D = -0.40 \times 10^4$; and $D' = -1.07 \times 10^4$.

With the foregoing constants of integration determined and substituted in Equation (7), the maximum value of h for $x = 30\,000$ is obtained by first placing $\frac{dh}{dt} = 0$, solving for t, and then computing h from Equation (7) for the value of t thus obtained, giving $h = \pm 6.10$ ft for $t = 20$ min after $\frac{T}{4}$ and $\frac{3T}{4}$. In other words, maximum high water and minimum low water at the lock site occur shortly after the time of occurrence of the corresponding levels at the entrance to Cape Cod Bay.

Similarly, with the constants of integration substituted in Equation (13), the maximum value of velocity, $\frac{d\xi}{dt}$, for $x = 0$, is obtained by placing $\frac{d^2\xi}{dt^2} = 0$, solving for t, and then computing $\frac{d\xi}{dt}$ from Equation (13) for the value of t thus obtained, giving $\frac{d\xi}{dt} = \pm 0.94$ ft per sec for $t = 0$ and $t = 22\,358.5$ sec. In other words, the maximum velocity at the entrance to Cape Cod Bay occurs close to the time of mid-tide.

Since the original objective in making the foregoing study was to determine the order of magnitude of the super-elevation and depression of tidal heights at the lock, rather than the refinement in the numerical result, slide-rule computations were used almost exclusively; and hence no claim is made to great accuracy in determining the critical times, which may be in error 10 min, or more. However, the following series of observations, brought to the attention of the writer by Lt. Col. Richard Park, Corps of Engineers, U. S. Army, then District Engineer at Boston, Mass., regarding a similar situation which occurred at the time when the Chesapeake and Delaware Canal was being lowered to sea level, tends to confirm the reliability of the method of analysis. The Delaware end of the Canal was opened to the tide for a distance of about three miles up to the lock at St. Georges. For a period of

TABLE 11.—COMPARISON OF OPEN AND DEAD-END CONDITIONS, CHESAPEAKE AND DELAWARE CANAL

Location	High-water lunitidal interval, in hours	Height, in feet	Low-water lunitidal interval, in hours	Height, in feet	Range, in feet
Reedy Island	10.83	5.96	5.46	0.46	5.50
Reedy Point	11.19	5.90	5.59	0.53	5.37
St. Georges (February, 1927)	11.47	6.01	5.96	0.45	5.56
St. Georges (April, 1927)*	11.44	5.57	5.95	0.67	4.90

* After canal was cut through.

about two weeks this was as far as the tide could be propagated, and advantage was taken of the opportunity to set a gauge in the lock, so as to permit comparison of its readings with those of the Delaware River gauges. As shown in Table 11, the St. Georges ranges were perceptibly higher until the canal was cut through.

E. C. Harwood,[16] Esq. (by letter).—Various interesting aspects of the Cape Cod improvement program have been mentioned by the discussers, and the only phase that seems to require further clarification is the reference by Mr. Jarvis to the change from earlier estimates in which, to use his words, "* * * the financial advantage was generally claimed for the lock canal."

Although no published data are available, various estimates, together with general assertions in reports, certainly justify Mr. Jarvis' statement that earlier estimates had claimed a financial advantage in favor of the lock canal. The writer's estimate for the open canal, 32 ft deep, with a bottom width of 540 ft throughout the land cut, was $23 172 000, whereas the corresponding estimate for a lock canal was $27 499 000. The marked change shown by the estimates presented in the paper is traceable to two major features, each involving approximately the same sums of money.

In the first place, estimates for twin locks complete (110 ft by 1 000 ft), as originally presented by the Engineer Department, allowed a total of $8 940 000 for these features of the work. Subsequent to the publication of those estimates[17], extensive model studies were undertaken and new estimates were prepared. The result was a revision upward of the estimated cost of such locks, especially taking into consideration the difficult foundation conditions as revealed during the construction of the new bridges at the Cape Cod Canal. The increase in this part of the estimate was $2 560 000, making the total estimated cost of the twin locks, $11 500 000. This is the value given in the paper.

The second important feature which resulted in a marked reduction of cost estimates for an open canal was a reconsideration of the possibilities of excavating "in the dry." A careful study of earlier data, including the paper on this subject presented by General Parsons,[2] led the writer to believe that the excavation "in the dry", to the maximum extent possible would result in marked savings. The estimates prepared with that idea in mind show a saving in the amount of $2 703 000.

It will be noted that the sum of the two amounts just mentioned (that is, $5 263 000) is substantially in excess of the saving shown for an open canal according to the estimates in the paper. Although the figures are not precisely the same as would be found in earlier studies, it is plain that in the absence of these changes the lock canal would have been the more economical of the two so far as first cost is concerned, at least.

Of course, it will always be impossible to test the accuracy of the estimates for the cost of constructing twin locks, but readers will be interested to know that the estimated rate of savings for excavation "in the dry" has actually been realized in the work done to date (1936).

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[16] Capt. Corps of Engrs., U. S. Army, Boston, Mass.

[17] House Doc. No. 795, 71st Cong., 3d Session.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 1954

THE ENGINEER AND HIS CODE

ADDRESS AT THE ANNUAL CONVENTION, PORTLAND, OREGON, JULY 15, 1936

BY DANIEL W. MEAD,[1] PRESIDENT AND HON. M. AM. SOC. C. E.

> "The President shall * * * deliver an address at the Annual Convention." (Society By-Laws, Article 3, Paragraph 2.)
>
> "The Engineer shall endeavor to assist the public to arrive at a correct general understanding of the technical phases of public questions. He shall discourage and challenge untrue, unfair, and exaggerated statements on technical subjects, especially when such statements may lead to unworthy or uneconomical public enterprise." (Society Code of Practice, Part V, No. 3.)

Great crises frequently bring into play high ideals and are met by great sacrifices, even from those from whom such action would be least expected. In the stir and excitement of such events the spirits of men are exalted to meet, undaunted, almost any contingency. But when the excitement subsides, when the crisis is over, the selfish spirit of personal advantage returns; the love of country again becomes subordinated to the love of self.

The persistent exercise of the higher ideals, as a basis of life and action—which is often of even greater importance than a supreme sacrifice to the continued development of civilization—requires continued and unremittent effort which human nature, in the main, is apparently too weak to sustain. It has been said that it is easier to die for one's country than it is to live for one's country.

War always seems to bring out both the best and the worst in the people of a nation. In the World War in which we engaged "to save the world for democracy", hundreds of thousands willingly took up arms; thousands gave their health and their lives; and millions laid aside their own affairs and eagerly accepted service for the common good. After the Armistice, in many cases these high ideals rapidly faded away; the motive of personal advantage returned almost immediately, often greatly accentuated; action and reaction were equal. In the back-wash of post-war events, the worst frequently came to the fore; sorrow, suffering, and sacrifice had exhausted the patience and forebearance of the people of many nations. Democracies were born and hastily abandoned because they did not immediately entail Utopia. In some nations liberty has been destroyed and dictatorships have been established, throwing additional doubts on an uncertain future.

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[1] **Prof. Emeritus, Hydr. and San. Eng.**, Univ. of Wisconsin; Cons. Engr., Madison, Wis.

In our own country, recovery and re-adjustment seemed at first to be rapid, but it was a false recovery, based on extravagant hopes and resulting speculation. Low moral standards seemed to have gained the ascendency in political, financial, social, business, and, to an extent, in professional life. Such low ideals can by no means be entirely attributed to the war and its consequences for many such ideas antedate that event by many years.

Corporation and financial ethics have been for many decades in a chaotic condition. The history of many corporations, dating back over a long period, brought to light corporate practices in perhaps their worst phases and, while the most reprehensible of the practices on which these organizations were erected, have been eliminated, further restraint seems necessary for the welfare of the nation.

This century, and the closing years of the last century, witnessed the creation of many corporations which at times secured such great financial power and political control that they were able to ride roughshod over the rights of individuals and communities. The impersonality of such corporations has led their officials to disregard human rights and frequently human laws and to exercise their powers, to the great detriment of the nation, in ways which even the members of such corporations would never as individuals have desired or have dared to attempt.

These examples have had a depressing effect on the ethical standards not only of those exercising these powers but of many business and professional men who have been associated with them or who have been affected by them until the "law of the jungle" seemed at the time to have become the rule of business.

During this same period, however, there has been a development of higher ideals in the thoughts of the best professional and business men. The idea of the necessity of "service" as a sound fundamental requirement of professional and business activities has become dominant. Many codes of ethics and rules of conduct have been adopted, and although many who have recognized the desirability and necessity of such codes have not always observed them, when they could be ignored to individual advantage without too severe a reaction, there has been a distinct improvement in ethical ideals Such codes always have been and perhaps always will be objective ideals to be approximated as nearly as practicable. Some of the principles declared in these codes, however, are so important that their observation is essential to common decency and honesty, and the lack of their observance should disbar the offender from his association with honorable men. Such are the principles embodied in the present code of the American Society of Civil Engineers.

During these last decades the Federal and the State Governments have passed regulatory laws and have organized utility, rate, railroad, and trade commissions which have been able to eliminate many objectionable corporation practices, and the way seemed open to a gradual and satisfactory adjustment of professional and trade practice.

The objectionable practices of the past, however, under the immediate excuse of the depression, are the apparent cause of the Federal Government attempting to take over the control of the business of the country, and are the only apparent excuse for the construction of competing power plants built by the Federal Government itself and by the Federal financing of competitive plants constructed by municipalities.

It must be remembered that these private corporations are organized under law. Their securities were issued under the supervision of the States, and such securities have been sold under the laws of the States to millions of innocent purchasers who have had nothing to do with the management of the corporations. While the States have made no guaranty of permanency or of adequate returns on investments, they did to all intents and purposes guarantee a fair deal and honorable treatment to the corporations and to the investors. It cannot be maintained that our Government has met these conditions on any higher ethical plane than that which previously had been used by business.

I hold no brief for those who favor the law of the jungle in finance, in business, or in government, and I am entirely in sympathy with every reasonable effort to control improper individual and corporate initiative for the benefit of our social advancement. I cannot remain neutral when I believe that recovery would respond immediately to the establishment of confidence in a fair and equitable governmental control and that under such condition unemployment would soon be a thing of the past.

Many engineers will find it difficult to square high ideals with conditions that now exist or with the new doctrine that they are now asked to accept. Although the very foundation of the Engineering Profession is based on the belief that improved production, improved facilities, and new inventions and discoveries will add to the happiness and welfare of the people of our nation and that such results brought about by the engineers of the country are worthy of the highest praise, the engineer is now asked to believe that, in the future the road to national advancement, to happiness, and to prosperity should, with the single exception of power supply, be based on a doctrine of scarcity and high prices. Under this doctrine the few will profit at the expense of the many, and the unfortunate will do without. Under this doctrine he who makes "two blades of grass to grow upon a spot of ground where only one grew before", is to be condemned rather than praised. Under this doctrine the days of development are past, and the engineer must confine his attention simply to maintenance and modification of the things that are. The doctrine is not only destructive to engineering ideals, but it is false and vicious. If I know the engineers of the United States, as I believe I do, they will not accept this doctrine of despair, this principle of pessimism.

Much still remains to be done for the further advancement and improvement of our country, for the protection and welfare of our people, for the happiness and contentment of our citizens. The work of the engineer is not completed; we shall go on to greater things in the future than the past has ever known.

The doctrine of plenty needs for its success only an adjustment in exchange and not a cessation of effort, in order that all may benefit. The problem of exchange can and will be solved, and with abundance the needy shall be fed not on the enervating basis of charity, except to the helpless, but on a fair basis of an exchange of effort that shall arouse a feeling of manhood and womanhood instead of apathy and dependence in a free and self-reliant people. There is even hope that the principles of economics and politics may some day be better understood and be placed on a uniform, sound, honest, and intelligent basis. On this doctrine of abundance the engineers of the nation will take their stand.

Since the birth of our nation there has never been a time when intelligence, sound judgment, high ideals, and an unwavering stand for honesty, fair play, and a square deal, have been more needed than at present, and such ideals and ideas should find their firmest support from the professional men of our nation.

The Code of Ethics of the American Society of Civil Engineers, which in effect is a code of laws for the members of our Society, and the codes of most other engineering societies, apply most directly to the small percentage of engineers in general practice, and only remotely to the large majority of the membership. It seems highly desirable that a code of ethics and of conduct, perhaps advisory rather than mandatory, should be adopted which will be applicable to our entire membership.

A code of ethics for the Engineering Profession is somewhat complicated by the fact that the profession contains many men who are not engaged exclusively in independent engineering practice nor employed by those in such practice; it includes also many engineers employed in public works—Federal, State, and municipal—as well as those who are officials of transportation, utility, manufacturing, or contracting corporations, or who are employed by such corporations. There are also engineers engaged in promotional and financial operations.

The American Institute of Architects requires that its members shall not personally be interested in the building trades or be under personal obligation to manufacturers or others whose products enter into work under their supervision. The American Institute of Consulting Engineers requires that its members shall be in independent practice and not be connected with contracting or promotion, except in a purely advisory or supervisory capacity.

The American Society of Civil Engineers cannot limit its membership in any such manner, as many of its most valued members comprise not only those who are engaged in independent engineering work, but also those in all the various activities just mentioned and in various combinations of those activities and who, in the course of time, change from one occupation to another. Therefore, it becomes difficult but even more essential to prepare a Code of Ethics which will be reasonably applicable to the entire fraternity of engineers.

Although high ideals are essential to moral progress in the development of civilization and for the future welfare of the nation and of the professions, it must be recognized that such ideals must be limited by the actual conditions of life. A supreme sacrifice is possible in a crisis, but long-continued service must involve some consideration of the one who serves if such service is to continue. "The laborer is worthy of his hire." Personal consideration of self, family, dependents, and the ordinary duties of life is also of primary importance in a practical world. The nice adjustment between service and self is the real difficulty in the way of the successful application of any ethical code. Such a code of ethics, therefore, should cover as far as possible the best thoughts on the duties of the engineer in all his activities. The profession is equally interested in the consulting engineer and the engineer in general practice, and in the engineer in public works, in the service of corporations, in various combined relations, in business, in expert legal work, and in the subordinate positions of the profession, and also in the student who is intending to enter the profession.

I am greatly interested in the engineering students for they are the men who will take the place of those of us whose professional work will soon be over. They are the men who in the future will make the profession great if they possess great ability and high ideals. The profession owes a duty to these young men to see that they appreciate the necessity of high attainments and that they are informed concerning both the privileges and the duties of our great profession. At the present time many of these young men are leaving their Alma Maters without any information, other than technical, as to their duties as citizens and as engineers, many are morally injured by unfortunate associates or employment, due to their ignorance of professional ideals.

The Educational Committee of the American Institute of Architects is apparently taking great pains to see that the important matters of professional relations are discussed in all architectural schools. While a knowledge of professional ideals is of great importance to all professional students, it seems to be doubly important to the student of engineering who is to enter a profession so complicated in its manifold relationships. The profession owes to these young students, to the public, and to the profession itself, a clear statement of the best thoughts and the highest practical ideals concerning the duties of the engineer in all his relations in life.

It is my greatest hope that by the careful preparation of such a code the opportunity will be available for all engineers to review frequently the ideals of their profession and improve their relationships with the public and with their fellow engineers and associates; that through the pressure of this Society and its other co-ordinate societies, an understanding of these ideals shall be brought to every student of engineering; and that ultimately the ideals of honor, integrity, and dependability shall predominate in all the relations of our great profession.

MEMOIRS OF DECEASED MEMBERS

CHARLES FREDERICK LOWETH, Past-President, Am. Soc. C. E.[1]

Died May 15, 1935

Charles Frederick Loweth was born on March 3, 1857, in Cleveland Ohio, the first son of Daniel and Mary Ann Pennington (Brown) Loweth. His education included attendance at the Bellevue High School and one year, 1876–77, at Oberlin College, Oberlin, Ohio, where he took courses in the natural sciences and mathematics. He began his engineering work in 1875, the year before he entered Oberlin, as a Chainman on surveys.

After his year in Oberlin, Mr. Loweth was employed as Rodman in 1877 and, later, became Assistant Engineer on the construction of the Wheeling Extension of the Cleveland, Tuscarawas Valley, and Wheeling Railway, later part of the Baltimore and Ohio Lines. In this work as Assistant to the Chief Engineer, the late William Warren Card, M. Am. Soc. C. E., he gained his first experience in tunnel work. During the year 1878, he was employed as a Draftsman by the American Bridge Company, of Chicago, Ill., under E. Hemberle, Engineer. Following that engagement Mr. Loweth was employed in making surveys for the entrance of the Wabash Railroad into Chicago, by the Chief Engineer, C. S. Masten. Then followed in 1879 and 1880 ten months as a Draftsman, "designing iron bridge and roof work", with the Edge Moor Iron Company, at Wilmington, Del.

From March, 1880, until May, 1881, he was with the Atchison, Topeka and Santa Fé Railroad Company, first as Draftsman, at Topeka, Kans., in the Department of Construction, and, later, in the Department of Track, Bridges, and Buildings. Mr. Loweth arrived at Topeka the day after Mr. George S. Davison, and both young men started as Draftsmen in the office of the late Howard Vernon Hinckley. The late Charles Le Roy Annan was also in the same office, and these four formed a congenial quartette. Later, they all became Members of the Society. Of this group George S. Davison, Past-President, Am. Soc. C. E., alone survives.

The following year, 1881–82, when he was only twenty-four years of age, Mr. Loweth was Principal Assistant Engineer, under the late Edward Cornelius Kinney, M. Am. Soc. C. E., Chief Engineer, of the St. Louis, Des Moines, and Northern Railway. In this position, he was in charge of the construction of High Bridge over the Des Moines River on the line of that railway. This structure was 2 020 ft long, and 101 ft above the water.

In May, 1882, he became Engineer for Raymond and Campbell, Bridge Builders, at Council Bluffs, Iowa, and, in 1883, Manager of the Branch Office of that firm at St. Paul, Minn. This engagement permitted him to enter into private practice, and Mr. Loweth became Consultant to cities and towns, including Mt. Pleasant and Fairfield, Iowa, and Eau Claire, Wis., making investigations, plans, and reports for water supply and sewerage. It was about this time that he made application for admission to the Society and was elected a Junior on January 3, 1883. He was then twenty-five years of age, but showed an early realization of the obligations of young engineers to their profession as well as the benefits they may gain by identifying themselves with the national organization connected with it.

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[1] Memoir prepared by Theodore L. Condron, M. Am. Soc. C. E., George S. Davison, Past-President Am. Soc. C. E., and Robert Ridgway, Past-President and Hon. M. Am. Soc. C. E.

Mr. Loweth continued, from 1883, to practice as a Consulting Engineer and maintained his office in St. Paul from that year until 1901, designing and supervising the construction of bridges, water-works, and sewerage systems for about one hundred municipalities, and for several railroad companies with lines extending from Michigan to the Pacific Coast. It was during this period that he acted, successively, as Chief Engineer of the South St. Paul Belt Railway Company and the Davenport, Rock Island and Northwestern Railway Company, and as Consulting Engineer for the Northern Pacific Railway Company, the Minneapolis, St. Paul and Sault Ste. Marie Railway Company, and others.

In going over some of Mr. Loweth's personal files, a specification for the superstructure of a bridge across the Rum River at Anoka, Minn., for the St. Paul Northern Pacific Railroad Company, was found, written in long hand in that fine penmanship with which all his friends and correspondents became familiar. Although this bridge was not a major structure, it was no inconsiderable undertaking in 1883 for a young man, twenty-six years of age. It was a span of 160 ft. The specifications provided that the design be based on the actual "static weight" and a moving load consisting of two consolidation engines weighing 81 tons each, coupled, and followed by a train weighing 3 000 lb per lin ft, the engine wheel loads being applied as shown by a diagram introduced into the manuscript, the lettering and drawing of which look like copper-plate work. This document is an example of the careful and painstaking manner which characterized Mr. Loweth's work from his youth to the time of his retirement, two weeks before his death. No more perfect examples of careful and beautiful draftsmanship can be found than the drawings of some of these earlier structures which were very elegantly made by him personally. It is worth recording that Mr. Loweth earned the money to purchase his first book on an engineering subject by chopping wood.

It is difficult to follow the number of engagements that Mr. Loweth fulfilled, or even to list the most important of these engagements. He was not given to indulging in publicity. However, from such records as are available, the following examples of his work have been selected.

In 1885, he designed the double-track railroad bridge over the Mississippi River, at Minneapolis, Minn., for the Northern Pacific Railway Company. This bridge had two deck spans of 245 ft each over the river, and an approach viaduct, 750 ft long, built on a curve, the total length of the structure being 1 240 ft, with the tracks 101 ft above the water. The Lafayette Avenue Bridge, at St. Paul, was built in 1887, with a roadway 40 ft wide, the total width being 60 ft, and the span, 90 ft. The St. Paul Union Station Train-

Shed was built on Mr. Loweth's design in 1889. This shed was 190 ft wide and 640 ft long. The steel trusses were carried on two lines of columns with overhanging cantilevers. In 1891, the bridge across the Mississippi River between Lyons, Iowa, and Fulton, Ill., was completed on his design. This bridge included five spans ranging in length from 200 ft to 360 ft, the total length of the structure, including approaches, being 2 617 ft, with the channel span 73 ft above low water. On May 1, 1895, the highway bridge over the Mississippi River, designed by Mr. Loweth and built by the City of Red Wing, Minn., was opened to traffic. The channel span of this bridge was 430 ft long and the total length of the steel structure, 1 725 ft. The channel piers were 69 ft high. In the same year Mr. Loweth designed the Mississippi River Bridge for the South St. Paul Railroad Company. This was a double-deck structure with a railroad track on the upper level and a highway floor on the lower deck. The revolving draw-span was 442 ft long and the total length of the steel structure, 1 633 ft.

All these structures were completed on designs made by Mr. Loweth before he was thirty-eight years of age. This indicates the surprising attainments of so young a man and the great responsibilities which had been entrusted to him.

In 1885, a bridge was proposed across the Mississippi River between Davenport, Iowa, and Rock Island, Ill., for which a charter was granted by Congress in that year. Mr. Loweth was retained early as Engineer for this project. It seems that there were many difficulties, financial and otherwise, which caused the project to drag on until 1895 when several contracts were entered into for the substructure, but work did not actually begin until April, 1896. A later substructure contract was made and work resumed in September, 1896. The contract for the river piers was not made until September, 1897, after which work on the substructure and superstructure proceeded continuously until the final completion in 1899. This bridge is quite a notable structure having a revolving draw-span, 442 ft long, seven spans over the river varying from 200 ft to 365 ft. Approaches on both sides of the river were on 7° and 8° curves, with 0.6% grade. The curvature caused the bridge to be known as the "Crescent" Bridge. This bridge is still in service and is used by both the Burlington Route and the Chicago, Milwaukee, St. Paul and Pacific Railroad Company. It has a total length of 2 639 ft.

Mr. Loweth made the design and supervised the construction of the bridge over the Mississippi River at Brainerd, Minn., erected in 1898. This steel highway bridge included three long span deck trusses and had several unusual and original features, such as cantilever extensions of the main trusses beyond the single tower bent and beyond the rocker bent at the other end of the third span. From a photograph, the spans of this bridge are of ten panels each, and each span is probably about 150 ft long. The expansion and contraction of the three spans are taken care of by tall rocker bents carried on slender masonry piers.

His noteworthy accomplishments led to his being called in 1901 by the Chicago, Milwaukee, and St. Paul Railway Company to fill the position of Engineer and Superintendent of Bridges and Buildings, to succeed Onward

Bates, Past-President, Am. Soc. C. E., who had resigned. Mr. Loweth's long connection with that Railway Company was a brilliant one. He was advanced to the position of Chief Engineer in 1910, upon the retirement of the late Don J. Whittemore, Past-President, Am. Soc. C. E. He held this position until, upon his own request, he was retired and appointed Consulting Engineer, only two weeks before his death.

During Mr. Loweth's services with the "Milwaukee", as Engineer and Superintendent of Bridges and Buildings, and as Chief Engineer, a great variety of important undertakings were designed and executed under his direction, including new lines, grade reductions, relocations, and double-tracking. Some of the grading done in connection with the double-track projects in 1912 involved earth quantities comparable in volume with those handled on the Panama Canal. The masonry construction and bridge erection were largely done by the Company's own forces under Mr. Loweth's direction, and during the period of greatest activity these forces numbered more than two thousand men.

Mr. Loweth was keenly interested in the early development of concrete construction for railroad structures and was one of the pioneers in the use of concrete slabs for subways, in the track elevation work at Evanston, Ill., in 1908. He also inaugurated the use of concrete ballast deck floors as early as 1904. Concrete trestle construction and reinforced concrete pipe for culverts were used by him at an early date, and a concrete pipe manufacturing plant was established at the Company's Bridge Shop.

He was called upon to design and replace, for heavier loading, bridges across all important rivers on the older parts of the System as well as new constructions for the Western Extension from Mobridge, S. Dak., to the Pacific Coast. These structures included crossings over the Mississippi, Missouri, Yellowstone, and Columbia Rivers, as well as many of their principal tributaries.

The location of the Pacific Coast Extension also required the construction of a number of trestles, many more than 200 ft in height, most of which have concrete floor ballast decks. These trestles aggregate a total length of three miles and their erection during the building of the new line developed interesting and difficult problems for which special derrick cars had to be designed and built.

The double tracks and relocation of the line through Iowa in 1911 to 1913 called for a great number of permanent structures, including several reinforced concrete arches and a concrete deck, double-track viaduct, 2 425 ft long and 145 ft high, which was probably the longest double-track viaduct in the West when it was built in 1913.

After the completion of the Pacific Coast Extension a number of the tunnels were given a permanent lining of concrete, the most interesting example of which is the St. Paul Pass Tunnel, 2 miles long, which was done by Company forces. Much of the concrete lining of this tunnel was placed by the pneumatic process, all the work being done during steam operation and prior to the electrification of this part of the line.

A striking example of emergency work handled by Mr. Loweth was the restoration of the line to Kansas City, Mo., when the flood of June, 1903, washed ont the west approach of the Missouri River Bridge. He took personal charge of the situation, bridging a gap, 1 300 ft long and 100 ft deep, in ten days.

He was more than an Engineer, frequently being consulted by the Executive Staff, for his wisdom and fine judgment in all matters involving both legal and engineering considerations. Notwithstanding his arduous duties in connection with this railroad, Mr. Loweth took great interest in all kinds of engineering and general improvement efforts, devoting his talents especially to the development of City Planning as a broad topic. He attended the meetings of the National Conference on City Planning, contributing to the discussions. One of his valuable contributions was his paper entitled, "The Place of the Railroad in City Planning", presented at the Eighteenth National Conference on City Planning, in March, 1926. He was a Charter Member of the Chicago Regional Planning Association.

One of Mr. Loweth's last contributions to general engineering and to a great national problem, was a paper by him entitled, "Some Economical Considerations Involved in the Construction and Use of Inland Waterways of the United States". This paper is dated July, 1934, but was revised by him after that date and before he was obliged to give up his active work. The material compiled by Mr. Loweth in this paper is of tremendous importance. Certainly, the statistics included therein show that there is a serious question as to the economic justification supporting the enormous expenditures of money raised by taxation for questionable benefits to the taxpayers at large.

Mr. Loweth's direction of the Society during the year that he was President, and his wise counsel and advice while he was still a member of the Board of Direction, are preserved in the records of the Society. Anything that affected the standing and welfare of this organization was of deep concern to him. His mind was distinctively of a constructive type. He labored unceasingly for the advance of his profession and for the benefit of all members of that profession.

The very traits that made him esteemed as an engineer made him loved and admired by those who came in close personal contact with him. His interest in civic and church activities was as marked as his leadership in the Engineering Profession. His devotion to his family surpassed his devotion to his profession, which seemed to those who knew him only professionally, to be the absorbing interest in his life.

Following his death the Board of Directors of the Chicago, Milwaukee, St. Paul and Pacific Railroad Company, passed a resolution expressing deep regret at his passing, calling attention to his long service with the road and stating further:

"Mr. Loweth's work was always conducted with great care and skill and his life was one of conspicuous achievement, marked by strict adherence to the highest ideals.

"*Resolved* further that this Board hereby records its deep appreciation of his keen business ability, splendid character and his long and faithful service to this Company and its predecessor."

D. J. Brumley, M. Am. Soc. C. E., Past-President of the American Railway Engineering Association, states that:

"Among the stalwarts who participated in the early history of the A. R. E. A. was Charles Frederick Loweth. He brought to the Association thirty years of accumulated experience on railroad and municipal engineering work in the middle west. This experience was obtained at a time when precedents were few and far between, and when the engineer was placed upon his own ingenuity and resources as to specifications, design, and construction procedure."

Mr. Loweth was a member of the American Railway Engineering Association (Treasurer, 1901–1910, and Director, 1911–1912 and 1913); member of the Western Society of Engineers (President, 1908, and Honorary Member since 1931); Charter Member and Past-President of the Civil Engineers Club, of St. Paul, Minn.; and a member of the Engineers Club, Union League, and University Clubs, of Chicago.

He received the Honorary Degrees of Civil Engineer from the University of Wisconsin, Madison, Wis., in 1915, and Doctor of Engineering from Rose Polytechnic Institute, Terre Haute, Ind., in 1926.

He was married in Wilmington, Del., on February 15, 1881, to Carrie T. Curtis who, with their four children, Mary Grace, Margaret, Frederick Curtis, and Robert Charles, survives him.

Mr. Loweth was elected a Junior of the American Society of Civil Engineers on January 3, 1883, and a Member on February 6, 1884. He served as a Director of the Society from 1910 to 1912; as Vice-President in 1914 and 1915; and as President in 1923.

FRANCIS LEE STUART, Past-President, Am. Soc. C. E.[1]

Died January 15, 1935

Francis Lee Stuart was born on December 3, 1866, in Camden, S. C., the son of Barnwell Rhett and Emma Croom (Lee) Stuart. Left an orphan at two years of age, he first lived with his maternal grandmother, who before her marriage had been a Miss Cooper, of Cooperstown, N. J., then with his paternal grandmother, of South Carolina, who died when he was eleven years of age. He then went to live with his uncle, the Reverend Dr. Albert Rhett Stuart, Rector of Christ Church, Georgetown, D. C.

Mr. Stuart was graduated from Emerson Institute, in Washington, D. C., in 1882, and entered the service of the Baltimore and Ohio Railroad Company in 1884. During the next thirteen years, he had active experience on the Baltimore and Ohio and other railroads in the East and South, in their Engineering, Maintenance, and Operating Departments, in coal mining, and in public works. In 1897, he secured a position on the surveys for the Nicaraguan Canal and became Assistant Engineer. Upon the appointment of the Isthmian Canal Commission, he was sent to Nicaragua to complete the organization for further surveys.

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[1] Memoir prepared by a Committee consisting of George H. Pegram, Past-President and Hon. M. Am. Soc. C. E., *Chairman*, Lincoln Bush, Past-President and Hon. M. Am. Soc. C. E., and George L. Lucas, M. Am. Soc. C. E.

Returning to the United States, Mr. Stuart re-entered the service of the Baltimore and Ohio Railroad Company and became unusually expert on location of lines. He was often called in on other railroads to make reconnaissances, examine locations, and report on the economics of grade reductions. He was largely responsible for establishing grades lower than 0.3% on the Trunk Lines.

In 1905, he was appointed Chief Engineer of the Erie Railroad Company and remained with that Company until 1910, during its greatest period of improvement and expansion, which included the four-track open-cut and tunnel entrance into Jersey City, N. J., and re-vamping of the station, water-front, and terminal tracks, the completion of the Erie and Jersey Railroad for a low-grade freight line, and the building of the Genesee River Railroad—a low-grade cut-off from Cuba, N. Y., to the Buffalo Division through difficult country. The improvements also included double-tracking and grade reduction on the Chicago Division, and on other Divisions.

In 1910, Mr. Stuart returned to the Baltimore and Ohio Railroad Company as Chief Engineer. As such he was in charge of the Company's large improvement plans, including an unusually heavy construction program of new lines, second-tracking, tunnels, grade reduction, and terminal work, until 1915, when he resigned to devote his time to private practice as a Consulting Engineer in New York, N. Y.

During the period he was Chief Engineer of the Erie and of the Baltimore and Ohio Railroads, he served on many technical and terminal committees in New York and other ports, in Chicago, Ill., and in Cleveland and Cincinnati, Ohio, and had a part in the conception of many improvements since made.

In 1913–14, he designed the Baltimore and Ohio Coal Pier at Curtis Bay, Md., and took out several patents to cover the principal features. It has two car dumpers ashore, and the coal is carried by belt and discharged into ships on either side of a long pier. It has loaded more tons per hour than any other coal pier in the world and handles material at a very cheap rate.

From 1915 to 1917, he was busied with varied consulting work. He was one of the three Technical Advisers to the War Board of the Red Cross in Washington, D. C., in an engineering capacity. He also helped to formulate standards of cantonment houses, barracks, and health methods which were adopted in the Army camps by the War Department.

In 1917, Mr. Stuart became Chairman of the Terminal Port Facilities Committee of the Storage Committee of the Council of National Defense, and assisted in outlining the plans for handling the Government supplies, from point of manufacture in the United States to point of distribution in France. He was also instrumental in the location and layout of the Army Bases at

Port Newark Terminal, in New Jersey, Philadelphia, Pa., Baltimore, Md., Norfolk, Va., Charleston, S. C., and some interior points. During this time, he was also a member of the Depot Board and other Committees in the Service.

As Chairman of the Budget Committee of the United States Railroad Administration, with the assistance of the Chief Engineers of the Baltimore and Ohio, the Pennsylvania, and the New York Central Railroads, he passed on the plans, estimates, and reasons for the proposed improvements by the railroad companies east of Chicago, and north of Norfolk, and recommended the expenditure, through the Regional Directors to the Director of Capital Expenditures, of more than $550 000 000 of the $800 000 000 spent for improvements by the railroad companies during the World War.

In 1920, Mr. Stuart was Consulting Engineer to the Hydro-Electric Power Commission of Ontario, Canada, on the canal from above the rapids of the Niagara River above the Falls to the 600 000-hp power-house below the Whirlpool. Some new construction methods and hydraulic improvements were evolved, and are serving as precedents on large undertakings.

During 1921 he was Engineering Expert for the Port Development Commission of Baltimore, and also a member of the Technical Advisory Board of the Port of New York Authority, in a consideration of the many plans for the development of New York Harbor. About this time, the Cunard Steamship Company proposed a $50 000 000 development south of Weehawken, N. J., and Mr. Stuart was retained as Consulting Engineer. In 1923, he was a member of the Transit Advisory Board of Philadelphia, Pa., and made a report on rapid transit matters in that city.

From January 1, 1923, to December 31, 1926, he was retained by the eleven trunk-line railroads which serve the Port of New York, to report on transportation matters in that Port as they affected the railroads collectively and individually before the New York Port Authority. During that time he made a joint study with the Port Authority and the forces of the railroad companies on lighterage and floatage methods and costs. He also was Chairman of the Committee of Engineers of all the railroads entering New York City, to make a joint study of the co-operative use of facilities. During 1927 he was retained by the eleven trunk-line railroads in New York City as Consultant in Port Authority matters.

During 1924–25 Mr. Stuart was Consulting Engineer to the Greater Harbor Committee of Two Hundred of the Los Angeles, Calif., Chamber of Commerce. The plan made was adopted by the Government, the railroad companies, and the City.

He was a member of the Board of Review of the Sanitary District of Chicago in the Lake Leveling Controversy and Remedial Program, in 1924, and was Chairman of the Committee on the Value of Diverted Water for Transportation from Lake Michigan to the Gulf of Mexico. He was also a member of the Committee on Great Lakes Regulation.

From 1920 to the time of his death, Mr. Stuart was Consulting Engineer of the North River Bridge Company, which proposes a bridge of 3 250-ft span at 57th Street, New York City, across the Hudson River, which, with the approaches, is estimated to cost $180 000 000, and also proposes large terminal projects on both ends of the bridge and changes in the present method of handling rapid transit and freight in New York City. Many plans of improvements in transportation in New York Harbor have been considered. At one time, a Committee of the Chief Engineers of the Trunk-Line Railroads, with the North River Bridge Company, developed a plan for bringing freight cars across the bridge and distributing them through a continuous seven-story warehouse from 57th Street to Chambers Street.

From 1915, to practically the time of his death, except during the World War, he was Consulting Engineer for several railroad companies. As a Consultant, he also gave counsel to many different projects in widely separated fields of endeavor.

Mr. Stuart was married on March 18, 1901, to Anne Morson Rives, when he was in Western Pennsylvania building a railroad over the mountains to get out coal. Their home was, successively, at Cumberland and Baltimore, Md., and Essex Fells, N. J.; then back to Baltimore and, finally, at Essex Fells.

They had four daughters, Anne Morson (Mrs. A. B. Sayre), Lee (Mrs. W. H. Sayre), Elizabeth Scott (Mrs. J. J. Whelan) and Rives (Mrs. James M. Newall) all of whom, with Mrs. Stuart, survive him. Two sons were born to them, who died in infancy—Francis Lee Stuart, Jr., born in 1907, and Barnwell Rhett Stuart, born in 1910.

Mr. Stuart was fond of engineering problems, but when in need of relaxation resorted to golf and the study and practice of horticulture, importing and planting many trees and shrubs. He gave to the Borough of Essex Fells a large collection of trees and shrubs.

He loved people and was rarely heard to speak uncharitably of any one, and then always citing extenuating circumstances if he could think of any.

The quality of Mr. Stuart's achievements speaks more for the character of the man than their number, great as it is. To be entrusted with the execution of great enterprises is an acknowledgment of merit, and where the enterprise has been initiated by such executive, or undertaken largely through his endorsement, the honor is indeed great. It is in this light that they must be viewed. Mr. Stuart was progressive and extended all the fields of activity which he entered. Bereft of both parents at the age of two, and without the advantages of professional education beyond the age of seventeen, he must perforce have been endowed with very superior qualities.

He was analytic, ingenious, and had very unusual clarity of vision, which, combined with his absolute honesty, led to his frequent engagement for investigations and reports on large financial problems. It was these qualities doubtless, which determined his selection as Chairman of the Budget Commission of the Eastern Railroads during the World War.

His inventions in methods of handling coal and freight at terminals simplified that work, and his engagement in the solution of so many great railroad and harbor terminal problems testified to his pre-eminence in that line.

Mr. Stuart's cheerfulness was proverbial. He was an optimist and carried the air of cheerfulness and optimism wherever he went.

One of his contemporaries, John E. Greiner, Hon. M. Am. Soc. C. E., writes of him, as follows:

> "I first met the late Francis Lee Stuart when he was an office boy in the Consulting Engineer's Office of the Baltimore & Ohio Railroad, back in 1885.
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> "He was a very handsome, attractive and ambitious boy with a keen sense of humor and a desire for adventure. His personality at that time caused everybody to like him, and that same personality and liking were retained by him throughout his career. He always had original and big ideas on questions of railroad locations and terminal improvements, and could see further into the future than most of the other railroad officials with whom he worked.
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> "I followed his career since his boyhood days and have never heard of any man who, after meeting him either in business matters or socially, had retained any feeling except confidence in his ability and fair dealing, and a desire to be his friend.
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> "He could not help becoming a prominent, nationally known, successful engineer, and it was his lovable personality just as much as his broad-minded engineering ability which contributed to his successful career."

While living in Baltimore Mr. Stuart became President of the Engineers Club, a member of the Maryland Club, and of the Elkridge Kennels. While living in New Jersey, he became President of the Country Club of Essex Fells and a Charter Member of the Fells Brook Club of that place, and of the Engineers Club, at Roslyn, Long Island. During the World War he joined the Chevy Chase Club, of Washington. He was Chairman of the Building Committee for Yeaman's Hall, a club fifteen miles from Charleston, S. C., which was built on a tract of land first owned by Sir Thomas Yeaman—his ancestor. He was a member of the South Carolina Historical Society, the Southern Society of New York, the Engineers Club of Philadelphia, and the Engineers Club of New York.

His connections with professional societies included the following: He was a Past-President of the United Engineering Trustees; a member of Engineering Foundation Board; the John Fritz Medal Board; American Engineering Council; American Railway Engineering Association; a Past-President of the Terminal Engineers of New York; a member of the Institution of Civil Engineers of Great Britain, and of the Engineering Institute of Canada.

Mr. Stuart was elected a Member of the American Society of Civil Engineers on May 3, 1899. He also served as Director of the Society, from 1909 to 1911; as Vice-President, in 1920 and 1921; and as President, in 1931.

PALMER CHAMBERLAINE RICKETTS, Hon. M. Am. Soc. C. E.[1]

DIED DECEMBER 10, 1934

Palmer Chamberlaine Ricketts was born in Elkton, Md., on January 17, 1856, the son of Palmer C. and Elizabeth (Getty) Ricketts. When he was a young boy, his parents moved to Princeton, N. J., and there he prepared for college, studying with a tutor from Princeton University.

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[1] Memoir prepared by J. A. L. Waddell, M. Am. Soc. C. E.

In his course at Rensselaer Polytechnic Institute, at Troy, N. Y., Mr. Ricketts was truly a plodder and a most successful one, as his official records there testify. He was absolutely faithful in his attendance, never wilfully "cutting" a recitation, and always on time and ready to recite. He was well liked by his classmates; but being so constantly occupied with his studies, he failed to make many acquaintances outside his Class. He did not play "Institute politics"; but in Class matters he always did his part. It was not until after graduation that his great popularity began to develop and eventually to attain what a large number of his friends deem to be absolute uniqueness.

Mr. Ricketts was graduated from Rensselaer Polytechnic Institute in June, 1875, and the following September he returned as an Instructor of Mathematics and Astronomy. In 1882, he was made Assistant Professor of Mathematics and Astronomy, and, two years later, became the William Howard Hart Professor of Rational and Technical Mechanics, taking the place of the late William Hubert Burr, Hon. M. Am. Soc. C. E., who passed to the beyond only two or three days after the death of Professor Ricketts.

Recognizing the necessity for practical experience in the engineering field, Mr. Ricketts sought and obtained it soon after his graduation, quickly becoming widely known as an efficient and successful engineer in charge of the design and construction of both bridges and hydraulic works.

In 1876, he was named Assistant Engineer with the Troy and Boston Railroad Company; and, in 1887, he became Consulting Bridge Engineer of the Rome, Watertown, and Ogdensburg Railroad Company, besides handling numerous small constructions of various kinds in the vicinity of Troy. His outside duties did not take him far from there, because he would not let them interfere with his pedagogical work.

In 1892, Professor Ricketts, at the age of 36, became Director of the Institute; and, in 1901, upon the death of President Peck, the two offices were combined, Professor Ricketts becoming both President and Director, a position which he held until his death.

Up to 1904, the Institute consisted of only a few scattered buildings, with the main building on the site of the present approach steps; and the number of students was quite limited. The present stately campus with its ten splendid main buildings, its dormitories, and club houses, the magnificent approach from Broadway, its many hundreds of students from the United States and from numerous distant lands, and its imposing list of Faculty members, was not even dreamed of. The great changes that have taken place in the last three decades are almost entirely due to the boundless energy and the indomitable perseverance of Director Ricketts.

In 1904, there fell upon Rensselaer what appeared to be a terrible catastrophe, but which proved to be a blessing in disguise. The Main Building at the head of Broadway burned to the ground, leaving a few small structures, which had outlived their usefulness, as all that remained of the

Institute. There were rumors to the effect that it would have to be moved from Troy to merge with some other school. Director Ricketts held a long conference with the late Robert Woolston Hunt, M. Am. Soc. C. E., a man of great prominence in the world of steel, and then went to interview Andrew Carnegie, whom he persuaded into giving $125 000 for a building; and soon thereafter he induced Mrs. Russell Sage to donate $1 000 000 as a memorial to her deceased husband. There was also considerable insurance money from the burned building.

A fitting appreciation of his character and succeeding accomplishments at the Institute was presented in the January, 1935, *Rensselaer Alumni News.* Quoting from this splendid tribute:

"When in June, 1904, the old Institute was practically wiped out by the destruction of its main building at the head of Broadway, only the personality of Director Ricketts stood between its historic past and dissolution. With an enthusiasm and zeal, born of whole-hearted devotion, he took up the Herculean task of rebuilding a truly great institution upon its ashes. Enlisting the hearty co-operation of an interested Alumni and tapping the sources of wealth which had devoted themselves to the development of higher education, he visioned anew the dreams of the earlier promoters, the hopes his predecessor, Dr. B. Franklin Greene, had for a true polytechnic, and enlarged that vision with new and higher impulses that welled within himself. Step by step through three decades he brought project after project to fruition, erected and equipped one modern laboratory and campus structure after another, until the Institute is complete in all its ramifications and departments encompassed by the field of modern science and engineering. Living to the ripe age of almost 79 years, he has seen brought to reality the full measure of his vision of a greater Rensselaer Polytechnic Institute, with a physical property of more than $10 000 000 valuation, a complete re-organization of homogeneous departments, and a faculty and student body more than five times what they were thirty years ago—his own pride, and himself a recognized peer in the scientific-education world.

"The Director made Rensselaer Polytechnic Institute his life work, with a singleness of purpose and devotion such as has seldom been exemplified. * * * it stands to-day as a personal monument to his wisdom, his scholarship, and his executive ability.

"Personally he disliked the term 'genius'; and yet his life experience demonstrated all the qualifications of a genius in his chosen field. The motto which he set for himself and for the thousands of students from all parts of the world who came under his direction and all-embracing personal interest, was 'work.' He demonstrated this with an energy that was tireless and unflagging, even with the passing years, with a capacity that indicated a quick grasp of the essential, with a broad-minded appreciation of the range of the human mind, and with an incisiveness and insistence that never brooked failure. He won the esteem, the confidence, and the friendly appreciation of all with whom he came into contact, and with none more devotedly than the Alumni who, during many decades, came under his direct and impelling influence.

"Although the Director counted honors lightly in comparison with concrete accomplishment, he was accorded several honorary degrees, was a prominent figure in many distinguished engineering societies, was often sought as a consultant in notable bridge and other engineering projects, and had conferred upon him the titles of Commander of the Order of the Crown of Italy and Commander of the Legion of Honor of France. He took a deep interest in both civic and educational affairs in his adopted city, in which he enjoyed a devoted home life, and served as Trustee or Director of educational, charitable, and financial institutions.

"Dr. Palmer Chamberlaine Ricketts left the impress of an unusually able and unique character upon the institution which he made so fully his own, and in the community in which he cast his lot and performed a truly great life work, the influence of which will remain permanently.

"As Director Ricketts' position in the world of science and engineering grew in importance, he was honored in many fields. In 1905, Stevens Institute of Technology conferred on him the honorary degree of Doctor of Engineering, and, in 1911, he was given the degree of Doctor of Laws by New York University.

"Although he confined his activities almost exclusively to the Institute, he was accorded national and international distinction because of his educational achievements. Upon him was conferred the rare tribute of Honorary Membership in both the American Society of Mechanical Engineers and the American Society of Civil Engineers. * * * He was also a member of the American Institute of Mining and Metallurgical Engineers, the Institution of Civil Engineers of Great Britain, and the American Philosophical Society.

"His 'History of Rensselaer Polytechnic Institute' and his contributions to technical journals won him wide recognition as an engineer and an author.

"Always intensely interested in civic affairs and business life, Dr. Ricketts was a Director of the National City Bank of Troy, a Director and a member of the Executive Committee of the Samaritan Hospital; Trustee of the Albany Academy; Trustee of the Dudley Observatory, at Albany; and Trustee and Vice-President of the Troy Public Library.

"He was a member of the Troy Club, the Union Club of New York, and the Schuyler Meadows Country Club of Albany. * * * In politics, he was an independent, never becoming affiliated with any party.

"On November 12, 1902, he married Vjera Renshaw, of Baltimore, Md. Mrs. Ricketts, a brother, and a sister are the only immediate family survivors."

The funeral services of Director Ricketts, held on the Institute grounds on December 13, 1934, were of a character befitting a man of his world prominence. Educational leaders from most of the principal institutions of learning in the East, engineers whose names are universally famous, business men, graduates, students, and former friends, came to pay their last respects and to do homage to the memory of one of America's truly great men.

Director Ricketts will be greatly missed by all the Alumni, and this will be true for many years to come—but why mourn him? He had lived a full life and had almost reached the Biblical extreme age limit of four score years, without in any manner losing his splendid mental faculties. He passed on quickly at the zenith of his career, instead of slowly failing in health and gradually losing his mental grip. It is far better to remember the gloriously fine fellow, the eminently useful citizen, the noted and honored scientist, the outstanding technical teacher, the kindly and courteous gentleman, and the brilliant man of affairs that he was, instead of thinking of him as a "has been." In his passing, the Engineering Profession has lost one of its most prominent and respected members; for he was, in every sense of the words, a great man, a true gentleman, and a famous engineer—than which there could be no higher praise.

As Director of the Institute, Dr. Ricketts was certainly a strict disciplinarian—he had to be such in order to compel obstreperous students to obey the rules of the campus, to maintain the high standard of scholarship which Rensselaer has enjoyed throughout its long career, and to keep peace between "town and gown." Although at times he had to be severe, nevertheless, he was universally conceded to be strictly just and reasonable, as well as the students' best friend—always ready to devote time and attention to their needs, to steer the wanderers back to the path of rectitude, and to help materially all who were worthy of his assistance. The indubitable proof of the correctness of this statement was his universal popularity among the students, the Alumni, and the Faculty of the Institute, as well as the citizens of Troy.

His outstanding generosity in the spending of his income both for needy students and in aiding the Institute to carry on during the long existing depression is a matter that, by his desire, has been kept in confidence. It has been talked of, especially among the Alumni, but it has not been mentioned in print.

In spite of having earned at the Institute the reputation of being of a rather retiring disposition, eventually Professor Ricketts developed, in a socio-scientific sense, into one of America's most prominent men of affairs in the field of engineering education.

In conclusion, the writer desires to say a few words about his old friend and classmate. Not only did he hold him in the highest esteem for more than half a century, but he truly loved him—and the affection was reciprocated. When it comes the writer's turn to pass on, he will be quite content if he can count even one-tenth as many true, loyal, and devoted friends as did Palmer Chamberlaine Ricketts.

Director Ricketts was elected an Affiliate of the American Society of Civil Engineers on February 3, 1886; a Member on October 5, 1887; and an Honorary Member on October 5, 1931. He served as a Director of the Society from 1899 to 1901, and as Vice-President in 1916 and 1917.

JONAS WALDO SMITH, Hon. M. Am. Soc. C. E.[1]

Died October 14, 1933

Jonas Waldo Smith died of a sudden heart attack at his home in New York City on October 14, 1933. He had not been in good health for several months yet, characteristically, at the moment of his death he was preparing to go to the golden wedding celebration of a life-time friend and associate of his early days at Holyoke, Mass. His remains were interred in the family plot at Lincoln, Mass., on a cloudless day amid the gorgeous foliage of autumn and, as one stood beside the grave, it seemed as though Nature had bedecked herself for the ceremony of taking back that which she had given.

J. Waldo Smith, as he always subscribed himself, was born at Lincoln, Mass., on March 9, 1861. He was the youngest son of Francis and Abigail Prescott (Baker) Smith. On his father's side his ancestry goes to John Smith, who came to Watertown, Mass., from England as a freeman in 1636. On his mother's side the line goes to John and Elizabeth Baker who came to America about 1720. From these roots he inherited those inherent qualities of intellect, of courage, of honesty, and of sincerity which shaped his life and made of him not only a man among men, but a leader and counselor of them.

At the age of 15, Waldo Smith helped on the construction of the water supply system of his home town and, at 17, had charge of its Pumping Station as Fireman, Engineer, and Superintendent. He attended the public schools and, when 18 years old, entered Phillips Academy, at Andover, Mass., from which he was graduated in the Scientific Course in 1881. Thereafter, he went to Lawrence, Mass., and was employed in the Engineering Department of the Essex Company. Here, he came into contact with those masters of hydraulics—Hamilton Smith, Jr., Mills, Boyden, Davis, Francis, Storrow, Herschel, and Freeman—and was inspired to further study at the Massachusetts Institute of Technology, from which he was graduated in 1887. During his summer vacations, he worked for the Holyoke Water Power Company and, for three years after graduation, he continued under the same ægis and under the influence of the men previously mentioned. These were the formative years of his life. From the living examples before him he learned the necessity for care and precision and the importance of research and experiment but, even more significantly, he also came to understand that human relations ranked at least equally with those of a technical nature. So it was that the traits of his ancestry came to bloom in the environment of his younger days and out of them grew the man of scientific vision and human understanding.

In 1890, a great water development was projected in the rapidly growing district of Northern New Jersey, and the late Clemens Herschel, Past-President and Hon. M. Am. Soc. C. E., who was in charge of these operations called young Waldo Smith to his aid. Thus began his association with the East Jersey Water Company where he learned the art and business of water supply from the standpoint of construction as well as from that of the maintenance and operation of the works which delivered the water to the ultimate consumer. In the field of construction he was engaged in the building of the Oak Ridge, Canistear, Clinton, Macopin, and Great Notch Reservoirs and on the laying of the main lines of steel pipe, 42 and 48 in. in diameter, for conveying the waters of these reservoirs to the City of Newark, N. J. Then, in 1900, he became Chief Engineer of the East Jersey Water Company and its affiliated companies, on the supplies of Paterson, Passaic, Montclair, and other communities in New Jersey. Soon thereafter it became evident that the raw water delivered by the Little Falls Pumping Plant direct from the Passaic River was rapidly becoming unsuitable for use, and that a remedy for the situation must be found. This condition led to the construction of the Little Falls, N. J., Filtration Plant which was the model and the forerunner of that great host of similar works which have since given to the people of America the best and most potable water supplies on earth. The Little Falls plant was due to the inspiration, the logic, and the insistence of Waldo Smith, and to the aid of George W. Fuller, William B. Fuller, Charles L. Parmelee, and others, to all of whom he would give due credit were this writing his own.

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[1] Memoir prepared by Thaddeus Merriman, and Frank E. Winsor, Members, Am. Soc. C. E., and Robert Ridgway, Past-President and Hon. M. Am. Soc. C. E.

Early in 1900, the interests which Mr. Smith represented undertook to complete the system that had been proposed for the supply of Jersey City, N. J. This project included the Boonton Dam, a masonry structure, 3 150 ft long and 150 ft high, and an aqueduct of cut-and-cover sections, tunnels and 72-in. steel pipe passing under the tidal waters of the Hackensack and Passaic Rivers and aggregating 22 miles in length. The Boonton Dam was one of the notable structures of its day. It was the first dam built of "cyclopean concrete" all of American-made Portland cement. Here thermo-couples were first installed for measuring the internal temperatures of the masonry and here was constructed the first hydraulic model of which there is record. This model of the spillway was tested, and the design of the final structure was based upon the results obtained from it. The Boonton Dam was also notable in that the water impounded by it was treated with chlorinated lime before being delivered into the aqueduct. Jersey City thus became the first American municipality to enjoy a sterilized water.

During the years 1890 to 1903, Mr. Smith, while occupied on the various undertakings outlined, never lost an opportunity for study, for research, or for investigation. He constantly experimented with his structures and subjected them to frequent operating trials. He tested the materials of construction and observed their behavior. Always had he in mind that improvement was possible, and that each succeeding construction should be better than the last. Indefatigable in his efforts, he shirked neither responsibility nor danger. During the great flood of October, 1903, on the Passaic River, he took personal charge of the Little Falls Pumping Station which the flood threatened to destroy. With a few men inside that structure, while the waters rose high against and around it, he strutted the walls to prevent their collapse. Had those walls failed there would have been no escape, but the station was saved and the pumps ran on.

This period of intense activity and achievement marked the second phase of Mr. Smith's development. The teachings he had learned and the character he had formed during his earlier days in New England grew and ripened and the Man was shaped by his own effort, by his contacts with men of affairs, by his constantly increasing responsibilities, and by his continued study of the human heart. As he was thus growing in stature his reputation was increasing day by day while he constantly made friends of his associates and endeared himself to all with whom he came in contact.

In 1903, the water situation in the rapidly growing City of New York had become most difficult, and the Metropolis, confronted with the gravest of problems, sought the solution by engaging Waldo Smith as Chief Engineer of its Aqueduct Commissioners. From the beginning he showed his quality by infusing new life into a situation which had practically reached an impasse. He quickly decided the questions which were delaying the operations on the New Croton Dam, then the largest masonry dam in the world; he resolved the difficulties which surrounded the Jerome Park Reservoir and the Muscoot Dam, and put the design for the Cross River Reservoir under way. Here, in a new environment and under public auspices totally different from those on the private enterprises with which he had been previously associated, he showed such high technical skill and administrative ability coupled with such indomitable energy that, through his keen grasp of the human factor in life, he was able to accomplish results and to have his decisions accepted and respected by all. Later, in 1905, when the City of New York needed a man to direct the destinies of its Catskill Water Supply System, the choice naturally fell to Waldo Smith, and, on August 1 of that year, he was designated as Chief Engineer of the greatest and the most costly water supply undertaking which, up to that time, had been conceived.

Entering upon that great task he did so with energy and determination, but with the future always in view. Around him he gathered a group of men upon whom he could rely. The work, he knew, was far beyond the capacity of any individual, and his immediate staff was augmented by a number of Consultants so that every detail would be made as nearly perfect as humanly possible. Waldo Smith was always the first to give credit to his associates, as he fondly called his staff, and, therefore, he would not approve this memoir were their names not mentioned. As Deputy Chief Engineer there were, from time to time, Charles L. Harrison, Merritt H. Smith, Alfred D. Flinn, and Thaddeus Merriman. As Department Engineers, there were Carleton E. Davis, George G. Honness, Robert Ridgway, Ralph N. Wheeler, Frank E. Winsor, William W. Brush, Walter E. Spear, and Charles M. Clark. The Senior Designing Engineer was Thomas H. Wiggin. The Consulting Engineers were John R. Freeman, Frederic P. Stearns, William H. Burr, Alfred Noble, Allen Hazen, and George W. Fuller. There was also a group of Consulting Geologists, James F. Kemp, W. O. Crosby, and Charles P. Berkey, whose duty it was to advise as to the geological features of the terrain in which the structures were to be built and through which the tunnels would pass. Waldo Smith was foremost among American engineers who felt and understood the need for close co-ordination between the scientific arts of geology and engineering.

As an important part of his philosophy Waldo Smith held that the relations between the Engineer and the Contractor were those of men associated on a common enterprise. He often said: "The Engineers and the Contractors on this work are partners. We must work together. We must respect each other's opinions and be fair one to another so that when the work is done it will be a credit to everybody as well as to the City." He infused this spirit into his men and into those directly engaged on the construction with the result that both Contractor and Engineer all strove together, each helping the other, to give to the "job" the best they had. The idealism of Waldo Smith was based on truth, on honesty, on fairness, and on his abiding faith in the loyalty of men.

The preliminary plans and estimates for the Catskill System were completed within two months after his appointment as Chief Engineer. Then followed a period of careful planning and sub-surface investigations so that, as Mr. Smith said, "both we and the contractors may know all about the work before it is begun." The first contract was let in the spring of 1907, and water was delivered to the city from the Esopus development in 1915. The second part of the System, comprising the Schoharie development, was completed in 1926. In the meantime, in 1922, Mr. Smith retired as Chief Engineer, but continued in the capacity of Consulting Engineer to his last day. His great love was the Catskill Water System and those who worked with him. He gave to it and to them all he had and was happy to see it to its conclusion.

He felt, however, because of the large growth in population from 1900 to 1930, that this System was not sufficient for a city as great as New York, that it provided too small a margin of safety, and that the only way of putting the water supply of the city on a sound basis was to increase it by developing a new source. The best of the possible supplies available was the Delaware, but, as the waters of this river were interstate between New York, New Jersey, and Pennsylvania, the line of approach presented difficulties of an unusual kind. It was Mr. Smith's firm belief that there was no reason why the situation could not be fairly composed, because the other States, sooner or later, would need Delaware River water for themselves and also because the plan of New York, looking as it did to an equitable apportionment, was fair, just, and reasonable. Beginning in 1921, with the hearty co-operation of the Commissioners of the Board of Water Supply, of which Mr. George J. Gillespie was President, of the late Mayor John F. Hylan, and of Governor Alfred E. Smith, the establishment of compact commissions in each of the three States was consummated. Mr. Smith was designated as Engineer to the New York Commission, and the three Commissions early in 1923 unanimously agreed on a form of compact under which the waters of the Delaware River might be used by each of the three States. This form was approved by New York State, but failed in the legislatures of the other States.

Nothing daunted, the attempt to reach a conclusion was continued and, in January, 1925, the Commissions once more unanimously agreed on a second form of compact, but history repeated itself, and the outcome in the legislatures was the same as in 1923. In the meantime, the situation was growing more acute and it seemed clear that early provision should be made for an additional supply of water. The City then, in 1927, the Hon. James J. Walker being Mayor, formally approved a plan for utilizing these Delaware waters. New Jersey thereupon sought to enjoin the project through an original action in the Supreme Court of the United States, while the Commonwealth of Pennsylvania intervened *pro interesse suo.* The issues were finally joined in 1929. Hearings before a Special Master were held in 1930, and the decision of the Supreme Court was handed down in May, 1931. This momentous decision written by the late Mr. Justice Holmes established a new doctrine relating to the use of interstate waters. Wrote Justice Holmes:

> "A river is more than an amenity, it is a treasure. It offers a necessity of life that must be rationed among those who have power over it. New York has the physical power to cut off all the water within its jurisdiction. But clearly the exercise of such a power to the destruction of the interest of lower States could not be tolerated. And on the other hand equally little could New Jersey be permitted to require New York to give up its power altogether in order that the river might come down to it undiminished. Both States have real and substantial interests in the River that must be reconciled as best they may be. The different traditions and practices in different parts of the country may lead to varying results but the effort always is to secure an equitable apportionment without quibbling over formulas."

Thus was the judgment and foresight of Mr. Smith confirmed, and so, out of the facts, did a new principle of the common law arise.

This memoir cannot attempt to describe the Catskill System, which was part and parcel of Mr. Smith's very life. That has been done in many papers by many authors, but it is essential that a few of the important developments of technical progress which grew out of this work should be stated briefly as a record for posterity to read. In the largest sense they were the work of Waldo Smith, because of the continual support, encouragement, and recognition he gave to his men. While he constantly urged them on to newer and better things, his marvelous insight enabled him to rule out the chaff and few indeed were the innovations he approved that were not successful.

First, should be mentioned the deep pressure tunnels and particularly that which passed 1 100 ft under the Hudson River. No tunnels of such size and operating under so great a head had previously been built. Equally noteworthy was the 18-mile City Distribution Tunnel, far below sea level, passing directly beneath the city and delivering its water through shafts into the distribution mains.

Second, was the use in the Olive Bridge and Kensico Dams of expansion joints with copper cut-offs, drainage wells, and inspection galleries. No previous dam had these features.

Third, was the consolidation by grouting under pressure with Portland cement of the foundations of all the dams as well as the linings of all the shafts and tunnels. It may be said that the modern art of grouting was developed under the direction of Waldo Smith.

Fourth, the use of bronze for all important valves and operating parts in all the dams and along the aqueduct. The pressure-reducing needle-valves at the Ashokan Reservoir and the riser valves and their control mechanisms in the City Tunnel, were all new and novel and involved special developments of design and in the art of bronze manufacture.

Fifth, the compressed lead joints in the flexible jointed pipe under The Narrows from Brooklyn to Staten Island.

Sixth, the beginning in 1912 of a research program looking toward a determination of the reasons for the deterioration of Portland cement concrete. This work is being continued.

Seventh, the adoption of Portland cement mortar as an interior lining for steel pipes of large diameter, together with the correlative outer concrete support for holding the pipes to true form.

Eighth, the use of models for many purposes, including stress determinations, hydraulic behavior, and other related phenomena.

Ninth, the form of contract specifications which divided the work into many definite items so as to present to the contractor as complete a picture of the work as possible—included under this heading was the prescribing of fixed and definite payment lines based on previous actual experience.

Many other items could be mentioned, but these must suffice. Mr. Smith was a man of natural research instinct and, because of his understanding of hydraulics and the behavior of flowing water, the details of the great system as a whole were always in the forefront of his mind.

The Catskill Water System was inaugurated and begun during the administration of Mayor George B. McClellan, whose term of office was from 1904 to 1909. There had been great objection to this project in the New York Legislature and Mayor McClellan led the fight in its favor almost single-handed. It is of interest, therefore, to note what he said at a dinner given on June 27, 1922, to mark the retirement of Mr. Smith from the position of Chief Engineer:

"The task that was mine was to appeal from Pilate to Cæsar. To create a public opinion so strongly in favor of the right to existence of the people of New York as to force a partisan legislature in very shame and self-defense to grant us the legislation we required. I should not have had the courage to carry the fight through had it not been for the constant help and encouragement, the professional knowledge and the technical skill, the friendship and the sympathy of Waldo Smith.

"Immediately after I became Mayor I met the Chief Engineer of the Croton Aqueduct Commission as Smith then was, and found that in developing the new Catskill Water Supply project he was just the man I needed. We soon learned to understand each other and to work together, so that ever since the Catskill project has been for me nothing but the concrete expression of the personality and of the genius of Waldo Smith.

"When a reluctant legislature had given us our needed legislation and when the commission had taken office, as a matter of course Waldo Smith was appointed Chief Engineer. And such he has remained until today, increasing always in the respect of his subordinates, in the affection of his friends and in the confidence of the public."

Following his resignation as Chief Engineer in 1922, Mr. Smith continued as Consulting Engineer to the New York Board of Water Supply. He also served in the same capacity for the Metropolitan Water District of Boston, Mass., on its new development of the Ware and Swift water-sheds and, among others, he gave advice and counsel to the Cities of Philadelphia, Pa., Providence, R. I., San Francisco, Calif., St. Louis, Mo., Hartford, Conn., Baltimore, Md., Kansas City, Mo., Kingston, N. Y., Asheville, N. C., Vancouver, B. C., Canada, and Havana and Santiago, Cuba. He was also Consulting Engineer on the Moffat Tunnel, in Colorado.

On April 17, 1918, the John Fritz Medal was conferred on Mr. Smith, "for achievement as engineer in providing the City of New York with a supply of water." Commenting on this award *Engineering News-Record,* in its issue of April 25, 1918, under the editorial caption of "J. Waldo Smith, Human Engineer", stated:

> "With utter modesty and self-effacement the recipient of the John Fritz medal last week brought out, as the previous speakers had failed to do, the chief reason for rejoicing that he should have been chosen for the highest honor possible for an American engineer. It is found in the fact that J. Waldo Smith never misses an opportunity to share his laurels with his subordinates. He did not forget them on the occasion of his designation as the fourteenth to receive this emblem of high engineering achievement. He asserted that it would be presumptuous for him to assume that the award was based on his individual work; and told of the ability and *esprit de corps* that had characterized his organization throughout. True enough, the spirit of the Catskill organization is far famed; but J. Waldo Smith is responsible for that spirit, and without it the ability of his staff would have counted far less."

At the Annual Meeting of the Western Society of Engineers in 1925 the Washington Award of that organization was bestowed on Mr. Smith in these words:

> "For Preeminent Services in Promoting the Public Welfare and for the Rare Combination of Vision, Technical Skill, Administrative Ability and Courageous Leadership in Engineering."

Columbia University conferred on him in 1918 its honorary degree of Doctor of Science and in that same year there came from Stevens Institute of Technology the honorary degree of Doctor of Engineering.

Mr. Smith was a member of the American Society of Mechanical Engineers, the Institution of Civil Engineers of Great Britain, the American Institute of Consulting Engineers, the American Public Health Association, the New Jersey Sanitary Association, the American Water Works Association, the Boston Society of Civil Engineers, the New England Water Works Association, the Chamber of Commerce of the State of New York, the Municipal Engineers of New York City, and the Franklin Institute of Pennsylvania. He was also a member of The Century Association, the Technology Club, and the Engineers Club, all of New York.

Mr. Smith was elected an Associate Member of the American Society of Civil Engineers on October 5, 1892, and a Member on April 5, 1899. He served as Director from 1906 to 1908, and as Vice-President in 1913 and 1914.

On October 1, 1928, he was elected an Honorary Member of the Society and, at the Annual Meeting in January, 1929, he was presented for the award by the late Clemens Herschel, Past-President and Hon. M. Am. Soc. C. E. Among others, Mr. Herschel used these words:

"But there is another feature in the work of Waldo Smith for New York City, of peculiar gratification to me who contributed to it, by transplanting him into this vicinity. It is the sterling honesty of purpose and of action with which his public work has invariably been conducted. He has left a broad trail of ability, efficiency, and of duty well done in public work, in New York City; an environment which tests its engineers as though by fire, not only in technical integrity, but also in great steadfastness of character and of upright action."

Mr. Smith contributed many discussions to the *Proceedings and Transactions*, and served the Society in many diverse ways on committees and as representative on many occasions.

The following paragraphs are from *Engineering News-Record*, of October 19, 1933:

"Mr. Smith's exceptional abilities and character are reflected more adequately in the following words, written at the request of *Engineering News-Record* by one of his intimate friends, associated with his work for many years past:

"'It is seldom possible from a categorical recital of the accomplishments of an individual to glean anything as to the true inwardness of the man. Every normal person admires success and feels respect for him who has earned it. The winning of distinction is significant of greatness and of quality. But from what roots does greatness spring and wherein lies its essence? I am writing these words because you have asked me out of my long friendship and love for Waldo Smith to analyze, as best I may, the reasons for the fulness of his life and the great measure of distinction he achieved in the hearts and minds of all with whom he came into contact. And this is most fitting because Waldo Smith fully as much as any engineer of the period exemplified the highest ideals of the profession. His mind and heart were a book as open as his office door, which was never closed. Nothing was ever said or done in that office which could not have been shouted from the housetops. His door was always open to those who came and no previous appointment was needed.

"'Sprung from a long New England ancestry, he inherited those qualities of fairness, of economy and of rugged honesty which controlled and characterized his every thought and action. No matter what question came to him, his primary concern was that his decision should be just and that the equities should be balanced. Had he followed the law he would have been a great judge. But Waldo Smith was an engineer by nature. He could determine the value of a plan almost at a glance and his mind at once went forward to the effects which would follow from the plan. To him the details of a structure pictured an operating whole; the plan for a project visioned the social, the economic and the political consequences to which it would give rise. Yet his reports were models of brevity, though full and complete.

"'As an engineer he worked not only with materials, but even more brilliantly with men—both the men who put the materials from the drafting board into the completed structure and the public officials who provided the necessary funds and approved his plans. Waldo Smith worked with men as one of them. As they had ability he recognized it, as they had faults he overlooked them. He was the engineer and director of men because he held the respect and confidence of all.

"'His qualities were brought into outstanding relief by his great personal modesty. Never did he seek publicity or notice for himself. Always did he insist that credit should go to the men who deserved it, and always did he particularly remember those who served with faithful distinction. The organization he built up on the Catskill Aqueduct was one of socialized service, and partly because of this fact its spirit and loyalty were boundless and its influence has gone far and wide. Each man of his staff as well as every one of the many contracting organizations which executed the work saw in him the exemplification of all that was good and noble and true. His friends were legion. They were his strength and his comfort'."

In order that a fitting memorial to the man and the engineer might remain as an outward and visible sign for the future, about 400 friends of Mr. Smith contributed to the erection of a tablet resting on the south face of the triangulation tower on a hilltop overlooking the Olive Bridge Dam and the Ashokan Reservoir, and bearing the following inscription:

In Tribute To

J. WALDO SMITH

Engineer of Water Supply
New York City 1903 to 1933

1861 1933

Man of Vision and of Courage
Leader and Counselor of Men
Student of the Human Heart
He Inspired His Associates
And Adorned His Profession
The Catskill Water System
Is His Enduring Monument

"NONE KNEW THEE BUT TO LOVE THEE
NOR NAMED THEE BUT TO PRAISE."
Halleck.

ERECTED BY HIS FRIENDS IN 1936

At the center of the tablet is a bronze medallion 19 in. in diameter showing a portrait relief of the features of the man in whose memory it stands. The tower is surrounded by a grove of beautiful pines. These trees were planted under Mr. Smith's direction as part of the forestation program around the Ashokan Reservoir. The tower is not far from "Menaltink", his country place on the shore of the reservoir, where he spent many happy hours entertaining his friends and associates both during and after the days of the Catskill operations. It was dedicated on June 25, 1936.

By his will and testament, Mr. Smith bequeathed to the Society the sum of $20 000, the income of which is to be used in the way of promoting and advancing the best interests of Engineering as the Board of Direction may determine. One of the first objectives is to be a memorial volume describing the Catskill Water Supply Project. For effecting this purpose the Board of Direction at its meeting in January, 1936, appointed a committee of Society members.

Such was the man, so was he honored in life, and so did he to the very last strive to discharge his debt to the profession. His whole life was devoted to that end—to help his fellow workers—to make his profession better and broader and of greater service to mankind.

These words are in tribute to his memory. He was a genius of skill and prophetic vision. His leadership was that of fearless honesty. He held his profession as a cause to be kept sacred and to this end he accomplished far more than falls to the lot of most. As his name is written high on the record of accomplishment so also is it inscribed at the top of the scroll of those who labored to promote both the welfare of the engineer as an individual and the ideals of his calling.

And withal, Waldo Smith was a man of exceeding gentleness. The modesty of greatness was his. He shunned publicity. As honors came to him he insisted that they were the due of his associates. He was kind and considerate always, but forceful and direct whenever occasion required. He loved the simple things of life. In his friends, his flowers, his pigeons, his dog, in music, and in poetry he found peace and inspiration and comfort. The autumn of each passing year brought to him rare delight in the delicate blue of the fringed gentian; and now, in another autumn, he sleeps under the sod of Massachusetts where in boyhood he learned to live the truth and where the gentians, as heralds of winter, will bloom in all the years to come.

Waldo Smith, we salute you. Your name and your influence will live through the years both as history records them and as the legion of your friends tells them to its children. *Sic itur ad astra.*

WILLIAM VALORUS ALFORD, M. Am. Soc. C. E.[1]

DIED AUGUST 2, 1935

William Valorus Alford was born in Windham, Ohio, on October 7, 1858. He was the only child of Darius Manley and Cathaline (Brewster) Alford, being, on his mother's side, the tenth generation direct from William Brewster who came to America on the *Mayflower*, and on his father's side, the ninth generation from Patience Brewster, the daughter of William Brewster.

He attended school in Garrettsville, Ohio, and was graduated from High School in 1879. As a boy on a farm, he looked forward to surveying and engineering and worked toward the goal which he determined to attain, and in which he achieved great success. Wherever Mr. Alford went, he continued his studies.

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[1] Memoir prepared from information supplied by the family and on file at the Headquarters of the Society.

When very young, his first engineering trip took him with a party into the Southwestern United States, where they found it necessary to hide in canyons in order to escape the raids of Apache Indians under Geronimo. In 1887, he was employed on the Nicaragua Canal Survey and was the first white man to go through the wilderness of the Isthmus of Darien. The late Rear-Admiral Robert E. Peary, U. S. Navy, who afterward attained fame as an Arctic explorer, was Chief Engineer at Nicaragua at that time, and his tireless energy and steadfastness of purpose won the sincere admiration of Mr. Alford, who always felt Admiral Peary to have been one of his truest friends.

After the taste of the tropics, Mr. Alford always longed for the southland, and much of his life was spent in Central America and South America, his work taking him into all the South American countries with the exception of Chile and the Guianas, and across the Isthmus of Panama thirty-five times. He especially enjoyed railroad construction work and from 1907 to 1910 was Engineer of Maintenance for the Cerro de Pasco Railway Company in Peru. This work included preliminary surveys, triangulation covering about 40 to 100 miles of surveys for hydro-electric power plants, etc. He made four reconnaissance trips into the Valley of the Amazon, investigating timber resources. He always took great interest in the life of the country through which he passed. He was a keen observer of both plant and animal life, and always made friends with the natives in order to study their primitive ways of living.

In 1911, he was Locating Engineer on about 350 miles of railway in Uruguay. In 1912, he was Government Engineer, on the Island of Santo Domingo, on highway location and construction of about 30 km of water-bound macadem. From 1913 to 1916, he was Chief Engineer for the United States Steel Products Company on railway location, in Salvador. This road was never built because of the World War.

The years 1917 and 1918 found Mr. Alford again in the High Andes as Chief Engineer for the American Vanadium Company in charge of all plans, maps, and estimates, as well as the location of five miles of aerial tram-line for conveying ore. Other trips into the Valley of the Amazon followed until 1921, when he went to the Orient in charge of construction work in Shanghai, China. While in the East, where he remained until 1923, he constructed the foundation for St. Luke's International Hospital in Tokyo, Japan. This site was abandoned after the 1923 earthquake, although the foundation withstood the crash, and was practically the only construction which remained in that part of Tokyo.

His last trip was in 1926 and 1927 when he was Chief Engineer for the Venezuela Gulf Oil Company, at Maracaibo, Venezuela. Ill-health made it necessary for him to return to the United States, and he never was well enough to undertake further extensive work, although he was engaged in land surveying and smaller projects at Garrettsville for several years.

Mr. Alford had a keen appreciation of beauty and the eye of a true artist. He realized the field for photographic work on his trips, and early in his career took time off to perfect himself in this interesting hobby. At one time he conducted a photographic gallery, but soon gave it up to continue his chosen profession. He always carried a complete photographic outfit with him wherever he went and brought back a thorough record of the country through which he had traveled and of the work which was being done. He carried this hobby to such perfection that his pictures have been published in many leading magazines and books. The Fifteenth Edition of the Encyclopædia Britannica carries a full page of illustrations on South America, and the Pan-American Union has in its files many fine pictures from his negatives. The *Nature Magazine* for September, 1935, contained two of his interesting pictures from the Rock Forest of Peru. He was so completely in love with photography that he never disposed of a negative; he did his own printing and sold (or gave away) only prints. His splendid collection of negatives was one of his greatest sources of joy.

Mr. Alford was a great lover of trees, flowers, and birds, and when his health became so poor that he was no longer able to engage in active outdoor duties, he found great pleasure in his own door yard and home. Tulips were his favorite flowers, and he cultivated and enjoyed them to the utmost.

He was a member of the Masonic Order. He was also a Fellow of the Royal Geographic Society of London, England.

He was married in 1919 to Georgia Lee Robinson, of Warren, Ohio, who survives him, and lives at their home in Garrettsville. Three children by a former marriage, Mrs. J. C. Archer, of New Haven, Conn., Mrs. D. D. Kellogg, of Warren, Ohio, and William Brewster Alford, of Cleveland, Ohio, are also left to mourn his passing.

Mr. Alford was elected an Associate Member of the American Society of Civil Engineers on November 1, 1910, and a Member on March 5, 1928.

THOMAS HENRY ALISON, M. Am. Soc. C. E.[1]

DIED JULY 2, 1932

Thomas Henry Alison was born in Toronto, Ont., Canada, on September 25, 1872. His parents, James and Mary (McCord) Alison, came from Belfast, Ireland, and settled in Toronto.

He was prepared for college at the Wellesley School, and at Jarvis Street Collegiate, in Toronto. He was graduated in 1892 from the School of Practical Science in Civil Engineering (the Engineering Department), of the University of Toronto. In 1892 the University established the degree of Bachelor of Applied Science open to graduates of the School; Mr. Alison took the extra year and obtained the degree of Bachelor of Applied Science in 1893. In 1898, the University conferred on him the degree of Civil Engineer. The degree of Civil Engineer is only granted at this University three years or more after graduation.

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[1] Memoir prepared by T. Kennard Thomson, M. Am. Soc. C. E.

The School of Applied Science of the University of Toronto was founded by Professor John Galbraith, who later became Principal, and, finally, Dean. It is now called the Faculty of Applied Science and Engineering. Professor Galbraith who was fifty years ahead of his time, thought it foolish to call a man a Civil Engineer until he had at least three years practical experience after graduation. He always said that he did not try to turn out full fledged engineers, but only endeavored to teach his students how to study. He realized that if they obtained actual work during the summer vacation, they would better understand what they were studying during the winter, and, therefore, made the college term six months, from October to April, with the vacation (for work) from April to October. Mr. Alison took advantage of this opportunity to obtain work with the Toronto Belt Line Railway Company in the summers of 1890 and 1891.

After finishing his post-graduate course in 1893, Mr. Alison was appointed Assistant Engineer for The Central Bridge and Engineering Company, at Peterboro, Ont., Canada, until October, 1895. From October, 1895, to March, 1896, he was Designing and Assistant Engineer for the New Jersey Steel and Iron Company, at Trenton, N. J., on the designs for cableways in Haiti, bridges and buildings, including theatres, etc., working with the writer much of the time.

From March, 1896, to February, 1897, Mr. Alison was with the firm of Post and McCord, in New York City, on steel work. In February, 1897, he accepted a position as Engineer for Brunner and Tryon, Architects, on steel buildings and foundations, leaving that work in April (at the request of the writer) to become Assistant Engineer for the Dutton Pneumatic Lock Company, which proved to be interesting work.

In July, 1897, Mr. Alison became associated with the late Augustus Smith, M. Am. Soc. C. E., Engineer and Contractor, in New York City, as his Chief Engineer, a connection which lasted until Mr. Smith died in March, 1932, a period of thirty-five years of mutual respect and friendship. Mr. Smith had left the management of The Bergen Point Iron Works largely to Mr. Alison for years, and the day before his sudden death he told the writer with satisfaction that although they were not making money in 1932, owing to the depression, they had not been discharging their men.

On Mr. Smith's death, from a heart attack, Mr. Alison became President as well as Chief Engineer of The Works, but, like Mr. Smith, he had long suffered from heart trouble, and the strain of losing his old friend and Chief of thirty-five years' standing was very severe, although, as usual, he faced the situation bravely.

It would be impossible to mention all the structures designed or built by Thomas H. Alison for Augustus Smith and The Bergen Point Iron Works, but some of these are: The West Bank Light House, 50th Street, in New York City; 129th Street Recreation Piers, New York City; New York Central and Hudson River Railroad Freight Terminal at Barclay and Desbrosses Streets, New York City; the bridge at 135th Street and Mott Haven Canal, for the Department of Bridges, New York City; the design and construction of coal pockets for various companies, and for the United States Navy Department, at Washington Navy Yard, etc.; coaling stations for the U. S. Navy Department, at the New York and Boston (Mass.), Navy Yards, and at Balboa and Cristobal, Canal Zone; the Narragansett Bay Coal Depot, concrete buildings and pockets, steel bridges, etc.; and buildings for sugar factories in Cuba and Puerto Rico.

Mr. Alison was an Engineer in the highest sense of the word, a born student, and fond of research work. He was a quick, accurate, and industrious worker, with the most necessary qualifications for a good Engineer—common sense, a strong sense of justice, and love for his fellow men.

While at college, he had been Captain of the Cricket Team and frequently played the game at the Old Athletic Grounds of the Bergen Point Section, of Bayonne, N. J. An Englishman says that anything dishonorable "is not cricket". Here, we say a good man "played the game". Mr. Alison always "played cricket" and "played the game".

He had a summer home at Orr's Island, Me., and had lived for many years in Bayonne, N. J., where he died.

Mr. Alison was married to Mary Lowery Moorhead, the daughter of the famous Dr. Joseph Moorhead, and their fine children are: James Moorhead Alison, and Janet and Mary Lowery Alison.

In spite of his strenuous engineering work, Mr. Alison found time for many other activities; he was an Elder of the Fifth Street Reformed Church of Bayonne; and Treasurer of the Bayonne Chapter of the American Red Cross. He was connected with the Young Men's Christian Association for about forty years, and was on the Board of Directors of the Central Young Men's Christian Association; a member of the Mutual Culture Club, the University of Toronto Club of New York (26 yr), serving as a Director, Vice-President, and then, for five years, as Secretary-Treasurer; and of the Canadian Club of New York for many years, serving as a Director and on many Committees. No matter how many organizations he belonged to, he gave the best of his ability and time to each.

Thomas Henry Alison was loved and held in the highest esteem by all who knew him. A man of great ability, quick, hard working, accurate, absolutely honest and faithful to his family, friends, and work, his passing at the age of 60 left a great void.

Mr. Alison was elected a Member of the American Society of Civil Engineers on October 7, 1908.

CHARLES KYES ALLEN, M. Am. Soc. C. E.[1]

DIED JULY 14, 1935

Charles Kyes Allen, the son of Hiram Kyes and Martha (Davis) Allen, was born in Massillon, Ohio, on February 20, 1867. Shortly after his birth, his parents moved to Missouri where his father was engaged in railroad contracting and, later, in agricultural interests near Columbia.

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[1] Memoir prepared by Mrs. Martha (Allen) Hyre, Louisville, Ky.

Charles Allen was educated at the State University of Missouri, at Columbia, from which he was graduated in 1890. In 1892, he entered the State School of Mines at Rolla, Mo., from which he was graduated with the degree of Bachelor of Science in Civil Engineering.

He commenced the practice of engineering as Resident Engineer in charge of the construction of a sewer system for Mexico, Mo. He was appointed City Engineer of Mexico and County Surveyor of Audrain County, holding both positions for several years. During this time he was married to Katye Lee Ringo, of Mexico. After completing his term as County Surveyor, he was employed for a few years in railroad location surveys and in the surveying of large ranches in the (then) Indian Territory. He was also employed on the precision level survey for the Mississippi River Commission.

In 1900, Mr. Allen moved to Kansas City, Mo., and was engaged in the drafting of designs of bridges for J. A. L. Waddell, M. Am. Soc. C. E., working in that capacity for about nine months. He resigned to become Chief Inspector in the Water Department of Kansas City, remaining in that position for about eight years, during the last half of which he was Assistant Engineer in active charge of the design and construction of all plants.

He returned to the office of Mr. Waddell in the latter part of 1908 and was Resident Engineer on the Congress Avenue Bridge, at Austin, Tex., several large lift bridges, in Portland, Ore., the Arroyo Seco Bridge, at Pasadena, Calif., and the Kaw River Bridge and the Blue River Bridge, in Kansas City. Later, he made surveys and borings of the Mississippi River at New Orleans, La., for Mr. Waddell.

In February, 1919, he entered the service of the United Iron Works of Kansas City as Designer for new steel sheds at several of its plants, but, in 1920, he again returned to Mr. Waddell's Office to take charge of the building of a highway bridge over Red River at Author City, Tex., and various other work being constructed by the firm, in the next few years.

In the fall of 1925, Mr. Allen went to Bath, Me., as Resident Engineer on the building of the large bridge over the Kennebec River at that place. On that bridge caissons were sunk 123 ft below high tide, which was 9 ft deeper than any pneumatic caissons had ever gone before. Upon the completion of this work he went to Charleston, S. C., to build the Cooper River Bridge, an exceptionally difficult piece of construction.

In October, 1929, he started the work on the Suspension Bridge over the Maumee River, at Toledo, Ohio, for Mr. Waddell, completing it in the fall of 1931. Mr. Allen's health compelled him to take a much needed rest at that time and he did not attempt to take up his work again until the summer of 1934, when he went to Chicago, Ill., to construct some of the plants for the new sewerage system built under Government supervision. He was only there a few months, however, when he was obliged to return to his home in Kansas City because of his health. His death occurred within a short time, on July 14, 1935.

He is survived by his widow, Katye (Ringo) Allen, a daughter, Mrs. Martha (Allen) Hyre, and a granddaughter Martha Kathryn Hyre.

This brief recital of Mr. Allen's work is rather inadequate, inasmuch as his whole life was wrapped up in his profession. His outstanding ability as an engineer, his devotion to his duty at all times, and his high ideals will be remembered by his many friends always.

Mr. Allen was elected a Member of the American Society of Civil Engineers on October 31, 1911.

JAMES PIERSON ALLEN, M. Am. Soc. C. E.[1]

Died September 11, 1934

James Pierson Allen, the son of Thomas Pierson and Sarah (Bell) Allen, was born at Charleston, S. C., on April 18, 1848. His early education was received at a private school in his native city. As a boy of sixteen, he saw service in the Confederate Army with the Citadel Cadets, remaining with the Corps until it was disbanded at the close of 1865. In 1866, Mr. Allen entered Furman University, at Greenville, S. C., from which he was graduated in 1869 with the degree of Bachelor of Philosophy. He attended the University of Virginia, at Charlottesville, Va., from 1869 to 1870.

In August, 1870, Mr. Allen entered the employ of the United States Engineer Department and for about five years was associated with its offices at Milwaukee, Wis., Detroit, Mich., and St. Paul, Minn. In the spring of 1875, he went to Peru where he was engaged on the construction of the Chimerlote, Hacuraz, and Pecŭay Railroad, in the Andes Mountains. He returned to the United States in December of that year and engaged in farming in South Carolina until the fall of 1876, when he went to San Antonio, Tex., to practice general surveying and engineering. In July, 1878, he accepted an appointment as Assistant Engineer in the United States Engineer Office at Rock Island, Ill. While there, he was engaged in mapping 115 miles of the Mississippi River, from Savanna, Ill., to Burlington, Iowa. He continued work on the Mississippi River until November, 1880, having local charge of improvements near Fountain City, Wis., and Winona, Minn.

On March 1, 1881, Mr. Allen was appointed Assistant in charge of surveys for the United States Engineer Office, at Charleston, S. C. Later, he became Principal Assistant Engineer, having direct supervision of projects for the improvement of Charleston Harbor, Wappoo Cut, Salkahatchie and Edisto Rivers, in South Carolina, and the Altamaha River and Brunswick Harbor, in Georgia. He also had supervision of the fortifications of Charleston Harbor, which included the modernization of the batteries at Forts Sumter and Moultrie. His chief work, however, was the construction of the jetty system in Charleston Harbor, which stands to-day as a monument to his untiring energy.

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[1] Memoir prepared by W. S. FitzSimons, Assoc. M. Am. Soc. C. E.

Shortly after the United States entered the World War, Mr. Allen was appointed District Engineer for the South Carolina District, and served in that capacity until January, 1919, when he resumed his duties as Principal Assistant Engineer.

He retired from the Government Service on August 20, 1920, at the age of 72, when the National Civil Service Law went into effect. In September, 1920, he opened an office in Charleston, for the practice of general engineering under the firm name of Allen and FitzSimons. This partnership continued for a period of about six years, after which Mr. Allen retired from active business.

Mr. Allen was recognized by all who knew him as a man of splendid character and high ideals. He loved his work and devoted practically all his time to engineering matters. Personally and professionally, he maintained a high standard of ethics, never swerving from a course he believed to be right. While he was severely just in his dealings with his subordinates, he inspired their respect by his integrity and impartiality. To those whose privilege it was to know him, his memory will always be cherished.

He was married on April 25, 1876, to Mary Malvina Bailey, of Charleston, S. C., and is survived by a son and three daughters.

Mr. Allen was elected a Junior of the American Society of Civil Engineers on March 5, 1879, and a Member on June 4, 1884.

WILLIAM WALLACE ATTERBURY, M. Am. Soc. C. E.[1]

Died September 20, 1935

William Wallace Atterbury was born at New Albany, Ind., on January 31, 1866. He was brought up in Detroit, Mich. His father was the Rev. John Guest Atterbury, D. D., who renounced the practice of law to become a Presbyterian minister. His mother was Catherine J. Larned, daughter of Gen. Charles Larned, who was Attorney-General for the Territory of Michigan under Governor Lewis Cass; his grandfather was Lewis Atterbury, and his grandmother, Catherine Boudinot, niece of Elias Boudinot, at one time President of the Continental Congress.

After receiving a liberal preparatory education, and because of a family tradition, Mr. Atterbury entered Yale University, at New Haven, Conn., in the Sheffield Scientific School, from which he was graduated in the Class of 1886 with the degree of Bachelor of Philosophy.

After graduation he entered the service of The Pennsylvania Railroad Company on October 11, 1886, as an Apprentice in the Altoona (Pa.) Shops. Upon completing his apprenticeship, from 1889 to 1896, he served, successively, as Assistant Road Foreman of Engines on various divisions of the Pennsylvania Railroad System, Assistant Engineer of Motive Power, on what was then known as the Company's Northwest System, and Master Mechanic of the Pennsylvania Railroad Shops at Fort Wayne, Ind.

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[1] Memoir prepared by a Committee of the Philadelphia Section, consisting of Edward Brinton Temple, *Chairman*, and Clark Dillenbeck, Robert Farnham, John Clifford Hodges Lee, and Wallace Nelson Mayhew, Members, Am. Soc. C. E.

On October 26, 1896, Mr. Atterbury was appointed Superintendent of Motive Power, and on October 1, 1901, he was advanced to General Superintendent of Motive Power, with jurisdiction over the Pennsylvania Railroad Lines East of Pittsburgh and Erie.

His wonderful executive ability was clearly demonstrated the following year (1902), during a period of great industrial activity, when he was called upon to clear a traffic jam which seriously affected freight and passenger movement through the Pittsburgh District. He was rewarded on January 1, 1903, by being appointed General Manager of the Pennsylvania Lines East of Pittsburgh and Erie. It was during this period that he came to understand both the needs and psychology of the employees, and contended thus early in his career that the employees had rights which were too often neglected.

Mr. Atterbury was appointed Fifth Vice-President of the Pennsylvania Railroad Company on March 24, 1909, and placed in charge of the Transportation Department. On March 3, 1911, upon a change in the organization of the Company, he was made Fourth Vice-President, and, on the same date, was elected a Director of The Pennsylvania Railroad Company. On May 8, 1912, when the practice of designating the Vice-Presidents numerically was discontinued, his title was changed to Vice-President in Charge of Operation. He occupied this position until November 15, 1924, when he was elected Vice-President of the Company, being advanced to its Presidency on October 1, 1925, on the retirement from active service of the late Samuel Rea, Hon. M. Am. Soc. C. E.

While Mr. Atterbury was Vice-President in Charge of Operation of the Pennsylvania System he was unanimously chosen to be President of the American Railway Association, on May 17, 1916. As the head of this Association he rendered invaluable service to the United States Government in connection with the transportation of troops and war material and supplies to the Mexican Border, as well as to the Atlantic Seaboard, thus, in a sense, paving the way for his later mission to Europe in the service of his country.

Shortly after the entry of the United States into the World War, the Commanding General of the American forces abroad included the following in a cable to the War Department:

> "Have made thorough study of railroad situation and am convinced that operation of railroads must be under man with large experience in managing commercial railroads at home. Successful handling of our railroad lines so important that ablest man in country should be selected."

The War Department thereupon chose Mr. Atterbury, and he was subsequently designated as Director General of Transportation of the American Expeditionary Forces, with the rank of Brigadier General. In this position he assumed charge of the details of organization and operation of the United States transportation requirements in France and their co-ordination with those of the other Allied Armies. His success in this difficult undertaking is attested not only by the testimonials of his superior officers in the Army, but also by the decorations which he received from his own and from foreign Governments.

In his autobiography, Gen. John J. Pershing, U. S. Army, said of Mr. Atterbury:

> "In our first conversation we ran over the problem in a general way. Much to my surprise, Mr. Atterbury seemed to be very familiar with the situation, and his personality, his force and his grasp of the difficulties of the task, and his willingness to understand it, appealed to me at once."

Mr. Atterbury sailed for Europe on August 18, 1917. He was commissioned Brigadier General in the United States Army on October 5, 1917, which appointment, three days later, was confirmed by the United States Senate. He returned to America on May 31, 1919, receiving honorable discharge from the Army the following day. By reason of his work with the American Expeditionary Forces he later received the following decorations:

United States..	Distinguished Service Medal.
France	Legion of Honor, Rank of Commander.
Great Britain..	Companion of the Most Honorable Order of the Bath.
Belgium.......	Commander of the Order of the Crown.
Serbia.........	Royal Order of the White Eagle.
Rumania	Grand Officer of the Order of the Crown.

Upon his return to the United States, Mr. Atterbury again took up his work with the Pennsylvania Railroad Company, and found himself in the midst of the greatest labor problems of his life. The interests of the Pennsylvania Railroad employees were always his interests. Consideration of the rights of those who labor was uppermost in his mind. In regard to this, he stated:

> "The World War brought home to me that what we fought for in '76, in 1812, in '61, in '98, and again in 1917-18, was real—that life, liberty and the pursuit of happiness were the inherent right of all. I had always known that it was my right, but I do not know that I had ever had so keen a realization that it was the right of every other fellow. I tried to put myself in the other fellow's place. What did that right mean to me, and what did it mean to him? It seemed to me that the following, at least, were essentials:
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> "1.—Steady employment;
> "2.—A good wage;
> "3.—Time for recreation;
> "4.—Opportunity to elevate myself in my employment;
> "5.—A voice in determining the rules and regulations under which I should work;
> "6.—A fair division of any profits after a reasonable wage had been earned and a sufficient amount had been paid to capital to attract it to an expanding business."

These were the principles upon which Mr. Atterbury founded the plan of employee representation, which was designed to give the employees of the Pennsylvania Railroad Company an equal voice with the Management in the handling of their mutual affairs. The effect of this new policy in management brought substantial benefits not only to the Pennsylvania Railroad Company, and its employees, but also, generally, to the railroad transportation industry of the United States. This was his most prideful accomplishment. It laid the foundation for a spirit of fair dealing and co-operation which was largely responsible for the success with which the Pennsylvania Railroad Company was able to meet the depression period through which the country was passing at the time of his death.

During Mr. Atterbury's absence in France the railroads had been taken over and operated by the Federal Government. After he resumed his position as an Executive Officer of the Pennsylvania Railroad Company he proceeded with the plans for the re-organization of the Operating Staff incident to the consolidation of the former East and West Lines of the Company, and for the creation of separate regions for the operation of the System lines upon the return of the railroads of the country to their owners. The success of these changes was amply demonstrated in the safe, efficient, and economic operation of the entire Pennsylvania Railroad System since the termination of Federal control on March 1, 1920.

Increasing competition of other forms of transportation, and the current tendency of bringing the Government more actively into business management, largely due to the depression and the difficulties of financing large corporations, required Mr. Atterbury's constant attention as head of the Pennsylvania Railroad Company. To these and similar problems he devoted himself with resourcefulness, vision, and courage. It was his frequently asserted view that the Company should not consider itself as a railroad enterprise alone, but rather as a general carrier ready to encourage, develop, and utilize any form of transportation, or any combination of whatever forms of transportation that may meet the public needs. Practical demonstration of this view materialized through the Company's interest in the development of air lines, motor-bus lines, trucking activities, and in the collection and delivery of less than carload shipments of freight.

His contribution, therefore, to the development of American transportation, while it is fairly well known and recognized to-day, will not be fully appreciated in all its far-reaching significance for years to come. He left a stamp of progressive development on American railroads that even his contemporaries who were intimately associated with him, have not yet fully realized.

Following the death of Mr. Atterbury on September 20, 1935, M. W. Clement, President of the Pennsylvania Railroad Company, said of him:

> "He was one of the most far-seeing business executives of the country, recognized not only in his own field of transportation, but in the industrial and economic field generally. He had the admiration and affection of every one who was privileged to know him."

During Mr. Atterbury's incumbency as President of the Pennsylvania Railroad Company the many new problems confronting the railroad industry crystallized the necessity for a co-ordination of the activities of the railroad companies. From his suggestion grew the Association of American Railroads, which was organized to promote and improve railroad service in the public interest and to maintain the integrity and credit of the industry. The Association became effective on October 12, 1934, and now enables the railroad companies to present a united front in dealing with such problems to be met by the industry as a whole.

His courage in the face of obstacles is well exemplified by his prosecution of the work of electrifying the lines of the Pennsylvania Railroad Company between New York, N. Y., and Washington, D. C. Begun in 1928, the work was halted by the difficulty of financing brought about by the world-wide depression. Upon the offer of the Government to advance funds, Mr. Atterbury did not hesitate. He accepted the offer and pushed the work to a conclusion so that the lines were opened for operation in February, 1935.

At the re-organization meeting of the Board of Directors of the Pennsylvania Railroad Company on April 24, 1935, following the Annual Meeting of the Stockholders, Mr. Atterbury, because of ill health, declined to permit the placing of his name in nomination again for the Presidency. Had he lived and continued in this capacity, he would have been retired automatically from active service under the pension regulations of the Company on February 1, 1936. The Directors reluctantly acceded to his request, which thus closed the active official career of one who was, according to a Minute adopted that day by the Board of Directors, "distinguished not only by his able and faithful administration of the affairs of this corporation, but for his outstanding services to the Nation." He, however, retained his position as a member of the Board of Directors, and regularly attended the meetings.

Mr. Atterbury died unexpectedly on September 20, 1935, in the Bryn Mawr Hospital. "Apoplexy, following arteriosclerosis", was given as the cause of his death, in a formal statement issued by the attending physicians.

The funeral services held on the afternoon of Monday, September 23, at the Protestant Episcopal Church of the Redeemer in Bryn Mawr, Pa., were largely attended by relatives, friends, and heads of the principal industrial and transportation companies of America, and a large number of Pennsylvania Railroad employees, among whom were many who had served side by side with him in his early days of railroading. Interment was private, in the cemetery of the Old St. David's Church, near Radnor, Pa.

Mr. Atterbury was a staunch Republican and for years was a prominent figure in his party's politics in the State of Pennsylvania. In 1928, he became a National Committeeman from his State, but this post he resigned in 1930.

His character throughout his life did not change with his steady rise to prominence. When he was President of the Railroad Company, his office was as simple as that of any employee, and his home tastes were similar. He lived at Radnor, a suburb of Philadelphia, in a remodeled farm house, and also owned a large farm in the Chester Valley, to which he frequently motored, and on which he took great pride in raising dairy cattle and draft horses, more or less as a hobby. He played golf with the same enthusiasm he always displayed at work. Although he liked to dance and play cards, he disliked banquets and ceremonies of all kinds. He was much sought as a public speaker, but because of his position he invariably read an address which he had prepared in advance.

Among his business associates he was known as a great fighter, yet he had the utmost respect for those who fought him. He did not respect a man who could be won over too easily. He never took a decision until he was thoroughly familiar with both sides of a case. Once having said "no", however, he was extremely difficult to change. He was most determined in every stand he took to see that justice and right prevailed.

Mr. Atterbury was actively identified throughout his lifetime with the leading engineering, philosophical, economic, scientific, business, historical, fraternal, and military societies of the United States. He was a member of the Academy of Natural Science of Philadelphia; American Academy of Political and Social Science; American Legion; American Museum of Natural History (New York); American Society of the French Legion of Honor; American Philosophical Society; American Society of Mechanical Engineers; Franklin Institute of Pennsylvania; Historical Society of Pennsylvania; Indiana Society; Sons of the American Revolution; Military Order of the World War; National Aeronautic Association; Pennsylvania Academy of the Fine Arts; and the Society of American Military Engineers.

He was also a member of the Engineers Club, Manufacturers and Bankers Club, Philadelphia Club, Rittenhouse Club, and Union League, of Philadelphia; the Union Club of New York; and an Honorary Member of the Union League of New York; a member of the Merion Cricket Club, Haverford, Pa.; Gulph Mills Golf Club, Gulph Mills, Pa.; Seaview Golf Club, Absecon, N. J.; Radnor Hunt, Radnor, Pa.; the Corinthian Yacht Club of Philadelphia, Essington, Pa.; and the Gibson Island Club, Pasadena, Md.

In the halls of learning Mr. Atterbury was publicly recognized as an outstanding national figure. He was twice honored by Yale University in conferring upon him in 1911, the degree of Master of Arts, and in 1926 the degree of Doctor of Laws. Similarly, the degree of Doctor of Laws was conferred upon him by the University of Pennsylvania, Philadelphia, Pa., in 1919, by Villanova College, Villanova, Pa., in 1927, and by Temple University, in 1929. In 1932, he received the degree of Doctor of Engineering at the Pennsylvania Military College, at Chester, Pa.

Few men in American industrial and economic life leave behind such indelible achievements as those handed down to posterity by William Wallace Atterbury. His death marked the passing of a dynamic personality, one of the Nation's outstanding leaders in transportation, a man who served his country faithfully and well, and one who in his later years was the President of one of America's foremost railroads.

Pre-eminent in his chosen field of endeavor and endowed with all the qualities of genuine leadership, Mr. Atterbury well deserved the international reputation which he enjoyed. His place was as high in the respect of the railroad fraternity as it was in the affections of his close associates and the army of co-workers which he so ably led.

The entire career of William Wallace Atterbury ran true to American standards. From shopman's apprentice to the head of the railroad which he served for half a century, his deeds will live on to furnish inspiration and guidance to the coming generations.

He was married in 1895 to Minnie H. Hoffman, of Fort Wayne, Ind., who died in 1910. In 1915 he was married to Mrs. Arminia Rosengarten MacLeod, of St. Davids, Pa. He is survived by his widow, three sons, Malcolm, George Rosengarten, and William Wallace, Jr., and a daughter, Mrs. Elizabeth Connelly, of Radnor, Pa.

Mr. Atterbury was elected a Member of the American Society of Civil Engineers on March 2, 1909.

ALMON BYRON ATWATER, M. Am. Soc. C. E.[1]

DIED MARCH 6, 1935

The Atwater family is an old one in American history, the first member having settled in America in 1637. Almon Byron Atwater was a direct descendant of this early settler. He was the oldest of a family of three and was born in Sheffield, Ohio, on November 19, 1845, the son of John Todd Atwater and Matilda E. (Hill) Atwater.

After attending the common schools, Kingsville Academy, at Kingsville, Ohio, and Austenbury Institute, at Austenbury, Ohio, he entered the Engineering Profession by way of railroad construction work. His first engagement was that of Resident Engineer with the Canada Southern Railway Company, from April, 1870, to March, 1874. Next, he was Assistant Engineer with the Port Dover and Lake Huron Railroad Company, from October, 1874, to November, 1876. Mr. Atwater often told of the changing conditions which occurred at this period and commented on a very trite observation, substantially as follows: "Many well informed people about 1873 said America would never have prosperity again." In view of the fact that he lived during the acute period of the 1929–1935 depression, and through the 1873, 1893, and 1907 depressions, his prediction in the early Seventies must be of much interest to the present generation.

Mr. Atwater served as Chief Engineer of the Stratford and Huron Railroad Company, from November, 1876, to March, 1877, and from the latter date until May, 1880, as Superintendent of the Port Dover and Lake Huron and the Stratford and Huron Railroad Companies. Subsequently, he was General Superintendent and Engineer of the Georgian Bay and Lake Erie Division of the Grand Trunk Railway until May, 1882, at which time he was made Chief Engineer of the Chicago and Grand Trunk Railway. He was Superintendent of the Grand Trunk Railway from May, 1885, until July 12, 1898. At this time, he left the service of the Grand Trunk Railway Company and became Assistant General Superintendent of the Michigan Central Railroad, at Detroit, Mich. After serving with this Company until July, 1902, he again returned to the Grand Trunk Railway Company as Assistant to the President, with headquarters at Detroit. As the ranking railroad official of the Grand

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[1] Memoir prepared by A. C. Everham, M. Am. Soc. C. E.

Trunk Railway Company in the United States, he had many duties to perform which gave him opportunities to show his executive ability, which was much needed during the period from December, 1902, to the date of his retirement, which was October 1, 1922.

Mr. Atwater's duties were largely executive and not engineering, although his many years of engineering training had particularly qualified him to guide the management of the Grand Trunk Railway Company in handling such difficult construction problems as important track-elevation work in Chicago, Ill., and Detroit, and other large cities. He also had charge of the completion of the Detroit and Toledo Shore Line Railroad from Toledo, Ohio, to Detroit and equipped it for steam service. He gave a great deal of his time to the building of the Grand Rapids Terminal Railroad; the new passenger station and bridge over the river at Bay City, Mich.; and also purchased considerable land at Detroit for terminal yard purposes. He gave considerable of his time also to the development of railroad facilities in Michigan, at the time of the expansion of the automobile business. Due to the fact that the Grand Trunk Railway was owned by English capital, it made his work additionally difficult, as it required careful negotiations to protect the railroad interests which were constantly being attacked by legislative and other utility controlling bodies.

Mr. Atwater was a civic worker of the highest type, but in no sense a handler of details. He knew very well how to delegate authority and have his railroad profit by this executive ability. On several occasions he had the opportunity and did escort Henry Ford and the late Thomas A. Edison on important trips. It is unfortunate that he did not record the very interesting history of these journeys as the philosophies that were exchanged by these two men, as described by Mr. Atwater, were unusually interesting. He regarded Mr. Ford as a remarkable executive. When the *Titanic* sank, it was Mr. Atwater's very sad duty to find and identify the body of the well-known President of the Grand Trunk Railroad System, Mr. C. M. Hays, who had been his life-long friend.

Mr. Atwater was a lover of the out-of-doors, having been a good golf player and continuing to play until he was 85 years of age. He was a Life Member of the Detroit Golf Club. He also belonged to, and was an active member of, a curling club in Detroit. He and Mrs. Atwater traveled extensively, having visited Iceland, Russia, Germany, Austria, France, and England during 1932 when he was 87 years of age. He was a great student and reader and left in his estate a very fine library of standard literature. He was, at one time, President of the Prismatic Club, of Detroit, an exclusive literary society.

He was a Mason, an Episcopalian, a conservative Republican, and an active member of the Sons of the American Revolution. His philanthropies were unusually quiet, carefully chosen, and of generous sums. He was one of the fortunate who profited by the development of the early air-brake patents and from this small beginning left an estate which was distributed among his nephews and nieces.

Mr. Atwater was married to Jane Thompson, of Welland County, Ontario, Canada, on July 2, 1872. They had no children. His only survivors are Dr. Carleton W. Atwater, of Indianapolis, Ind., Mrs. Maude Gerow, of Cleveland, Ohio, Mrs. A. C. Everham, of Kansas City, Mo., and Mrs. Barbara Leach, of Painesville, Ohio. He died March 6, 1935, in Pasadena, Calif., where he lived after his retirement from the railroad service on October 1, 1922.

He was a man of very definite ideas, but kindly. His executive ability, his spirit of fairness, and his clear thinking in dealing with those with whom he came in contact commanded their respect and admiration. Although quiet in his manner, he was firm in his decisions and positive in his ideas as to how work should be done. He will be remembered with gratitude by many engineers and railroad officials who owe to him their opportunities in life.

Mr. Atwater was elected a Member of the American Society of Civil Engineers on May 5, 1886.

HERBERT HOWARD BASSETT, M. Am. Soc. C. E.[1]

Died April 26, 1935

Herbert Howard Bassett was born in New Britain, Conn., on October 21, 1877. His parents, Milton Humphrey Bassett and Prudence Maria (Butler) Bassett, were long-time residents of that city, his father, a veteran of the Civil War, having been trained in the profession of the law. They were both of New England ancestry, devoted to their family and, at the same time, actively interested in the affairs of church and school and the general welfare of their city.

Herbert Howard Bassett prepared for college in the New Britain High School where he maintained a high standing. He entered Cornell University, at Ithaca, N. Y., in September, 1896, in the College of Civil Engineering, from which he was graduated in June, 1900.

His first work was with the American Bridge Company, in East Berlin, Conn., where he remained until January, 1903, when he joined the office force of Frink and Hazen, Consulting and Construction Engineers, in Baltimore, Md., a connection that was terminated by the great fire of February, 1904. For about six months, to September 1904, Mr. Bassett acted as Sewer Inspector for the Town of West Hartford, Conn., leaving this work to join the Berlin Construction Company, with which he remained until 1908. In August, 1908, he went with the Syracuse Bridge Company, at Syracuse, N. Y., for four years as Chief Engineer, changing to the Groton Bridge Company, at Groton, N. Y., in 1912, with which he remained until 1916.

In November, 1916, he was made Manager of the Worcester, Mass., Office of the Berlin Construction Company, and held this position until he was commissioned, in 1918, a Major of Engineers of the Construction Division of the United States Army, for service in the World War. In the fall of 1919, he returned to the Groton Bridge Company as Chief Engineer and remained with that Company until it went out of business in 1923.

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[1] Memoir prepared by H. N. Ogden, M. Am. Soc. C. E.

During these twenty years, Mr. Bassett had intimate and direct contact with all the ordinary problems of structural design and had become thoroughly familiar with shop construction and field erection. He was well equipped, therefore, for his entry into a changed field that occupied him almost continuously for the remainder of his life. Retaining his residence in Groton, he acted as Consulting Structural Engineer to architects practicing in cities other than Groton, but within reasonable distance. He acted for Gibb and Waltz, of Ithaca, in developing the plans for the Baker Chemical Laboratory of Cornell University, a $1 000 000 building, involving many unusual and complicated steel details. He designed the steel for a number of buildings for which Mr. S. E. Hillgar, of Auburn, N. Y., was the Architect.

Of late years, Mr. Bassett's work had been almost entirely with Mr. C. W. Clark, of Cortland, N. Y., on the design of school buildings. His last work was with Mr. Clark in the preparation of plans for the State and Government project of the office which will serve as a Regional Wholesale Market for Central New York, located on a 60-acre plat in the City of Syracuse.

As Chairman of the Building Committee of the Ithaca City Hospital, the writer had occasion to estimate the ability of Mr. Bassett, both as a Designer and as a tactful co-worker. When the latter acted with Mr. J. Lakin Baldridge, Architect, of Ithaca, in the design of the five-story Nurses Home for the City Hospital, Mr. Bassett was rigidly insistent on ample strength, particularly at joints, and could not be moved from his position either by motives of economy or architectural composition. Mr. Clark says of him:

> "Aside from being a careful, experienced, competent engineer, Bert Bassett had a personality never to be forgotten by those who were so fortunate as to know him. His kindliness, high idealism, public spirit and friendliness will be remembered by me as long as I live."

In these few words Mr. Clark has characterized admirably Mr. Bassett. His work tended to isolate him from his fellows. His ability to become absorbed in his work and deaf to his surroundings was noticeable. Equally characteristic, however, was his friendliness and social gifts. He was a regular attendant at the evening meetings of the Ithaca Section of the Society, driving from Groton to Elmira, or Binghamton, and returning home, apparently indifferent to time or weather.

During the World War, he was as stated a Major in the Construction Division of the Army Construction Corps, and after the war he was an active member of Carrington-Fuller Post No. 800, American Legion, at one time Chaplain of the Local Post, and a Past Commander of the Legion.

Mr. Bassett was a prominent Mason, a Past Master of Groton Lodge, No. 496, F. and A. M.; Past Assistant Grand Lecturer of the Cayuga-Tompkins District; Excellent High Priest of St. John the Baptist Chapter, Royal Arch Masons, of Moravia, N. Y., and a member of Cortland Lodge of Perfection, A. A. S. Rite.

Mr. Edwin L. Harman gives this tribute to his activities in the Masonic Body:

> "I never knew him to refuse any request for aid or assistance. His keen mentality, his wonderful memory, his ability as a lecturer, his clear concise judgment was ever at the service of the lodge or a brother in distress. He believed from the depths of his heart in brotherly love, relief, and truth, and endeavored to exemplify them in his daily life."

Mr. Bassett was a member of the Committee of Troop 10 of the Boy Scouts of Groton, and was active in the interests of that organization.

He was a member of the Congregational Church, regular in attendance, and for some years leader and teacher in the Men's Class in the Church School. The Rev. J. A. Goodrich writes of him:

> "It was a joy for him to render any service and he never refused to do his part in any good work. Disappointment and reverses were met in a philosophical spirit and enabled him to live victoriously. A man of clear vision and sound judgment, he had the courage of his convictions and frankly gave expression to them. One of his outstanding characteristics was to see the good in others and he never spoke ill or unkindly of any one."

In 1906, he was married to Helen L. Davis who survives him, together with a daughter, Prudence, and a son, Ethan, of Springfield, Mass.

Mr. Bassett was elected a Member of the American Society of Civil Engineers on November 28, 1916.

ROGER DERBY BLACK, M. Am. Soc. C. E.[1]

Died April 12, 1936

Roger Derby Black was born at West Point, N. Y., on January 18, 1883, at which time his father was on duty at the United States Military Academy as Instructor in Practical Military Engineering and Commander of the Engineer Detachment. He was an Army boy and came from a family of West Point graduates and Regular Army officers. He was the son of the late Major General William Murray Black, U. S. Army, M. Am. Soc. C. E., the World War Chief of Engineers, and Daisy Peyton (Derby) Black. He was the grandson of Captain George H. Derby, Topographical Engineers, U. S. Army, who under the *nom de plume* of John Phœnix wrote "Phœnixiana" and "Squibob Papers", among the first books of American humor.

Roger Derby Black's early education was obtained at various Army posts and stations in the United States to which his father happened to be assigned. His secondary education was obtained at the Central High School, in Washington, D. C., in 1896 and 1897, and at St. Paul's School, Concord, N. H., in 1898 to 1900. He received an "at large" appointment to West Point in 1900 and entered the Military Academy on August 1 of that year. While at West Point he demonstrated his ability in both scholastic and military matters, being a first section man in one, resulting in his assignment to the Corps of Engineers upon graduation, and in the other, wearing chevrons for three years, as a Cadet Corporal, the Cadet Sergeant Major, and one of the six Cadet Captains in the Corps of Cadets. He was graduated from West Point in the Class of 1904 and was commissioned a Second Lieutenant in the Corps of Engineers.

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[1] Memoir prepared by Colonel R. T. Ward, Corps of Engrs., U. S. Army.

Lieutenant Black served in the Corps of Engineers, United States Army, passing through all the grades: Second Lieutenant, 1904 to 1906; First Lieutenant, 1906 to 1912; Captain, 1912 to 1917; Major, May to August 1917; and September, 1917, Lieutenant Colonel, reaching that of Colonel of Engineers, National Army, on February 1, 1918. In July, 1919, he resigned his commission in the Regular Army to engage in engineering in civil life.

From his graduation in 1904 to the declaration of war against Germany, Colonel Black performed the varied military and civil duties chargeable to the officers of the Corps of Engineers, including engineering works performed by Engineer troops and river and harbor operations under the Engineer Department at Large.

As Second Lieutenant, his first service was with Company B of the 1st Battalion of Engineers, at Fort Leavenworth, Kansas, and in the Philippine Islands, at Fort William McKinley, where he was engaged upon military surveys, road, target range, and water supply works, and various strictly military duties. He left the Philippine Islands in the summer of 1906 and spent the next year as a Post-Graduate Student at the Engineering School, then at Washington Barracks, D. C. He was promoted to the grade of First Lieutenant in September, 1906. The year, 1907–08, was spent in garrison duty at Fort Leavenworth, with Company M, 3d Battalion of Engineers. From 1908 to 1910, he was engaged in survey, land defense, and fortification work with a detachment of Company M in the vicinity of San Francisco, Calif.

From 1910 to 1914, Lieutenant Black was on river and harbor work with station at Albany, N. Y. He was in immediate charge of the Northern Section of the First New York Engineer District, having charge of river and harbor projects in the vicinity of Albany, Lake Champlain, and the Upper Hudson River. In this work he developed and tested a detailed cost-accounting system, one of the first of its kind used in the Engineer Department and now in standard use in its essential features. He was promoted to the rank of Captain in February, 1912.

For the next two years Captain Black again served in the Philippine Islands as Commander of Company L and, later, as Adjutant, of the 3d Engineers. In addition to his company and regimental duties, he was in charge of a District of the Military Survey of the Island of Luzon and of topographical surveys of the environs of Manila.

In the spring of 1917, while on duty in the Office of the Chief of Engineers and as a member of the Board on Engineer Troops, he prepared the manuscript for the Engineer Training Manuals used during 1917 and 1918 by the divisions in training in the United States and overseas. He was promoted to the rank of Major in May, 1917, and in September, 1917, to that of Lieutenant Colonel.

During the World War, Colonel Black rendered invaluable service due to his detailed and intimate knowledge of military organization and military training methods. He was responsible for much of the training and organizational literature used by the troops, and participated actively in the training of Staff Officers and of Engineer troops, both in the United States and in France.

Among the first American troops called for service in France were Engineer Railway Regiments. These regiments had to be raised, equipped, trained, and sent overseas in a very short period of time and practically the entire officer personnel and all the enlisted personnel came, of course, from the American railroads. The very few Regular Engineer officers available for assignment to the units had to be specially qualified men. Major Black was one of the officers selected for this work in May, 1917. As Adjutant of the 13th Engineers, recruited in Chicago, Ill., he had charge of the multitudinous details of organization and training of the regiment and accompanied it overseas within three months after the initial organization. He served with the regiment until September, 1917, when the need for his organizational and training knowledge called him to service with the Expeditionary Staff at General Headquarters.

From September, 1917, to January, 1918, he was immediate Assistant to the Chief of Engineers of the American Expeditionary Force, in charge of personnel and training, and was employed upon matters relating to Engineer Staff organization. From January to May, 1918, while Colonel, commanding the 116th Engineer Regiment, he selected, organized, and operated the Engineer Replacement Training Center at Angers, France.

In June, 1918, being detailed a member of the War Department General Staff, War Plans Division, he returned to the United States and served at the Army War College, in Washington, D. C., acting as Instructor in the School for Divisional Staff Officers. After the Armistice, he was active in the drafting of plans for the re-organization of the War Department and the Staff Services which finally were, to a great extent, incorporated in the National Defense Act of 1920. He resigned from the Army in July, 1919, while a Colonel on the War Department General Staff.

The remaining years of Colonel Black's life were spent in the practice of civil engineering. Notwithstanding the exigencies of his private practice he soon found that his twenty years of military training could not be forgotten, and, in 1921, he applied for, and was commissioned as, a Lieutenant Colonel in the Engineer Section of the Organized Reserves; the following year he was promoted to the grade of Colonel. As such, Colonel Black was one of the strongest supporters and most enthusiastic members of the Engineer Reserve Corps. He was assigned to the command of the 342d Engineers in 1921, and retained command of this regiment until December, 1934, when having moved to Washington, outside the limits of the Second Corps Area, he had to relinquish command. Returning to New York City in the spring of 1936, he was again assigned to the command of the 342d Engineers, the official orders being issued on April 10, 1936, just two days before his decease.

Colonel Black was a valuable asset in the organization and training of the Engineer Reserve Officers in New York City. His regiment was of high efficiency and extreme loyalty. He was a man of unusual brilliancy and talent and had an engaging personality, which made him hosts of friends and attracted able personnel to his regiment. Much of the activity in military matters among the engineers in New York City is due to his enthusiasm and training ability. He never missed an opportunity to attend the Reserve Training Camps and his work and personality will long be remembered by the Engineer Reserve Officers in the Metropolitan Area. As a drill master he was equalled by few officers in the Army. His two-man squad drill, which he developed for the use of Reserve Officers, has been used repeatedly in camp and in inactive training with most excellent results. He was always willing to, and did, take an active and wholehearted part in having to do with the problem of national defense.

From the time of his resignation from the Regular Army in 1919 until his death, he spent the major portion of his time in and around New York City as a practicing civil engineer. He was an expert on waterways and was employed on a number of studies and reports as a Consulting Engineer in this specialty.

His first work in private practice was for H. L. Cooper and Company, and consisted of a special study and report on the economic value of improving the St. Lawrence River for deep-draft navigation and on the proper channel and lock characteristics to accommodate the anticipated commerce, without undue initial investment or necessity for future reconstruction.

From 1920 to 1924 he was President of, and organized, financed, and operated, the Roger Black Company, Incorporated, Engineering and Construction, and engaged in general contracting and speculative building projects in New York City and Long Island. He later became interested in the promotion of construction projects and engaged in real estate and building operations on Long Island. He was a Director of the Long Island Realty Investors Corporation. As partner in the firm of Leaycroft and Black, and, later, as Vice-President and Director of Colprovia Process, he was active in the development and promotion of "colprovia", a special form of bituminous paving, tested and used on various highways throughout the United States. He made special studies of Mississippi River revetments for the Merritt, Chapman and Scott Corporation. In 1932, he was engaged by the Engineer Department at Large of the U. S. War Department to make a special study on the relative merits of a high-level ship canal in comparison with a sea-level ship canal which was then under consideration for the New York-New Jersey Section of the Intracoastal Waterway, Raritan Bay to the Delaware River. This was a thorough and extensive study, including detailed analysis of possible tonnage, types of vessels, canal capacities, and general waterways economics, and is a most valuable addition to the voluminous studies and reports made upon this proposed waterway.

Upon completion of this work for the War Department, Colonel Black was engaged upon the initial organization and subsequent operation of the

Station Department of the recently completed Independent Subway System in New York City until 1934, when he was called to Washington and served as Chief of the Management Branch of the Housing Division of the Public Works Administration. Shortly before his death, he was transferred to New York City as Project Engineer of the Wards Island Sewage Plant.

Colonel Black was a man of vision and imagination. He was specially skilled in organization. His technical reports and studies are clear and complete and very valuable. He was a man of tireless energy and great initiative and had a wonderful faculty of making friends, being especially able to attract and understand the younger men. His sudden and untimely death at the age of 53 was a great loss to the Civil and Military Engineering Professions.

He was a Past-President of the New York Post of the Society of American Military Engineers; a member of the Loyal Legion; Past Commander, New York Chapter of the Military Order of the World War; a Mason; and belonged to the Army and Navy Club, of Washington, D. C. He was active in the Episcopal Church and was for some years a Vestryman of St. Marks-in-the-Bouwerie, in New York City. For his services in France he was decorated by the French Government as "Officier de L'Ordre de L'Etoile Noire."

Colonel Black died in New York City on Easter Sunday, April 12, 1936, of coronary and general arteriosclerosis from which he had suffered for a number of years. He was buried at West Point, N. Y., with full military honors, on April 14, 1936. The escort consisted of the U. S. Military Academy Engineer Detachment and the colors of his Reserve Regiment, the 342d Engineers, borne by officers of the regiment.

In 1907, he was married to Margaret Eveleth Smith, of Portland, Me. He is survived by his widow; a daughter, Helen Townsend (Mrs. Alan Gray), of New York; three sons, Lieutenant Roger Derby Black, Jr., Field Artillery, U. S. Army (Rhodes Scholar), John Murray Black, and Richard Winthrop Black; and by two brothers, William Murray Black, of Warrentown, Va., and Major Percy Gamble Black, Field Artillery, U. S. Army.

Colonel Black was elected a Junior of the American Society of Civil Engineers on January 3, 1905; an Associate Member on January 7, 1913; and a Member on September 9, 1919.

ALBERT NELSON BURCH, M. Am. Soc. C. E.[1]

DIED FEBRUARY 21, 1936

On the night of February 21, 1936, Albert Nelson Burch, a Member of the Society since 1929, passed away at his home in Sacramento, Calif. Recovering from an attack of pneumonia, his weakened body was unable to overcome the effects of complications that developed.

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[1] Memoir prepared by W. A. Perkins, M. Am. Soc. C. E.

Born at Fayette, Iowa, on June 21, 1864, Mr. Burch was the son of Benjamin and Elizabeth (Rafter) Burch. He received his education in the public schools of that State and at the Peru, Nebr., State Normal School. The teaching profession for which he had prepared was not attractive to him, however, and he subsequently entered his real life profession with a special course at the University of Wisconsin, at Madison, Wis.

California attracted him in the early Nineties, and on January 1, 1898, he became Superintendent of the Stanislaus Water and Power Company, at Oakdale, remaining in this position for nearly ten years. On October 1, 1907, he was appointed Irrigation Engineer in the United States Reclamation Service and assigned to the Umatilla Project in Oregon, being transferred later to the Orland Project in his adopted State. He became Project Manager shortly thereafter, and under his régime this project was developed into one of the most successful of those within the scope of the Reclamation Service.

On October 1, 1921, Mr. Burch resigned his position on the Orland Project and was appointed a Consulting Engineer by the Reclamation Service. For the next six years, he was retained as a Consultant by the Service, and also was employed by the State Engineer of California from March 1, 1922, to December 1, 1923, as a Consultant and Special Investigator of problems connected with the study of the water resources of the State. During 1924 and a part of 1925, he was employed as Chief Engineer of the Hollister Irrigation District. Late in 1927, he returned to the State service as Supervising Hydraulic Engineer in charge of irrigation district matters and remained in that position until his last illness.

On October 9, 1898, at Oakdale, Calif., Mr. Burch was married to Ivy May Sivley who, with their daughter, Katherine, survives him.

He was a recognized authority in irrigation problems and did much in improving practice of the art. Naturally of a retiring nature, he was respected by all who knew him for his wide knowledge and sound common sense and loved for his unfailing kindliness and subtle sense of humor. His passing leaves a place among his brother members in the Society, and among his friends and associates, that can never be filled.

Mr. Burch was elected a Member of the American Society of Civil Engineers on July 15, 1929.

WILLIAM STODDERT CARUTHERS, M. Am. Soc. C. E.[1]

DIED NOVEMBER 1, 1935

William Stoddert Caruthers, the son of William Caruthers and Mary Jane (Stoddert) Caruthers, was born at Jackson, Tenn., on January 22, 1863.

Of his early education it is known that he attended the University of the South, at Sewanee, Tenn., from 1877 to 1880, and, later, was a student in engineering at the University of Missouri, at Columbia, Mo., which he attended in 1883 and 1884. At intervals during his University studies, Mr. Caruthers was engaged in engineering work as Rodman and Leveler.

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[1] Memoir prepared by C. S. Pope, M. Am. Soc. C. E.

Following his studies at the University of Missouri he was engaged as Resident Engineer in charge of the construction on the Yazoo and Mississippi Railroad, completing a section fourteen miles in length during the years 1884 to 1886. His next engagement was as Resident Engineer during 1886 and 1887 on the Chicago, Rock Island, and Pacific Railway.

In 1888, Mr. Caruthers was engaged in general surveying practice in California, and during the same year he became Resident Engineer in charge of the construction of twenty-eight miles of track for the Oregon Railroad and Navigation Company, and on similar work, in 1889, for the Alabama Midland Railroad Company.

From 1890 to 1894, he was engaged in business in the State of Washington, and, from 1894 to 1898, was Chief Clerk of the United States Surveyor-General's Office, at Olympia.

He again returned to railroad work in 1899 as Resident Engineer in charge of line-revision surveys and construction between North Yakima and Spokane, Wash., for the Northern Pacific Railway Company, leaving this Company in the same year to become Resident Engineer for the McDonald Construction Company, on railroad construction.

In 1900, Mr. Caruthers was Engineer in charge of the substructure of a bridge over the Illinois River on the Illinois, Indiana, and Iowa Railway, under August Ziesing, M. Am. Soc. C. E. Also, during 1900, he was employed as Assistant Engineer on maintenance of way on the Union Pacific Railway.

In 1901, he was Resident Engineer in charge of the construction of fifty miles of the El Paso and Southwestern Railway, leaving this work to engage in private practice in Seattle, Wash., during 1902.

In 1903, he served as Resident Engineer in charge of construction of fifteen miles on the White River Line of the Missouri Pacific Railway, including tunnel and drawbridge work, and, in 1903 and 1904, he was Locating Engineer on the Memphis and Gulf Railway. He continued in railroad engineering work in 1904 as Locating Engineer on the Tennessee Central Railway.

In 1905, Mr. Caruthers was Assistant Engineer for the Kansas City Southern Railway Company, and, later, as Locating Engineer in Mexico on the Rio Grande, Sierra Madre, and Pacific Railway.

From 1906 to 1908, he was Assistant Engineer on the Western Pacific Railway in charge of location, and, later, of heavy construction in the Feather River Canyon, in California, including the bridges.

In 1909, he became Construction Superintendent on the Los Angeles Aqueduct, but, in 1910, he returned to railroad work as Division Engineer in charge of the construction of twenty-five miles on the Des Chutes Line, Oregon-Washington Railroad and Navigation Company.

In 1911, he again became Assistant Engineer with the Western Pacific Railway Company, and, later, Locating Engineer for the Dozier Construction Company on the West Side Railway in Sacramento Valley, California.

Mr. Caruthers' health was a constant source of concern, and it was necessary for him on many occasions to resign from agreeable work to seek a location where the climatic conditions were better suited to his well-being.

In January, 1912, he was appointed Division Engineer for the California Division of Highways at San Luis Obispo, Calif., and in August, 1912, assumed the duties of Division Engineer of Division III. He held this position from 1912 to 1920, and during the latter part of this period he also served as Highway Commissioner for the County of Sacramento, then engaged in the construction of a County Highway System. In 1920, he assumed the duties of Chief Engineer of the Sacramento County Highway Commission, and on the completion of this assignment, he again joined the California Division of Highways and was appointed Assistant Engineer on special assignments in highway location during the years 1923 to 1926. In October, 1926, he left the service of the State to engage in private practice in Oakland and Berkeley, Calif.

Mr. Caruthers' diversified experience with general construction in numerous changes of location, due largely to the fact that he never acquired a vigorous constitution, brought him into contact with a great many engineers, and especially with the younger men of the profession in whom he always evinced great interest and with whom he kept in contact throughout the greater part of his career. He was a man of unfailing courtesy and consideration, a true exponent of the old Southern school of gentlemen, and a consistent supporter of institutions for the advancement of the interests of young men, such as the Young Men's Christian Association, and other such organizations. His secret charities and the financial assistance he furnished to many people less provident than himself were guessed at, but never fully revealed until his death. He died as the result of a fall in an elevator during which he sustained a fracture which eventually proved fatal.

Mr. Caruthers was not married, but devoted himself to the care of his mother and sister who formed his family for the greater part of his life. His kindly personality made him respected and considered with the kindest feeling by all with whom he came in contact. He is survived by his sister, Mrs. Frances Wimberly, of Berkeley, Calif.

He was a member of the American Association of Engineers. He was a Mason, a Democrat, and a member of the Protestant Episcopal Church. He was also for many years a member of the Sutter Club of Sacramento.

Mr. Caruthers was elected a Member of the American Society of Civil Engineers on January 7, 1913.

JAMES ALANSON CHILDS, M. Am. Soc. C. E.[1]

DIED APRIL 13, 1935

James Alanson Childs was born in Fergus Falls, Minn., on April 15, 1886, the son of Henry W. and Alberta (Hakes) Childs. His father was a prominent attorney in Minnesota and served as Attorney General of the State. After receiving his preliminary education in St. Paul, Minn., Mr. Childs entered the University of Minnesota from which he was graduated in 1909 with the degree of Bachelor of Science in Civil Engineering.

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[1] Memoir prepared by Frederic Bass, William N. Carey, and George M. Shepard, Members, Am. Soc. C. E.

Upon graduation, Mr. Childs entered the employ of the Minnesota State Board of Health as Assistant Engineer. In 1914 he was made Engineer, Division of Sanitation, Minnesota State Board of Health and, except for a short period during the World War, he continued in this capacity until 1927. In the Army he was commissioned a Captain in the Sanitary Corps, and served as Camp Sanitary Engineer at Camp Jackson, South Carolina.

His work with the State Board of Health was chiefly of a supervisory character, pertaining to water systems, sewerage systems, and water-pollution problems. It was during this period of service with the Board that the annual typhoid fever death rate in Minnesota was reduced from about 350 to less than 50, due quite largely to improvements in public water supplies brought about by the activities of the Board of Health. Although credit for this great improvement must be distributed among several persons, Mr. Childs was proud of the fact that he was associated with, and had a part in, this undertaking.

In 1927, Mr. Childs was selected to serve as Chief Engineer and Secretary of the Metropolitan Drainage Commission of Minneapolis and St. Paul, and had responsible charge of the investigation of the sewage disposal problem of the "Twin Cities." This work he has recorded in considerable detail in the four reports of the Metropolitan Drainage Commission, published in bound volumes, for the years 1928, 1929–30, 1931–32, and 1933.

In November, 1933, the Minneapolis-St. Paul Sanitary District was organized by authority of an Act of the State Legislature, with orders to proceed with the construction of a sewage disposal project. Because of poor health, Mr. Childs declined to be considered for the position of Chief Engineer of the new District, but was appointed Sanitary Engineer, a position which he held until the time of his death. For many years he had been deeply interested in the problem of sewage disposal for Minneapolis and St. Paul, and, in this connection, had served on the Committee on Metropolitan Affairs, of the Northwestern Section of the Society which Committee considered this question among others. His studies of river pollution, particularly as applied to the Mississippi River, in Minnesota, are well known to sanitary engineers.

In addition to the positions of responsibility mentioned, Mr. Childs was Assistant Professor in the Department of Preventive Medicine and Public Health at the University of Minnesota. He was also Collaborating Engineer, United States Public Health Service. He held memberships in Sigma Xi and Tau Beta Pi Fraternities, and the Central States Sewage Association, being on the Executive Board of the latter. He rendered service as a member of Committees of the Sanitary Engineering Division of the Society and was President of the Northwestern Section in 1929.

A recital of Mr. Childs' work and achievements yields an inadequate estimate of his life. He regarded his engineering work as a means of promoting public health and welfare. His influence in the community in which he lived extended beyond the visible results of his labor; it was always creative and constructive not only by outward manifestations, but also by its force in the lives of his associates. His unswerving devotion to his duty as he saw it, his honesty of purpose and action, and his high ideals will long be remembered by his many friends.

In 1912, he was married to Muriel Tappan, who, with a daughter, Muriel, and a son, Tappan, survives him.

Mr. Childs was elected an Associate Member of the American Society of Civil Engineers on April 18, 1916, and a Member on January 16, 1928.

ERNEST WILDER CLARKE, M. Am. Soc. C. E.[1]

DIED APRIL 24, 1935

Ernest Wilder Clarke was born at Syracuse, N. Y., on May 1, 1868, the son of William M. and Ella (Wilder) Clarke. He was educated in the public schools of Syracuse, Chicago, Ill., and Boston, Mass., and completed his education by graduation in 1887 from the English High School, at Boston.

Mr. Clarke was prepared and fitted for his future engineering career by employment with various engineers and City Departments in Boston and surrounding towns. This work gave him training and experience in city surveying, sewers, water supply, electric railroads, and rapid transit, and covered a period from 1887 to 1889, varying in length in proportion to the magnitude of the projects. The variety of this training fitted him for the more important work of his later life.

In 1900, Mr. Clarke went to New York, N. Y., where he was employed on the first rapid transit subways, on changing sewer locations.

From July, 1904, to June, 1906, he was with the Panama Canal Commission on sanitary work, first at Cristobal and later at Panama City; as Division Engineer he had charge of municipal sanitation work in Panama, Colon, and all towns across the Isthmus. Upon his return from Panama in 1906 he was engaged on water-works construction at St. Stephens and at St. Johns, N. B., Canada.

In 1906, Mr. Clarke joined the Engineering Corps of the Board of Water Supply of New York City, then actively engaged on the construction of the Catskill System. He was first assigned as Senior Section Engineer to the Croton Division which covered the Catskill Aqueduct from Peekskill to the Kensico Reservoir; later, he was for a short time Assistant to the Division Engineer on the construction of the Kensico Dam. When the White Plains Division was organized, he was promoted to the position of Division Engineer and completed the Aqueduct included in that Division. Thus, he was intimately familiar with the Catskill Aqueduct construction from Peekskill to Yonkers and, in addition, the preliminary construction of the Kensico Dam and its appurtenant works.

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[1] Memoir prepared by George G. Honness, M. Am. Soc. C. E.

With this work completed, Mr. Clarke was engaged with the Quartermaster Corps, Construction Division, United States Army, in 1917. He was connected with the construction of Camp Upton on Long Island, the Fox Hill Hospital on Staten Island, and other World War projects in the Metropolitan District. From this connection, he went with the United States Shipping Board, with headquarters at Baltimore, Md., where he remained for three years.

In 1923, he joined the organization of Nicholas S. Hill, Jr., M. Am. Soc. C. E., as Designing Engineer and continued with Mr. Hill until 1932. Mr. Clarke then joined the New York City Board of Sanitation as Designing Engineer and served in that capacity until April 1, 1935, when he retired because of ill health.

He was a member of the New England Water Works Association, and of the New York State Sewerage Works Association.

Mr. Clarke was an active member and a Warden of St. John's Protestant Episcopal Church, of Pleasantville, N. Y., where he made his home.

He was married in 1893 to Ethel Lawrence Jones, of Boston, and is survived by his widow, a daughter, Mrs. George Knowles, two sons, Robert L. and John M. Clarke, and a sister, Mrs. George H. Tryon.

Mr. Clarke was elected a Member of the American Society of Civil Engineers on January 4, 1905.

RUEL KEITH COMPTON, M. Am. Soc. C. E.[1]

Died June 21, 1935

Ruel Keith Compton was born in Charles County, Maryland, on July 8, 1869, the son of Dr. Ruel Keith Compton and Rachel (Dement) Compton. He was graduated from Saint Johns Academy, at Alexandria, Va., when he was 19 years of age. In 1916, he received the honorary degree of Civil Engineer from Maryland State College.

Immediately following his graduation from Saint Johns Academy in 1888, Mr. Compton was engaged in surveying and other engineering work which added to his experience and outlook in his chosen profession. In 1890, he was made Assistant Engineer of Sewer Construction for the City of Baltimore, Md., which position he held for eight years. From 1898 to 1901, he held various engineering positions with, and supervised construction for, the United Railways and Electric Company, of Baltimore; the Pittsburgh and Alleghany Telephone Company, of Pittsburgh, Pa.; and the Johnstown Telephone Company, of Johnstown, Pa. In January, 1901, he returned to his former position as Assistant Engineer of Sewer Construction for the City of Baltimore, and remained in that capacity until May, 1905, when he was appointed Special Engineer in charge of the reconstruction of highways within the "Burnt District" of Baltimore, following the devastating fire of 1904, in which position he continued until November, 1906, when he was made Assistant City Engineer of Baltimore in charge of highway maintenance.

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[1] Memoir prepared by G. M. Bowers, M. Am. Soc. C. E., and P. A. Rice, Assoc. M. Am. Soc. C. E.

His outstanding ability and knowledge gained in the field of street paving and highway construction merited Mr. Compton's appointment in May, 1911, as Chairman and Consulting Engineer of the Paving Commission of the City of Baltimore, in which capacity he was charged with the responsibility for all construction, administrative, and executive work of the Commission, and for the expenditure of millions of dollars for all classes of standard paving construction.

He resigned his Chairmanship of the Baltimore Paving Commission to enter the World War when he was commissioned in July, 1917, as a Major in the Corps of Engineers, U. S. Army, and assigned to Camp Meade, Maryland, where he was placed in charge of the construction of roads and buildings. Later, he was transferred to Curtis Bay (Md.) Ordnance Depot in charge of general construction work and was also made Disbursing Officer at that station. Subsequently, he was transferred to and placed in charge of roads at the Quarantine Station, at Baltimore; he also served as Executive and Summary Court Officer and, later, as Acting Construction Quartermaster at the Acid Plant, Little Rock, Ark. He was honorably discharged from the Army in June, 1919, and promoted to the rank of Lieutenant Colonel of Engineers, in the Reserve Corps.

After his discharge from the Army, Colonel Compton immediately returned to Baltimore and was re-appointed, in June, 1919, as Chairman and Consulting Engineer of the Paving Commission of that city. In 1924, he received from the Mayor of Richmond, Va., the offer of the position of Director of Public Works of the City of Richmond and again resigned his position as Chairman of the Paving Commission of Baltimore to accept Richmond's call. After having served nearly eleven years in this position, his death followed a sudden heart attack. It was a shock to the entire community in which he lived and worked. He was well known and beloved by a large circle of friends, far and near, for his characteristics of heart and manner, and as an engineer of recognized ability.

Throughout his official career, Colonel Compton developed a rare spirit of public relations that inspired the confidence of all those with whom he was associated or came in contact. As a public official he brought to his position real and lasting accomplishments through his professional ability, tact, and dignity.

His worth and standing in his profession is evidenced in the fact that he was a Past-President and Director of the American Society of Municipal Engineers; Past-President of the American Road Builders' Association; Past-President of the Engineers Club of Baltimore, and, at the time of his death, President of the Virginia Section of the American Society of Civil Engineers. He was also, at the time of his death, Chairman of the Paving

Brick Committee (SPR1) Simplified Practice, United States Department of Commerce, and Chairman of the Asphalt Committee (SPR4) Simplified Practice, United States Department of Commerce.

Colonel Compton was connected with the following social organizations and clubs: A member of the Commonwealth Club, of Richmond; Past-President, Southern Maryland Society, of Baltimore; member, Central Engineers Club, of Richmond; and a member of the Society of the Cincinnati.

He was married in 1894, to Elinor S. Hough, of Baltimore. He is survived by his son, R. Keith Compton, Jr., of Richmond, and by two sisters, Mrs. Roger Manning, of Accokeek, Md., and Mrs. Canfield Jenkins, of Charles County, Maryland.

Colonel Compton was elected a Member of the American Society of Civil Engineers on November 12, 1913.

ROBERT FULTON COWLES, M. Am. Soc. C. E.[1]

Died November 26, 1935

Robert Fulton Cowles was born in Cleveland, Ohio, on September 14, 1887, the son of Edward H. and Ella C. (Hill) Cowles. His family moved to Colorado and, in 1896, to California. His formal education was received in the Grammer Schools of San Diego and Los Angeles. His technical education was obtained through home study and experience, supplemented by an uncompleted course as a "Special Student" at Leland Stanford University.

Mr. Cowles' first engineering assignment was on the Los Angeles Aqueduct, in 1908. Beginning as a Draftsman, he became Assistant to the Location Engineer, and, later, Assistant to the Division Engineer in charge of construction, having responsible charge of engineer crews, cement testing, and concrete mixing and curing.

In 1912, he was appointed Chief Engineer of the Casa Grande Valley Water Users Association, in Arizona, on the survey and construction of a main canal, with a capacity of 1 000 cu ft per sec. At the conclusion of that work in 1913, he was employed by J. B. Lippincott, M. Am. Soc. C. E., in Los Angeles, on the survey and design of water and irrigation systems, including a water system at South Oceanside, Calif., and on hydraulic studies of surface and underground waters.

Mr. Cowles was retained by the Bureau of Light and Power, of the City of Los Angeles, in 1915, to take charge of the collection and preparation of data and the compilation of reports on seven potential hydro-electric projects in the Owens Valley and Mono Lake regions, in California. This assignment involved stream measurement and run-off studies, as well as the location of proposed reservoirs, dams, power plants, and transmission lines.

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[1] Memoir prepared by William H. Nanry, Esq., San Francisco, Calif.

In 1916, he was employed by the late M. M. O'Shaughnessy, M. Am. Soc. C. E., on the Hetch Hetchy project, for the City of San Francisco, Calif., in charge of a topographical survey (40 sq miles), selection of core-drilling sites, tracing serpentine stratum, and locations for a power-house and the penstocks thereto, at Moccasin Creek.

Soon after the United States entered the World War in 1917, Mr. Cowles was commissioned as First Lieutenant of Engineers, U. S. Army, and served more than a year in France. As Commanding Officer of Engineers at Vau Claire, under Hugh L. Cooper, M. Am. Soc. C. E., he had charge of the rebuilding of an old monastery as a hospital, as well as the operation of water, steam heat, and electric power supplies. He was also in charge of the water supply, drainage, and sewage disposal at a large military hospital at Beau Desert.

On his return to civilian life in 1919, Mr. Cowles engaged in general engineering practice with offices in Los Angeles. He was retained to make a topographical survey of the 60 000-acre Gillespie Project, in Arizona, and extensive hydraulic studies of the holdings of the Sweetwater Water Company in San Diego County, in California.

In 1921, he was placed in charge of topographical surveys for the Arrowhead Lake Project, in California, and in 1922, he was appointed Chief Engineer of the South Coast Land Company, of Los Angeles, in general charge of land development and sub-division, the design, construction, and operation of extensive water supply and irrigation systems, and the supervision of street lighting and sewerage system installations.

After designing sewerage and water supply systems for Normal Heights, San Diego, Mr. Cowles was appointed, in 1925, as Engineer-in-Charge of the Bureau of Water Development, of the City of San Diego. During his régime he had charge of the design and construction of a 17-mile pipe line, 3 ft in diameter.

In 1928, he was retained by the J. P. Campbell Irrigation Company, California, for the design and construction of overhead irrigation systems. In 1929, for the Water Department of the City of Los Angeles, he served as Topographer on the Colorado River Project and participated in the extensive studies involved in that gigantic undertaking. For a short time in 1930, as Associate Engineer, United States Bureau of Reclamation, he participated in the study of the water resources of the Sacramento Valley, in California (Central Valley Water Project).

Mr. Cowles was appointed in 1930, Assistant Hydraulic Engineer, State Division of Water Resources, California, in connection with the administration of the Water Commission Act, and for investigations of the availability and use of water in the Napa and Santa Clara Valleys particularly, and elsewhere throughout the northern part of the State. It was during this period, that his health began to fail due to the ailments that finally caused his death.

He next, in 1934, served as Supervising Engineer, United States Coast and Geodetic Survey, at Kennett Reservoir, California (Central Valley Water Project). This was Mr. Cowles' last active professional work, although while hospitalized at Yountville, Calif., in 1935, he made some volunteer water studies for the hospital authorities of the proposed Rector Dam project.

On November 26, 1935, he passed away at the United States Veterans' Hospital, at Yountville. During his life he worked, with a high degree of professional ability, on several of the really great engineering projects in the West and with some great engineers.

Mr. Cowles was elected a Member of the American Society of Civil Engineers on August 16, 1929.

FRANCIS ELIHU CRANE, M. Am. Soc. C. E.[1]

Died August 13, 1935

Francis Elihu Crane, the son of N. Martin Crane and Julia A. (Harris) Crane, was born at Elizabeth, N. J., on April 4, 1861. He was graduated, in 1885, from Union College, at Schenectady, N. Y., with the degrees of Bachelor of Arts and Civil Engineer.

Following his graduation, Mr. Crane taught in his Alma Mater. In 1887, he went to Amsterdam, N. Y., as Assistant Engineer on sewer construction. In 1889, he was made the City Engineer of Amsterdam and continued as such for about twenty-five years. This was at a time when the city was growing rapidly.

Afterward, he opened an office in Amsterdam, and engaged in private practice along sanitary and appraisal lines. He made plans for sewers and disposal works for several towns, and was called in on many appraisal cases, in which his fairness and clear thinking were of great value.

Mr. Crane was an active and highly esteemed member of the Presbyterian Church and filled responsible positions in it.

For years, he was a member of the County Historical Society, the local Board of Trade, and the Board of Education. Of him, the local paper truly stated,

> "It was characteristic of the man to do what fell to his lot efficiently, but without fuss or great show of effort. A Christian gentleman in every sense and a man of unusual attainments in his chosen profession, he commanded the respect and admiration of all with whom he came in contact.
>
> "Amsterdam is a better community for Mr. Crane's having been a member of it, and that is a fine thing to be able to say for any citizen."

His ability, kindly disposition, and good judgment endeared him to those who knew him, and they became his friends.

Mr. Crane was a member of the Society of Engineers of Eastern New York, and served on several of its committees. For years he represented his Class on the Graduate Council of Union College.

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[1] Memoir prepared by William B. Landreth, M. Am. Soc. C. E.

On October 20, 1892, he was married to Emma W. Myers, of Amsterdam, who, with a daughter, Elizabeth, and a sister Sarah E. Crane, survives him.

Mr. Crane was elected a Member of the American Society of Civil Engineers on October 1, 1902.

JOHN FRANCIS CUSHING, M. Am. Soc. C. E.[1]

Died October 7, 1935

Born in Arapahoe, Nebr., on May 21, 1882, John Francis Cushing, the son of Francis Cushing and Flora (Patchen) Cushing, after receiving his preliminary education, including studies at the University of Nebraska, at Lincoln, Nebr., entered the University of Notre Dame, at Notre Dame, Ind., in 1902, from which he was graduated in June, 1906, with the degree of Civil Engineer.

The narrative of Mr. Cushing's progress and success in the engineering and business world should be inspirational to any reader. Born the son of a village blacksmith, learning this trade, and that of a cooper, while pursuing his elementary and High School studies, his educational ambition and industry mastered all obstacles, and secured to him university training for his splendid participation in the complex affairs of the modern world.

After his graduation from Notre Dame, Mr. Cushing entered the employ of the Barber Asphalt Paving Company as Engineer and Superintendent, accepting more responsible employment shortly thereafter as Field Engineer on the construction of a utility tunnel for the City of Chicago, Ill., upon the completion of which, he entered the ranks of the Great Lakes Dredge and Dock Company on October 14, 1907, and devoted his great talents to the affairs of that Company for the remainder of his life.

During his twenty-eight years of loyal service to the Great Lakes Dredge and Dock Company, as a consequence of his sturdy character, unusual industry, keenness of mind, and noteworthy application, he advanced persistently to the pinnacle of this great national organization, as follows: 1907 to 1910, Timekeeper and Field Engineer; 1910 to 1912, Construction and Cost Engineer; 1912 to 1918, Division Engineer; 1918 to 1921, Assistant General Manager; 1921 to 1922, Vice-President; and 1922 to 1935, President and Director.

Mr. Cushing's earlier work comprehended field construction, the development of cost systems and estimates, the formulation of tenders, and the investigations and reports related to construction progress. In later years, his efforts were concentrated upon executive problems, of which one of the outstanding evidences is a modern marine construction equipment, developed under a deeply thoughtful, long-term plan, the efficiencies and capacities of which are of international recognition. Such a development was, perforce, paralleled with the equally difficult task of selecting and training a highly skilled personnel, not only for design and construction, but for the safe and efficient operation of many radically new machines and devices. The passing of direct authority and responsibility to dependable individuals, the establishment of a runner-up system for each key position, and myriad other general administrative activities, were given Mr. Cushing's keen and constant supervision. Throughout his arduous life, a great interest in, and loyalty to, his associates and employees were manifestly uppermost in his mind. His encouragement and support were extended infallibly to the younger men whose aptitude and ambition for advancement attracted his attention. His striking accomplishments in these many and diverse phases of engineering and executive responsibilities insure him a most enduring and living memorial.

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[1] Memoir prepared by a Committee consisting of J. E. Cahill, *Chairman*, E. M. Markham, Edward J. Kelly, W. W. DeBerard, and William P. Feeley, Members, Am. Soc. C. E.

Mr. Cushing's exceptional faculty for organization clearly indicated him as the outstanding leader in his industry, invariably respected as a strong character of definite aims, and honored by the friendship and confidence of all with whom he came into business association. His activities and energy were boundless, comprehending the entire scope of the marine construction business. He formed and led many groups, such as the River and Harbor Improvement Association, and others, representing the marine contractors in their many considerations related to labor unions, code affairs, and other matters of general welfare. Difficulties merely stimulated him to increased efforts and greater accomplishments. He was a man's man, deeply devoted to his family, and to his unusually wide circle of friends and associates.

Mr. Cushing was President of the Hydraulic Dredging Company, Incorporated, of Oakland, Calif., which venture he controlled as an individual. He was a Director of the Independent Pneumatic Tool Company, and of other business enterprises. He held memberships in the Western Society of Engineers, the Rotary Club of Chicago, the Edgewater Golf Club, and the Illinois Athletic Club, of which he served as President for two successive terms. His Alma Mater, the University of Notre Dame, in recognition of his achievements, selected him to serve as a member of its Board of Lay Trustees. He was a communicant of the Roman Catholic Church, and a member of the Knights of Columbus.

Some of the more recent outstanding projects completed under Mr. Cushing's direct supervision were the Chicago River straightening, Damen and Ogden Avenue improvements, South and Lincoln Park extensions, A Century of Progress and Skyride, at Chicago, extensive dredging contracts at New York, N. Y., Cape Cod Canal, in Massachusetts, Buffalo, N. Y., Toledo and Monroe Harbors, Ohio, the Calumet and Indiana Harbor steel breakwaters, in Indiana, and a multitude of river and harbor improvements throughout the Great Lakes and along the Atlantic and Pacific Coasts.

Mr. Cushing was the author of the technical paper, "Economical Design of Hydraulic Pipe-Line Dredge",[2] which gave complete data on the design of the 30-in. Diesel electric hydraulic dredge, *New Jersey*, built in 1930.

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[2] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 625.

In 1931, he donated to the University of Notre Dame the beautiful structure known as the John F. Cushing Hall of Engineering, intending that it should represent his concrete expression of gratitude and appreciation of the assistance extended to him while a student. His benefactions to the worthy were many, with none finer than his financial assistance to numerous students struggling for an education.

The letter from Mr. Cushing, of April 16, 1931, to the President of the University of Notre Dame, in making his gift of the College of Engineering, is typical of his clear thinking and planning for the future. The following extracts are quoted therefrom:

"I conceive that an Engineer, speaking particularly but not exclusively of my own field of civil engineering, must be a man with a conscience in his profession. The gravest of all natural responsibilities are borne by him. He will be responsible for human safety and human lives. He must know his materials and his principles of construction. He must respect the one and be faithful to the other. The seriousness of this needs no elaboration. It is a matter of primary and essential honesty of workmanship but it has its roots in the character of the man. The engineer must know how to do it and have the strength of character to do it right.

"Engineering has written a glorious chapter in the history of progress in our time. The glory is not all of the past, it has not all been won. There are still great opportunities for those prepared to seize them. The field of service, indeed, is constantly broadening. Electricity, for example, seems always to be in its infancy. Nobody knows what the next day will produce. The same is true in regard to the extension of the principles of mechanics. We want adventuring minds in all departments of engineering. Where shall we look for these results except in men developed in schools like Notre Dame? The native genius, inventor, or discoverer is, no doubt, born, not made, but he alone would not get very far. Our best hope is the high level of professional intelligence and professional character developed in the colleges and technical schools of the land. It is there the thought needs to be instilled that men owe it to their profession, not to lay it down, finally, exactly as it was when they took hold of it, but to pass it on a finer thing, enriched and advanced, and more valuable to the world because of the use they have made of it."

In the John F. Cushing Hall of Engineering, directly opposite the entrance is a gray marble tablet treated in gold and colors in the manner of an illuminated manuscript, and bearing the inscription:

"With honor and gratitude Notre Dame cherishes the name of John F. Cushing, whose sympathy with the ideals of the University and generous interest in the high purposes of scholarship are enduringly recorded in the gift of this building for the service of learning and the greater glory of God."

Mr. Cushing's sudden death at the early age of 53, resulting from an airplane accident at Cheyenne, Wyo., on October 7, 1935, is a tragic and irreparable loss to his associates and multitude of friends.

On September 12, 1906, Mr. Cushing was married to his boyhood sweetheart, Harriet M. Webber. Mrs. Cushing and his seven beloved children, Francis, Paul, Jerome, Gregory, Mary, Martha, and Vincent, two sisters, Mrs. L. V. Jenkins and Mrs. Mary Mackenzie, and a brother, Marvin Noble, mourn their great loss, but fondly cherish his memory.

Mr. Cushing was elected a Member of the American Society of Civil Engineers on October 10, 1927.

ROBERT BENJAMIN DAVIS, M. Am. Soc. C. E.[1]

DIED JULY 31, 1934

Robert Benjamin Davis was born in Jersey City, N. J., on May 20, 1856, the son of Judge James N. Davis and Jane (Russell) Davis. He received his early education in the public schools of that city, completing preparation for his college course at Hasbrock's Classical and Commercial Institute in 1874. In 1878 he was graduated from New York University, New York, N. Y., at which time he delivered the Philosophical Oration and received the degrees of Bachelor of Science and of Civil Engineer. He was a member of Phi Beta Kappa.

While a student at the University Mr. Davis' interest centered in its musical activities. A voice of unusual quality, the inspiration received from his instructor, Dr. Leopold Damrosch, father of Dr. Walter Damrosch, led him toward a musical career. The demands of his professional practice soon proved this to be impossible, although for pleasure and relaxation he always turned toward the best that music afforded.

After his graduation from New York University, Mr. Davis began his life work in the Public Works Department of Jersey City, where he remained until 1880 when he became Resident Engineer for the Toledo, Cincinnati, and St. Louis Railroad Company. This work required surveys in Indiana and Illinois, necessary cross-sections for cuts and fills, designs for timber trestles over waterways, etc., with estimates for the work. Illness contracted in the swamps of that country compelled Mr. Davis to resign that position in 1881.

Upon recovery, he became Assistant Division Engineer on the West Shore and Buffalo Railroad, from 1881 to 1883. His duties in this position included general office work, the preparation of plans, profiles, calculations, estimates, etc. In 1883, he became associated with Frank H. Earle, a Civil Engineer, of Jersey City. To the general requirements of such an office were added surveys for road construction and the development of "New Town," in Roseville, N. J.—under Mr. Davis' supervision.

In 1884, he accepted the position of Assistant Engineer, with the Brooklyn Railway Company. As such, he made surveys of property adjacent to routes for elevated structures; designs for timber trestles in the yard of the Elevated Railroad; and plans for a machine shop. He also designed a coaling plant for locomotives and made the preliminary general drawings with detail steel drawings for the elevated structures.

In 1886, Mr. Davis became Principal Assistant to the Chief Engineer, the late George B. Cornell, M. Am. Soc. C. E., when responsibility was added to the work of designing foundations and structural steel, calculations and plans for track work, designs for station buildings, supervision of all plans for, and the construction of, various classes of work.

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[1] Memoir prepared by Mrs. Robert B. Davis, Brookline, Mass.

In 1888, in order that he might be more closely associated with his friend, the world-renowned engineer, the late Charles C. Schneider, Past-President, Am. Soc. C. E., Mr. Davis accepted the position of Manager of the Bridge Department of the Pencoyd Iron Works, Philadelphia, Pa. This position required both professional and business knowledge in arranging the routine of work through the drafting-rooms and shops, with its supervision in all stages until shipped.

In 1894, Mr. Davis decided to enter private practice. Resigning from the Pencoyd Iron Works, the firm of Davis and Barnes was established, which continued until 1897. During that time, among various outstanding works, were the designs of the buildings comprising the plant of the Sessions Foundry Company, at Bristol, Conn., and the supervision of the construction. The main foundry building was at that time the largest of its kind east of Michigan. These buildings were of brick, with steel roof trusses and concrete floors. The work included the planning of the heating, plumbing, and ventilating systems and the supervision of their installation.

In 1897, Mr. Davis closed his private business in order that he might concentrate on that branch of his profession that for him held the greatest interest. Mr. Cornell, still Chief Engineer of the Brooklyn Elevated Railway Company, suggested to Mr. Davis that past experience in the Company, with accrued architectural knowledge, would be of value and once more he became Principal Assistant to Mr. Cornell. During that time Mr. Cornell frequently gave expression to his feelings of confidence in Mr. Davis' ability and of the conscientious performance of work which seldom if ever called forth criticism or discussion. The extension of the Elevated Railway had become urgent, and designs for the structural steel work and foundations were necessary, including direct connections of the Brooklyn Elevated with its New York Terminal. Mr. Davis took great pride and interest in this work.

In 1899, he was called to Boston, Mass., by Gen. William A. Bancroft, then President of the Boston Elevated Railway Company, for expert consultation on plans and equipment necessary for the extension of that road. These conferences required only a limited time, but mutual esteem and confidence developed which resulted in Mr. Davis becoming an outstanding figure in the construction and development of the Boston Elevated Railway, covering a period of more than thirty years. Beginning in 1899, Mr. Davis was associated with the late George A. Kimball, M. Am. Soc. C. E., Chief Engineer, as Designing Engineer in charge of plans covering a large percentage of structures. The items of this work are numerous: Bridges of various descriptions; power stations; elevated footwalk to the North Station of the Boston and Maine Railroad Company; and changes and alterations from inter-track and "island" stations to side-track stations. From 1900 to 1902, some of the most important works designed by Mr. Davis were the Atlantic Avenue Division of the original elevated structure, and the Lincoln Coal-Handling Plant, the entire design of which (except the towers) being under his direct charge and the construction under his personal supervision. The coal-handling plant included a coal-pocket of 5 000 tons capacity, connecting with a steel and concrete ash-pocket. The coal-pocket was to support towers, which necessitated a substantial wharf to hold such loading, and a dock of sufficient dimensions to accommodate coaling steamers. To obtain the latter required the construction of a heavy sea wall.

Other miscellaneous work came under Mr. Davis' supervision, such as elevated stations and platform extensions; changes and additions to the Sullivan Square and Dudley Street Elevated Terminals; the personal inspection of all bridges over which surface cars passed, as well as the solution of problems concerning the determination of safe loads and types of cars to be operated over certain bridges. During this period Mr. Davis occasionally spoke on the subject of "Elevated Railroads" to the students at the Massachusetts Institute of Technology.

From 1905 to 1912, as Engineer of Elevated Structures, he was in charge of plans and structures, such as car barns, machine shops, and extensions. His designs were used for the Forest Hills Extension of the Elevated Railway and for structures over Arbor Way. This work was done under his personal supervision. Other important works were the south approach to Washington Street Tunnel; reconstruction under operation of the City Square and Dover Street Elevated Stations; design of steel details for the Everett Extension of the Elevated Railway from Sullivan Square; calculations with estimates necessary for final judgment on the bridge over Mystic River and the lift-bridge over Malden River; and inspection for car safety in operation over highway bridges from 1914 to 1918.

After the death of Mr. Kimball in 1912, Charles S. Sergeant, Vice-President of the Elevated Railroad Company, assumed charge of the Department, Mr. Davis continuing his work under the title of Engineer of Structures. From 1918 to 1920 he was Consulting Engineer for the Company. These years covered work of a varied character, such as would demand the attention of professional experience and judgment. Mr. Sergeant, learning of the death of Mr. Davis, wrote:

> "It was a pleasure to be associated with so skillful an engineer who was also an accomplished gentleman. In his work I felt great reliance for it was competent, careful and safe and I can recollect no instance of error or failure in his designs. It was always a regret that he had not a wider field than was afforded with one company, one in which his capacity could have been better known. That he was of great assistance to us in the design and building of our elevated structures goes without saying and I shall always remember our pleasant association."

Edward Dana, Executive Vice-President and General Manager, writes:

> "Mr. Robert B. Davis will always be remembered by those who were associated with him on the construction of the elevated lines of The Boston Elevated Railway as a man of the highest moral character, strict integrity, and an engineer of unusual ability, ever willing and ready to offer advice and assistance in a most friendly and courteous manner."

Although repeatedly urged to enter broader fields of his profession, the lure of publicity or professional honor was not attractive to Mr. Davis. His unobtrusive personality is strikingly expressed in a tribute offered by his friend, J. R. Worcester, M. Am. Soc. C. E.:

> "Mr. Davis was over modest about his own ability and for that reason did not become well known among engineers as a leader, but his ambition evidently lay in doing faithfully and well everything that he undertook. I have always regarded him as a gentleman whom I was proud to call a friend."

These tributes convey to some extent the disregard Mr. Davis held for personal aggrandizement. A permanent home, the simpler forms of pleasure and relaxation after a day of service in worthwhile achievements, were to him sufficient. In 1919 there were evidences of failing health, and Mr. Davis retired from active professional work. Buying a farm in Wayland, Mass., he decided to spend his remaining years in the restful environment of country life. In a few years, however, with improved physical health, he felt an urge for old interests, and visions of subways and tunnels disturbed the quiet of the country.

In 1924 he became Designing and Consulting Engineer in the Transit Department of the City of Boston. From 1924 to 1933 some of Mr. Davis' outstanding works were: The Grosvenor Square Extension of the Boston Street Subway; the Dorchester Rapid Transit Extension; and the Sumner Traffic Tunnel under Boston Harbor, from the city proper to East Boston. The Dorchester Rapid Transit Extension was an extension of the old Dorchester Tunnel to Mattapan, mostly over the right of way of the Shawmut Branch of the New York, New Haven, and Hartford Railroad. The length of the Extension was about six miles; it consisted of substantially all new construction and was partly underground. Mr. Davis planned and supervised the relocation of the four tracks of the New Haven Railroad between Columbia Road and Savin Hill Playground, while maintaining traffic on four tracks for the Railroad Company. He also planned the relocation of the present steel span of Savin Hill Avenue Bridge, designing a new steel and concrete span and a temporary bridge for single passage of vehicles and foot traffic while alterations and additions were being made. On the Dorchester Rapid Transit Extension he designed two power sub-stations, one at Fields Corner and the other at Ashmont. These were followed by designs for the Charles Street Station, in Boston, the structural steel work for the alterations of a portion of the present elevated structure, including footbridges and stairways to stations, alterations to the Cambridge Bridge, in connection with the proposed station, and a design for a timber trestle for the use of elevated trains during new construction.

Mr. Davis designed the Harrison Sub-Station Building complete, including architectural as well as steel details; the Forest Hills Station galleries, stairways, and footbridges connecting with the New Haven Railroad Station at Forest Hills; the Ashmont Signal Tower and the Yardmen's Building in the Train Storage Yard on Codman Street at Ashmont; the over-pass at Columbia Station, Mattapan Superstructure; the under-pass of the Elevated Railway, at South Station; subway to South Station; and Mattapan Station, including all architectural, steel, and concrete details, with estimates for all the work mentioned. He designed the canopies over the platforms at numerous stations, including Columbia, Savin Hill, Milton, Crescent Avenue, and Mattapan Stations.

The Boylston Street Subway Extension was a mile in length entirely underground. On this subway Mr. Davis' work was chiefly the design of the steel work. He also designed the power sub-station at the North Station of the Washington Street Tunnel, and the one at Newbury Street of the Boylston Street Subway. It was during the construction of the Sumner Traffic Tunnel that Mr. Davis' health again began to fail and, in the early spring of 1933, it was obvious he could no longer tax his strength. The summer spent at Chatham, on Cape Cod, was of great benefit. The severe winter following proved this to be only temporary. Again, he was taken to the Cape, but grew rapidly weaker and on July 31, 1934, passed on as one sleeping.

The many tributes to his memory stress the exceptional attributes of his character, his high ideals, his faithfulness of purpose in the exacting demands of his profession, and to every duty he assumed.

Thomas F. Sullivan, M. Am. Soc. C. E., of the Transit Department of the City of Boston, wrote: "We had known each other for a long time and I admired him very much. Mr. Davis' passing is a source of regret to the entire organization."

Wilbur W. Davis, M. Am. Soc. C. E., Chief Engineer of the Transit Department, writes:

> "I wish to express my own comprehension of Mr. Davis' personal qualities as well as those of the other associates with whom he worked here in the Transit Department during his last nine years. * * * We found him to possess a personality so fine that the like is seldom met with. He was regarded with a high degree of respect by all. Mr. Davis' every act showed that which I can best express as a gentleman. * * * He was very modest, mild-mannered, and courteous to every one. His loss is deeply felt."

Thomas N. Ashton, a lawyer and engineer, for a time associated professionally with Mr. Davis, offers this tribute to his memory:

> "His journeying on ahead of us does indeed leave me with a sense of personal loss. It was my extremely good fortune to come into contact with Mr. Davis very early in my engineering activities. The examples which he set for me and the sound principles which he so generously and patiently imparted to my immature mind have left me with an inner feeling that he has been to me a 'father' in my engineering life. * * * I have met many leading engineers during my thirty years of engineering labors in different parts of Eastern United States, but these other leaders in engineering thought, have never somehow measured up to the enviable standard set by Mr. Davis in being just his own individual self as an engineer. * * * The structures which he designed have never failed to bespeak a more artistic and substantial product than those of his contemporaries. * * * At least, this thought has ever been my opinion as I weighed one structure against another during my periodic journeys about the several States."

On January 1, 1883, he was married to Effie Potter Wilkes, of Toms River, N. J. Mr. and Mrs. Davis celebrated their Golden Wedding in their home, at Wayland, Mass., on January 1, 1933. The surviving members of his family are his widow; a daughter, Mrs. Anita Davis-Chase, of Boston; a son, Paul B. Davis, of Wayland, and one grandson, Robert Laurence Davis.

Although reared under the influence of a Methodist Episcopal Church and home, Mr. Davis became a member of the Protestant Episcopal Church, at the time of his marriage.

After Mr. Davis had passed on, there was found in a billfold he always carried, the following lines of Josiah Gilbert Holland, carefully preserved on cardboard, but blurred by time. The sentiment fully evidenced Mr. Davis' attitude of mind:

> "Men whom the lust of office does not kill;
> Men whom the spoils of office cannot buy;
> Men who possess opinions and a will;
> Men who have honor; men who will not lie;
> Men who can stand before a demagogue
> And damn his treacherous flatterings without winking;
> Tall men, sun-crowned, who live above the fog
> In public duty and in private thinking."

This was his standard of manhood. As such he lived, as such he died.

Mr. Davis was elected a Member of the American Society of Civil Engineers on October 12, 1925.

BENJAMIN CURTIS DONHAM, M. Am. Soc. C. E.[1]

Died January 15, 1936

Benjamin Curtis Donham was born on August 4, 1873, in Rockland, Mass., the son of George E. and Sara A. (Studley) Donham. He was graduated from the Rockland High School in 1891 and from the Massachusetts Institute of Technology, at Boston, Mass., in 1895, with the degree of Bachelor of Science in Civil Engineering.

Following his graduation, Mr. Donham entered into active engineering work which soon took him to the Pacific Coast where he spent nearly three years on estimates and designs for the San Francisco Bridge Company. Later, he spent a year in irrigation engineering for the Spreckels Sugar Company, at Soledad and Spreckels, Calif., where he designed and constructed ditch and check systems for irrigating sugar-beet ranches, and superintended the construction of about 4 miles of 30-in. steel pipe line. In the six months, November, 1899, to May, 1900, he perfected and completed an irrigation system on the Gonzales Ranch, of the Gonzales Water Company.

In May, 1900, Mr. Donham went to Seoul, Korea, as Chief Engineer for the Collbran, Bostwick Development Company, in charge of the estimates, design, and construction of the Imperial Highway to the Queens' Tomb; the Seoul Water-Works (the first in Korea); the maintenance and extension of the Seoul Electric Railway which included 3 miles of construction extension and 12 miles of design extension; many important brick buildings, including those for the German Consulate, Young Men's Christian Association, etc. He made the complete designs and estimates for the new buildings for the United States Legation, which were accepted by the State Department in Washington, D. C., but the buildings were never constructed, owing to the Russian-Japanese War and the subsequent change from a Legation to a Consulate.

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[1] Memoir prepared by N. C. Grover, M. Am. Soc. C. E.

In 1909, Mr. Donham returned to the United States and entered the employ of J. G. White and Company as Construction Supervisor in charge of the following work: Cuba Eastern Railroad—reconstruction of the railroad from Boquerm to San Lois, Cuba; San Joaquin Light and Power Company—hydro-electric development of 20 000 hp, at North Fork, Calif. (a few months of the early work); Tri-State Land Company—reconstruction and extension of the irrigation system to cover 57 000 acres; after May, 1909, he served as Secretary of the latter Company and was practically in charge of the management of its affairs, including an active selling campaign for the disposal of water rights for the entire area, as well as 24 000 acres of land which was owned by the Company; Upper Columbia Company—a company organized for developing lands in the Upper Columbia Basin, in Washington, of which he became Vice-President after May, 1910, in charge of its management as well as of the construction of its irrigation system covering Tract No. 1, Upper Columbia Orchards.

In 1912, Mr. Donham organized and was Senior Partner in the engineering firm of B. C. Donham and Company, the name of which was changed in 1919 to Donham-Adams Company. This Company took over from the J. G. White Engineering Corporation the management contract with the Tri-State Land Company of which Mr. Donham continued as Secretary until the formation of a Bondholders Committee. As member and Secretary of the Bondholders Committee, he had complete charge of the re-organization and refunding of the bond issue, purchase of stored water rights in the Pathfinder Reservoir from the United States Reclamation Service, formation of the Farmers Mutual Canal Company, and, later, the Farmers Irrigation District which took over all water rights from the natural flow of the North Platte River, as well as the storage rights from the Pathfinder Reservoir. In 1926, he completed the successful refunding of the Farmers Irrigation District bond issue and settled all differences between the Tri-State Bondholders Committee and the District.

In 1926, Mr. Donham organized and was President of the Air Filtration Corporation and took out important patents covering improvements in the art of dry-air filtration. This Corporation was sold to the American Air Filter Company in 1929. Thereafter, he was Vice-President and, later, President of the Electro Anti-Corrosion Corporation. As indicated by the trend of his professional work, he had in a marked degree the combination of qualities that produce sound engineering and successful business. He was a thorough engineer, an exceptional negotiator and diplomat, and a successful business executive.

Mr. Donham retired from active business about three years before his death because of failing health. For many years, he resided in Glen Ridge,

N. J., where he was actively engaged in civic affairs. He served six years on the Glen Ridge School Board, three as President, retiring in 1928. He resigned in January, 1936, from the Glen Ridge Board of Adjustment, on which he had served for eleven years. He was a member of the Borough Building Committee from 1927 to 1933, and the Municipal Plan and Art Commission from 1927 to 1930.

In addition to his engineering and business characteristics, Mr. Donham had those personal qualities of mind and heart that endeared him to a host of friends. Among that host was an unusually wide circle of intimate friends who were sincerely devoted to him because of his spirit of optimism, helpfulness, and loyalty. By his interest, advice, and assistance in many ways—tangible as well as intangible—he helped many over the rough spots with which the path of life is strewn. Those who knew him thus intimately will long miss him.

He was married on August 4, 1902, to Edith Alexandria McKean, of Alameda, Calif., who survives him. He is survived also by two sons—George Alden and Winfield Benjamin—and by a daughter, Dorothy Evelyn (Mrs. Frederick Page), all residents of Glen Ridge; also by two brothers, G. Herman and Wallace B. Donham, and by a sister, S. Agnes Donham.

Mr. Donham was elected an Associate Member of the American Society of Civil Engineers on January 2, 1901, and a Member on March 31, 1908.

GEORGE ALEXANDER MILLER ELLIOTT, M. Am. Soc. C. E.[1]

Died December 23, 1934

George Alexander Miller Elliott was born on October 21, 1880, at Wellington, New Zealand, the son of Robert Elliott and his wife, Isabel H. Elliott, *née* Booth.

As a child he came to San Francisco, Calif., with his parents, and attended the public schools in Berkeley, Calif. His collegiate education was obtained at the University of California, at Berkeley, and the University of Colorado, at Boulder, Colo. He was graduated from the latter in 1904 with the degree of Bachelor of Science in Electrical Engineering.

Immediately following graduation Mr. Elliott entered the Testing Department of the General Electric Company, at Schenectady, N. Y., where he took the requisite ten months' course, returning to San Francisco upon its completion.

During the next four years his engineering work was for the most part in connection with the design and construction of high-tension electric transmission lines in New Mexico and California. In the latter State he made surveys and acted as Construction Foreman for the Pacific Gas and Electric Company, secured the right of way for a high-tension tower transmission line from the Feather River to San Francisco Bay for the Great Western

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[1] Memoir prepared by F. C. Herrmann and George W. Pracy, Members, Am. Soc. C. E.

Power Company, and was Superintendent of Construction of this line for Viele, Blackwell, and Buck.

In June, 1909, Mr. Elliott accepted the position of Superintendent of Operation and Maintenance of the Spring Valley Water Company, serving domestic water to the City of San Francisco and its environs. Subsequent to this date, his work was in hydraulics, either as Engineer, Constructor, or Executive. He remained with the Spring Valley Water Company for twenty-one years, serving, successively, as Superintendent, Chief Engineer, Vice-President, and Manager, with outstanding ability.

When the City of San Francisco acquired the properties of the Spring Valley Water Company in 1930, Mr. Elliott declined to continue as Manager of the System, and entered private practice as a Consulting Engineer, associating himself with Edward F. Haas and Frederick C. Herrmann, Members, Am. Soc. C. E., with offices in the Merchants Exchange Building, in San Francisco, where he was busily engaged in the practice of his profession to the time of his unexpected death.

The Spring Valley Water Company made many large additions and improvements under Mr. Elliott's direction as Chief Engineer. Among these was the construction of the Calaveras Dam and Storage Reservoir, with the Upper Alameda Diversion; a number of large pumping plants and transmission mains; the development of the underground waters of Livermore Valley; and a revamping of the distribution system in San Francisco. He was with the Spring Valley Water Company during a critical period of its history—times fraught with rate-fixing and litigation, together with the threat of the installation of a competitive municipal system. It was then that his characteristic fairness, firmness, and sound judgment were strikingly displayed. He was direct in methods and reasoning, never permitting non-essentials to make a problem or situation complex.

In 1917, he acted as Supervising Engineer, United States Army, on the construction of Camp Fremont, at Palo Alto, Calif., and of the tri-nitrotoluene (T.N.T.) plant, at Giant, Calif.

During his career Mr. Elliott was called in as Consulting Engineer at one time or another by practically all the privately owned public utilities of California.

From 1914 to 1930, he was engaged as Consulting Engineer for the San José (Calif.) Water-Works. He supervised the plans and construction of the principal improvements made thereto, including the pumping stations and four reservoirs.

Beginning in 1920 he served as a member of the Board of Consulting Engineers for the State Engineer of California in the study of the water resources of the State and their comprehensive development, which, in 1931, culminated in the proposed State Water Plan for the great Central Valley of California, with an estimated cost of $170 000 000.

While in private practice Mr. Elliott was called upon to serve in many undertakings. Among the important engagements were the following: Investigating the safety of many dams for the State of California; serving as Consulting Engineer for the Los Angeles County Flood Control District in the construction of the San Gabriel River Dam estimated to cost about $15 000 000; acting as Arbiter in the case of Rolandi *versus* the State of California to fix payments for the construction of the Klamath River Bridge; making an exhaustive study and report for the City of San Francisco on the future water requirements of the San Francisco Bay region, the water productivity of near-by sources of supply, and their utilization in conjunction with the Tuolumne River supply.

At the time of his death Mr. Elliott was actively engaged as Consulting Engineer for the Orange County Flood Control District in the preparation of plans for large flood-control reservoirs.

Mr. Elliott was very active in the affairs of the San Francisco Section of the Society, serving as its President during the year 1923. He served on the Committee of the Section which effectively and efficiently handled its funds for the aid and relief of engineers in distress in the San Francisco Bay region during the depression.

He was a member of the American Water Works Association, organizing the California Section of this Association in 1920, and serving as its President for the first three years thereof.

Mr. Elliott was a member of the University Club, the Family Club, and the Engineers Club, all of San Francisco. He held membership in the following fraternal organizations: Alpha Tau Omega College Fraternity; Berkeley Lodge No. 363, Free and Accepted Masons; California Bodies of the Scottish Rite; and Islam Temple of San Francisco.

Always interested in civic affairs, he was an active member of the Industrial Committee of the San Francisco Chamber of Commerce.

Mr. Elliott was married on October 12, 1909, to Rosemary Zeile, who survives him.

He was a staunch citizen, notably fair and firm, but withal generous, kindly, and understanding, having a high conception of proper ethics in his chosen profession.

Greatly loved by his many friends, and highly respected by his brother engineers, the Engineering Profession on the Pacific Coast suffered a great loss in his untimely demise.

Mr. Elliott was elected an Associate Member of the American Society of Civil Engineers on September 5, 1911, and a Member on June 24, 1914.

FRANK WILLIAM FLITTNER, M. Am. Soc. C. E.[1]

DIED OCTOBER 26, 1935

Frank William Flittner was born at Visalia, Calif., on August 14, 1893, the son of Henry Flittner and Mary (Ames) Flittner Trimble. He attended the Lowell High School, at San Francisco, Calif., and was graduated from the University of San Francisco (St. Ignatius College), with the degree of Bachelor of Science in Civil Engineering, in 1916. During the last few years he had taken a Correspondence Course in Law with the Blackstone Institute and received the degree of Bachelor of Laws in 1933.

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[1] Memoir prepared by C. H. Sweetser, M. Am. Soc. C. E.

After his graduation from the University of San Francisco, Mr. Flittner worked with the Shell Oil Company for a time. On May 8, 1917, he entered the United States Army as Second Lieutenant, 4th Engineers. In the Army he served at Vancouver Barracks, Washington, Camp Greene, North Carolina, and in France, where he was wounded during an engagement at Ypres, in August, 1918.

Following his discharge from the Army as First Lieutenant, on May 8, 1919, Mr. Flittner had engagements with F. Rolandi, Contractor; the Standard Oil Company of California; and the Hawaiian Sugar Refining Corporation, on engineering construction. He was also employed by the California Railroad Commission, and the Cities of Berkeley, and San Diego, Calif., on valuation and assessment work. In 1929 and 1930, he was engaged in private practice on various surveying and estimating projects.

He entered the service of the United States Department of Agriculture, Bureau of Public Roads, on February 2, 1931, as Associate Construction Engineer, with headquarters at Phœnix, Ariz. In this service, he was assigned as Supervising Engineer on highway work in Southern Arizona and Southeastern California. During his connection with the Bureau, he was careful and thorough, and continually strove for high standards on all work coming within his supervision.

Frank Flittner will be long remembered in District No. 2 of the Bureau of Public Roads for his engineering ability, sterling character, and agreeable personality. In spite of periods of ill health and physical weakness, he carried on his work with diligence.

Mr. Flittner was a Mason, a Knight Templar and a member of Islam Temple Shrine, of San Francisco, Calif. He was also a member of the American Association of Engineers and the Society of American Military Engineers.

Death came unexpectedly after only a short illness, although he had had periods of poor health for several years. He is survived by his widow, Genevieve Perrin Flittner, to whom he was married on May 9, 1917, and by his mother, Mrs. Mary Trimble.

Mr. Flittner was elected a Junior of the American Society of Civil Engineers on November 21, 1921, an Associate Member on March 12, 1923, and a Member on October 30, 1933.

WESTON EARLE FULLER, M. Am. Soc. C. E.[1]

DIED JUNE 22, 1935

Weston Earle Fuller, the son of William F. and Dora A. (Howard) Fuller, was born in Phillips, Me., on July 27, 1879. He received his early training in the public schools of Portland, Me., and his engineering education at Cornell University, at Ithaca, N. Y., which in 1900 conferred upon him the degree of Civil Engineer. He was a member of the Alpha Tau Omega Fraternity.

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[1] Memoir prepared by Chester M. Everett, M. Am. Soc. C. E.

After graduation he served as Instructor of Civil Engineering at Cornell from 1900 to 1902, and was then employed as Assistant Engineer of the Ithaca Water Company from 1902 to 1903. It was in connection with the building and operation of this very early example of a mechanical filter plant that he first came in contact with the late Allen Hazen, M. Am. Soc. C. E., and this association was destined to shape the course of the early part of his engineering career.

From Ithaca, Mr. Fuller went to Watertown, N. Y., as Resident Engineer for Hazen and Whipple on the construction of the mechanical filtration plant in that town, and upon the completion of this work he was transferred to Poughkeepsie, N. Y., to superintend the building of a slow sand filtration plant for the State Hospital for the Insane.

As a result of the excellent work done for Hazen and Whipple in the field supervision of these various plants, he was brought to the New York Office as Chief Draftsman in charge of all details of design. His progress there was rapid and, at the end of two years, in 1907, he became a Junior Partner and began to assume more and more responsibility, extending his activities from the direct supervision of design to the making of field inspections for new projects and the preparation of reports and direct contact with clients.

His acquaintance in the water-works field and his experience increased rapidly, and on January 1, 1915, the special value of his services was recognized by the inclusion of his name in the firm, which became Hazen, Whipple and Fuller, and this arrangement continued until September 1, 1922, when he resigned, and accepted the Professorship of Civil Engineering at Swarthmore College, Swarthmore, Pa.

The period of his association with Mr. Hazen was one of development and progress, first devoted to the practical problems of design and the construction of new works, and, later, to work in valuation and rate-making. During this portion of his life Mr. Fuller traveled over most of the eastern part of the United States and Canada, studying the water and sewerage problems of various communities and companies. He had much to do with the design of the slow sand filtration plant at Toronto, Ont., Canada, and prepared, on the ground, the plans and specifications for two new mechanical filtration plants at Ottawa, Ont., Canada. Neither of the latter was built, however, due to political controversy.

In the United States Mr. Fuller had full charge of the design and building of the new water-works, at Ogdensburg, N. Y., additional works at Watertown, the slow sand filters at Peekskill, and at Yonkers, N. Y., and took a prominent part in the design of the new water supply for Springfield, Mass., and the new 42-in. main from Lake Sebago to Portland, Me. He also developed all the main features of a report on the Passaic Valley Sewer, in New Jersey, which was actually built some years later, and of another on a large sewage disposal plant for Pittsburgh, Pa., proposed for location on Neville Island.

During this time Mr. Fuller took over all the work of the firm in Florida and designed the first filtration plant, of the gravity rapid sand type, at West Palm Beach, with the pumping station and other additions to this water-works system; he served the Miami Water Company for several years in the solution of its problems, and laid out the water and sewerage works for the newly developed Miami Beach area. He was responsible for the interesting and unique method of disposing of sewage at Miami Beach by pumping it into the rapidly flowing outgoing current of the sea channel at ebb tide.

In 1913, Mr. Fuller presented two papers of great value and interest to those engaged in water-works practice, the first of which, entitled "Flood Flows"[2] was presented before the Society, and the second, entitled "Loss of Head in Bends"[3], was read at a meeting of the New England Water Works Association. His paper on "Flood Flows", for which, in 1914, he received the Fuertes Graduate Medal from Cornell University, represented the result of much careful research, being an analysis of all the data available at that time as to the extent and frequency of the occurrence of floods. Empirical formulas were deduced, giving a concrete expression to the results of these analyses, which have been quite generally used throughout the United States and are known internationally. It is perhaps through this research on floods and the formulas for estimating their probable size, that Mr. Fuller had become best known throughout the Engineering Profession.

Although the early years of his work with Mr. Hazen were spent mostly in the practical problems of design, construction, and operation of water-works plants, Mr. Fuller found his greatest interest and was most successful in valuation and rate-making cases. He began this work at a time when the practice was relatively new and ideas regarding such matters, particularly the proper allowances for depreciation, were not nearly as well crystallized as at present. Mr. Fuller was one of the pioneers in establishing a more general understanding of these abstruse questions, and, as the years went on, he devoted more and more of his time to them. The first case of importance carried through directly by him for his firm was the rate case of the Hackensack Water Company. Another case of importance in which Mr. Fuller was a principal expert on the side of the consumer was that against the Commonwealth Water Company of New Jersey, and also one against the Interurban Water Company, of Westchester, N. Y. Most of Mr. Fuller's valuation and rate-making work was as an expert in the interest of the municipality or consumer, but in Durham, N. C., he represented the Durham Water Company and he also did considerable work for the Miami Water Company, in Florida. During the later years of his association with Mr. Hazen, Mr. Fuller's time was almost entirely given to valuations, rate cases, and damage suits.

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[2] *Transactions*, Am. Soc. C. E., Vol. LXXVII (1914), p. 564.

[3] *Journal*, New England Water Works Assoc., Vol. 27 (1913), p. 509.

During the early part of this country's participation in the World War, Mr. Fuller served on the Staff of the Adjutant General of the State of New York, ranking as Captain in the Engineer Corps.

Later, representing his firm—Hazen, Whipple and Fuller—he headed a Staff of Engineers at Camp Dix, Wrightstown, N. J., on the design and supervision of construction, of the water and sewerage systems.

Following acceptance in 1922 of an invitation to head the Civil Engineering Department at Swarthmore College, he was made Chairman of the Division of Engineering of that college in 1924. During this period he continued to do substantial outside work largely in valuation and rate-making cases and including services on the side of the municipality in the case of the City of Elizabeth, N. J., against the Elizabethtown Water Company, of certain other communities in New Jersey against the Middlesex Water Company, and in the case of the Village of Mamaroneck, N. Y., and other communities against the Inter-Urban Water Company. During this time, also, he did substantial work for the Legal Department of the Board of Water Supply of the City of New York, and he also represented the North Jersey District Water Supply Commission in suits arising as a result of the building of the Wanaque Reservoir.

Although successful as a Professor at Swarthmore College, the increasing demand for his services as Expert Adviser in such cases as those just mentioned, led him, in June, 1929, to resign from the Faculty of Swarthmore College and once more to devote himself entirely to private practice. From this time until May 1, 1931, he carried on a private consulting practice with headquarters at Swarthmore. Much of his work was for New York City, but perhaps the largest and most interesting project was the taking over of the properties of the Passaic (N. J.) Consolidated Water Company by the newly formed Passaic Valley Water Commission. In this case his services and those of several other engineers resulted in obtaining from the Condemnation Commission an award closely approximating that given in their valuation. Out of this association came a contract between Mr. Fuller and Frank A. Barbour, M. Am. Soc. C. E., and the Passaic Valley Water Commission for engineering services in the design and construction of substantial additions to the plant, including a complete rehabilitation of the pumping station.

From 1928 to 1930, Mr. Fuller represented the State of Massachusetts in the Massachusetts-Connecticut case on the Connecticut River. In 1930, when the Tri-State case on the Delaware River came before Special Master Bruch, representing the Supreme Court of the United States, Mr. Fuller was retained to represent the water power interest for New York State.

On May 1, 1931, Mr. Fuller formed a partnership with Chester M. Everett, M. Am. Soc. C. E., the remaining partner of the firm of Hazen and Everett on the death of Mr. Hazen in July, 1930. This partnership with Mr. Everett continued until the time of Mr. Fuller's death.

During his association with Mr. Everett, Mr. Fuller's activities were mostly in connection with the work of revamping the properties acquired by the Passaic Valley Water Commission, at a total cost of more than $2 000 000, and in looking after several power-damage suits for the New York Board of Water Supply and the Metropolitan Water Board of Boston, Mass. He was also employed during this period as Engineering Expert to advise the Water Rate Case Committee on a water-rate case against the Commonwealth Water Company of New Jersey. The work was still in progress at the time of his death.

Mr. Fuller was a member of the American Water Works Association, Sigma Xi, Cornell Engineers, Engineers' Club, of Philadelphia, Pa., and Alpha Tau Omega, of New York.

In 1907 he was married to Bertha Palmer, of Ogdensburg, N. Y. Mrs. Fuller and his daughter, Ada Palmer Fuller, survive him. A son—Weston Earle Fuller, Jr.—died in 1927, at the age of 14.

Mr. Fuller was elected an Associate Member of the American Society of Civil Engineers on June 7, 1905, and a Member on December 3, 1912.

HARRY CARTER GARDNER, M. Am. Soc. C. E.[1]

DIED JANUARY 23, 1935

Harry Carter Gardner was born on December 11, 1881, in Adams County, Nebraska, the son of William H. and Eva H. (Carter) Gardner. After his academic schooling, Mr. Gardner entered the University of Colorado, at Boulder, Colo., from which he was graduated with the degree of Bachelor of Science in Civil Engineering in 1906. In 1909, he received the degree of Civil Engineer from the University.

Mr. Gardner began his professional career as a Structural Steel Draftsman with the American Bridge Company, followed by similar employment with the J. R. Bowles Structural Steel Works. He was afterward Assistant Engineer on irrigation surveys in Eastern Oregon, and, on the completion of this work, he was Structural Steel Draftsman with C. G. Sheely, Bridge Contractor, at Denver, Colo.

In September, 1908, Mr. Gardner returned to the University of Colorado where he was a Graduate Student and Assistant Instructor. From September, 1909, until June, 1912, he was Instructor in Civil Engineering at the University of Pennsylvania, at Philadelphia, Pa. Mr. Gardner was engaged in designing and estimating on all kinds of structural steel buildings and bridges with Lewis F. Shoemaker and Company, Pottstown, Pa., from June, 1912, until February, 1915.

He became Chief Engineer with the firm of John H. Wickersham, Engineering and Construction, Lancaster, Pa., in February, 1915. He acquired an interest in this business, and his intimate association with the writer continued until his death.

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[1] Memoir prepared by John H. Wickersham, M. Am. Soc. C. E.

Throughout his life Harry Carter Gardner proved the value of a Christian engineer. His technical opinions and designs were always sound, and his integrity was always above question.

On July 2, 1910, he was married to Jane Graham, at Philadelphia, Pa., and to them were born the following sons and daughters: Harry Graham, Richard Carter, Betty Jane, Barbara Mary, and Carter Samuel, who, with his widow, survive him.

Mr. Gardner was elected a Junior of the American Society of Civil Engineers on July 1, 1909; an Associate Member, on November 12, 1913; and a Member, on June 12, 1917.

ALLAN VINAL GARRATT, M. Am. Soc. C. E.[1]

DIED APRIL 30, 1932

Allan Vinal Garratt came from very early American stock. On his father's side, Peter Hallock reached this country from London, England, in 1640; the Garratts were but little later, while on the distaff side there were several generations of Vinals before the Revolutionary War, and representatives on both sides of the family were prominent in Colonial affairs and in the Revolutionary and Civil Wars.

Born in Boston, Mass., on December 11, 1857, the son of Dr. Alfred Charles Garratt, a Boston physician, surgeon, and medical author of note, and of Martha Bowker (Vinal) Garratt, Allan Garratt, while by no means an invalid, was not particularly strong, and his early education was acquired largely at private schools, notably Chauncy Hall. From 1876 to 1879 he was a Special Student at Massachusetts Institute of Technology and Lawrence Scientific School, following this with special studies at Harvard University in botany and geology under the famous Professor Nathaniel S. Shaler. He accompanied Professor Shaler, as his Assistant, on a geological survey of Kentucky.

On his return, the combined effects of the death of his mother, to whom he was particularly devoted, and the unfortunate outcome of a love affair, brought on a dangerous attack of what in those days was diagnosed as "brain fever." While still so ill that he had to be carried on shipboard, he sailed on the historic barque, *Sarah*, to the Azores. There he regained his strength, and then sailed to Maderia, where he made important botanical studies. Here, too, he made the acquaintance of an African sea captain and trader, supposedly dying. A born doctor, although without the degree, Mr. Garratt nursed the sick man back to health, and, in 1880 and 1881, accompanied him on a trading and exploring expedition up the Congo River in Africa. Here

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[1] Memoir prepared by Alden B. Hoag, Esq., Holliston, Mass.

Mr. Garratt heard rumors of Alfred Aloysius Smith, later to become famous as "Trader Horn", but although the two expeditions came within a short distance of each other, the men did not actually meet.

The African trip was followed in 1883 and 1884 by an engagement to develop a salt mine in Santo Domingo. On that Island he made the original examination and report on the mine, including preliminary surveys for eighteen miles of railroad and dockage facilities at Barahona, and opened up some of the benches. The Company had many setbacks and difficulties, but eventually seemed on the verge of success, when, in the course of one of the periodical revolutions on the Island, a party of insurrectionists seized the plant just after the supply boat had left, and stripped it of everything portable, including the food. Mr. Garratt, who had suffered a broken leg during the affray, rode horseback ninety miles without rest across the Island, and succeeded in intercepting the boat, and getting it to return to the plant, although his horse had to be shot. This, coupled with the loss of an important shipment, caused the Company to abandon the mine.

Returning to the United States, Mr. Garratt set up as a Consulting Engineer, with offices and interests in Boston and New York, N. Y. During this period—from 1885 to 1887—he also represented the Volta-Pavia Battery Company, for which he made a number of installations; edited the periodical, *Modern Heat and Light*; and contributed technical articles to the *Electrical Review.*

His writings led to his selection as the first Secretary and Treasurer of the then National Electric Light Association, later the Edison Electric Institute, which position he held from 1887 to 1890, but he was more concerned with the duties of the office than its politics, so that in the latter year he failed of re-election. About this time he served as Chairman of the first Committee to draw up an Electrical Safety Code and of the first Committee on Wire Gauge Standards.

On October 20, 1887, he had married Mary Charlotte Wilder, of Boston, and with her he shared many of his next adventures, the scenes of which were in the deep West, then very much a frontier country. He went first to Spokane, Wash., as Chief Engineer of the proposed Spokane and University Heights Electric Railroad. Crossing the country during the last important Indian uprising, Mr. and Mrs. Garratt saw settlers fleeing with their goods across the plains, and on one memorable night the glow of burning ranches could be seen.

The sudden death of the head of the group of bankers handling the securities resulted in the abandonment of the railroad project, and Mr. Garratt became connected with the Portland, Ore., Office of the Thomson-Houston Electric Company, installing for that Company a hydro-electric plant for Waterville, Wash. It was during this installation that he first realized the need for an accurate water-wheel governor for such plants, and he began the study and research that led to his great contribution to Hydraulic Engineering. While still in the West, he revamped the electrical installation of the Palace Hotel, in San Francisco, Calif.

Returning to Boston in 1895, his next move was an interesting example of the part chance appears to play in life. Hearing that the Simplex Wire Company was looking for a Plant Superintendent, Mr. Garratt went to see about it. The position had already been filled, but in the course of a conversation following this discovery, it appeared that one of the executives was interested in a water-wheel governor a Mr. Lombard was developing. Mr. Garratt, immediately realizing that his work along this line would admirably complement the Lombard studies, offered to effect a combination, and the Lombard Water-Wheel Governor, long the outstanding American device of the kind, was the outcome.

From 1898 to 1920, he was Chief Engineer for the Lombard Governor Company, in the last eight years of that period being General Manager. He was in responsible charge of the design of the governors, of which about fifty types were developed, including the "actuator" design. So far as elements pertaining to regulation were concerned, many were of considerable intricacy and magnitude, involving much design of flumes, stand-pipes, surge tanks, regulating chambers, fly-wheels, etc. During his fifteen years with the Company Mr. Garratt guided all construction and erection engineers in the field.

Refusing to establish a monopoly by withholding from the Engineering Profession the details of his research—a monopoly which could have been enormously profitable—Mr. Garratt freely placed at the disposal of all other workers in his field the result of his unique work, and by this means made possible the development of similar devices by the larger water-wheel makers. The result was that it became the custom for governors to be produced as an integral part of the wheel unit, and the demand for separate governors was greatly diminished. The chief credit for the basic studies, however, belongs to Mr. Garratt, a fact evidenced both by the Gold Medal awarded him personally at the St. Louis Exposition "for notable work in water-wheel governing", and by his articles on the subject, notably the chapters contributed to Mead's "Water Power Engineering" and to Lof and Rushmore's "Hydro-Electric Power Stations", and various papers before technical societies and in the technical press.

During the World War he served in Washington, D. C., and at the Naval Proving Grounds in Boston Harbor in experimental work on artillery fire-control devices, developing particularly a firing mechanism for anti-aircraft guns. He also served as Public Safety Commissioner, Fuel Administrator, and Treasurer of the Young Men's Christian Association fund of his home town, Holliston, Mass.

Following his departure from the Lombard Governor Company in 1920, Mr. Garratt served Lockwood, Green and Company, of Boston, as Construction Engineer, working on surveys, reports, and preliminary designs of a number of proposed water-power developments, including preliminary work on eight propositions to develop about 380 000 gross hp. Later, he returned to private consulting practice taking also the work of Eastern States Representative of the Redwood Manufacturers Company, of San Francisco, for which he designed redwood pipe lines. He was thus engaged up to within a short time of his death.

For Allan Garratt an adverse fate seemed always to have set definite bounds which were wholly without reference to his ability, energy, or courage. A brilliant engineer, whose contribution to his special field of hydraulics afforded notable and permanent advances, a man of untiring energy and enthusiasms, of unshakable courage and an astonishingly wide range of knowledge, a great executive and a born leader, and with marked business ability in many directions, he suffered, nevertheless, the misfortune through life of repeatedly becoming connected with projects which gave every promise of a most prosperous outcome, only to fail more or less completely because of some unanticipated and indeed unforeseeable turn of fortune entirely beyond his control.

His professional experience, his extensive travels, and his wide range of reading, coupled with an excellent memory, provided him with a rich fund of information, and gave his opinions, professional or otherwise, a well-deserved weight. As an acquaintance once put it: "Garratt is always sure of himself and the worst of it is, he is always right."

Excelling in sports calling for agility and skill rather than great physical strength, he was a superb horseman, a fine fencer (having studied under Zalinski, at the Massachusetts Institute of Technology), and an expert marksman. He also enjoyed fishing and sailing. Up until a year or two of his death he always appeared much younger than his actual years, and in the last months of his life his splendid courage in overcoming the infirmities of his physical condition was an inspiration to many who were ready to give up with less to contend with. He died almost "with his boots on", ready almost to the very last hour to go to his office and deal with the engineering problems he knew awaited him there.

A member at the time of his death, of the American Institute of Electrical Engineers, of the American Society of Civil Engineers, of the Harvard Engineering Society and of the Society of American Military Engineers, he had previously held membership, and, as was characteristic of his membership in any organization, had been very active in the Boston Natural History Society, in the Boston Scientific Society, in the Boston Society of Arts, and in a number of other scientific and social societies. A Mason, he was a member of Mount Hollis Lodge, in Holliston.

Besides his widow, Mr. Garratt left two daughters, Helen Lucy (Mrs. Alden B. Hoag) and Myrtis Manning (Mrs. Albert L. McManus), and four grand-children.

The records of hydro-electric development in the United States bear witness to his professional ability; of him as a man no better characterization can be given than the expression at the time of his death by a number of friends, widely separated and obviously quite independently: "He was a gallant gentleman."

Mr. Garratt was elected a Member of the American Society of Civil Engineers on March 7, 1921.

XANTHUS HENRY GOODNOUGH, M. Am. Soc. C. E.[1]

Died August 10, 1935

Xanthus Henry Goodnough, the son of Xanthus and Kate (Harley) Goodnough, was born in Brookline, Mass., on October 23, 1860. He prepared for college at the Brookline High School and was graduated from Harvard University in 1882. During his college course, not only was his scholastic record good, but he was also prominent in athletics, for four years being stroke of his Class crew which is reputed to have won all its races.

After his graduation Mr. Goodnough was employed for a short time by the Boston (Mass.) Main Drainage Commission, and then in general railroad work in the West. Returning to Massachusetts in 1885, he worked with the Massachusetts Drainage Commission on investigations concerning the North Metropolitan Sewerage System. In 1886, he entered the employ of the State Board of Health as an Assistant to the late Frederic Pike Stearns, Past-President, Am. Soc. C. E., Chief Engineer, and upon Mr. Stearns' resignation, in 1895, Mr. Goodnough was made Chief Engineer of the State Board of Health. This position he held until 1914 when upon the re-organization of that Board he became Chief Engineer and Director of the Division of Sanitary Engineering in what is now the Department of Public Health, and continued in this position until, owing to age limitation, he was retired in October, 1930. Upon his retirement, Mr. Goodnough invited Bayard F. Snow, M. Am. Soc. C. E., to join him in forming an engineering organization which, under the name of X. Henry Goodnough, Incorporated, has continued engineering practice, principally in the fields of water supply and sewerage. In this way, Mr. Goodnough kept in touch with engineering and sanitary matters up to his final illness.

During the thirty-five years that he was Chief Engineer of the Board of Health and the Department of Public Health, he guided that Board and Department ably and efficiently in matters of water supply, sewerage, sewage disposal, improvement of rivers, etc., and served as Engineer of some of the largest and most important public health investigations and projects ever undertaken by the State. His work of this character began, as stated previously, as an Assistant to the late Mr. Stearns during the investigations and the development of the plan for the establishment of the Metropolitan Water Supply System for Boston and the surrounding cities and towns. Thirty years later, from 1919 to 1922, he was Chief Engineer of a Joint Board which investigated the enlargement of this Metropolitan Supply by extending the system to the Ware and Swift Rivers. His recommendations relative to this extension were contained in an elaborate and able report. His plan for enlarging the System was at first rejected, and a second commission proposed an entirely different plan. Finally, after extended legislative hearings and much controversy between the proponents of these two plans, Mr. Goodnough's plan was adopted. Throughout these hearings he displayed the same able and adroit generalship always evident in his work. Upon the formation of the Metropolitan District Water Supply Commission to carry out this project, he was appointed one of its Consulting Engineers and after his retirement in October, 1930, through X. Henry Goodnough, Incorporated, he continued to render services in that capacity, even to the time of his death.

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[1] Memoir prepared by Arthur D. Weston and Frank E. Winsor, Members, Am. Soc. C. E.

Among many other investigations made by Mr. Goodnough, the following may be mentioned as typical of his work: In 1895, he had charge of engineering work for improving the Concord and Sudbury Rivers and the rebuilding of bridges over these streams; from 1895 to 1897, he had charge of the investigations for the improvement of the Neponset River and the Fowl Meadows, and from 1911 to 1915, he was Engineer for the construction of this work; in 1896, he had general charge of an investigation for a sewerage system for Salem and Peabody, Mass.; in 1899 and 1900, he had charge of further investigations for a main sewer for the South Metropolitan District and a new outlet into Boston Harbor; in 1906 and 1907, he had charge of studies for the improvement of the Mystic River and Alewife Brook as well as designs for the improvement of the water supply of the City of Lynn, Mass.; in 1914, he made plans for the extension of the Metropolitan Sewerage System in the Charles River Valley; in 1916, he, jointly with the City Engineer of Lynn, planned a new sewerage system for that city; from 1923 to 1927, he was in charge of various special investigations for water supplies for Lawrence, Lowell, and Methuen, Mass.; and during the same years he had charge of planning a general sewerage system for Salem, Peabody, Beverly, and Danvers, Mass.

Apart from his work for the Commonwealth of Massachusetts, Mr. Goodnough was from time to time Consultant for various municipalities and water and sewerage districts concerning water supply, sewerage, and other sanitary engineering matters; among these, during 1907 and 1908, for the City of Boston, in regard to municipal waste disposal; during 1914 for the City of Providence, R. I., in regard to water supply, and for the Metropolitan Sewerage Commission of New York, N. Y.; and during 1926 for the County of Los Angeles, California, concerning sewerage.

On account of his intimate knowledge of all water supplies, sewerage systems, and rivers of the State of Massachusetts and his direction of engineering studies for the solution of a great variety of sanitary problems, he was pre-eminently known as an expert in public health engineering. His mastery of methods within his field made him a commanding figure in legislative committees before which he was often called for consultation and advice, and he was alert and skillful in presenting his point of view before these committees. He was an efficient executive and understood the delicate art of human engineering, how to deal with men and how to get things done. He was a lover of mountains, forests, streams, and outdoor life, an ardent fisherman, and a good companion.

Mr. Goodnough was the author of papers on various engineering subjects, including rainfall, stream flow, freshets, yield of water-sheds, ground-water supplies, drainage, disposal of sewage and municipal refuse, and, in

1921, was the recipient of the Dexter Brackett Memorial Medal of the New England Water Works Association for his paper on "Rainfall in New England."[2]

He was an Honorary Member of the American Water Works Association, and the New England Water Works Association; Past-President of the Harvard Engineering Society; and a member of the Boston Society of Civil Engineers, the Society of American Military Engineers, American Meteorological Society, and the American Geographic Society.

On October 5, 1892, he was married to Maria T. Dyer, who died on January 13, 1925.

Mr. Goodnough was elected an Associate Member of the American Society of Civil Engineers on May 6, 1896, and a Member on June 4, 1902.

WILLIAM CHARLES GOTSHALL, M. Am. Soc. C. E.[1]

DIED AUGUST 20, 1935

William Charles Gotshall, Engineer and Scientist, was born in St. Louis, Mo., on May 9, 1870, the son of Daniel H. and Minnie (Wortmann) Gotshall. His great-great-grandfather, Michael Gotshall, of Lancaster County, Pennsylvania, was attached to Washington's Headquarters at Valley Forge and while on military duty lost his life in crossing the Schuylkill River. His grandfather, Daniel Gotshall, was an officer in both the Mexican and Civil Wars. His father served in Company F, 4th Ohio Volunteer Infantry, in the Civil War and, later, was first a newspaper editor and afterward a publisher of law books in St. Louis.

William Charles Gotshall worked in the office of his father's Publishing Company as a copy-holder and type-setter. Meanwhile, he took an intensive course of study under private tutors in mathematics, electricity, and engineering, and began his professional career as a Draftsman in the Metzger Iron Works, of St. Louis. Later, he was engaged as an Engineer in railway location and construction and, in 1892, entered the employ of the Missouri Electric Light and Power Company, of St. Louis, one of the pioneer alternating-current electric light and power plants (1 000-volt) in the United States. It was at this time that he laid the foundation for his career as an Electrical Engineer and engaged in much important original research and development work. While with this Company Mr. Gotshall made the first 1 000-hr incandescent lamp test for life and efficiency, using a lamp board starting with 1 000 sixteen candle-power incandescent lamps.

Through the wide interests of the Company's President, Samuel Dodd, Mr. Gotshall branched into the field of electric railways and soon became a recognized expert and authority on design, construction, and operation. The graphic method of showing construction and operation variations and activities was first utilized by him; it soon came into general use and is now the universal practice in construction work in almost every field of industrial and scientific activity.

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[2] *Journal*, New England Water Works Assoc., Vol. 35, 1921, pp. 228–293.

[1] Memoir prepared by James J. Reilly, Esq., New York, N. Y., Prof. Frederick K. Morris, Mass. Inst. Tech., Cambridge, Mass., and Prof. William F. Bade, Univ. of California, Berkeley, Calif.

During 1893 he was a Government Engineer in charge of the work of riprapping and protecting 150 miles of banks on the Mississippi River. In 1894, as Chief Engineer for the Union Depot Railway Company, of St. Louis, he rebuilt and electrified the road, making the pioneer installation of the three-wire system on electric railroads. In this system, which can only be used on double-track roads, the track serves as the neutral wire. The current passes from one wire over one track downward through the car to the rails beneath, then rises through the rails and car on the parallel track to the negative wire above, thus completing the circuit in balanced condition and giving to each over-head wire and track a voltage of about 550 with a total line voltage of about 1 100. Although the system made a large saving in copper and eliminated electrolytic action on gas and water pipes, and was successful in St. Louis, the necessity of using dynamos in series and other mechanical difficulties, militated against its general adoption.

Later, Mr. Gotshall had charge of the location of the St. Louis and Eastern Railroad; the rebuilding and operation of the Cairo (Ill.) Electric Railway; the design and construction of the Belleville (Ill.) Electric Railway; the Marshalltown (La.) Railway and Lighting Plant; the Muncie (Ind.) Electric Railway; and the Grand Avenue Railway, in St. Louis. He was also Consulting Engineer on the construction of the St. Charles Railway, in New Orleans, La. During 1897 and 1898 he had charge of the conversion of the Second Avenue Railway, New York, N. Y., from a horse-car to conduit electric system, and then became President and Chief Engineer of the New York and Port Chester Railroad, which he designed and constructed, introducing pioneer improvements in high-speed electric traction. The Port Chester line, costing $25 000 000, was the first high-speed electric railway in the United States built entirely on its own right of way and entirely without grade crossings. After the completion of this railway in 1912, Mr. Gotshall engaged in the purchase and rehabilitation of railroad properties in the United States, Europe, and Africa, designing among others, a road from Cairo to Helouan, in Egypt.

Notwithstanding his varied and widespread professional interests Mr. Gotshall devoted much time to scientific activities and traveled extensively in the promotion of biological, geographical, and archæological research and exploration. Some of his most important work of this character was in exploration and development activities in Alaska, in behalf of the United States National Park System, and also in the Near East. Beginning in 1925 he was associated with Dr. William F. Bade in archæological explorations and excavations in Palestine where the site of the ancient Biblical City of Mizpah of Benjamin was unearthed eight miles north of Jerusalem; and, in 1926, a terra cotta head of Astarte, the Babylonian Venus, was found, the age of which is reckoned at nearly 3 000 yr. The limestone and clay wall of this ancient city was found to be 35 ft thick at the bottom, 29 ft thick at the top, and approximately 40 ft high. A narrow inner wall was also unearthed, and pottery of the Iron Age, storage bins, an Israelitish sanctuary, and a new double-cup type of Semitic pottery were found. The most important find was a jar handle on which was stamped a seal with three Hebrew letters meaning Mizpah, proving definitely that there had been discovered the correct ancient site of the Biblical Mizpah of Benjamin.

Mr. Gotshall was also active in athletic sports, rowing, fencing, boxing, and wrestling. He was one of the most skillful fencers in the United States, having won many medals with both the dueling sword and saber, including the National Championship in each. He was also a big game hunter, indulging in that sport in China, Mexico, Alaska, British Columbia, and the Canadian Rockies, and many exhibits of his skill are on view at the New York Athletic Club, Travers Island, New York, given to the Club by Mr. Gotshall. He had climbed all the glaciers in the Canadian Rockies and had ascended Yungfrau Yoch, in Switzerland, as well as the Chamonix Mountain, in France. He was also interested in aviation, and was an expert swimmer, being a recipient of the United States Life Saving Medal.

Among his diversions was the collecting of paintings and first editions of books of which he had a splendid collection, and which were left by will to the State Library, at Albany, N. Y.

In the World War period, he was a Major of Engineers, U. S. Army, serving as a Technical Transportation Expert in the laying out and construction of military roads, in railroad design and construction, and in the construction of Army camps.

He was the author of "Electric Railway Economics" (1904); the article on Railway Economics in the Encyclopedia Americana; and of contributions to technical and scientific journals. He was also deeply interested in heliotherapy and in two pamphlets, "Fasting and Efficiency" and "Heliotherapy with Fasting", he pointed out the virtues of abstemious eating with occasional periods of fasting and frequent sun baths.

Mr. Gotshall was a Fellow of the American Museum of Natural History and the American Geographical Society. He was also a member of the American Institute of Electrical Engineers and the American Association for the Advancement of Science. He was a 33d Degree Mason, and a life member of the Explorers, Lotos, and New York Athletic Clubs; the Automobile Club of America; Olympic Club, of San Francisco, Calif.; the Royal Automobile Club, of London, England; the Union Interalliée, of Paris, France; and the Army and Navy Club, of New York City. He was an accomplished linguist, speaking and writing German, French, Italian, and Spanish, in addition to his native language.

Dowered with a strong body to support any degree of hardship and effort, he had fertile vision and bold energy in carrying out his purposes. Such a career as his calls out a powerful opposition from many able men and well-intrenched interests. Risking everything in an uphill fight, he struggled forward against overwhelming odds to success, and to many successes. He was not bitter during the fight, nor boastful in victory. He opposed his enemies without hate; loved his friends loyally; he valued money for the service it enabled him to render, and he valued fame not at all. His world was an active world of realities, and everything interested him, calling forth dynamic response. Human problems interested him and he mastered five languages to deepen his contacts with his fellow men.

Largely self-educated, Mr. Gotshall lectured at several universities and aided their research progress—notably Rensselaer Polytechnic Institute, at Troy, N. Y., and Lehigh University, at Bethlehem, Pa. His lectures on the economics of electric railway engineering were published in the book previously mentioned, which filled a long-felt want in engineering literature. At the Massachusetts Institute of Technology, he was one of the Aldred Lecturers, and became a valued friend of President Stratton. In his will, he generously remembered schools and educators. His love of life urged him to study the art of living; and he wrote many papers on health and diet. All who knew his able mind and vigorous personality will feel a lack in their lives that no other friend may replace. The great frame and warrior face, with its bold earnest expression—the record of a fighting life—formed an inspiring picture.

He was married on September 15, 1897, to Countess Adelaide von Rathgen, who died on August 29, 1935.

Mr. Gotshall was elected a Member of the American Society of Civil Engineers on May 7, 1902.

FREDERICK PASSMORE GUTELIUS, M. Am. Soc. C. E.[1]

Died September 12, 1935

Frederick Passmore Gutelius was born in Mifflinburg, Pa., on December 24, 1864, the son of Jacob and Mary (Passmore) Gutelius. He was a descendant in the fifth generation of John Peter Gutelius, Physician and Surgeon, who came from France in 1750 and qualified at Philadelphia, Pa.

Mr. Gutelius received his preparatory education in the public schools of Mifflinburg, and, later, attended Lafayette College, at Easton, Pa., receiving his degree of Civil Engineer in 1887. In 1914, the degree of Doctor of Science was conferred upon him by the same college. He was Captain of the football team in 1886.

He began his career in 1887 as Assistant Engineer on the construction of the Englewood, N. J., Sewerage System, joining the Pennsylvania Railroad Company in 1888 as Assistant Engineer in charge of the reconstruction of the Tuscarawas Tunnel in Ohio. Later, he was attached to the Maintenance-of-Way Department of that Company until June, 1890, when he went West, spending five years in Montana, first as Assistant Engineer on the Flint Creek Power Development, at Butte, where he remained for a year.

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[1] Memoir prepared by J. M. R. Fairbairn, M. Am. Soc. C. E.

For a short time during 1891, he was Assistant Engineer with the Butte City Water Company, and during 1892 was in the private practice of Civil Engineering in that city. For three years, from 1893 to 1895, he was County Surveyor of Silver Bow County, Montana.

Mr. Gutelius went to Canada in November, 1895, as General Superintendent of Construction and Operation of the Trail Creek Tramway, at Trail, B. C., in which capacity he remained until the end of 1896, after which he was, from January, 1897, to March, 1898, General Superintendent in charge of construction and operation of the Columbia and Western Railway. In March, 1898, he was appointed Superintendent, Canadian Pacific Railway Company, in charge of operation and maintenance of the Columbia and Western Division, and from June, 1899, to June, 1900, he served as Resident Engineer, at Nelson, B. C. From 1900 to 1902 he was on special work, reporting to the Vice-President of the Canadian Pacific Railway Company, at Montreal, Que. His progress thereafter with the Canadian Pacific was rapid. He was appointed Engineer of Maintenance of Way of the System in 1902; four years later he was promoted to be Assistant Chief Engineer of the System, in which position he remained until August, 1908, when he was appointed General Superintendent of the Lake Superior Division, at North Bay, Ont., and subsequently, for a short period, held the same position in charge of the Eastern Division of the System at Montreal.

In January, 1912, Mr. Gutelius was selected as one of the Commissioners of a Royal Commission appointed to investigate expenditures and other matters in connection with the construction of the National Transcontinental Railway, and on the abolition of the Government Railways Managing Board, in May, 1913, he was appointed General Manager of the Canadian Government Railways, with all powers usually vested in the executive of railway corporations, reporting to the Minister of Railways and Canals of Canada.

He resigned this position in May, 1917, on his appointment as Vice-President of the Delaware and Hudson Company, with headquarters at Albany, N. Y. On the taking over of the management of the railways by the United States Railroad Administration during the World War, he was appointed Federal Manager of the Delaware and Hudson Railroad Company. On the relinquishment of that control on March 1, 1920, Mr. Gutelius was appointed Vice-President and General Manager of the Company and its allied properties. In the latter part of 1920, he was retained by the Ontario Government Commission inquiring into the proposed hydro-radial railway system. In 1923, he was appointed Resident Vice-President of the Delaware and Hudson Railroad Company at Montreal, in which capacity he remained until his death.

During his occupancy of this latter position, Mr. Gutelius established many friendships, and his loss is felt most keenly. He was a gentleman of quiet bearing, but had great decision of character, and had earned an international reputation as an engineer and railway executive. He was a capable, useful, and respected citizen, and his passing will be profoundly regretted.,

Mr. Gutelius is survived by his widow, Anna Walker Eaton, to whom he was married at Mifflinburg, on April 21, 1892, by a son, Frederick Passmore Gutelius, Jr., and by three married daughters, all resident in Canada.

He was a member of the Engineering Institute of Canada, and was prominent in Masonic circles at Mifflinburg, Pa.

Mr. Gutelius was elected a Member of the American Society of Civil Engineers on October 3, 1906.

JOHN VENABLE HANNA, M. Am. Soc. C. E.[1]

Died April 30, 1935

John Venable Hanna was born at Plattsmouth, Nebr., on January 1, 1864, the son of Thomas King and Judith Joyce (Venable) Hanna. When he was two years old his father, a wholesale merchant, moved his business and his family to Kansas City, Mo. John Venable Hanna lived in Kansas City until he entered Sheffield Scientific School, Yale University, at New Haven, Conn., from which he was graduated in 1885, with the degree of Bachelor of Philosophy. He again became a resident of Kansas City in 1906 and continued to make it his home until his death at the age of 70.

His father's wholesale business has been continued to this day. When his father died, his mother disposed of the family interests in the business, as, after a short trial, John Venable Hanna decided that he was not greatly interested in mercantile pursuits. From that time he was thoroughly and completely absorbed with construction work.

His experience took him through a variety of engagements, beginning as an Axeman on the Chicago, Burlington, and Northern Railway, which position he held from August, 1885, to July 1886. From July to November, 1886, he was employed as Assistant Engineer with the Colorado Railway Company. For the next two years, or more (May, 1887, to January, 1889), he was Assistant Engineer with the Current River Railroad Company and the Kansas City, Fort Scott and Memphis Railroad Company, following which, for six months (January to July, 1889), he served as Shop Inspector for the materials for the Thames River Bridge for the Providence and Boston Railroad Company. In April, 1890, he went West again and until November of the same year was engaged in mining in Colorado.

In January, 1891, Mr. Hanna returned to railroad work and until August, 1899, he was Assistant Engineer on the lines of the Kansas City, Memphis, and Birmingham Railroad Company. He then accepted a position in March, 1901, with the Fort Scott and Memphis Railroad Company as Assistant Chief Engineer which he held until September of the same year. Mr. Hanna then accepted a position as Assistant Chief Engineer with the St. Louis and San Francisco Railroad Company (the "Frisco Lines").

In December, 1906, he was appointed Chief Engineer of the Kansas City Terminal Railway Company, with headquarters at Kansas City, which position he retained until his death on April 30, 1935. As Chief Engineer of the Kansas City Terminal Railway Company, Mr. Hanna represented twelve trunk-line railroads, in connection with the construction of a $50 000 000 modern railroad passenger-freight terminal, a large part of which was built in a new location in Kansas City, Mo. and Kansas City, Kans. The amount of work done, its quality, the economical costs of the many different types of construction, and the speed with which the work was accomplished, all bear tribute to the care with which he supervised this gigantic undertaking. The importance of the work may be gained from the following figures: More than 50 viaducts valued in excess of $8 000 000 were built or rebuilt; 1 500 000 cu yd of steam-shovel excavation was moved several miles; a main-line railroad was lowered a maximum of 40 ft under traffic; a $5 000 000 passenger station second in size (at that time) to only one other, was constructed; a $1 000 000 sewer was built in the heart of the city; freight-houses; cach yard; a steam tunnel, a mile long; round-house; team tracks; main tracks; modern signal plants, all were constructed and brought to completion in an orderly scheduled manner creditable to all who were parties to the work and particularly to the Chief Engineer, John Venable Hanna.

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[1] Memoir prepared by a Committee of the Kansas City Section, consisting of A. C. Everham, E. E. Howard, and C. E. Johnston, Members, Am. Soc. C. E.

The construction of this Terminal called for frequent and unending conferences with the Chief Engineer's Committee which represented the twelve property lines. At these meetings Mr. Hanna presided, and his ability to remember details, facts, and important contractual and ordinance obligations, made the work of the members of the Committee unusually simple and efficient.

The contacts with the City officials with whom Mr. Hanna dealt, were always of a nature to bring respect from those politicians who would ask for many more and greater concessions than they expected to get. His ability to command their respect when he declined to grant many of these unwarranted requests coming from these sources, gave him a position of diplomacy as well as of engineering.

Mr. Hanna was a lover of detail and was so thorough in his understanding of what he was ready to discuss, report, or present, that it was seldom he had to call for his associates to help him. His work and his method of handling it brought to him many friendships. Engineers, contractors, railroad officials, politicians, and prominent citizens believed John Venable Hanna was absolutely honest in all his dealings with his fellow men. He had many problems which tested his sterling character, and he did not fail. His monumental work in Kansas City is a splendid achievement and one of which his family should be proud.

His hobbies were few. In recent years, he had been much interested in painting, and painted quite a few pictures in water colors. At the time of his death, he was working on a painting of the Station site and Station building. He had been an ardent golfer for many years; a man of small physique, he did not play among the low par shooters, but he always played a fair game and was always pleasant company.

Mr. Hanna was a member of the Kansas City Country Golf Club, and the Yale Alumni Club of Kansas City; he was Past-President of the Engineers'

Club of Kansas City, and of the Kansas City Section of the Society. He was very active in the affairs of the Safety Council, and served as Chairman for many years on its most important Committees.

He was married in Kansas City, Mo., on November 2, 1892, to Marguerite Vaughan. He is survived by two daughters, Mrs. Margaret (Hanna) Lang and Mrs. Judith (Hanna) Gregory, and one grandson, John Hanna Lang, two sisters, Mrs. Luther Welsh, and Edith Hanna, and one brother, T. K. Hanna.

Mr. Hanna was elected an Associate Member of the American Society of Civil Engineers on May 3, 1893, and a Member on April 2, 1902.

MORRIS HANSFORD, M. Am. Soc. C. E.[1]

Died July 17, 1935

Morris Hansford was born in St. Albans, W. Va., on December 6, 1865. He was the son of James Frazier and Annie (Noyes) Hansford who resided at the family home, which was known as "Ravenswood," overlooking the Kanawha River Valley. His father was engaged in farming and coal mining.

It was the custom at that time for parents to have tutors for their children, and Mr. Hansford received his first schooling from this source. He afterward attended the free schools until 1878. One of his first teachers was the Hon. William E. Chilton, former United States Senator from West Virginia, and present owner and publisher of the *Charleston Gazette*. He studied next under Professor P. B. Reynolds, from 1878 to 1883, at Shelton Academy at St. Albans. Although Mr. Hansford did not have the opportunity to gain a college education, he did acquire a thorough knowledge of engineering in the field and by constant study at home.

He began his practical experience on the location of the Upper Division of the New York Central Railway to Gauley Bridge, W. Va., under the direction of Mr. William C. Reynolds, serving as Rodman, Leveler, and Transitman from May, 1885, to November, 1886. Railroad and coke-oven construction consumed considerable of his time also. Mr. Hansford was employed as Topographer and Leveler on railroad location and construction from November, 1886, to September, 1888.

He entered the field of Mining Engineering in July, 1889, when he became associated with Mr. O. A. Veazey, of Pratt, W. Va. Until July, 1895, they were engaged in general engineering, in the location and construction of railroads, tipples, and inclines, and in the laying out of mining plants.

Mr. Hansford began his engineering career in the early days of the industrial development of the Kanawha River Valley which he lived to behold as one of the remarkable industrial centers of this generation.

From July, 1895, until April, 1899, Mr. Hansford was Mining Engineer for the Mt. Carbon Company, Limited, of Powellton, W. Va. As such he had full charge of the location and construction of tipples, inclines, and the opening of mines. In April, 1899, he joined the Charles M. Pratt Company, at Pratt, W. Va., and was occupied with the location of the Kanawha and Pocahontas Railroad and surveyed and divided 26 000 acres of land, continuing in that capacity until October, 1901.

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[1] Memoir prepared by Howard A. Levering, M. Am. Soc. C. E.

Mr. Hansford was engaged in private practice, at Pratt, between October, 1901, and July, 1905, including railroad, tipple, and bin construction. During a part of this period, the firm was known as the Hansford and Burrell Engineering Company. It employed several engineers and draftsmen and performed a considerable volume of work. He became Superintendent of the Great Kanawha Colliery Company and the Eureka Colliery Company, in July, 1905, and completed that assignment in November, 1908, when he returned to private practice at Pratt, as a Mining Engineer, retaining the latter office until July, 1910.

He was made Superintendent of the Mt. Carbon Company, Limited, in July, 1910, and remained at Powellton, until December, 1914, when he accepted his first highway position as an Engineer for the Kanawha County Court, serving under P. J. Walsh, Assoc. M. Am. Soc. C. E., from June, 1915, to May, 1916, and participating in the construction of surfaced highways. From January, 1917, to 1923, Mr. Hansford was Chief Engineer with the W. E. Deegans Coal Interests, in charge of twenty-three coal mines, the location of forty miles of railroad, and the planning of various buildings, tipples, etc.

He then returned to private practice in which he was engaged from 1923 until September, 1925, when he joined the Engineering Staff of the State Road Commission in the Second District, with headquarters at Huntington, W. Va., under H. J. Spelman, M. Am. Soc. C. E., then Division Engineer. Except for a period of six months, Mr. Hansford served continuously with the Commission until the time of his death. During this period he was engaged in construction within the District and for a time was in charge of a survey party operating from the Charleston Division of Plans and Surveys.

Mr. Hansford was a splendid gentleman and served his profession faithfully and efficiently. He took a keen interest in every task and every assignment which confronted him in the pathway of life, and met them graciously. As an Engineer, he was extremely accurate, thoroughly dependable, and strictly honorable.

He was a Mason, a Knight Templar, and a member of Beni Kedem Shrine, Charleston, W. Va. He was also a member of the West Virginia Society of Professional Engineers, and a Director of Huntington Chapter. He was a Registered Professional Engineer in West Virginia. He was a member of Johnson Memorial Methodist Episcopal Church, in Huntington.

On March 26, 1890, he was married to Willia Lauck, who survives him. He is also survived by a son, Russell L. Hansford, of Marmet, W. Va., and a sister, Mrs. Philip Doddridge, of Berea, Ohio.

Mr. Hansford was elected a Member of the American Society of Civil Engineers on January 14, 1924.

FREDERICK NATHANIEL HATCH, M. Am. Soc. C. E.[1]

Died November 13, 1934

Frederick Nathaniel Hatch was born in Chartiers Borough, Pa., on June 12, 1885, the son of the late Frederick Thomas Hatch, M. Am. Soc. C. E., and Alice Gertrude (Hill) Hatch.

After completing the High School course, in Logansport, Ind., he entered Rose Polytechnic Institute, at Terre Haute, Ind. By taking extra subjects during the first year and passing some of the first year's work by examination, he was enabled to complete the four-year course in Civil Engineering in three years and was graduated in 1906 with the degree of Bachelor of Science in Civil Engineering. Five years later, he received the degree of Civil Engineer from the Institute.

Mr. Hatch's father was for years Chief Engineer of the Pennsylvania Lines West of Pittsburgh, and it was quite natural, therefore, that the son should follow in his footsteps. Prior to his graduation in 1906, he had acquired about a year's experience in railroad engineering work, and his engagements for several years thereafter were on projects akin to railroading.

From July, 1906, to February, 1907, Mr. Hatch was Rodman and Transitman with the San Pedro, Los Angeles, and Salt Lake Railroad Company, at Salt Lake City, Utah, and from February to August, 1907, he was Field Engineer and Assistant to the Superintendent of Construction on a large concentrating works for the Steptoe Valley Smelting and Mining Company, at McGill, Nev. Although the pioneer days were supposed to have vanished before the advance of civilization, nevertheless in this desert country, far from human habitations, an occasional "bad man" loaded with bad whiskey could still produce thrills for a young engineer.

Later, he was Engineer on Maintenance of Way and Grade Reduction Projects of several other railroad companies until 1910, when he took a responsible position with Westinghouse, Church, Kerr, and Company (later, Dwight P. Robinson and Company) on the supervision of design and construction of industrial plants, railway locomotive terminals, shops, etc. Prior to the entrance of the United States in the World War, this firm constructed a number of plants for the production of munitions for the Allies.

During the World War, Mr. Hatch served with the American Expeditionary Force in France, as Captain of Engineers, first with the 312th Engineers at Camp Pike, Arkansas, and, later, with the 35th Engineers at Camp Grant, Illinois, and at La Rochelle, near Bordeaux, in France. The 35th Engineers was a shop regiment, recruited to build American freight cars. Although Captain Hatch greatly preferred to be at the front, he served his country faithfully far back in the interior, where his previous military training and experience were especially valuable to his regiment.

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[1] Memoir prepared by James W. Skelly, M. Am. Soc. C. E.

Following his discharge from the Army in August, 1919, he was promoted successively to Major and Lieutenant Colonel, in the Engineer Reserve. After the war, he was employed in the office of the Building Commissioner of the City of St. Louis, Mo., until 1927, after which he was engaged as Construction Engineer with the Southwestern Bell Telephone Company until his death.

Colonel Hatch, as he was later known to his associates, gave much time and attention to military pursuits. He was President of the St. Louis Chapters of the Society of American Military Engineers, the Military Order of the World War, and the Reserve Officers Association. He was President of the Missouri Department of the Association in 1930. He was alert, well informed, and was considered an excellent engineer and soldier. He was positive, both in speech and action, and his writings show clarity and breadth of thought.

Colonel Hatch leaves a stepmother, Mrs. Nola P. Hatch; a sister, Alice K. Hatch; and a brother, William S. Hatch. He was never married. A side light on the kindly nature of a man who may have appeared to be cold and distant, even to his friends, is gained from the instance that on the death of his father, in 1920, he gave up his position in New York City and came to St. Louis that he might care for his stepmother, and this devotion continued to the end of his life.

In 1933, he suffered a severe attack of pneumonia, from which after several months he seemed to recover; but early in 1934, other serious maladies, diagnosed as abscesses of the brain and liver, made their appearance, and after a long battle, including several operations, he finally succumbed on November 13, 1934.

Colonel Hatch was elected a Junior of the American Society of Civil Engineers on June 4, 1907; an Associate Member, on February 4, 1913; and a Member on April 26, 1921.

DAVID CHRISTIAAN HENNY, M. Am. Soc. C. E.[1]

DIED JULY 14, 1935

A board of consulting engineers had been called to one of the large projects of the United States Bureau of Reclamation for the consideration of some of the important engineering phases of the project. They had been in session several days—days of trying discussion on points for which no precedents existed—days of give and take, for a report, unanimous if at all possible, must be made. All were men of ripe experience, technically trained; all with at least one, and some with several, scholastic degrees. The final *piece de resistance* hinged about certain phraseology in the report, and the one who had prolonged the discussion with righteous tenacity finally surrendered with, "If you want to know English correctly you've got to get it from a Hollander who couldn't talk a word of it when he came to this country." This, of course, was an exaggeration as Mr. Henny had studied English in college before he came to the United States in 1884.

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[1] Memoir prepared by J. C. Stevens and J. L. Savage, Members, Am. Soc. C. E.

David Christiaan Henny was born at Arnheim, The Netherlands, on November 15, 1860, and was proud of his Dutch ancestry. His parents were David and Bernedina (Lorentz) Henny, his mother having been a cousin of the famous physicist, H. A. Lorentz.

He was educated at the Polytechnic School, at Delft, Holland, and was graduated with the degree of Civil Engineer in 1881. He had kept contact with his native country by membership in the Royal Institute of Engineers and by occasional visits there during the later years of his life.

Mr. Henny first came into national prominence in the early Nineties in connection with the manufacture of wood stave pipe. He served as General Manager and Chief Engineer of the Excelsior Wooden Pipe Company for ten years, and then for three years as General Manager of the Redwood Manufacturers Company, both of San Francisco, Calif.

Fresh from successful achievements in the redwood field he was taken into the organization of the United States Reclamation Service in February, 1905, and served as Supervising Engineer for the Pacific Division for four years, whereupon he was made Consulting Engineer for the same organization on intermittent service, available on call. The remainder of his life was devoted to private practice as a Consulting Engineer, in which capacity he maintained an office in Portland, Ore., from February, 1909, until his death.

Mr. Henny served as a member of a great many boards of Consulting Engineers for both governmental and private agencies in the United States and abroad on land reclamation, power, flood control, and allied problems. He was an authority on the design and construction of dams and there is scarcely an important dam in the West that has not been made safer and more serviceable as a result of his studies and advisory services. He was an indefatigable worker and no task was put off. Things were done now. Procrastination was not in his dictionary, which perhaps is the secret of his great capacity for accomplishment.

Two tasks will keep his memory ever green in the annals of engineering. One is his paper on "Stability of Straight Concrete Gravity Dams",[2] for which the Norman Medal was awarded to him posthumously by the Society, at the Annual Meeting on January 15, 1936, his son, Arnold L. Henny, Assoc. M. Am. Soc. C. E., being present to receive it. The other is the invention of the Henny Shear Joint, a special joint between the up-stream and down-stream faces of concrete blocks used in the construction of massive concrete dams. It was designed while he was on the Board of Consulting Engineers for the Boulder Dam. This shear joint is of such great importance that in order to keep it from being exploited he secured a patent on it and immediately gave the U. S. Bureau of Reclamation royalty-free use of his patent rights. Boulder Dam was the first in which the Henny shear joint was used, but it is being incorporated in the design of the Grand Coulee Dam and doubtless will be used on future massive concrete dams, particularly those in which the load is largely taken by cantilever action, as in the high, straight gravity type.

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[2] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1041.

Mr. Henny was small in stature but of impressive dignity, and yet could relax on appropriate occasions into a delightful banter of sparkling wit and pertinent anecdote. An Engineering Board was examining the site of the Owyhee Dam in Eastern Oregon. After a "boiling" hot day in July, climbing over the still hotter rocks, the party found themselves at the water's edge. Mr. Henny began tugging at his shirt with, "We've examined everything else about this dam site except the water—let's go in swimming—and the last one in's a sissy." And they all did—raw.

At the entrance of the United States into the World War Mr. Henny offered his services to go over seas or serve elsewhere in any capacity in which he could be useful. His letter offering his services closed with the remark that he "would never be able to do enough for a country that had done so much for him." The reply he received was splendid and to the effect that such loyalty were better exercised in his chosen field at home than anywhere else, that there were plenty of younger men to do the fighting.

Oregon State College honored itself and him by conferring upon him a degree of Doctor of Engineering in 1933, and in the same year he was made an Honorary Member of the Royal Institute of Engineers of Holland.

Mr. Henny was most happily married in 1893, at San Francisco, to Julia Antoinette Hermanie Watzel-Brown. To them were born four children: David—who died in his senior year in college at Madison, Wis.; Frances Bernedina, George Christiaan, and Arnold Lorentz. Mr. and Mrs. Henny were great pals. She accompanied him on nearly all his trips whether in or out of the United States, and was probably the best known woman in civil engineering circles in the country. When she died on May 26, 1935, he died also, although he continued to walk around about his business, outwardly smiling and genial as usual, for two months longer.

The *Oregon Journal*, of July 16, 1935, paid a most worthy tribute to his memory:

> "D. C. Henny saw this vision of a greater Oregon. He saw Oregon like the Holland of his birth, fertile in its gardens and fields, and related to the world in trade. Yet, he saw it as Oregon alone can be, in the bold beauty of its mountains and shore and in the promise of its future.
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> "So D. C. Henny worked with rivers and the land as a weaver would work with threads, weaving a bright and rich pattern of opportunity for to-day and for the generations that are to come.
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> "In his recent leadership as Chairman of the Oregon State Planning Board, Mr. Henny had reached the climax of a life distinguished internationally as a hydraulic and reclamation engineer. For what many paid him large fees as a consultant, he gave without price to Oregon. Into the decisions and plans of Boulder Dam, Grand Coulee, and Bonneville, to mention only a few, went his penetrating wisdom. And he was Oregon's Commissioner in the Pacific Northwest Regional Planning Commission."

He was a member of the Committee of Engineering Foundation on Arch Dam Investigation and took an active part in supervising the construction and experimental measurements at the Stevenson Creek Test Dam, near Fresno, Calif., the results of which were published by the Society.[8] He was also Chairman of the Special Committee on Irrigation Hydraulics of the Society during the ten years of its existence.

Mr. Henny was elected a Member of the American Society of Civil Engineers on September 7, 1887. He served a term as Director for the 12th District beginning in 1920, and a term as Vice-President from Zone IV beginning in 1932.

VIVION ROSE IRVINE, M. Am. Soc. C. E.[1]

Died December 16, 1933

Vivion Rose Irvine, the son of Hugh Rose and Mollie (Vivion) Irvine, was born in Dallas, Tex., on April 4, 1885. He was a descendant of that William de Irvine who married a daughter of Robert the Bruce of Scotland. The Irvine family came to America in the early part of the Eighteenth Century and settled first in Pennsylvania and then moved on to the Valley of Virginia. It furnished soldiers to Washington for service in the Indian Wars and in the Revolution. Descendants moved on into Tennessee and eventually settled in Texas when that region was a Republic.

Mr. Irvine's primary schooling was obtained at public and private schools in Texas; at Bingham Military Academy, Asheville, N. C.; and at the Lawrenceville School, Lawrenceville, N. J.; later, he attended the University of Texas, at Dallas.

After leaving the University of Texas in 1906, he entered the service of the Southern Pacific Lines as Rodman on a locating party. He was soon promoted to the position of Instrumentman and, later, to that of Assistant Engineer on Construction. For two years, 1908 and 1909, he was in private practice at Marshall, Tex., engaged in general engineering work, but in 1910 he was back in railroad work where he was to remain until 1917. During this seven-year period he served as Locating Engineer on the New Iberia and Northern Railroad; as Assistant Engineer with the Florida East Coast Railroad Company, on maintenance of way and on the construction of its Kissimee Valley Branch; and, finally, with the Texas and Pacific Railroad Company as Assistant Engineer on Maintenance of Way and Valuation.

From May, 1917, to October, 1919, he was in the Army. Mr. Irvine had applied for a commission in the Reserve Corps, United States Army, in 1916; he was commissioned a Captain, Engineer Section, Officers Reserve Corps, on February 23, 1917, six weeks before the United States entered the World War, and was ordered to report to the Citizens Training Camp, at Leon Springs,

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[8] *Proceedings*, Am. Soc. C. E., May, 1928, Pt. 3.

[1] Memoir prepared by L. M. Gray, M. Am. Soc. C. E.

Tex.; he was transferred on May 16, 1917, to the 7th Reserve Engineers (later, designated the 17th Railway Engineers), and was relieved from active duty on May 19, 1917. He was re-assigned to active duty on September 2, 1917, and ordered to report to the Engineer Training Camp, at Fort Leavenworth, Kansas; he was transferred on December 10, 1917, to Company A, 35th Engineers; was relieved from duty with the 35th Engineers and ordered to Hoboken, N. J., on January 22, 1918; sailed from the United States for overseas service on January 31, 1918, on the S. S. *Adriatic*; was assigned to the 116th Engineers; was transferred, on March 3, 1918, to the 14th Engineers and assigned on June 13, 1918, as Commanding Officer, Company A, 14th Engineers. He participated with his organization in the Somme Defensive and the Aisne-Marne and Oise-Marne Offensives; was relieved from duty with the 14th Engineers on September 27, 1918, and returned to the United States, October 10, 1918, on the S. S. *Agamemnon*. He was ordered to report to the Commanding Officer, at Camp A. A. Humphreys, Virginia, and was assigned to duty with the 2d Engineer Training Regiment, on October 21, 1918. He was transferred, on February 25, 1919, to the office of the Chief of Engineers, at Washington, D. C., and was honorably discharged on October 13, 1919, at Washington, with the rank of Captain, Engineers, United States Army.

While on duty in the office of the Chief of Engineers, Captain Irvine revised and rewrote the Manual on Light and Standard-Gauge Railroads as a textbook for Army instruction. After his discharge from the Army he was appointed Major, Engineer Section, Officers Reserve Corps, on December 12, 1919; he accepted on December 24, 1919; and was re-appointed in the same rank on December 12, 1924, but declined the appointment.

From October, 1919 to April, 1920, Major Irvine served with the Department of State in charge of the selection of railroad personnel for service with the Russian Railway Service Corps in Siberia; from May, 1920, to June, 1921, he was Supervising Engineer with the Quartermaster Corps, U. S. Army, in charge of the construction of water-works, buildings, and sewers at Camp Bennings, Georgia, a $1 250 000 project.

In July, 1921, he accepted an assignment to go to Colombia, South America, as Representative of the Seldon Breck Construction Company, of St. Louis, Mo., and thereafter remained in that country continuously until 1932. He was made Vice-President of the Breck-Parrish Engineering Corporation (incorporated September, 1921, under the laws of Colombia, with headquarters at Barranquilla) and had charge of all the engineering and construction undertaken by that Company, including the design and construction of a 100-bed hospital, a reinforced concrete dam, water-works system for Barranquilla, the development of an extensive new residential addition to the city, the paving of the down-town section, and other work in connection with the modernization of this city of 150 000 inhabitants.

The complete transformation of this sleepy, tropical city, with unpaved streets, inadequate water and sanitary facilities, into a thoroughly modern municipality with well paved streets, a plentiful supply of pure water, adequate sewers, and ample residence sub-divisions, was affected in the 10-yr period from 1921 to 1931. Most of this work was initiated by Mr. Karl C. Parrish, and the greater part of the construction work was done by his Company of which Major Irvine was Chief Engineer. In 1925, the Breck-Parrish Company became associated with Winston Brothers, of Minneapolis, Minn., in the construction of the Antioquia Railroad to the Cauca River and Major Irvine went up country to assist on this work. He was associated with this project, which involved some of the heaviest construction in the Andes, until its completion. In 1929, he went to Cali, to become Manager of an urbanization project in that city, and was there until 1932 when the condition of his health necessitated his return to the United States. In the fall of 1933, he accepted a position as Engineer Inspector with the Public Works Administration, and was assigned to Dallas, where he died shortly after his arrival.

In Colombia, where Major Irvine lived for eleven years and where his most important work was done, he was one of the best known of foreigners; speaking Spanish fluently he was as popular with the Colombians as with the Anglo-Americans. He was an engineer of exceptional ability, and in addition to his engineering qualifications he possessed the human qualities that endeared him to his associates and friends. He had the happy faculty of meeting people and of enjoying social intercourse; it is doubtful whether anything he ever did gave him greater pleasure than the laying out and building of the nine-hole golf course for the Barranquilla Country Club. Whenever distinguished Americans visited Colombia, Major Irvine was invariably selected to head the Committee appointed to look after their entertainment. In a country where entertaining is a fine art, he was the perfect host. Major Irvine's death is a loss to the Engineering Profession, and he is mourned by a host of friends.

He was a Mason, a Shriner, and a Knight Templar; a member of the Anglo-American and Country Clubs of Barranquilla, Colombia, and of the Union and Campestra Clubs of Medellin, Colombia.

He was married on June 10, 1911, in St. Augustine, Fla., to Frances Elizabeth Ransom, who survives him; he also left a daughter, Mary Delia Irvine.

Major Irvine was elected a Member of the American Society of Civil Engineers on May 28, 1923.

EDWARD CLARENCE JORDAN, M. Am. Soc. C. E.[1]

DIED SEPTEMBER 15, 1935

Edward Clarence Jordan was born in Westbrook, Me., on March 17, 1846, the son of Samuel and Eunice Q. (Seal) Jordan. His father was a prominent citizen of Westbrook and, at one time, was Postmaster of Portland, Me.

After receiving his early education in the public schools and at Westbrook Seminary, Mr. Jordan entered the office of Anderson and Bonnell, in Portland, to study for an engineering career. One of the partners, Mr. John Anderson, was a prominent Railroad Engineer and, at one time, Railroad Commissioner of Maine. Two years later, Mr. Jordan entered Union College, at Schenectady, N. Y., where he studied under Professor W. M. Gillespie who at that time was conceded to be one of the leading Engineers in the United States. Mr. Jordan was graduated from Union College in 1868, with the degree of Bachelor of Science in Civil Engineering. While at college he was elected to Kappa Alpha Fraternity.

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[1] Memoir prepared by Henry I. Jordan, Esq., Portland, Me.

Immediately after his graduation Mr. Jordan went West to enter the employ of the Central Pacific Railroad Company in its early construction across the country; it was the first railroad line to connect the Atlantic and Pacific Oceans. His first position was that of Leveler on location, the work starting from Salt Lake City, Utah, westward, and from California, eastward, to a meeting point. He soon became Transitman and, later, in 1869 and 1870, Assistant Engineer on location and construction with the Company, in California, Utah, and Nevada, and had the good fortune to be in charge of the Division where the rails met—an occasion for an imposing ceremony. The last tie laid was of highly polished California laurel wood with a silver inscription. The last railroad spike was of solid gold, and a silver hammer was used to connect the telegraph wires. When the President of the Railroad Company struck the wire a corresponding click was heard all over the United States, announcing that the railroad from the Atlantic to the Pacific was finished.

After this interesting experience, Mr. Jordan became Locating and Resident Construction Engineer with the Northern Pacific Railroad Company in Minnesota, North Dakota, and Montana, from 1870 to 1872, when he returned to Portland, Me. In 1873 he opened an office for general engineering practice in Portland, specializing largely in hydraulics. One of his first experiences was on harbor work for the Federal Government.

He designed and constructed a sewerage system for the Town of Old Orchard, Me., in 1883 and 1884. In 1891 and 1892 he laid out and supervised the construction of the Maine Mile Track, called Rigby, at that time considered to be one of the best race tracks in the country. He designed and constructed the dam at North Gorham, Me., for the Portland Electric Light Company, from 1899 to 1901, and the dam at Mallison Falls, at Windham, Me., in 1905, for the Robinson Woolen Mills. He also designed and installed a water supply system for the Hebron Sanatorium, in 1909. At the famous Poland Springs, owned by Hiram Ricker and Sons, by whom Mr. Jordan was employed for all the firm's important projects, he laid out the sewerage system, water supply system, much of the road system, a private railroad, and a golf course. He had to do with many other hydraulic and sanitary engineering projects in his long practice of more than sixty-two years.

However, the largest proportion of Mr. Jordan's practice was in municipal and real estate surveys, many of which were contested boundary line cases, and his vast knowledge of real estate law enabled him to satisfy and settle many such disputes without recourse to the Law Courts. He was known to remark many times that it took more research, study, and judgment to settle such difficult cases properly than it did to design a dam. A quotation from a newspaper refers to him as follows:

> "Mr. Jordan has a marvelous memory for facts and figures, and he knows the history of this section as probably no other man does for the reason that he is the oldest engineer now in service, and has had occasion to survey and map most of this country during his experience, hereby actually working out by his own hands much of the data that others have to trust to records to find."

While abroad in 1881, and again in 1892, to inspect and study sanitary conditions in European cities, Mr. Jordan learned much that was helpful to him in his professional career. It was while he was in Europe the second time that he was appointed City Engineer of Portland, much against his will. He was prevailed upon to accept the office, however, and during his tenancy brought about many needed reforms. He made surveys and gave expert testimony in Court proceedings on several flowage cases incidental to the raising of dams.

For many years, he was the Engineering Member of the State Board of Health and, at one time, served as President of the Board. He was a member, until it ceased to exist, of the Maine State Water Storage Board created in 1909 to facilitate the building of reservoirs. He was also a member of the Commission for the Improvement of Back Cove which resulted in the construction of Baxter Boulevard; a member of the American Public Health Association; President for some years of the Portland Fraternity, a local benevolent association; a Charter Member of the Maine Association of Engineers; and a member of the Portland Civic Club, the Cumberland Club, the Portland Country Club, and the Portland Economic Club. He was also a member of the State Commission appointed to investigate and report on Portland Harbor. In 1896, he was a Delegate to Indianapolis, Ind., in the Palmer and Buckner campaign.

During the World War Mr. Jordan was appointed an Associate Member of the Naval Consulting Board by the Secretary of the Navy, the Hon. Josephus Daniels, and was Chairman for the State of Maine of the Board arranging for Organized Industrial Preparedness.

Mr. Jordan was a man who took a vital interest in all civic and State improvements, and was a respected and constant writer to the newspapers in regard to them. That he spared neither time, strength, nor money to fight for a cause which seemed right to him, was clearly evidenced by his flying trip home to Portland from Ormond, Fla., where he had gone for his usual winter vacation and golf, to oppose a proposition of the railroad to build a new terminal within the limits of the City of Portland, which, had it been allowed, would have created an appalling smoke nuisance and real estate damage to one of that city's most promising residential sections.

Mr. Jordan was a faithful attendant at the First Parish Unitarian Church to which he gave freely of his time and professional skill as well as his money. Thoroughness in research was the means of discovering for his Church the fact that its Trustees could justly claim a few feet of land in the rear of their lot, about which they did not know, but which they needed very much indeed for an addition to their Parish House.

No biography of this Engineer would be at all complete without reference to his very wide knowledge and appreciation of the best in literature. His books were his companions, especially so after the death of his wife. His ability to quote from so many standard works was an accomplishment in itself. He was outstanding as a high type of gentleman, scholar, and engineer.

He was married to Eliza Payson Thomas, in 1874, who died two years later. In 1882, he was married to Marcia Dow Bradbury, who died in 1923. He is survived only by his nephews and their families.

Mr. Jordan was elected a Member of the American Society of Civil Engineers on May 7, 1890.

IRVING PATTERSON KANE, M. Am. Soc. C. E.[1]

Died November 20, 1935

Irving Patterson Kane was born in Gittings, Md., on January 2, 1886, the son of James G. Kane and Leonore (Patterson) Kane. His early life was spent in Gittings, a suburb of Baltimore, Md. He attended the public schools of Baltimore County and, in 1903, entered St. John's College, at Annapolis, Md., from which he was graduated in 1907, with the degree of Bachelor of Science. He then entered the Massachusetts Institute of Technology, at Boston, Mass., and was graduated in 1910 with the degree of Bachelor of Science in Sanitary Engineering.

After leaving the Massachusetts Institute of Technology, Mr. Kane was employed for a short time with the Baltimore and Ohio Railroad Company and the Lehigh Valley Railroad Company. In May, 1912, he entered the service of the Sewerage Commission of the City of Baltimore and was engaged until February, 1915, on designs, estimates, and specifications, in particular, for intercepting sewer work in Baltimore. From February to December, 1915, he served as Assistant District Engineer on the International Joint Commission under Professor Earl B. Phelps, making preliminary layouts, with cost estimates, for intercepting sewerage systems for Detroit, Mich., and for other cities and towns on the Detroit and St. Clair Rivers, as well as treatment-plant studies. This work was done in connection with the studies of remedies of the pollution of boundary waters, and the results of the investigation were published in the report made by Professor Phelps to the International Joint Commission.

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[1] Memoir prepared by Langdon Pearse and Leslie C. Whittemore, Members, Am. Soc. C. E.

In 1916 and 1917, Mr. Kane served as Assistant Engineer with the Seaboard Air Line Railroad Company, and then with the Western Maryland Railroad Company. In May, 1917, he entered the United States Army, serving as Captain and then as Major and Commanding Officer, Company A, 305th Engineers, and also, later, with the 45th Engineers, for fourteen months in France. He was mustered out in August, 1919.

From November, 1919, continuously until the time of his death, he was employed first as Assistant Engineer and then as Engineer of Treatment Plant Design with The Sanitary District of Chicago, Chicago, Ill., the major part of the time being in immediate charge of design and plans of treatment plants, in particular, the Calumet, North Side, West Side, and Southwest plants, as well as the Northbrook, Glenview, and Morton Grove plants, exclusive of electrical, mechanical, and architectural work. In this connection, Mr. Kane was also interested in the design of several intercepting sewers. At the time of the litigation between the Lake States and the hearings before the Master, the Hon. Charles Evans Hughes, Mr. Kane was in charge of preparing general studies and estimates for the entire sewage treatment program of the Sanitary District, and also with plans and estimates for the Engineering Board of Review. He was active in developing various special features of the plants, in particular, a machine for stripping sludge on a large scale from air-drying beds. A new type of bed was developed, in units 80 ft wide and 1 000 ft long.

Mr. Kane was not a writer, but was intensely interested in the details of design and the development of sound data. His chief interest was in his work, in which he was very thorough and painstaking. One of his principal hobbies was financial research and in this field he took a lively interest in economics and various current problems. He had a keen mind and thought problems through with good judgment.

Never having married, and having lived with his bachelor brother, in Chicago, during the last fifteen years of his life, Mr. Kane took a deep interest in his mother's welfare, spending his vacations at the family estate at Gittings, near Baltimore. Although quiet and reserved, and not socially inclined, he was well liked by his associates. He was always interested in the outside activities of the younger men in the office, although not actively participating in them.

During the last eight years of his life, Mr. Kane suffered from a very serious eye ailment which, at times, threatened to destroy his sight. Notwithstanding this affliction, he applied himself energetically and patiently to his work and few, even of his most intimate associates, knew the extent of the handicap under which he was laboring. His general health began to fail during 1935, and on October 31, he applied for leave of absence and entered the Presbyterian Hospital in order to recuperate and endeavor to restore his health. Unfortunately, it proved too late. He died very suddenly on the evening of November 20, 1935.

Mr. Kane was elected an Associate Member of the American Society of Civil Engineers on December 6, 1915, and a Member on December 3, 1928.

ERNEST AVERY LAMB, M. Am. Soc. C. E.[1]

DIED FEBRUARY 1, 1935

Ernest Avery Lamb was born in Hastings, N. Y., on December 6, 1865, the son of Lucian B. and Eliza Ann (Caulkins) Lamb. He received his preparatory school education at Mexico Academy, Mexico, N. Y., after which he taught in the public schools for several years. In 1889, Mr. Lamb entered Union College, at Schenectady, N. Y., and was graduated in 1893, with the degree of Civil Engineer.

He entered the service of the State of New York as Rodman in September, 1894, and was assigned to the Middle Division Office of the State Engineer's Department, at Syracuse, N. Y., where he was employed on the improvement of Madison County canal feeders, the re-building of three locks on the Black River Canal, preliminary surveys and estimates for special canal work, and construction work at Seneca Lake Outlet. In 1896, Mr. Lamb was promoted to the grade of Leveler and was engaged principally on the improvement of the Erie Canal, known as "The Nine Million."

He left the service of the State in May, 1898, and until May, 1900, was employed as Engineer for a contractor on the construction of the Albany and Southern Railway (third-rail electric). This work involved, among other features, a masonry dam, a long viaduct, and two steel penstocks $\frac{3}{4}$ mile in length.

Mr. Lamb then returned to State employment at Albany, N. Y., and was engaged in making preliminary surveys and estimates for the proposed Barge Canal Improvement. In January, 1901, he went to Pittsburgh, Pa., where he was employed for eighteen months by the American Bridge Company on structural work. He then accepted a position with W. G. Wilkins Company, Engineers and Architects, as Superintendent and Engineer on plant construction for the W. W. Lawrence Company, Paint Manufacturers, in Pittsburgh.

In April, 1903, Mr. Lamb returned to the New York State service as Assistant Engineer. In June, 1906, he was appointed Resident Engineer and, for the next three years, was employed principally on highway work. In August, 1909, he took charge of the Mohawk River Residency which included fifty-four miles of Barge Canal construction. From 1915 to 1919, he was in charge of Barge Canal Residency No. 4, covering the territory between Schenectady and the Oneida County Line, comprising about fifteen construction contracts.

In 1919, Mr. Lamb was appointed Engineer-Secretary of the New York-Pennsylvania Joint Interstate Bridge Commission, with headquarters in Albany, a position which he held at the time of his death.

On November 29, 1894, he was married to Margaret M. Boylon, of Worcester, N. Y. Mrs. Lamb died in October, 1919. On December 6, 1922, Mr.

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[1] Memoir prepared by Charles S. Sterling, M. Am. Soc. C. E.

Lamb was married to Kathryn McKay Gilbert, of Albany, who survives him, with three children by his first marriage: Marion E., Mildred A., and Joseph M. Lamb. There is also one granddaughter, Shirley E. Lamb.

Mr. Lamb was a member of the Madison Avenue Reformed Church, of Albany, Ancient City Lodge No. 452, F. and A. M., and the Albany Sovereign Consistory. He was also a member of the National Geographic Society, the National Society of Professional Engineers, and the Berlin Mountains Fish and Game Club.

He had been in failing health for some time prior to his death. He was able to attend to business, however, until late in November, 1934, and died rather suddenly at his home in Albany, on February 1, 1935.

Ernest A. Lamb was a quiet, unassuming Christian gentleman. He was kind and sympathetic, and interested in people whose pathways were beset with difficulties and troubles. His passing marks the end of a life of service. He is missed and mourned by his circle of friends.

Mr. Lamb was elected an Associate Member of the American Society of Civil Engineers on September 7, 1904, and a Member on September 9, 1919.

WILLIAM CASWELL SMITH LEMEN, M. Am. Soc. C. E.[1]

Died April 7, 1935

William Caswell Smith Lemen was born at Collinsville, Ill., on December 19, 1873. He was the son of Clarence J. and Sarah Catherine (Smith) Lemen, and his forebears were pioneers in the State of Illinois. He first attended Union Academy, at Morganfield, Ky., and on graduation from that school he entered the College of Engineering, University of Illinois, at Urbana, Ill. There, he received the degree of Bachelor of Science on completing the course in Civil Engineering in 1895.

Mr. Lemen began his engineering career as a Recorder and Instrumentman for the Mississippi River Commission, with headquarters at St. Louis, Mo. For the first year, he was a member of the Drafting Department of the Commission, and for the next three years, was engaged in field and office work. From July, 1899, until August, 1900, he was a Topographer in the United States Engineer Office, at St. Paul, Minn. Here, he was engaged in the work of surveying the head-waters of the Mississippi River.

Returning to St. Louis in August, 1900, he again joined the Mississippi River Commission as Gauge Inspector on the Lower Mississippi. From April, 1901, to November, 1902, he returned to his home State to act as City Engineer and Superintendent, of the Municipal Electric Light and Water-Works, at Paris, Ill. At Paris, he constructed street paving, sanitary sewers, and water mains, and installed a new pumping plant which represented a city investment of approximately $500 000. After this interesting experience in a new field, he again resumed work in the Mississippi Basin, this time as Chief of Party on a cadastral survey of the reservoirs of the head-waters of the Mississippi River.

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[1] Memoir prepared by Col. Charles W. Weeks, U. S. Infantry, Fort Benning, Ga.

In the following year, however (November, 1903), Mr. Lemen entered the sphere of engineering activity that was to occupy the next fourteen years of his life. By transfer he became Junior Assistant Engineer to the Principal Assistant Engineer, U. S. Engineer Office, Savannah, Ga. In this position, he had charge of surveys and various construction works. In August, 1910, he became Assistant Engineer, and in April, 1912, Principal Assistant Engineer.

In this office, he had general charge of the supervision of all improvements in the District, and also had direct charge of the Savannah Harbor and Inside Waterway. His supervision also included repairs and construction of bank-protection works, training walls, stone jetties, and wharves, the design and construction of hydraulic dredges, and snagging and construction plants. He was also responsible for the monthly and annual reports and the proper expenditure of all appropriated public funds for his work, including one large special project on the Inland Waterway System, at Brunswick, Ga. He was also the author of numerous special reports.

In 1917, when the United States entered the World War, Principal Assistant Engineer Lemen became, on June 21, Captain Lemen, of the Corps of Engineers, U. S. Army. In July, 1918, he became a Major, and on October 24, a Lieutenant Colonel. During this period, also, he became the Officer in charge of the Kearny United States Engineer Depot, at Newark, N. J., and Depot Engineer and Assistant Storage Officer of the Port of New York, N. Y. He afterward prepared a paper on this work entitled "Construction and Reorganization of U. S. Engineer Supply Depot, Kearny, New Jersey."[2]

Honorably discharged as an emergency officer in October, 1919, he was again commissioned a Major in the Regular Army, Corps of Engineers, as of July, 1920.

Major Lemen's army career continued to be as full of interest as his previous years. From November, 1920, until September, 1922, he was United States District Engineer, at Jacksonville, Fla., for which position he was particularly fitted by his experience in the near-by Savannah District. From late in 1922, and during the ensuing year, Major Lemen attended the Engineer School at Fort Humphrey, Virginia. He then served from July, 1923, to April, 1925, as Engineer Supply Officer and Assistant Department Engineer, at Fort Hayes, Ohio.

Then followed a tour of foreign service, and with it a change from staff to command duty. For the year from April, 1925, to March, 1926, Major Lemen was Commanding Officer, 2d Battalion, 3d Engineers, at Schofield Barracks, Hawaii. His second and third years of foreign service, however, found him, first as United States District Engineer, at Honolulu, Hawaii, and Department Engineer, Hawaiian Department, and from September, 1926, to August, 1928, he devoted his entire time to the latter position.

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[2] *Military Engineer*, May–June, 1930.

On leaving Hawaii, Major Lemen became Assistant Professor of Military Science and Tactics, Engineer Unit, Reserve Officers Training Corps, University of Illinois, Champaign-Urbana, Ill., his old school. Here, among familiar scenes, he continued active duty until 1932, when, after a sick leave of several months, he was retired with his war-time rank of Lieutenant-Colonel, on December 31, 1932. Lt. Col. Lemen spent the greater part of his remaining years at Interlachen, Fla. He died in that city on April 7, 1935.

He was married at Evansville, Ind., on December 19, 1896, to Olive Belle Jutton. He is survived by his widow and the following children: Catherine Neil Lemen (Mrs. A. B. Rowe), Wilhemina Lemen, and Orian Gregg Lemen.

During his career before the World War and as an officer in the Corps of Engineers, Colonel Lemen was always public-spirited, ever ready to lend his aid and to co-operate to the full in undertakings in which he could be of assistance. He was most agreeable socially and altogether a fine type of citizen. He was an ardent and proficient golfer, and both official and companionable contacts with him were always pleasurable. His home was always open to his friends.

He was a member of the American Association of Engineers, and the American Society of Military Engineers. He was a Royal Arch Mason, and Past Exalted Ruler of the Benevolent and Protective Order of Elks, and a member of the Presbyterian Church.

Colonel Lemen was elected an Associate Member of the American Society of Civil Engineers on July 10, 1907, and a Member, on March 14, 1916.

FREDERICK WILLIAM LOVELL, M. Am. Soc. C. E.[1]

DIED JULY 25, 1934

Frederick William Lovell was born at Bloomington, Ill., on January 26, 1872. His father, Thomas Lovell, whose birthplace was Swineshead, Bedfordshire, England, came to Canada as a young man and, later, migrated to Towanda in Central Illinois, where he met and married Mary Ellen Ricker. In 1883, the family moved to Iowa and, later, to California.

Frederick William Lovell was the second of five children born on the Towanda farm. He attended the primary schools at Towanda and Gibson, Ill., and at Humboldt, Iowa, and was graduated in 1891 from the Iowa City Academy. He studied Civil Engineering at the State University of Iowa, at Iowa City, Iowa, from which institution he was graduated in 1894, having completed the course in three years with high honors.

He spent the years from his graduation until 1898 as Draftsman with the firms of Purdy and Henderson, of Chicago, Ill.; Brown-Ketcham Iron Works, Indianapolis, Ind.; and the Elmira Bridge Company, Elmira, N. Y.

In 1898, Mr. Lovell began his connection with the McMyler Manufacturing Company, of Cleveland, Ohio, which, later, became the McMyler Interstate Company. For this Company he designed all structural work and had supervision of the details of such in the office. Typical of the work of the Company at this time was the building of twelve ore-conveying bridges, each 450 ft long, supported on two towers, for the Lorain Steel Company, and the construction of car-dumping machinery for the Wheeling and Lake Erie Railroad Company. From 1901 until 1922, Mr. Lovell served as Secretary, as Chief Engineer, and as Vice-President of the Company, with supervision of all engineering work, both structural and mechanical, covering a wide range of problems in the design of engines, machinery, steam plants, and electrical apparatus. Many patents in the United States and foreign countries covering improvements in coal and ore-handling machinery, such as car dumpers, locomotive cranes, bridge conveyors, buckets, etc., were issued to him. The outstanding developments in coal and ore-handling devices took place largely within his life time, and he made important contributions to them. The Company also designed and fabricated structural steel for bridges and buildings.

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[1] Memoir prepared by E. W. Crellin, M. Am. Soc. C. E.

Throughout his professional career, Mr. Lovell was a conscientious and tireless worker. His persistence and success in solving unfamiliar technical problems were remarkable. A notable instance of his resourcefulness in time of great stress occurred during the World War, when without previous experience his Company made 8-in. shells for the British and American Governments and became the largest producer of such shells in the United States.

From 1922 until his death, Mr. Lovell was engaged in consulting practice, making a specialty of the design of car-dumping machinery and other appliances for the rapid and economical handling of coal, ore, and similar materials. As already has been intimated, he was a Mechanical Engineer as well as a Civil Engineer, and had from his long experience an intimate knowledge of the mechanical as well as of the structural and economic problems involved in the design of such machinery. He spent much of his time in travel in the United States, Alaska, Mexico, and South America, and in Europe. His wide experience in his own field and his discernment in the study of commercial and political backgrounds gave his observations great value for his friends and business associates.

Much of his enthusiasm, after the World War and during the depression, was given to assisting young people to obtain their education and training. For many of them Mr. Lovell made possible a college education and an artistic career. A special fondness for young people made him a kindly figure in private life, as his experiences and travels made him a dominating personality as an engineer and executive.

During his entire residence in the City of Greater Cleveland, he was a devoted member of the First Baptist Church. His gifts thereto were numerous, and he gave both of his time and his work, unstinted, during the erection of the new church building in 1929.

His death from heat prostration while traveling in Iowa came very suddenly during the severe summer of 1934. Hs is survived by his widow, Eva (Glass) Lovell, to whom he was married in 1899, and by one son, Wheeler Glass Lovell, a member of the Engineering Profession. He is also survived by his brothers, Walter Danville Lovell, M. Am. Soc. C. E., Robert Lovell, and Judson Lovell, and a sister, Lucy Lovell.

Mr. Lovell was elected a Member of the American Society of Civil Engineers on June 6, 1906.

IRA WELCH McCONNELL, M. Am. Soc. C. E.[1]

DIED JANUARY 7, 1933

Ira Welch McConnell, the son of James Calvin and Cecilia (Welch) McConnell, was born on October 17, 1871, of a family of pioneers who settled in Shell City, Mo., soon after the Civil War. This country was then on the frontier. Mr. McConnell's mother, who is now (1935) in her ninety-seventh year and still enjoys an active life, has told and written of the many interesting experiences of the pioneer family of those days. His father, in addition to being a farmer, operated a general store in Butler, Mo. In his boyhood, Ira McConnell helped his father in the store and gained his preparatory school training at Butler Academy.

After several years in the United States Railway Mail Service as a Postal Clerk, Mr. McConnell had accumulated a fund which was to take him to college. He was then 22 years old. Shortly after he entered the College of Civil Engineering of Cornell University, at Ithaca, N. Y., in 1893, the bank in which his money was deposited failed. Having embarked on his college career he would not stop. From that point forward he earned his entire college expenses by doing odd jobs, and as the first General Manager of the Cornell Co-Operative Society, which was formed during his Junior year.

The traits of character which later made him eminent in his profession were soon recognized by all with whom he came in contact. In his quiet and effective way he became a leader, not only among the civil engineering students, but in his entire Class. Before his Senior year it was already a foregone conclusion that he would be selected as Chief Engineer of the Civil Engineering Survey, a recognition by his fellow students of his qualities as an organizer. His work in this connection was outstanding. Throughout his college course his scholastic standing was always good, and he was elected to membership in Tau Beta Pi and Sigma Xi during his last year in college.

A friend and classmate, Charles F. Hamilton, M. Am. Soc. C. E., has written of him:

> "As I review Mac's career at Cornell, there is most evident the splendid character and high qualities which in his after life brought him to the head of his chosen profession.
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> "As a friend and classmate he was without an equal—kind, considerate, thoughtful, self-effacing, thoroughly dependable under all circumstances, gifted with an unusual brain, perfect poise, and splendid sense of values, always seeking and enjoying the higher and better things of life. To have

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[1] Memoir prepared by George Schobinger, M. Am. Soc. C. E.

intimately known him and worked with him is one of the finest memories I treasure of my college days."

After his graduation from Cornell University in 1897 as a Civil Engineer, Mr. McConnell served as Instructor in that Department for two years. During these six years at Cornell he formed many of the friendships and associations which lasted throughout his life. These associations were extended to the under-classmen and students who felt his leadership and, later, became associated with him or had business relations with him in his engineering work. There is probably no Cornell graduate of the past forty years who was better known to Cornell men than Ira McConnell.

He left Cornell to become a Contractor's Superintendent. During the following three years, he served in that capacity in various parts of the United States. He then became Professor of Engineering at Missouri School of Mines, at Rolla, Mo. After one year in that institution, he embarked on his active business career, being then thirty-two years of age.

Mr. McConnell entered the United States Reclamation Service as Project Engineer in charge of the Gunnison Tunnel, one of the first large works of that Service, and, at the time, one of the longest and largest tunnels in the world. Within a few years he became a Supervising Engineer, having charge of the Central District including Colorado, Wyoming, South Dakota, Utah, and adjacent States. He remained in the U. S. Reclamation Service for a total of six years and then resigned in 1909 to become Chief Irrigation Engineer of the J. G. White and Company, which Company at that time was active in irrigation work. Louis C. Hill, M. Am. Soc. C. E., an associate during this period, writes as follows:

"I have known McConnell for about 25 years or more. He was an able organizer, and he had a sense of humor that carried him by, and everybody else with him, any kind of a disaster. He played the game on the dead square. His Uncompaghre tunnel was one of the most difficult tunnels to drive. * * * I think he was the ablest man we ever had in the Reclamation Service."

J. H. Quinton, M. Am. Soc. C. E., an associate in the U. S. Reclamation Service, says of him:

"I first met Mr. McConnell when he was 32 and I was 53 years old. He had been employed as assistant to the Engineer of the Uncompaghre Project of the U. S. Reclamation Service. He had just completed the field survey for a location of the Gunnison Tunnel, the main feature of that Project. I was then a Consulting Engineer in the Service, and * * * was allowed to select the engineer to be placed in charge of the construction. I selected McConnell, and my choice proved well founded.

"I early recognized his fitness for the position, and gave him a 'free rein'. He is entitled to all the credit for the engineering, and successful construction, of the Gunnison Tunnel—six miles long, and a most difficult piece of work. He was in charge of it from beginning to end, and it is a lasting monument to his ability in construction.

"* * * He had all the qualities that go to make a successful engineer: A good disposition, sound common sense, magnanimity, integrity, ambition to excel, and untiring energy in the discharge of his duty. He was a most lovable character, and his comparatively early death was a distinct loss to the engineering profession."

In 1910, Mr. McConnell became Vice-President and General Manager of the Idaho Irrigation Company, a private project. Shortly afterward he became Hydraulic Engineer for Stone and Webster Engineering Corporation. He remained with that Company for about five years, becoming its Chief Engineer in 1917. Mr. George O. Muhlfeld, an associate during this time, speaks of him as follows:

"Mr. McConnell was highly skilled and broadly experienced on engineering and construction work, and in addition to an excellent business ability he possessed and utilized an unusual amount of common sense. He was painstaking and untiring in his work and he had the faculty of not only remaining unruffled even under the most exacting and turbulent conditions but of smoothing out difficult and strained situations. He was a man of sterling character and excellent personality, was eminently fair in all his dealings and loyal to his associates. Mr. McConnell was liked by all who knew him, and the news of his sudden death in Buenos Aires brought a keen sense of loss to the large group of individuals who were proud to be known as his friends."

In 1918, Mr. McConnell became General Manager for the American International Shipbuilding Corporation at the Hog Island Shipyard, Philadelphia, Pa. Few people can appreciate his accomplishment in that position. Industry in the United States had become almost chaotic as a result of the sudden expansion from European war orders and the disjointed control in the early days after the United States entered the World War. Hog Island was no better and no worse than others, but through his efforts it became a smoothly working production machine. If the war had continued, this organization would have been the largest production plant in the world, but the Armistice brought the work to a sudden stop.

Late in 1918, Mr. McConnell joined in the formation of Dwight P. Robinson & Company, Incorporated, as Senior Vice-President and Chief Engineer with headquarters in New York, N. Y. This association continued for about ten years, when the Company was merged with others to form United Engineers and Constructors, Incorporated. He continued as Senior Vice-President with this Company until 1932.

His activities as Executive Vice-President of Dwight P. Robinson & Company, Incorporated, cover the entire program of that Company for the 10-year period from 1919 to 1928. A careful and thorough organizer and leader of men, a diplomat *par excellence,* a highly gifted engineer, and, at the same time, an extremely practical and hard-headed construction man, he was also thoroughly versed in finance and in the negotiations which precede and make possible the development of large projects. A catalog of his activities during these ten years is beyond the scope of this memoir, but the cost of the work, for the conduct of which he was responsible during this period, amounted to more than $150 000 000.

From 1921 to 1924, Mr. McConnell handled the work done by Dwight P. Robinson & Company, Incorporated, for the Brazilian Government. A leading newspaper of Fortaleza, the capital of Ceara, Brazil, speaks of him as follows:

"From the time when Mr. McConnell first became active among us in 1922 in the preparatory work for construction of the great irrigation systems in the southern part of this State, Mr. McConnell became known as a great friend of Brazil, dedicating himself with great success to the study of our most important problem and becoming thoroughly acquainted with our economic situation. We consider his death a great and sad loss for Brazil, since in our opinion his co-operation would have been for us most useful and necessary in the ultimate solution of our most important problem. Few Americans could attain the high level which Mr. McConnell held with relation to the interests which join Brazil to North America. We convey through these lines our sympathy and condolences to Dwight P. Robinson & Company."

During this period he was also responsible for the construction of the United States Embassy in Rio de Janeiro. The Hon. Butler Wright, later Minister of the United States to Uruguay, was Commissioner of the United States on this work. Mr. Wright says of him:

"I welcome with appreciative gratitude the opportunity which you afford me to pay tribute to my good friend the late Ira W. McConnell, with whom it was my good fortune to be associated at the time of the construction of the permanent and temporary buildings erected by the Government of the United States in Rio de Janeiro at the time of the Brazilian Centennial Exposition—the former of which is now the American Embassy in that capital. Mr. McConnell was at that time engaged in superintending, on behalf of the firm of which he was so distinguished a member, the erection and completion of these buildings—which proved to be a particularly difficult and delicate task due to the fact that buildings intended for exposition purposes were ultimately to be transformed into the Embassy residence and office and, further, to countless irritating complications arising from changes in plan, adaptation to changing circumstances, and other annoyances which can only be fully appreciated by those who have had to contend with similar circumstances. In my capacity of a Commissioner of the United States, charged with oversight of the entire undertaking, individual and official co-operation and intimate personal friendship were the essential considerations for success.

"In the enjoyment and fruition of these relations I owe much to Mr. McConnell, for during our association from August, 1922, to March, 1923—or thereabouts—I found his wide experience, his integrity and honesty of thought and purpose, his cheery philosophy and his delightful sense of humor contributed in a most unusual degree toward the final success of an undertaking of which all those who participated therein may well be proud. In short, the association was an unusual and pleasurable experience for me, and from it grew a friendship in which I took great pride and pleasure."

During the four years (1928–1932), in which he was a resident in Buenos Aires, Argentine Republic, Mr. McConnell attained a unique position in that community. There was probably no other American who was so widely known and thoroughly admired on the entire East Coast of South America. During this period perhaps the major accomplishment was the development of the second and largest subway system in that city constructed for the Central Railroad and Terminal Company. This work seemed to be a natural outgrowth due to Mr. McConnell's early interest in foreign developments, in the course of which a number of buildings, as well as industrial and utility plants, were constructed in South America by Dwight P. Robinson & Company, Incorporated, under his direct supervision. His admirable organizing ability was never better shown than in his handling of foreign construction. This activity, however, kept him away for long periods from his associations in the United States.

When the subway was completed, Mr. McConnell was appointed Director and General Manager of the Central Railroad and Terminal Company of Buenos Aires. He occupied this position at the time of his death.

His associates in the Buenos Aires Central Railroad and Terminal Company prepared and engrossed on parchment, and sent to Mrs. McConnell, the following resolution expressing their admiration for him and their grief at his loss:

"In the City of Buenos Aires, on the 28th day of the month of January, 1933, the Board of Directors of the Company met under the Chairmanship of the President, Dr. Teofilo Lacroze. There being present Directors Louis A. Rocca, C. P. Billings, Graham Steel, Emilio A. Godoy, and F. O. Cartwright. The meeting was called to order at 17 o'clock.

"The President stated that it was the unanimous decision of the directors that this meeting be held for the exclusive purpose of rendering a tribute of gratitude, respect and affectionate remembrance to the late Mr. Ira W. McConnell, former director-general manager of the Company.

"The President stated that I. W. McConnell, a man of great capability and kindness, leaves a vacancy in the company difficult to fill, and in the hearts of his friends memories which will endure throughout their lives as pleasant remembrances of a great man and friend, who knew how to win the affection of all who had the privilege of association with him.

"These sincere sentiments caused the President to propose that the Board honor the memory of their late associate, Mr. Ira W. McConnell, by rising and rendering one minute of silent homage, have engraved on parchment a copy of the minutes of this special meeting, to be signed by all the Directors and sent to his wife and sons."

Charles W. Comstock, M. Am. Soc. C. E., a lifelong friend and associate, writes of Mr. McConnell as follows:

"The most outstanding trait of Mr. McConnell's character, the one that more than anything else made him universally loved and respected, was his innate and invariable sense of fairness. He was quick to grasp and to appreciate other views than his own and to give them full and sympathetic consideration. He did not consider it his duty nor was it ever his purpose to out-trade another in any negotiation. He aimed only to reach a conclusion which should be fair to all concerned.

"This attitude was not paraded for effect; it was instinctive—an integral part of his nature. It was evident at all times whether the matters under consideration were relatively trifling or of great moment.

"If one looks beyond Mr. McConnell's profound and extensive knowledge of technical engineering for some further explanation of his professional success, perhaps the most important element was his skill in building an organization. He did not expect to find men without faults. He was as quick to recognize their weak points as their strong ones, and he was a master hand at so placing them as to minimize the effects of their weaknesses while allowing full play to the exercise of their special abilities.

"He never emphasized his own part in any work. Always he aimed at success for the organization and if some mischance later gave credit for that success to an assistant he was never resentful."

H. G. Balcom, M. Am. Soc. C. E., a lifelong friend, says of him:

"I first met him in the fall of 1894 when we were sophomores in the College of Civil Engineering at Cornell University. He was then very jolly and care free and readily drew people to him and made them like him, in some cases almost to the point of idolizing him. Underneath his general exterior we all knew and admired his sound judgment. In school work, it always seemed to me that he was more interested in learning essentials than he was in high marks. It is small wonder that the Senior Class elected him Chief Engineer and that the faculty recommended him for Sigma Xi. Even in college he had a vivid and forceful way of expressing his ideas.

"After graduation I saw very little of him for a number of years. When he came east, * * * he called upon me. He was the same jolly fellow I knew in college, but I also saw he had developed into a hard-headed business man. He told me what his title was to be in his new position. I asked him how much salary that counted for. In his characteristic way he said 'Not a damn cent'.

"We were thrown most closely together at Hog Island during the War, and for several months he was my direct superior. In his painstaking and judicial way, he studied department after department and perfected a smooth-working organization. At last he was put in charge of 'shop design and production' which covered all engineering features of the ships and also the purchasing, inspecting and bringing the ship material to the yard. Here he perfected an organization that was a model of excellence, in spite of the fact that he was constantly subjected to the criticism of the Shipping Board. No man could have accomplished this who did not have a much higher regard for work well done than he did for holding his position. His whole career at Hog Island impressed me as that of an almost super-man. * * * The least and perhaps the most I can say is: 'He was a brilliant and an honest man, a genial companion and a loyal, helpful friend.'"

Perhaps no one was more intimate with Mr. McConnell in personal and engineering matters than John C. Hoyt, M. Am. Soc. C. E., who thus epitomizes his estimate of his lifelong friend:

"If I were to sum up the secret of Ira McConnell's success I would say that it was due to unusual native ability supplemented by a broad education obtained both through study in schools and by experience. With this he had the fullest wholeheartedness with which his frank and unassuming ways attracted people to him."

No more fitting summary of Mr. McConnell's philosophy toward life and his fellow man could be given than a few words which were included in his will as advice to his sons, John and Charles: "Honors to the unrighteous are empty. For him who pursues his duty in honor and fidelity there can be no failure."

On January 9, 1933, the *New York Herald-Tribune* printed on its editorial page an appreciation which in a few words gives a full and rounded appraisal of his qualities as an engineer and as a man:

"The death of Ira W. McConnell in Buenos Aires removes a figure known to few of his home-keeping fellow countrymen beyond the engineering fraternity. Though he built the long Gunnison Tunnel in Colorado and the large Pathfinder Reservoir in Wyoming, he had the engineer's dislike of publicity, and the sight of great waters diverted or impounded for the service of men was all he asked of fame. Like the Roman engineer who left his name on a single stone in the bridge of Alcantara, he was satisfied to leave the glory of his achievement to those for whom he worked.

"In South America, the field of his later activities, few Americans have been so well known or so honored. Not only were his consummate technical abilities recognized all along the east coast of the continent, but a rare personality that made him more than a highly skilled engineer won him the strong confidence of governments and laborers alike. For he was diplomat as well as engineer, and had an extraordinary facility for working with strange peoples without offending their susceptibilities. When he set up his camps in the drought-ridden lands of northeastern Brazil, which he was to turn into another Egypt by a series of cyclopean irrigation works, and sufficient workers failed to appear, he summoned the famous Padre Cicero, whose influence alone could move the minds of the people throughout the dry country of the sertão. It was through the friendship and understanding which engineer and priest instinctively felt for each other that he was able to marshal the army of workers necessary for the vast enterprise. That it finally failed of fruition and left the desert country strewn with uncompleted dams was no reflection on the masterly hand of McConnell or the loyalty of his devoted associates, but was due to the exhaustion of the inadequate funds * * *.

"Ten years ago, as a diversion from his larger undertakings he built the American Embassy building at Rio de Janeiro. The crowning achievement of his career was the completion last year of the Lacroze Subway in Buenos Aires, and when he died, a few days ago, it was in the harness as supervising director of the road which he had dug beneath the Argentine metropolis. The engineer, with the friendly twinkle in his shrewd eyes and the laugh that could disarm the suspicious reserve of high officials or the timid awe of a simple laborer, was in his way a more effective envoy of the best in his country than the generality of ambassadors hedged about by the formalities of protocol."

Mr. Dwight P. Robinson, who knew him intimately from 1908 until his untimely death, pays him this tribute:

"No words can adequately picture Ira Welch McConnell. One had to know him to realize his splendid worth. While even casual contact with him left a favorable impression, one had to be associated with him in the discharge of common responsibilities to attain a true measure of his fineness, of the crystal clearness and cleanness of his soul. He had in overflowing measure everything that his loving friends have attributed to him—courage, ability, winsomeness, integrity, selflessness—but away beyond those he had that something so rare that made him a man in God's likeness. To have been his partner and to have won his trust is an inspiration for all time."

Mr. McConnell is survived by his widow, Grace (Bowerman) McConnell, and his two sons, John and Charles.

Although he did not claim membership in any church, his religion was Christian in its highest sense and recognized the brotherhood of man; it was a religion broader in its concept and more far-reaching in its effect than that which is bound by creed and dogma.

He was a member of the following associations and clubs: Cornell Society of Engineers, New York; American Association of Engineers; Society of Marine Engineers and Naval Architects; Sigma Xi, Tau Beta Pi, Kappa Sigma; Cornell University Club, Engineers Club of New York, Uptown Club, and the American Club of Buenos Aires. He was also a member of the Masonic Fraternity and of the Benevolent and Protective Order of Elks.

Mr. McConnell was elected an Associate Member of the American Society of Civil Engineers on December 7, 1904, and a Member on September 3, 1913. He also served as a Director of the Society from 1921 to 1923.

LUIS MATAMOROS, M. Am. Soc. C. E.[1]

Died January 18, 1934

Luis Matamoros was born in the little Town of Atenas, in the west central part of Costa Rica, Central America, on June 20, 1859. The oldest child of six and the only son of Juan Matamoros, Lobo, and his wife, Josefa Sandoval Rodriguez, he attended the graded school in Atenas and then went to San José, the Capital, for his High School work. Upon his graduation, he entered the Technical Institute at San José and so great were his talents in the profession he had decided to follow, that during his last two years as a student at the Institute, he was made Assistant Instructor in Mathematics and Physics. He was graduated with the degree of Bachelor of Arts and as a Land Surveyor.

Young Matamoros sought higher engineering knowledge and selected the Ecole d'Ingénieurs de Université de Lausanne, in Switzerland, as his Alma Mater, from which he was graduated with the degree of Civil Engineer on December 12, 1882. He returned to Costa Rica in 1883.

From 1883 to 1886, Mr. Matamoros was General Manager of the Pacific Railroad of Costa Rica, and in the latter year he was asked to act as Chief Engineer for Costa Rica on the Boundary Conference between Costa Rica and Nicaragua, on which work he remained until 1889. In this year he was made Director General of Public Works of Costa Rica. In this same year, he was sent to Washington, D. C., to advise the Costa Rican Legation on the question of the Nicaraguan Canal, which was then up for important discussion.

In 1893, while still Director of Public Works, Mr. Matamoros had one of the thrills that come once in a life time and which he recited to the writer as late as 1933. The Intercontinental Railroad Commission, of which the late Alexander J. Cassatt was President, had been created and field parties, aided by the United States Government and in charge of that incomparable locating engineer, the late William F. Shunk, were headed for South America from the United States, and, of course, they must pass throughout the length of Costa Rica from Nicaragua to Colombia (later the Republic of Panamá).

Mr. Matamoros lost no time in contacting Mr. Shunk and was as quickly rewarded by being placed in charge of a section of the work in Costa Rica. A genius in picking men and getting the best out of them with a few kind words, Mr. Shunk conferred on Mr. Matamoros this honor, which the latter loved to recall in his retirement and which unquestionably went with him, in great satisfaction, to the very end.

In an official report on the Intercontinental Survey which was printed in permanent book form by the United States Government, much credit is given to Luis Matamoros for the successful completion of the Costa Rican Section of this important undertaking.

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[1] Memoir prepared by John K. Flick, M. Am. Soc. C. E.

Mr. Matamoros' tenure as Director General of Public Works continued until 1902, when he was elected to the National Congress of Costa Rica, which office he held for a four-year term.

Returning to active duty in his profession, he was Chief Engineer in charge of the construction of the Pacific Railroad, as well as Municipal Engineer for the Capital City, San José, from 1906 to 1910. Although he had served his country in the National Congress for a time he never neglected his private practice of civil engineering, which, during the more active years of his life, was of considerable scope.

During the ensuing four years, 1910 to 1914, the Boundary Commissioners again sought Mr. Matamoros' advice and engineering skill, this time in the parley between Costa Rica and the new Republic of Panamá. As Chief Engineer for his country he was called to the Legation in Washington on important missions, both pertaining to engineering and diplomacy in the interest of Costa Rica.

Having been compelled to spend much of his life away from his home and his family, Mr. Matamoros now decided to settle down in San José and devote more of his time to his growing private practice. Therefore, until 1921, he accepted no active engagement which would take him away from his loved ones for any length of time.

Again heeding the call of his country, Mr. Matamoros was appointed and accepted the post of Chief Engineer and Member for Costa Rica, of the International Commission for the location of the boundary line between Costa Rica and Panamá, in accordance with the award of the late Chief Justice White, of the United States Supreme Court. This assignment lasted nearly two years.

With advancing years, Mr. Matamoros wisely declined to consider any more offers calculated to overtax his faculties and retired to his home and study. His engineering library was the best in Costa Rica, and nothing pleased him more than to spend his days reading and chatting with a few callers. He loved to tell of the visit to San José of the late Leonard Metcalf, M. Am. Soc. C. E., for the purpose of investigating and reporting on the water supply of the Capital.

Mr. Matamoros was the author of the following pamphlets: "Memoir on the Figure of the Celestial Bodies" (First Prize at the Academy of Lausanne); "Theory of the Formation of the Mountains of the Globe"; and"Theory of Earthquakes"; as well as several pamphlets on Mathematics and Physics. He was a member of the Associacion Amicale des Ingenieurs, the Boston Society of Civil Engineers, the New England Water Works Association, and Past-President, Facultad de Ingenieros en Costa Rica.

On May 8, 1887, Luis Matamoros was married to Angelina Loria Yglesias, in San José, who bore him six children, three boys and three girls, of whom a boy, Luis, died in infancy. There are sixteen grandchildren. One son, Juan Matamoros, is a graduate in Civil Engineering from Lehigh University, at Bethlehem, Pa.

Mr. Matamoros was elected a Member of the American Society of Civil Engineers on October 4, 1905.

MARSHALL MORRIS, M. Am. Soc. C. E.[1]

DIED MARCH 24, 1935

Marshall Morris was born at Jefferson Mines, Ala., on September 16, 1877. His father was the late Marshall Morris, Sr., M. Am. Soc. C. E., and his mother was Tinie (Witherspoon) Morris, of Longview, Tex.

Most of Marshall Morris' early life was spent in Louisville, Ky. After graduating from the Manual Training High School, at Louisville, he attended Rose Polytechnic Institute, at Terre Haute, Ind., and Vanderbilt University, at Nashville, Tenn., Class of 1899. He was a member of Chi Chapter of Kappa Alpha.

Mr. Morris started his business career as a Rodman with the Corps of Engineers, United States Army. Later, he was with the Breckenridge Asphalt Company, and the Louisville Bridge and Iron Company. He also served as Assistant Engineer for the Louisville and Nashville Railroad Company, at Louisville, and in the same capacity on the Memphis, and on the Cincinnati, Divisions of the System. These activities extended from 1897 to 1906. From 1906 to 1909, Mr. Morris was employed as Efficiency Engineer for the J. B. Speed Company Coal Mines in Kentucky.

In 1909, he moved to Texas, turning his attention to the building construction field, and in this endeavor (in addition to his private engineering practice), served as General Superintendent of various large building projects. Some of the more prominent buildings on which he was engaged are: The Mills Building, at El Paso, Tex.; The Blind Institute Buildings and Confederate Womens Building, at Austin, Tex.; Hodges-Neal Building, and Gholson Hotel, at Ranger, Tex.; and the San Antonio Arsenal (1918 addition), the Blue Bonnet Hotel, the Incarnate Word College Buildings (1928 additions), and the San Antonio Municipal Library, at San Antonio, Tex.

Mr. Morris specialized in reinforced concrete design and construction and was known throughout the State as an authority on the subject. He studied constantly and kept abreast of the advancement of concrete design from its earlier stages. He was particularly appreciated by the Building Industry for his knowledge, thoroughness, and painstaking execution of details. His handling of men was as noteworthy as his accomplishments in construction. While erecting a building he was also modeling and shaping men of character and giving them a deeper, keener insight into their chosen profession. He was never too busy to give a helping hand.

At the time of his death, on March 24, 1935, he was serving as Resident Engineer Inspector for the Public Works Administration on projects at Dallas, Tex.

Although quiet and unassuming, Mr. Morris always made and retained a host of friends wherever he went, who recognized his honest, straightforward, clear-thinking, friendly manner.

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[1] Memoir prepared by Messrs. Preston P. Brooks, and J. F. Johnson, Austin, Tex.

On May 21, 1919, he was married to Margaret Groos, of Kyle, Tex. Besides his widow, he leaves a sister, Mrs. George A. Haynes, of Indianapolis, Ind., and a brother, John F. Morris, of New York, N. Y.

Mr. Morris was elected a Member of the American Society of Civil Engineers on February 1, 1910.

ROBERT BROOKS MORSE, M. Am. Soc. C. E.[1]

Died January 31, 1936

Robert Brooks Morse, Chief Engineer of the Washington Suburban Sanitary District, Hyattsville, Md., died on January 31, 1936, of septicemia, in a hospital at Washington, D. C. His death marked the close of a career exemplifying the finest qualities associated with the Engineering Profession. Diffident in approach, generally silent of speech, and imbued with the highest sense of public service, Mr. Morse carried on a long professional career productive of more permanent usefulness to the State of Maryland than perhaps any single individual resident therein. As is always the case with persons of great modesty, the State of Maryland perhaps would be the last to recognize the validity of this statement, when more outspoken, more popular, and less effective individuals absorb the public spotlight.

The contributions of an engineer in public life such as Mr. Morse, devoting an entire lifetime to the development of public administration for the welfare of the citizens of the State, are generally buried in the archives of technical societies and of legislative halls. Fortunately, history discloses these contributions, and the permanent imprint of men of this type assumes the significance which those familiar with Mr. Morse's career believe it deserves.

Robert Brooks Morse was born on September 13, 1880, at Montpelier, Vt., and was the eldest son of the late Professor Harmon N. Morse, of Johns Hopkins University, at Baltimore, Md., and Caroline Augusta (Brooks) Morse. He was educated at the Baltimore City College and was graduated from Johns Hopkins University, in 1901, with the degree of Bachelor of Arts. He was a student at the University of Maine, Orono, Me., in 1902, and, in 1904, he received the degree of Bachelor of Science from the Massachusetts Institute of Technology.

He began his engineering experience as a Draftsman in the Bureau of Construction and Repair, United States Navy Department, Washington, D. C., in 1904 and 1905. Later, he became Draftsman, Assistant Engineer, and Assistant Division Engineer, of the Sewerage Commission of the City of Baltimore from 1905 to 1910, and then Assistant Sanitary Engineer of the Metropolitan Sewerage Commission of New York, N. Y., from 1910 to 1912, on plans and studies for eliminating the pollution of New York Harbor.

In 1912, when Mr. Morse was appointed Chief Engineer of the Maryland State Department of Health, he started an organization responsible for the control of the water supply and sewerage systems in Maryland. That Division became one of the pioneers in the sanitary engineering field in the United States. He held this office from 1912 to 1922, and during that period wrote the Maryland Water and Sewerage Law which has stood the test of time for twenty-two years and has served as a model for legal enactments in a number of other States. It remains on the legislative books to-day as one of the important factors in sanitary administration in the State of Maryland.

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[1] Memoir prepared by Abel Wolman, M. Am. Soc. C. E.

During this period, he collaborated in the preparation of the Washington Suburban Sanitary District Act, in 1918. This act, too, was drawn with little or no precedent for guidance in creating the pioneer sanitary district in Maryland. It, too, has been used as a model in the formation of similar districts, both within and without the State. In the direction of the Washington Suburban Sanitary District, Mr. Morse showed the highest degree of originality and skill in the many intricate engineering details involved in the administration of that agency.

Always on the alert to depart from orthodox principles and methods, where consistent with sane and logical engineering practice, he conceived and patented the design of a concentric steel filter plant assembly, recently constructed at Burnt Mills, in the Washington Suburban Sanitary District. The plant has attracted international attention as one of the most novel and ingenious water filtration plant structures in the world. It is mentioned here solely to indicate the breadth of vision, the originality of thinking, and the ever-present desire to avoid being hampered by precedent.

Mr. Morse was also the co-holder of patents for efficient methods of pumping sewage and for a special cast-iron cut-in fitting for use in making sewer line connections.

He was a conferee of the New Annex Advisory Commission, Baltimore (1919); a member of the Annapolis, Md., Sewerage Commission (1920); of the Maryland Park and Planning Commission (1927); of the Regional Water Supply, and Drainage and Sewerage Committees of the National Capital Park and Planning Commission, Washington, D. C. (1929–1930); of the Water Resources Commission of Maryland (1931–1933); and a member of the Sub-Committee on the Maryland-Washington Region of the State Planning Commission of Maryland. Mr. Morse had been in private engineering practice since 1910, serving as Consulting Engineer to the Baltimore Sewer Department, on sewer system valuation; to Annapolis and Laurel, Md., on improvements to the water systems; and to a number of other communities and private enterprises in matters of water supply, sewage disposal, and land development. He testified a number of times as an expert witness in matters pertaining to sewerage, sanitation, water supply, and power cases.

As a member of the American Water Works Association, Mr. Morse was Chairman of the Publication Committee and a member of the Committee on Water Service Pipes. He represented the American Society of Civil Engineers on the Committee for Standardization of Manhole Frames and Covers; was a member of the Committee on Clay and Cement Sewer Pipe of the American Society for Testing Materials; and a member of the Committee on Municipal Contract Forms of the American Society of Municipal Engineers.

He was the author of the section on "Metropolitan Water Districts", and co-author of the section on "Relation Between Filtered Water Storage and Filter Capacity", of the Manual of American Water Works Practice, as well as the author of a number of papers on water and sewerage matters in technical journals. He was a member of the American Water Works Association, the New England Water Works Association, American Society of Municipal Engineers, National Conference on City Planning, American Association of Engineers, Engineers Club of Baltimore, Phi Beta Kappa Fraternity, Manor Club, Beaver Dam Club, and a Fellow of the American Public Health Association.

Indefatigable in work, strong in character, keen in intellect, endowed with a sense of personal responsibility for public service, Mr. Morse leaves the State permanently indebted to his activities. Ever helpful to those associated with him, in a friendly and unassuming manner, he leaves behind him a legion of friends in all walks of life.

He was married in 1902 to Caroline E. Ross, of Nashua, N. H., and is survived by his widow, a daughter, Mrs. Julian Ashton Devereux, of Ruxton, near Baltimore, a grand-daughter, Caroline Vail Devereux, and a sister, Dr. Mary Elizabeth Morse, of Baltimore.

Mr. Morse was elected an Associate Member of the American Society of Civil Engineers on June 30, 1910, and a Member on September 2, 1914.

WILLIAM MULHOLLAND, M. Am. Soc. C. E.[1]

Died July 22, 1935

The morning of July 22, 1935, marked the closing of the glorious career of one of the truly great members of the Engineering Profession.

William Mulholland was born in Belfast, County Antrim, Ireland, on September 11, 1855, the son of Hugh and Ellen (Deakers) Mulholland. His early life was spent in Dublin, where his father was connected with the Government mail service. He began his education in the National Schools and, later, attended the Christian Brothers College in that city. At the age of fifteen, after preparing himself by a special course of study of mathematics and navigation at the hands of a private tutor, Mr. Mulholland went to sea. After four years before the mast, he became dissatisfied with a sea-faring life and determined to come to the United States to take up a different career in a new country, having visited some of its Atlantic Coastal cities during his nautical years.

Landing in New York City, on June 9, 1874, Mr. Mulholland went directly to the Great Lakes where he resumed his career as a sailor during the summer and worked in the lumber camps near Manistee, Mich., in the winter. The following year, 1875, he went to Pittsburgh, Pa., where he made his home with his uncle, and was employed in the latter's dry goods store.

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[1] Memoir prepared by H. A. Van Norman, M. Am. Soc. C. E.

During his residence in Pittsburgh, he read a book, Nordhoff's "California", which so aroused his youthful interest that, in December, 1876, shortly after his twenty-first birthday, he and his brother set sail from New York City, for the land of his dreams. Arriving at Colon, Panama, they left the ship and, to save the $25 railroad fare, walked the forty-seven miles across the Isthmus, following the railroad to the City of Panama. There the brothers joined the crew of a ship that sailed to Acapulco, Guerrero, Mexico. At that port another ship was taken which bore them to their destination, San Francisco, Calif., where they landed in 1877. After a few days in San Francisco, they started for the San Joaquin Valley the land of wonder and beauty that had been so vividly portrayed to Mr. Mulholland by the book he had read in Pittsburgh. At Martinez, they obtained horses and rode down through the Valley, passing through Bakersfield, and finally reaching Los Angeles.

This little Spanish city (at that time of less than 9 000 inhabitants) surrounded by vineyards and citrus groves through which meandered a tiny willow-banked river (which later came to play such an important part in his life) so impressed him that he determined to make this place his home; and near the little Town of Compton, on the outskirts of Los Angeles, he began his long career of water production and distribution by digging Artesian water wells with a hand-drill, subsequently installing the first water system for the town which is now the City of Long Beach. The following winter months were spent prospecting near Erhenberg, Ariz., and there Mr. Mulholland first saw the Colorado River which, forty-five years later, he planned to utilize as a new source of water supply for the City of Los Angeles.

Returning to Los Angeles in the spring of 1878, he went to work for the Los Angeles City Water Company, one of the companies which at that time had a franchise to supply the city with water. The Los Angeles River was the source of supply, and the distribution system consisted of one small reservoir with a capacity of 3 000 000 gal, together with a series of open ditches, which in Spanish were termed "zanjas." Here, really began that which later developed into one of the most unique and successful lives of professional and public service that a man could desire. Starting as a humble "zanjero", or ditch-tender, whose duty was to keep the ditches clear of brush and foreign material, his talents and qualities of leadership were soon recognized, culminating in his appointment as Superintendent in 1886. In 1898, the franchise under which this Company operated expired, and Mr. Mulholland continued to serve as Superintendent through the following years of litigation that preceded the ultimate purchase of the existing works by the City of Los Angeles in 1902. After the transfer to the new ownership, Mr. Mulholland continued to serve in his former capacity.

During his fifty years of meritorious service to the people, he achieved a distinction unprecedented in the history of water development and waterworks construction. Keeping pace with his city of adoption, realizing its potentialities, anticipating its needs, he by his foresight and dynamic leadership, brought about in his lifetime, by the construction of one of the largest and most unique water systems in the world, the unrestricted development of an overgrown village to the fifth largest metropolis in the United States.

From an original distribution system containing a few thousand feet of open ditches, supplied by a single reservoir with a capacity of less than 10 acre-ft, to a system of more than 3 800 miles of pipe and 65 reservoirs and tanks capable of storing more than 53 000 acre-ft of water (the most of which was conceived and completed during his last thirty years of service) was, in itself, an achievement which, no doubt, will never be duplicated.

In addition, in 1908, Mr. Mulholland began the construction of that remarkable project that was to bring to him world-wide recognition and appreciation by the commercial world as well as by his professional associates. Completed in 1913, in less than the estimated time of construction and below its estimated cost, the Los Angeles Aqueduct, the largest structure of its kind in the world. established a successful precedent for long-distance water transportation for the profession for generations to come. This system in addition to its 240 miles of conduit and siphon, contains 68 000 acre-ft of primary and control reservoir storage. The difficulties to be overcome and the daring and unusual features incorporated in the construction of this monumental piece of work have long been familiar to the profession and, consequently, they will not be discussed herein. However, mention must be made, in connection with this work, of that great quality of leadership and ability possessed by Mr. Mulholland—which, unfortunately, is so infrequently possessed by other great engineers—a leadership that enabled him to appear before and verbally present his idea to a more or less disinterested laity; arouse and convince it that the sum of $24 000 000 which he was asking for the construction of this project, and which in those days was considered an unheard of amount of money, was really based on sound, far-sighted judgment, and was not merely an engineer's dream; and that the successful completion of this project was feasible and vitally necessary for the future growth of the city.

Mr. Mulholland also gained great distinction by his successful construction of large earth dams for the creation of required reservoir storage, establishing a precedent thereby for the profession in the matter of high earth dams. In all, he personally conceived and supervised the construction of twenty-seven dams of this type.

In 1923, realizing that in a few years the potential water supply that could be obtained from the Sierra Nevada by means of the Los Angeles Aqueduct would prove inadequate to provide for the needs of the rapidly growing city, he turned for a new source of supply to the Colorado River, which he had first seen in the year of his arrival in Los Angeles. Under his direction, 60 000 sq miles of territory theretofore unmapped were surveyed by plane-table and studied for the location of possible routes for an aqueduct having a capacity of 1 500 sec-ft. The result of this work has been expressed in the construction of an aqueduct by The Metropolitan Water District, to which all maps, surveys, studies, and data were given upon the formation of that organization, an association of thirteen Southern California cities.

Mr. Mulholland acted as Consulting Engineer to the Cities of Sacramento, San Francisco, Oakland, and the East Bay District, in California, and for Seattle, Wash., during the development of their respective water supplies. He also acted in a similar capacity on numerous smaller projects throughout the Southwest, as well as Consultant for the City of Los Angeles with respect to the construction of its Outfall Sewer. He served as a member of the Engineering Advisory Board on Water Resources and Development of the State of California.

Of the personal side of Mr. Mulholland, enough good cannot be said. Self-schooled for the most part, this remarkable individual early in life learned that Mother Nature was the one Great Engineer whose precedents could be followed safely. This, coupled with his keenly observant nature and remarkably retentive memory, contributed largely to the success of his career. In the process of digging his first well in Southern California he encountered a tree and various fossils more than 600 ft below the surface. This inspired a study of geology which gave him authoritative recognition in that branch of science later in life. The desire to protect and beautify the reservoirs and dams he had built, led to an intensive study of botany that left him in possession of authentic knowledge of the flora indigenous to California as well as to those of foreign countries of similar climatic and soil characteristics.

A seldom disclosed but nevertheless ever-present deeply poetic nature, the birthright of his Celtic forebears, made Mr. Mulholland a diligent and appreciative student of the classics. The works of Pope, Carlyle, and Shakespeare gave him his great understanding of mankind and human nature. "Hamlet," his favorite play, provided him with the following precept on which he patterned his life and from which there was no deviation:

> "This above all: to thine own self be true,
> And it must follow, as the night the day,
> Thou canst not then be false to any man."

These habits of self-application and a highly developed sense of observation, coupled with the most remarkably retentive memory with which the writer has ever been in contact, no doubt contributed largely to his tremendous fund of natural and technical knowledge.

It was the writer's great privilege to have enjoyed the professional and social association of Mr. Mulholland for more than twenty-five years, during the period of his greatest activity. It was his remarkable will and personality that made his tremendous accomplishments possible. His absolute fairness, his honesty, and his sincerity were outstanding. He enjoyed a loyalty from those who worked with him and under him that was a tribute to the man he was.

Despite his extraordinary capabilities and his tremendous accomplishments, he was a man of extreme modesty. The writer vividly recalls the day of the dedication of the Los Angeles Aqueduct. After years of appearing before the public and speaking in behalf of the necessity and feasibility of this project; after bearing the burden of heckling and skeptical opposition during the formative years of the planning of this work when considerable doubt was cast that water would actually be brought to the city as he said it would, as well as during the period of construction when the tremendous responsibilities of overcoming the seemingly insurmountable difficulties encountered engendered a load that no one man should have shouldered—yet his dedicatory address delivered when the gates were opened and the first water flowed from the Owens Valley down the Cascades into the San Fernando Reservoir was, "There it is. Take it."

In later years an incident that occurred shortly before he was stricken was very typical of Mr. Mulholland. The writer and he were standing on the San Fernando Dam overlooking the San Fernando Valley that had been transformed from a desert waste into the garden spot that it is to-day as a result of the water he had brought there by means of the Los Angeles Aqueduct. Silently he gazed out over the verdant expanse for a few minutes, and then he said, "The fact that the Valley has developed as it has is compensation enough for all of us." That terse statement was characteristic of him. If the job was well done, that was all he asked.

On July 3, 1890, Mr. Mulholland was united in marriage to Lillie Ferguson, to which union five children were born. He is survived by his children: Rose Mulholland, Mrs. Lucille Mack, Mrs. Ruth Wood, and Thomas and Perry Mulholland.

In December, 1928, he retired from active service, but still continued to serve the City in an advisory capacity until the time he was stricken.

As a tribute to his work, the University of California in 1914 conferred upon him the honorary degree of Doctor of Laws. Mr. Mulholland was a Charter Member, and acted as third President, of the Los Angeles Section of the Society; a member of the Pacific Association of Consulting Engineers; an Honorary Member of the American Water Works Association; a Charter Member of the Engineers and Architects of Southern California; a member of the Seismological Society of America; and an Honorary Member of the National Association of Power Engineers. He was also an Honorary Member of the Tau Beta Pi Fraternity, and a member of the California, Sunset, and Celtic Clubs.

At its meeting on July 29, 1935, the Board of Directors of the Los Angeles Section of the Society unanimously passed the following resolution:

"*Whereas,* July 22, 1935, the Supreme Engineer of the Universe engraved another name on his roster of public servants, that being William Mulholland; and,

"*Whereas,* during his lifetime he constructed a greater monument to his capabilities than can now be erected to his memory; and

"*Whereas* throughout his long career he combined his faculties with human qualities, which made him universally loved as well as respected, and by his devotion to the common welfare reflected credit upon his chosen field of endeavor; and

"*Whereas,* the Engineering Profession, in its entirety, and in particular the Los Angeles Section of the American Society of Civil Engineers, of which he has been a member since its inception and its third President, has suffered an irreparable loss; now, therefore,

"*Be it Resolved,* that the Board of Directors, acting for the Los Angeles Section of the American Society of Civil Engineers, express its profound grief and convey to the members of his family its deepest sympathy for the deprivation of his counsel and friendship and that a copy of this resolution be forwarded to the members of his family."

Mr. Mulholland was elected a Member of the American Society of Civil Engineers on February 6, 1907.

EDWARD TOWLER MURCHISON, M. Am. Soc. C. E.[1]

Died May 8, 1935

Edward Towler Murchison was born in Ripon, Wis., on November 22, 1888, the son of Murdock William Murchison and Clara (Eddy) Murchison. He received his primary education at Englewood High School, at Chicago, Ill., from which institution he was graduated in 1906. Circumstances at the time prevented his completing his education which was started at the University of Washington, at Seattle, Wash. His educational career, although not completed at the University, was supplemented by intensive study after the usual working hours, while engaged in his several engineering positions.

As a Rodman from June, 1906, to July, 1907, Mr. Murchison was employed on the Topographical Survey of the Illinois River Valley for the Sanitary District of Chicago, Ill. This engagement was followed by his employment as a Rodman on the construction of the Grand Trunk Pacific Railroad, on the main line through Manitoba and Saskatchewan, Canada, and following that work as a Topographer on the location of the Canadian Northern Pacific Railroad on the main line in British Columbia. In June, 1909, Mr. Murchison had his first position in responsible charge of work when he was made Assistant Engineer of the Powell River Paper Company, 80 miles north of Vancouver in British Columbia, on its heavy mill, dam, and railroad construction program. On this assignment he had charge of the field work and was in entire charge of the project in the absence of the Chief Engineer. On the completion of this project, he became Assistant Engineer for Ray and McKean, Engineers, at Seattle, Wash., and had charge of surveys and construction on a real estate development at Factoria, Wash.

From April to December, 1911, Mr. Murchison was an Assistant Engineer with the San Benito County Highway Commission, in California, after which he became associated with the South Park Commissioners of Chicago, as Assistant Engineer on tunnel construction. In January, 1913, he entered the service of the City of Chicago, and for four years was actively engaged in the design and construction of sewers, in the Southern Division of the city. In December, 1916, he was transferred to the Technical Staff of the Committee on Finance of the City of Chicago, as one of the engineers who made engineering reports and investigations. He was engaged in this capacity when he was called to serve his country in the World War. After entering the Service in 1917, he became a candidate engineer in the Officers' Training Camp of the United States Army, was commissioned a First Lieutenant of Engineers, and following duty with various organizations in the United States, went overseas as Engineer Officer in Charge of Construction of the 39th Division, American Expeditionary Force.

Following his discharge from the Army, Mr. Murchison returned to Chicago, in January, 1919, and re-entered the service of the City with the

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[1] Memoir prepared by the late George C. D. Lenth, M. Am. Soc. C. E.

Technical Staff of the Department of Finance. In December, 1919, he was assigned as Junior Engineer on sewer design and construction in the South Division of the Board of Local Improvements of the City of Chicago, which position he retained until April, 1921, when he became Assistant Engineer for the Sanitary District of Chicago, in charge of sewer construction, reconstruction of canal banks, and the preparation of engineering reports and sewer design. In March, 1923, Mr. Murchison resigned his position with the Sanitary District of Chicago and became Assistant Superintendent of Building Construction with the Commonwealth Edison Company, of Chicago. He had charge of the field work and the construction of several large electric power plants in Chicago.

From May, 1925, to May, 1926, he was in private practice as a Civil Engineer in Florida. Returning to Chicago he was, from May, 1926, to January, 1929, Assistant Chief Engineer of the Krenn and Dato Construction Company in charge of the design and construction of municipal utilities, including complete charge of sewer and water contracts as Construction Superintendent in various parts of the Central West. Following his work with the Krenn and Dato Construction Company, Mr. Murchison became associated with the Century of Progress Exposition, at Chicago, and, for four and one-half years, he was in charge of the design and construction of all underground utilities; plumbing within huildings; water and sewerage pumping stations; water and sewerage systems; and the pavements for all roadways.

Throughout his career as an engineer, he was honored and respected by all his friends and engineering associates, and was a tower of strength to all with whom he came in contact. On his last work at the Century of Progress Exposition, at Chicago, Mr. Murchison had the courage to pioneer in many phases of this important work, and, as a result, effected economies in design and construction that saved enormous sums of money.

In 1917, he was married to Maude Louise Jansen, who survives him, with a son, Robert, and a daughter, Ruth.

Mr. Murchison was elected an Associate Member of the American Society of Civil Engineers on April 3, 1922, and a Member on December 26, 1933.

FRANK AMENDE MUTH, M. Am. Soc. C. E.[1]

DIED JANUARY 31, 1936

Frank Amende Muth was born in Kirksville, Ky., on January 27, 1887, the son of Edward Godfrey and Elizabeth Carroll (Tevis) Muth. After his preparatory education, he attended the University of Missouri, at Columbia, Mo., from 1906 to 1910, and received the degree of Bachelor of Science in Civil Engineering from that institution. He also did post-graduate work at the University of Missouri, as well as at Tulane University, in the College of Commerce, at New Orleans, La.

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[1] Memoir prepared by a Committee of the Louisiana Section, consisting of E. S. Lanphier and B. H. Grehan, Members, Am. Soc. C. E.

Mr. Muth began his engineering work as Surveyman with the United States War Department in 1909, for the period from June to September. From June to September, 1910, he was Surveyman, with the United States Reclamation Service, on the Milk River Project, at Malta, Mont. From September, 1910, to July, 1911, he taught for one term in the Civil Engineering Department, at the University of Missouri, and from July, 1910, to July, 1911, he served as Assistant to Dean H. B. Shaw, Construction Engineer on the water-works system, at Columbia, Mo.

As an Aide in the United States Lighthouse Service, at Staten Island, New York, Mr. Muth made surveys, designed for signal layouts, clockwork, and mechanisms, made photometric tests, inspected engineering materials for steel buoys and vessels, at steel mills, including the supervision of the construction of a concrete lighthouse at Huntington Harbor, New York, as well as a concrete and steel casing around a 167-ft brick tower, at Fire Island, New York. His work also included tests of oil engines and air compressors, the construction of coast-protection devices and riprap protection at lighthouses, and the rebuilding and restoring of lighted structures. From December, 1914, to July, 1919, he was Assistant Superintendent, Office, Lighthouse Inspector (this designation later was changed to Superintendent of Lighthouses), Eighth Lighthouse District, at New Orleans, La., where he was engaged on the design and supervision of construction of lighthouse structures, consisting of lighthouses, beacons, dwellings, and wharves. From July, 1919, to November, 1935, he was First Assistant Superintendent, Eighth Lighthouse District, with increase in responsibility in design, and in charge of the supervision and control of field repair and construction forces. At the time of his death, Mr. Muth was serving in the capacity of Superintendent of Lighthouses, Seventeenth District, at Portland, Ore.

From June 14 to December 8, 1924, Mr. Muth was granted six months leave of absence by the Lighthouse Service to perform the work of revising the Building Code of the City of New Orleans.

He was elected a member of the Louisiana Engineering Society on October 1, 1915, and was appointed Secretary of that Society on January 20, 1923, in which office he served continuously for nearly thirteen years, until his departure from New Orleans, in November, 1935.

For many years Mr. Muth served as a member of the Board of State Examiners for Civil Engineers practicing in the State of Louisiana. He was also a member of the Board of Appeals on Building Code.

He was a member of the Propeller Club, of New Orleans; a Commander of the Boy Scouts of America, New Orleans Council (1926 to 1931); and a member of the Rotary Club of New Orleans.

He was married on July 16, 1913, to Allie Lewis Frank, and is survived by his widow, two sons, and a daughter.

Mr. Muth was elected an Associate Member of the American Society of Civil Engineers on October 10, 1916, and a Member on January 18, 1926. He served as President of the Louisiana Section of the Society in 1927 and 1928, and as Secretary for many years prior to April 25, 1925, when he was elected Vice-President of the Section.

ARTHUR O'BRIEN, M. Am. Soc. C. E.[1]

DIED APRIL 24, 1935

Arthur O'Brien was born in Forestport, N. Y., on January 2, 1870, the son of Jeremiah and Catherine (Hennessy) O'Brien. He attended the local schools and received as good an education as the town then afforded. After leaving school, Mr. O'Brien taught in the district school at Floyd, N. Y. Soon afterward he went into the lumbering business in which his knowledge of the work and his natural aptitude for mathematics secured rapid promotion.

In April, 1894, he entered the Department of the State Engineer and Surveyor of New York State where he served as Chainman, Rodman, Leveler, and Assistant Engineer. In 1896, he was placed in charge of surveys, estimates, etc. Later, he had charge of the construction of a sea-wall and breakwaters, at Geneva, N. Y.; seven miles of the Erie Canal improvement, at Montezuma, N. Y.; dredging the harbor and canal, at Cayuga, N. Y.; building causeways across Otisco Lake, at Otisco, N. Y.; and other canal work.

He also took charge of surveys, plans, and estimates for highway improvement, at Syracuse, N. Y.; the relocation of the boundary line between New York State and Connecticut and Canada; the construction of a lift-bridge, at Utica, N. Y.; and a fixed-span bridge, at Rome, N. Y. These seven years of State work gave Mr. O'Brien not only a wide knowledge of public work, but also an extensive acquaintance with public officials, contractors, and others engaged in the construction industry.

In April, 1902, together with Messrs. Campbell W. Adams, former State Engineer, of Utica, and Herschel Roberts, of Albany, N. Y., Mr. O'Brien sailed for Norway to take charge of the building of a railroad and harbor at Mo, for the Dunderland Iron Ore Company, Limited, of London, England. After the completion of this important piece of work, Mr. O'Brien traveled on the Continent and studied some of the European engineering projects.

Upon returning to the United States, he engaged in private practice in Albany, N. Y., until 1906, when he took the position of General Manager of the Fort Orange Construction Company. In this capacity he had charge of the construction of three great locks on the Barge Canal, at Waterford, N. Y.

In the fall of 1907 Mr. O'Brien was asked to accept the position of City Engineer, of Utica, in his home county. He was the first City Engineer under the second-class cities charter. In 1910 and 1911, he was again engaged in canal work, being employed by the Receiver of the McDermott Construction Company, at Fulton, N. Y.

In January, 1912, Mr. O'Brien was appointed by the Board of Supervisors of Oneida County as County Superintendent of Highways, which position he held continuously until his death, except for one year—August, 1919, to August, 1920. Under his direction more than 500 miles of County roads were constructed, giving employment during the last few years to between 600 and 700 men, at an annual expenditure of about $500 000. He supervised the work done on town roads as well as County roads and organized and supervised the work of snow removal throughout the County. He was known as one who spent public money as conscientiously as his own, saving the County thousands of dollars.

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[1] Memoir prepared by H. C. Seubert, Esq., Utica, N. Y.

Mr. O'Brien possessed an unusual knowledge of highway law and was considered an authority on the subject. He was known throughout the State as an excellent authority on road construction and maintenance. Throughout his life, he was a constant reader of worth while books. This trait, combined with his excellent memory, gave him a knowledge of the best in literature which few possess.

For many years he was a member of the Fort Orange Club, of Albany, and the City Club, of Utica. He also belonged to the Rome Club and the Republican Club, of Utica. He was never married.

The *Utica Daily Press* of April 27, 1935, contained the following editorial notice of Mr. O'Brien's death:

> "Much regret will be felt over the death of Arthur O'Brien, County Superintendent of Highways, which followed an illness of only a few days. He was a quiet, unassuming man, courteous in his attitude toward his fellow men. There was a reserve about him, however, which gave him personality and strength of character. His demeanor and actions won him the confidence and trust of those with whom he had to deal.
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> "These qualities, combined with his ability as an engineer, had enabled him to hold the office of County Superintendent of Highways for twenty-two years. Merit, rather than politics, retained him in office. * * *; he served the County faithfully and honestly. Whatever factional differences prevailed in the Board of Supervisors, to which he owed his appointment, they rarely, if ever, involved him. This was because all regarded him as an honest and efficient official, whose services were valuable to the county. He had created a following upon a stronger base than politics. He set a good example for those who are in office, or who may be contemplating entering public life. He had a busy and useful career and by industry and perseverance had won a high rank in his profession as an engineer."

Mr. O'Brien was elected a Member of the American Society of Civil Engineers on July 1, 1909.

SIR FREDERICK PALMER, M. Am. Soc. C. E.[1]

DIED APRIL 7, 1934

Frederick Palmer, son of George Palmer, Railway Contractor, was born in South Wales, Great Britain, on January 31, 1862. He was educated at Neath and in 1876 became an articled pupil on the Great Western Railway, serving it as Assistant Engineer after the completion of his four years of pupilage, until 1883.

In that year Mr. Palmer was appointed Assistant Engineer on the East Indian Railway. He was selected for the post by the late Sir Alexander

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[1] Memoir prepared by Sir Robert Richard Gales, M. Am. Soc. C. E.

Rendel, then Consulting Engineer to the Government of India, who nearly thirty years later, invited him to become his partner.

In India Mr. Palmer soon began to distinguish himself. After experience in charge of the Drafting Room and as personal Assistant to the Chief Engineer, in Calcutta, he was promoted in 1889 to Resident Engineer and placed on the survey of the Grand Chord Line in the Hill Section. In 1893, he became District Engineer, at Allahabad, in charge of maintenance. In 1896, he was appointed Engineer in Charge of the Moghalserai-Gaya Survey and subsequently became District Engineer in charge of the construction of the line which comprised 126 miles of new railway, including a bridge over the Sone River consisting of 93 spans of 100 ft, which incidentally was and still is the longest bridge in India.

On the completion of this work in 1900, Mr. Palmer had spent about seventeen years on railway engineering on the East Indian Railway. He then took up and devoted himself for twelve years exclusively to dock and harbor work.

About this time the trade of Calcutta was outgrowing the accommodation and the Port Commissioners required an experienced engineer to prepare plans for the development of the Port and to carry out the extensions. Mr. Palmer applied for the position and, in 1901, was appointed Chief Engineer. He was soon actively engaged in the construction of additional jetties in the Hooghly, with transit sheds and equipment and new berths in the Kidderpore Docks to meet the most pressing needs. At the same time he prepared a comprehensive report dealing with the increase in traffic and in the size of ships and including proposals and plans for new docks and jetties at Garden Reach, but it was not until after the World War that these projects were undertaken.

Meanwhile in 1909 Mr. Palmer, being then on leave in England, was offered and accepted the appointment of Chief Engineer to the newly constituted Port of London Authority. The problem here was similar to that at the Port of Calcutta but on a larger scale. It was in fact to modernize the existing accommodation and equipment and to provide for the future growth of the largest port in the world. In 1910 he presented a report which included a program of deepening the channel of the River Thames for 47 miles from London Bridge to the Nore Lightship and proposals for improvements in existing docks, consisting of new enlarged entrance locks, installations of impounding pumps, dry docks, enlarged transit sheds and warehouses, and improved equipment and communications by road and railway, together with new docks with larger entrance locks and new dry docks at the Victoria and Albert Dock group and at Tilbury, where also a floating passenger landing-stage was to be provided at which the largest liners using the Port could go alongside at any stage of the tide. Many of the works of improvement had been carried out when in 1913 Mr. Palmer resigned the post of Chief Engineer to become a partner in the firm of Rendel, Palmer and Tritton, Consulting Engineers, of which, on the death of Sir Alexander Rendel in 1918, he became the head. In 1925, however, he was appointed Consulting Engineer to the Port of London Authority to carry out major works at the West India Docks and at Tilbury. The floating passenger landing-stage, the last of these works to be completed, was brought into use in 1930, by which time £20 000 000 had been spent on the improvements and extensions which he had initiated.

During the World War, 1914 to 1918, Mr. Palmer gave his services voluntarily to the Ministry of Munitions, under which he became Director General of Contract Finance and Chairman of the Munitions Works Board, and to the Board of Trade by which he was appointed Chairman of a Committee to consider the future of English railways. The conclusions of this Committee formed the basis of the legislation subsequently adopted on this subject.

After the war, he directed with ability the activities of his firm, as Consulting Engineers to the Government of India, including the State Railways of India, as well as most of the company-owned railways in that country; to a number of Native States; to several Indian Harbor Boards; to the Crown Agents for the Colonies; to the Port of Glasgow, Scotland; to the Tyne Commissioners; to the River Mersey Conservators; to the Mersey Harbor Board; and to other public bodies.

In 1920 Mr. Palmer was elected a member of the Commission Consultative Internationale de Travaux du Canal de Suez, in succession to Sir William Matthews, and continued to serve on that Commission to the time of his death.

In 1919 he was asked to advise on facilities at the Port of Karachi, in India, and in 1920 after two visits to that port he presented a valuable report embodying a plan of extension and improvement marked by his then well-known foresight and good judgment. The plan was accepted and the works have since been largely completed. In 1921 Mr. Palmer represented the Foreign Office on the Shanghai Harbor Committee and in 1923 he reported to the Government of China on Yangtse River improvements. In the same year, he advised the Anglo-Persian Oil Company on the possibility of improving the Shatt-al-Arab between the Anglo-Persian Oil Company's depot at Abadan and the open sea. His recommendations have been carried out and have resulted in the depth of water on the bar about 15 miles from Fao being increased from 10 to 20 ft at low water, thus enabling oil tankers and other large vessels to navigate the river. In this year also, at the request of the Colonial Office, he visited Palestine to advise the Government on the question of a harbor. His recommendations were adopted and the construction of a harbor at Haifa by the Palestine Government, with his firm as Consulting Engineers, was begun in 1929 and completed in 1933.

In 1924, he was asked by the Secretary of State for the Colonies to report on Takoradi Harbor, Gold Coast, and the firm was subsequently appointed Consulting Engineers for the completion of the construction. The harbor was opened in 1928.

In 1926, Mr. Palmer became President of the Institution of Civil Engineers and it was while holding this important office that he was invited by the Canadian Government to advise on the selection of the terminal port for the Hudson Bay Railway, projected to afford a more direct route to England for wheat and other products from Western Canada. His recommendation of Churchill in preference to Nelson was adopted and the Port of

Churchill was opened for traffic in September, 1931, by the loading of the *Farnworth* with 227 000 bushels of wheat for London. In 1927, he advised the Canadian Government on a plan for a £10 000 000 terminal station for the Canadian National Railways, at Montreal, Que.

In 1927 his firm was asked by the Colonial Office to prepare a project for a bridge over the River Zambesi to give uninterrupted rail communication between Nyasaland and the seaport of Beira. The bridge, which is 12 064 ft in length and is believed to be the longest bridge in the world over a river, was constructed in collaboration with the firm of Livesey, Son and Henderson, Consulting Engineers.

In association with Aktiebolaget Vattenbyggnadsbyran, of Stockholm, Sweden, his firm was responsible for the hydro-electric power installation at Chenderoh, on the Perak River, in the Federated Malay States. This was put into commercial operation in 1930.

Mr. Palmer was closely connected with questions concerning the bridges of London. In December, 1924, he reported to the London County Council on the condition of Westminster Bridge. From 1926, he was concerned with the various projects for rebuilding or reconditioning and widening the much discussed Waterloo Bridge, but it was not until after his death, in 1934, that the demolition and rebuilding of the bridge was entrusted to his firm as Consulting Engineers. Following the extraordinary flood which occurred in the Thames on the night of January 6, 1928, he was appointed, with Sir George Humphreys, to report on the future standard of flood prevention works in the County of London. In 1929, he was appointed by the Council as Engineer for the Charing Cross Bridge project. This scheme would have involved an expenditure of £12 500 000, but it was rejected by the Select Committee of the House of Commons in May, 1930. Mr. Palmer was also appointed Engineer, in 1931, for the reconstruction of Chelsea Bridge and, subsequently, in 1933, this work was entrusted to his firm as Consulting Engineers.

In 1929, his firm was appointed Consulting Engineers for the bridge over the River Hooghly, at Calcutta; for the construction of the Victoria Docks Road forming a main artery of communication for this group of Port of London docks; and in collaboration with the firms of Livesey, Son and Henderson and Mott, Hay and Anderson, the firm was appointed by the Ministry of Transport to report on the Channel Tunnel scheme.

Mr. Palmer was appointed a Companion of the Most Eminent Order of the Indian Empire (C.I.E.) in 1907 for his work in India and, in 1930, he received the Honor of Knighthood and was invested with the Insignia of a Knight Commander of the Most Distinguished Order of St. Michael and St. George (K.C.M.G.) for services to the Colonies.

In 1930 and 1931, his firm located a new railway line, about 700 miles long, from Haifa, the port in Palestine, to Baghdad, the capital of Iraq.

Sir Frederick Palmer was a man of outstanding personality, great vitality, broad views, and sound judgment. He was held in high repute for the many large projects with which he was connected, and he was greatly esteemed by all who knew him. At his death, he was honored with a memorial service in Westminster Abbey.

He held the rank of Colonel in the Engineer and Railway Staff Corps and the Efficiency Decoration of the Territorial Army was conferred upon him in December, 1931. He was a Member and Past-President of the Institution of Civil Engineers, and a Fellow of the Royal Geographical Society.

He was married, in 1903, to Florence, only child of the late John Elliott Mason, of San Francisco, Calif., and is survived by Lady Palmer and a son and two daughters.

Sir Frederick was elected a Member of the American Society of Civil Engineers on October 4, 1899.

HARRY de BERKELEY PARSONS, M. Am. Soc. C. E.[1]

DIED JANUARY 26, 1935

Harry de Berkeley Parsons was born on January 6, 1862, in New York, N. Y., the son of William Barclay Parsons and Eliza Glass (Livingston) Parsons. He was descended from distinguished Colonial stock, long identified with the development of New York City. Among his direct ancestors were Robert Livingston, First Lord of the Manor, who came to this country from Scotland in 1673; Cadwallader Colden, Lieutenant-Governor of the Province of New York in 1760; and Colonel Thomas Barclay, who was appointed the British Consul General for New York, after the American Revolution. A brother who distinguished himself as an engineer and a soldier of the United States was the late William Barclay Parsons, Hon. M. Am. Soc. C. E., at one time a Director of the Society, who was Chief Engineer of the Rapid Transit Railroad Commission during the construction of the first Rapid Transit Subway for New York City, and, later, Colonel of the 11th Engineers, American Expeditionary Forces, which served so notably in France during the World War.[2]

In 1870, Mr. Parsons went to Europe with his family and during the following four years studied under private tutors, while traveling in France, Germany, and Italy. He entered Columbia College, in New York City, in 1878, from which he was graduated with the degree of Bachelor of Science in 1882. He then entered Stevens Institute of Technology, in Hoboken, N. J., from which he was graduated in 1884 with the degree of Mechanical Engineer. In 1926, Stevens conferred on him the Honorary Degree of Doctor of Engineering.

In 1885, Mr. Parsons established an office in New York City as a Consulting Engineer and maintained it there until his death. His field of practice was an unusually diversified one. To recite all the activities of his fifty years of experince as a Consulting Engineer would make too formidable a list to publish herein, but one who has studied them is impressed with their number, and with the versatility of mind of this quiet conscientious man. Among the earlier projects with which he was connected was the construction of the Fort Worth and Rio Grande Railway, in Texas, including the erection of a bridge over the Brazos River, and, in 1889, the completion of the water supply for Stevens Point, Wis. In 1893, he was Consulting Engineer for the Nicaragua Canal Construction Company and some of its related organizations.

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[1] Memoir prepared by Robert Ridgway, Past-President and Hon. M. Am. Soc. C. E., and Lynne J. Bevan, M. Am. Soc. C. E.

[2] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1485.

From 1892 to 1907, he was Professor of Steam Engineering at Rensselaer Polytechnic Institute, Troy, N. Y., and from the latter date until his death was its Emeritus Professor of Practical Engineering.

In 1898 he was consulted on the adequacy of the foundations for the Protestant Episcopal Cathedral of St. John the Divine in New York City. At a later date, he designed and installed the heating system for the Cathedral, and also had to do with the design of some of the appurtenant structures.

In the period, 1901 to 1903, Mr. Parsons designed the Spiers Falls Dam and Power House, on the Hudson River, at Spiers Falls, N. Y.—1 570 ft long, with a maximum height of 154 ft—and, in the period, 1921 to 1923, the Sherman Island Dam on the same river, near Glens Falls, N. Y., for the International Paper Company. He also acted as Consulting Engineer while these projects were being constructed.

Between 1898 and 1914 he was a member of the New York State Voting Machine Commission, and was Chairman of the Mayor's Committee to Report on Street Cleaning and Waste Disposal, in New York City, in 1906 and 1907. From 1908 to 1914, he served as a member of the Metropolitan Sewerage Commission which prepared a monumental report on the Storm and Sanitary Drainage of New York City; and from 1898 to the time of his death Mr. Parsons was a Consulting Engineer for the New York Zoological Society.

His activities included the designing of water and steam power plants for mills; roof trusses, heating plants, and other features of church buildings; water supply and sewerage projects for cities; many designs for hydro-electric developments, and numerous appraisals of industrial projects, water powers, and railroads in many States. He designed foundations and underpinning for many structures for construction contractors, in connection with the building of the Rapid Transit Subway in New York City; and in 1918 and 1919, as District Appraisal Officer at Detroit, Mich., Mr. Parsons represented the United States Government in the settlement of World War contracts connected with the Air Service and aircraft production. He acted as Consulting Engineer for the Cramp Ship and Engine Building Company on reports and appraisals; for the Pressed Steel Car Company; for The Consolidated Gas Company of New York; numerous paper companies; the Seaboard Air Line; and the New Hampshire Traction Company. He was Consulting Engineer for the Department of Street Cleaning of New York City through five administrations, during which he designed and built two rubbish destructor plants and prepared the design for a third for the building of which the City did not appropriate the necessary funds. For a time, 1905–1906, he was the Consulting Engineer for the Fire Department and for the Department of Docks and Ferries, of New York City, and designed and constructed a number of fire-boats and ferry-boats for these Departments.

For many years Mr. Parsons was the Consulting Engineer for the Corporation of Trinity Church, the Rhinelander Estates, the Corn Exchange Bank, the Mechanics and Metals Bank, and the Bush Terminal, all of New York City. The Delaware and Hudson Company retained him to report on and appraise some of its electric railway properties, the construction cost of which amounted to more than $14 000 000. The City of New York employed him to appraise the damages to the Ulster and Delaware Railroad by the flooding of the Ashokan Reservoir, for the properties taken for the large Chelsea Dock improvement, on the Hudson River front, and for the new County Court House—the condemnation for all of which amounted to many millions of dollars.

He regarded as his most important accomplishments the Spiers Falls and the Sherman Island Dams, the study of tidal flows in the Harbor of New York for the Metropolitan Sewerage Commission, and the appraisals at the end of the World War for the U. S. Army Air Service plants, at Detroit.

Mr. Parsons was a liberal contributor to the publications of the Society, as well as to those of other engineering and scientific bodies, both in original papers and in discussions of those of other authors. He was awarded the Thomas Fitch Rowland Prize by the Society in 1925 for his paper on "Sherman Island Dam and Power-House",[3] and, in 1930, was given the J. James R. Croes Medal for his paper on "Hydrostatic Uplift in Pervious Soils."[4] Mr. Parsons was also the author of "Disposal of Municipal Refuse and Rubbish Incineration"[5]; "Tidal Phenomena of the Harbor of New York"[6]; "Some Soil Pressure Tests",[7] and many other papers, including an original research on the "Displacement Curves of Fish". His book entitled "Steam Boilers, Their Theory and Design" went through five editions (first edition, 1903; fifth edition, 1917).

He was a member of the Delta Psi Fraternity, of the American Society of Mechanical Engineers, the Society of Naval Architects and Marine Engineers, and the American Institute of Consulting Engineers, of which he was the President in 1926. He served as President of the Alumni Association of Stevens Institute in 1895 and 1896, and as Alumnus Trustee from 1896 to 1899. He was Chairman of the Professional Engineers Committee on Unemployment, which represented the four Founder Societies, in 1931 and 1932, and was on its Advisory Board in 1932 and 1933. In addition, Mr. Parsons was active in civic affairs, and was a member of the New York

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[3] *Transactions*, Am. Soc. C. E., Vol. 88 (1925), p. 1257.

[4] *Loc. cit.*, Vol. 93 (1930), p. 1317.

[5] *Loc. cit.*, Vol. LVII (1910), p. 45.

[6] *Loc. cit.*, Vol. LXXVI (1913), p. 1979.

[7] *Loc. cit.*, Vol. 100 (1935), p. 1.

Zoological Society and of its Board of Trustees, the Chamber of Commerce of the State of New York, the Merchants Association of New York, the Metropolitan Museum of Art, New York, and was a Trustee of the United Hospital of Portchester, N. Y., and Chairman of its Endowment Fund.

He belonged to the Protestant Episcopal Church and was a member of the vestry of the Church of the Incarnation, in New York City, where his funeral services were held, and of Christ Church, at Rye, N. Y., where his beautiful summer home is located. He belonged to the following clubs in New York City: Union, Engineers, New York Yacht, and the Downtown Association, and to the American Yacht Club, and the Apawamis Club, in Rye, N. Y.

Mr. Parsons was a keen yachtsman and as a young man gained much experience at the helms of his own boats and of his father's yacht. He was much interested in this clean sport, and for many years subsequent to 1895 served as Chairman of the Race Committee of the American Yacht Club. From 1904 to 1922, he was also a member of the Race Committee of the New York Yacht Club, being Chairman from 1907 to 1922. The *New York Times* in an article published concerning him on January 27, 1935, quotes this statement of his made in 1917:

> "Yachting has a peculiar charm which is difficult to describe. I fancy it comes from the feeling of freedom and from the intimate companionship which is so pleasant. Men who are fond of the sea are usually fearless, frank and good sportsmen, qualities which make for staunch friendships and the pleasantest associations."

One feels how inadequate is this recital of Mr. Parson's deeds as a measure of what he accomplished during his life. When meeting him, one always felt that one was with a man who, by instinct and training, was a finished engineer. Always courteous, his quiet refinement of manner, his sincerity of purpose, his modesty, and his grounding in the fundamentals of his profession distinctly impressed one. A cultured gentleman possessed of broad knowledge, his life was unusually full and useful. In spite of his modesty time will appraise the real value of his contributions to his profession and the Society. An engineer associate—who was close to him and his activities for many years—writes of Mr. Parsons:

> "As I knew him I came to feel that he had rendered his best services to the profession in his very many contributions of lucid thought to the literature of the Society, and his modest assistance to his fellow engineers. These services have rarely commanded the spotlight, yet in the aggregate they are far more valuable to mankind than the occasional spotlight performance of less modest men."

He was married on December 16, 1890, to Frances Thompson Walker, of New York City, who died in 1917. He is survived by a son, Livingston Parsons, and a daughter, Katherine de B. Parsons, and three grand-children.

Mr. Parsons was elected a Member of the American Society of Civil Engineers on February 3, 1897.

WILLIAM MERIT PENNIMAN, M. Am. Soc. C. E.[1]

DIED AUGUST 26, 1934

William Merit Penniman, the son of Merit Farnum and Lavina (Damon) Penniman, was born at Windsor, Vt., on February 19, 1868. His boyhood was spent on his father's farm, near the banks of the Connecticut River.

He attended the public schools of Windsor and Hartland, Vt., and was graduated from Kimball Union Academy at Meriden, N. H., in 1888. Mr. Penniman entered Dartmouth College, at Hanover, N. H., in 1889, and was graduated therefrom in 1893, with the degree of Bachelor of Science. In college, he was distinguished for scholastic standing, especially in higher mathematics.

During summer vacations, beginning May 1, 1891, Mr. Penniman entered the service of the United States Government, in the United States Engineer Department, St. Louis, Mo., District, and was engaged on the improvement of the Mississippi River. Shortly after leaving college in 1893, he was re-employed by the St. Louis District, and remained in that service in various grades practically without intermission until his death, his last engagement being as Senior Engineer.

From August, 1907, to June, 1909, he was engaged as Principal Assistant Engineer on the survey of the Mississippi River between St. Louis, and Cairo, Ill., for the Board of Examination and Survey of the Mississippi River (14-ft waterway, Lakes to the Gulf). His review of methods of complete river regulation by bank protection, permeable dikes, and submerged dams, published as Appendix No. 5, to the report of that Board,[2] was so comprehensive that it is still used as a reference.

In later years, a formula for the flow of water in open channels was derived by Mr. Penniman, but was never published. The formula has been widely used in monograph form in the St. Louis Engineer District and in other districts, where the ability and work of the originator are known. Locally, it is known as the "Penniman" formula.

It is difficult to state wherein Mr. Penniman was of greatest value to the Engineer Department. Because of his unusual knowledge and technical skill, he was employed almost constantly during his later years on special investigations and reports. His services were utilized to aid the decisions reached by Boards of Engineers on problems beyond the bounds of the St. Louis District. His training of younger engineers to produce accurate and reliable work has proved to be of major value to the Engineer Department, since many of his former assistants now hold responsible positions in the St. Louis District, and in other Engineer Districts.

Mr. Penniman's entire service was characterized by his loyalty to the Department which he served so well. His open-mindedness and ability to keep informed on current developments in hydraulic and other engineering theories, made him a man whose counsel was frequently sought when important decisions were to be made. Since his death, as a tribute to his high standing, the St. Louis District has renamed a large steam towboat, *Penniman*.

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[1] Memoir prepared by a Committee of the St. Louis Section, consisting of J. W. Skelly, E. C. Constance, C. M. Daily, and J. B. Dean, Members, Am. Soc. C. E.

[2] H. R. Doc. No. 50, 61st Cong., 1st Session.

His long experience embraced a wide range of phases of open-river improvement, including surveys, hydrography, bank protection, contraction works, and the design and operation of hydraulic dredges, and other floating plant. He was trained under Captain D. M. Currie and the late William Selby Mitchell, M. Am. Soc. C. E., who were engineers and gentlemen of the old school. They lived in a day when Government funds were sparingly granted, and when projects were most critically examined for their engineering and economic merit.

Mr. Penniman loved the outdoor life, and during the first ten years of his service, which was largely spent in the field, he took a keen delight in baseball and hunting. In manner, he was affable, although quiet and reserved, and to a stranger might have even seemed to be taciturn. In his later years, he weathered several attacks of sickness, which however gave no premonition of the malady which brought a useful life to what seems an untimely end. About two years prior to his death he suffered a paralytic stroke from which he partly recovered. On August 23, 1934, he was fatally stricken and three days later passed peacefully away. He was buried in Bellefontaine Cemetery, at St. Louis, on a hillside overlooking the Mississippi River which he loved so well.

He was married on August 6, 1901, to Idell Lombardi, of Galveston, Tex. He is survived by his widow, a daughter, Catherine Lavina (Mrs. T. D. Storie), and a son, Allen Damon. He was a member of the Masonic Fraternity and of the Society of American Military Engineers.

Mr. Penniman was elected a Member of the American Society of Civil Engineers, on May 28, 1923.

EUGENE EVERETT PETTEE, M. Am. Soc. C. E.[1]

DIED MARCH 17, 1935

Eugene Everett Pettee, the son of Captain Lemuel and Mary (Westall) Pettee, was born on January 15, 1870, at Jefferson, Tex., where his father, who had served throughout the Civil War in the Regular Army, was stationed at the time, on guard duty against Indians.

During his boyhood Mr. Pettee lived with his parents in many parts of the United States and his schooling was obtained from the public schools wherever he happened to be located. He was finally fitted for the Massachusetts Institute of Technology in the Whitman (Mass.) High School, and entered the Institute in 1888. For various reasons, he was unable to complete the course, and, in 1891, left school to enter the Drafting Office of the Boston Bridge Works, where his unusual ability was recognized and where he remained for three years.

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[1] Memoir prepared by J. R. Worcester, M. Am. Soc. C. E., and Thomas Worcester, Esq., Boston, Mass.

In 1894, he accepted a position in the City Engineer's Office, of Newton, Mass., where he was employed until, in 1896, he was called into the office of one of the writers, J. R. Worcester, M. Am. Soc. C. E., to assist in completing the design of the train-shed (since removed) of the South Station of the New York, New Haven and Hartford and the Boston and Albany Railroads.

The association thus begun was to continue throughout Mr. Pettee's life. In 1906, he became a partner in the newly organized firm of J. R. Worcester and Company, and until February, 1934, when he suffered a paralytic stroke, he was actively engaged in the design and supervision of construction of many bridges, buildings, and other structures, and by his frankly honest, yet courteous and genial bearing, won many enduring friendships with other engineers, with architects and contractors, as well as with many public officials. As an expert witness, he was convincing and effective.

Among the projects with which Mr. Pettee was most closely identified were: The elevated structure of the Boston Elevated Railway, including the Charles River Bridge; three bridges over the Connecticut River, at Woodsville, N. H., Bellows Falls, Vt., and Walpole, N. H.; the bridge across Portland Harbor, between Portland and South Portland, Me., to which he gave his personal attention, living in Portland for two years; and, toward the end of his life, the bridge now (1935) under construction over the Saugus River between Revere and Lynn, Mass. For some years he was engaged in examining all the bridges over which the lines of the Boston and Northern and Old Colony Street Railways had tracks.

Among his other gifts Mr. Pettee had a strong artistic sense, rare among engineers, and he found his greatest satisfaction in producing, with the help of architects, structures having real architectural merit. Numerous small bridges and over-passes bear witness to his success in this endeavor. He was an expert photographer and delighted in thus catching and preserving beautiful natural scenery.

He was a member of the Boston Society of Civil Engineers, the Appalachian Mountain Club, the Boston City Club, the New-Church Club, and the Wellesley Country Club.

Descended from two grandfathers who were New-Church (Swedenborgian) ministers, Mr. Pettee was naturally a faithful member of that church, in which he was a prominent layman, and held a number of important official positions.

In 1909, he was married to Margaret Edith Babcock, who survives him. He is also survived by one brother, Charles L. W. Pettee, of Hartford, Conn., and by three nephews and three nieces.

Mr. Pettee was elected an Associate Member of the American Society of Civil Engineers on September 3, 1902, and a Member on March 2, 1909.

AUGUSTUS LYON PHILLIPS, M. Am. Soc. C. E.[1]

DIED JANUARY 31, 1935

Augustus Lyon Phillips was born in Manchester, England, on October 6, 1860, the son of George Ricketts Phillips and Elizabeth Catherine (Robinson) Phillips. He was educated at the Edinburgh Academy, Edinburgh, Scotland, and studied Civil Engineering at Owens College, Victoria University, Manchester, England, for two years, 1877–1878.

In 1879–80, he was articled to T. B. Farrington, Borough Engineer, Colwyn Bay, North Wales, and was engaged in surveying, the design and construction of streets, water-works, sewerage systems, etc. In 1881, Mr. Phillips was articled to G. L. Waring, Chief Engineer, West Lancashire Railway, in England, as Junior Engineer on location and construction, design of coffer-dams, etc. In 1882, he was Contractor's Engineer, on the replacement of a two-track, with a five-track, plate-girder bridge (under traffic), over the intersection of three streets, at Stockport, England, for the Midland Railway Company, and on the widening of part of the main line (two additional tracks), of the London and North Western Railway; he also designed the falsework to carry traffic during the reconstruction of a bridge over the Mersey River.

From 1883 to June, 1884, he was Assistant Engineer for the British Government on a railroad in Jamaica, for which he had charge of the construction of a 25-mile section, with 5 miles in rough mountain country, two short tunnels, long retaining walls, river diversions, small concrete bridges, etc. These bridges were among the earliest constructed of concrete.

From June, 1884, to March, 1885, Mr. Phillips was Assistant to Samuel McElroy, Consulting Engineer, of New York City, and was engaged in office and field work on the farms of the Insane Asylum for Brooklyn, N. Y.

For nine months in 1885, he was engaged in designing and drafting heating and ventilating layouts, for the Creque Manufacturing Company, of New York City. From January, 1886, to June, 1887, Mr. Phillips was Assistant Engineer (under the Chief Engineer), for the Leroy and Caney Valley Air Line Railroad, on prospecting, surveys, location, and construction; he also designed the Howe truss bridges and timber trestles (one more than 500 ft long and 47 ft high). His work for the Leroy and Caney Valley Air Line Railroad was field work through Kansas and Indian Territory on what became later a part of the Missouri Pacific System. Some of his experiences as an Eastern tenderfoot, but lately out of Europe, in a West which still retained a great deal of the frontier aspect, were very amusing; such as his carefully taking along a revolver and then, after observing some of the local talent pick out the six spots from a six of spades in about 2 sec, deciding that he would be safer unarmed and promptly giving away his revolver. Then there was the local rodman who would cheerfully ford all rivers less than a couple of feet in depth, but when they came to the Arkansas, which at that ford at that season was about 3½ ft, he was panic-stricken, as he had always heard that if he ever got any water on his back, he would take cold and die (Mr. Phillips said his neck bore out his strict faith in this theory), so they could only get him across standing up in the cook wagon. Of course, there was the matter of building lots; promoters were laying out fictitious towns far ahead of the rail-head on what they hoped would be the line, and everybody was speculating in lots, Mr. Phillips included. Anybody could always sell him a lot, and he invariably made the down payment, and then found that the lot was worthless; at one time or another, he must have made down payments on most of the New Jersey coast between Sandy Hook and Cape May (and between the extreme low-water mark and the coast of Spain). So, of course, he bought a block of lots in one of these paper towns, made his down payment and according to rule never did anything more about them. Years later, he discovered the lots were at what later became the business center of Wichita, Kans. One morning in Indian Territory they sighted the last remnant—about thirty head—of the Southern Herd of buffalo, which only a few years before had numbered millions.

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[1] Memoir prepared by T. Kennard Thomson, M. Am. Soc. C. E., and J. H. H. Muirhead, Esq.

From June, 1887, to September, 1888, he served as Draftsman, for the Union Elevated Railroad Company, in Brooklyn, N. Y., designing and detailing under the Chief Engineer, the late Othniel Foster Nichols, M. Am. Soc. C. E.

From October, 1888, to March, 1902, Mr. Phillips was with the Bridge and Construction Department, of the Pencoyd Iron Works (later, the American Bridge Company), at Pencoyd, Pa., having been given the position by the Chief Engineer, the late Charles Conrad Schneider, Past-President, Am. Soc. C. E., who was familiar with his work in Brooklyn. Mr. Schneider always had the highest opinion of Mr. Phillips. On this work Mr. Phillips was Checker and Squad Boss, on buildings and bridges of all kinds and sizes, subway and elevated work, etc. It has been well said that he was an exceptional Checker, inasmuch as he was extremely accurate and thorough, and, at the same time (and this is very unusual) able to point out the errors and have them corrected, without giving offense to the Draftsmen, many of whom still have the highest admiration and affection for his memory.

From April, 1902, to March, 1908, he was Assistant Engineer, with the Philadelphia Rapid Transit Company, under William S. Twining, M. Am. Soc. C. E., Chief Engineer. Mr. Twining had the highest opinion of Mr. Phillips, who, he states, was no small factor in the design of that work. Mr. Phillips spent 1½ yr on the preliminary investigations and on the design of the subways and elevated structures; 4½ yr in charge of the Engineering Office, designing and detailing for 2½ miles of subway; a four-track bridge over the Schuylkill River; 5½ miles of two-track elevated, etc., the approximate cost being $20 000 000.

From April, 1908, to March, 1909, he was Supervising Engineer for the Interstate Engineering and Supply Company of Philadelphia, Pa., building a large power plant for the Metropolitan Electric Company at Reading, Pa., changing and rebuilding the old power plant (under operation) into substation, etc. (cost, $2 000 000).

From April, 1909, to 1923, Mr. Phillips was in private practice as a Consulting Engineer, in Philadelphia. As such he made investigations, surveys, estimates, layout, and plans for the Philadelphia and Suburban Railroad Company, for 8 miles of two-track subway, and 32 miles of two-track elevated railroad (estimated cost, $43 000 000; not built); he also made the preliminary surveys and layout for a system of docks on the Delaware River, below Philadelphia, for the Hughes Terminal Company (estimated cost, with 4 miles of connecting railroad, $5 000 000); he made surveys, and located and constructed 4½ miles of single-track railroad across the salt meadows for the Wildwood and Delaware Bay Short Line Railroad Company, including three trestle bridges (1 200 ft) and a single-track, single-leaf Scherzer rolling lift bridge over the inland waterway, with concrete piers in 20 ft of water at low tide (cost, $250 000); made surveys and plans for a high-grade land development suburb (100 acres) of Philadelphia, with central heating and domestic service plant, all pipes, wires, etc., in sidewalk conduits, and roads in park effect (estimated cost, $400 000); as well as investigations and reports on several proposed interurban trolley lines and miscellaneous minor work.

Mr. Phillips was one of those forward-looking engineers who are years ahead of their time. He advocated the abandonment of surface trolleys and their replacement by double-deck busses, at a time when the present system of bus transportation was practically unknown in the United States.

He was the originator of the project, and one of the incorporators of a transportation company formed for the purpose of operating arterial north and south subway and elevated lines in Philadelphia, with lateral bus-line feeders, and connecting elevated lines to the principal suburbs. He completed all the preliminary engineering work on this comprehensive system. Financial backing for the project was arranged for, and a charter was obtained from the State of Pennsylvania, but the opposing influences in the City Council proved too powerful to enable the Company to secure the necessary franchise.

However, the wide publicity given these projects, and the consequent strong public demand, practically forced the City Council into the building and the operation of the present city-owned Subway and Elevated Railway System. Mr. Phillips was also an early advocate of two-level streets in the center of the city whereby the vehicular traffic would be carried at the street level and the pedestrian traffic in sub-surface concourses with shop entrances and display windows at both levels for the accommodation of both classes of patrons. This has only been done to a limited extent as yet, but may be realized in the near future.

Among his non-engineering activities was his life-long devotion to vocal music. He was almost constantly engaged as choir master or tenor soloist in some church, the most famous of which was "Old St. Peter's" in Philadelphia.

He always took a great interest in athletics, having been captain of the football team at his college, and an active member of the cricket team, of which his brother was captain. He was one of the organizers and charter members of the Wissahickon Cricket Club of Philadelphia. He was also very expert with boxing gloves, which he frequently enjoyed using—during lunch hour—in the Drafting Room, at Pencoyd. He maintained a keen interest in doing carpenter work and gardening about the house, and was an avid stamp collector.

Besides his connection with the Society (of which he was very proud), Mr. Phillips was a member of the Engineers' Club of Philadelphia, Roxborough Lodge No. 135 F. & A. M., The Society of the Sons of St. George, and the Veteran Athletes' Association of Philadelphia.

Mr. Phillips was much liked by all—even the draftsmen whose numerous mistakes he had to correct. One of them writes, after 46 years, of what a fatherly care Mr. Phillips took of him, a young chap just out from Scotland, and how Mr. Phillips had visited him in the Philadelphia Hospital every Sunday for 25 weeks, while he had typhoid fever. Several others wrote that he was the best friend they ever had. Mr. Twining wrote affectionately of his high character, and sound engineering judgment.

In short, Augustus Lyon Phillips was a man whose absolute integrity, great ability in Engineering, and kind-heartedness, won for him the undying love and admiration of all who knew him. He died of arterio-sclerosis after years of suffering, in the beautiful Lincolndale (Westchester), home of his sister, Miss J. Annette Phillips, where he had received the best of care and loving attention.

He was married to Janet Macpherson Fulton, in Brooklyn, N. Y., on June 23, 1887, who survives him. He is also survived by two sons, George William Macpherson Phillips, and Charles Stuart Phillips, one daughter, Jean Bannerman Phillips, one grandson, Alfred Gordon Phillips, one granddaughter, Janet Fulton Phillips, two brothers, Henry H. S. Phillips, and John R. Phillips, and three sisters, J. Annette Phillips, Margaret I. C. Phillips, and Ellen A. G. Phillips.

Mr. Phillips was elected a Member of the American Society of Civil Engineers on June 23, 1916.

LOUIS HENRY PRELL, M. Am. Soc. C. E.[1]

DIED MARCH 18, 1936

Louis Henry Prell was born in Cincinnati, Ohio, on March 28, 1874, the son of Ludwig H. Prell and Maria (Goetze) Prell. He received his preparatory education in the public schools at Cincinnati, and at Mt. Healthy, Ohio.

Mr. Prell began his career in engineering with Frank B. Betts, Civil Engineer, in 1893. During the period, 1893 to 1898, he continued in the service of Mr. Betts, and was engaged in land surveying, sub-division work, and railroad layouts. During this period he was employed in various capacities as Rodman, Chainman, Transitman, and Draftsman. Following his engagement with Mr. Betts, he entered the Ohio Normal University, at Ada, Ohio, in 1899, from which he received the degree of Civil Engineer in 1902.

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[1] Memoir prepared by H. H. Peters, Esq., U. S. Engr. Office, Cincinnati, Ohio.

In 1901, Mr. Prell was employed as Rodman by the United States Engineer Department, at Cincinnati, Ohio, during his school vacation. On completing his University studies, he re-entered the service of this Department in July, 1902, and from that date until his death, he was continuously employed by the Department, except for short intervals in 1904 and 1905. During this time, his work and accomplishments pertained principally to the canalization of the Ohio River by the construction of a series of locks and dams extending from Pittsburgh, Pa., to Cairo, Ill. In this work, he had a conspicuous part, and rendered capable and valuable service. In the period from July, 1902, to June, 1905, he was engaged in the field, on drafting, and on computations of the original Ohio River survey which established a basis for the present system of locks and dams on the Ohio River.

From July, 1905, to June, 1910, Mr. Prell was Assistant to the Engineer in charge of the design and construction of Lock and Dam No. 37, Ohio River, at Fernbank, Ohio, and from July, 1910, to June, 1913, he was Assistant Engineer in field charge of the completion of that lock and dam, power-house, dwellings, and other features of the reservation. Upon completion of the work at Lock and Dam No. 37, Ohio River, he was assigned as Assistant Engineer in field charge of the construction of Lock and Dam No. 35, Ohio River, which was built under contract. He was engaged on that work during the period from July, 1913, to June, 1916.

In July, 1916, he was transferred to Lock and Dam No. 39, Ohio River, which was under construction by hired labor. He was placed in field charge, and the work was completed under his supervision in December, 1922. In addition to his assignment at Lock and Dam No. 39, Mr. Prell was also the Assistant Engineer in field charge on the construction of Lock and Dam No. 38, Ohio River, during 1921 and 1922.

In February, 1923, he was appointed to the position of Assistant Engineer in charge of the United States Engineer Depot and Repair Yards of the Cincinnati Engineer District, at Fernbank, Ohio. In connection with this project, he completed the construction work on Lock and Dam No. 36, Ohio River, during 1924 and 1925. He was promoted to the position of Senior Engineer on April 4, 1930. The activities of the U. S. Engineer Depot and Repair Yards included the maintenance and repair work on completed Ohio River Locks and Dams Nos. 29 to 39; Kentucky River Locks and Dams Nos. 1 to 14; and the upkeep of steamboats, dredges, and other floating plant of the District. He was engaged on this work at the time of his death.

Through his long and varied experiences on Ohio River lock and dam construction, Mr. Prell acquired special and valuable knowledge of under-water work. He was a useful and respected citizen, a capable engineer, and his passing will be keenly felt by friends and by those who sought his counsel.

Mr. Prell is survived by his widow, Elizabeth Seiter Voris Prell, to whom he was married at Lawrenceburg, Ind., on January 17, 1917, and his step-son, William T. Voris. He is also survived by a sister, Mrs. Emma (Prell) Schroth, and a brother, Frederick Prell.

He was a member of the Masonic Order, McMakin Lodge No. 120, F. and A. M. and Kilwinning Chapter No. 97, R. A. M., at Mt. Healthy, Ohio.

Mr. Prell was elected a Member of the American Society of Civil Engineers on August 31, 1915.

LOUIS E. RITTER, M. Am. Soc. C. E.[1]

DIED JULY 3, 1934

Louis E. Ritter was born on March 14, 1864, at Cleveland, Ohio. After his preparatory schooling he entered the Case School of Applied Science, at Cleveland, from which he was graduated in 1886.

His early experience in engineering was obtained on miscellaneous railroad surveys and construction. This was followed by hydrographic work with the United States Corps of Engineers and the Mississippi River Commission, the rigid training school for so many engineers who later rose to prominence in the profession.

In 1892, Mr. Ritter became associated with the firm of Jennie and Mundie, Architects, of Chicago, Ill., as Structural Engineer. From that time until his death on July 3, 1934, he devoted himself to the problems of building design and construction. The development of the modern skyscraper was taking place during all this period and to Louis E. Ritter, as a pioneer, must be given credit for such development, probably to as great an extent as to any engineer in the United States.

In 1897, he entered private practice as a Consulting Civil Engineer, and, in 1899, became associated with Mr. A. D. Mott under the firm name of Ritter and Mott. Mr. Mott retired in 1917, at which time Mr. Ritter resumed practice under his own name. During the thirty-eight years of his private practice, many of the most important buildings in the United States were designed and constructed under his direction, as Consulting Engineer for practically every prominent architect in the country.

Mr. Ritter possessed a deeply philosophical and analytical mind and was regarded, for many years, as the Dean of the Engineering Profession in the Middle West. He was the unofficial mentor and adviser to the younger men in the profession, who came to him for advice and help on personal as well as on professional matters.

He was a great believer in the necessity of proper organization among engineers to combine for the common good, and was a member of practically all the major professional societies, with which he did yeoman work. The social side of the engineer also received Mr. Ritter's very active encouragement and support. As a prominent member of the Chicago Engineers' Club, his efforts in this direction are lovingly remembered by his many associates.

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[1] Memoir prepared by I. F. Stern, M. Am. Soc. C. E.

He was married in 1899, and is survived by his widow, Mrs. Mary (Stair) Ritter, and by two children, Louis Stair Ritter, and Frances Hamilton Ritter.

Mr. Ritter was elected a Member of the American Society of Civil Engineers on October 4, 1905.

STEPHEN ALLEN ROAKE, M. Am. Soc. C. E.[1]

Died February 2, 1935

Stephen Allen Roake was born in Peekskill, N. Y., on August 10, 1874, and received his formal education there. He was graduated from the Peekskill High School in June, 1891.

Mr. Roake entered the Engineering Profession in 1892 by way of a job as Rodman for the City and County of New York, continuing in that work until 1895. At that time, October, 1895, he was employed as a Draftsman by the Elmira (N. Y.) Bridge Company, at the Southport Works. This plant later became known as the Elmira Plant of the American Bridge Company and was commonly referred to as the "Old South Works" after the construction of the new plant at Elmira Heights. Between 1895 and 1900 he was engaged in various assignments as a Structural Draftsman, in and out of the Elmira Bridge Company, working for longer or shorter periods with the Union Bridge Company, at Athens, Pa., with Levering and Garrigues, of New York, N. Y., the Pittsburgh Bridge Company, at Pittsburgh, Pa., and the Lehigh Valley Railroad Company.

On April 18, 1901, Mr. Roake was transferred to the Athens, N. Y., Plant of the American Bridge Company, and, on June 30 of the same year, he was sent to the Company's Pencoyd (Pa.) Plant. On January 4, 1904, he was again transferred to the Ambridge (Pa.) Drawing Room of the Company, and on June 4, 1906, he was sent to its Trenton (N. J.) Drawing Room, where he became Squad Boss in charge of all movable bridges assigned to that office.

From this last date (June 4, 1906) to May 15, 1915, although he was normally assigned to the Trenton Plant, Mr. Roake spent several short periods in the New York Designing Office, of the Company. He also received leave of absence for a short period to do special work for Boller, Hodge, and Baird, Consulting Engineers, of New York City. On May 15, 1915, he left the American Bridge Company, severing a connection of fifteen years' standing. He had acquired a thorough knowledge of steel construction, a specialized knowledge of movable bridges, and a wide circle of friends in the steel business.

From 1915 to 1917 Mr. Roake worked for the Celluloid Company, of Newark, N. J., as a Concrete Designer. For a short period in 1917 he was Chief Draftsman at the Bergen Point Iron Works, at Bayonne, N. J. In 1917 he entered the employ of the Southern Pacific Company in the office of the late John Dove Isaacs, M. Am. Soc. C. E., then Consulting Engineer for that Company, in New York City. At first Designer, then Chief Draftsman, he later became Assistant Consulting Engineer, continuing in that capacity until 1925. From 1925 to 1928 Mr. Roake was Office Engineer in the General Office in New York City. After 1928, changes in the organization of the Company resulted in his transfer to San Francisco, Calif., where he acted as Chief Designing Engineer for the Suisun Bay Bridge, built by the Southern Pacific Company to replace car ferries which had long been a "bottle-neck" between San Francisco and the north. Upon the completion of the Suisun Bay Bridge in December, 1931, he entered the office of the San Francisco-Oakland Bay Bridge as a Supervising Design Engineer, in which position he remained until ill health caused him to take a leave of absence in August, 1934.

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[1] Memoir prepared by J. R. Kelsey, M. Am. Soc. C. E.

"Steve" Roake was christened Stephen Allen Roake, but his friends neither used the long form of his first name, nor realized that he had a second. His structural experience was invaluable to the organization of which he became a part. It was largely through his efforts that the many varied and unprecedented problems which arose in connection with the 1 400-ft cantilever structure (one of the largest of such structures in the world), between Yerba Buena Island and Oakland, Calif., were solved. Many of the larger joints of this structure are his designs. In the face of increasing physical disability he prosecuted this work until he was able to feel that the essentials of it were behind him. During this time he worked with the knowledge, imparted by previous illness, that overstrain was dangerous to his health. Throughout his life "Steve" had been handicapped by extremely poor eyesight which rigidly limited his pleasures and which made his measure of success the more remarkable. Those who worked with him marveled at the energy with which he overcame his inability to see properly.

He is survived by his widow, Mrs. Katharine Griffes Roake, and two sons and two daughters.

Mr. Roake was elected an Associate Member of the American Society of Civil Engineers on June 6, 1911, and a Member on March 13, 1917. He was also a member of the American Railway Engineering Association.

JAMES WINGATE ROLLINS, Jr., M. Am. Soc. C. E.[1]

Died November 19, 1935

James Wingate Rollins, Jr., was born in West Roxbury, Mass., on October 17, 1858, the son of James Wingate and Sophia Webb (Hutchings) Rollins. Although born in Massachusetts, Mr. Rollins belonged to the well known New Hampshire family of that name, which traces its lineage to the earliest settlers. James Rawlins, the founder of the family in America, immigrated to this country from England in 1632 and first settled in Ipswich, Mass., but moved shortly afterward to Dover, N. H.

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[1] Memoir prepared by Clarence Blakeslee, M. Am. Soc. C. E.

Mr. Rollins was in direct descent from the Hon. Ichabod Rollins who held various public offices in Revolutionary days and was a member of the Convention when, in January, 1776, it resolved itself into an independent State Government. He was the first Judge of Probate under the new Government and held the position until 1784. It was in his generation that the spelling of the name was changed from Rawlins to Rollins. Mr. Rollins' father was graduated from Dartmouth College, at Hanover, N. H., and practiced law in Boston, Mass., for many years. His own education was obtained in the public schools, and upon graduation from the High School at West Roxbury, he entered the Massachusetts Institute of Technology, taking the Civil Engineering Course, and was graduated in the Class of 1878. During his school and college years, he was an enthusiastic baseball and football player.

Following his graduation from college he was employed by the Massachusetts Central Railroad Company on survey and construction work. In 1882, he went to Virginia as Chief Engineer for the Atlantic and Danville Railroad Company, which position he held until 1886. After leaving the service of this Company (later part of the Southern Railway System in Virginia), Mr. Rollins entered that of the Union Pacific Railway Company through his acquaintance with the late Charles Francis Adams, then President of the Company, and was in charge of important construction in Kansas. He then was transferred to the Union Pacific Railway in the Northwest, where he was engaged on very difficult location in the Bitter Root Mountains.

Mr. Rollins was next sent to Wyoming, and started construction of a branch line of the Union Pacific from Fort Steele up the North Platte River toward the North Park of Colorado. When this work was suspended, Mr. Rollins was called into Omaha, Nebr., to settle with the contractors. This being accomplished, after a few weeks he organized a locating party and was sent to Wyoming to continue the location of the line up the North Platte River to the North Park of Colorado—a most difficult piece of engineering.

After the completion of this work, he was sent with his party of engineers to the top of the Continental Divide to re-locate the Denver, South Park, and Pacific Railway, a narrow-gauge branch of the Union Pacific. This work started near Leadville, Colo., and continued down Ten-Mile Creek nearly to Breckenridge, Colo. The party was then taken to Omaha and disbanded.

Shortly afterward, Mr. Rollins spent some time making surveys for a projected railroad in the Adirondack Mountains, and then returned to Boston and entered the service of the Old Colony Railroad Company (later a part of the New York, New Haven and Hartford System). During his employment with the Old Colony Railroad Company he prepared plans for the four-track work from the Back Bay Station, in Boston, out through Forest Hills, and, later, was Resident Engineer on this work when it was started. He remained there until he was transferred to take charge of grade-crossing elimination work at Brockton, Mass.

In 1897, Mr. Rollins left the Old Colony Railroad Company to join the organization of Holbrook, Cabot, and Daly; in 1900, he became a partner and the firm name was changed to Holbrook, Cabot, and Rollins. He was President of this Corporation from 1900 until its dissolution in 1928.

In the early days of his contracting career he developed and patented several unique and highly successful pieces of construction equipment, notable among which was a submarine pile-cutting device for pile foundations, which operated in depths as great as to 40 ft, and also buckets for use in placing concrete under water. He had the ability and foresight to surmount the most unusual construction difficulties. He liked to attempt the most perplexing operations and on numerous occasions would be the only bidder on difficult foundations.

From 1900, the firm of Holbrook, Cabot, and Rollins, under Mr. Rollins as President, successfully constructed many large structures in New England and New York State, among the most important being the West Boston Bridge, over the Charles River; the Charles River Dam and Locks at the lower end of the Charles River; the four-track drawbridge at Fort Point Channel, for the New York, New Haven and Hartford Railroad Company; the Neponset Railroad Bridge; the Lechmere Viaduct; the Boston Fish Pier; the Larz Anderson Bridge, near the Harvard Stadium; the Thames River Railroad Bridge, at New London, Conn.; the Portland-South Portland Bridge, in Portland, Me.; and the Boston Dry Dock, the largest dock of its kind on the Atlantic Coast.

To quote from an article on his career that appeared in 1917:

> "If Mr. Rollins has a hobby it is the surmounting of difficulties by the use of resourceful expedients. He uses keen judgment and has inaugurated many improvements in the work under his supervision.
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> "J. W. Rollins has a slogan: 'Best plant, best work, best results.' An important factor in the success of Mr. Rollins is his principle of profiting by failures or disappointments if they occur, and this fact should be an inspiring example to other contractors, especially the young ones.
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> "The surest road to success is to do the best piece of work possible and then welcome suggestions for possible improvement. This was characteristically demonstrated on recent construction where it was necessary to re-design and rebuild: Mr. Rollins cheerfully explained the original mistakes and their corrections, satisfied in having done as well as possible at the beginning, yet ready to share with others the knowledge acquired by experience. Among his recent improvements is one that will be standard in future construction, a detachable concrete bottom for floating caissons, materially improving the footings of many submerged bridge piers."

From 1912 until the World War years, many important projects were completed by the firm in New York, N. Y., among them Dry Dock No. 4 at the Brooklyn Navy Yard and an important section of the New York Subway System, at Times Square.

Mr. Rollins' activities during the war years included the construction of the wharf areas of the Boston Army Base and the New Orleans (La.), Army Base; and the Squantum Destroyer Plant, at Quincy, Mass. His firm was also one of the many contracting firms engaged at the famous Hog Island Shipyard, near Philadelphia, Pa.

Following the World War came the foundations for the Memorial Bridge across the Piscataqua River, at Portsmouth, N. H., a most difficult undertaking of sinking two pneumatic caissons in 90 ft of water, with a 5-mile per hr tidal current running in alternate directions.[2] Other work followed, including the outfall end of the Passaic Valley Sewer Tunnel, at Bayonne, N. J., which was the last large contract accepted by the Holbrook, Cabot, and Rollins Corporation before its dissolution.

In 1924, Mr. Rollins became associated with C. W. Blakeslee and Sons, of New Haven, Conn., and the Blakeslee Rollins Corporation was founded. He served as Vice-President and Treasurer of this firm until his death. This Corporation did considerable work under his direction, including the construction of water-front properties for the Bethlehem Shipbuilding Corporation; piers for the Mid-Hudson Bridge, at Poughkeepsie, N. Y.; underpinning of the Boston Public Library; and the construction of piers for the highway and railroad bridges over the Cape Cod Canal.

During the construction of the Mid-Hudson Bridge, one of the cassions weighing about 20 000 tons listed to an angle of 45° and was almost wholly submerged in the waters of the Hudson River. The successful righting of this caisson, conducted under Mr. Rollins' direction, is an epic in construction history. Many engineers said that it could not be done and urged that the caisson be destroyed, but after a year's work and tremendous expense, Mr. Rollins proved his contention, and the bridge now stands on this pier which he was urged to abandon.

He tried to do work quickly and well, and to assure his client, whether it was public or private work, of what he termed a "good job." He did this to the end, even though at great personal loss to himself. He earned that confidence which so few contractors ever gain, in that he never failed to complete a project, nor to give the owner the best possible job.

Mr. Rollins was the author of several papers describing his construction activities, one of which, describing construction of the Mid-Hudson Bridge piers, was awarded the Desmond FitzGerald Medal by the Boston Society of Civil Engineers as being the best of the year. His articles on the "Relationship Between Engineers and Contractors" were widely read and contained much homely philosophy on this subject.

Mr. Rollins was a member of the Boston Society of Civil Engineers, to which organization he was elected in 1895. He was at one time President of the University Club of Boston; Vice-President of the Boston City Club; and President of the Boston Society of Civil Engineers, Technology Alumni Association, and Technology Clubs, Associated. He had twice served as a term member of the Corporation of the Massachusetts Institute of Technology. At the time of his death he was also a member of the Engineers Club of Boston, the Milton Club, and the Hoosic Whisick Club. For many years previously he had belonged to the Brookline Country Club, the Cohasset Golf Club, the Algonquin Club, and the Automobile Club, and had been President of the Crow Point Golf Club, in Hingham, Mass., where he had a summer home for twenty-seven years. He was a member of the Board of Managers of the

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[2] "Difficult Foundation Problems for Piscataqua Bridge at Portsmouth, N. H.", by James Wingate Rollins, Jr., M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol LXXXVI (1923), p. 443.

Boston Boys Club, and was also much interested in the civic welfare of his home town, Milton, Mass., where he had served for a great many years as Chairman of the Sewer Commission, and also on the Milton Committee on Employment.

In 1892, Mr. Rollins was married to Clara Boyden Clark, the daughter of the Rev. Nathaniel G. and Elizabeth Sargent Worcester Clark. Mrs. Rollins, their daughter, Elizabeth Sargent Rollins (Mrs. O. H. Saunders), and their son, Wingate Rollins, survive him.

Among Mr. Rollins' outstanding characteristics were his never-failing loyalty, his sense of fair play, his sincerity, and his charity of judgment. He had an inherent belief in the honesty of his fellow men, and was reluctant to believe evil of any one. His devotion to his highest ideals, both in his profession and in his daily living, was unfaltering. His courage was great, and he would fight to the end for a cause he believed to be just. Yet when disappointments came, he always met them bravely, with a gallant spirit that harbored no bitterness. Added to these qualities were his modesty, his natural optimism, his sense of humor, and his sympathetic, genial ways. He was indeed, as one of his friends has said, "not only a builder of works, but a builder of friends", and the same friend also added that,

> "Out of this life of active adventure and strenuous work a character had developed seemingly strangely at variance with the rigorous circumstances of his exacting responsibilities. His innate nature was both delicate and sensitive. His heart responded instinctively to the emotions of pity and sympathy for the distressed and the unfortunate, and he gave fully of his resources to the relief of those in need. He loved the beautiful and admired the finer qualities of others. His spirit remained youthful to the end, and attracted to him a host of friends from every walk of life. It is most significant that those who knew him best loved him most. His passing leaves an aching void in many hearts that will never be completely healed."

Mr. Rollins was elected a Member of the American Society of Civil Engineers on November 7, 1900.

HENRY BEDINGER RUST, M. Am. Soc. C. E.[1]

DIED JANUARY 17, 1936

Henry Bedinger Rust was born on December 13, 1872, at "Rockland", near Leesburg, Va., the family home built by his grandfather, General George Rust, about 1820. He was a son of the late Colonel Armistead T. M. Rust and his wife, Ida Lee, and a descendant of two of the pioneer families of Virginia, both of which had taken an active part in public life.

His grandfather, General Rust, served as a member of the Virginia Legislature from 1819 to 1824, and as Superintendent of the United States

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[1] Memoir prepared by The Koppers Company, Pittsburgh, Pa.

Arsenal at Harpers Ferry from 1829 to 1837. His father was graduated from the United States Military Academy at West Point, N. Y., in 1842, and served at Fort Gibson, Indian Territory, as a Lieutenant in the First Dragoons. He resigned from the Army about three years later and engaged in farming near Leesburg, but, in 1861, he was commissioned a Colonel in the Confederate Army and served throughout the Civil War. Mr. Rust's mother was a member of the Lee family of Virginia, and a cousin of General Robert E. Lee. Two members of this family, Richard Henry Lee and Francis Lightfoot Lee, were signers of the Declaration of Independence.

Mr. Rust was one of a large family which, like most Virginia families, had suffered serious reverses during the long struggle between the North and the South. With money scarce and educational facilities very limited, he was able to obtain but a meager education in the local country schools, although his private study continued throughout his lifetime.

At the age of 18 he secured a position as Axeman on an Engineer Corps in the employ of the City of Pittsburgh, Pa., which was the start of his long and successful engineering career. While on this work his spare time was devoted to the study of engineering subjects, and he used every opportunity to become familiar with the transit and the level. By private study and close application to work he became, successively, Chainman, Rodman, and Transitman, and, at the age of 21, Assistant City Engineer.

At this time the City of Pittsburgh was starting the development of Schenley Park, a rugged, hilly tract of about 600 acres located in the eastern section of the city. Mr. Rust was in charge of this work, which included the location, grading, and paving of roadways, the erection of steel arch bridges with heavy masonry approaches, and the construction of park buildings and sewers. The design and construction of an ornamental masonry bridge with a single arch of 150-ft span, carrying an 80-ft roadway, was perhaps the outstanding feature of this work.

On completion of the work in Schenley Park, Mr. Rust was transferred to the Bureau of Construction, which had charge of designing and building pavements, sewers, and bridges for the City. One of his duties required a careful study of materials and methods with a view to improving the quality of asphalt streets. Following this, he revised the contract forms and prepared complete specifications and standard designs for sewers, pavements, and other requirements of the City. At various times he was in charge of different sections of the city work, such as the construction of sewers, field work in connection with new bridges, paving of streets, and the completion of Bigelow Boulevard. For a time he had charge of the inspection of materials and workmanship on all construction projects.

In 1901, the Colorado Fuel and Iron Company initiated a program for greatly expanding its works at Pueblo, and Mr. Rust was employed as Superintendent of Construction. The existing plant included three small blast furnaces and auxiliary equipment. The new Superintendent of Construction was charged with the responsibility of removing the old blast furnaces, boiler houses, etc., and of building three new blast furnaces, with the necessary boiler and engine houses; also new rail and Bessemer mills. Many of the new buildings replaced existing structures, and the work presented many difficulties. However, Mr. Rust so organized and co-ordinated construction activities that the improvements were carried out without delay to regular operations.

He returned to Pittsburgh early in the summer of 1903 and succeeded his brother, Edwin G. Rust, as Vice-President and General Manager of the Rust Boiler Company, which the latter had organized early that year with the support of Messrs. H. C. and W. C. Fownes, for the manufacture and sale of the Rust Water Tube Boiler, designed and patented by him. Mr. Rust shared the office of the Messrs. Fownes and began his task of promoting the Rust boiler. He first arranged with the Riter-Conley Manufacturing Company to manufacture this boiler, at its Leetsdale Plant, and then built up an engineering, erection, and sales organization. He did most of the selling himself and as the Rust boiler was especially adapted to utilize blast-furnace gas as a fuel, many steel plants purchased large installations.

The Rust boiler was just becoming established in the trade when a competitor instituted suit for infringement of patents. This suit lasted about two years, but, finally, Mr. Rust won the decision. By this time the business had grown sufficiently in volume to require a plant of its own, and, in 1906, the Company purchased a site at Midland, Pa., and built a large modern boiler shop. The organization and volume of business continued to grow and by 1908 was quite a factor in the boiler industry, so much so that its principal competitor started negotiations for the purchase of the Company. The business was sold late that year to the Babcock and Wilcox Company, with which Company Mr. Rust became associated as Special Representative, with headquarters in Pittsburgh.

In 1914, with Messrs. H. W. Croft and Hamilton Stewart, he developed a plan to acquire the H. Koppers Company, an engineering and contracting company, with headquarters in Chicago, Ill. This Company was engaged in designing and building by-product coke, benzol, and tar-distilling plants under patents of H. Koppers, of Essen, Germany. Mr. Rust saw very clearly the great possibilities for the development of this business, and with the co-operation and backing of Messrs. A. W. and R. B. Mellon, it was taken over the latter part of 1914. Mr. Rust was made President and a few months later the offices were moved to Pittsburgh, bringing a total of 82 men from Chicago. It developed rapidly into the outstanding organization in its particular field, which includes designing and building of by-product coke, gas, tar, and other industrial plants, mining and production of bituminous coal, and the manufacturing and merchandising of coke, tar, gas, and other coal by-products. Later, the corporate name was changed to The Koppers Company and, at the end of 1935, employees of the Company and its subsidiary and affiliated interests numbered more than 28 000.

The business grew to be one of the most successful enterprises of Pitts-burgh and one of the great engineering and manufacturing organizations of the United States. The interests of Messrs. Croft and Stewart were acquired by Messrs. C. D. Marshall and H. H. McClintic in 1919. Early in 1929, the Company moved into its new 32-story office building. Mr. Rust continued as President of The Koppers Company until the latter part of 1933, when he was made Chairman of the Board, which position he held at the time of his death.

The by-product coke oven has now practically replaced the old-fashioned bee-hive coke oven, thereby effecting great savings through by-products, such as tar, motor fuel, creosote, benzol, toluol, etc., which were formerly wasted. As many of these products entered directly into the manufacture of explosives and other war materials, the development of this industry under the direction of Mr. Rust and his associates at this particular time was of material help to the United States and the Allies in the World War.

Mr. Rust was a man of vision and great constructive ability. He believed that no industry could succeed except through leadership in its own field, and by leadership he meant not the development of a large organization, but rather the development of the technical aspects of its business. In 1916, he was convinced that The Koppers Company should build and operate plants of its own, and two such plants were built—the Seaboard By-Product Coke Company, at Kearny, N. J., supplying gas to the Public Service Corporation of New Jersey, and the Minnesota By-Product Coke Company, at St. Paul, Minn., supplying gas to the St. Paul Gas Light Company. Similar plants were built later at Montreal, Que., Canada, Boston, Mass., New Haven, Conn., Brooklyn, N. Y., Philadelphia, Pa., Hamilton, Ohio, and Chicago. The coke from these plants was marketed largely for domestic fuel.

In 1920, to assist the Peoples Gas Light and Coke Company to overcome difficulties due to conditions in the gas industry, he organized the Chicago By-Product Coke Company, to finance, build, and operate a large by-product plant in Chicago. This plant was largely instrumental in enabling the Gas Company to meet adequately the rapidly growing demands for its service.

Mr. Rust's progressive spirit was typified by his support of Joseph Becker's design of a new coke oven, since known as the Becker oven. At the time the Koppers oven was at the peak of its popularity, he recognized the merit of the Becker oven and, with his support, this oven was developed.

Soon after the Koppers organization had moved to Pittsburgh, a Fellowship was established at the Mellon Institute of Industrial Research to study coal carbonization and utilization of the by-products. With this Fellowship as a base, a strong research organization was developed, with the necessary laboratory and field forces of men with good training and analytical mental processes. This organization developed processes of marked value, for many of which patents were secured. These processes covered the entire field of liquid purification of gases and resulted in basic patents through which The Koppers Company was recognized as a leader in this field. Fertilizer research and soil and crop production were given careful study, and an experimental farm was placed in operation.

The testing and evaluation of coal for by-product qualities proceeded both at laboratory and plant; methods were developed for making good coke from what had hitherto been regarded as poor coal; studies were made of various coals and their qualities, through which conclusions were reached as to the best coals for a given plant from an economic standpoint. The tar industry was carefully surveyed, new products were introduced, and the manufacture of pitch coke was brought to a high standard of quality. Benzol plants were improved in output and quality of product and, during the World War, a process was developed for the recovery of toluol from water-gas operation that was invaluable to the Nation.

Starting with a small designing and constructing organization, the new business grew, through Mr. Rust's foresight and vision, to a complete and integrated operating unit. In the operation of by-product coke ovens, coke, gas, and various by-products are produced. The coke is marketed for blast furnace use and for domestic heating. To secure the quality of coke needed for these and other specific purposes, certain grades of coal and certain mixtures of high and low volatile coals are required. Mr. Rust realized that to insure an adequate supply of the proper kinds of coal the Company should have its own coal-producing properties. Such properties were acquired from time to time until the Company now ranks second in the United States in the commercial production of bituminous coal.

The gas is sold to distributing companies, but various objectionable substances must first be removed in order to make it suitable for domestic or commercial use. Processes for such purification of gas were developed by The Koppers Company and are now used throughout the gas industry.

Tar obtained in the purification of coke oven gas is refined into many widely used products. The Koppers Products Company purchases the crude tar from Company plants, as well as from many independent plants, refines it into various saleable products and markets them. One of the products is creosote oil, used principally in timber preservation, and another subsidiary, the Wood Preserving Corporation, is the principal user of this creosote oil.

From time to time, as seemed expedient, other businesses were acquired which would fit into the operation as a whole, either as producers of materials required by other units, or as users of products for which a market was needed.

Mr. Rust was outstanding in his character, integrity, and loyalty. He had a remarkable understanding of human nature and was deeply interested in his fellow men and their problems. He gave unstintingly of his time and strength in the interest of any employee or friend who needed help. Perhaps his greatest achievement lay in the confidence which he inspired in others through his ability to understand, to sympathize, and to help.

He was impressed with the importance of placing responsibility on the shoulders of younger men in order to develop them for greater usefulness. In appointing operating heads for subsidiary companies he endeavored to select young men of real capacity to whom practically the entire management of their operations could be entrusted. His relations with employees, and especially with labor, were so eminently fair that practically no labor difficulties arose during the period of his management.

In recognition of Mr. Rust's character and attainments Syracuse University, Syracuse, N. Y., conferred on him the honorary degree of Doctor of Engineering in 1930, and, in 1934, Oglethorpe University, Atlanta, Ga., conferred on him the honorary degree of Doctor of Commercial Science.

In addition to his activities in business and industry, Mr. Rust took a substantial but unassuming part in nearly all movements for the good of his community and his fellow men. He was especially interested in St. Barnabas' Free Home, at Gibsonia, Pa., the National Cathedral, at Washington, D. C., and many other good works. He was a member of Calvary Protestant Episcopal Church, at Pittsburgh, and served on its Vestry for many years. He was also a member of St. James' Church, at Leesburg, Va., and served on its Vestry, as did his father before him. Deeply religious himself, he was a great influence for good, particularly among the younger people whom he met and counted his friends.

Mr. Rust was married to Elizabeth S. Watkins, of New York, N. Y., in 1901. He is survived by his widow and one daughter, Mrs. Stanley N. Brown, of Pittsburgh; three grandsons, Henry B. R., Stanley N., and Fitzhugh Lee Brown; two sisters, Mrs. T. W. Edwards, of Leesburg, Va., and Mrs. John D. Follett, of New York City; and six brothers, Captain Armistead Rust, U. S. Navy (*Retired*), of Boston; George Rust, of Salt Lake City, Utah; William F. Rust, of Leesburg; E. J. Lee Rust, of Birmingham, Ala.; E. Marshall Rust, of Washington, D. C.; and S. Murray Rust, of Pittsburgh.

Services were held at Calvary Church, Pittsburgh, on Sunday, January 19, 1936, Dr. Edwin J. van Etten officiating. Following this service Dr. van Etten accompanied the family to Leesburg, where he again conducted short services at "Rockland", the home of Mr. Rust, and at Union Cemetery. These services were attended by men prominent in the industrial and financial affairs of the country and by many friends and representatives of his business organizations, who came from distant points.

Mr. Rust was elected an Associate Member of the American Society of Civil Engineers on April 6, 1898, and a Member on April 1, 1903.

CHARLES SCHULTZE SAMPLE, M. Am. Soc. C. E.[1]

DIED SEPTEMBER 25, 1935

Charles Schultze Sample, the son of John and Maria (Kraft) Sample, was born at Philo, Ill., on June 1, 1873. His family having moved to Indiana during his boyhood, it was natural that when he determined upon an engineering career, he should choose Purdue University, at Lafayette, Ind. At Purdue he became one of the now famous "Dorm Devils", the membership of which included so many of the University's noted sons. Solving the engineering problems connected with the famous "tank scraps" were not the only difficulties encountered at Purdue, for young Sample played tackle and halfback on the University football team. Those were the days of the flying wedge, mutton chop whiskers, and Mike and Katy Golden, a brother and a sister who came from Ireland to Purdue and whose interest in athletics, particularly football, was outstanding.

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[1] Memoir prepared by a Committee of the St. Louis Section, consisting of Baxter L. Brown and W. J. Burton, Members, Am. Soc. C. E.

After his graduation from the School of Civil Engineering in 1899, he served for two years as Rodman and Transitman on construction for the Chicago and Northwestern Railway Company.

In 1901, Mr. Sample was employed by the St. Louis Valley Railway Company on the location of its lines from East St. Louis to Thebes, Ill., and, later, as Resident Engineer on the construction of the road. On this work he demonstrated his ability to such extent that upon the purchase of the railway, in 1902, by the Missouri Pacific Railway Company, he was retained by that Company and upon the completion of the work, was transferred to the construction of its Gurdon and Fort Smith Branch, in the semi-mountainous district of Western Arkansas. Later, he was in charge of important terminal construction at Hoisington, Kans., and, in 1913, was placed in charge of the location and construction of the 42-mile line in Eastern Arkansas, known as the Marianna Cut-Off. The construction of this line was especially difficult because of high water in the Mississippi River, and, at one time, before the line was finally completed, it was possible to make a trip over the right of way by steamboat.

For brief intervals in 1908 and 1915, Mr. Sample left the Missouri Pacific Railway Company to engage in railroad construction for the Dalhoff Construction Company and for the Garden City Sugar and Land Company, the latter work being in Western Kansas. Following the latter engagement, Mr. Sample returned to the Missouri Pacific to become Chief Pilot Engineer in connection with the inventory and valuation of the lines being made by the Interstate Commerce Commission.

When an extensive program of improvement work was undertaken by the Missouri Pacific Railway Company in 1921, Mr. Sample was made Construction Engineer. His experience in dealing with contractors, his characteristic ability to secure the loyal support of subordinates, his common sense, good judgment, and fairness, all made him peculiarly fitted for this responsibility. Among the larger projects under his direction as Construction Engineer were the $18 000 000 double-track project between St. Louis and Jefferson City, in Missouri, and the heavy line and grade revision work between Kansas City, Mo., and Pueblo, Colo., which cost more than $5 000 000.

In 1902, Mr. Sample was married to Helen Brewster, of Corydon, Ind. Their two sons, Dr. Charles S. and Robert Brewster, and their daughter, Mrs. Mary Baxter Conrad, survive him. Mrs. Sample died in 1923. In 1926, Mr. Sample was married to Alice Alter, of Kirkwood, Mo., the St. Louis suburb where he made his home, who also survives him. He died on September 25, 1935, and was buried in Oak Hill Cemetery, at Kirkwood.

Mr. Sample was a member of the American Railway Engineering Association and the St. Louis Railway Club. He served as Alderman of Kirkwood for five years, and as a member of the Board of Public Works. He was a Mason (Chester, Ill.), and an active member of the First Presbyterian Church of Kirkwood.

He will be remembered by his many friends and associates for his thoughtful and kindly concern for all with whom he came in contact, and his mindfulness of the teachings found in the Sixth Chapter of the Gospel according to St. Matthew.

Mr. Sample was elected a Member of the American Society of Civil Engineers on December 22, 1930.

WILLIAM LUTHER SIBERT, M. Am. Soc. C. E.[1]

Died October 16, 1935

William Luther Sibert was born at Gadsden, Ala., on October 12, 1860, the son of William J. and Marietta (Ward) Sibert, and was the descendant of three generations of pioneers. His great-grandfather was a South Carolina Revolutionary soldier; his grandfather moved from South Carolina to Alabama with the typical covered wagon of pioneer days; his father was a Confederate soldier; all sturdy stock who obtained their livelihood primarily from the soil.

Young Sibert's boyhood was spent on the farm, and he became accustomed early to physical labor. His education in the country schools was meager, as there were no high school facilities in Gadsden at that time. However, as the advantages of a higher education were desired for the boy, a tutor was engaged, and William Sibert entered the University of Alabama, at University, Ala., in 1878.

After two years at the University of Alabama, he entered the United States Military Academy, at West Point, N. Y., from Alabama, on July 11, 1880. As a Cadet he was studious, reserved, tolerant, and loved by all; he became known affectionately as "Goliah" due to his splendid physique and the fact that he roomed with the late Colonel David Gaillard, U. S. Army (*Retired*): "David and Goliath living together." He was graduated on June 15, 1884, standing seventh in a class of thirty-seven members, and was commissioned a Second Lieutenant in the Corps of Engineers.

On the completion of his graduation leave, in September, 1884, Lieutenant Sibert reported at Willets Point, N. Y., for duty with the Battalion of Engineers, and to pursue the course in the Engineer School of Application from which he was graduated in 1887. He left Willets Point on July 6 of that year, and from this time until the Spanish-American War, was engaged almost exclusively on river and harbor work.

After a period of duty as an Assistant to the Officer in Charge of the District at Louisville, Ky., from July, 1887, to April, 1888, he served in a similar capacity at Cincinnati, Ohio.

On April 7, 1888, he was promoted to the grade of First Lieutenant. The following December, Lieutenant Sibert was placed in local charge of improvement works on the Green and Barren Rivers, near Bowling Green, Ky., where he remained until August, 1892. The next two years were spent at Detroit, Mich., where the "Soo" Locks and channels connecting the Great Lakes were under construction. He was placed in local charge of the construction of the ship channel from the foot of Lake Huron to the head of Lake Erie. In August, 1894, he was transferred to Arkansas and placed in charge of the River and Harbor District at Little Rock, one of the first officers to be placed in charge of a district while in the grade of First Lieutenant. His responsibilities in this position included all improvements on the Arkansas, White, Black, Current, and St. Francis Rivers, in Arkansas, Missouri, and Indian Territory. During this period of duty he was promoted to the grade of Captain on March 31, 1896.

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[1] Memoir prepared under the direction of Major General E. M. Markham, Chf. of Engrs., U. S. Army, M. Am. Soc. C. E.

In September, 1898, the Army took advantage of the wide experience in practical engineering which had been afforded Captain Sibert in the preceding ten years, and he returned to Willets Point, as an Instructor in Civil Engineering in the School of Application; he was also placed in command of Company B of the Battalion of Engineers. Shortly after the declaration of war against Spain, Captain Sibert officially requested that he be assigned duty at the front with the field forces, but it was not until July 5, 1899, that he left the station at Willets Point for field service. On this date he left the post in command of Company B, Battalion of Engineers, en route to the Philippine Islands for operations against the insurgents.

Much was crowded into the brief period between his departure from Willets Point and May 31, 1900, when he returned to the United States. During this period Captain Sibert participated in several minor actions in the field operations in Luzon and distinguished himself as an engineer soldier, being highly commended by his superior officers not only for his professional work, but for his actions in the firing line. He served in both the First and Second Divisions of the Eighth Army Corps, and from September 2, to November 25, 1899, was Chief Engineer of the Department of the Pacific, in addition to commanding the Engineer Battalion. He was also for some time Chief Engineer of the Eighth Army Corps. He accompanied General Schwan's Brigade against the insurgents in Northern Luzon, from October 7 to 14, 1899; and in his campaign through Cavite, Laguna, Batangas, and Tabayas Provinces, in Southern Luzon, from January 2, to February 5, 1900. In regard to his work, General Schwan in his official report on the latter operations stated: "I cannot commend too highly the work of this officer; his constancy, his tact, his professional skill are worthy of special recognition." At a later date, this same officer in a letter to the Adjutant General, U. S. Army, wrote:

> "I know of no individual officer who contributed more to the success of the movement than did Captain William L. Sibert, Corps of Engineers. Not only as an engineer officer, in the surmounting, removal, and avoidance of road obstacles, were his services brought into requisition: His good work extended to and had the effect of facilitating and expediting every operation that was undertaken in the course of the campaign. To me he proved, as he did on a former expedition, a safe and most valuable prop. I cannot sufficiently empha-

> size my appreciation of the services of this accomplished, discreet, and withal modest officer.
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> "Though my acquaintance with the officers of the Army is quite extensive, I know of none who possesses the qualities requisite in the command of a volunteer regiment in a higher degree than Captain Sibert, or who, by reason of his past service, better merits the appointment of colonel of volunteers. It is hoped that the War Department may see its way clear in the future to recognize in some marked way the excellent work he has rendered in the field and with troops."

From February 10 to April 20, 1900, Captain Sibert was Chief Engineer and General Manager of the Manila and Dagupan Railway, during which period he reconstructed, re-organized, and operated the line until it was transferred back to its owners. He sailed on the transport, *Meade,* for the United States on May 5, 1900.

Following his return to the United States, Captain Sibert was assigned as District Engineer, at Louisville, Ky., and took over his duties in July, 1900. For the next seven years he was to be actively engaged on engineering works. In December, 1901, he was transferred to Pittsburgh, Pa., to become District Engineer of that important District, in which capacity he served until March, 1907. He was largely responsible for the adoption and development of the 9-ft channel on the Ohio River. The proof of his outstanding engineering ability and high character are amply borne out by letters from residents of the District in which he served written to officials of the Government and which are on file in the War Department. While in charge of this District, he was promoted to the grade of Major on April 23, 1904.

During a severe flood of the Allegheny River in January, 1907, Major Sibert's prompt and energetic action in demolishing part of Dam No. 3, at Springdale, Pa., was undoubtedly the means of saving considerable property. The following letter from the President of the Heidenkamp Mirror Company to the Engineer Department is of interest in this connection:

> "We desire to express our appreciation of the prompt and strenuous measures used by your department to prevent damage to property by the recent washout around Dam No. 3 in the Allegheny River at Springdale, Pennsylvania. The river had washed out around the north end of the dam and for two days it continued to carve out a new channel, the force of the water steadily growing in power until it seemed as if all property below the dam, between the river and the railroad, for the distance of a mile, must be swept into the river. It also seemed as if the railroad communication between here and the city would be officially cut off.
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> "We cannot too strongly commend the energy and zeal of Major William L. Sibert during this crisis. His recommendation that the top of the dam be blown off with dynamite in order to divert the current, was most wise and resulted in saving a large amount of valuable property. Our factory seemed to be right in the river's path, and while we have lost considerable land and several of our smaller buildings, yet our main buildings are intact and our factory has continued in operation.
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> "We feel that words of highest commendation are due to these men who have labored so earnestly and so wisely to keep the river within its proper bounds."

The Chief of Engineers, in referring to this commendation in a communication to the War Department, declared:

> "The Chief of Engineers wishes to add that this commendation of Major Sibert is well deserved, and that in the entire conduct of this difficult and dangerous matter Major Sibert has shown ability and judgment of a high order."

In March, 1907, Major Sibert, selected as one of the three Army engineers to build the Panama Canal, was appointed a member of the Isthmian Canal Commission. This assignment can be considered as a most definite proof of his ability and integrity. He served as a member of this Commission until April 1, 1914, a period of seven years. On September 21, 1909, he was promoted to the grade of Lieutenant Colonel.

As Division Engineer of the Atlantic Division of the Panama Canal he was directly in charge of the construction of the Colon Breakwater, the channel excavation from Gatun to the Atlantic Ocean, the Gatun Dam, and the Gatun Locks. He had a fanatical zeal for his work, appearing daily on the job, and instilled a tremendous spirit of competition in the force operating under his supervision. He took an active interest in the life and welfare of his subordinates of all classes. It is a tribute to Colonel Sibert's ability as an engineer that the Atlantic Sector was the first part completed, particularly as it had been estimated by the Chief Engineer that the construction of the Gatun Locks and Dam would require two years longer than the remainder of the Canal.

In 1914, as the result of a joint resolution of Congress, the Chief of Engineers was called upon to recommend an officer of the Corps of Engineers for appointment as Chairman of a Board of Engineers to assist the Republic of China in making an examination and report on the reclamation of the Huai River for the prevention of floods and resultant famines. With due appreciation of the responsibility of such an assignment, the great difficulties involved in the conduct of the work to be undertaken, and the necessity of appointing an officer of national repute, Colonel Sibert was recommended. His appointment was highly satisfactory to the American Red Cross which was sponsoring and assisting in the work; and he was granted leave of absence to enable him to accept the appointment. This work engaged his attention from June 11 to October 15, 1914, although the outbreak of the World War disrupted the Red Cross plan.

Following his return from China, Colonel Sibert assumed the duties of Division Engineer, Central River and Harbor Division, and District Engineer, 2d Cincinnati District. He was again active in the solution of the problems of the Ohio River and became a member of a Board on the construction or modification of locks and dams on that river, and also a member of a Board on flood conditions in the valley of the Ohio River and in the drainage area of Lake Erie.

On March 4, 1915, Colonel Sibert received the thanks of Congress for having "rendered distinguished service in constructing the Panama Canal", an honor of which he was justly proud. In addition, he was at the same time rewarded by being promoted to the rank of Brigadier-General of the Line. He was then ordered to the Pacific Coast where he assumed command of the Pacific Coast Artillery District, on March 4, 1915. He remained on this duty until shortly after May 15, 1917, when he was promoted to the rank of Major General.

General Sibert's first duties following his relief from assignment on the Pacific Coast were in Washington, D. C., with what later became the Headquarters of the American Expeditionary Force. On June 8, 1917, he assumed command of the First Division and shortly afterward sailed for France with the first American troop units to enter the theater of active operations. He remained in command of the Division through the arduous preliminary training period and brought it to a high state of effectiveness.

He participated in action with the Division in the Luneville Sector, October 21 to November 20, 1917. In the words of his Chief of Staff, Major General Hanson Ely:

> "His sterling character was the admiration of all his subordinates. He commanded the division most efficiently, without fear, favor, or affection; while working it to the limit under the circumstances. He tempered justice with mercy and showed such discriminating consideration for his subordinates that all were sorry to see him go. Those who knew his excellent work under the many handicaps of the early training were convinced that the division suffered a great loss in his being called to what may have been considered more important work."

On December 14, 1917, General Sibert released command of the Division to General Bullard and returned to the United States to assume command of the South Eastern Department, with headquarters at Charleston, S. C. He was relieved from this duty on May 16, 1918, and brought to Washington to organize what became the Chemical Warfare Service. This was an entirely new service in the Army, and it is due largely to the energy and administrative ability of General Sibert that an able and efficient service was brought into being. In recognition of this accomplishment he was awarded the Distinguished Service Medal with the citation: "For services in the organization and administration of the Chemical Warfare Service, contributory to the successful prosecution of the war."

On March 1, 1920, he left the Chemical Warfare Service and was assigned command of the Fifth Division and Camp Gordon, Georgia. He retired from active service upon his own request on April 4, 1920, under the provisions of the Panama Canal Act.

Following retirement from active service, General Sibert was intimately connected with and engaged in engineering work until 1932. His native State of Alabama turned to him, as an able engineer and administrator, to act as Chief Engineer and General Manager of the Alabama State Docks Commission which, under a $10 000 000 appropriation, constructed an ocean terminal at Mobile. Consequently, the City of Mobile can now boast of one of the most modern and well-equipped ports of the South. In 1928, he was also appointed Chairman of the Boulder Dam Commission, a Commission of Engineers and Geologists appointed under an Act of Congress to investigate and report on the Boulder Dam project. From 1929 to 1930, he was President of the American Association of Port Authorities.

In 1932, General Sibert retired from active engineering and spent the final years of his life on his farm near Bowling Green, Ky., where he died on October 16, 1935. He was buried with full military honors in Arlington National Cemetery, on October 18, 1935.

General Sibert's active life was devoted to public service, not only during his thirty-six years in the Army Corps of Engineers, but also for twelve years after his retirement. During his life he was many times singularly honored —the thanks of Congress; the Distinguished Service Medal; a Commander in the Legion of Honor of France; the degree of Doctor of Laws from the University of Alabama (1919); the degree of Doctor of Engineering from the University of Nebraska (1919); the many important positions of trust—all bear witness to the worth of his service and the regard of those for whom he devoted his life, his fellow men. Like all men who have done great things, General Sibert knew men and how to use them. Particularly did he understand young men and how best to develop them. The heritage which he left in the hearts and minds of the many young men who served under him and were taught by him is being transmitted by them to other young men and will be continued as long as the Corps of Engineers endures. He was loyal and zealous in his work, efficient without being harsh, loved and respected by all.

He was married to Mary Margaret Cummings, of Brownsville, Tex., in September, 1887, Juliette Roberts, in 1917, and Evelyn C. Bairnsfather, of Edinburgh, Scotland, in June, 1922. He was survived at the time of his death by his widow, Evelyn C. Sibert, and by William, Franklin, Harold, Edwin, Martin, and Mary (Mrs. E. S. Smith) Sibert, children of his first wife.

General Sibert was elected a Member of the American Society of Civil Engineers on June 2, 1897.

EUGENE ADALBERT SILAGI, M. Am. Soc. C. E.[1]

DIED APRIL 5, 1935

Eugene Adalbert Silagi was born at Rozsno, Hungary, on August 24, 1876. He was educated at the Joseph Royal Technical University, at Budapest, from which he was graduated in 1898 with the degree of Civil Engineer. For a period of fourteen years following his graduation, he served as Lieutenant and First Lieutenant in the Hungarian Army, teaching mathematics, surveying, and mapping in the Military Academy at Pecs, Hungary, during the last two years of this service.

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[1] Memoir prepared by the late E. G. Bradbury, M. Am. Soc. C. E.

From 1902 to 1904, Mr. Silagi was engaged as Supervising Engineer on the construction of reinforced concrete stations and roundhouses for the Trans-Siberian Railways, at Tamsk (or Omsk), Siberia. This was followed by a two-year engagement as Designing Engineer for concrete structures of the Chemin de Fer d'Algerique, at Oran and Saida, Alger, North Africa.

In 1907, Mr. Silagi came to the United States and was employed by the Trussed Concrete Steel Company, at Detroit, Mich., his work including the construction of numerous warehouses, bridges, tall buildings, and factories. This connection was continued until 1913, when he became Chief Engineer for the United Engineers, Limited, of Singapore, Straits Settlements, and, for two years, he was engaged in the design and construction of sea walls, retaining walls, wharves, heavy foundations, bridges, and buildings.

During 1915, Mr. Silagi made an investigation for the Trussed Concrete Steel Company of its agencies in Manila, Philippine Islands, Shanghai, China, and Honolulu, Hawaii, making reports and recommendations as to future policies. Late in the same year he returned to America, and became Manager of the Columbus, Ohio, Office of the Truscon Steel Company, directing designs, estimates, and sales, and personally designing a number of warehouses, schools, hotels, and factories. In 1919, while still managing the Truscon Company's Office, he acted as Consulting Engineer in connection with the design of a large refinery for the Ohio Cities' Gas Company, and a number of other projects.

His connection with the Truscon Steel Company was continued until Mr. Silagi's death. In 1920, he left Columbus and until 1922 his headquarters were at the Company's New York Office, but he traveled extensively, his duties taking him to Calcutta and Rangoon, in India, Singapore, Batavia, Dutch East Indies, and Shanghai; from 1923 to 1927, he was the Company's General Manager at Shanghai; in 1928, he went to Manila, as its representative, associated with the Pacific Commercial Company, and remained in this capacity until 1933, when he became Sales Manager for the China United Engineering Corporation, Truscon Agents at Shanghai, holding this position for the remainder of his life. In all these positions he was engaged largely in the design and construction of important buildings and engineering works.

Mr. Silagi was a remarkable linguist, speaking nine languages, namely, English, Hungarian, German, French, Italian, Austrian, Chinese, Japanese, and Hindustani. During the World War he rendered valuable service to the Columbus, Ohio, Chamber of Commerce as interpreter and by addressing, in their own languages, gatherings of the foreign population, ironing out factional differences and possibly averting serious trouble. He also organized, and acted as Captain and Drillmaster of Company A, Columbus Reserve Guard, which was made up of members of the Engineers' Club of Columbus, of which he was an active and valued member. He was a member of the Masonic Order and of the Benevolent and Protective Order of Elks.

He was possessed of a very pleasing personality and of a frank, loyal, and generous character, and was highly regarded by those with whom he had business or social relations. He was an enthusiastic lover of horses and dogs, and sometimes rode his own horses in races in the Far East. He was also fond of hunting.

Mr. Silagi was elected an Associate Member of the American Society of Civil Engineers on June 11, 1917, and a Member on March 9, 1920.

HENRY CLEMENT SMITH, M. Am. Soc. C. E.[1]

DIED JANUARY 11, 1936

Henry Clement Smith was born in Philadelphia, Pa., on February 2, 1860, the elder son of A. Lewis Smith and Rebecca Levis (Wood) Smith. He obtained his college preparation at the Friends Central School, in Philadelphia, and was graduated from the University of Pennsylvania as a Civil Engineer in 1881.

His early engagements were with the Pennsylvania Railroad Company. the Pennsylvania, Schuylkill Valley Railroad Company, and the Philadelphia, Germantown and Chestnut Hill Railroad Company. He served also with the Department of Surveys of Philadelphia, in the capacities of Draftsman, Rodman, and Instrumentman.

In 1885, Mr. Smith was with the Little Rock and Fort Smith Railroad Company, at Little Rock, Ark., as Chief Clerk in the General Manager's Office. He was obliged to resign, however, and return home in the course of a year as the result of a severe illness. Throughout the remainder of his life, he remained in the Philadelphia District.

During the following two years he was in the employ of the Pennsylvania Railroad Company as an Assistant Engineer in charge of shop and warehouse construction, track construction, and a preliminary survey of the Maurice River Railroad, in New Jersey. Mr. Smith then went to the old Philadelphia, Wilmington, and Baltimore Railway Company as an Assistant Engineer. Later, he became the Division Engineer of its Central Division.

He remained with the Pennsylvania Railroad System, until about 1905, when he resigned to become Division Engineer of the Philadelphia Division of the Philadelphia and Reading Railroad, with offices in Philadelphia. During his long service with the Reading System, Mr. Smith, in addition to discharging the onerous duties of a Division Engineer, was often consulted by his associates on matters concerning improvements or the design and erection of important structures needful in the expansion of a great railroad system. In the latter part of his life, he at times shouldered responsibilities incident to the uninterrupted operation of his Division, which severely taxed his strength and undoubtedly enfeebled his retiring years.

In his bearing, Mr. Smith was quiet and dignified, but firm in the discharge of his duties. He was conscientious to a marked degree, well informed in his profession, and had the respect and good will of his men. He was a member of the Engineers Club of Philadelphia.

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[1] Memoir prepared by William Easby, Jr., M. Am. Soc. C. E.

He was married on October 9, 1889, to Anna P. Cheyney, of Media, Pa., and is survived by his widow and a son, Lewis Cheyney Smith.

Mr. Smith was elected a Member of the American Society of Civil Engineers on October 2, 1901.

JONATHAN PARKER SNOW, M. Am. Soc. C. E.[1]

Died September 4, 1933

Jonathan Parker Snow was born in Concord, N. H., on November 19, 1848, the son of Jonathan and Lydia Ann (Parker) Snow. His paternal ancestors were sea-faring men, of Cape Cod, Mass., and his maternal ancestors, English immigrants prominent in the early settlement of the Massachusetts Bay Colony. His early education was obtained at a private academy at Contoocook, N. H., and he was graduated from the Thayer School of Civil Engineering, Dartmouth College, at Hanover, N. H., in 1875, with the degree of Civil Engineer.

At that time schools or colleges for teaching Civil Engineering were few, and most of them were privately endowed. State universities of repute could be counted on the fingers of one hand; high schools were rare; engineering textbooks were few and inadequate; recourse was had to English and French treatises; and, consequently, American practice conformed in many respects to that abroad. In spite of handicaps, however, by zealous application Mr. Snow obtained a thorough knowledge of the science of engineering, and a retentive memory enabled him to make full use of his education.

At the time of his graduation, Mr. Snow was in his twenty-seventh year, but more mature than is usual at that age. His first employment was as Principal Assistant to the City Engineer of Fitchburg, Mass.; and then, on the Manchester and Keene Railroad, in New Hampshire, as Transitman and as Assistant Engineer in charge of location, and as Division Engineer in charge of construction. He also fulfilled some engagements in North Carolina and Pennsylvania. Recalled to the Thayer School as an Instructor during one year, he conducted field work and prepared useful notes on carpentry, explosives, rock-drills, and quarrying, which served temporarily until suitable textbooks appeared. His next engagement was as a Surveyor on map work at Prince Edward Island. Entering the Boston Bridge Works, in November, 1879, he served for four and one-half years as First Assistant to the late David Herbert Andrews, M. Am. Soc. C. E., the Proprietor, in charge of designs and estimates, at a time when it was customary for railroad companies, municipalities, and architects to depend upon contractors for engineering designs. In 1884, he entered the office of the late John Waldo Ellis, M. Am. Soc. C. E., at

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[1] Memoir prepared by the late Robert Fletcher, M. Am. Soc. C. E., and J. R. Worcester, M. Am. Soc. C. E.

Woonsocket, R. I. In addition to a large private practice, Mr. Ellis served as Engineer of the Providence and Worcester Railroad Company, and Mr. Snow had charge of the Bridge Department for four years. In July, 1888, he was apointed Bridge Engineer of the Boston and Maine Railroad Company, which office he held until, twenty-one years later, he was promoted to be Chief Engineer. After two years as Chief Engineer, a re-organization brought in a controlling influence of the New York, New Haven, and Hartford Railroad Company, and he was released from the office. Thereafter, he carried on a consulting practice, with an office in Boston.

In this connection with the railroads, Mr. Snow devoted himself to the service with whole-hearted zeal and the knowledge gained by years of previous experience. His notebooks indicated the condition of every bridge on main lines and branches, not only in general, but by computed strength of each member. On occasions, he conducted small groups of students on a tour for bridge inspection. The bridges with which he had to deal included practically all types of construction, among the most unique of which were several "jack-knife" draw-bridges, such as those over the Charles River, in front of the Boston Terminal. In this construction the track stringers were hinged at the fixed ends, while the outer ends were suspended from a rigid frame. In operation, the stringers while remaining parallel to each other, were swung around by a side-haul so as to shut together like a jack-knife, leaving a clear passageway.

The majority of the bridges for which he was responsible were built with timber, chiefly the lattice and Howe truss types, with or without timber arches. Mr. Snow advocated conserving these bridges as long as they could be made assuredly safe under the loads then allowed. He held strongly to the conviction that, in the development of types of timber trusses and combination trusses, the merits of the Towne lattice with built-in timber arch were not fully appreciated. He pointed to the facts that they required no elaborate framing, used only timber in less bulky shapes, were more easily erected, had greater lateral stability, and that experience of thirty years in designing and maintaining such structures had shown that they continued to carry—without over-strain—loads far heavier than those for which they were designed. In a paper entitled "Wooden Bridge Construction on the Boston and Maine Railroad",[2] he described such a truss with drawings in detail and convincing arguments.

It was a time of experimentation with various types of wrought-iron trusses or those in which timber and wrought iron were combined. Steel had not then been adopted in bridge construction. The Bollman, Fink, and Post trusses, etc., bore the names of their inventors, while others were named from the bridge shops which produced them. Those were the days of controversy between American engineers who designed pin-connected trusses to gain the utmost economy of material, and English engineers who preferred riveted types with less concern for the greater weight of metal used. Mr. Snow's preference was for the riveted type, particularly to gain more rigidity, and in this preference he was forestalling the trend of more recent times.

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[2] *Journal*, Assoc. of Eng. Societies, Vol. XV, 1895, p. 31.

Mr. Snow was a faithful member of the Society and contributed liberally to its publications. Previous to 1895, he had furnished discussions on subjects, such as the life of iron bridges, railway bridge design, thin floors for bridges, tests of bridge members, wind-bracing on high buildings, etc. His final contribution to the *Transactions* was as co-author of a paper entitled "A History of the Development of Wooden Bridges"[3], prepared during the years 1931–33. Of this paper a distinguished member of the Society wrote: "It is not only a history of wooden bridges; it is practically a treatise on the design of wooden bridges of the more recent types." Its authoritative and practical value as a treatise is due to a large extent to Mr. Snow. He died within two weeks after the issue of the complete reprint.

Mr. Snow was a member of the American Railway Engineering Association from the time of its organization, and, for many years, a regular attendant at its Annual Conventions, where he took part in the discussions. He was Chairman of the Committee on Iron and Steel Structures during the time that this Committee was formulating its first Specification for Steel Railway Bridges. He was adverse to the law under which the Interstate Commerce Commission made the valuation of the railroads, claiming that the cost of rails, roadbed, buildings, and rolling stock had little significance if the financial return was inadequate; and that the proper basis is revenue received. He was a member of the Special Committee of the Society which made the Final Report to Formulate Principles and Methods for the Valuation of Railroad and Other Public Utilities.[4]

He was also a member of the American Association of Superintendents of Bridges and Buildings, the American Society for Testing Materials, the Boston Society of Civil Engineers, and the Thayer Society of Engineers, as well as the following clubs: New England Railway, Engineers, and New Hampshire Club, of Somerville, Mass.

In 1880, Mr. Snow was married, in Boston, to Mrs. Marietta B. Eaton, of Wilton, N. H., who died in June, 1928. In June, 1929, he was married to Mrs. Delia Hardy Osgood, who died in September, 1932. During the later years his sister, Josephine E. Snow, kept house for him at his home in Somerville, Mass. He was a Republican in politics, and a Universalist in faith.

Mr. Snow was an original thinker, not bound by precedent; he could visualize the result of his work, and his object was to attain the result without wasting time or expense in the process. While he had a proper respect for beauty, he did not consider it to be his job to produce it; he seemed to feel that in engineering structures an evident suitability for the use was all that appearance demanded. His practical experience with shop and field work made him ever watchful of the practicability of constructing and erecting whatever he designed. His open-mindedness is evidenced by a brief talk to members of a graduating class, in which he cautioned them to "restrain their knowledge until it was needed," and assured them that they would find much to be learned from the workmen on the job. Also, that the safe policy was to say little and, in the meantime, to make full use of their eyes and ears.

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[3] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 314.

[4] *Loc. cit.*, Vol. LXXXI (1917), p. 1311.

In his contacts with men he was blunt, frank, and outspoken, taking a seeming pleasure in the use of colloquial expressions, but he was never discourteous. He was always open to argument and free to listen to his opponent. As a result, he was respected and beloved by those who were privileged to have dealings with him. He was too modest to set a proper value on his own opinions, and gave freely of advice that to others was priceless.

The impression which Mr. Snow's personality made on his associates is happily expressed by one of his colleagues, a prominent member of the Society, who writes as follows:

> "We were all very fond of Mr. Snow, and had great respect for his ability as an engineer and his character as a man. I feel that * * * his sound judgment, common sense, and practical attitude toward all * * * problems made him one of the most valuable counsellors * * *.

Mr. Snow was elected a Member of the American Society of Civil Engineers on June 3, 1885. He served as a Director of the Society from 1911 to 1913.

EARL EUGENE SPERRY, M. Am. Soc. C. E.[1]

Died August 17, 1935

Earl Eugene Sperry, the son of John Eugene and Julia (Paul) Sperry, was born in West Monroe, N. Y., on May 27, 1883. His preliminary education was obtained at Manlius Military Academy, at Manlius, N. Y., from which he was graduated in 1900 at the head of his Class. In 1904, he was graduated from the Rensselaer Polytechnic Institute, at Troy, N. Y., with the degree of Civil Engineer. During his course at college he demonstrated that exceptional ability which proved such an asset during his business career and, on graduation, received the much coveted insignia of Sigma Xi, representative of the highest type of scholarship. He was a member of the Delta Kappa Epsilon Fraternity.

Following his graduation from Rensselaer, Mr. Sperry, in July, 1904, entered the Engineering Department of the Solvay Process Company, in Syracuse, N. Y. In 1907, he was promoted to the position of Assistant Civil Engineer and, in 1921, he was given the position of Chief Civil Engineer which he held until his resignation in 1924.

Much could be recorded regarding his varied and intricate engineering work during his years with the Solvay Company. Among his activities were included the design, construction, and operation of immense quarry and rock-crushing plants, water and sewerage systems, brine wells, steam power plants, water, air, and brine pipe lines, and plant railroads for the Company's plants at Syracuse, Detroit, Mich., and Kansas City, Mo. His most notable professional work consisted of a unique development of deep-well pumping which revolutionized the methods then in use.

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[1] Memoir prepared by William P. Creager, M. Am. Soc. C. E.

In 1924, he entered into private business as a Retail Coal Merchant and a Concrete Products Manufacturer, with an office and works at Oneida, N. Y., which business he conducted until his death on August 17, 1935.

Mr. Sperry endeared himself to all with whom he came in contact, not only because of his engaging personality, but also for his exceptionally high standard of sportsmanship. His business associates admired him not only for his great ability as an engineer, but for his friendly spirit and wonderful cheerfulness and fortitude under extreme stress. Had he continued within the technical branch of his profession, he undoubtedly would have attained the highest pinnacle in the field of industrial engineering.

During his late years, he successfully hid from even his closest associates the continued agony of the heart affliction which not only handicapped him in every way, but ultimately was the cause of his death. He was a member of the Congregational Church, of Munnsville, N. Y.

He was married to Elizabeth Burns, of Troy, N. Y., on October 3, 1904, who, with his daughter, Margarete Elaine, one brother, and two sisters, survives him.

Mr. Sperry was elected a Member of the American Society of Civil Engineers on December 15, 1924.

OSCAR VAN PELT STOUT, M. Am. Soc. C. E.[1]

Died August 4, 1935

An awkward, overgrown country lad once enrolled in the Civil Engineering Department of the University of Nebraska, at Lincoln, Nebr. Early in his second year he presented himself at the Dean's office one day after class. A year at college had added but little to his urbanity. His movements were still gawky, his hands and feet were too large, and he could not find any place to put his hat.

He finally explained, hesitatingly, that his folks had told him he had better have a talk with the "perfesser" to see whether there was any use in his taking any more engineering. He said he liked the study and the surveying all right, but "did the 'prof' think he could ever be an engineer?" Dean Stout's reply was brief and pertinent: "If I were you I'd try to join a 'frat', mix a little, and learn something about life."

What a diviner of things human! Such an one becomes infinitely more than a mere teacher—he becomes an institution, a human guide-post!

Dr. Stout began his career in railroad work during college and subsequent to his graduation from the University of Nebraska in 1888. He left the "Burlington" early in 1890, to accept an appointment as City Engineer of Beatrice, Nebr., but was called back to his Alma Mater in the fall of 1891 as Instructor in the Civil Engineering Department. Within two years he was placed in charge of the Department and, in 1912, was made Dean of the Engineering College which position he held until he resigned in 1920.

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[1] Memoir prepared by J. C. Stevens, M. Am. Soc. C. E., and C. H. Purcell, Assoc. M. Am. Soc. C. E.

As his students finished their Freshman and Sophomore years and learned about railroads from his experience (in the early days railroading was about the only engineering experience available for civil engineers), the stern professor relaxed, particularly toward those who were following engineering in the summer during their vacations. They were called in and afforded those intimate contacts which, even more than the technical training, everlastingly endeared him to them.

He had a most subtle way of pointing out the human elements in engineering—lessons not found in books, but of which he was a past master. One of his epigrams has become famous—"the engineer is a resourceful man."

Those who were graduated under him carried with them the conviction that engineering was an office of trust and a profession of honor—a conviction so firmly instilled that it could never be forgotten in the practice of their profession.

At the time of receiving his Bachelor's degree he was awarded the scholastic honor of membership in Phi Beta Kappa. He was also a Charter Member of the Nebraska Chapter of Sigma XI, organized in 1897. The professional degree of Civil Engineer was conferred upon him in 1897, and that of Doctor of Engineering in 1932, both by the University of Nebraska. His college fraternity was Beta Theta Pi. He was made an honorary member of Sigma Tau at the founding of that engineering honorary fraternity at the University of Nebraska in 1904.

In honor of Cyrus Hall McCormick a Gold Medal Award was created in the American Society of Agricultural Engineers to be given each year "for exceptional and meritorious engineering achievement in agriculture." Dr. Stout was chosen almost without competition as the first recipient of that award which was made at Columbus, Ohio, in 1932, at the Annual Convention of the Society.

During the period of teaching he was also engaged in the general practice of engineering. He did considerable work for the Water Resources Branch of the United States Geological Survey, and for what is now the Bureau of Agricultural Engineering of the Department of Agriculture. Some of his private engagements included work for the Ingold Placer Mining Company, of Colorado, Tristate Land Company, of Scottsbluff, Nebr., and Costillo Estates Development Company, in the San Luis Valley, Colorado. His professional advice was frequently sought in matters of reclamation, drainage, and other fields of hydraulic engineering. He had a most happy faculty of combining thorough theoretical knowledge with simple and practical ways of doing things. One of his pungent remarks long remembered is, "You should never place your entire organization on an experimental basis."

In connection with his water supply work for the Government he devised a simple and effective method of determining the daily discharges from a limited number of measurements on erratic streams, such as those with unstable beds, or those affected by growth of aquatic plants. This method is still used by hydrographers and is known as "Stout's Method for Shifting Stream Beds."

As soon as the United States entered the World War, Dean Stout volunteered for service, secured a leave of absence from the University, and received a commission as Major of Engineers in the Reserve Corps, United States Army. He spent two months in a training camp, four months with the Construction Division, and seven months as Battalion Commander. To most of his contemporaries he is still lovingly referred to as "Major Stout."

In the midst of the war excitement of 1917 and 1918, an old Nebraska engineering graduate alighted one evening at a hotel in Washington, D. C., and looking across the lobby saw Dean Stout dressed in the uniform of a Major of Engineers. In answer to the "grad's" question as to what he was doing, he very modestly deprecated the very important service he was then rendering, but showed himself as restless as any 21-yr old recruit to get over to France. To one who had been out of school for fifteen years, and who as a student had looked upon the Dean as a man of mature age, it was amazing and most refreshing to find him so full of vigor and "champing at the bit" to get into action.

In 1919 he resumed work at the University of Nebraska, but resigned within a year to engage in private practice. Land reclamation projects which looked promising at the time "went on the rocks", and he turned to irrigation work for the Government in which service he continued until his death. His last work was in the Division of Irrigation of the Bureau of Agricultural Engineering, with headquarters at Berkeley, Calif. It consisted of special research work carried on in connection with the California Agricultural Experiment Station and in co-operation with the California State Department of Public Works.

At the time of his death he was engaged in a special study of the Platte River in Nebraska for the United States Bureau of Reclamation. He was taken ill on duty and died in Denver, Colo., after only a few days' sickness.

Late one hot summer afternoon in 1932, a Nebraska engineering graduate driving in the Sacramento Valley, stopped beside a car parked in the shade of a tree, only to find Dean Stout in the midst of one of his numerous investigations, sitting in his car making notes. Pleasantries were exchanged and the former student learned that the Dean expected to travel about ninety miles yet to reach home that night, and even then was planning what he was going to do early the next morning—and the Dean had been thought of as elderly by his students, thirty years before!

His knowledge of human nature was most cogently demonstrated as an expert witness in an important water-right case affecting very valuable properties in Southern California. The case, which was being tried before a jury of farmers, had been hopelessly muddled by a metropolitan attorney and the witnesses he had produced up to that time. The Dean's frank and clear statements were an education to that farmer jury, to which they responded graciously in their verdict which prevented the public from suffering the loss of many thousands of dollars on a purely trumped-up claim. After the verdict every member of the jury insisted on shaking hands with the Dean and lingering longer to talk with him.

Dr. Stout was born in Jerseyville, Ill., on November 14, 1865. He was married to Edith Forbes, at Beatrice, Nebr., in 1890. To them were born six children—Richard Forbes, Donald John, Oscar Charles, Marian Edith, Harris Pinkerton, and Burt Elihu. A granddaughter lived for a number of years at his home in Oakland, Calif. At the age of seven, one day her rejoinder to something a playmate had said was overheard, "Aw, grandfathers don't get cross at yu!"

Dr. Stout had a burning interest in every young man that passed through his institution. He never tired of meeting "his boys." He knew the names of most of them and could tell with astonishing detail the career of each since leaving college. Probably one of the proudest moments in his life was the occasion of his escorting Charles H. Purcell, Assoc. M. Am. Soc. C. E., one of "his boys", to the platform to receive a degree of Doctor of Engineering at the Commencement of 1935, after the Dean had been away from the University for seventeen years.

A bit of advice he often gave to young engineers was, "Any engineer who ever expects to amount to anything should join the American Society of Civil Engineers and do it before some one gets it in for him." It is well known that he held the Society in great reverence, but for some reason never applied for membership until he resigned as Dean. In 1922, he had his application papers made out and was just on the verge of sending them in when his reclamation "bubble" burst, and the application was never filed.

On November 5, 1926 a special dinner meeting was called of the University of Nebraska Alumni at San Francisco, Calif., and about two hundred sat down to the banquet at the Stewart Hotel. Fred M. Hunter, now Chancellor of Higher Education in Oregon, was Toastmaster. Dean and Mrs. Stout and available members of his family were present. The Dean was presented with a handsomely bound and suitably inscribed volume of letters that had been gathered from "his boys" from all corners of the world—about 160 of them had written a message of affection and cheer. They had also made up a fund and presented to the Dean the grade of Member in the American Society of Civil Engineers[2], with all fees and dues paid. A small balance remained in that fund at the time of his death, which was immediately turned over to Mrs. Stout.

Dean Stout was elected a Member of the American Society of Civil Engineers on April 18, 1927.

CHARLES FREDERICK SWIGERT, M. Am. Soc. C. E.[1]

DIED JUNE 23, 1935

Charles Frederick Swigert, one of the best known Engineer-Executives of the Pacific Coast, was born at Swanton Village, Ohio, on December 1, 1862.

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[2] The story of this episode is contained in *Civil Engineering*, September, 1935, p. 603.

[1] Memoir prepared by James H. Polhemus, M. Am. Soc. C. E.

He was the son of Charles Frederick Swigert and Elizabeth (Gorrill) Swigert, and was the only boy in a family of four children. His father died when Mr. Swigert was quite young, and he went with his three sisters to live with his grandparents. He received his education in the public grade and high schools.

Mr. Swigert's career identifying him with engineering, construction, and operation of various types on the Pacific Coast began in June, 1881, with the Pacific Bridge Company of San Francisco, Calif. In 1888, he moved to Portland, Ore., to establish a branch of the Pacific Bridge Company and his first major undertaking was the building of the original Morrison Street Bridge which was the first to span the Willamette River, at Portland.

He was still identified with the Pacific Bridge Company at the time of his death, although in the interim of 54 yr, he had been actively concerned, mostly in an executive or proprietary capacity, with many activities. These included extensive construction works as well as in giving the benefit of his abilities in unremunerative public service.

Among his earlier activities in addition to his work with the Pacific Bridge Company, he was Consulting Engineer for the Albina Light and Water Company from 1892 to 1898; between 1892 and 1905, he acted as Consulting Engineer, then as Chief Engineer, for the City and Suburban Railway Company and, later, became Vice-President and Manager of that Company. When that Company's interests were sold to a new company which absorbed three electric railway companies and the power and light company serving Portland, Mr. Swigert retired from active management, but became a Director of the consolidated company, then known as the Portland Railway, Light and Power Company. When Mr. Swigert relinquished the active management of the Street Railway Company mentioned, he devoted the greater part of the year, 1906, to travel in various parts of the world, viewing on this trip not only ancient and modern engineering works of the European countries and on the borders of the Mediterranean, but acquiring a comprehensive understanding of the world's economic relations.

Mr. Swigert's service in a public capacity was as a member of The Port of Portland Commission from 1901 to 1911, except for a part of 1905 and 1906, while he was traveling in foreign countries. Shortly after his appointment to The Port Commission he was elected President of the Commission and continued to be its leader during his tenure. This Commission is a body created by the Legislature for the purpose of establishing and maintaining a channel in the Willamette and Columbia Rivers between Portland and the sea, capable of accommodating ocean commerce. The Commissioners serve entirely without remuneration and the leadership in this endeavor was well suited to the creative talents and executive abilities of Mr. Swigert. At the time he entered upon these public duties river and harbor appropriations by the Federal Government for the improvement of the Columbia River channel were not large. The channel in the Columbia and Willamette Rivers for a distance of about 100 miles, stretching between the sea and Portland, was inadequate in depth and width for the increasing commerce of the Port, and its maintenance against annual sedimentation due to freshets was uncertain. Even adequate equipment was lacking. Mr. Swigert was largely instrumental in formulating and carrying out a program by which definite results were obtained.

In large construction work, Mr. Swigert's activities increased with his years. He was already in his seventies when he and his associates in the Pacific Bridge Company undertook and carried out, before his death, the construction of the foundation piers of the Golden Gate Bridge, as well as participating in similar construction on the San Francisco Bay Bridge. Bridge foundation work was a specialty with Mr. Swigert. Among the other comparatively recent bridge foundation structures placed by the Company of which he was President, are the Interstate Bridge, over the Columbia River, at Vancouver, Wash.; the Lake Union Bridge, at Seattle, Wash.; the Longview Bridge, over the Columbia River; the Burnside and Ross Island Bridges, over the Willamette River, at Portland Harbor; and the St. Johns Suspension Bridge over the Willamette River. In past years the Company did the pier work on practically all the bridges at Portland and its vicinity and, at times, carried out similar work in other parts of the United States.

The Pacific Bridge Company was one of the original members of the six companies which built the Boulder Dam and which were awarded the contract for the Parker Dam on the Colorado River. The Company is also associated with the General-Shea Company which was awarded the contract for the Bonneville Power House.

Among the industrial activities in which Mr. Swigert engaged at various times during his career, some of which were still receiving his attention at the time of his death, were manufacturing in such lines as iron and steel products, wood products, paints, and wood preservatives. The best known institutions in this line are the Electric Steel Foundry of Portland, and the Willamette Hyster Company. To list in detail the various companies with which Mr. Swigert was connected either as an active executive, or in a consulting or advisory capacity, would take much space.

To mention so many varied activities encompassing works of such extraordinary size tends to give the impression of a man in whom what are called the human or homely characteristics might be expected to be crowded out. This was emphatically not the case with Charles F. Swigert. He numbered personal friends and acquaintances by thousands and his life was as filled with friendly human contacts as it was with business activities. He was never the distant impersonal executive but, to the limits of human capacity, he had a personal knowledge of and interest in the personnel of the organizations with which he was connected, not overlooking the humblest employee.

He passed away unexpectedly in San Francisco, Calif., on June 23, 1935, having died during his sleep, in his seventy-second year. He is survived by his widow, Rena G. Swigert and three sons, Charles Frederick, Jr., Earnest Goodnough, and William Gorrill Swigert.

Mr. Swigert was elected a Member of the American Society of Civil Engineers on June 4, 1928.

CHARLES VOETSCH, M. Am. Soc. C. E.[1]

Died February 7, 1935

Charles Voetsch, the son of Jacob and Anna (Huonker) Voetsch, was born at Waldenbuch, Germany, on August 8, 1877. His early education was obtained in that city where his father was a Public Attorney. His technical education was extensive, covering both Mechanical and Electrical Engineering. He received his degree of Mechanical Engineer from the Technical University of Stuttgart, Germany, and, later, his degree of Electrical Engineer from the Technical University of Darmstadt, Germany. As part of the requirements for these degrees, Mr. Voetsch on his first work obtained considerable shop experience as a machinist.

Upon coming to America in 1904, Mr. Voetsch was employed by the I. P. Morris Company, of Philadelphia, Pa., as Designer of hydraulic turbines. While with this Company he designed the 10 500-hp runners for the Shawinigan Water Power Company, of Quebec, Que., Canada. His next work was with the Allis-Chalmers Manufacturing Company, at Milwaukee, Wis., where he was employed from November, 1904, to September, 1906. During this period he designed the 13 500-hp turbines for the Great Northern Power Company, and the 10 000-hp turbines for the California Gas and Electric Company.

From September, 1906 to October, 1907, Mr. Voetsch was associated with Sellers and Rippey, Consulting Engineers, of Philadelphia, engaged on the design of hydro-electric power developments. He was next with the Bidwell Electric Company, of Chicago Heights, Ill., until October, 1908, as Chief Engineer in charge of design of electric motors, dynamos, and carbonic acid ice machines, and Superintendent in charge of the Company's shops.

From April, 1909, to October, 1910, he was employed by the Indiana Steel Company, in Chicago, as Designing Engineer. His next work, continuing to October, 1911, was with the Falkenau Electric Construction Company, of Chicago, on the design of the hydro-electric power plant for the Davis and Weber County Canal Company, of Ogden, Utah, and sub-stations and steam-generating stations at Ogden, and at Salt Lake City, Utah, including also the electric installations on these projects. Thereafter, until February, 1913, he was Chief Engineer in charge of designs and shops for the Camden Hydraulic Turbine Corporation, of Camden, N. Y.

Mr. Voetsch served as Electrical, Hydraulic, and Mechanical Engineer for the Utica Gas and Electric Company, of Utica, N. Y., from April, 1913, to August, 1922, in charge of reconstruction of steam and hydro-electric plants, and planning and designing power-plant extensions, sub-stations, transmission lines, and new power developments. While with this Company, he prepared designs for the Trenton Falls Plant, consisting of three 9 000-hp turbines operating under a 265-ft head. In the latter part of this period, he was temporarily transferred to the staff of the Hooven-Owens-Rentschler Company, to re-design and build two 10 000-hp turbines for the Utica Gas and Electric Company and the Cohoes (N.Y.) Power and Light Company.

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[1] Memoir prepared by J. L. Savage, M. Am. Soc. C. E., assisted by Henry J. Tebow, Jun. Am. Soc. C. E., and M. H. Fresen, Esq., all of the U. S. Bureau of Reclamation, Denver, Colo.

From August, 1922, to June, 1924, Mr. Voetsch was with the Adirondack Power and Light Company, of Schenectady, N. Y., in charge of the design and installation of a 9 000-hp unit and of a 114 000-volt outdoor sub-station at Spier Falls, N. Y., on the Hudson River.

In 1925, he was in Hamburg, Germany, where he supervised the designs of a pulverized fuel boiler plant for the Vickers-Spearing Boiler Company, of London, England. From July, 1925, to April, 1927, he was engaged in a consulting engineering practice particularly on the design of special machinery, with offices in Cleveland, Ohio.

Mr. Voetsch entered the Government service in April, 1927, and from then until June 5, 1928, he was employed in the United States Engineer Office, at Chattanooga, Tenn., as Hydro-Electric Engineer on the power surveys of the Tennessee River and its tributaries.[2] This survey included a comprehensive study of the hydro-electric possibilities of the Tennessee River Basin, involving the investigation of about 150 proposed storage projects, run-of-river plants, and auxiliary steam plants. In connection with this work, Mr. Voetsch developed new and more direct methods for determining the combined power of this complicated system with the most economical use of water for maximum firm power development. In connection with this survey, furthermore, he made preliminary layouts and designs for the Cove Creek Project, on the Clinch River, which later became the Norris Dam Project of the Tennessee Valley Authority. Subsequently he was transferred to the United States Engineer Office, at Nashville, Tenn., where on November 1, 1929, he was advanced to the position of Senior Hydro-Electric Engineer and assigned to a power survey of the Cumberland River and its tributaries.

On November 23, 1929, he was transferred to the Madden Dam Project of the Panama Canal Zone where he prepared preliminary designs and estimates for the Madden Dam and Power Plant. He also prepared the final designs for the 66-kv transmission line, from Madden Dam to Summit. In February, 1931, Mr. Voetsch was transferred to the United States Bureau of Reclamation at Denver, Colo., to assist in the preparation of final designs for the Madden Dam and Power Plant. After the completion of these designs, he was engaged on hydraulic designs for the Boulder Canyon, Grand Coulee, and Seminoe Power Plants, of the Bureau of Reclamation, and also for the Norris and Wheeler Power Plants of the Tennessee Valley Authority.

Those who are familiar with Mr. Voetsch's professional work attribute his successful and useful career to certain distinctive qualifications. He was an indefatigable student and worker. He had a keen mind, devoted much time to reading technical literature, and was able to converse in four different languages, English, German, French, and Spanish, which gave him an unusual ability to study technical problems. His early technical training, together with his broad experience and sound engineering judgment, made him unusually well qualified to handle important design work.

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[2] H. R. Doc. 328, 71st Cong., 1st Session.

He was exceptionally well informed in precedents and standard practice in the design, construction, and operation of American and European hydroelectric developments, and resourceful in the adaptation of these practices to the problems at hand. Mr. Voetsch contributed generously to the store of American engineering knowledge by translating articles on engineering and research work done in Europe. He was the author of several papers on hydroelectric design and construction.

He was a member of the American Society of Military Engineers, the Masonic and Shrine fraternities, and the Benevolent and Protective Order of Elks; and he was a Licensed Engineer and Land Surveyor of the State of New York.

Mr. Voetsch was modest, unassuming, and friendly. He always dealt with people with courtesy and with instinctive consideration for their feelings. He was highly respected by his associates who prized his sincere and unaffected friendship. He was thoughtful of those working under him and was always ready to assist and encourage his associates. Notwithstanding his extensive technical knowledge, he was open-minded and considered carefully the suggestions and opinions of others.

His principal hobby, aside from study, was the collection of curios, specimens, pictures, and especially engineering data. He was a skilled violinist and occasionally delighted his friends by playing classical music.

The following are quotations from letters that have been received from some of Mr. Voetsch's friends and associates:

Lt.-Col. Lewis H. Watkins, General Staff, U. S. Army, formerly District Engineer, U. S. Engineer Offices, Chattanooga and Nashville, Tenn., writes:

"Mr. Charles Voetsch was employed by me from 1927 to 1929, on the surveys of the Tennessee River and its Tributaries and the Cumberland River and its Tributaries for their combined development for navigation, power, flood control, irrigation, prevention of soil erosion, and other useful purposes.

"These were the first comprehensive surveys covering the complete economic development of a large river and its tributaries ever made by the Federal Government. They involved a large amount of mapping, research work, developments of principles governing the best and most economic methods of developing the river and its tributaries for the combined purposes, and the preparation of general plans and estimates of cost for many engineering projects.

"As a result of these surveys a project for the combined development of the Tennessee River and its Tributaries was adopted by Congress in 1929. The construction of the project is being carried out by the Tennessee Valley Authority.

"In these surveys, Mr. Charles Voetsch rendered valuable assistance in research work, in the development of principles governing the combined development of the river and its tributaries, and in the preparation of general plans and estimates of cost. He soon became my leading man on those phases of the survey. Whenever information or data was desired on any engineering subject, Mr. Voetsch either had it on hand in his personal library or soon acquired it. He showed great initiative and excellent judgment in the development of principles governing the combined development of the river and its tributaries. He was untiring in the preparation of plans and in the computation of the great mass of data required for the survey.

"In all this work Mr. Voetsch proved himself to be a man and an engineer of the highest type and a patriot, serving his Government untiringly, loyally and to the best of his ability, often working overtime at night, regardless of his own personal affairs.

"On account of his excellent work on the survey of the Tennessee and Cumberland rivers, Mr. Voetsch was recommended for and assigned to duty on the construction of the Madden Dam in Panama.

"I consider his death a great loss to the engineering profession and to the Federal Government."

C. M. Hackett, Senior Engineer, U. S. Engineer Office, Nashville, writes:

"The death of Mr. Voetsch has elicited many expressions of sincere regret and sorrow from those associated with him in the Chattanooga and Nashville United States Engineer Offices, at the loss of one whom they regarded as a friend. They remember him as one who was genial and ever willing to co-operate and assist in the solving of any problem that might develop, whether it related to them personally or to their work."

Byron E. White, M. Am. Soc. C. E., of the Syracuse (N. Y.) Lighting Company, Incorporated, formerly with the Utica (N. Y.) Gas and Electric Company, writes:

"We who worked with him daily for nine years knew him as Carl. I must say that the news of his death came as a distinct shock to those in our organization who had long been associated with him.

"Mr. Voetsch personified the traits of patience, accuracy, extreme thoroughness, common sense, tenacity and the rearing of every structure and every conception upon solid foundations, building up block by block, testing each by the square of the fundamental truths, resulting in a finished, workmanlike structure. His was the mind of a scientist rather than of an executive; of a designer rather than of the master of men, materials, and circumstances, who rides roughshod over all difficulties in the rough and tumble of construction and operation.

"He was modest and unassuming, yet his convictions were strong and tenaciously held against all opposition. Like a true scientist, he was honest with himself and others and he appeared to take a peculiar pleasure in imparting knowledge to others or assisting them in a solution of their problems."

George A. Orrok, M. Am. Soc. C. E., Consulting Engineer, New York, N. Y., writes:

"I am most sorry to hear of the death of Mr. Charles Voetsch, whom I knew very well in the past.

"I first became acquainted with Mr. Voetsch * * * when he was employed by a turbine company near Watertown, N. Y. He made for me a number of good wheels which we used at the Dolgeville Plant of the Utica Company. Shortly after, he came in to the Utica Company and was regularly employed there until I believe 1922. During this time, he did a quantity of very good work for the Utica Company for which I was Consulting Engineer, both in the hydraulic line and in mechanical and structural work covering the building of several hydro plants, a steam station, sub-station and much other work.

"Mr. Voetsch was very careful and conscientious and thoroughly competent, was always able to take his end of an argument and present the facts in logical order. I knew of his work with the Government where a designer of his capacity would be appreciated."

E. S. Randolph, M. Am. Soc. C. E., Construction Engineer, Madden Dam, Panama Canal, Balboa Heights, Canal Zone, writes:

> "It was a shock to hear of Mr. Voetsch's death. I had grown so well acquainted with him down here and also his wife and children that his passing was very regrettable to me. Personally, Mr. Voetsch impressed me as a very bright and attractive personality, an extremely clever engineer, a congenial friend, and a fond parent to his children."

In 1919, Mr. Voetsch was married, in Brooklyn, N. Y., to Aletha Johnson, of Utica, N. Y. Mrs. Voetsch died in Denver, in February, 1932. He died on February 7, 1935, at St. Anthony's Hospital, in Denver, as a result of septicæmia following rheumatic arthritis and influenza. Two children, Ruth, and Robert Otto, to whom he was very devoted, survive him. He is also survived by a brother, Otto Voetsch, living at Friedrichshafen, Germany.

Mr. Voetsch was elected a Member of the American Society of Civil Engineers on November 27, 1917.

GEORGE CLINTON WARD, M. Am. Soc. C. E.[1]

Died September 11, 1933

George Clinton Ward was born at White Plains, N. Y., on January 9, 1863, the son of James and Elizabeth (Ennis) Ward. At an early age he removed to Boonville, in Oneida County, New York. He attended the public school of that town, and when only seventeen, taught in a rural school to help defray his expenses at Phillips Academy, at Andover, Mass., where he studied engineering for two years, graduating in the Class of 1882. His first position, in the summer of 1882, was with the Mohawk Division of the West Shore Railroad, later a part of the New York Central System.

During the three years following, Mr. Ward was engaged in the construction of iron bridges in the northern part of the State, among which was the lift bridge, at Utica, N. Y., the first known of its kind.

In 1886, he was made Assistant Engineer of Location and Construction of the Carthage and Adirondack Railroad. He filled the office of Supervisor of Oneida County for two terms—in 1886 and 1887. During the latter year, he was Assistant Engineer on the Susquehanna Division of the Erie Railroad, and in the succeeding years he conducted the preliminary surveys on various roads, all later incorporated in the New York Central Lines.

From 1890 until 1892, Mr. Ward was engaged in the construction of the Mohawk and Malone Railway, and from 1895 to 1897 served as Superintendent of the Black River Canal. In 1897, he became Assistant Engineer for the New York State Canals, resigning therefrom to accept a position with the Furnaceville Iron Company, and took up his residence at Utica, where he lived until 1902. In 1898, occurred the final construction of the Mohawk and

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[1] Memoir prepared by H. W. Dennis, M. Am. Soc. C. E.

Malone Railway through the Adirondack Mountains to Racquette Lake. Of this road Mr. Ward had complete control until the death of Collis Potter Huntington, F. Am. Soc. C. E., in August, 1899, when it was leased to the New York Central Railroad Company.

During the years following, Mr. Ward constructed the Glenfield and Western Railroad, in New York State, and, in 1902, went to Washington Court House, Ohio, at the instigation of Henry Edwards Huntington, the nephew of the late Collis P. Huntington. Mr. Huntington owned the water supply system at Washington Court House which up to the time of Mr. Ward's association with it had never produced a sufficient supply of water to serve the needs of the community. He developed new wells and left the plant in a much improved condition when he came to California in the spring of 1905, at the request of Mr. Huntington.

In Los Angeles, he was associated with the Huntington interests for many years, first as Assistant General Manager, later as General Manager of the Huntington Land and Improvement Company and the various subsidiary companies, developing water properties, and laying out subdivisions, in addition to his managerial duties.

One of the most ambitious of Mr. Huntington's ventures was the Pacific Light and Power Corporation, of which Mr. Ward was elected Vice-President in 1911. It was this Company which made the original hydro-electric development in the Big Creek Region northeast of Fresno, Calif. In 1917, the Pacific Light and Power Corporation and the Southern California Edison Company were consolidated, bringing the partly completed Big Creek project into the new ownership and also bringing to the staff Mr. Ward and many of his associates. This great engineering project carried on over a period of years was finally completed in 1928, and attracted widespread attention everywhere in the United States because of its magnificent proportions and completeness of development. During the construction of the Big Creek project, 56 miles of railroad were built through the mountains about 70 miles northeast of Fresno, for the purpose of hauling construction material and men, and huge camps were operated for the thousands of employees. Dams, tunnels, power houses, and transmission lines were built. Something in excess of 500 000 hp was thus developed for use in Central and Southern California.

During this period Mr. Ward was Vice-President in Charge of Engineering and Construction of the Southern California Edison Company and this great project was carried to completion by him. In the fall of 1931, he became Senior Vice-President of the Company, and upon the death of Mr. Russell H. Ballard in August, 1932, he became its President, in which capacity he served until his death.

To provide a supplementary source of power to meet possible emergencies caused by adverse weather conditions, the Edison Company carried on the program of steam-plant construction under Mr. Ward's direction. The site of these supplementary plants is near Long Beach, Calif., where there is an abundance of cooling water, ready access of fuels, and a short transmission distance to the principal areas where electricity is sold. This steam-plant program culminated with the erection of two highly efficient units with a combined capacity of 268 095 hp. Mr. Ward's insistence upon the highest known type of construction in these plants was justified by their performance during the earthquake of March, 1933, when they maintained their service with only a slight interruption.

Mr. Ward's work was recognized by the Western Division of the United States Chamber of Commerce at its meeting in Seattle, Wash., on December 7, 1925. At that time in commemorating the completion of the Florence Lake Tunnel, the Chamber of Commerce presented him with a large bronze medallion mounted on a granite block taken from the excavation of that tunnel. In making the presentation Vice-President Shoup of the Chamber of Commerce paid tribute to the engineering genius of Mr. Ward and his associates and declared that their achievement was a distinct contribution to the welfare and prosperity of the nation. The medallion is centered with a bas-relief of Mr. Ward with this significant phrase, "By their works ye shall know them."

Mr. Ward received the Honorary Degree of Doctor of Engineering from the University of Southern California, at Los Angeles, on June 4, 1927, "in recognition of distinguished achievement in the field of applied science."[2]

Dr. Ward had conferred on him the further Honorary Degree of Doctor of Science by Oberlin College, Oberlin, Ohio, in 1928. In presenting Dr. Ward at the ceremonies Dr. L. W. Taylor, of Oberlin, said of him:

> "He is responsible for the engineering aspects of what has been termed the greatest conservation project in history. The construction of huge water reservoirs at an altitude of more than a mile; the installation of a great system of dams, conduits, tunnels, and power stations; the provision of irrigation facilities for hundreds of square miles, and a power supply for millions of population; the design and development of the most economical system of power distribution known to the engineering world; such accomplishments require scholarly attainment in technical fields, combined with wise and far-seeing administration."

Posthumously, Dr. Ward has been further honored. The Southern California Edison Company, Limited, caused his name to be used as a permanent designation of the tunnel between two of its major reservoirs. This tunnel connects Florence Lake Reservoir with Huntington Lake Reservoir as a part of the Big Creek project. It is 13 miles in length, at an altitude of 7 000 ft, and was constructed through the granite of the Sierras at a cost of $17 000 000. During construction days it was called the Florence Tunnel, but now is officially known as the Ward Tunnel, a name which has been confirmed by the Division of Geographic Names of the United States Department of the Interior. Looking down upon the intake of the Ward Tunnel at the Florence Lake Reservoir is a massive pile which appears upon the maps as an unnamed mountain. It includes three peaks at altitudes of 10 395, 10 500, and 10 800 ft, respectively, above sea level. This mountain has been named Ward Mountain, also by action of the Division of Geographic Names.

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[2] From the minute book in the office of the President of the University of Southern California.

Dr. Ward was married in 1886 to Katherine L. Schweinsberg, of Boonville, N. Y. Mrs. Ward and their daughter, Mrs. Louise (Ward) Watkins, survive him.

He had many interesting hobbies, among them the collection of rare books and manuscripts and of relics of the Indians of the Southwest. He was a member of several organizations, including the California, the Jonathan, the Sunset, and the Zamorana Clubs; and was a Knight Templar and a Thirty-Second Degree Mason. He was an Associate Member of the Board of Trustees of the California Institute of Technology, and a contributor to the philanthropic and civic organizations of Southern California.

Dr. Ward was elected a Member of the American Society of Civil Engineers, on February 10, 1930.

CHARLES ALFRED WILSON, M. Am. Soc. C. E.[1]

DIED JUNE 3, 1935

Charles Alfred Wilson was born in Conowingo, Md., on April 4, 1855, the son of John Howard and Anna Catherine (Cheyney) Wilson.

Charles Alfred Wilson entered railway service in 1869 at the age of 14 and during his entire professional career was engaged in railroad engineering. A High School education, at West Chester, Pa., supplemented by technical training obtained while working with his father, who was a practising surveyor and civil engineer, fitted him for his early start as a Flagman with the Engineer Corps of the Philadelphia and Reading Railroad Company.

In those early days, railroad engineering meant mostly railroad construction, and positions as Rodman on construction, with the Pennsylvania and Delaware Railroad Company, and the Atlantic and Great Western Railway Company, and as Resident Engineer, with the South Mountain and Boston Railroad Company, brought him up to February, 1875.

From that time until 1878, Mr. Wilson served as Engineer and Roadmaster, with the Philadelphia, Newtown, and New York Railroad Company, and as Engineer of Surveys, with the Atlantic and Lake Erie Railroad Company. Also, within that period, in 1876, he served on the Bureau of Awards, for the United States Centennial Commission, at Philadelphia, Pa.

During 1879 and 1880 he was Resident Engineer on the construction of the Ohio and West Virginia Railway, which was the Ohio River Division of the Columbus, Hocking Valley, and Toledo Railway, later a part of the Chesapeake and Ohio Railway.

Mr. Wilson's long service with the Wheeling and Lake Erie Railway Company began in 1881 with the duties of Principal Assistant Engineer on Construction. In 1883, he was made Chief Engineer, at the age of 28, and continued in that position until January 1, 1896. In addition, he held the title of General Superintendent from March, 1893, to September, 1895.

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[1] Memoir prepared by O. E. Selby, Jun. Am. Soc. C. E.

On January 1, 1896, Mr. Wilson entered the service of the Cincinnati, Hamilton, and Dayton Railway Company as Chief Engineer and Assistant to the President. The latter title was dropped eight years later, but he continued as Chief Engineer until displaced by reason of a change in management in 1909.

In 1909, Mr. Wilson opened an office in Cincinnati, Ohio, as a Consulting Engineer and Railway Expert. He accepted retainers as Special Consulting Engineer for the Erie Railroad Company and the Chesapeake and Ohio Railway Company, and conducted investigations and made reports on the valuation and other features of properties for a number of Eastern railroad companies. His principal activities, however, beginning in 1912, were in studies for the improvement of passenger and freight terminal facilities in Cincinnati, resulting in the completion, in 1933, of the magnificent passenger station and other facilities of the Cincinnati Union Terminal Company. Until the formation of that Company for the purpose of building the new terminal in 1927, Mr. Wilson was employed as Consulting Engineer for various Committees of Engineers of the seven railroads entering Cincinnati, engaged in making studies of the various aspects of the terminal problem as related to both passenger and freight service.

The Board of Directors of the Cincinnati Union Terminal Company appointed him Consulting Engineer to the Chief Engineer on November 23, 1927. He continued in that capacity during the construction period, until May, 1933, after which he retained a nominal connection as Consulting Engineer, until his death.

At this point it seems fitting to describe briefly Mr. Wilson's technical society activities. He was a frequent attendant at the Annual Meetings and Annual Conventions of the Society, was an active member of the Local Section, at Cincinnati, Ohio, and, in other ways, showed his interest in the Society and its affairs.

He was one of the promoters and a Charter Member of the American Railway Engineering Association. He was proud of his record of having attended every Annual Convention of that Association. At the meeting in March, 1935, he was cited for that record and for his part in the inception of what has come to be one of the outstanding organizations of railway men.

In his remarks in response to that citation, Mr. Wilson disclosed the history of the formation of the Association and the reasons that actuated him in promoting it. He said that in his relations with the officers of the Motive Power and Car Departments of the railroad companies on which he represented the Engineering and Maintenance-of-Way Department, he found himself at a disadvantage because those men belonged to their respective associations of master mechanics and master car builders, and those organizations had standards and rules of practice, whereas there was nothing like that for maintenance-of-way engineers.

This state of affairs led him to discuss with others the formation of an association of engineers engaged in railway maintenance. A meeting at Chicago, Ill., was called for that purpose in 1898, at which Mr. Wilson nominated the Chairman. The outcome was the formation, in 1899, of what is now the American Railway Engineering Association.

A man of sociable habits and free conversational traits, Mr. Wilson nevertheless was modest to the point of shyness with respect to himself. He discussed men and things and policies, but not his personal affairs. Several of his friends who were in close business relations with him for years have been able to recall for the writer of this memoir only a few incidents or anecdotes relating to his personal life.

He was a man of sterling character and simple habits, with the highest ideals of duty to his employers and to society. Holding strong convictions on many subjects, he discussed them freely, but never was known to show vindictiveness toward any one who held other views. He resided for many years in Wyoming, a suburb of Cincinnati, where he was active in social and civic affairs and formed many friendly relations. In addition to his technical society memberships previously described, he was a member of the Wyoming Golf Club and the Cincinnati Torch Club.

He was married to Carree Ione Stevens on November 21, 1883. Two sons, Cheyney Stevens and Hamilton, were born of this union. Mrs. Wilson died on October 21, 1919. He is survived by Mrs. Edna Sutherland Wilson, whom he married on September 20, 1929, and his two sons.

Mr. Wilson was elected a Member of the American Society of Civil Engineers on April 1, 1891.

HENRY FELIX WILSON, M. Am. Soc. C. E.[1]

DIED MARCH 2, 1935

Henry Felix Wilson was born in Mobile, Ala., on October 31, 1867, the son of Henry Felix Wilson, Sr., and Virginia (Clark) Wilson. He received his preparatory education in private schools, and then entered the University of Alabama, at Tuscaloosa, Ala., from which he was graduated in 1886 with the degree of Bachelor in Engineering.

Immediately after his graduation from college, Mr. Wilson was employed from July, 1886, to March, 1888, as Assistant Engineer and Draftsman with the late Arthur Owen Wilson, M. Am. Soc. C. E., at Birmingham, Ala. As such, he was engaged in general engineering practice, on the location and construction of railroads, municipal engineering, land and topographical surveys, foundations, etc.

Mr. Wilson then entered the railroad field as Assistant Engineer and Draftsman with the Georgia Pacific Railroad Company (later, a part of the Southern Railway System). From May to December, 1888, he served as Assistant Engineer on the construction of the Yazoo River Crossing in Mississippi, which bridge consisted of one 300-ft draw-span, two 100-ft approach spans, about 1 000 lin ft of pile trestle approaches and also 5 miles of grading. On the completion of the bridge, Mr. Wilson was transferred to the Superintendent's Office, in Greenville, Miss.

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[1] Memoir prepared from information supplied by the family and on file at the Headquarters of the Society.

From January to March, 1889, he was Assistant Engineer in charge of heavy construction on the Huntsville and Monte Sano Railway; and from April to December of the same year, he was Assistant Engineer and Draftsman on the construction of the Briarfield, Blocton, and Birmingham Railroad (later, part of the Southern Railway System). In this position, Mr. Wilson made designs and drawings for wooden Howe trusses, checked estimates, etc.

As Engineer for the Montevallo Coal and Transportation Company from January to April, 1890, he made surveys and mapped the Company's lands, located coal seams, etc. He then returned to the Briarfield, Blocton, and Birmingham Railroad Company and was engaged in making trial right-of-way maps, etc., in the Chief Engineer's Office, in Birmingham. From October to December of that year, Mr. Wilson served as Engineer for the Woodward Iron Company, with headquarters at Woodward, Ala. In this capacity he surveyed and mapped the Company's property, and also did some railroad location and construction.

In January, 1891, Mr. Wilson opened an office in Birmingham and engaged in private practice there until April, 1892, in general engineering. From May, 1892, to September, 1896, he served with J. W. Worthington and Company, of Birmingham. This firm was the largest contractors of iron ores and limestone for furnace fluxing in the District, and Mr. Wilson was engaged in construction, such as railroads, hydraulic pumps, etc.

He then returned to railroad work and from October to December, 1896, he served as Resident Engineer on heavy construction for the Mobile, Jackson, and Kansas City Railroad Company in Mobile County, Alabama, with headquarters in Mobile. In 1897, Mr. Wilson went to the Southwest and was engaged as Chief Engineer of the Raton Coal and Coke Company, at Raton, N. Mex. In this capacity, he designed and constructed railroads, coke ovens, power houses, etc., as well as opening mines and equipping them with electricity.

Still in the Southwest, Mr. Wilson spent the year 1898 as Draftsman and Computer with the International (Water) Boundary Commission between the United States and Mexico. He made large maps of surveys of the Rio Grande, checked the notes of the surveys preparatory to platting, etc. Leaving the Government service, he was employed from January to May, 1899, as Assistant Engineer on the construction of the El Paso and Northeastern Railroad, with headquarters at Alamogordo, N. Mex. He also was engaged in the construction of a mountain timber road.

Returning to Birmingham, Ala., Mr. Wilson was employed from June to September, 1899, as Assistant Engineer and Draftsman with the Southern Bridge Company. He also had charge of all contract lettings for the Company in the South from Virginia to Texas.

In October, 1899, he again entered private practice, with an office in Birmingham, until December, 1908. During this time he was engaged in railroad location and construction; coal and iron ore mining; timber and concrete construction; tests and reports; served on boards of arbitration, as expert witness in Courts in railroad cases, etc. Mr. Wilson represented the Pittsburgh Testing Laboratory, of Pittsburgh, Pa., in the Birmingham District for several years; he also made tests of creosoted timber for the New York State Barge Canal, etc.

In 1909, he closed his office in Birmingham, and went to Goldfield, Nev., during the gold strike there. In 1910, however, he returned to Alabama and was employed as Assistant Engineer in charge of timber railroad construction in Middle Alabama for the Louisville and Nashville Railroad Company. During 1911, and the first three months of 1912, the South was in the grip of an industrial depression, and Mr. Wilson was employed for only three months as Engineer in charge of railroad location from Memphis, Tenn., toward Pensacola, Fla.

In April, 1912, he went to Cuba and until June of that year served as Assistant Engineer with the Guantanamo and Western Railroad Company, with headquarters at Guantanamo. He had been engaged to superintend the reconstruction of the entire road, but the Revolution of 1912 stopped all work and the engineering force was laid off.

Mr. Wilson was employed as Assistant Engineer with the Cuban Central Railways, Limited, at Sagua la Grande, Cuba, from July, 1912, to June 1914. He had charge of the location and construction of various lines, deep-water harbor improvements at Nuevitas, Cuba, including docks, terminals, water-works, etc.

He then returned to the United States and again entered the service of the Southern Railway Company. Among other works, he had charge of the construction of the Freight Terminal, in Mobile. In 1917 and 1918, he was again in Cuba, for eight months, as Engineer of Bridges and Buildings of the Cuban Central Railways Company, Limited, and from June to October, 1918, on special surveys with the Spanish-American Iron Company, at Felton, Cuba.

From November, 1918, to December, 1919, Mr. Wilson was employed on design and construction for the Pratt Consolidated Coal Company of Birmingham. He then entered the service of the Alabama State Highway Department and for six months in 1920 he was engaged on field surveys and in charge of the Drafting Office.

In April, 1921, went to Central America and served on construction and operation with the Cuyamel Fruit Company, in Spanish Honduras. In 1922, he again returned to Birmingham and entered the employ of Martin J. Lide, Consulting Engineer. He retained this position until September of that year, and was engaged in drafting, designing, and construction work. From October, 1922, to July, 1923, Mr. Wilson was with the engineering firm of Black, McKinney, and Stewart, of Washington, D. C., as Chief Draftsman and Chief Engineer of Construction.

In 1924, he entered the employ of the Tennessee Coal and Iron Company, as Draftsman, and remained with this Company until 1933. From that time until his death he was an Engineer for the Federal Government on work for the Civil Works Administration and the Federal Emergency Relief Administration.

He was a member of the Birmingham Engineers Club, the Kappa Alpha Fraternity, and the Presbyterian Church.

On December 23, 1890, Mr. Wilson was married to Corinné Estelle Harris and is survived by his widow, and a son, Harry F. Wilson, of Little Rock, Ark.

Addressing the Editor of *The News,* Mr. E. C. Crow, of Birmingham, Ala., a college classmate, fraternity brother, and long-time neighbor, writes of Mr. Wilson, as follows:

> "In the death of Henry F. Wilson recently, Birmingham has lost another of its pioneer builders."

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> "I am confident that Mr. Wilson's business associates and other college friends would join me in testifying to his professional integrity, and his conscientious and efficient services for his city and country."

Mr. Wilson was elected an Associate Member of the American Society of Civil Engineers on April 2, 1902, and a Member on May 2, 1905.

JACKSON FRANKLIN WITT, M. Am. Soc. C. E.[1]

Died April 2, 1935

Jackson Franklin Witt was born at Dallas, Tex., on November 5, 1869, the son of John T. and Nannie (Johnson) Witt. His early education was received in the grade schools of East Dallas.

Through necessity, Mr. Witt left school when he was twelve years of age and began work as a Chainman with a railroad location party. After this first job, he spent three years at home attending the public schools. When he was sixteen years old he was employed as a Chainman on Federal Government work.

In 1886, Mr. Witt was with the Gulf, Colorado, and Santa Fé Railroad Company, as Rodman and then as Levelman. He was with the St. Louis and Southwestern Railroad Company as a Transitman in 1887, and, in 1888, he served as Transitman on a railroad location from Lampasas, to Llano, Tex. In 1889, he was Transitman on a location survey for the Dallas and Northwestern Railway.

After completing the survey for the Dallas and Northwestern Railway, in 1889, Mr. Witt was employed as City Engineer for East Dallas, Tex. Late in 1890, when he was 21 years of age, he opened an office, and engaged in engineering and surveying. He maintained this office until 1905, when he became Assistant to the County Engineer of Dallas County Texas. During the time he was Assistant County Engineer Mr. Witt assisted in the first program of constructing hard-surfaced roads in Dallas County. This was one of the first programs of its kind in Texas.

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[1] Memoir prepared by A. P. Rollins, M. Am. Soc. C. E.

In 1907, he was appointed County Engineer of Dallas County, which position he held until 1919. It was while he was County Engineer, in 1911 and 1912, that the old Oak Cliff Viaduct was designed and constructed. This viaduct, one mile long, composed of 80-ft concrete arches, on pile foundations, and with a clear roadway of 44 ft, was outstanding at that time. Several other large bridges were constructed, and many miles of surfaced roads were built. As County Engineer, Mr. Witt was a pioneer in introducing asphalt surfacing for highways in Texas.

During the World War, he served the Government as a member of the Highway Transport Board, at $1 per yr. In 1919, Mr. Witt became a member of the Nagle, Witt, Rollins Engineering Company, which Company operated from 1919 to 1928. During this time he managed the affairs of the Company and had personal supervision of construction amounting in value to more than $10 000 000.

From 1928, until his death, Mr. Witt remained active in private practice in the engineering and construction field. His experience as a Consulting Engineer was broad, including such projects as toll bridges across the Rio Grande and the Red River; a highway program for Hopkins County, Texas; laying out and supervising the construction of a highway plan for Dallas County, Texas; and highway plans for Potter and Stephens Counties, Texas.

His life, as indicated by this record, was a continuous struggle, the opportunity for training having been denied him. His success as an engineer was due to his courage and indomitable will.

Mr. Witt's ability to make new friends and to retain his old ones, his constant desire to serve these friends and his willingness to participate unselfishly in civic affairs brought him wide recognition as a friend and a citizen. He was a member of the Presbyterian Church.

He was married on December 4, 1889, to Lucia Matella Lagow. He is survived by Mrs. Witt and a daughter, Mrs. Elsie Eria Clark.

Mr. Witt was elected an Associate Member of the American Society of Civil Engineers on October 31, 1911, and a Member on September 21, 1915. During his active engineering career, he served for a number of years as Secretary and Treasurer of the Texas Section of the Society.

GEORGE STEVENSON YATES, M. Am. Soc. C. E.[1]

DIED NOVEMBER 1, 1935

George Stevenson Yates, the son of James and Elizabeth (Stevenson) Yates, was born in Leith, Scotland, on November 11, 1875. After his preparatory education, he entered Edinburgh University, in 1894, where he studied engineering for three years, and was Assistant Demonstrator in the Engineering Laboratory for two years.

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[1] Memoir prepared by William Stewart, Esq., London, England.

His first important appointment was with the Caledonian Railway Company, at Glasgow, Scotland (later a part of the London, Midland, and Scottish Railway Company). Under the late Sir Donald Mathieson—then Chief Engineer but afterward General Manager of the Company—Mr. Yates took part in the rebuilding of the Central Station, at Glasgow.

In 1903, he transferred to the Engineering Staff of Dorman, Long and Company, Limited, at Middlesbrough, England, as a Designer and Estimator. After serving with this firm for four years Mr. Yates received an appointment as Chief Engineer and Draftsman at the works of Redpath, Brown and Company, at Greenwich, England. Late in 1918, he opened a Branch Office for the firm in Brussels to take part in the reconstruction of the war areas in Belgium. From Brussels, he was transferred, in 1920, to the works of the Skinningrove Iron Company, Limited, in Yorkshire, in which Redpath, Brown and Company had acquired a financial interest. At Skinningrove, Mr. Yates was in charge of the Materials Yard and had control of the distribution of materials to the various branches of Redpath, Brown and Company throughout Great Britain. On the termination of the arrangement with the Skinningrove Iron Company, Redpath, Brown and Company entered into a financial arrangement with Bolckow, Vaughan and Company, Limited, at Middlesbrough. Here, again, Mr. Yates had control of the Materials Yard and the distribution of materials to all the branches of Redpath, Brown, and Company, as well as of a small construction plant.

His next move, in 1930, was back to the Greenwich Works of Redpath, Brown and Company where, following an amalgamation between Dorman, Long and Company, Limited, and Redpath, Brown and Company, Limited, he had charge of a new Department in which the purchase and control of tools and equipment for all branches of Redpath, Brown and Company and Dorman, Long and Company, in Great Britain, were centered.

At the time of his death, Mr. Yates was a member of the Mechanical Industry Committee of the British Engineering Standards Institution, dealing with derrick cranes.

During the course of his career, he was responsible for the design and erection of many important bridges, factories, office buildings, and clubs, one of his latest designs being for the twelve-span floating bridge, at Mombasa, Kenya Colony, British East Africa. He was a fine mathematician, and had an extensive knowledge of French and English literature.

Of a generous and genial disposition, Mr. Yates had hosts of friends, particularly in London and on the Northeast Coast of England, and was highly esteemed by all who knew him. His death was due to angina pectoris, to which he succumbed after a short illness.

He was married on December 26, 1902, to Margaret Stewart, at Glasgow, Scotland, and is survived by his widow, a daughter, and two sons.

Mr. Yates was elected a Member of the American Society of Civil Engineers on May 12, 1919.

GEORGE WASHINGTON ARMITAGE, Assoc. M. Am. Soc. C. E.[1]

DIED APRIL 24, 1935

George Washington Armitage was born in Chicago, Ill., on February 22, 1876, the son of William and Delia Armitage. He received his education in the public schools of his native city, after which, in 1896, he removed to New York, N. Y., where he was employed as a Draftsman in the Construction Division of the Board of Education.

Two years later Mr. Armitage went to the Yukon on a prospecting expedition, and a few months after his return, in 1899, obtained a position under the late General William M. Black, U. S. Army (*Retired*), M. Am. Soc. C. E., as a Civil Engineer in Cuba during the First American Intervention. He remained there until the end of the intervention (May 20, 1902), and during that time took an active part in many construction projects. Upon his return to the United States, he obtained a position as a Civil Engineer and Superintendent of Construction in the Quartermaster Corps of the United States Army. He served in that capacity in Chickamauga Park, Georgia, Fort Lincoln, North Dakota, and other posts.

In 1907, at the beginning of the Second American Intervention in Cuba, his services were especially requested by the Cuban Government. Accordingly, he was granted a leave of absence by the War Department and became Chief of the Department of Civil Construction in Cuba. He remained in that post throughout the Second Intervention and at its close, in 1908, returned to the United States and resumed his duties with the War Department.

In 1909 Mr. Armitage was one of the engineers charged with supervising the construction of the Army piers at Fort Mason, San Francisco, Calif. In the latter part of 1912, he was ordered to Schofield Barracks, Hawaiian Islands, as Superintendent in Charge of Construction. He also planned and supervised the construction of Fort Kamehameha.

After war was declared by the United States against Germany, he accepted, in October, 1917, a commission as Captain in the Quartermaster Corps of the National Army, and in the spring of 1918 was ordered to the Port of Embarkation, at Hoboken, N. J. During that year he was promoted to the rank of Major. In 1920, he was commissioned a Captain in the Regular Army. From 1920 to 1924, he was Construction Officer for the Second Corps Area, with headquarters on Governor's Island, New York. His next station was Washington, D. C., where he remained until 1927. During part of his tour of duty there he was a member of the General Staff and did valuable work in connection with the Housing Program of the Army.

In 1928 Captain Armitage was again sent to the Hawaiian Islands and, after a brief tour of duty as Utility Officer at Fort Armstrong, was appointed

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[1] Memoir prepared by George Armitage, Esq., San Francisco, Calif.

Constructing Quartermaster for the Hawaiian Department. While serving in that capacity, his principal work was planning and supervising the construction of Wheeler Field. That project was dear to him, and the barracks and quarters there are considered models throughout the Army.

In 1933, he became Constructing Quartermaster for the Ninth Corps Area, with headquarters at the Presidio of San Francisco. In addition to his regular duties in connection with construction, repair, and alteration in the Area, Captain Armitage planned and supervised the erection of about 600 camps for the Civil Conservation Corps.

On April 24, 1935, he passed away at the Letterman General Hospital, Presidio of San Francisco, following an operation. He was buried at Arlington National Cemetery, in Virginia, with full military honors.

Captain Armitage leaves his widow, Marie Edna Smith, to whom he was married in New York City, in 1902, and a son, George. He is also survived by three sisters and a brother: M. Theresa Armitage, Mrs. S. H. Richman, Mrs. Dom F. Zito, and William Armitage.

Through his long and eventful career with the War Department, Captain Armitage earned the reputation of being extremely able in his profession. He had the faculty of getting things done despite all obstacles, and his determination was one of his outstanding characteristics. He was regarded as an engineer of marked ability and his loss to the Service will indeed be felt. He leaves many monuments to his memory throughout the United States, including many Army Posts, the two most notable of which are Schofield Barracks and Wheeler Field, in the Hawaiian Islands.

Captain Armitage was elected an Associate Member of the American Society of Civil Engineers on April 2, 1902.

FRANK KARR ASHWORTH, Assoc. M. Am. Soc. C. E.[1]

DIED JULY 24, 1935

Frank Karr Ashworth was born on September 11, 1885, at Osceola, Iowa, the son of Robert H. and Ada (Chapman) Ashworth. On his mother's side he was a descendant of Peter Bulkeley, famous in early Colonial history, and also of the Chapman family, well known throughout New England. His paternal grandmother was a member of the Murray family prominent in the establishment of the Universalist Church in the United States. His father was a Civil Engineer, a graduate of the University of Ohio, at Athens, Ohio.

In 1890, the family moved to Colorado Springs, Colo., where for many years the father was City Engineer. In this enviroment Frank Ashworth early took on interest in engineering work and when old enough spent his vacations learning the rudiments of the profession. He went through the grades and High School in Colorado Springs and distinguished himself in mathematics. At this time, also, he became intensely interested in music and determined to perfect himself in his chosen instrument, the piano. To that end he went to the Bush Temple Conservatory of Music, in Chicago, Ill., in the autumn of 1904. His work there was outstanding and he won a scholarship. However, in 1906, when he was obliged to choose between music and engineering as his life work, the latter won although the former remained a source of great pleasure and inspiration throughout his life.

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[1] Memoir compiled from information on file at the Headquarters of the Society.

Since a university or technical school training was not possible, Mr. Ashworth early formed the habit of studying by himself, a habit which remained with him and which resulted in his becoming a highly educated and exceptionally well-read man.

At the beginning of 1906, Mr. Ashworth entered the office of the City Engineer in Colorado Springs as Transitman and by 1908 was in charge of the design and construction of many municipal improvements. He remained in Colorado Springs, gaining broad and valuable experience, until late in 1911, when he went to San Diego, Calif., to join the Staff of the San Diego County Highway Commission and, later, the San Diego Securities Company, in charge of general sub-division improvements.

In the early spring of 1913, Mr. Ashworth went to Florida to join his father who had gone there to take charge of the development of a large project, and they opened an office for private practice in the City of Miami. This partnership continued until the death of the father in 1916, after which Mr. Ashworth continued the business alone. In the rapidly growing community of which Miami was the center, he built up a large practice, interesting and diversified, including the design and construction of drainage systems, roads, sea-walls, and various types of municipal improvements. This general practice was interrupted in January, 1917, when he became Chief Engineer for the Chevalier Corporation and spent his entire time in planning the development of its holdings in Southern Florida, amounting to several thousands of acres. He planned a drainage system and a railroad to tap the Great Cypress Swamp.

This project was interrupted by the World War. On September 1, 1918, Mr. Ashworth went to Camp A. A. Humphreys, in Virginia, where he was commissioned a First Lieutenant of Engineers. On his return to civil life in December, 1918, he resumed his private practice in Miami and specialized in sub-division design and supervision. In that community this included drainage, sea-walls, and dredging, in addition to the usual municipal improvements. During the years of Florida's most rapid development, he designed and supervised the work on many of the most beautiful sub-divisions of Miami, including the famous Miami Shores, the plans for which included the expenditure of more than $5 000 000.

In 1930, Mr. Ashworth decided to return to the West and went to Los Angeles, Calif. He did some special work for that City in the Park Department, and then became interested in valuation engineering. He was employed by the Southern California Gas Company, the City of San Diego, and, later, the State Board of Equalization, in this capacity. In April, 1935, he was offered a position with the State Highway Commission, and was stationed at Redding, Calif. A serious attack of bronchial pneumonia in January had gravely affected his heart, however, and on July 24, 1935, he passed away in his sleep. He is survived by his widow, formerly Myrtle English, to whom he was married on March 24, 1915, and by one brother, Raymond C. Ashworth, Assoc. M. Am. Soc. C. E., Valuation Engineer for the Los Angeles Railway Company.

Mr. Ashworth was a Mason, and while in Florida was a member of the Florida Engineering Society, the Florida Natural History Society, the Civitan Club, Miami City Club, and the Miami Section of the Society.

Although quiet and somewhat shy in manner, Mr. Ashworth made many close friends, and he had established an enviable reputation for absolute integrity and loyalty in both personal and business associations. His mental equipment was far beyond the average, and his fine spirit never faltered in the face of adversities, both material and physical, which would have conquered a less courageous soul.

Mr. Ashworth was elected an Associate Member of the American Society of Civil Engineers on June 18, 1919.

THERON McCABE BROWN, Assoc. M. Am. Soc. C. E.[1]

Died March 3, 1935

Theron McCabe Brown was born at Winfield, Kans., on February 19, 1897, the son of Lewis and Orlena L. (Walrath) Brown. His elementary education was obtained in the public schools of Winfield and Hutchinson, Kans., after which he studied at Lombard College, Galesburg, Ill., where he specialized in mathematics. In 1917 he enrolled at the University of Cincinnati, Cincinnati, Ohio, where he was a student of engineering until September, 1918, when he entered the Army Training Corps at the University. Like many college men of this period, his studies were seriously interrupted by the war in Europe, but he was able eventually to complete his technical training through extension courses with the University of Wisconsin, Madison, Wis.

After the World War Mr. Brown was employed as Rodman and Instrumentman on highway and drainage surveys under County Engineer W. B. Harris, at Hutchinson. He then became Secretary-Treasurer of the Southwestern Engineering Company, of Hutchinson, organized to engage in highway and bridge engineering. In December, 1920, Mr. Brown accepted the position of Assistant Engineer to O. N. Powell, County Engineer, at Corpus Christi, Tex., where he was engaged on the design of the Corpus Christi Causeway until the early part of 1921. On the completion of this work he returned to private practice at Hutchinson for about a year. In April, 1922, he was employed as Designer and Construction Engineer in the office of Horner and Wyatt, Consulting Engineers, Kansas City, on industrial development and reclamation projects, where he remained until March, 1924.

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[1] Memoir prepared by C. L. Williford, M. Am. Soc. C. E.

From April, 1924, to November, 1927, Mr. Brown was Designer and Office Engineer for Nagle and Thompson, Consulting Engineers to the City of Dallas, Tex., on the design and construction of the Garza Reservoir, a $5 000 000 water-supply project, and one of the largest municipal undertakings in the Southwest, which positions he filled with high credit. On the completion of this project, he became Designing Engineer in the Department of Water Supply of the City of Dallas, of which John B. Winder is Superintendent and Chief Engineer. In this position, Mr. Brown was engaged, until April, 1930, in the design and construction of new municipal pumping plants, costing approximately $800 000.

On the completion of his work with the City of Dallas, Mr. Brown became associated with the Frank Parrott Construction Company, of Dallas, as Engineer and Construction Superintendent. This Company was then engaged in the construction of a large viaduct across the Trinity River, in Dallas, and several important grade separation structures in Dallas County. Mr. Brown handled the field operations on these structures to successful completion, and they stand as monuments to his industry.

From April, 1932, to November, 1933, Mr. Brown was connected with the Engineering Staff of the State Highway Department at Dallas, on special designs and construction of important highway bridges. At this time he also inaugurated preliminary studies for the Department and the City of Dallas on the proposed Elm-Main-Commerce Street grade separation at the Union Terminal Railway which is now under construction at a total cost of nearly $1 000 000. He, no doubt, would have been placed in responsible charge of this notable project, but while these studies were in progress he was drafted by the Director of Public Works in Texas, of the National Administration, to supervise all Public Works Administration construction work in the Dallas Area. It was while engaged on this work that he suffered severe injuries in an automobile accident which caused his death a week later, on March 3, 1935. Mr. Brown's untimely death at his prime was a severe shock to his many friends and associates, and most of all to his family.

He was married on June 14, 1920, to Pauline A. Lockhart, of Dallas, who, with their daughter, Marie Adell, and a son, Lewis, survives him. He is also survived by his father and mother, and two sisters, Ruth Evelyn Brown, of Hutchinson Kans., and Rhoda A. Brown, of Los Angeles, Calif.

He was a Mason and a Shriner, and for many years had been an active member of the Episcopal Church, in Dallas. In his home, he was a true husband and father. By those who knew him, he was most highly esteemed as an upright citizen, a Christian gentleman, and a thorough and competent engineer. He was laid to rest among the scenes of his boyhood at Hutchinson, Kans.

Mr. Brown was elected an Associate Member of the American Society of Civil Engineers on November 12, 1928.

GEORGE BUTLER, Assoc. M. Am. Soc. C. E.[1]

Died April 16, 1935

George Butler was born on February 6, 1864, at Boston, Mass., the son of Major George Butler, United States Marine Corps, and Alice (Kierulff) Butler. He was educated in the public and private schools of the region, and at St. Johnsborough Academy, in Vermont. In early youth he suffered a severe accident which interrupted his schooling, and the major part of his engineering education was obtained by constant home study and reading.

In 1887, Mr. Butler moved to Colorado Springs, Colo., where he entered the City Engineer's Office as Draftsman and Instrumentman, later transferring to the County Surveyor's Office of El Paso County, Colorado. In 1888, he entered the employ of the Hartford Loan and Trust Company as Computer and Draftsman on irrigation projects and, in 1889, became Assistant to the Chief Engineer of this Company.

In 1890, Mr. Butler entered into private engineering practice in and around Denver, Colo. In 1891, he joined the United States Indian Bureau, Department of the Interior, under the late Walter Haden Graves, M. Am. Soc. C. E., as Draftsman for the Superintendent of Irrigation and Special Disbursing Agent. In 1892, Mr. Butler was made Principal Assistant to the Superintendent and placed in charge of the design of wooden and masonry irrigation structures, the location of canals, and in immediate charge of the construction of irrigation works in the Crowe and Navajo Indian Reservations. In 1895, he was placed in charge of this work as Superintendent of Irrigation and Special Disbursing Agent.

In 1899, Mr. Butler was appointed General Superintendent of Irrigation with the Indian Bureau in charge of examining, reporting upon, designing, and constructing irrigation, water supply, sanitary, and other engineering works. This work took him to the various Indian reservations in Arizona, New Mexico, Colorado, Wyoming, Nevada, and California.

In 1907, he resigned from the Indian Bureau to enter private engineering practice in and around the City of San Diego, Calif., and, in this same year, he was appointed to the office of County Surveyor of the County of San Diego.

It was during this period that his ability for re-organization and foresightedness stood him in good stead. The use of the automobile for transportation developed rapidly during these years, and the necessity of designing and constructing roads to meet this new method of transportation became apparent, roads had to be relocated and widened, sharp curves and grades eliminated, and drainage facilities provided. This necessitated the mapping of a comprehensive plan of road development and the re-organization of the office to meet these demands. This general use of automobile transportation also brought with it the subdivision of many outlying tracts of land for sale, and Mr. Butler at once realized that some regulations would have to be placed upon the promiscuous sale of lots to the public; it was due in a large part to his efforts that County ordinances and State laws were passed requiring the subdivider to meet certain regulations before his map could be filed and he was allowed to sell the property. This regulation of subdivisions in later years led to the formation of a County Planning Commission. In assuming the functions of County Highway Engineer and Planning Commissioner the once obscure office of County Surveyor became one of the most important departments of the County and was handled very ably by Mr. Butler.

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[1] Memoir prepared by Paul R. Watson, M. Am. Soc. C. E.

In 1923, he again entered into private engineering practice, specializing in the legal aspects of land surveying, and he was considered an authority on boundary disputes.

As a hobby, Mr. Butler took up photography and his pictures of some of the rugged scenery of Arizona compare favorably with any professional work of this nature. He also took a great interest in the rites, ceremonies, and customs of the American Indian with whom his work with the Indian Bureau brought him in close contact, and he possessed a valuable collection of Indian relics and handicraft.

Mr. Butler was a member of the Masonic Order. He was the first President of the San Diego Section of the Society, and it was during his régime that the licensing of practicing engineers first came to the front in California. It was not until 1929, however, that a law was enacted by the State of California licensing the practice of Civil Engineering in the State.

He was married in 1893 to Bessie F. Wilstack, who survives him. He was a detail man, working from early boyhood under a great physical handicap, patiently and well. The one who knew him best states that whatever the game, however the stress, he always played fair and tried to help the other fellow.

Mr. Butler was elected an Associate Member of the American Society of Civil Engineers on September 6, 1910.

LEVI OSCAR COLEMAN, Assoc. M. Am. Soc. C. E.[1]

Died July 9, 1935

Levi Oscar Coleman was born at Paradise, Ky., on March 19, 1887, the son of P. S. and Arvilla Coleman. After passing the usual Grammar and High School courses, he entered the College of Engineering of the University of Kentucky, at Lexington, from which he received the degree of Bachelor of Civil Engineering in 1914.

During the summer of 1912, he began his engineering career as a Rodman and Levelman for Mr. John B. Wilson, of Hartford, Ky. His 1913 summer vacation was spent as Resident Engineer on five miles of sanitary sewer constructed by the City of Madisonville, Ky.

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[1] Memoir prepared by A. H. Riney, Assoc. M. Am. Soc. C. E.

After receiving his degree in 1914, young Coleman went with the Delaware, Lackawanna, and Western Railway Company, as Rodman and Inspector on the construction of its Buffalo, N. Y., Terminal. However, the salary offered by the Railroad Company was not sufficient to hold the young engineer, and when in August of the same year a better offer was received from the Interstate Commerce Commission, it was promptly accepted, and from that time until the entry of the United States into the World War, Mr. Coleman worked out of Chattanooga, Tenn., on railroad valuation surveys. Starting as a Tapeman, he had progressed through the grades of Chainman and Rodman to that of Junior Engineer when war was declared.

At the outbreak of the World War Mr. Coleman enlisted, and, of course, was assigned to the Engineers. From May to August, 1917, he was in training—first at Fort Benjamin Harrison, Indiana, and then at Fort Leavenworth, Kansas; and, in August, he was commissioned Second Lieutenant of the 23d Engineers, and stationed at Camp Meade, Maryland, where he was engaged in training engineer troops. In February, 1918, he sailed for France, and during the remainder of the war—a period of sixteen months—he served as Roads Service Officer, building and repairing roads in the zone of the armies of the American Expeditionary Forces. Most of the roads were of waterbound macadam, and Lieutenant Coleman's work included the supervision of the operation of several quarries and stone crushers, about fifty trucks, several rollers, and from 100 to 350 men, including soldiers, labor troops, and prisoners of war. The commission of First Lieutenant was received in September, 1918. He took part in the St. Mihiel Offensive, the Meuse-Argonne Offensive, and the activities in the Toul Sector.

Lieutenant Coleman returned to the United States in July, 1919, and after receiving his honorable discharge in August, went with the Illinois Division of Highways, and was stationed in District 4, with headquarters at Peoria. Starting as a Resident Engineer, his summers were spent in supervising grading and drainage projects, concrete paving, and railroad grade-crossing construction, and during the winters he conducted surveys, made field and office locations, and designed reinforced concrete culverts.

After a time as a Resident Engineer, Mr. Coleman was made Construction Engineer of District 4, which called for the supervision of the various Resident Engineers. During one season, he had more than 100 miles of concrete pavement under construction. One of the most interesting jobs that he supervised was the building of 1½ miles of "sea wall" to protect pavements from being flooded by the Illinois River. The winter months during this period were spent in supervising the location parties operating in the twelve counties of the District.

While with the Illinois Division of Highways, Mr. Coleman began taking a series of Highway Short Courses for graduate engineers at the University of Illinois, at Urbana, Ill., covering Highway Administration, Finance, and Economics. These courses were taken in 1920, and in each of the years from 1922 to 1926.

His final position with the Illinois Division of Highways was as District Maintenance Engineer of the Peoria District, directing the maintenance and all force account construction for additions and betterments of several hundred miles of all types of highways.

In April, 1923, he accepted an offer from the L. S. Kuhn Construction Company, of Bloomington, Ill., and for the next eighteen months he was Construction Superintendent for that firm, in charge of grading, drainage, concrete pavement, and bridge construction projects, using an industrial railway outfit for hauling the aggregate, tractor-drawn elevating graders, and pile-driving outfits. His duties included not only the supervision of construction, but the making of bidding estimates, designing of plant layouts, and the keeping of cost-account records.

This cost-accounting experience was put to profitable use in August, 1924, when it became necessary to leave the Construction Company due to its failure to obtain additional contracts, and he accepted the position of Assistant Highway Engineer in charge of the office of the Illinois Division of Highways, at Springfield. In this capacity, Mr. Coleman supervised about twenty employees, paying bills, and keeping cost-account data on all maintenance and force account construction in the State. Costs and methods in the various Districts were compared in order to promote uniformity of methods and reduction of costs. While in charge of this office, he installed the cost-accounting system now used, and designed many of the devices employed in collecting maintenance cost data. He also collaborated in the design and installation of the maintenance cost-accounting system that was adopted later by most of the States of the Mississippi Valley Highway Association.

When he was offered a higher salary with the State Highway Department of Kentucky, in April, 1926, Mr. Coleman left Illinois to return to his native State as Assistant District Engineer, at Louisville. Here, he was in direct charge of all work done by contract in the District, which, during the three years he held the position, amounted to about $6 000 000. The work included heavy grade and drainage projects, paving, and other types of surfacing, and many concrete structures. Mr. Coleman's duties included also the supervision of surveys, the making of contractors' payment estimates, and the preparation of estimates preliminary to requesting bids. While in this position, he supervised the construction of 3½ miles of napped, stone base, water-bound, macadam road by force account.

In January, 1929, Mr. Coleman left the Highway Department of Kentucky to accept a more remunerative position as Vice-President, and Stockholder, of the Southeastern Construction Company, of Eminence, Ky. Acting as General Field Superintendent, his duties included the making of bidding estimates, the purchase of materials, and the superintendence of cost accounting. In 1929 and 1930, the Company constructed about $1 000 000 of projects, consisting of 75 miles of heavy grade and drainage work, together with bridges. Four steam shovels, ten caterpillar tractors, four scraper outfits, steam pile-driving equipment, and one hundred horses and mules were utilized.

Feeling that the equipment sales business had more possibilities than construction supervision, his interest in the Southeastern Construction Company was sold in April, 1931, and Mr. Coleman went with the Roy C. Wayne Supply Company, of Louisville, selling contractors' equipment in Kentucky. However, the depression caused such a severe curtailment in contractors' equipment purchases that commissions soon dropped below running expenses and shortly after the first of 1932, the sales position was abandoned.

In July, 1932, Mr. Coleman returned to the Kentucky Highway Department, as Superintendent of force account construction, in charge of the building of oiled roads and the construction of heavy grade and drainage projects. He remained with the Highway Department until April, 1935, when he left road building to return to the life that he had loved—that of the Army. He had received the commission of Captain in the Reserve Corps of Engineers on April 12, 1929, and in April, 1935, he was offered the opportunity of going into active service in charge of a camp of the Civilian Conservation Corps which he accepted. He was sent to Fort Knox for preliminary training, preparatory to taking charge of a camp, and then to Camp Benton, Kentucky. He was there until June 3, 1935, when he received an order of discharge because of a physical disability. He underwent an operation and was well on his way to recovery in the Jewish Hospital, at Louisville, when, on July 9, he died suddenly of a blood clot in the heart.

Captain Coleman had a high reputation in his profession, being known not only as a very careful, but also as a successful, engineer. He was especially proficient on work requiring high fills and one of his last jobs was the supervision of the construction of a section of concrete road on the Dixie Highway entering Louisville from the south, which was especially difficult because of the great fill. He never avoided responsibility and was well liked by his associates, having a very pleasing personality.

He was deeply interested in Army affairs from the time of his war service, and was very active in the Reserve Corps, of which he was a Captain in the 380th Engineer General Service Regiment. He often drove many miles to attend the Training School of the Regiment.

He was married at Owensboro, Ky., on May 29, 1920, to Eddith Duke. He is survived by Mrs. Coleman, a son, Bruce Sutton Coleman, a daughter, Anne Coleman, three brothers, and three sisters. He was buried at Owensboro, Ky.

Captain Coleman was elected an Associate Member of the American Society of Civil Engineers on October 15, 1923.

HENRY LEO CONNELL, Assoc. M. Am. Soc. C. E.[1]

Died March 25, 1935

Henry Leo Connell was born in New York, N. Y., on December 18, 1879, the son of Hugh G. and Ellen (Spence) Connell, and throughout his lifetime was identified with New York City. He was educated in its schools, and obtained his scientific education in that efficient school—Cooper Union.

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[1] Memoir prepared by Lazarus White, M. Am. Soc. C. E.

Mr. Connell entered the service of New York City in 1900 which service continued until his death, except for a period when he was in business for himself as a Contractor.

Entering the City service as Axeman, Mr. Connell served as Axeman, Rodman, Leveler, Transitman, and Section Engineer on subway construction for the Rapid Transit Commission and with the Bridge Department. When the Board of Water Supply undertook the task of bringing Catskill water to New York City in 1905, Mr. Connell was placed in charge of the surveying of the Esopus and Rondout water-sheds, for which work he was well fitted by experience and physique, having been a notable oarsman in his youth. Later, he was in charge of the construction of part of the Ashokan Dam and also a part of the Catskill Aqueduct in New York City.

Ambitious to have an independent business, Mr. Connell, in 1917, organized a contracting firm under the name of The Tunnel Construction Corporation. This Corporation was active for fifteen years and constructed many important works, among which were foundations for the great "Skyway" across the New Jersey meadows, sewage treatment works for the Cities of Lynn, Mass.; and Lakewood, Ridgefield, and Teaneck, N. J. He was President of the Corporation to within a few years of his death.

In 1923, Mayor Hylan, anxious to secure an engineer whose integrity could not be questioned, induced Mr. Connell to accept the appointment as Engineer Member of the Board of Standards and Appeals. This Mr. Connell accepted as a public duty, although it took a great deal of time from his own business. The Board of Standards and Appeals was a most important body, having the power of review of decisions of the Building Superintendents, interpretations of the Zoning Laws, Building Code, etc. As this was a period of intense building activity, great pressure for favorable decisions was brought to bear upon the members of the Board. Mr. Connell manfully accepted his responsibility and mastered the complicated provisions of the Building Code and practices of New York City. When at length charges were brought against the Board—which were never legally substantiated—no suspicions were directed toward Mr. Connell, and he was made Acting Chairman of the Board, serving in this difficult position and in the difficult time of 1930 to 1932, with honor. In spite of his ill-health, Mr. Connell was active to the end of his life.

He was married to Geneveve M. Martinez, on October 18, 1915, who, with two sisters, M. Katherine Connell and Louise H. Connell, and one brother, Jasper Spence Connell, M. Am. Soc. C. E., survives him.

Leo Connell well typified in his life's work a large number of engineers who devote a lifetime to the service of great cities, bearing their burdens, expecting little reward except the satisfaction of service well done.

Mr. Connell was elected an Associate Member of the American Society of Civil Engineers on January 4, 1910.

ERNEST HARRY CORNELIUS, Assoc. M. Am. Soc. C. E.[1]

Died May 8, 1934

Ernest Harry Cornelius, the son of Harry B. and Georgia P. Cornelius, was born at Hastings, Nebr., on October 28, 1890. He attended the public schools of his native town and was graduated from the High School in 1907. He then entered Northwestern University, at Evanston, Ill., but due to an injury received as the result of a fall during his Junior year, he was obliged to leave school. After a residence of approximately eighteen months in California, in order to regain his health, Mr. Cornelius returned to his home in Nebraska and entered the State University, at Lincoln, from which he was graduated in 1913, with the degree of Bachelor of Science.

From September, 1913, to January, 1914, Mr. Cornelius acted as Salesman and Representative of the Jacques Steel Company, in Northern Texas. In January, 1914, he went to Kansas City, Mo., where he had charge of the appraisal of the building of the Kansas Light and Power Company made for the Armour Interests of Chicago, Ill.

On the completion of this work in May, 1914, Mr. Cornelius entered the employ of the Metropolitan Street Railway Company, of Kansas City, as Inspecting Engineer. As such, he was engaged on the construction of new trackage, building and car designs, etc., until April, 1916. From that time until November of the same year he served as Assistant to the Bridge Engineer of the Kansas City Southern Railway Company.

In November, 1916, Mr. Cornelius removed to Tulsa, Okla., where he became associated with the Oklahoma Structural Steel Company, first as Contracting Engineer in charge of sales, etc., and, later, as Chief Engineer and Sales Manager. In January, 1918, he entered private practice as a Consulting Engineer, with headquarters at Tulsa. He was engaged chiefly on the construction of steel buildings in the oil fields, incorporating in 1919, as the Industrial Construction Company.

In 1921, Mr. Cornelius founded the Oklahoma Steel Castings Company which, however, did not become active until 1922. Although Mr. Cornelius had many other interests, he devoted his time and energies as President to this Company. Due to his untiring efforts the Oklahoma Steel Castings Company is now classed as one of the important foundries of its kind, covering practically all trade in steel castings in the southwestern part of the United States; it is also nationally known.

Mr. Cornelius was active in the various Masonic bodies of his city and State, a Past-President of the Rotary Club of Tulsa, and a member, and Chairman of the Financial Committee, of the Boston Avenue Methodist Episcopal Church. He was also a member of the Board of Trustees of Tulsa University.

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[1] Memoir prepared from information supplied by Burtner Fleeger, Pres., Oklahoma Steel Castings Co., Tulsa, Okla., and from data on file at the Headquarters of the Society.

He was greatly interested in civic affairs and the development of industry within Oklahoma. He was a Past-President and also a Director for years of the Associated Industries of Oklahoma; a Director of the Chamber of Commerce; and was always active in the Tulsa Community Fund, and other charitable organizations of the city. Previous to his last illness, Mr. Cornelius had been appointed to the National Code Commission for the Steel Castings Industry.

His social clubs included the Tulsa Club, the Tulsa Country Club, and the Sigma Nu Fraternity. He was also a member of the American Society of Mechanical Engineers.

Mr. Cornelius was greatly interested in aviation and, in addition to his many activities, found time to take up flying first as a hobby and, later, as a means of business transportation.

He was an ardent lover of Nature and of his fellow men, and no one ever came to him for aid and was turned away. His associates in business and his employees knew him as the kindest and most considerate of employers, always ready to listen to their troubles, advise them, and help in every manner possible. During the depression although he could have saved considerable money by closing down his plant, he kept it in operation, with the characteristic statement, "because of the hundred families dependent on us for a living. * * * We cannot forget our responsibility to those who are less fortunate."

Mr. Cornelius died in Cleveland, Ohio, on May 8, 1934, from complications following an operation, and was buried at Rose Hill Cemetery, in Tulsa.

He was married on October 17, 1914, to Virginia Moseley, of Lincoln, Nebr. Mrs. Cornelius died in 1923. He is survived by three children, Virginia, Marjorie Ann, and Ernest Cornelius, Jr.

Mr. Cornelius was elected an Associate Member of the American Society of Civil Engineers on July 11, 1921.

CHARLES WESLEY ERISMAN, Assoc. M. Am. Soc. C. E.[1]

DIED JUNE 15, 1934

Charles Wesley Erisman was born in Lancaster, Pa., on October 31, 1887, a son of Clement S. and Salome (Kleckner) Erisman. His father was a successful contractor and builder in the vicinity of Lancaster for many years.

Mr. Erisman received his education in the public schools of Lancaster and was graduated from the High School. His engineering knowledge he acquired in home study and through association and experience with engineers and engineering construction, having worked his way up from the position of Rodman to stations of responsibility in engineering and business fields.

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[1] Memoir prepared by Ralph L. Kell, Assoc. M. Am. Soc. C. E.

His first engineering experience was obtained in 1905 in the office of F. H. Shaw, a special engineer engaged by the City of Lancaster to design and supervise the construction of certain water-works and sewerage improvements. In this position Mr. Erisman performed the duties of Rodman, Chainman, and Draftsman. After completing his engagement with Mr. Shaw in 1906, he secured employment with the Pennsylvania Railroad Company with which he served for a period of six months as Rodman, Chainman, and Instrumentman.

In March, 1907, Mr. Erisman made an engineering connection with the Pennsylvania Department of Highways in which field he was destined to direct his energies for the remainder of his life. For a period of nine years he was identified with the Bureau of Township Highways, serving in the capacities of Rodman, Chainman, Instrumentman, Chief of Party, Draftsman, Inspector, and Assistant Engineer. His service as Assistant Engineer covered a period of more than two years, during which time his activities were directed toward the betterment of rural roads that were under the care of Township Road Supervisors. He had under his charge the supervision of the construction of roads and bridges by Township Supervisors over the entire State of Pennsylvania, and was instrumental in organizing Township Boards into County Units, after which Schools of Instruction were formed in the various counties, with sessions held annually or semi-annually. These schools were the means of bringing the construction of rural roads under a uniform standard, resulting in great improvement in appearance and general utility.

In 1916, Mr. Erisman was promoted to the position of District Engineer in the Pennsylvania Department of Highways, in which capacity he had charge of the construction and maintenance of State highways in several districts of the departmental organization, the types of roads involved being water-bound macadam, bituminous macadam, brick, and concrete.

During Mr. Erisman's connection with the Pennsylvania Department of Highways the organization grew from a small unit at the beginning of the automobile era to the vast proportions now found in various Commonwealths of the Union. He was thus brought into close contact with the pioneering and experimental work required in building up a system of highways from the early dirt roads and turnpikes to the present type of road designed for high-speed and heavy traffic, and thereby acquired a first-hand knowledge of road engineering and materials which was a valuable asset in his later work.

In 1921 he severed his connection with the Pennsylvania Department of Highways to engage in the private practice of Highway Engineering and the distribution of road machinery and bituminous and tar road materials, with headquarters in Lancaster. His engineering service took the form of surveys, plans, and designs for roads, streets, and bridges, together with a consulting service for individuals and road-building authorities. He was the local distributor for "Ugite" road tars, his territory embracing several counties in Southeastern Pennsylvania. In 1925, he built an office and warehouse in Lancaster to improve his facilities for handling his business. In 1931, he added a bulk storage plant with a capacity of 50 000 gal which enabled him to develop an extensive business in the distribution of oils and tars. Increasing demands required him to enlarge his storage facilities by 10 000 gal in 1933, at which time he also constructed a mixing plant for the manufacture of bituminous cold-patch material used in repairing macadam roads.

Mr. Erisman was not able to reap the full benefit from the expansion of his business facilities, as failing health necessitated the curtailment of his activities. Although he was still in the prime of life, being only 48 yr of age, he was unable to check the decline in his physical condition, and death followed at his home in Lancaster, on June 15, 1934. The community in which he lived had learned to love and respect him. He was a man of sterling character, such as the world cannot afford to lose. Through his straightforward business dealings and honorable practices, he had built up a large circle of social and business friends who mourn his untimely death. He took an active interest in community affairs and was affiliated with the Masonic Fraternity, the Royal Arcanum, the Lancaster Monarch Club, and the Lancaster Chamber of Commerce.

On November 23, 1909, Mr. Erisman was married to Barbara M. Rogers, a daughter of Augustus and Mary (Young) Rogers, of Intercourse, Lancaster County, Pa., who survives him. Surviving also is a daughter, Lucretia M. (Mrs. Walter C. McMinn, Jr.), and one son, Charles A. Erisman.

Mr. Erisman was elected an Associate Member of the American Society of Civil Engineers, on August 28, 1922.

RALPH DANIEL HAYES, Assoc. M. Am. Soc. C. E.[1]

Died February 20, 1935.

Ralph Daniel Hayes was born in Hawkinsville, N. Y., on June 24, 1881, the son of Oscar Willis Hayes and Emma (Jones) Hayes. On his mother's side he was a direct descendant of Ethan Allen, who assisted eminently in establishing American independence. His father was born in Hawkinsville in 1853, and his great-grandfather was a pioneer in that section. His mother, who died there a few weeks after her son's death, was also a native of Hawkinsville, with which her family had long been prominently identified.

Mr. Hayes' early life was spent in Hawkinsville, but he was educated in the public schools of Boonville, N. Y. In 1900, he entered Clarkson College, at Potsdam, N. Y., from which institution he was graduated in 1904 with the degree of Bachelor of Science in Civil Engineering. During his college years, he worked with the United States Lake Survey, on topographic and hydrographic surveys of the St. Lawrence River. In this work—his first engineering assignment—he served under F. C. Shenehon, M. Am. Soc. C. E.

After his graduation from Clarkson College, Mr. Hayes was employed for a short time with the United States Geological Survey in the Adirondack Mountains, and then joined the staff of the then New York State Engineer,

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[1] Memoir prepared by Frank C. Boes, M. Am. Soc. C. E.

Henry A. Van Alstyne, M. Am. Soc. C. E.; thereafter, he was engaged in highway work from 1904 to 1907.

In 1908 he transferred to the New York State Barge Canal Department, where he was importantly engaged in the construction of that system of waterways, remaining on this work until 1923, when he was transferred to the Operating and Maintenance Division in the Albany District of the Department of Public Works.

For the next four years he remained in Albany continuously, being actively engaged in various branches of this great waterway, and thus familiarizing himself with every detail of its operation. In 1927, he was appointed Deputy Commissioner, and Engineer Assistant to the Commissioner, Division of Canals, Major Thomas F. Farrell. During his long experience in the construction, operation, and maintenance of the Barge Canal, Mr. Hayes acquired a knowledge concerning this Department that was invaluable to the State of New York. This knowledge, and his ability to inspire full confidence, co-operation, and respect in his assistants and in the public with whom he came in contact, were recognized by his superiors and by the Hon. Franklin D. Roosevelt, then Governor of the State of New York, and, on February 1, 1930, although he had never engaged in politics, Mr. Hayes was appointed Commissioner of Canals and Waterways in the Department of Public Works, to succeed Major Farrell, who was made Chief Engineer of the Department of Public Works at that time. Mr. Hayes served in this position with distinction, looking after every detail of this 500-mile system of canals until his death. He had faithfully served his State for thirty years.

All his appointments and promotions were under the New York State Civil Service. He was the first man to be promoted step by step through the Civil Service from the grade of Leveler to the high post of Commissioner of Canals and Waterways.

Frederick Stuart Greene, M. Am. Soc. C. E., Superintendent of Public Works of New York State, remarked that in all the thirty years' service that he gave to the State, not one hour or one minute of that time did Mr. Hayes swerve from his duty. He so impressed both Governor Roosevelt, and Governor Lehman, that on several occasions they saw fit to show publicly their deep regard for him.

In his college days Mr. Hayes was an earnest student and an honor man, with a full realization of his duty to his parents to make the most of his opportunities and thus show to them his appreciation of their efforts to fit him for an important position in life. At that early age, he possessed unusual dignity, sympathy, and kindliness, and these characteristics were outstanding with him throughout his life. He showed courtesy and thoughtfulness toward others to an unusual degree. His quiet sense of humor was delightful and drew to him many friends who held him in deep affection.

His ethical standards were above question and he was always alert to the responsibilities of his position and his profession. At his death Mrs. Hayes received hundreds of messages of sympathy and condolence from his many friends. Flags along the entire route of the Barge Canal flew at half-mast.

Governor Lehman, with Mrs. Lehman, attended the funeral, as well as many others prominent in the Engineering Profession and in public life.

Governor Lehman issued this statement concerning Mr. Hayes:

> "I regarded him as one of the finest and most valuable men in the State service. He was untiring in his work and uncompromising in his devotion to duty. He gained public good-will because he impressed people with his fairness and integrity. I mourn the loss of a loyal, efficient, and devoted public servant."

The *Knickerbocker Press*, of Albany, N. Y., of February 20, 1935, published the following editorial:

> "To say that Ralph D. Hayes was an excellent public official does not give him his full meed of merited praise. He was more than an official. As Commissioner of Canals and Waterways, he went far beyond mere duty. To him, office was a public trust. He was modest and he felt himself a servant of the people and he joyed in giving more than mere service.
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> "Probably few men were held in such high personal regard as was he. To know him was to know a man to whom honesty and fairness was a creed and duty a religion. Throughout the State he served is a feeling of real loss in his death, loss of a man with high ideals of citizenship, a man with an unusual sense of honor and obligation. Ralph Hayes was an exemplary man and official, whose whole life was an inspiration to others."

It goes without saying that such spontaneous and sincere tributes from great and small, showed that here was a most unusual man. They indicate Mr. Hayes' real character and worth far more clearly than volumes of technical accounts of his achievements. His death was a shocking loss to hundreds of friends who will never cease to honor his memory. The writer himself, treasuring many memories of Mr. Hayes from boyhood together, cannot conceive how any one could have come in contact with him without having been made better by the experience.

Mr. Hayes died in the Post Graduate Hospital, in New York, N. Y., as the result of an operation from which he had almost fully recovered, but from which he unexpectedly suffered a relapse. He was buried at Hawkinsville, N. Y., in the family plot, in a cemetery established for the use of his community by his great-grandfather.

In September, 1908, he was married to Genevieve Deneen, of Fort Covington, N. Y., who survives him.

Mr. Hayes was elected a Junior of the American Society of Civil Engineers, on June 4, 1907, and an Associate Member on August 31, 1909. He was also a member of the Albany Society of Civil Engineers.

CHARLES JAY HOGUE, Assoc. M. Am. Soc. C. E.[1]

DIED MAY 28, 1935

Charles Jay Hogue, the son of Dan C. and Rebecca (Starr) Hogue, was born at Watsontown, Pa., on July 12, 1870. After graduation from the Watsontown High School he pursued special studies at Lehigh University, at Bethlehem, Pa., and, in 1890, entered the field of railroad engineering as Draftsman for the Central Railroad Company of Pennsylvania, subsequently removing to Wheeling, W. Va., where he held a similar position with the Wheeling and Citizens Street Railway Company.

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[1] Memoir prepared by William P. Starr, Esq., Hyattsville, Md.

In May, 1892, Mr. Hogue entered the service of the Short Line Improvement Company as Transitman and Draftsman on the location of a line of railroad from Berwick, to Williamsport, Pa. From May to November of the following year, he was employed as Draftsman and Assistant on Track Work with the Cleveland, Cincinnati, Chicago, and St. Louis Railroad Company. In November, 1893, he entered the service of the Central Pennsylvania and Western Railroad Company as Instrumentman and Chief of Party, engaged on location surveys from Berwick to Stroudsburg, Pa. During 1895, Mr. Hogue was employed as Instrumentman on the Montauk Extension of the Long Island Railroad, from Bridgehampton to Fort Pond Bay, Long Island, under the direction of Frederic Molitor, M. Am. Soc. C. E., Engineer in Charge; he later served as Chief of Party on extensive topographical surveys of Montauk Point.

From 1896 to 1902, Mr. Hogue was again attached to the corps of Mr. Molitor, then Chief Engineer of the Choctaw, Oklahoma and Gulf Railroad. During this time he located the 50-mile extension, from El Reno to Weatherford, Okla., also a part of the 130 miles from Howe, Okla., to Little Rock, Ark. As Division Engineer, he had charge of the rehabilitation of the Choctaw and Memphis Railroad (formerly the Little Rock and Memphis Railroad). This work involved the raising of grades to conform with the St. Francis Levees along the Mississippi River, in Arkansas; relaying 132 miles of heavier steel; moving buildings; laying out yards; designing shore protection, sewer, and water lines; and supervising the erection of shop buildings.

In 1903, Mr. Hogue became Engineer of Maintenance of Way for the Rock Island System, leaving that Company in 1904 to assume charge of a party on the location of the Midland Valley Railroad from Muskogee to Tulsa, Okla. In the following year, he went to New York, N. Y., and was attached for a short time to the office of the Designing Engineer of the New York Central and Hudson River Railroad Company.

In 1906 Mr. Hogue entered the service of the Philippine Railroad Company, acting in the capacity of Principal Assistant Engineer, on the location and construction of approximately 120 miles of railroad on the Islands of Panay and Cebu, Philippine Islands. In 1909, he returned to the United States where he was engaged for a short time in private work.

Returning again to the Philippine Islands, he entered the service of the Manila Railroad Company as Engineer in Charge of Construction (1911–13). Most of this time was spent on a 37-km extension from Aringay to Baguio, Luzon Island, where the mountain route was very difficult. For 11 km, 12% grades were used, and in places the curves were as sharp as 115-m radius. The total rise above sea level was 1 450 m. Four tunnels and one hundred and twenty bridges were required. Owing to the difficulty of this mountain route, the use of rack rail and special locomotives was contemplated.

Upon his return to the United States, Mr. Hogue was employed by the Cumberland and Pennsylvania Railroad Company to make surveys and prepare maps required by orders of the Interstate Commerce Commission, Bureau of Valuation. In the spring of 1915, he purchased an interest in the firm of Brownworth and Company, engaged in the design and manufacture of light structural steel work. He was elected President of that Company, which position he held until the time of his death.

Mr. Hogue was an engineer of resourcefulness and ability. He was a tireless worker, energetic and thorough in everything which he undertook; no assignment was too small to receive his most careful consideration. Although the earlier years of his life were spent in various phases of railroad work, he had an exceptionally keen eye for country and its topography, and his most important service was in the field of railroad location.

By his kindly and genial manner he won and held the friendship of his associates, by whom he was highly esteemed for his integrity and fine personal qualities. Mr. Hogue, although unmarried, enjoyed the quiet simplicity of home life, and welcomed the opportunity which, in 1915, was afforded him to enter business in Philadelphia, Pa., where he resided until on May 28, 1935; he passed away after an illness of six months. His passing has been felt with a distinct sense of loss by his many friends and associates.

Mr. Hogue was elected an Associate Member of the American Society of Civil Engineers, on December 7, 1898.

EDWARD LOUIS KOCH, Assoc. M. Am. Soc. C. E.[1]

Died April 29, 1935

Edward Louis Koch was born at Huntingdon, Pa., on August 17, 1883. He was the son of the late Ferdinand B. and Sue (McHugh) Koch. His father was a prominent business man in Altoona, Pa., having served as Manager for the Standard Supply and Equipment Company at that place.

The Koch family originally came to the United States from Germany in the early years of the Nineteenth Century, and the name is now well known in that part of Huntingdon County, Pennsylvania, where they first settled.

Edward Louis Koch was educated in the public schools of Altoona and was graduated from the local High School in 1902. Unfortunate enough not to be able to take a course at one of the larger universities in his favorite subject of engineering, he made up for this deficiency by night school work, and this resulted, later, in his becoming a qualified Civil Engineer.

After his graduation from High School, he worked for a short time with the State Highway Department of Pennsylvania, but having been brought up in a railroad town, where interest in the Pennsylvania Railroad was paramount, his ambition was to enter railroad work with that Company.

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[1] Memoir prepared by J. M. Fox, Div. Engr., P. R. R., Baltimore, Md.

From October, 1902, to July, 1904, Mr. Koch was employed as Chainman by the Pennsylvania Railroad Company on construction work.

In November, 1904, he was transferred to the Maintenance-of-Way Department where his particular interest lay. By gradual stages he advanced from Chainman, to Rodman, to Transitman, and, in April, 1910, received his first official appointment as Assistant Track Supervisor on the old Cresson Division, with headquarters at Cresson, Pa. This Division has since been absorbed as a part of the Pittsburgh Division.

Two years later (1912) Mr. Koch was transferred to the Conemaugh Division with headquarters at Pittsburgh, Pa., and, in June, 1913, he was advanced to a job on the Main Line at Pittsburgh in the same capacity.

In 1917, a real chance came to show his knowledge of practical track work, when he was first made Supervisor of Track on the Maryland Division, at Media, Pa. Later in the same year, he was advanced to the West Jersey and Seashore Railroad, at Camden, N. J., and, in 1920, he returned to the Maryland Division as Supervisor on a Sub-Division of the Main Line between New York, N. Y., and Washington, D. C.

In this last capacity Mr. Koch had outstanding success in winning the much sought after prizes offered by the Pennsylvania Railroad Company for excellence in line and surface. For two consecutive years he was awarded the Improvement Prize given to the Supervisor of the Eastern Region whose Division showed the most marked improvement in riding and general condition of track, a feat which had not been equalled prior to this time. Other prizes seemed to be awarded continually to him, indicating a complete mastery of practical track work.

In August, 1929, Mr. Koch was made Assistant to the Division Engineer on the Baltimore Division, with headquarters at Baltimore. During 1934 and 1935, his work in Baltimore necessitated active participation in the improvement program in connection with electrification, and his knowledge of track work, together with his engineering ability, made Mr. Koch invaluable to his Company.

He was married to Mary Price, of Altoona, on November 9, 1907, and is survived by his widow and seven children, two girls and five boys.

Mr. Koch was elected an Associate Member of the American Society of Civil Engineers on November 26, 1918.

FRANKLIN EDWARD LELAND, Assoc. M. Am. Soc. C. E.[1]

DIED JULY 8, 1935

Franklin Edward Leland was born at Ashfield, Mass., on June 29, 1889, the son of George Edward and Mary (Smith) Leland. He was educated in the public schools of Ashfield and at Sanderson Academy. Later, he continued his studies by taking various evening courses along engineering and general business lines.

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[1] Memoir prepared by Waldo F. Pike, Esq., Boston, Mass.

For several months, in 1909, Mr. Leland was employed as Rodman with C. E. Brown, Civil Engineer, in Greenfield, Mass., following this with a year as Draftsman for Wells Brothers Company, also of Greenfield, and a year with the Chase Turbine Manufacturing Company, at Orange, Mass.

From September, 1911, to November, 1912, he was employed by K. S. Putnam, of Northampton, Mass., on architectural detailing, and followed this by a period of seven months of similar work with the firm of Kilham and Hopkins, of Boston, Mass., and two and one-half years with Allen and Collens, of Boston, on architectural drafting and some structural design.

In 1915, Mr. Leland was employed by the Boston and Albany Railroad Company, at Boston, on the valuation of structures as required under the Federal Valuation Act. From 1915 to 1918, he served as Checker, Designer, and Assistant Squad Chief with the Stone and Webster Corporation, in Boston, and was responsible for detail design of various types of industrial structures, chiefly steam power plants.

During the World War, Mr. Leland was engaged as a Designer by the firm of Fay, Spofford, and Thorndike, of Boston, and was responsible for a considerable part of the design of the superstructure of the Boston Army Supply Base. At the termination of the war, he became Assistant Designing Engineer with the Aberthaw Construction Company, of Boston, returning, later, to Fay, Spofford, and Thorndike with which firm he remained until 1922, as Designer and Resident Engineer on construction.

In 1922, Mr. Leland began an independent practice in engineering design and management, and maintained an office for the following nine years. During this time he handled successfully a number of large projects, principally industrial. He designed and supervised all the work done at the Boston Airport, in 1929, aggregating in value approximately $500 000.

In 1933, he was employed by the firm of Metcalf and Eddy, of Boston, where he assisted in the design of a large sewage disposal project and several water treatment plants. He was in the employ of this firm at the time of his death.

For two or three years prior to his death, Mr. Leland was engaged, as a member of the Committee, in the preparation of the draft of the new Building Code for the City of Boston. He was also a Trustee of the Kendall Square Building, in Cambridge, Mass., and, at one time, had served as a member of the Chamber of Commerce, of Cambridge. He was a member of the Boston Society of Civil Engineers.

Mr. Leland was a capable engineer, a clear thinker, accurate and thorough in his work, and he was highly respected by those with whom he came in contact. He had a high sense of honor and upheld the ethics of his profession, and he took great pride in his work.

He was married to Flora Paramenter Ash, of Orange, Mass., who, with a daughter, Elizabeth Foxwell Leland, survives him. He also leaves a sister, Edith Leland, and two brothers, Archie V. Leland and Harold A. Leland.

Mr. Leland was elected an Associate Member of the American Society of Civil Engineers on June 6, 1921.

HARRY CLIFFORD McCLURE, Assoc. M. Am. Soc. C. E.[1]

DIED NOVEMBER 9, 1935

Harry Clifford McClure was born in Mansfield, Ohio, on August 15, 1884, but he grew to manhood in Cawker City, Kans., where his parents moved when he was between one and two years of age. His father, Robert McClure, was of Scotch ancestry, while his mother, Sarah (Hudson) McClure, was a direct descendant of Henry Hudson, the English navigator and explorer.

He received his formal training for the Engineering Profession at the University of Kansas, at Lawrence, Kans., where he was enrolled as a student in the School of Engineering from 1905 to 1909 and during the year 1913–1914, and at the University of Michigan, at Ann Arbor, Mich., where he did graduate work in 1914–1915. His professional degrees were Bachelor of Science (in Civil Engineering) and Civil Engineer from the University of Kansas in 1909 and 1914, and Civil Engineer from the University of Michigan in 1915.

Mr. McClure's first actual engineering job after his graduation from the University of Kansas was with the Riggs and Sherman Company, of Toledo, Ohio, a noteworthy engineering firm of that time and place which, in addition to distributing regular salary checks, gave effective if somewhat informal post-graduate training in practical engineering to a hand-picked group of young men fresh from the blackboard and testing laboratory. The two fruitful years (June, 1909, to October, 1911) spent with this firm were occupied chiefly by a wide variety of office duties and, later, by work in the field as Engineer-Inspector on a concrete dry dock for the Toledo Shipbuilding Company. His next engagement (November, 1911, to December, 1915) was with the Toledo Board of Education, first as Staff Engineer in the Department of Architecture, but later as Chief Engineer in charge of structural design and construction. This was during a period of active school building in Toledo in which several large ward and High Schools of thoroughly modern type were built and equipped.

He left the service of the Board of Education of Toledo in January, 1916, to become Commissioner of Engineering and Construction in the Department of Public Works of the City of Toledo, under the Hon. David Goodwillie, Director of Public Service. This position was held until February, 1921, when Mr. McClure became Works Manager and Superintendent of New Construction with the Toledo Scale Company. The position with the City of Toledo extended over a period of intense activity in city planning and improvement, covering general plans for the removal of sewage and industrial wastes from Ten-Mile Creek, Swan Creek, and the Maumee River (both sides); the extension and straightening of Summit Street; the reclamation of land for industrial areas; and various water-front and harbor improvements.

Mr. McClure's engagement as Commissioner of Engineering of Toledo was interrupted by a year's leave of absence in 1918 when, commissioned

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[1] Memoir prepared by W. C. Hoad, M. Am. Soc. C. E.

Captain in the Ordnance Department of the United States Army, he was assigned to duty investigating sites for nitrate plants, and, later, was put in charge of the construction of roads, water supply, and sewers. Afterward he served as Intelligence Officer and Assistant Director of Operations, at U. S. Nitrate Plant No. 4, near Cincinnati, Ohio. This plant had not been finished at the close of hostilities, but Captain McClure was held for a time at this station to assist in closing up the affairs of the plant.

From the Toledo Scale Company, Mr. McClure was called in April, 1923, to be City Engineer and, later, Director of Public Works, of Flint, Mich. As City Engineer he had charge of the design and construction of pavements, sewers, sidewalks, bridges, buildings, and miscellaneous structures (not including water-works), together with the maintenance and repair of such services and structures; he was also in charge of the system of sewage disposal, and managed the collection and disposal of the city's garbage and other city wastes. While Director of Public Works he was also in supervisory charge of the Water-Works Division of Flint. During this period the City's construction work, under Mr. McClure's direct supervision, amounted to approximately $15 000 000, practically all of which was done by force contract. The number of men employed in the Public Works Department varied from about 250 during the routine winter maintenance and snow-removal operations to as many as 1 200 during construction seasons. This was a period of active building in the city to keep pace with the continued expansion of its industries and the rapid growth in its population, a period of consolidating earlier gains, and of welding all improvements together into a smoothly functioning whole. In many ways, it was the period of transition of Flint from the condition of a large town to the status of a well-organized city, and in this transition Mr. McClure bore a notable part. His northwest corner office on the second floor of the City Hall was the recognized center for constructive planning and a never-failing source of forward-looking ideas.

Because of a rapidly developing political opposition, Mr. McClure resigned the position of Director of Public Works of Flint in April, 1931, leaving in May. The following month, he became a member of the State Public Utilities Commission of Michigan, by appointment of Governor Wilbur M. Brucker. He remained in this service until March, 1934, when he, along with other members of the Commission, was removed from office by Governor William A. Comstock.

Mr. McClure's work on the Utilities Commission was about what should normally be expected of a high-minded engineer, mature in years and in constructive thinking, and well-seasoned in experience. He tried to get at all the significant facts of a situation before making a final decision. While he held to a pronounced view of the obligations of the public utilities to the public which they served, he likewise had an unusually clear understanding of the difficulties by which the utility companies themselves were beset. He was known as an official who was always alert and vigorous in the public interest, but always fair. At a time when an enormously expanded State Highway System was stimulating trucking and bus services, when changes in economic conditions were forcing a multitude of rate changes, and when the rapid discovery and exploitation of the gas and oil resources of the State were creating new problems of broad public concern, Commissioner McClure's knowledge of engineering as well as his sound views of public responsibility were of outstanding value to the State.

Mr. McClure was married, in 1912, to Hazel Carter, of Toledo, while he was with the Board of Education of that City. Mrs. McClure and their two daughters, Suzanne and Sallie, survive him.

The beginning of the malady to which Mr. McClure finally succumbed undoubtedly dates back a number of years, but this trouble became more pronounced in the early summer of 1934 and, following a partial recovery, still more severe in 1935. He died suddenly at East Lansing on November 9, 1935, of a heart attack. The funeral services were held November 12, 1935, interment being in Deepdale Cemetery, at Lansing. To one so accustomed to constant and vigorous action, the rest periods prescribed by the physician seemed tedious and difficult to bear and even the light work permitted during his apparent recovery in 1934 fell far short of satisfying his energetic nature; but with his family his buoyancy of spirit and his irrepressible good humor never flagged and much of his time during this period of partial health was happily spent in studying and contriving for their best interests.

He was a member of the Michigan Engineering Society; a member of Sigma Xi and of Tau Beta Pi; he was a Rotarian; a Shriner; a Republican; and a member of the Methodist Episcopal Church. He was also a member of a number of local clubs and civic associations.

Mr. McClure was elected a Junior of the American Society of Civil Engineers on March 1, 1910, and an Associate Member on September 3, 1913.

OSCAR WILLIAM MELIN, Assoc. M. Am. Soc. C. E.[1]

Died March 29, 1935

Oscar William Melin, the son of August Melin and Christine (Sundell) Melin, was born in Moline, Ill., on April 22, 1888.

After attendance at the local schools of his birthplace, he went to the University of Wisconsin, Madison, Wis., from which he was graduated in the Class of 1910, with the degree of Bachelor of Science in Civil Engineering His first duties were the typical tasks of the young engineer seeking to perfect himself in his field: Instrumentman for the United States Reclamation Service, the Illinois Central Railroad Company, and the Madison Street (Chicago, Ill.) Railway Company; a year in the field on concrete construction; and two years as Instructor in Civil Engineering at the University of Kansas, Lawrence, Kans.

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[1] Memoir prepared by O. C. Spurling, M. Am. Soc. C. E.

In 1914, Mr. Melin entered the Engineering Offices of the Illinois Central Railroad Company and during the ensuing four years worked as Draftsman, Chief Draftsman, and Group Chief in various Departments of the Company. Following a brief engagement in 1918 as an Inspector for the Government on power plant construction, he returned for a short time to railroad work with the Baltimore and Ohio Railroad Company, Western Lines, as Assistant Engineer of Building.

Meanwhile, the chief opportunity he was to enjoy to use his abilities in his too brief career had been slowly taking form at the Hawthorne (Ill.) Works of the Western Electric Company, faced with heavy requirements for new manufacturing facilities brought about by the increasing demands of the nation for telephones.

Mr. Melin joined the Western Electric Company in 1919 as Structural Engineer in the office of the Engineer of Plant. His work was soon regarded favorably, and he was placed in charge of the structural design of a warehouse more than 100 000 sq ft in area. Following this in rapid succession came both single and multi-story buildings, including the complex structural design of large rod and wire-drawing mills with their accompanying heavy machine loads.

Mr. Melin had a keen intelligence and his work was marked primarily by drive and initiative but, always pushing himself harder than he did any of his men, he was a popular leader. In 1923, when the Western Electric Company chose the site for its Kearny (N. J.) works, Mr. Melin was transferred to New York, N. Y., to assume charge of the structural design of the buildings required there. Here, he had charge of the taking of bids and preparation of construction contracts, and from 1924 to 1929 he supervised the structural design and handled the contractual relations of building projects comprising about 900 000 sq ft of floor space.

Fundamentally honest with himself and all those with whom he came into contact, he was satisfied only with the most careful analysis of a problem and demanded of himself and others painstaking and sound engineering practice in its execution. He drew men to him more by their own recognition of his character and ability than by any conscious effort on his part to win them.

Recognizing the need of a "yardstick" to measure building construction performance by the Engineer of Plant, he developed in 1927 from a quantity survey of typical Western Electric buildings, a Building Construction Cost Index which is still in use by the Company. It is only one example of the natural ability and intense application by which Mr. Melin is best remembered by his many friends in the profession.

Mr. Melin was married to Florence Roach on November 27, 1915, at Fennimore, Wis. He is survived by his widow and one daughter, Marjorie Jane Melin, of Pasadena, Calif. He is also survived by three sisters, Emma and Hilma Melin, of Moline, Ill., and Ethel Melin of Pueblo, Colo.

Mr. Melin was elected an Associate Member of the American Society of Civil Engineers on October 14, 1919.

FRANK HURD PICKETT, Assoc. M. Am. Soc. C. E.[1]

Died July 4, 1935

Frank Hurd Pickett was born in Toledo, Ohio, on July 13, 1888, the son of Edward and Mary (Watkins) Pickett. He attended school in Toledo until 1905, at which time he entered the University of Colorado, at Boulder, majoring in Civil Engineering. Subsequently, he was a student at Purdue University, La Fayette, Ind., where for one year he received training as a Cadet, and was made a member of Theta XI Fraternity. His education was complete in every detail, because of his eagerness always to acquaint himself with every new method in the constant problematical and progressive field of engineering.

In 1915, Mr. Pickett attended the Brooklyn (N. Y.) Evening Technical School for special work; and, again, in the fall of 1920, he entered the American Rolling Mills School, at Middletown, Ohio, for a six weeks' term. He next took a short course given by the California Corrugated Culvert Pipe Company, at Berkeley, Calif., during the months of January and February, 1935. The completion of this course preceded his last illness and death, by only one month.

Mr. Pickett's practical experience began in 1908, on various irrigation projects in Colorado, as Chainman, Levelman, Transitman, etc. From 1908 to 1910, he served as Assistant Engineer for the Colorado Irrigation Construction Company, at Denver, Colo. This work consisted of preliminary and final location surveys, plane-table topography, and estimates of cost of water duty on a 60 000-acre project near Denver. Afterward, he was appointed Assistant Engineer for the Rock Creek Conservation Company, Rock River, Wyo., on the location and construction of an irrigation system of a 100 000-acre project.

In 1911 and 1912, Mr. Pickett took a position with the F. W. Dodge Company, of New York, N. Y., his work being entirely in the Construction Reports Department. From 1912 to 1913, he was associated with W. H. Gahagan, Incorporated, of Brooklyn, N. Y., for which Corporation he was in charge of the construction of a power plant and of brick and concrete mine buildings, for the Delaware, Lackawanna, and Western Railroad Company, at Nanticoke, Pa. From 1913 to 1916 he was employed as Engineer for J. and F. Kelly, Dock-Building Contractors, of Brooklyn, N. Y. This detailed work consisted of estimates, designs, and the construction of piers, bulkheads, jetties, and foundations.

Mr. Pickett then accepted a position with the American Graphaphone Company with which he remained during 1916 and 1917, as Assistant Construction Plant Engineer. As such, he had general supervision of the building of the New Record Plant, consisting of brick and steel factories and power plants. For several months prior to the entrance of the United States into the World War, he served as Assistant Construction Engineer for the

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[1] Memoir prepared by Morris D. Simpson, Esq., New York, N. Y.

Remington Arms Union Metallic Cartridge Company, in charge of the general maintenance and construction of dry kilns.

From 1917 to 1919, Mr. Pickett served in the United States Army as First Lieutenant of Engineers. On August 7, 1918, he received his commission as Captain of Engineers, and held that rank in the Reserve Officers Corps until his death. While in active Army service, he was in charge of troops and Military Trade Schools, and was stationed at Camp Lee in Virginia, Washington Barracks, in the District of Columbia, Fort Benjamin Harrison, in Indiana, Camp Dodge, in Iowa, and Camp Humphreys, in Virginia. He was granted an honorable discharge from the Army at Fort D. A. Russell, in Wyoming, on September 5, 1919.

After his discharge from the Army, Mr. Pickett renewed his engineering career and business interests by accepting a position as Engineer for J. A. Whiting, of Cheyenne, Wyo., on the construction of a sewerage system for the City of Cheyenne, and the Hawks Springs Reservoir Dam. In 1920 he joined the Engineering Corps of the Hardesty Manufacturing Company of Denver, Colo., with which he remained until 1932. He was made Sales and Consulting Engineer. This Company manufactured the Armco iron culverts, flumes, head-gates, meter-measuring devices, etc. In 1932, The California Corrugated Culvert Company purchased the Idaho, Utah, and Colorado territories from the Hardesty Manufacturing Company, thus bringing eleven Western States, and New Mexico, into one Company group. Mr. Pickett maintained supervision of the State of Idaho. Owing to his various contacts with highway officials, contractors, boards of directors of road construction, county commissioners, city councils, etc., his business was made very interesting.

Mr. Pickett was widely known for his "Irish wit" intermingled with a deep and sincere kindness. Especially did he take great pride in scattering sunshine and kindness to those who were less fortunate than himself. He was honest, honorable, and dependable in all great matters, as well as in matters of lesser importance. His keen appreciation of music and literature also added zest to his wonderful and honorable character. The great loss of such a splendid man will be felt by all who knew him.

On June 10, 1921, he was married to Muriel Elizabeth Simpson, who survives him. He is also survived by a sister, Mrs. Milo R. Foley, of Denver, Colo.

Mr. Pickett was elected an Associate Member of the American Society of Civil Engineers on October 9, 1917.

WILLIAM THOMAS REED, Assoc. M. Am. Soc. C. E.[1]

Died April 28, 1935

William Thomas Reed was born at Lynn, Mass., on August 30, 1872, the son of Thomas P. and Esther (Richardson) Reed. He received his education in the public schools of Lynn, and was graduated with one of the earliest classes from the local English High School, in 1889, nearly at the head of the Class. He helped to organize the School Alumni Association and was elected the first President. His interest in the Alumni Association was great and steadily maintained during all the subsequent years until his death.

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[1] Memoir prepared by Harry E. Sawtell, M. Am. Soc. C. E.

He was unable to attend a technical school or college, but gained a considerable knowledge of engineering principles by intensive home study and application. The work that led Mr. Reed into the field of structural design, and afterward into engineering contracting, was his first engagement in November, 1895, with the Norton Iron Company, at Everett, Mass., where he worked both in the structural shop and in the field. In the early part of 1897, he went into the Drafting Room of the Boston Steel and Iron Company, Boston, Mass., as a Draftsman, working on steel stairs as well as on structural design. Owing to his natural ability, hard study, and application, he rose rapidly in the Designing Room of this growing Company and filled, successively, the positions of Chief Designer, Chief Draftsman, and Engineer in Charge of Design. His work was so outstanding that he was transferred to the Estimating Department, and was soon placed in charge of that work as Contracting Engineer. While in that position he was responsible for the design of the steel structures of many important buildings in Boston, and in other parts of New England. About this time construction work in reinforced concrete was increasing rapidly and he concentrated on the application of the principles of mechanics to reinforced concrete structural design.

Early in 1904 Mr. Reed left the Boston Steel and Iron Company and, in May of that year, became associated with the Eastern Expanded Metal Company, of Boston, and, later, the Eastern Concrete Construction Company, a subsidiary, for which he designed many reinforced concrete buildings and supervised the construction of a number of them. Mr. Reed continued with this Company until March, 1910, when he with others formed the New England Concrete Construction Company, of Boston, of which he was the President, Treasurer, and Manager. This Company was organized for contracting, engineering, designing, and erecting structures of reinforced concrete, brick, steel, or timber. The Company specialized in industrial plant buildings, warehouses, and other similar structures.

In May, 1921, the Company was re-organized and named the William T. Reed Company, and Mr. Reed again became President, Treasurer, and Manager. Under his direction, the Company continued to perform the work of designing and constructing many important industrial buildings until his death on April 28, 1935. He had the honor of working on one of the first steel-frame buildings to be erected in the City of Boston, as well as one of the first reinforced concrete structures constructed in that vicinity. His activities are thus linked with the pioneer growth in these fields.

Mr. Reed was very successful in creating and maintaining a reputation for the highest character and honor in his work. Being satisfied with nothing short of the best, he was often spoken of as being too honest for a money maker. He always maintained that a reputation for doing good work and being fair to his clients and workmen was worth more to him than a fortune in money without them.

He succeeded in placing his contracting work on a high professional plane. His Company did work for literally hundreds of the best known firms in New England, and Mr. Reed took considerable pride in the fact that he seemed never to lose a client once he did his work, this being shown by the fact that of nineteen selected names of owners on his books, the average number of repeat orders was nine, and one company gave him at least twenty-three separate jobs, aggregating in cost approximately $1 000 000.

His religious principles were carried into his secular work, and he leaves many friends among the laborers of his Company, as well as business and professional associates, who regret his passing.

On October 8, 1902, Mr. Reed was married to Grace Edith Bickford, of Lynn, who passed away May 18, 1918. They had no issue, and Mr. Reed never married again. A brother, Frank C. Reed, survives him.

His interests were varied as shown by the following list of organizations of which he was a member: First Congregational Church, of Lynn; Sons of American Revolution; Massachusetts Charitable Mechanics Association; Lynn Historical Society; and Reade Society for Genealogical Research.

Mr. Reed was elected an Associate Member of the American Society of Civil Engineers on February 28, 1911.

HARRY ASHTON ROBERTS, Assoc. M. Am. Soc. C. E.[1]

DIED APRIL 6, 1935

Harry Ashton Roberts was born in Strawn, Ill., on April 2, 1876, the son of Edward H. and Eliza Roberts. At the age of twenty-six, he was graduated with honors from the University of Illinois, at Urbana, Ill., receiving the degree of Bachelor of Science in Civil Engineering. In 1916, the University of Illinois conferred upon him the degree of Civil Engineer. He was a member of Tau Beta Pi.

During vacation periods, between 1899 and 1901, Mr. Roberts was employed on various railroads as Rodman, Draftsman, and Transitman. In 1901, he was in charge of a party making townsite surveys for the United States Department of the Interior in Indian Territory. From 1902 to 1904 he was, successively, in the service of the Oregon Short Line Railroad Company, Delaware, Lackawanna and Western Railroad Company, and the Sante Fé System, his last assignment being in charge of a residency on the construction of the Newkirk-Pauls Valley Line, which involved heavy grading, masonry, and bridge work.

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[1] Memoir prepared by Samuel Murray, M. Am. Soc. C. E.

In 1904, he re-entered the service of the Oregon Short Line Railroad Company, as Draftsman, shortly thereafter being promoted to be Assistant Engineer on Maintenance of Way, on the Montana Division, and in February, 1906, he was appointed Assistant Engineer on construction. The remainder of 1906 was a period of great activity for Mr. Roberts, the construction of yards, freight-houses, terminals, water stations, and various other railroad facilities at Nampa, Boise, Caldwell, and other points in Idaho, being carried out under his jurisdiction.

In November, 1906, he was promoted to the position of Division Engineer and Assistant Superintendent of the Oregon Short Line Railroad Company, with headquarters at Pocatello, Idaho, and although for the next eight years he was actively engaged with important maintenance and operating problems, Mr. Roberts found time for a deep interest in civic matters as a member of the Pocatello School Board, and a Director of the Young Men's Christian Association.

In September, 1914, he was called to the University of Kansas, at Lawrence, Kans., as Assistant Professor of Civil Engineering. In this position he spent probably the most pleasant days of his professional career, until America entered the World War and he joined the Service as a Captain in the Engineering Corps of the American Expeditionary Force. He was mustered out in February, 1919, with a record of conspicuous service overseas, returning to the University of Kansas in February, 1919. In the succeeding September he re-entered the service of the Oregon Short Line Railroad Company as Division Engineer, where he remained until 1920, when he was transferred to the Oregon-Washington Railroad and Navigation Company, a unit of the Union Pacific System, at Portland, Ore., as Engineer of Maintenance of Way. Here, he served until his death, which occurred unexpectedly at his country home at Aloha, Ore.

Harry Roberts was a competent and dependable engineer, and a fine example of American gentleman. He possessed high qualities of leadership, and inspired devoted loyalty from his subordinates. A strong adherence to principle was tempered with an unfailing sense of fairness and active sympathy for the "under dog." He had a host of friends who mourned his passing, among them many to whom he had endeared himself by his never failing willingness to help where help was needed.

In 1903, he was married to Henrietta Henderson, who, with his daughters, Henrietta M. Roberts and Mrs. George R. Barlow, and his sons, Donald H. and H. Kenneth, survives him. He is also survived by his sisters, Mrs. Harvey L. Beightol and Mrs. Miriam E. Brannum, and his brothers, Dr. A. J. Roberts and Dr. E. N. Roberts.

He was a member of the American Railway Engineering Association, the Society of American Military Engineers, the Reserve Officers' Association, and of the Congregational Church.

Mr. Roberts was elected an Associate Member of the American Society of Civil Engineers on April 4, 1904.

WILLIAM WHITE ROUSSEAU, Assoc. M. Am. Soc. C. E.[1]

Died September 28, 1935

William White Rousseau was born in Troy, N. Y., on April 18, 1873, the son of William White Rousseau, Sr., and Jeannette (Parker) Rousseau. He received his early education in the public schools of Troy and at the Troy Academy. He entered Rensselaer Polytechnic Institute in 1891, and was graduated in June, 1895, with the degree of Civil Engineer.

Following his graduation, Mr. Rousseau was employed in the Engineering Department of the Delaware and Hudson Railroad Company, at Albany, N. Y., until June, 1896. From June, 1896, to January, 1897, he was in charge of dike construction at Corning, N. Y., under the late Palmer Chamberlaine Ricketts, Hon. M. Am. Soc. C. E. From 1897 to the date of his death he was a member of the Faculty of Rensselaer Polytechnic Institute, holding, successively, the positions of Instructor, Assistant Professor, and Associate Professor of Surveying, Railroad Engineering, and Forestry. For the past fourteen years Professor Rousseau had been Faculty Adviser of Athletics, controlling the financial and eligibility conditions of athletics, maintaining cordial relations between the Faculty and the students under trying conditions, and gaining the love and esteem of both.

Professor Rousseau was Superintendent of Construction of the Troy Water-Works from February, 1907, to January, 1913. He was also a member of the firm of Breese, Rousseau, and Company, Real Estate and Insurance Agents. He was a member of the Rotary Club, of the Troy Chamber of Commerce, having served, at one time, as one of its Directors, and of various Masonic organizations. He was a member of the Rensselaer Society of Engineers, Sigma Xi, and Tau Beta Pi.

He was the Treasurer and Organist of the Church of the Holy Cross, at Troy, which is unique in having had until this year (1935), only two rectors and two organists from the time of its organization, Professor Rousseau succeeding his father as organist, their combined services covering a period of ninety-one years.

Professor Rousseau was married on June 5, 1901, to Frances Hardy, of Troy, who, with their two children, Parker H. and Ruth Rousseau, survives him.

His kindly nature and willingness to help all with whom he came in contact made for him a host of friends who mourn his passing. As a teacher, a citizen, and a Churchman, he gave himself to the service of his fellow men, and has doubtless received the reward: "Well Done, Thou Good and Faithful Servant, Enter Thou into the Joy of Thy Lord."

Professor Rousseau was elected an Associate Member of the American Society of Civil Engineers on December 1, 1908.

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[1] Memoir prepared by E. R. Cary, M. Am. Soc. C. E.

WILLIAM DWIGHT SAMPLE, Assoc. M. Am. Soc. C. E.[1]

DIED JANUARY 5, 1935

William Dwight Sample was born in Granville, Ohio, on October 13, 1881. He was the son of John H. and Virginia F. (Hughes) Sample. He was graduated from Denison (Ohio) University in the Class of 1902, and was a member of the Mu Chapter of Sigma Chi and a Phi Beta Kappa.

His entire life was spent in an engineering atmosphere, his father having been for many years an official in the Engineering Department of the Pennsylvania Railway Company. His mother was a sister of Lieutenant Hughes, U.S.N., who was in command of the U.S.S. *Petrel* at the Battle of Manila Bay.

After his graduation from Denison, Mr. Sample was employed in the Engineering Department of the Cleveland and Pittsburgh Division of the Pennsylvania Railroad and, later, he was engaged with the Southern Railway Company, in Alabama and Mississippi, on new railway location.

In 1907, together with several of his brother engineers, he was called to Brazil for the location and construction of the São Paulo-Rio Grande Railway, a subsidiary of the Brazil Railway System. The work consisted of location through a little known country south of the Rio Iguassú and down the Rio do Peixe, through Indian country, to the River Uruguay.

On the completion of this location in 1909, Mr. Sample left this Company to assist in the location of the Madeira-Mamoré Railway, in the Upper Amazon country, between Porto Velho and Guajará Mirim, which was then under way. He remained on this work until the location was finally finished to the end of the line at Guajará Mirim, on the Mamoré River, in 1910.

In 1910 and 1911, he was engaged in the construction of the Mexico North Western Railway in the State of Chihuahua as Resident Engineer and, later, as Assistant to the Chief Engineer. This period was during the first Mexican Revolution under Francisco Madeira and Pancho Villa, and it was due in great part to his efforts that the construction work on the railway could be carried on during this time.

In 1912, Mr. Sample returned to Brazil, and was connected with the location of the railways in the State of Rio Grande do Sul and, in 1913, he went to São Paulo, Brazil, in the capacity of Field Engineer with The São Paulo Tramway, Light, and Power Company, Limited, on the construction of the Sorocaba Power Plant.

When war was declared in 1914 he was among the first to volunteer for service and left São Paulo for England with a group of his English friends. On their arrival in London, Mr. Sample enlisted in the famous Rifle Brigade and served with this Regiment throughout the World War. He was in active service on the western front in France from early in 1915 until the spring of 1918 when he was badly wounded and sent to a hospital in England.

He received his discharge in 1919, and returned to the United States where he underwent several operations on his leg which had not healed properly and, in fact, left him lame for the remainder of his life. He was able, however, to return to his engineering work.

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[1] Memoir prepared by Walter Charnley, M. Am. Soc. C. E.

He went back to Brazil with The São Paulo Tramway, Light, and Power Company, Limited, in 1920 as Field and Construction Engineer, and assisted in many hydro-electric developments, among others, the large Parahyba River Plant, at Ilha, State of Rio de Janeiro, and for the past ten years he had been connected with the great Serra Development, in the State of São Paulo.

During his long period of residence in Brazil Mr. Sample made a host of friends, and his passing is deeply felt by all who knew him.

His entire active life was spent in the field amid the hardships usually encountered in this class of work, especially in wild and unsettled country; but through it all he remained a keen active engineer, up to date in his profession and loved by his associates and friends in all walks of life for his kindliness, loyalty, and his ready helping hand to those in trouble.

He never married and was the last of his immediate family. The epitaph on his tombstone in São Paulo, Brazil, where he lies, reads:

"In Memoriam
A Soldier and Engineer"

Mr. Sample was elected an Associate Member of the American Society of Civil Engineers on December 1, 1908.

GORDON EARL SISSONS, Assoc. M. Am. Soc. C. E.[1]

Died July 31, 1935

Gordon Earl Sissons was born in Vina, Calif., on March 14, 1892. His parents were of French descent, and after finishing his earlier education, he attended The Polytechnique School of Engineering, in Paris, France, from which he was graduated in 1914 with the degree of Bachelor of Science in Civil Engineering.

Upon his return to California, Mr. Sissons secured a position with the Vina Ranch Company on a pumping and irrigation project. As Assistant to the Chief Engineer, he was responsible for surveying and mapping the territory involved, and for the design and construction of a pumping station and the various works connected with the irrigation of a tract of approximately 6 000 acres.

This assignment was completed at about the time of the outbreak of the war between France and Germany. Responding to the call of the French blood in his veins, Mr. Sissons enlisted with the Canadian Engineers, went overseas with the first contingent, as Lieutenant, and was among those subjected to the first gas attack of the war. He completed his war service in November, 1918, as Inspector and Testing Engineer with The Imperial Munitions Board of Canada.

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[1] Memoir prepared by Timothy S. Williams, Assoc. M. Am. Soc. C. E.

From that time, Mr. Sissons' experience was wide and varied, and included the supervision of design and field work connected with the construction of hydraulic developments, paper mills, steam and hydro-electric power layouts, and foundations of all kinds, with many of the large engineering concerns of the United States and Canada. He also served with The New York and New Jersey Bridge and Tunnel Commission (later, the Port of New York Authority), on the Holland Tunnel, and was engaged on maintenance work in mechanical plants, etc.

During the construction of The Pennsylvania Railroad Terminal improvements, in Philadelphia, Pa., from 1928 to May, 1931, he acted as Chief of Design Squads of the force of The United Engineers and Constructors, engaged on the study and design of a large section of this work. The construction involved the use of open caisson foundation work, and heavy steel and concrete superstructures. Considerable study was necessary in developing methods by means of which the work could be executed without interfering unduly with train movements.

Since 1933 Mr. Sissons had held the position of Senior Engineer in charge of the Department of Health of the Borough of Richmond, New York, projecting, supervising, and in responsible charge of all Department activities in the Borough. His work received especial commendation from his superiors as being well and economically planned and executed, under most difficult conditions.

The gassing he received during the World War had continuously undermined his health, but he refused to heed the doctor's warning that he must have a period of complete rest or a collapse would result. His conscientious insistence on carrying on with his work 100% during the winter's cold hastened his end which he fought on his feet as long as strength permitted.

Mr. Sissons was a keen judge of men, and expected of those working with him their fullest co-operation (however, he never asked of them more than he would give himself). He had little patience with shirkers, but was always ready to help, in any possible way, the man who was giving of his best.

He will be remembered by his associates, not only as an able engineer, conscientious and painstaking to an unusual degree, but as one who dealt fairly and squarely alike with superiors and subordinates.

He was married on July 2, 1921, to Lillian Rickaby, of Windsor, Ont., Canada, and is survived by his widow and a son.

Mr. Sissons was elected an Associate Member of the American Society of Civil Engineers on October 14, 1930.

ALBERT ORANGE SMITH, Assoc. M. Am. Soc. C. E.[1]

DIED JUNE 30, 1934

Albert Orange Smith, the son of Charles Albert and Isabella (Dana) Smith, was born at West Hartford, Vt., on August 4, 1877. After his preparatory education, he entered the University of Vermont, at Burlington, Vt., from which he was graduated with the degree of Bachelor of Science in Civil Engineering in 1902.

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[1] Memoir prepared by H. T. Tuthill, Esq., Westhampton Beach, N. Y.

Mr. Smith worked as a Rodman for the New York Central Railroad Company from May, 1897, to June, 1898, and as Transitman for the City Engineer of Barre, Vt., from June to September, 1898. From May, 1899, to September of the same year, he was Chief of Party for the Vermant Marble Company, and from July to September, 1900, he was employed as Transitman by the Rutland-Canadian Railroad Company. From June to September, 1901, Mr. Smith served as Draftsman for W. R. Wilcox, Architect. He was with the Lehigh Valley Railroad Company as a Draftsman from May, 1902, to February, 1903, after which, until October of that year, he was engaged as a Transitman for the Erie Railroad Company. From October, 1903, to May, 1905, he served as Transitman for the Ontario and Western Railroad Company.

It was in May, 1905, that Mr. Smith came to Suffolk County, New York, and began his work for C. P. Darling, Civil Engineer, at Huntington, Long Island.

In March, 1906, he decided to open an office for the private practice of engineering and contracting in Greenport, N. Y. Three years later, he was appointed County Superintendent of Highways by the Board of Supervisors of the County of Suffolk, with headquarters at Port Jefferson, N. Y., which position he held until 1914. In 1918, Mr. Smith was again named for that office and continued as Superintendent of Highways of the County of Suffolk until May 31, 1934, when he was retired.

His work in this position was quite varied and demanded a knowledge of all types of road and bridge construction, drainage conditions and their remedies, and the ability to approach people, in order to acquire rights of way for all State highways and county roads in the County.

The towns built their own concrete pavements from 1918 to 1930, under the direct supervision of local engineers, with the approval of plans, etc., by the County Superintendent. In 1930, a County Road System was inaugurated and, from that time, about 17 miles of reinforced concrete pavement was built each year. The plans and specifications for all this work were prepared under the direct supervision of Mr. Smith, and the contracts were let by him. His staff had increased to thirty-one in 1932, and Mr. Smith had established a very efficient organization.

Thirty-four bridges were built in Suffolk County under Mr. Smith's direct supervision. Plans and specifications were also prepared under his direction for the construction of two high-level highway suspension bridges. One of these bridges had a 700-ft main span and two 350-ft side spans. The other had a 1 000-ft main span, and two 600-ft side spans.

Mr. Smith had direct charge of the construction of the interlocking steel bulkheads, dredging, and rip-rapping, in Shinnecock Canal, an important connecting link in the waterways of Long Island, the total cost of the work amounting to approximately $200 000. He also prepared the specifications for an air map of Suffolk County, an area of 976 sq miles.

He had many friends, and his fair and honorable way of handling every situation won for him a fine reputation throughout Suffolk County. For more than twenty years he had served as Clerk of the Board of Education. He was Past Master of the Port Jefferson Lodge, F. and A. M.; former President of the Suffolk County Association of Highway Superintendents, which he had also served as Vice-President; and Vice-President of the Suffolk County Chapter of the New York Society of Professional Engineers. He was a member of the First Presbyterian Church, of Port Jefferson, N. Y.

Mr. Smith took an active interest in all civic affairs and was always working for the betterment of the community and for the country at large. His passing was a great loss to the community and to those who had the good fortune to be closely associated with him during his active career. The members of his staff feel most keenly the loss of his guiding spirit and his kindly interest in their welfare. He will always be remembered for his sterling qualities as an engineer and a gentleman.

He was married on October 30, 1906, to Susan Whiteman, of Burlington, Vt. Mrs. Smith died on January 29, 1933. He is survived by two daughters, Eleanor F. and Ruth C. Smith, and his mother Mrs. Isabella D. Smith, of Barre, Vt.

Mr. Smith was elected a Junior of the American Society of Civil Engineers on May 2, 1905, and an Associate Member on February 2, 1909.

EDWARD KING SMITH, Assoc. M. Am. Soc. C. E.[1]

DIED MARCH 8, 1935

Edward King Smith, the son of Thomas H. and Nellie (King) Smith, was born in Chicago, Ill., on July 31, 1891. His early life was spent in Beloit, Wis., where he attended Beloit College. His professional education was obtained at the University of Wisconsin, at Madison, Wis., from which he was graduated in 1914 with the degree of Civil Engineer.

Following his graduation Mr. Smith was appointed Instructor at the Summer Surveying Camp of the University of Wisconsin and placed in charge of hydrography and triangulation work. In the summer of 1914, he was appointed Assistant City Engineer of Beloit. In this capacity he had charge of all field work for major municipal improvements and, in addition, was engaged in the detailed design of storm sewers.

In July, 1917, upon the entrance of the United States in the World War, he entered the Army and was appointed Second Lieutenant in the Signal Corps. While on active service in France, he was placed in responsible charge of building camps, schools, hospitals, water supply structures, and repairing roads and buildings. He spent some time in the Signal Corps Photo Laboratory, in Paris, France, and took a keen interest in this phase of the work.

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[1] Memoir prepared by W. F. Tempest, Esq., Chicago, Ill.

After his return to the United States, Mr. Smith again took up his duties as Assistant City Engineer of Beloit, and was in charge of plans and field work for special sewer construction and other improvements. He was responsible for the design of a new sanitary sewerage system for the City of Beloit, the estimated cost of which was $400 000. He was also active in the preparation of a local Building Code which he wrote in collaboration with the State Building Engineer.

In 1925, he was chosen to fill a vacancy as Assistant Manager of the Highways and Municipal Bureau of the Portland Cement Association, in Chicago. In this executive capacity Mr. Smith was active in engineering society work and prepared numerous articles for publication in technical journals based upon his field observations of projects in the United States and Canada. He took a special interest in airport improvement to which field he contributed valuable information.

In spite of a busy life, Mr. Smith was always unusually active in the organizations to which he belonged and held numerous offices. He served on many committees and to each contributed tirelessly of his time and energy.

He was a member of the American Society of Municipal Engineers, the Society of American Military Engineers, the National Reserve Officers' Association, and Captain, Engineers Reserve, U. S. Army. Mr. Smith was also a member of Advertising Men's Post No. 38, American Legion, and served on many important State and National Committees. He was a member of the Hamilton Club, of Chicago, and the Sons of the American Revolution. His fraternal affiliations included the Masonic Lodge, the Order of the Eastern Star, the Knights of Pythias, and the Benevolent and Protective Order of Elks.

His sudden death, while visiting his mother at Beloit, was a severe shock to his many friends to whom he was affectionately know as "E.K." He is survived by his widow, Arline M. Smith, and his daughter, Nancy Frazer.

Mr. Smith was elected a Junior of the American Society of Civil Engineers on November 4, 1914, and an Associate Member on September 10, 1923.

ASAHEL CLARK TOLL, Assoc. M. Am. Soc. C. E.[1]

DIED DECEMBER 22, 1934

Asahel Clark Toll, the son of Truman Mitchell and Harriet (Clark) Toll, was born in Baldwinsville, N. Y., on September 26, 1880, where he received his Grammar and High School education. After his graduation from High School, he entered Leland Stanford, Jr., University, at Palo Alto, Calif., for the course in Civil Engineering, from which he was graduated in 1906, with the degree of Bachelor of Arts.

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[1] Memoir prepared by Manuel Font, M. Am. Soc. C. E.

After his graduation from college, Mr. Toll was employed for two years by the Bay Cities Water Company on surveys and preliminary studies of water supplies for San Francisco, Calif., and irrigation in the San Joaquin Valley. Subsequently, for two years, he was in the City Engineer's Office, in San Francisco, on preliminary studies for the Hetch Hetchy Water Supply and on the design of a high-pressure water system and supplementary cisterns for fire protection, as well as the design and installation of the main intercepting sewer for the city. During that time, he also worked privately for the Consulting Engineer to the City Attorney on an appraisal of the properties of the Spring Valley Water Company in connection with a suit for change in water rates.

Unfortunately, in 1911, due to impaired eyesight, Mr. Toll had, practically, to give up his professional activities, although he continued to engage in surveying and building, while devoting most of his time to fruit-growing in Puerto Rico where he had made his home.

Shortly before his death Mr. Toll had returned to the practice of his profession, and had been engaged by the Puerto Rican Emergency Relief Administration as District Engineer for Guayama. While in this position, through exposure to a tropical downpour, he developed pneumonia which caused his death.

Mr. Toll was a man of quiet manners, sedate, amiable, and friendly, and his death was felt by all those who knew him.

In 1912, he was married to Jane Bean, and is survived by his widow and two daughters, Harriet A. Toll and Jane B. Toll.

Mr. Toll was elected a Junior of the American Society of Civil Engineers on March 5, 1907, and an Associate Member on October 29, 1912. He was a prominent member of the Puerto Rican Section of the Society, having been its President during the year prior to his death.

JOSEPH INGHAM FRANCIS, Jun. Am. Soc. C. E.[1]

DIED AUGUST 22, 1934

Joseph Ingham Francis was born at Remsen, N. Y., on September 22, 1905, the son of Mr. and Mrs. George M. Francis. He was graduated from the local High School in 1923, and following some post-graduate work, entered Rensselaer Polytechnic Institute, at Troy, N. Y., choosing the four years' course in Civil Engineering.

After his graduation in 1928, Mr. Francis went with the Utica Gas and Electric Company as a Computer. The spring of 1929 found him a member of the Maintenance Survey Corps of the New York Central Railroad Company, at Weehawken, N. J., where he seemed to find the work he enjoyed most.

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[1] Memoir prepared by C. E. Chase, Asst. Div. Engr., The N. Y. C. R. R., Weehawken, N. J.

It was a combination of field measurements on tracks and bridges, buildings, and docks, with farm drainage and topographical surveys, from which the data are taken into the office for compilation, design, and maintenance reconstruction.

Although he had not passed his thirtieth year, he was already a man of breadth and of foresight—serious-minded and conscientious to a marked degree. He seemed bent on leaving no stone unturned to acquire the studious habits of thought, observation, and performance which would mould his career into the form of the greatest constructive value to himself and to others.

Industrial upsets of 1930 and the following years threatening to engulf young men before they had started, seemed to make but a small dent in the determination of "Joe" Francis to face the cold facts and come through in spite of them. He reasoned that it was no fault of his, and he pushed on without the spur of the extra financial appreciation which previous generations of young graduates had expected and experienced. He planned ahead and worked for the day when the young man worth while would come into his own, and he did not relax in the effort as did many of his discouraged contemporaries. He never faltered, but carried well the burden of uncertainty. His faithfulness in the performance of the more important duties was matched only by his devotion to the demands of the minor projects.

In his life outside his profession, Joseph Francis had been very active. At every turn he had joined hands with others to help brighten their prospects and pave a smoother way. No wastrel of time or of opportunities to aid, he may well be assumed to have done his level best and succeeded. His memory is held in the highest esteem by those whose lives touched his.

Associates and co-workers testify to the vitality of their colleague, and to the extraordinary alertness of his mind, acquisitive and inquisitive in all the best senses of the words. Untiring energy, a buoyant spirit, and a cheerful smile were his trade symbols, added to a healthy personality and an honest effort to elevate others to his plane of thought—that was Joseph Francis. His early death seemed incompatible with the wholesomeness of his life.

In addition to his parents, he is survived by two sisters, several other near relatives, and a host of friends to mourn his passing.

Mr. Francis was elected a Junior of the American Society of Civil Engineers on October 26, 1931.

FRANCIS ALBERT LANDRIEU, Jun. Am. Soc. C. E.[1]

Died August 9, 1935

Francis Albert Landrieu, the son of Victor Firmin and Cerintha C. (Mackey) Landrieu, was born at New Orleans, La., on April 10, 1904. He received his early education in St. Philip School and the Warren Easton High School, at New Orleans, and, in 1925, was graduated from Tulane Univversity, at New Orleans, with the degree of Bachelor of Engineering in Civil Engineering.

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[1] Memoir prepared by W. T. Hogg, M. Am. Soc. C. E., and A. F. Jacobi, Assoc. M. Am. Soc. C. E.

Mr. Landrieu was employed from June, 1925, to November, 1928, as Draftsman for the Board of Commissioners of the Port of New Orleans. From November, 1928, to September, 1930, he was a Designing Draftsman in the Engineering Service Division of the United States Navy, at its Portsmouth (Va.) Yard, and from September, 1930 to February, 1931, he was again employed by the Board of Commissioners of the Port of New Orleans, as a Structural Designer.

From February, 1931, until his death in August, 1935, he was employed in the offices of the United States Engineers at New Orleans and at La Crosse, Wis. In the First New Orleans District, he served as Assistant Engineer in charge of bridge construction on the Intercoastal Canal from February, 1931, to October, 1933, when he was transferred to the St. Paul, Minn., District, where he was engaged as Assistant Engineer on lock and dam construction at Dam No. 8, on the Mississippi River below La Crosse, Wis. On August 1, 1935, just a few days prior to his death, Mr. Landrieu was promoted to the rank of Associate Engineer. His knowledge of harbor structures, his exceptional ability as a structural designer, and his conscientious devotion to his work, made his career as a structural engineer very promising, and he should have attained a prominent place in his chosen profession, had his life been spared.

He was married in 1926 to Ethel Ann Sinclair, of New Orleans, who was drowned with Mr. Landrieu when the approach span of the bridge across the Mississippi River, at La Crosse, collapsed after the end-post of the truss had been struck by a car in which Mr. and Mrs. Landrieu were passengers. They are survived by a son, Francis Albert, Jr., and a daughter, Charmigne Ann Landrieu.

His disposition and pleasing personality endeared him to all those with whom he came in contact, and his death is sincerely regretted by his many friends and associates.

Mr. Landrieu was elected a Junior of the American Society of Civil Engineers on October 1, 1926.

PHILIP LANAHAN WELKER, Jun. Am. Soc. C. E.[1]

DIED DECEMBER 16, 1935

Philip Lanahan Welker was born in Baltimore, Md., on June 23, 1905, the son of Philip A. and Gertrude (Lanahan) Welker. After his preparatory education, he entered Cornell University, at Ithaca, N. Y., from which he received the degree of Civil Engineer upon his graduation in 1928.

In June, 1929, Mr. Welker was appointed to an engineering position in the office of the District Engineer of the Corps of Engineers, United States

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[1] Memoir prepared by William Bowie, M. Am. Soc. C. E.

Army, at New Orleans, La. He served with that organization until January, 1930, having been engaged on various classes of field work, notably as an Inspector on the Willow Grove Levee, the Lucy Goldmine Levee, and the Convent Levee, and as Inspector, Instrumentman, and Chief of Party on the Bonnet Carré Spillway, at Sellers, La.

Shortly after resigning from the office of the District Engineer at New Orleans, he became associated with the Sanford and Brooks Company, Engineers and Contractors, of Baltimore, Md., Norfolk, Va., and Charleston, S. C. He was engaged as a Field Engineer on a number of projects, including the Pennington Avenue Bridge, over Curtis Creek, at Baltimore, Canton Railroad Pier No. 11, for the Baltimore Mail Steamship Company, the Nanticoke River Bridge, at Vienna, Md., the plant of the Arundel-Brooks Concrete Corporation, on the Patapsco River, and the construction of the Bohemia River Bridge, in Maryland. In addition to his field work, Mr. Welker spent some time in the office of Sanford and Brooks on designs and estimates. He severed his connection with that firm on January 1, 1933.

From December, 1933, to March, 1934, he served as a Junior Civil Engineer on a Civil Works Administration project at the District of Columbia Penitentiary, at Lorton, Va., being engaged on the design of buildings, plumbing, heating fixtures, etc. In addition, he did the necessary field work, involving the layout and leveling, and also made maps showing the location of sewers, grades, etc.

On May 9, 1934, Mr. Welker became a member of the Division of Geodesy of the United States Coast and Geodetic Survey, and was engaged in making computations of the field observations secured in triangulation, leveling, and geodetic astronomy. For a brief period he served in the field as a Recorder and Observer on a triangulation party operating in the vicinity of the District of Columbia.

Because of his marked proficiency in the various classes of geodetic work involving field observations, mathematical computations, and adjustments of observations, he was one of the two engineers selected by the Director of the Coast and Geodetic Survey to carry on triangulation along the Guatemala-Honduras Boundary. The assignment of these two engineers was requested through the State Department by Sidney H. Birdseye, Chief Engineer on the Guatemala-Honduras Boundary Commission. After a furlough in January, 1935, from the Coast and Geodetic Survey, Mr. Welker proceeded to Guatemala, and reported for duty to Lieut. Joseph P. Lushene, the other engineer of the Coast and Geodetic Survey assigned to the Guatemala-Honduras triangulation.

Lieut. Lushene's report on the field operations carried on in Guatemala rated Mr. Welker very highly in professional ability, leadership, and personal qualities. The field work on the triangulation along this boundary involved the measurement of two base lines and the determination of the astronomical longitude and latitude at each of two of the triangulation stations near the ends of the arc.

Lieut. Lushene returned to the Washington, D. C., Office of the Coast and Geodetic Survey, but Mr. Welker was retained by the Boundary Commission to make all the computations and adjustments involved in the arc of triangulation.

He was engaged on this work until late in the summer of 1935, when a throat infection required treatment at a local hospital. The throat infection apparently subsided, or became less active, and it was believed that he was on the road to speedy recovery. However, within a short time, he had a relapse which so alarmed his wife who was with him and the attending physicians that he was placed aboard a steamer sailing for New Orleans. Upon arrival, he was taken immediately to the Touro Infirmary where he received the very best of medical attention. In spite of all that could be done, he passed away on Decmber 16, 1935. His body was sent to Baltimore, for burial in Greenmount Cemetery.

Mr. Welker's untimely death was a great shock to his many friends and associates who considered him a young engineer of great promise, and who admired him for his many attractive personal qualities and high sense of honor. In spite of his having been engaged in Geodetic Surveying only a comparatively short time, he mastered some of the most intricate phases of it. It is noteworthy that he majored in Geodesy during his course at Cornell University. He failed to secure, upon graduation, an appointment to the field force of the Coast and Geodetic Survey, for which he had applied, because of a defect in his sight which required the use of glasses. Since much of the service of a field engineer of the Coast and Geodetic Survey must be given aboard surveying ships, practically perfect eyesight is required on the part of those receiving appointments.

Mr. Welker was a member of the Society of American Military Engineers and the Engineers Club of Baltimore.

He was married to Flora Mary, of New Orleans, in December, 1929, who survives him. He is also survived by his mother, Mrs. Gertrude Lanahan Welker.

Mr. Welker was elected a Junior of the American Society of Civil Engineers on May 12, 1930.

CHARLES GUY ELDREDGE, Affiliate, Am. Soc. C. E.[1]

DIED NOVEMBER 9, 1933

Charles Guy Eldredge was born in Sabula, Iowa, on January 10, 1886, the son of Charles Guy Eldredge and Mary Elizabeth Eldredge. He was graduated from Cornell College, at Mt. Vernon, Iowa, in 1908, where he had specialized in inorganic research and water purification.

He began his professional career as Emergency Chemist with the Clinton Sugar Refining Company, of Clinton, Iowa, under Dr. A. P. Bryant, Chief Chemist. In 1909 and 1910, Mr. Eldredge served as Chemist of the Alturas

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[1] Memoir prepared by John B. Hawley, M. Am. Soc. C. E.

Gold Mining Company, at Oakland, Calif., and in 1910 and 1911, he was Chemist in Charge of the Water Purification Plant, at South Pittsburgh, Pa. As Chemist in Charge of the Water Purification Plant, he served the City of Fort Worth, Tex., from 1912 to November, 1913. From November, 1913, to April 15, 1914, he was in the Canal Zone in charge of the Purification Plants for the Isthmian Canal Commission.

Returning to Fort Worth Mr. Eldredge again took up the duty of Chemist in Charge of the Water Purification Plant there, until 1918. During the World War, from January, 1918, to May, 1919, as First Lieutenant, Sanitary Corps, U. S. Army, he was in charge of several water purification plants of the American Expeditionary Force, at Base Section No. 1, in Western France. After his discharge from the Army he returned again to Fort Worth, where from November, 1919, to June, 1921, he served as Chief Chemist of the Chemical Department of Terrell's Laboratories. Mr. Eldredge then moved to Vacaville, Calif., where from 1921 to 1931 he devoted himself to horticulture and to chemical work for various public works concerns, and municipal plants.

Guy Eldredge, a modest, retiring, taciturn man, a true scientist, skilled to near perfection in the details of his chosen vocation, was respected by all who knew him, and warmly beloved by all his associates and intimates.

A pertinent anecdote will not be amiss: Sometime in the summer of 1918, after much correspondence and lavish cutting of "red tape", as to the "loan" of Scientist Guy Eldredge to the writer's "Section" (in order that he might take charge of the various filtration and sewage disposal plants in Brittany, or "Base Section Number One"), this writer and the famous war correspondent, Peter Clark MacFarlane, returning together from an inspection trip on water and sewer works in the "Section", were walking up the principal thoroughfare of St. Nazaire, when the writer incontinently left "Mac" and started on a keen, undignified (for a Field Officer) run, toward a tall, square-shouldered silhouette that appeared under a street light. When MacFarlane caught up, he found the writer tightly hugging the subject of this memoir. The distinguished correspondent, Lieutenant Eldredge, and the Field Officer in charge of Water Supply and Sewerage then dined together (and "Mac" got fond of Eldredge during the meal). "Mac" went to his hotel, and the Field Officer took Lieutenant Eldredge to his own camp quarters, not merely for the night, but "for the duration of the war", as Army parlance has it.

The intimate contact during that war period, cumulative of several years theretofore, fortifies the writer in paying this posthumous tribute to a good citizen, a devoted husband and kind father, a splendid scientist, an efficient operator of all public works entrusted to his care, and an Army officer without blemish on his record.

On April 19, 1914, Mr. Eldredge was married to Esther Sharpe, of Vacaville, Calif. The result of their ideal union is three children: Angie Beth, Charles, and Doris, who, with his widow, survive him.

Mr. Eldredge was elected an Affiliate of the American Society of Civil Engineers on June 3, 1915.

SAMUEL MAGEE GREEN, F. Am. Soc. C. E.[1]

Died April 25, 1936

Samuel Magee Green was born in Philadelphia, Pa., on August 9, 1851, the son of John and Mary (Pugh) Green. He received his education at Stephen Girard Institute (later, Girard College) at Philadelphia, having been a member of the Class of 1866.

In 1866 when he was 15 years of age, Mr. Green went to Milwaukee, Wis., where, in 1870, he became Manager of the railway equipment firm of Pierce and Whaling.

In 1872, Mr. Green entered the railroad contracting business with the late Stephen A. Harrison F. Am. Soc. C. E., under the firm name of Harrison and Green, with an office in Milwaukee. This firm constructed a large part of the lines of the Wisconsin Central, Northwestern, and "Soo" Railroads in Wisconsin and Minnesota. The firm also built and owned the first horse-car street railway in Milwaukee.

Mr. Green became greatly interested in gold, silver, and copper mining in the late Eighties and early Nineties, and with Mr. Otto Mears built and owned the Durango and Leadville Railroad, in Colorado. They also constructed the original Durango Smelters, at Durango, Colo., now owned by the American Smelting and Refining Company.

During this period, and until 1900, Mr. Green was actively engaged in the real estate business in Milwaukee. He was also interested in a Foundry and Furnace Company in that city.

In 1901, he again entered the railroad contracting business and built the Herrs Island Stock Yards, at Pittsburgh, Pa., for the joint use of the Pennsylvania Railroad Company and the Pittsburgh Provision and Packing Company. From 1904 to 1908, he constructed the extension of the Monongahela Railroad from Ache Junction to Brownsville, Pa., and jointly with the Crossan Construction Company, the extension of the same railroad from Brownsville to Rices Landing, Pa.

Immediately after the completion of this work, Mr. Green became a member of the firm of Green and Read. This firm constructed a street railway line for the Irwin Traction Company, at Irwin, Pa.

During 1908 and 1909, Mr. Green constructed a part of the Long Island Railroad, from Jamaica to Far Rockaway, Long Island, N. Y., and in 1910 and 1911, he built an extension of the Chicago, Milwaukee, and St. Paul Railroad, at Dedham, Iowa. This latter line was the last railroad contracting work that he undertook.

From 1911 until his retirement from active work in 1921, Mr. Green was engaged in the real estate business in Milwaukee.

He was married on April 25, 1872, to Harriet L. Harrison, of Milwaukee, and is survived by his widow and three sons, Harrison S., Gardiner D., and

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[1] Memoir prepared by Raymond W. Green, Esq., Oconomowoc, Wis.

Raymond W., and two daughters, M. Emma, and Bessie L. Green. He died on their sixty-fourth wedding anniversary.

He was well known in Milwaukee and was one of three surviving life members of the Milwaukee Elks Club. He was a former Commodore of the Oconomowoc (Wis.) Yacht Club.

Mr. Green was elected a Fellow of the American Society of Civil Engineers on September 5, 1888.

HENRY CODDINGTON MEYER, F. Am. Soc. C. E.[1]

DIED MARCH 27, 1935

Major Henry Coddington Meyer, Civil War Veteran, Editor, and Pioneer in Sanitary Engineering, died at his home in Montclair, N. J., on March 27, 1935, after a short illness. He was 91 years old.

Henry Coddington Meyer, the son of Meyer Henry and Anne Maria (Price) Meyer, American citizens temporarily residing in Europe, was born in Hamburg, Germany, on April 14, 1844. His maternal grandfather, Eliphilet Price, was a Presbyterian clergyman who held a pastorate at Wappingers Falls, N. Y., while his father, a native of Denmark, was a merchant in New York City. He received his training in private schools at Montclair, N. J., and at Tarrytown and Yonkers, N. Y., and then entered the office of his uncle, T. B. Coddington, a manufacturer of New York City.

At the outbreak of the Civil War his parents would not allow him to enlist and he could not do so until he became eighteen years of age. In 1862, he enlisted in the Second New York Cavalry, known as the Harris Light, and served through most of the Civil War, emerging with the rank of Major and a wound from which he never entirely recovered.

During the war he served with his regiment through Pope's Campaign until after the Battle of Fredericksburg and, in 1863, received a saber wound at Brandy Station. In February, 1864, he was commissioned a Second Lieutenant in the Twenty-fourth New York Cavalry, and, later, was made a Captain, in the Wilderness Campaign. He saw service in the Battles of the Wilderness, Fredericksburg, Gettysburg, Spottsylvania, North Anna River, Cold Harbor, and the first assault on the works of Petersburg, Va., on June 17. It was during this engagement that he was badly wounded. After an assault on the enemy's works in which the Union forces were repulsed, it was reported to Captain Meyer that an officer of his Company had been shot. Captain Meyer went back over the battle-field in which corn had grown to the height of about a foot, to render assistance. He found the man desperately wounded and lying face down in the dirt. He turned him over, brushed the dirt from his mouth and nose, propped up his body on a corn hill, and left him. Captain Meyer was unable to carry the officer as he had been suffering from intermittent fever and was too weak to do so. As he was about to leave him and return to the Union breastworks, he was shot in the back, the bullet passing through his saber belt and up through his body to his shoulders. Captain Meyer crawled back to the Union lines under fire from the enemy. On reaching the breastworks, he was too weak to crawl over them and called out for help. Two soldiers jumped to the top of the breastworks, grabbed Captain Meyer by the arms and hauled him to a place of safety. As a result of his wound he spent eleven months in a hospital, and did not recover until some time after the war was over. Shortly after he was wounded Captain Meyer was brevetted Major and was subsequently discharged for disability.

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[1] Memoir prepared by Henry C. Meyer, Jr., New York, N. Y.

For this deed, Major Meyer was awarded the Congressional Medal of Honor for "distinguished gallantry in action," the citation reading: "During an assault on the enemy's works, this officer rendered heroic assistance to a helpless brother officer in the face of a heavy fire, thereby saving his life, and in the performance of this gallant act sustained a severe wound."

After his recovery, Major Meyer returned to the office of his uncle, in New York City, then a leader in the New York Shot and Lead Company, manufacturers of shot and lead pipe. In 1868, he founded the Henry C. Meyer Company to deal in plumbing, and gas and steam fixtures, which later became the Meyer-Sniffen Company. In 1883, he gave up active connection with the concern, maintaining, however, his stock holdings and a position on the Directorate.

In 1877, Major Meyer and other members of his household suffered an attack of diphtheria. He subsequently traced the disease to the plumbing in his house, which had no vents for the escape of sewer gases. Becoming interested in the problem he devised a system of plumbing to remedy the defect. The next year, 1878, he founded *The Sanitary Engineer* which later became the *Engineering Record.* This was an authoritative publication in its field, and was designed specifically for the discussion of just such improvements for greater public health and comfort. One of its first projects was a competition for architects, in which plans for improving the type of tenement structures on the standard 25 by 100-ft lots were submitted. This contest led to a degree of public interest in the subject which resulted in the Tenement Act of 1879, a considerable improvement over the preceding regulations. This Act by the New York State Legislature restricted the area of the standard lot to be occupied by a building to 65% of the lot, instead of 90%, previously permitted. Major Meyer still stressed his first interest, that of improving plumbing and embodying these improvements in a series of regulations which would make them mandatory. His interest in this subject led to the Plumbing Laws of 1881, which, effective in New York State, were the first of the kind in the United States, and still serve as the basic structure on which such laws are drawn.

Major Meyer then turned to another important question of his day, the heating of railway cars by steam pipes from the locomotive. The railroad companies opposed him with the claim that it was impossible or impractical; he demonstrated that it could be done. In those days, railway cars were constructed of wood, and were heated by stoves. Some railway operators allowed their roads to deteriorate to the point where wrecks were frequent and disastrous. The wooden cars often were set on fire by overturned stoves, and the passengers died in the flames. Again, as in the instance of plumbing, New York State led the nation in legislation forbidding the use of stoves in cars, and Major Meyer was one of its principal advocates.

In 1884, Major Meyer published a book called "Water Waste Prevention." This work, and constant discussion of the necessity for water conservation, did much to minimize the prejudices met by water-works authorities along lines of eliminating waste.

The *Engineering Record* was under Major Meyer's supervisory Editorship for twenty-five years until, in 1903, it was sold to Mr. James H. McGraw. As the *Engineering News-Record*, it is still published by the McGraw-Hill Company.

Within a few years of his return from the Civil War, Major Meyer was married to Charlotte English Seaman. Mrs. Meyer died in 1915, and some time later, Major Meyer married her cousin, Mrs. Gertrude Seaman Merrill, Mrs. Meyer and two sons, Henry Coddington Meyer, Jr., and Francis Thurber Meyer, survive him.

He was a member of the Century and Union League Clubs, of New York City, and the Military Order of the Loyal Legion.

Major Meyer was elected a Fellow of the American Society of Civil Engineers on October 22, 1885.

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